Theoretical Assignment
DeepBayes Summer School 2018 (deepbayes.ru)

Markovich Alexander

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Problem 1. The random variable ξ has Poisson distribution with the parameter λ. If $\xi = k$ we perform k Bernoulli trials with the probability of success p. Let us define the random variable η as the number of successful outcomes of Bernoulli trials. Prove that η has Poisson distribution with the parameter $p\lambda$.

Solution

Let $\xi \sim \text{Pois}(\lambda)$ and $\eta \,|\, \xi \sim \text{Bin}(\xi, p)$. We need to prove that $\eta \sim \text{Pois}(p\lambda)$.

Let $t \leqslant k$ and use law of total probability, otherwise $p(\eta) = 0$:

$$
\begin{aligned}
p(\eta) &= p(\eta = t \,|\, \xi = k) \cdot p(\xi = k) \\
&= p^t (1-p)^{k-t} \binom{k}{t} q \cdot \frac{\lambda^k}{k!} \exp(-\lambda)
\end{aligned}
$$

Thus :

$$
\begin{aligned}
p(\eta) &= \sum_{t=k}^{\infty} \left[\frac{\lambda^k}{k!} \exp(-\lambda) \cdot \binom{k}{t} p^t (1-p)^{k-t} \right] = \left\{ k - t = u, k = t + u \right\} \\
&= \sum_{u=0}^{\infty} \frac{\lambda^{t+u}}{(t+u)!} \exp(-\lambda) \frac{(t+u)!}{t!(t+u-t)!} p^t (1-p)^u \\
&= \frac{\lambda^t \exp(-\lambda) p^t}{t!} \sum_{u=0}^{\infty} \frac{(1-p)^u \lambda^u}{u!} \\
&= \frac{\lambda^t \exp(-\lambda) p^t}{t!} \exp(\lambda - p\lambda) = \frac{(\lambda p)^t \exp(-p\lambda)}{t!} = \text{Pois}(p\lambda)
\end{aligned}
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Problem 2. A strict reviewer needs t_1 minutes to check assigned application to Deep—Bayes summer school, where t_1 has normal distribution with parameters $\mu_1 = 30$, $\sigma_1 = 10$. While a kind reviewer needs t_2 minutes to check an application, where t_2 has normal distribution with parameters $\mu_2 = 20$, $\sigma_2 = 5$. For each application the reviewer is randomly selected with 0.5 probability. Given that the time of review $t = 10$, calculate the conditional probability that the application was checked by a kind reviewer.

Solution

Let $p(t \mid \text{strict}) \sim \mathcal{N}(t \mid 30, 100)$, $p(t \mid \text{kind}) \sim \mathcal{N}(t \mid 20, 25)$ and $p(\text{type}) = 0.5$. We need to find $p(\text{type} = \text{kind} \mid t = 10)$

Use lovely Bayes' theorem and law of total probability:

$$
\begin{aligned}
p(\text{type} = \text{kind} \mid t = 10) &= \frac{p(t = 10 \mid \text{type} = \text{kind})p(\text{type} = \text{kind})}{p(t = 10)} \\
&= \frac{p(t = 10 \mid \text{type} = \text{kind})p(\text{type} = \text{kind})}{p(t = 10 \mid \text{type} = \text{kind})p(\text{type} = \text{kind}) + p(t = 10 \mid \text{type} = \text{strict})p(\text{type} = \text{strict})} \\
&= \frac{\mathcal{N}(10 \mid 20, 25)}{\mathcal{N}(10 \mid 20, 25) + \mathcal{N}(10 \mid 30, 100)} = 0.67
\end{aligned}
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