Chemical Equations

In the process of photosynthesis, plants use radiant energy from sunlight to convert carbon dioxide (CO_2) and water (H_2O) into glucose ($C_6H_{12}O_6$) and oxygen (O_2). The chemical equation of the reaction is of the form

$$x_1CO_2 + x_2H_2O \rightarrow x_3O_2 + x_4C_6H_{12}O_6$$

To balance the equation, we must choose x_1, x_2, x_3, and x_4 so that the numbers of carbon, hydrogen, and oxygen atoms are the same on each side of the equation. Since

carbon dioxide contains one carbon atom and glucose contains six, to balance the carbon atoms we require that

$$x_1 = 6x_4$$

Similarly, to balance the oxygen, we need

$$2x_1 + x_2 = 2x_3 + 6x_4$$

and finally, to balance the hydrogen, we need

$$2x_2 = 12x_4$$

If we move all the unknowns to the left-hand sides of the three equations, we end up with the homogeneous linear system

$$\begin{aligned} x_1 \quad\quad\quad\quad\quad\quad - \ \ 6x_4 &= 0 \\ 2x_1 + \ x_2 - 2x_3 - \ \ 6x_4 &= 0 \\ 2x_2 \quad\quad\quad - 12x_4 &= 0 \end{aligned}$$

By Theorem 1.2.1, the system has nontrivial solutions. To balance the equation, we must find solutions (x_1, x_2, x_3, x_4) whose entries are nonnegative integers. If we solve the system in the usual way, we see that x_4 is a free variable and

$$x_1 = x_2 = x_3 = 6x_4$$

In particular, if we take $x_4 = 1$, then $x_1 = x_2 = x_3 = 6$ and the equation takes the form

$$6CO_2 + 6H_2O \rightarrow 6O_2 + C_6H_{12}O_6$$

Linear Algebra. Application 3. Matrix and system of equations Name:

Traffic Flow

In the downtown section of a certain city, two sets of one-way streets intersect as shown in Figure 1.2.2. The average hourly volume of traffic entering and leaving this section during rush hour is given in the diagram. Determine the amount of traffic between each of the four intersections.

Solution

At each intersection the number of automobiles entering must be the same as the number leaving. For example, at intersection A, the number of automobiles entering is $x_1 + 450$ and the number leaving is $x_2 + 610$. Thus

$$x_1 + 450 = x_2 + 610 \qquad \text{(intersection } A\text{)}$$

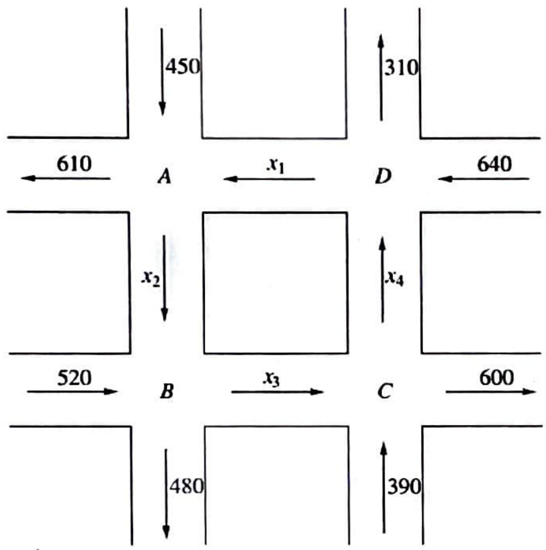

Figure 1.2.2.

Similarly,

$$x_2 + 520 = x_3 + 480 \qquad \text{(intersection } B\text{)}$$
$$x_3 + 390 = x_4 + 600 \qquad \text{(intersection } C\text{)}$$
$$x_4 + 640 = x_1 + 310 \qquad \text{(intersection } D\text{)}$$

The augmented matrix for the system is

$$\left(\begin{array}{rrrr|r} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ -1 & 0 & 0 & 1 & -330 \end{array}\right)$$

The reduced row echelon form for this matrix is

$$\left(\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 330 \\ 0 & 1 & 0 & -1 & 170 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The system is consistent, and since there is a free variable, there are many possible solutions. The traffic flow diagram does not give enough information to determine x_1, x_2, x_3, and x_4 uniquely. If the amount of traffic were known between any pair of intersections, the traffic on the remaining arteries could easily be calculated. For example, if the amount of traffic between intersections C and D averages 200 automobiles per hour, then $x_4 = 200$. Using this value, we can then solve for x_1, x_2, and x_3:

$$x_1 = x_4 + 330 = 530$$
$$x_2 = x_4 + 170 = 370$$
$$x_3 = x_4 + 210 = 410$$

Economic Models for Exchange of Goods

Suppose that in a primitive society the members of a tribe are engaged in three occupations: farming, manufacturing of tools and utensils, and weaving and sewing of clothing. Assume that initially the tribe has no monetary system and that all goods and services are bartered. Let us denote the three groups by F, M, and C, and suppose that the directed graph in Figure 1.2.4 indicates how the bartering system works in practice.

The figure indicates that the farmers keep half of their produce and give one-fourth of their produce to the manufacturers and one-fourth to the clothing producers. The manufacturers divide the goods evenly among the three groups, one-third going to each group. The group producing clothes gives half of the clothes to the farmers and divides the other half evenly between the manufacturers and themselves. The result is summarized in the following table:

	F	M	C
F	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
M	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$
C	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$

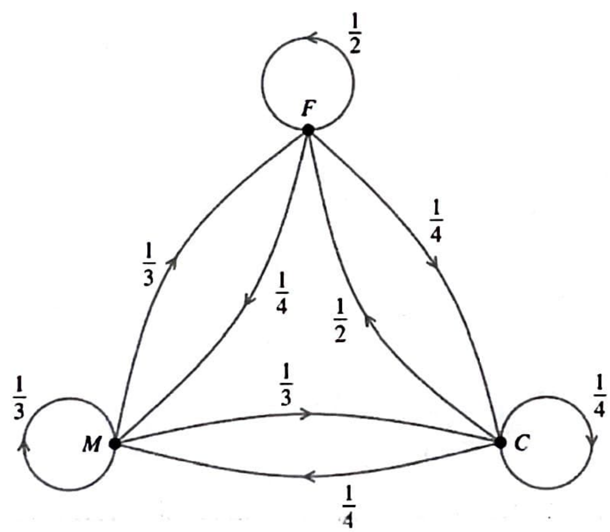

Figure 1.2.4.

The first column of the table indicates the distribution of the goods produced by the farmers, the second column indicates the distribution of the manufactured goods, and the third column indicates the distribution of the clothing.

As the size of the tribe grows, the system of bartering becomes too cumbersome and, consequently, the tribe decides to institute a monetary system of exchange. For this simple economic system, we assume that there will be no accumulation of capital or debt and that the prices for each of the three types of goods will reflect the values of the existing bartering system. The question is how to assign values to the three types of goods that fairly represent the current bartering system.

The problem can be turned into a linear system of equations using an economic model that was originally developed by the Nobel Prize-winning economist Wassily Leontief. For this model, we will let x_1 be the monetary value of the goods produced by the farmers, x_2 be the value of the manufactured goods, and x_3 be the value of all the clothing produced. According to the first row of the table, the value of the goods received by the farmers amounts to half the value of the farm goods produced, plus one-third the value of the manufactured products, and half the value of the clothing goods. Thus the total value of goods received by the farmer is $\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3$. If the system is fair, the total value of goods received by the farmers should equal x_1, the total value of the farm goods produced. Hence, we have the linear equation

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 = x_1$$

Using the second row of the table and equating the value of the goods produced and received by the manufacturers, we obtain a second equation:

$$\frac{1}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = x_2$$

Finally, using the third row of the table, we get

$$\frac{1}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = x_3$$

These equations can be rewritten as a homogeneous system:

$$\begin{aligned} -\tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{2}x_3 &= 0 \\ \tfrac{1}{4}x_1 - \tfrac{2}{3}x_2 + \tfrac{1}{4}x_3 &= 0 \\ \tfrac{1}{4}x_1 + \tfrac{1}{3}x_2 - \tfrac{3}{4}x_3 &= 0 \end{aligned}$$

The reduced row echelon form of the augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There is one free variable: x_3. Setting $x_3 = 3$, we obtain the solution $(5, 3, 3)$, and the general solution consists of all multiples of $(5, 3, 3)$. It follows that the variables x_1, x_2, and x_3 should be assigned values in the ratio

$$x_1 : x_2 : x_3 = 5 : 3 : 3$$

This simple system is an example of the closed Leontief input-output model. Leontief models are fundamental to our understanding of economic systems. Modern applications would involve thousands of industries and lead to very large linear systems.