CSC236 Problem Set 3

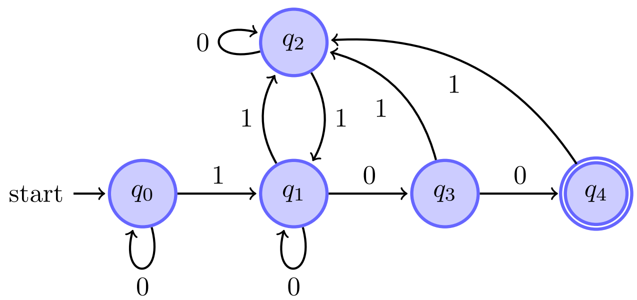
Jacob Nazarenko, James Currier, Mark Abdullah

April 4, 2018

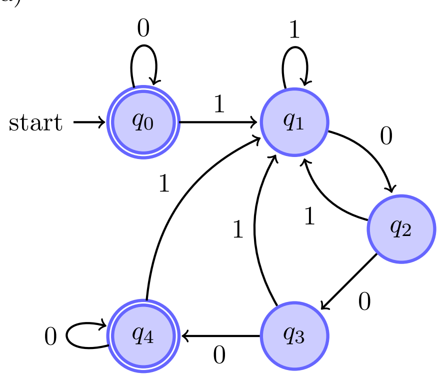
1. a) (11)(0+01+011)*(01+011)

   b) ((1+00)(0+1)*(0*+10+11))+0+1+10+11+00

2.

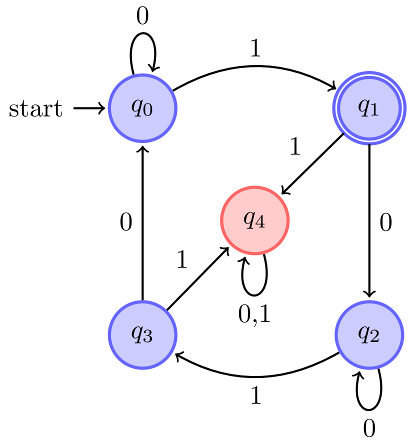


3. a)



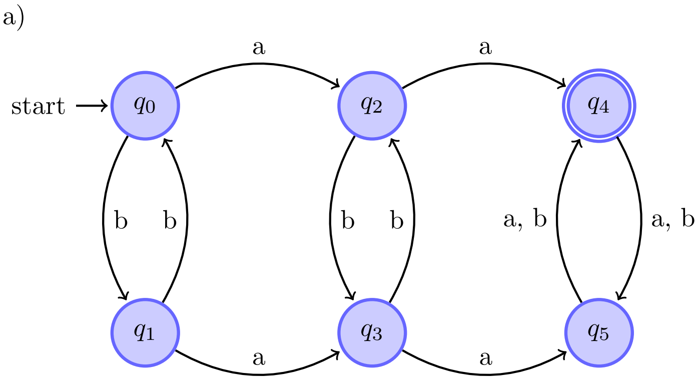
$$P(x): \quad \delta^*(q_0, x) = \begin{cases} q_0, & \text{if } x \text{ represents 0 (is all 0s)} \\ q_1, & \text{if } x \text{ ends in a 1} \\ q_2, & \text{if } x \text{ ends in a 1 followed by one 0} \\ q_3, & \text{if } x \text{ ends in a 1 followed by two 0s} \\ q_4, & \text{if } x \text{ ends in a 1 followed by three or more 0s} \end{cases}$$

b)



$$P(x): \quad \delta^*(q_0, x) = \begin{cases} q_0, & \text{if } x \text{ has an even \# of 1s and ends with 0} \\ q_1, & \text{if } x \text{ has an odd \# of 1s and ends with 1} \\ q_2, & \text{if } x \text{ has an odd \# of 1s and ends with 0} \\ q_3, & \text{if } x \text{ has an even \# of 1s and ends with 1} \end{cases}$$

4. a)



$$P(x): \quad \delta^*(q_0, x) = \begin{cases} q_0, & \text{if } x \text{ has even length and zero a's} \\ q_1, & \text{if } x \text{ has odd length and zero a's} \\ q_2, & \text{if } x \text{ has odd length and one a} \\ q_3, & \text{if } x \text{ has even length and one a} \\ q_4, & \text{if } x \text{ has even length and two or more a's} \\ q_5, & \text{if } x \text{ has odd length and two or more a's} \end{cases}$$

b) Let $\mathcal{L}(D)$ represent the language accepted by the DFA above. We want to prove that $L = \mathcal{L}(D)$
Let us begin by proving $P(x)$ holds $\forall x \in \sum^*$

Proof (by induction):
Base case: $x = \epsilon$

$$\delta^*(q_0, x) = \delta^*(q_0, \epsilon) = q_0 \implies P(\epsilon) \text{ holds}$$

Inductive Step:
Let $x \in \sum^*$ and $m \in \sum$. Assume P(x) [IH].

Case 1: Assume m = a

Case 1.1: x has even length and zero a's.
Then xm has odd length and one a.
By [IH] $\delta^*(q_0, x) = q_0$
By definition,

$$\begin{aligned}\delta^*(q_0, xm) &= \delta^*(\delta^*(q_0, x), m)\\ &= \delta^*(q_0, m)\\ &= \delta^*(q_0, a)\\ &= q2\end{aligned}$$

Therefore $P(xm)$ holds.

Case 1.2: x has odd length and zero a's.
Then xm has even length and one a.
By [IH] $\delta^*(q_0, x) = q_1$
By definition,

$$\begin{aligned}\delta^*(q_0, xm) &= \delta^*(\delta^*(q_0, x), m)\\ &= \delta^*(q_1, m)\\ &= \delta^*(q_1, a)\\ &= q3\end{aligned}$$

Therefore $P(xm)$ holds.

Case 1.3: x has odd length and one a's.
Then xm has even length and two a.
By [IH] $\delta^*(q_0, x) = q_2$
By definition,

$$\begin{aligned}\delta^*(q_0, xm) &= \delta^*(\delta^*(q_0, x), m)\\ &= \delta^*(q_2, m)\\ &= \delta^*(q_2, a)\\ &= q4\end{aligned}$$

Therefore $P(xm)$ holds.

Case 1.4: x has even length and one a's.
Then xm has odd length and two a.

By [IH] $\delta^*(q_0, x) = q_3$
By definition,

$$\begin{aligned}
\delta^*(q_0, xm) &= \delta^*(\delta^*(q_0, x), m) \\
&= \delta^*(q_3, m) \\
&= \delta^*(q_3, a) \\
&= q5
\end{aligned}$$

Therefore $P(xm)$ holds.

Case 1.5: x has even length and two or more a's.
Then xm has odd length and greater then two a.
By [IH] $\delta^*(q_0, x) = q_4$
By definition,

$$\begin{aligned}
\delta^*(q_0, xm) &= \delta^*(\delta^*(q_0, x), m) \\
&= \delta^*(q_4, m) \\
&= \delta^*(q_4, a) \\
&= q5
\end{aligned}$$

Therefore $P(xm)$ holds.

Case 1.6: x has odd length and two or more a's.
Then xm has even length and greater than two a's.
By [IH] $\delta^*(q_0, x) = q_5$
By definition,

$$\begin{aligned}
\delta^*(q_0, xm) &= \delta^*(\delta^*(q_0, x), m) \\
&= \delta^*(q_5, m) \\
&= \delta^*(q_5, a) \\
&= q4
\end{aligned}$$

Therefore $P(xm)$ holds.

Case 2: Assume m = b

Case 2.1: x has even length and zero a's.
Then xm has odd length and zero a.
By [IH] $\delta^*(q_0, x) = q_0$
By definition,

$$\begin{aligned}
\delta^*(q_0, xm) &= \delta^*(\delta^*(q_0, x), m) \\
&= \delta^*(q_0, m) \\
&= \delta^*(q_0, b) \\
&= q1
\end{aligned}$$

Therefore $P(xm)$ holds.

Case 2.2: x has odd length and zero a's.
Then xm has even length and zero a.

By [IH] $\delta^*(q_0, x) = q_1$
By definition,

$$\begin{aligned}\delta^*(q_0, xm) &= \delta^*(\delta^*(q_0, x), m)\\ &= \delta^*(q_1, m)\\ &= \delta^*(q_1, b)\\ &= q0\end{aligned}$$

Therefore $P(xm)$ holds.

Case 2.3: x has odd length and one a's.
Then xm has even length and one a.
By [IH] $\delta^*(q_0, x) = q_2$
By definition,

$$\begin{aligned}\delta^*(q_0, xm) &= \delta^*(\delta^*(q_0, x), m)\\ &= \delta^*(q_2, m)\\ &= \delta^*(q_2, b)\\ &= q3\end{aligned}$$

Therefore $P(xm)$ holds.

Case 2.4: x has even length and one a's.
Then xm has odd length and one a.
By [IH] $\delta^*(q_0, x) = q_3$
By definition,

$$\begin{aligned}\delta^*(q_0, xm) &= \delta^*(\delta^*(q_0, x), m)\\ &= \delta^*(q_3, m)\\ &= \delta^*(q_3, b)\\ &= q2\end{aligned}$$

Therefore $P(xm)$ holds.

Case 2.5: x has even length and two or more a's.
Then xm has odd length and two or more a.
By [IH] $\delta^*(q_0, x) = q_4$
By definition,

$$\begin{aligned}\delta^*(q_0, xm) &= \delta^*(\delta^*(q_0, x), m)\\ &= \delta^*(q_4, m)\\ &= \delta^*(q_4, b)\\ &= q5\end{aligned}$$

Therefore $P(xm)$ holds.

Case 2.6: x has odd length and two or more a's.
Then xm has even length and two or more a.

By [IH] $\delta^*(q_0, x) = q_5$
By definition,

$$\begin{aligned}\delta^*(q_0, xm) &= \delta^*(\delta^*(q_0, x), m)\\ &= \delta^*(q_5, m)\\ &= \delta^*(q_5, b)\\ &= q4\end{aligned}$$

Therefore $P(xm)$ holds.

We have now shown that $P(xm)$ holds in every case.
In conclusion, by the principle of Structural Induction, $P(w)$ holds $\forall w \in \sum^*$.

We will now show that,
1) $L \subseteq \mathcal{L}(D)$
2) $\mathcal{L}(D) \subseteq L$

1) Let $w \in L$, Then w has even length and at least two a's.

$$\begin{aligned}P(w) \text{ holds} &\implies \delta^*(q_0, w) = q4 \quad \text{(an accepting state)}\\ &\implies w \in \mathcal{L}(D)\\ &\implies L \subseteq \mathcal{L}(D)\end{aligned}$$

2) Let $w \in \mathcal{L}(D)$

$$\begin{aligned}P(w) \text{ holds and } \delta^*(q_0, w) = q4 &\implies w \text{ has even length and two or more } a\text{'s}\\ &\implies w \in L\\ &\implies \mathcal{L}(D) \subseteq L\end{aligned}$$

Then

$$\mathcal{L}(D) \subseteq L \text{ and } L \subseteq \mathcal{L}(D) \implies L = \mathcal{L}(D)$$

Therefore we have proven the language L and the language accepted by the DFA are equal, and hence the DFA is correct. ■

5. a) **Proof (by Contradiction):**
We want to prove that L_1 is not a regular language.
Assume for contradiction that L_1 is a regular language. Then we will use the pumping lemma, which tells us there exists a pumping length p.
Let $s = a^{3p}b^p$ be a string in L_1.
Then by the pumping lemma there exists strings x, y, z that satisfy the following,

1. $s = xyz = a^{3p}b^p$
2. $|y| \geq 1$
3. $|xy| \leq p$
4. $\forall i \in \mathbb{N}, xy^iz \in L_1$

By this we know the the only possibility for y is a^k for $1 \leq k \leq p$, since $|xy|$ must be less than p, and we know the first $3p$ characters of s are all a's.
By number 4, $xy^2z = a^{3p+k}b^p \in L_1$, however this cannot be true since this does not satisfy the condition of the language L_1 as stated in the question.
Therefore we have reached a contradiction, and L_1 cannot be a regular language. ■

b) **Proof (by Contradiction):**
We want to prove L_2 is not a regular language.
Assume for contradiction that L_2 is a regular language.

Define $L' = L_2 \cap a^*b^* = \{a^nb^m | m, n \in \mathbb{N}, n \leq m\}$

The only way there are no c's in a string in L_2 is if $m = n$ such that $a^nb^mc^{m-n} = a^nb^mc^0 = a^nb^m$. Then the complement of L_2 intersected with a^*b^* must be in the form as described by the set notation of L'.

Since regular languages are closed under intersection, and we know a^*b^* is regular as it is represented by a regular expression, we get that L' must also be regular, under the assumption that L_2 is regular. However it was proven in tutorial 10, question 2, that the set represented by L' is not regular. We have reached a contradiction!

Therefore, L' not regular $\implies$ L_2 not regular.

We have thereby proven L_2 is not a regular language. ■