Landmark-based Localization

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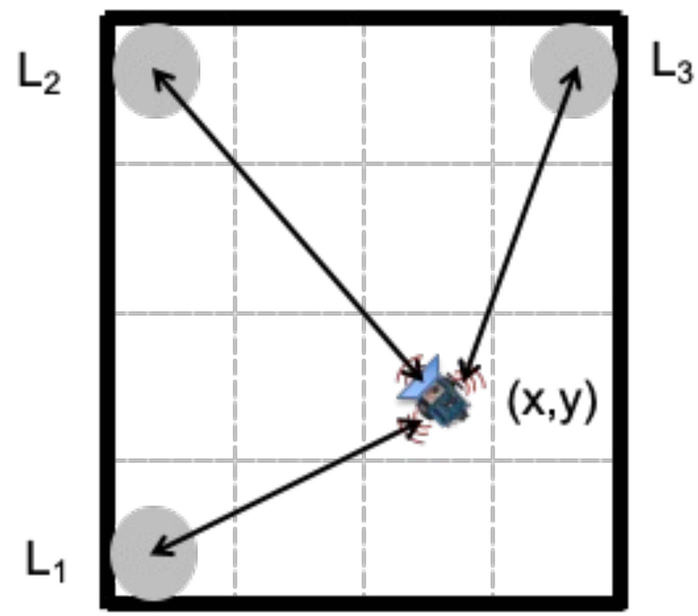
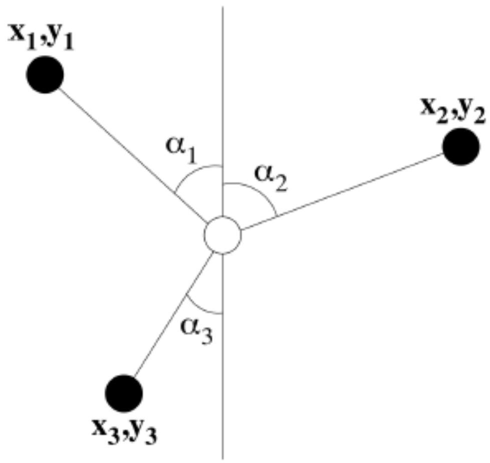
Landmark-based Localization

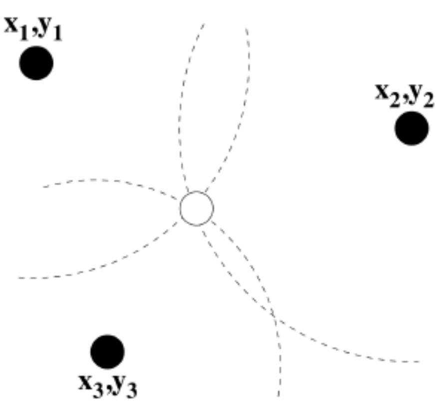
- Estimating location in 2D (or 3D) in relation to landmarks (L)
  - From measured ranges (distances)
  - From measured bearings (directions)
  - How to deal with measurement errors or noise?



Landmark-based Localization

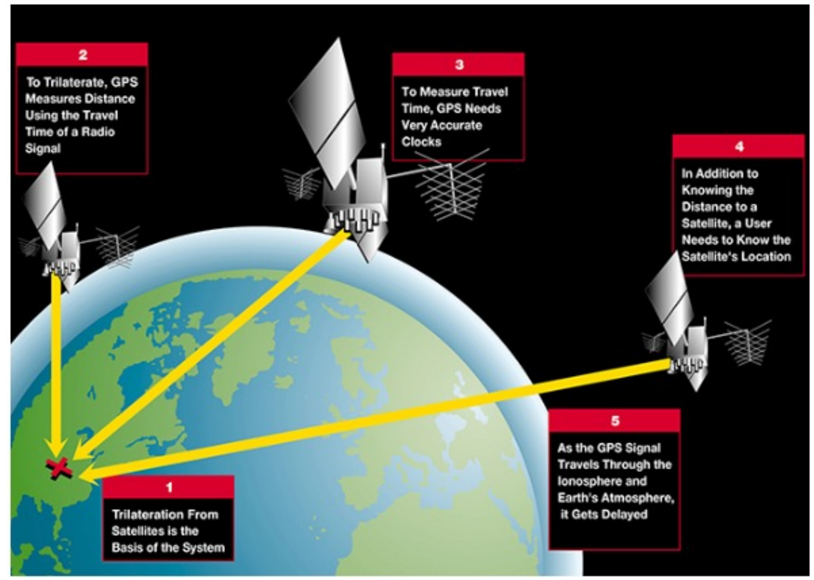
- Assume known location of anchor nodes (landmarks or beacons).
- Use the geometric properties of triangles (or circles) to estimate location
- Use angle (bearing) measurements
- Minimum of two bearing lines are needed for two-dimensional space





GPS

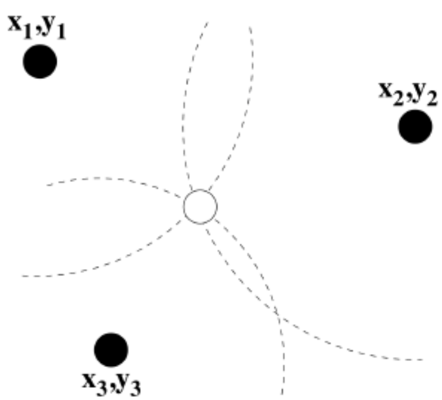
- Satellite Navigation is based on a global network of satellites that transmit radio signals from medium earth orbit based on 31 Global Positioning System (GPS) satellites developed and operated by the United States. Each of the 31 satellites emits signals from at least four satellites to determine their location and time.
- The basic GPS service provides users with approximately 7.0 meter accuracy, 95% of the time.
- GPS satellites carry atomic clocks that provide extremely accurate time. The receiver uses the time difference between the time of signal reception and the broadcast time to compute the distance, or range, from the receiver to the satellite.
- With information about the ranges to three satellites and the location of the satellite when the signal was sent, the receiver can compute its own three-dimensional position. However, by taking a measurement from a fourth satellite, the receiver avoids the need for an atomic clock. Thus, the receiver uses four satellites to compute latitude, longitude, altitude, and time.



https://www.faa.gov/about/office_org/headquarters_offices/ato/service_units/techops/navservices/gnss/gps/howitworks

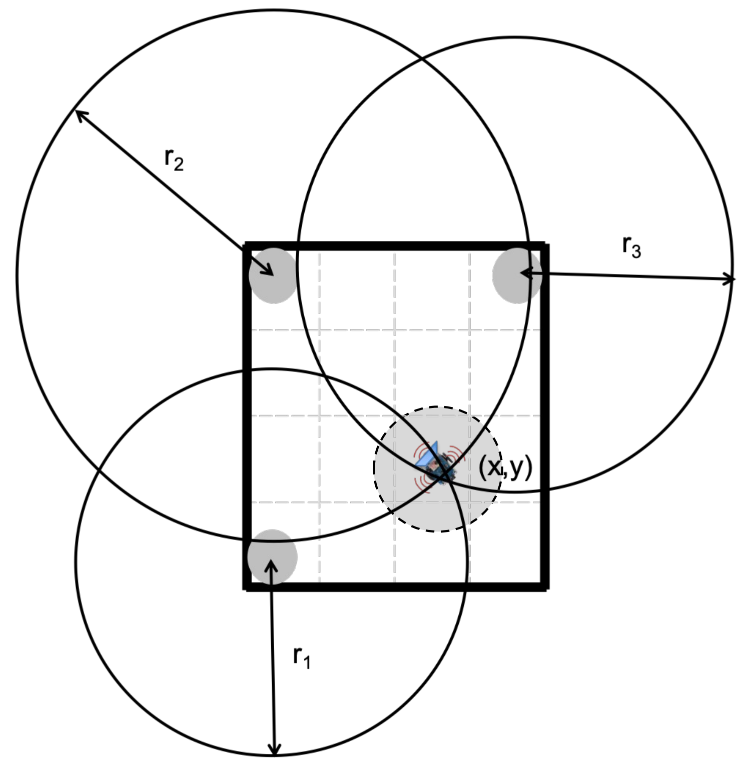
Trilateration

- Trilateration-based localization
  - Localization based on measured distances between a node and a number of anchor points with known locations
  - Basic concept: given the distance to an anchor, it is known that the node must be along the circumference of a circle centered at anchor and a radius equal to the node-anchor distance
  - In two-dimensional space, at least three non-collinear anchors are needed and in three-dimensional space, at least four non-coplanar anchors are needed



Trilateration

- In the case of unreliable angle calculations, robot localization (*x,y*) can be computed using trilateration methods based on the determination of absolute or relative measured distances to at least 3 known beacon or landmark locations.
- Intersection of 3 circles formed by the distances r_1, r_2, r_3, of the robot to the 3 beacons.
- Intersection at robot position may not be a single point since distance readings to beacons or landmarks may be noisy.
- If distance measurements are noisy, robot localization (*x,y*) will become imprecise, extending to a probability function (in gray).



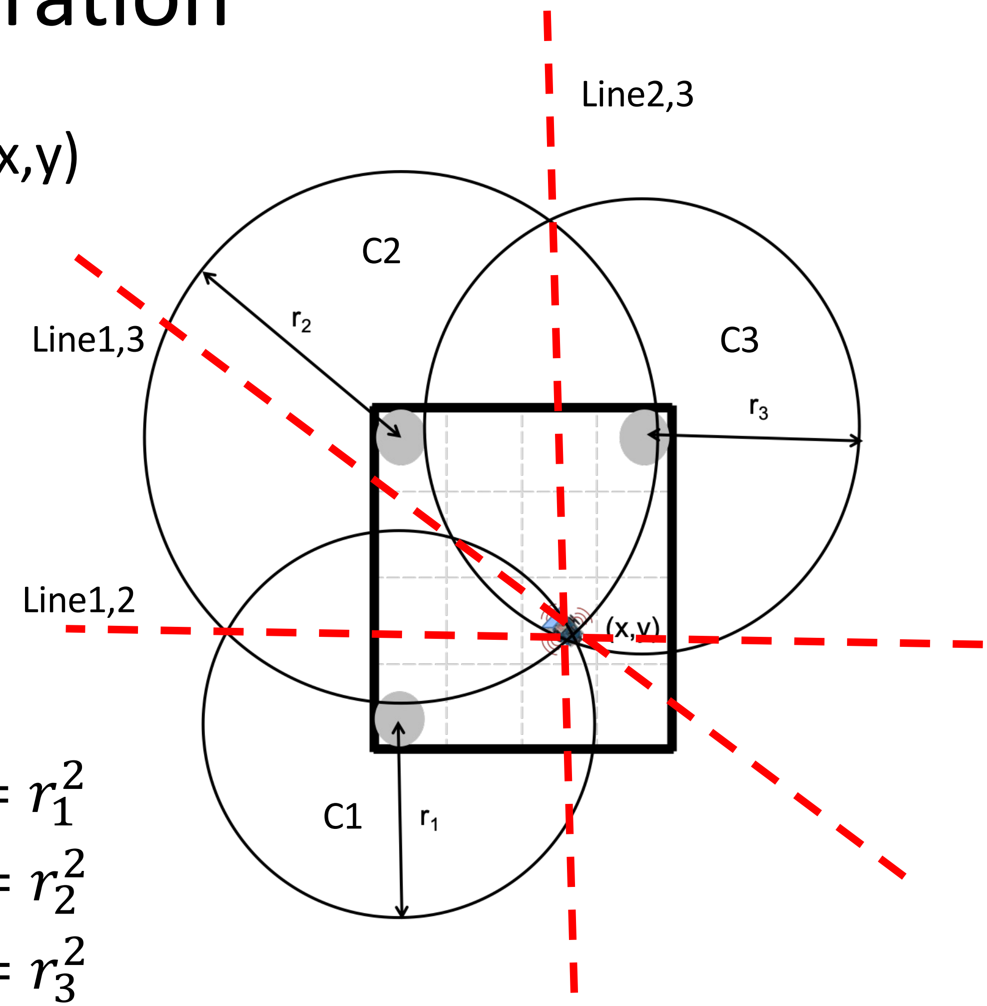
Trilateration

- Circles intersections to compute robot (x,y) location.
- The circles equations are given by:

$$C_1: (x - x_1)^2 + (y - y_1)^2 = r_1^2$$
$$C_2: (x - x_2)^2 + (y - y_2)^2 = r_2^2$$
$$C_3: (x - x_3)^2 + (y - y_3)^2 = r_3^2$$

- The expanded circle equations are:

$$C_1: x^2 - 2x_1 x + x_1^2 + y^2 - 2y_1 y + y_1^2 = r_1^2$$
$$C_2: x^2 - 2x_2 x + x_2^2 + y^2 - 2y_2 y + y_2^2 = r_2^2$$
$$C_3: x^2 - 2x_3 x + x_3^2 + y^2 - 2y_3 y + y_3^2 = r_3^2$$



Trilateration

- Subtracting two circle equations generates two lines:

$$C_1\text{-}C_2\text{: } (-2x_1 + 2x_2)\, x + (-2y_1+2y_2)\, y = r_1^2 - r_2^2 - x_1^2 + x_2^2 - y_1^2 + y_2^2$$

$$C_2\text{-}C_3\text{: } (-2x_2 + 2x_3)\, x + (-2y_2+2y_3)\, y = r_2^2 - r_3^2 - x_2^2 + x_3^2 - y_2^2 + y_3^2$$

- These two lines will intersect at (*x,y*) to generate an approximate robot location (without orientation information) given by the following two lines (note that a third line may generate a different (*x,y*) intersection due to trilateration computation imprecisions):

$$C_1\text{-}C_2\text{: } Ax+By=C$$

$$C_2\text{-}C_3\text{: } Dx+Ey=F$$

$$C_1\text{-}C_2\text{: } A=(-2x_1 + 2x_2),\ B=(-2y_1+2y_2),\ C= r_1^2 - r_2^2 - x_1^2 + x_2^2 - y_1^2 + y_2^2$$

$$C_2\text{-}C_3\text{: } D=(-2x_2 + 2x_3),\ E=(-2y_2+2y_3),\ F= r_2^2 - r_3^2 - x_2^2 + x_3^2 - y_2^2 + y_3^2$$

- Coordinate (*x,y*) is given by the intersection of the two lines (Note that there is an exception when EA=BD):

$$x=\frac{(\mathrm{CE}-\mathrm{FB})}{(\mathrm{EA}-\mathrm{BD})},\ y=\frac{(\mathrm{CD}-\mathrm{AF})}{(\mathrm{BD}-\mathrm{AE})}$$

Trilateration

- Assume landmarks: $x_1 = -2, y_1 = -2;\ x_2 = -2, y_2 = 2; x_3 = 2, y_3 = 2$
- Assume robot measurements to landmarks: $r_1 = 2.9, r_2 = 2.1, r_3 = 2.9$
- The circles equations are given by:

$$C_1: (x-(-2))^2+(y-(-2))^2 = (2.9)^2 = 8.4$$
$$C_2: (x-(-2))^2+(y-(2))^2 = (2.1)^2 = 4.4$$
$$C_3: (x-(2))^2+(y-(2))^2 = (2.9)^2 = 8.4$$

- The expanded circle equations are:

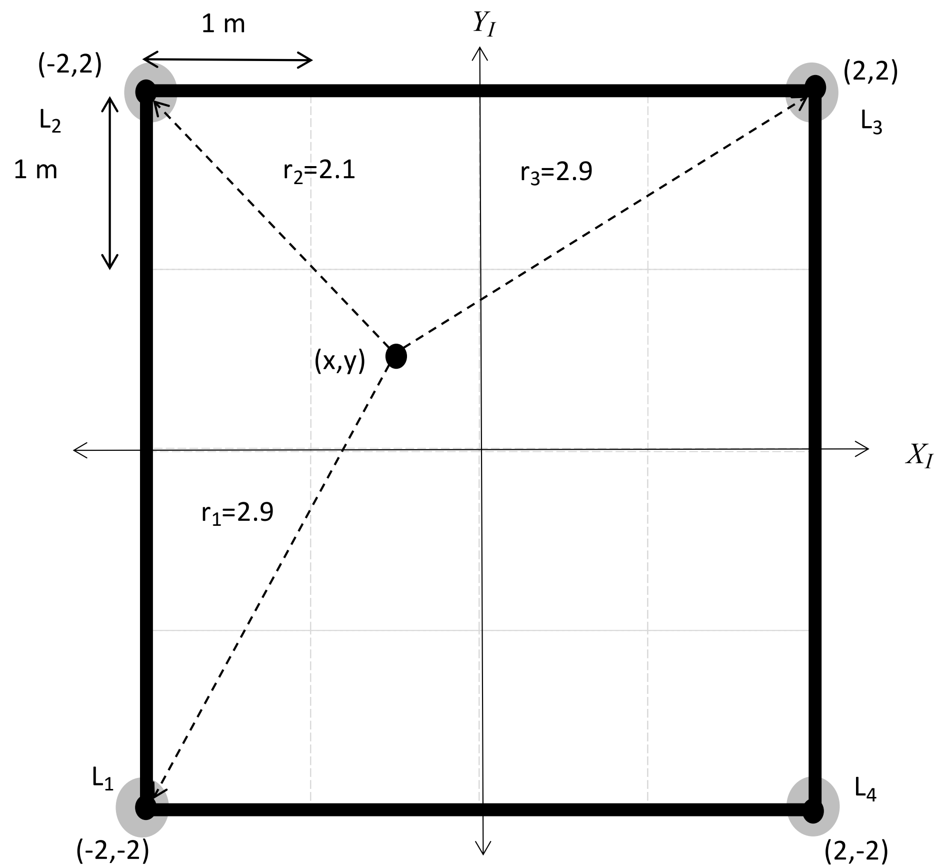
$$C_1\text{-}C_2: A=(-2x_1+2x_2), B=(-2y_1+2y_2), C= r_1^2-r_2^2-x_1^2+x_2^2-y_1^2+y_2^2$$
$$C_2\text{-}C_3: D=(-2x_2+2x_3), E=(-2y_2+2y_3), F= r_2^2-r_3^2-x_2^2+x_3^2-y_2^2+y_3^2$$

- The expanded circle equations are:

$$A=(-(-4)+(-4))=0, B=(-(-4)+4)=8$$
$$C= 8.4-4.4-4+4-4+4=4$$
$$D=(-(-4)+4)=8, E=(-4+4)=0, F=4.4-8.4-4+4-4+4=-4$$

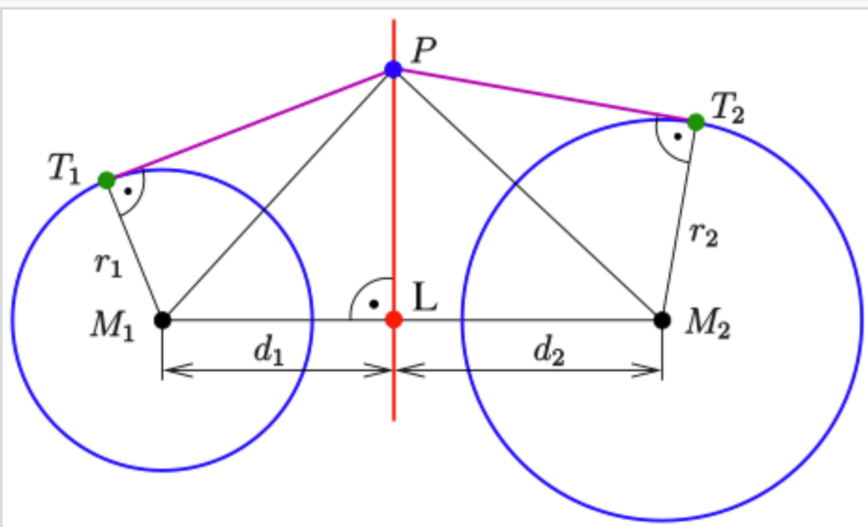
- Coordinate (*x,y*) is given by the intersection of the two lines (Note that there is an exception when EA=BD):

$$x=\frac{(CE-FB)}{(EA-BD)}=\frac{(4*0-(-4)*8)}{(0*0-8*8)}=\frac{32}{-64}=-0.5,\ y=\frac{(CD-AF)}{(BD-AE)}=\frac{(4*8-0*(-4))}{(8*8-0*0)}=\frac{32}{64}=0.5$$

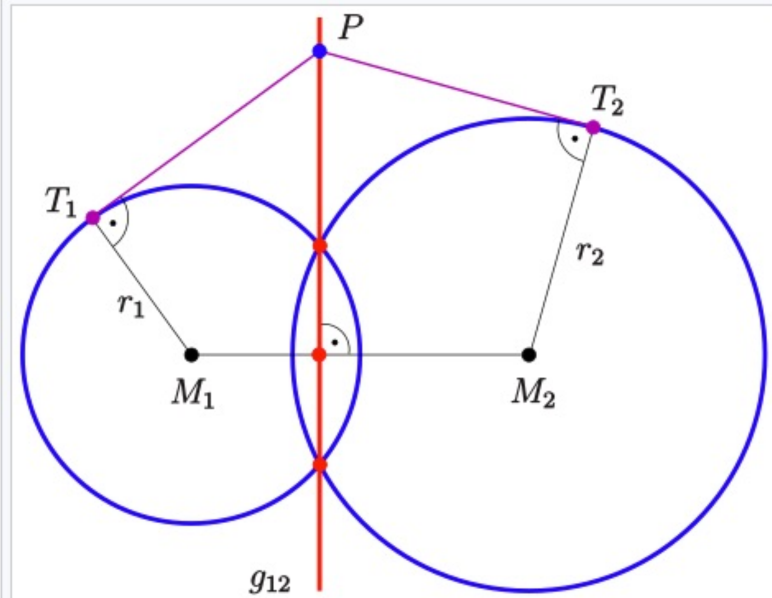


Radical Axis or Power Line

- The **radical axis** (**power line** or **power bisector**) of two non-concentric circles is the set of points whose *power* with respect to the circles are equal, i.e. a real number that reflects the relative distance of a given point from a given circle.



Definition and calculation of d_1, d_2

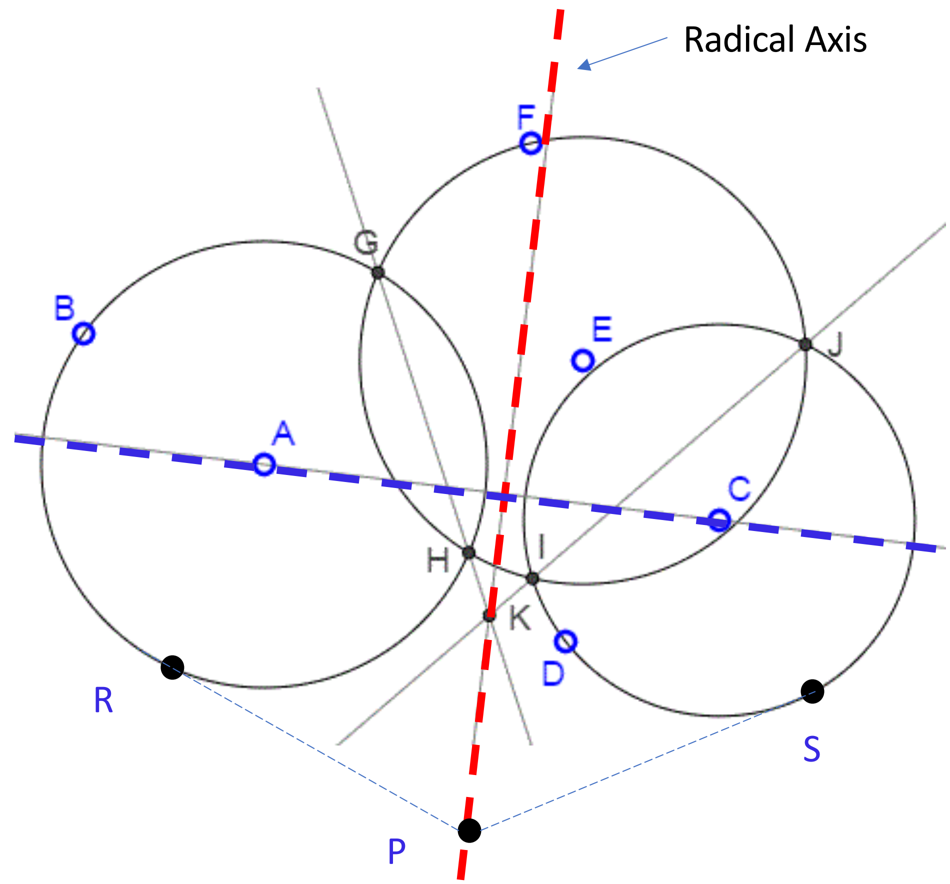


Two circles, centered at M_1, M_2
Radical axis, with sample point P
Tangential distances from both circles to P
The tangent lines must be equal in length for any point on the radical axis: $|PT_1| = |PT_2|$. If P, T_1, T_2 lie on a common tangent, then P is the midpoint of $\overline{T_1T_2}$.

Ref: https://en.wikipedia.org/wiki/Radical_axis

Radical Axis or Power Line

- Given two circles A(B) and C(D), with centers at A and C and passing through points B and D, respectively.
- The radical axis of two circles (in red) is the locus of points (P) from which the tangents to the two circles are equal (|PR|=|PS|).
- To construct a radical axis, draw any circle E(F) that intersects both A(B) and C(D).
- Intersection points are G,H, and I,J, respectively. Let K be the intersection of GH and IJ.
- The radical axis of A(B) and C(D) is (in red) the line through K perpendicular to (in blue) the line of centers AC.

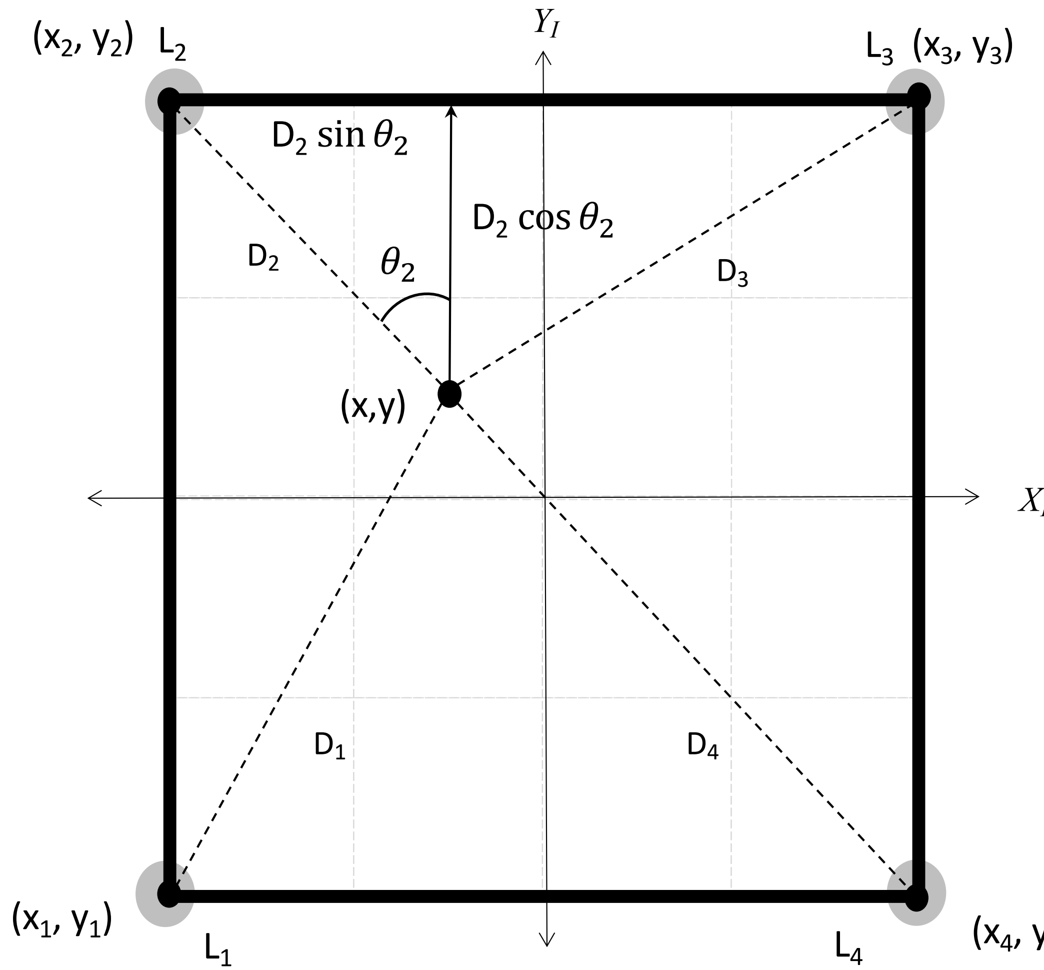


Triangulation

- Assume known distances D_i and angle bearings θ_i.
- Compute robot pose (x, y):

L$_1$: $x = x_1 + D_1 \cos(\theta_1)$
L$_2$: $x = x_2 + D_2 \cos(\theta_2)$
L$_3$: $x = x_3 + D_3 \cos(\theta_3)$
L$_4$: $x = x_4 + D_4 \cos(\theta_4)$

L$_1$: $y = y_1 + D_1 \sin(\theta_1)$
L$_2$: $y = y_2 + D_2 \sin(\theta_2)$
L$_3$: $y = y_3 + D_3 \sin(\theta_3)$
L$_4$: $y = y_4 + D_4 \sin(\theta_4)$



Trilateration

- Assume landmarks: $x_1 = -2, y_1 = -2;\ x_2 = -2, y_2 = 2; x_3 = 2, y_3 = 2$
- Assume robot measurements to landmarks: $r_1 = 2.9, r_2 = 3.0, r_3 = 2.1$
- The circles equations are given by:

$$C_1: (x-(-2))^2+(y-(-2))^2= (2.9)^2= 8.4$$
$$C_2:(x-(-2))^2+(y-(2))^2= (3.0)^2 = 9.0$$
$$C_3:(x-(2))^2+(y-(2))^2= (2.1)^2 = 4.4$$

- The expanded circle equations are:

$$C_1\text{-}C_2: A=(-2x_1 + 2x_2), B =(-2y_1+2y_2), C= r_1^2-r_2^2-x_1^2+x_2^2-y_1^2+y_2^2$$
$$C_2\text{-}C_3: D=(-2x_2 + 2x_3), E=(-2y_2+2y_3), F= r_2^2-r_3^2-x_2^2+x_3^2-y_2^2+y_3^2$$

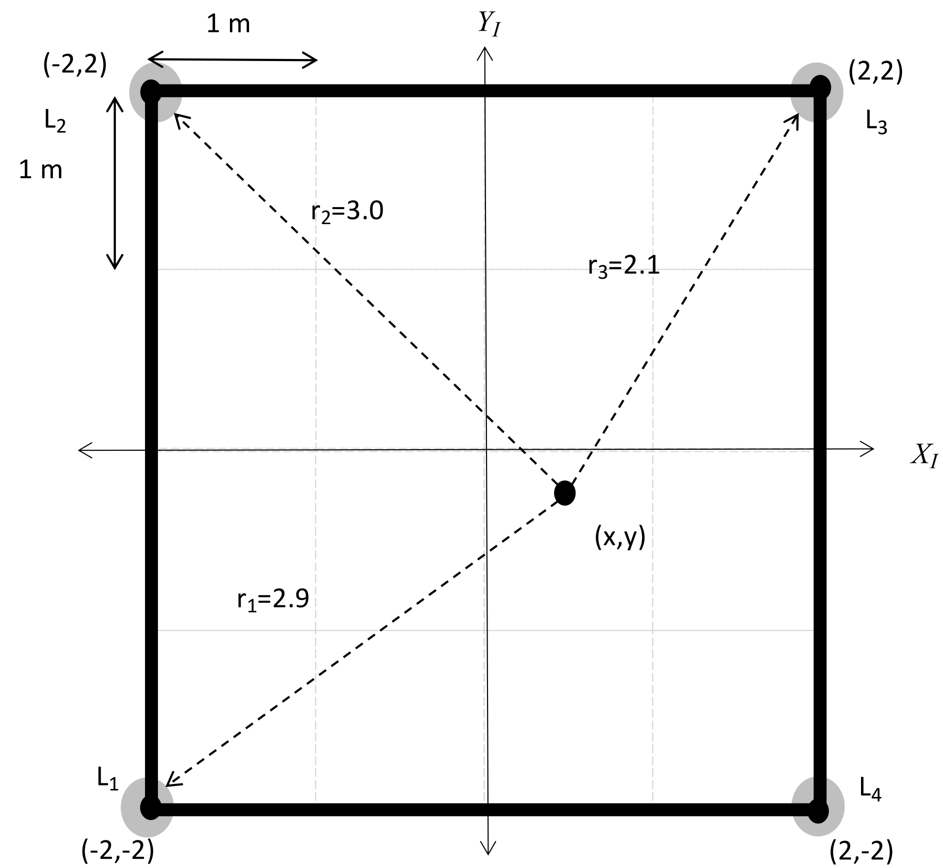
- The expanded circle equations are:

$$A =(-(-4) + (-4)) = 0, B =(-(-4) + 4) = 8$$
$$C= 8.4 - 9.0-4+4-4+4 = -0.6$$
$$D=(-(-4) + 4)= 8, E=(-4+4)=0, F= 9.0-4.4-4+4-4+4 = 4.6$$

- Coordinate (x,y) is given by the intersection of the two:

$$x=\frac{(CE-FB)}{(EA-BD)}=\frac{((-0.6)*0-(4.6)*8)}{(0*0-8*8)}=\frac{-36.8}{-64} = 0.575$$
$$y=\frac{(CD-AF)}{(BD-AE)}=\frac{((-0.6)*8-0*(4.6))}{(8*8-0*0)}=\frac{-4.8}{64} = -0.075$$

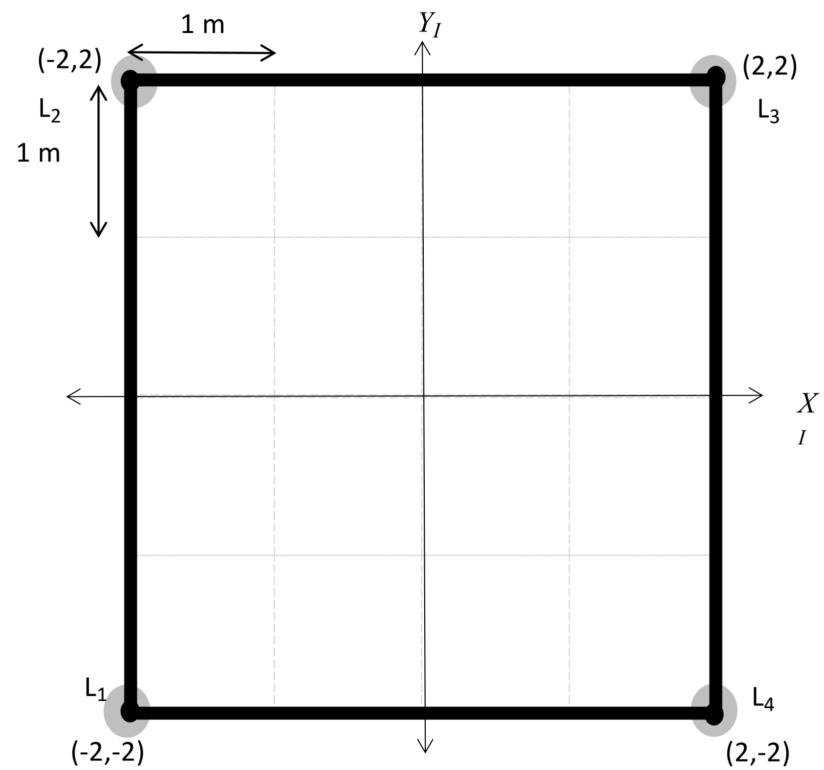
- Is this point on all 3 circles?



Localization with Landmarks

- What happens when measurements to landmarks are not precise, can we still use triangulation, i.e. p(s|z)?
- Assume the robot measures distances to each landmark as follows:

  $z_{R\text{-}L1} = 2.9$, $z_{R\text{-}L2} = 3.0$, $z_{R\text{-}L3} = 2.1$, $z_{R\text{-}L4} = 1.5$
- Where is the robot located?
- Can we compute instead the probability of the robot measurement given the cell, i.e. p(z|s)?
- Does the triangulation algorithm help obtain robot localization?

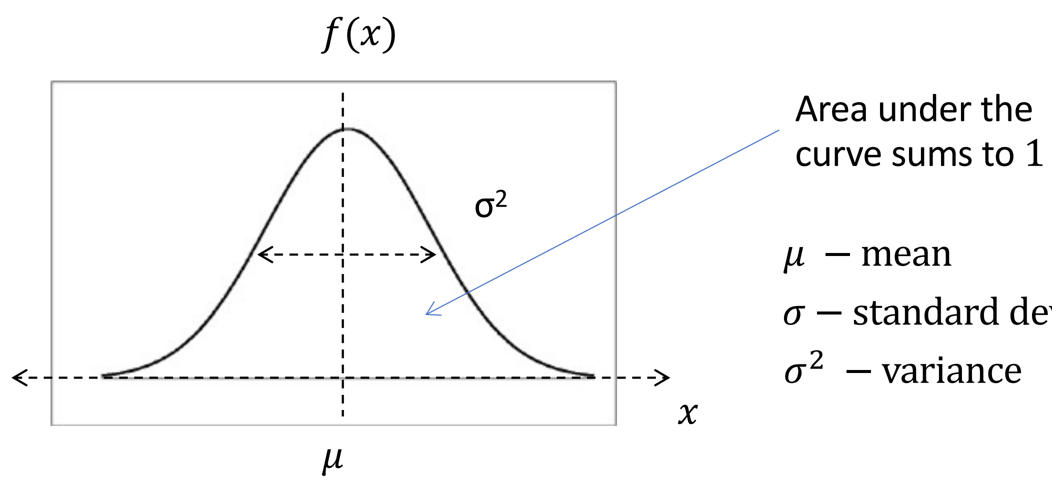


Localization with Landmarks
Normal Probability Distributions

- Normal Probability Distribution Function (NORMDIST in *Excel*):

  $f(x)$ = NORMDIST(x,μ,σ)
- $f(x)$ represents the normal probability distribution function, $\mathcal{N}(\mu, \sigma^2)$
- x represents the state variable
- μ (mean) represents the state with the highest probability value
- σ (sigma) represents the standard deviation of the probability distribution function

Note:
- Smaller $(x - \mu)$ results in higher probabilities.
- Highest probability is at $x = \mu$



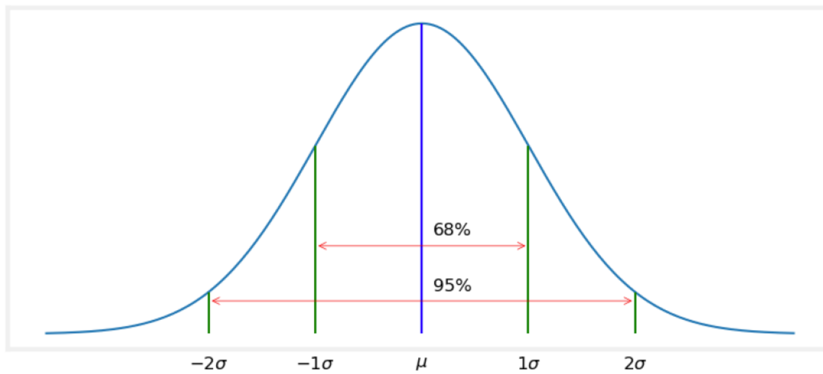
μ – mean

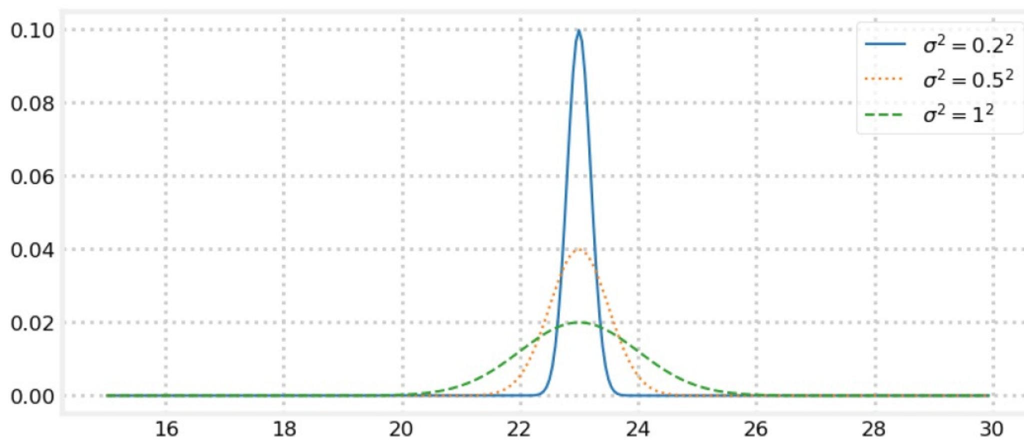
σ – standard deviation

σ^2 – variance

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \mathcal{N}(\mu, \sigma^2)$$

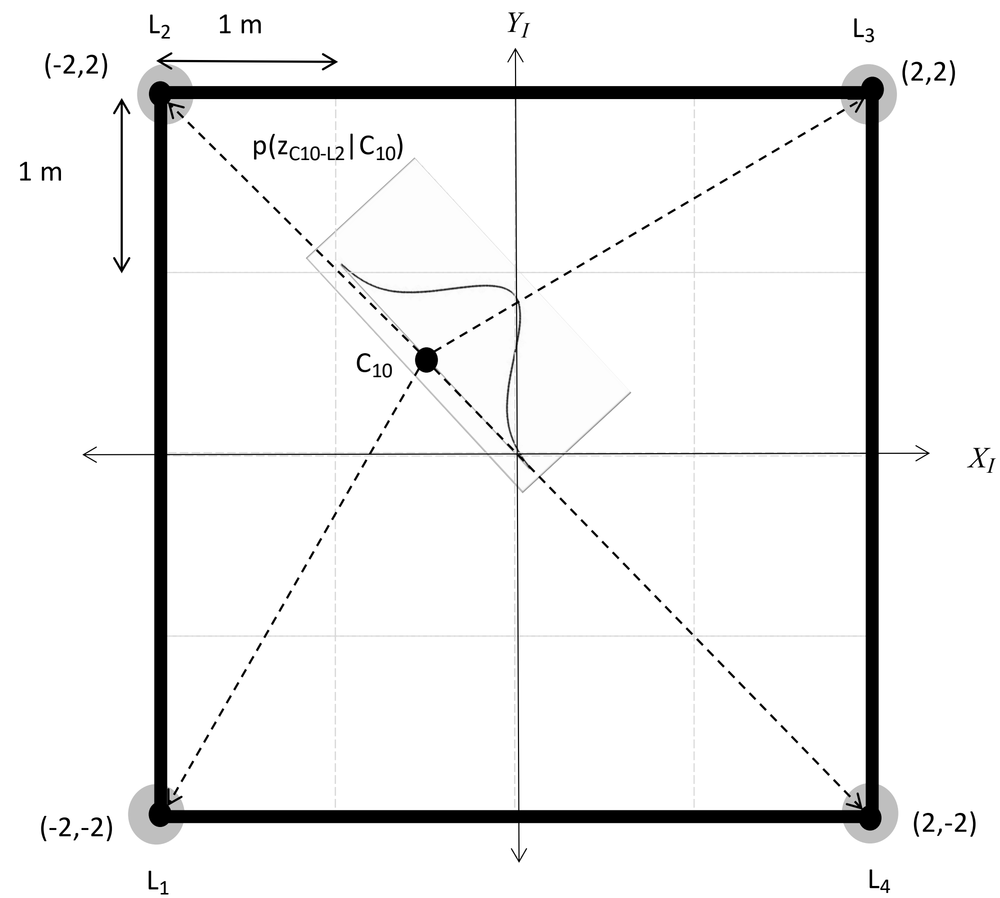
Probability Density Function (PDF)
(Gaussian, Normal Distribution)





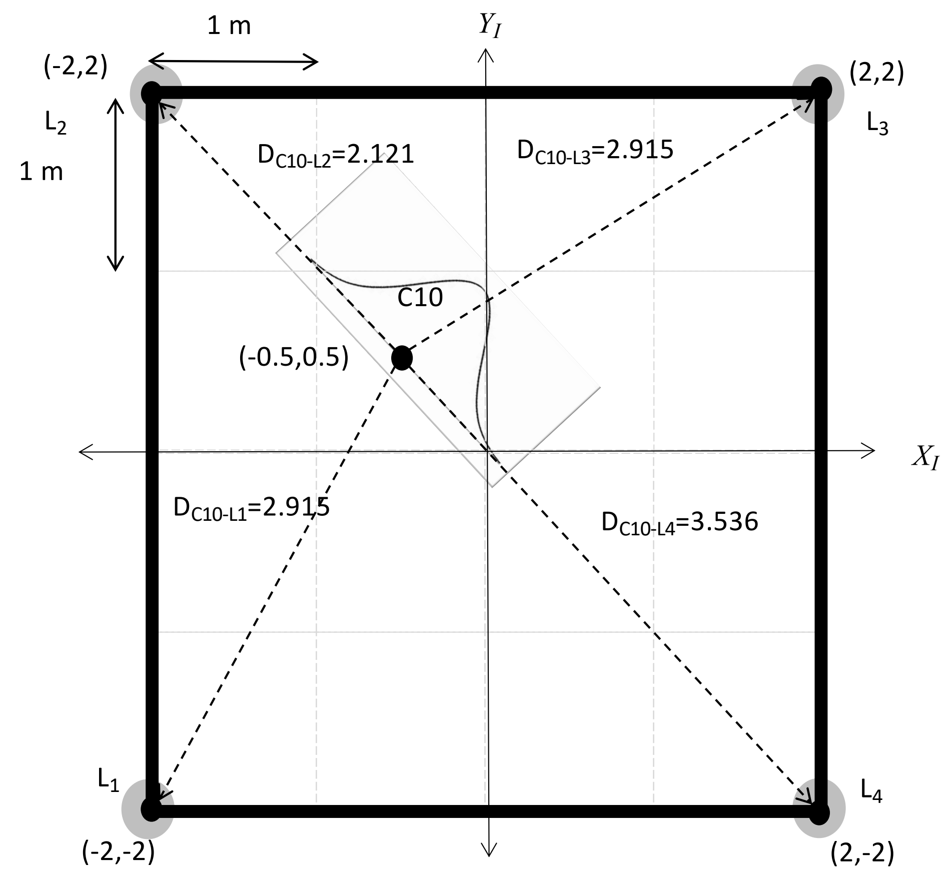
Localization with Landmarks

- What is the probability of robot being in C_{10} if the distance measured by the robot to each landmark is:
  $z_{L1} = 2.9$
  $z_{L2} = 3.0$
  $z_{L3} = 2.1$
  $z_{L4} = 1.5$
- Normal Probability Distribution Function (PDF) (NORMDIST):
  $$p(z|s = \mu) = \text{NORMDIST}(z,\mu,\sigma)$$
- Compute localization using a Normal distribution function for each measurement in relation to the center of state C_{10}.

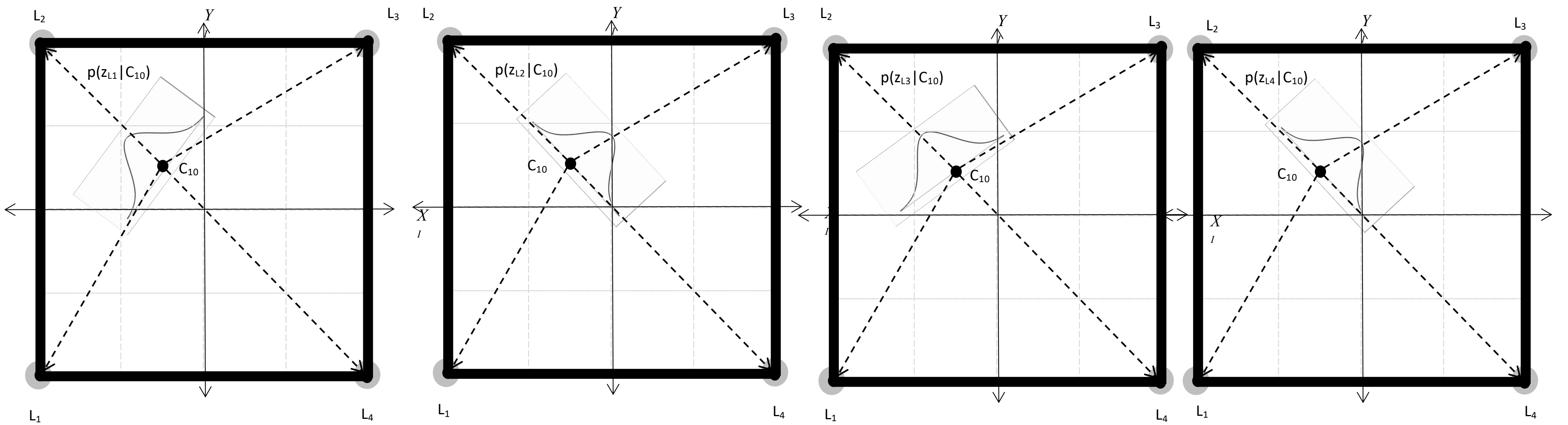


Localization with Landmarks

- What is the probability of robot being in C_{10} if the distance measured by the robot to each landmark is:
  $z_{L1} = 2.9$
  $z_{L2} = 3.0$
  $z_{L3} = 2.1$
  $z_{L4} = 1.5$
- Distance measurements to landmarks from cell C_{10}:
  $D_{C10\text{-}L1} = \text{sqrt}(1.5^2+2.5^2) = 2.915$
  $D_{C10\text{-}L2} = \text{sqrt}(1.5^2+1.5^2) = 2.121$
  $D_{C10\text{-}L3} = \text{sqrt}(2.5^2+1.5^2) = 2.915$
  $D_{C10\text{-}L4} = \text{sqrt}(2.5^2+2.5^2) = 3.536$
- Normal distribution for measurements in relation C_{10}:
  $N_{Z\text{-}C10\text{-}L1}$ = NORMDIST(2.9, 2.915, 1)=0.399
  $N_{Z\text{-}C10\text{-}L2}$ = NORMDIST(3.0, 2.121, 1)=0.271
  $N_{Z\text{-}C10\text{-}L3}$ = NORMDIST(2.1, 2.915, 1)=0.286
  $N_{Z\text{-}C10\text{-}L4}$ = NORMDIST(1.5, 3.536, 1)=0.050



Localization with Landmarks

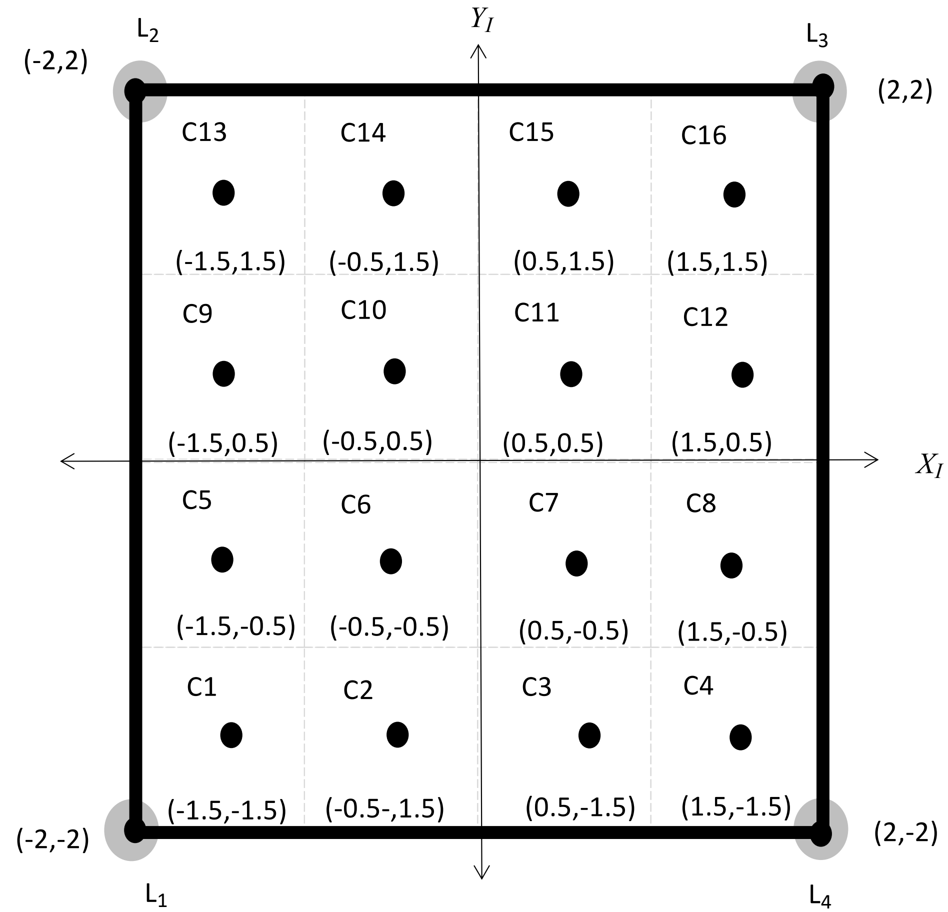


- The distance from C_{10} (-0.5,0.5) to each landmark is as follows:
  - $D_{C10\text{-}L1}$ = sqrt(1.5^2+2.5^2)=2.915
  - $D_{C10\text{-}L2}$ = sqrt(1.5^2+1.5^2)=2.121
  - $D_{C10\text{-}L3}$ = sqrt(2.5^2+1.5^2)=2.915
  - $D_{C10\text{-}L4}$ = sqrt(2.5^2+2.5^2)=3.536

- Normal probabilities for z in relation to C_{10} to each L_i:
  - $N_{Z\text{-}C10\text{-}L1}$ = NORMDIST(2.9, 2.915, 1)=0.399
  - $N_{Z\text{-}C10\text{-}L2}$ = NORMDIST(3.0, 2.121, 1)=0.271
  - $N_{Z\text{-}C10\text{-}L3}$ = NORMDIST(2.1, 2.915, 1)=0.286
  - $N_{Z\text{-}C10\text{-}L4}$ = NORMDIST(1.5, 3.536, 1)=0.050
- Weighted normal probabilities for C_{10} to L_i:
  - $W_{Z\text{-}C10}$=$N_{Z\text{-}C10\text{-}L1}$*$N_{Z\text{-}C10\text{-}L2}$*$N_{Z\text{-}C10\text{-}L3}$*$N_{Z\text{-}C10\text{-}L4}$=0.002

Localization with Landmarks

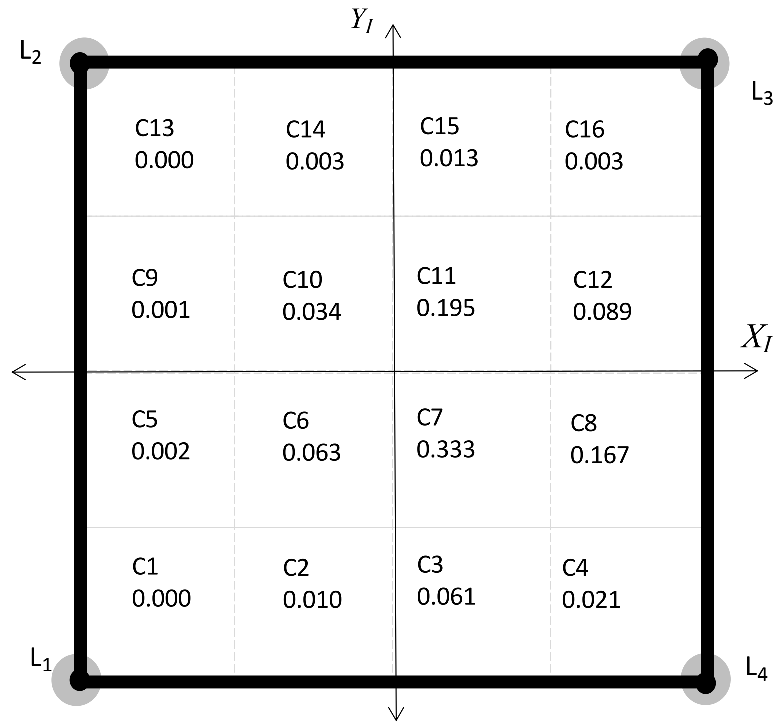
- What are the probabilities of the robot being in any of the cells if the distance measured by the robot to each landmark is:
  $z_{L1} = 2.9$, $z_{L2} = 3.0$, $z_{L3} = 2.1$, $z_{L4} = 1.5$
- Compute distances from the center of each cell to each landmark:
  $C_1(-1.5,-1.5)$, $C_2(-0.5,-1.5)$, $C_3(0.5,-1.5)$, $C_4(1.5,-1.5)$
  $C_5(-1.5,-0.5)$, $C_6(-0.5,-0.5)$, $C_7(0.5,-0.5)$, $C_8(1.5,-0.5)$
  $C_9(-1.5,0.5)$, $C_{10}(-0.5,0.5)$, $C_{11}(0.5,0.5)$, $C_{12}(1.5,0.5)$
  $C_{13}(-1.5,1.5)$, $C_{14}(-0.5,1.5)$, $C_{15}(0.5,1.5)$, $C_{16}(1.5,1.5)$
- Need to compute weighted probabilities for z with respect to all cells and landmarks.



Localization with Landmarks

- Normalized weighted probabilities for each cell:

  $W_{Z\text{-}C1}=N_{Z\text{-}C1\text{-}L1}*N_{Z\text{-}C1\text{-}L2}*N_{Z\text{-}C1\text{-}L3}*N_{Z\text{-}C1\text{-}L4}=0.000$
  $W_{Z\text{-}C2}=N_{Z\text{-}C2\text{-}L1}*N_{Z\text{-}C2\text{-}L2}*N_{Z\text{-}C2\text{-}L3}*N_{Z\text{-}C2\text{-}L4}=0.010$
  $W_{Z\text{-}C3}=N_{Z\text{-}C3\text{-}L1}*N_{Z\text{-}C3\text{-}L2}*N_{Z\text{-}C3\text{-}L3}*N_{Z\text{-}C3\text{-}L4}=0.061$
  $W_{Z\text{-}C4}=N_{Z\text{-}C4\text{-}L1}*N_{Z\text{-}C4\text{-}L2}*N_{Z\text{-}C4\text{-}L3}*N_{Z\text{-}C4\text{-}L4}=0.021$
  $W_{Z\text{-}C5}=N_{Z\text{-}C5\text{-}L1}*N_{Z\text{-}C5\text{-}L2}*N_{Z\text{-}C5\text{-}L3}*N_{Z\text{-}C5\text{-}L4}=0.002$
  $W_{Z\text{-}C6}=N_{Z\text{-}C6\text{-}L1}*N_{Z\text{-}C6\text{-}L2}*N_{Z\text{-}C6\text{-}L3}*N_{Z\text{-}C6\text{-}L4}=0.063$
  $W_{Z\text{-}C7}=N_{Z\text{-}C7\text{-}L1}*N_{Z\text{-}C7\text{-}L2}*N_{Z\text{-}C7\text{-}L3}*N_{Z\text{-}C7\text{-}L4}=0.333$
  $W_{Z\text{-}C8}=N_{Z\text{-}C8\text{-}L1}*N_{Z\text{-}C8\text{-}L2}*N_{Z\text{-}C8\text{-}L3}*N_{Z\text{-}C8\text{-}L4}=0.167$
  $W_{Z\text{-}C9}=N_{Z\text{-}C9\text{-}L1}*N_{Z\text{-}C9\text{-}L2}*N_{Z\text{-}C9\text{-}L3}*N_{Z\text{-}C9\text{-}L4}=0.001$
  $W_{Z\text{-}C10}=N_{Z\text{-}C10\text{-}L1}*N_{Z\text{-}C10\text{-}L2}*N_{Z\text{-}C10\text{-}L3}*N_{Z\text{-}C10\text{-}L4}=0.034$
  $W_{Z\text{-}C11}=N_{Z\text{-}C11\text{-}L1}*N_{Z\text{-}C11\text{-}L2}*N_{Z\text{-}C11\text{-}L3}*N_{Z\text{-}C11\text{-}L4}=0.195$
  $W_{Z\text{-}C12}=N_{Z\text{-}C12\text{-}L1}*N_{Z\text{-}C12\text{-}L2}*N_{Z\text{-}C12\text{-}L3}*N_{Z\text{-}C12\text{-}L4}=0.089$
  $W_{Z\text{-}C13}=N_{Z\text{-}C13\text{-}L1}*N_{Z\text{-}C13\text{-}L2}*N_{Z\text{-}C13\text{-}L3}*N_{Z\text{-}C13\text{-}L4}=0.000$
  $W_{Z\text{-}C14}=N_{Z\text{-}C14\text{-}L1}*N_{Z\text{-}C14\text{-}L2}*N_{Z\text{-}C14\text{-}L3}*N_{Z\text{-}C14\text{-}L4}=0.003$
  $W_{Z\text{-}C15}=N_{Z\text{-}C15\text{-}L1}*N_{Z\text{-}C15\text{-}L2}*N_{Z\text{-}C15\text{-}L3}*N_{Z\text{-}C15\text{-}L4}=0.013$
  $W_{Z\text{-}C16}=N_{Z\text{-}C16\text{-}L1}*N_{Z\text{-}C16\text{-}L2}*N_{Z\text{-}C16\text{-}L3}*N_{Z\text{-}C16\text{-}L4}=0.003$



Localization with Landmarks

- Since states are infinite, should we compute for infinite (x,y)?
- How do we compute localization if the four cylinders are non-unique (same color and size)?