

Self-Fulfilling Debt Crises, Revisited¹

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Abstract

We revisit self-fulfilling rollover crises by exploring the potential uncertainty introduced by a gap (however small) between an auction of new debt and the payment of maturing liabilities. It is well known (Cole and Kehoe, 2000) that the lack of commitment at the time of auction to repayment of imminently maturing debt can generate a run on debt, leading to a failed auction and immediate default. We show the same lack of commitment leads to a rich set of possible self-fulfilling crises, including a government that issues *more* debt because of the crisis, albeit at depressed prices. Another possible outcome is a “sudden stop” (or forced austerity) in which the government sharply curtails debt issuance. Both outcomes stem from the government’s incentive to eliminate uncertainty about imminent payments at the time of auction by altering the level of debt issuance. In an otherwise standard quantitative version of the one-period debt model, including such crises increase the default probabilities by a factor of five and the spread volatility by a factor of twenty-five.

1 Introduction

Sovereign debt crises involve high spreads and, at the extreme, failed auctions that lead to an inability to service maturing debt. These crises seem only weakly tied to fundamentals and also appear to exhibit some degree of cross-country contagion. A standard approach to understanding these events is to treat them as arising from a self-fulfilling run on sovereign debt; specifically, a failed auction that precipitates an immediate default on maturing debt. The canonical analysis of Cole and Kehoe (2000) formalizes this notion by relaxing the assumption (implicit in Eaton and Gersovitz (1981) and much of the sovereign debt literature) that a government can commit to repay maturing bonds prior to auctioning new bonds. The Cole and Kehoe (2000) model essentially treats the auction and the decision to repay as occurring simultaneously, since the default decision is perfectly predictable given the price of the government debt at the current auction. Moreover, it generates two possible prices for that debt in equilibrium, either the price implied by assuming that upcoming repayments will be made for sure, or the (zero) price implied by assuming that they will not be repaid for sure. As a result, the Cole and Kehoe (2000) model has a fairly limited scope for debt crises; in particular, a crisis always coincides with a zero price and immediate default. Hence, it suggests that an alternative approach is necessary to understand the many real-world crises in which governments successfully auction new bonds but at steeply depressed prices, or in which governments significantly restrict the amount of bonds they issue in order to avoid a failed auction.

To develop this alternative approach, we start by sharpening the focus on the gap in time between a debt auction and the repayment decision for maturing bonds. Participants in sovereign debt auctions understand that the funds raised will be used, at least in part, for near-term interest and principal payments. However, the auction and repayment are distinct events. This asynchronicity opens up the realistic possibility of there being shocks to the government's payoff to default or repayment that are not known at the time of the auction. We show that this uncertainty (which can be arbitrarily small) generates a rich set of self-fulfilling equilibria beyond the two extremes focused on by Cole and Kehoe (2000). Moreover, because these outcomes include successful auctions, the government potentially has an incentive to manipulate quantities to avoid extreme outcomes. We show that there are situations in which a government can avoid a failed auction by either significantly restricting the amount of bonds issued at auction to secure a favorable price, or by issuing larger quantities at fire-sale prices in order to have sufficient cash on hand to cover upcoming liabilities. The additional predictions from this alternative approach conform more readily to many of the important debt crises that we see in the data.

Our model starts with the familiar Cole and Kehoe (2000) framework in which a coordination failure can lead to a failed auction and subsequent default.¹ We then extend their framework by assuming that the participants in sovereign debt auctions understand that the funds raised will be used, at least in part, for near-term interest and principal payments and that there is uncertainty about the government’s relative payoff between default and repayment. This uncertainty may involve the costs (political or economic) to default or the size of a bailout the government receives from a third party, like the International Monetary Fund or European Union. We show that even if this uncertainty is arbitrarily small, it can lead to a rich set of self-fulfilling debt crises.

To build intuition, assume that the uncertainty is over a random benefit (or cost) of default that is drawn from a distribution with finite support. Also, hold future equilibrium behavior constant. Assume the state variables (the amount of debt due and the endowment) are such that if creditors coordinate on the best possible price at today’s auction, the government repays maturing liabilities with probability one; that is, regardless of the realization of the random return to default, the government honors maturing liabilities. However, if creditors coordinate on a price of zero, the government defaults with probability one, as repayment is too painful absent new auction revenue regardless of the realization of the random return to default. The latter scenario is the traditional Cole-Kehoe crisis. Now suppose creditors focus on the intra-period risk, where “intra-period” refers to the gap in time between the auction of new debt and repayment of maturing debt. We show that there also exists an intermediate price that is supported by a threshold for the random return to default. If the random return is higher than this threshold, then the government defaults on maturing debt. If the random return is lower, the government repays. The price at auction is therefore the probability that the random return is lower than the threshold times the best possible price. That is, bond prices, while strictly positive, are discounted to reflect intra-period risk. The concern about intra-period risk is self-fulfilling, as there is a price that can be supported by alternative creditor beliefs that removes all such risk. Moreover, the multiplicity is static in nature – holding constant future beliefs, there are multiple outcomes that can occur at today’s auction. Such an intermediate equilibrium price survives even as

¹There are two main traditions in the self-fulfilling debt crisis literature, one associated with Calvo (1988) and the other with Cole and Kehoe (2000). Loosely speaking, the former tradition focuses on the link between prices today and budget sets (and incentives to default) tomorrow. See Lorenzoni and Werning (2013) and Ayres et al. (2015) for recent papers in the Calvo tradition. The Cole and Kehoe (2000) model features multiple pairs of prices and contemporaneous default decisions that satisfy equilibrium conditions, with multiplicity reminiscent of a bank run. Recent papers in this tradition include Conesa and Kehoe (2011) and Aguiar et al. (2015). In an antecedent to this paper, Aguiar and Amador (2013) construct off-equilibrium mixed-strategy prices in a Cole-Kehoe framework to address the possibility of buybacks of long-term debt during a failed auction.

we take the variance of the intra-period uncertainty to zero.

This raises the question of what is the government's best response to such intermediate beliefs at auction. In the canonical rollover crisis, this is never an issue as the government raises zero revenue for any level of debt issuance. However, under the alternative set of beliefs, the government faces a depressed but positive price schedule.

To understand the government's optimal debt issuance, suppose that under the best-case price, the government would issue some amount of debt B^* . We show under fairly general conditions that when facing a discounted (but not zero) price schedule, the government chooses among two options. The first of these is to sufficiently reduce its borrowing so as to eliminate intra-period uncertainty and generate a price consistent with a zero probability of default in the upcoming settlement. This outcome is a "sudden stop" (or forced austerity), in which the crisis is manifested by a sharp reduction in the amount of debt the government issues. This choice is optimal if the level of maturing debt is relatively low; otherwise, the reduced amount of auction revenue is too small relative to maturing debt to credibly repay with probability one.

The second option is to issue *more* debt than B^* . In particular, the government raises a sufficient amount of revenue at auction that it repays maturing debt with probability one. By generating sufficient funds at auction, the government finds it relatively painless to repay immediately maturing bonds, eliminating intra-period uncertainty. However, the higher level of debt issuance raises the probability of default in future periods. Hence, the government trades intra-period certainty for a prolonged period of greater risk going forward. We show that this is indeed preferable to compensating lenders for intra-period risk. In this scenario, the government responds to low prices by borrowing more. This choice is optimal if the level of maturing debt is relatively high.

Note that because of the government's best response, we may not observe intra-period risk playing a direct role *along the equilibrium path*. Rather, such risk steers the government away from its otherwise optimal debt-issuance policy to a level at which it avoids paying the cost of such risk. If it over issues relative to the best-case benchmark, then the high spreads reflect default risk in future periods, not compensation for intra-period uncertainty. This result is established for small levels of potential intra-period risk. If such risk is sufficiently large, then the government may not have the ability to eliminate all such risk by altering its debt issuance decision. In that case, the intermediate price may become the only possible outcome that can be supported in equilibrium.

The same logic applies if the uncertainty is about the size of a potential bailout at the time of repayment. In particular, there may be a threshold size of bailout that separates repayment from default. If creditors focus on this aspect of the government's decision, such

a threshold becomes a self-fulfilling outcome. Hence, uncertainty over bailouts can become the central focus of the debt auction. This possible scenario adds an interesting perspective to the events in Europe around the 2010-2012 debt crises.

The gap in time between auctions of new debt and scheduled repayments, regardless of how short, lies at the heart of our model as well as many real life debt crises. As motivation, consider the canonical rollover crisis of Mexico. The Mexican government entered 1995, having just devalued the peso to stem the sudden reversal in credit inflows and the loss of international reserves.² This was coupled with a significant share of short-term dollar-indexed bonds (*tesobonos*) coming due. In the first quarter of 1995, the government had only \$8 billions dollars in foreign reserves, while nearly \$10 billion in tesobono debt was due to mature in the first quarter and an additional \$19 billion before the end of 1995. Tesobono auctions were occurring weekly, and by early January 1995 it became clear that creditors were hesitant to purchase enough bonds to roll over maturing debt. The Clinton administration's initial proposed response met resistance from the U.S. Congress, adding to the uncertainty. By the end of January, the administration had successfully circumvented the need for Congressional approval and teamed up with international financial institutions to rescue Mexico. This crisis dynamic is captured in our environment by modelling creditors – at the time of an auction – as uncertain about whether the government can raise sufficient funds from a third-party to cover any short fall due to a potentially under-subscribed auction.

This concentration of maturity and rapid sequence of events as the crisis worsened seems quite common. Similar sequences of events played out in the European crisis of the 2010s. For example, Portugal had financing needs of 56.5 billion euros in 2011, nearly 30 billion of which was maturing debt. They were only able to raise 16.7 billion in bond markets and turned to the EU and IMF for another 37.8 billion. To take another EU example, Italy had over 53 billion euros maturing in the month of September, 2011, alone (relative to average monthly tax revenue of 57 billion). When Italian yields started to spike in the summer of 2011, whether this debt could be rolled over (with or without assistance) become a central concern. Italy did not default and avoided a direct bailout (although it indirectly benefited from a host of European Central Bank activities), and ended 2011 with more debt outstanding than at the start of the year. This outcome of successfully navigating a crisis but at the cost of additional debt will be an equilibrium outcome in our model. The key factor in all these episodes is high-frequency uncertainty about events between today's auction and the next bond payment.

²Right after the first devaluation on December 20th, the government lost \$4.5 billion in reserves in a single day defending the peso. This led the government to announce that it was abandoning targeting the exchange rate on December 22. At the December 27 tesobono auction only \$28 million of the \$416 million bonds on offer were sold. See Whitt Jr. (1996) for a description and detailed analysis of events.

The paper initially establishes these insights in an analytical model. We then explore how the presence of intra-period risk (both on and off the equilibrium path) changes the model’s quantitative predictions. To do this in a transparent way, we start with a standard one-period quantitative model in the spirit of Eaton and Gersovitz (1981), Aguiar and Gopinath (2006), and Arellano (2008). It is well known in the one-period bond model that absent a highly non-linear cost of default, which essentially forgives default in low endowment states, the government rarely defaults in equilibrium. We show that allowing creditors to coordinate on the “desperate deal” intermediate price schedule raises the default frequency by a factor of five. More strikingly, it increases the volatility of spreads by a factor of 25. In this exercise, the additional spread volatility is driven entirely by the government ratcheting up its borrowing in response to the depressed price schedule. The “sudden stop” response to the crisis occurs at lower debt levels, that is, while the government is building up its debt after a default, while the typical best response near the ergodic mean level of debt is to issue additional debt during a crisis. We also show that restricting crisis beliefs to coordination on the canonical rollover outcome of a zero price leads to essentially zero defaults. This is because the extreme severity of such a crisis is sufficient to deter the government from ever borrowing enough to be vulnerable to a run. We then proceed to the case where debt is long-term and establish that many of the insights from the one-period bond model carry over, provided that the resources the sovereign can command to repay its debt do not vastly exceed its maturing obligations.

2 Model

2.1 Environment

We consider a single-good, discrete-time environment. There is a small open economy that receives a stochastic endowment and (initially) has access to international capital markets. We assume that the economy’s aggregate consumption and saving decisions are made by a sovereign government.³ The economy is small in the sense that its endowment realizations and its fiscal policy do not affect the world risk-free interest rate.

The economy receives a stochastic endowment $Y_t \in \mathbb{Y} \equiv [\underline{Y}, \bar{Y}]$, with $0 < \underline{Y} < \bar{Y} < \infty$. For this section, we assume Y_t is i.i.d. over time. The government’s preferences over the

³That is, we assume that the government has enough instruments to implement any feasible consumption sequence as a domestic competitive equilibrium. We therefore abstract from the problem of individual residents of the domestic economy. This does not mean that the government necessarily shares the preferences of its constituents but rather that it is the relevant decision maker viz-à-viz international financial markets.

sequence of aggregate consumption $\{C_t\}_{t=0}^\infty$ is given by:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t),$$

where $\beta \in (0, 1)$ and $u : \mathbb{R}^+ \rightarrow \mathbb{R}$ is continuously differentiable, strictly increasing, and strictly concave.

The rest of the world is populated by risk-neutral lenders who discount at the rate $R^{-1} = (1 + r^*)^{-1}$. The government has access to a one-period, non-contingent discount bond on which it can default. Let B denote the outstanding stock of bonds at the start of a period; note that $B > 0$ indicates the government is a net debtor, and $B < 0$ a net creditor. To rule out Ponzi schemes, we place an upper bound on debt, $\bar{B} > 0$; that is, $B \in (-\infty, \bar{B}]$. We assume \bar{B} is strictly greater than the natural borrowing limit, and hence never binds in equilibrium.

2.2 Timing

The timing of events within a period is depicted in Figure 1. The government enters with a debt payment due in the current period of B . At the start of the period, the endowment Y is realized. A sunspot that coordinates creditor beliefs, ρ , is also realized. The sunspot will be discussed in detail in Section 2.8. After observing the states $s \equiv (Y, \rho, B)$, and given an equilibrium price schedule $q(s, B')$, the government decides how much debt to issue (or assets to buy), denoted by B' .

After the auction is completed, the government decides whether or not to repay the outstanding debt B . The fact that the default/repayment decision takes place after the auction is crucial to the analysis. This timing follows Cole and Kehoe (2000) and differs from Eaton and Gersovitz (1981) and the existing quantitative literature. However, relative to Cole-Kehoe, we enrich the setting by allowing for additional information to arrive between the auction and repayment. The original Cole-Kehoe analysis had perfect foresight within a period.

In particular, let $V^R(s, B')$ denote the value of the government if it repays B after issuing B' in state s . The value functions will be defined below.

If the government defaults, it receives $V^D(s, B') + \sigma\epsilon$, where $\sigma > 0$ is a parameter and ϵ is a shock that affects the returns to default. This shock only becomes known after the auction, but prior to the default decision.⁴ A natural interpretation of ϵ is the high frequency

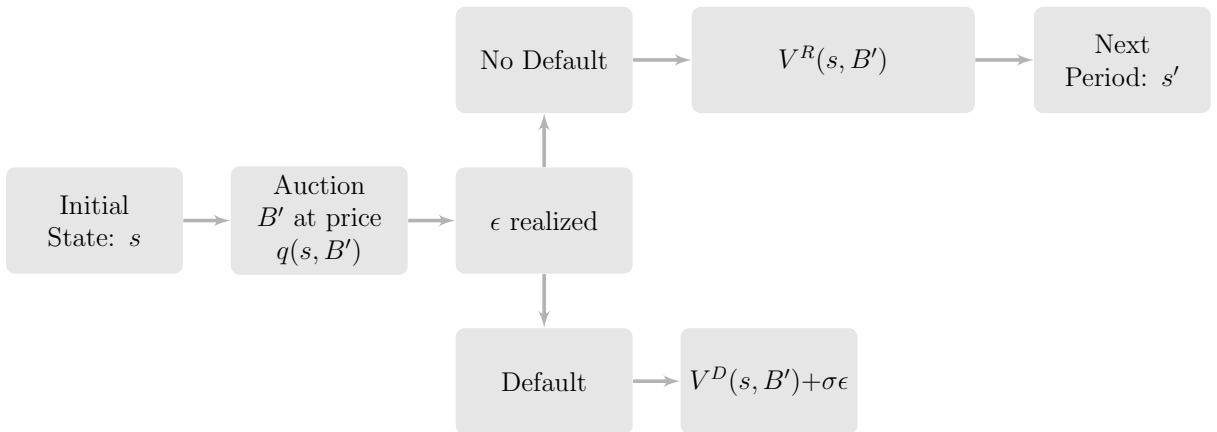
⁴There is a long history of such random payoffs to default. See the handbook chapters of Eaton and Fernandez (1995) and Aguiar and Amador (2014a) as well as more recent papers by Aguiar et al. (2016) and

realization of political payoffs to default that are orthogonal to output and the quantity of debt due. In Section 3.4 we show that we can recast the shock as a bailout of random size from a third party like the IMF or the ECB.

We assume ϵ is distributed i.i.d. over time with cdf F and a continuous density on support $[0, 1]$.⁵ We assume $F'(\epsilon) \geq \alpha > 0$ on $[0, 1]$ for some strictly positive constant α . We shall show that as $\sigma \rightarrow 0$, and the uncertainty over payoffs becomes negligible, we do not necessarily converge to the perfect-foresight equilibria of Cole and Kehoe (2000). That is, an arbitrarily small amount of uncertainty about default payoffs opens a door to a richer set of self-fulfilling crises than that studied by Cole and Kehoe (2000).

Much of the novel economics in this paper stems from thinking carefully about the gap in time between the auction of new bonds and repayment of old bonds. Given its importance in the analysis, it is useful to say a few things about the timing assumption. In the one-period-bond Eaton-Gersovitz model, prior to auctioning new bonds the government can commit to repayment of maturing bonds as well as to the amount of bonds to be auctioned that period. This represents within period commitment over B' as well as the repayment vs. default decision. Introducing longer maturity bonds relaxes both forms of commitment, and gives rise to dilution risk as well as multiplicity.⁶ It is instructive to abstract from dilution risk while maintaining the gap between the issuance of new debt and repayment of old. The timing in Figure 1 does exactly that.

Figure 1: Timing Within a Period



Chatterjee et al. (2016).

⁵Given the parameter σ , the (bounded) size of the support of ϵ can be normalized to one without loss. The location of the support shifts the value of default by a constant.

⁶See Stangebye (2016) and Aguiar and Amador (2016).

2.3 The Government's Problem

We shall consider equilibria such that the equilibrium outcome is a function of $s = (Y, \rho, B)$ and the credit history. If the government has not defaulted in the past, it faces an equilibrium price schedule for new debt B' given by $q(s, B')$. The government takes the schedule as given but recognizes that the price of its debt may vary with B' . In that sense, the government is large in its own debt market.

Given an equilibrium price schedule q , let $V(s)$ denote the value of the government in good credit standing conditional on $s = (Y, \rho, B)$ and prior to the period's auction. We can characterize the government's problem recursively by iterating on V . In particular, let $V_0(s)$ describe the conjectured government's continuation value starting from the *next* period. Working backwards through the current period, if the government repays, it obtains value:

$$V^R(s, B') = u(Y + q(s, B')B' - B) + \beta \mathbb{E}[V_0(s')|B'] \quad (1)$$

if $Y + q(s, B')B' - B \geq 0$. If $Y + q(s, B')B' - B < 0$, then repayment is infeasible, and we set $V^R(s, B') = -\infty$; obviously, default will always dominate this value. Note that the expectation in (1) incorporates that $B' \in s'$ is in the government's information set at the time of repayment in the current period. The expectation is not explicitly conditional on s and ϵ due to the i.i.d. assumptions.

If the government defaults, it consumes its endowment and the revenue from the period's bond auction and then remains in financial autarky thereafter. In this section, we abstract from reentry to financial markets. Letting $\underline{V}^D \equiv \mathbb{E}u(Y')/(1 - \beta)$, we define:

$$V^D(s, B') \equiv u(Y + q(s, B')B') + \beta \underline{V}^D. \quad (2)$$

Strategic default implies that the government repays if and only if:

$$\epsilon \leq \frac{1}{\sigma} \cdot \underbrace{[V^R(s, B') - V^D(s, B')]}_{\equiv \Delta(s, B')}, \quad (3)$$

where the expression in brackets defines $\Delta(s, B')$ as the net benefit of repayment when $\epsilon = 0$. Conditional on (s, B') , the equilibrium probability of repayment within the period is therefore $F(\sigma^{-1}\Delta(s, B'))$.⁷

⁷Note that if the government has assets ($B < 0$), we have $\Delta > 0$. Nevertheless, from (3), if ϵ is large enough, the government may prefer to walk away from its asset position. In that event, we assume that the counter-party continues to pay out on the assets (in accordance with the risk-free interest rate paid on

At the time of auction, the government's problem is to choose $B' \in (-\infty, \overline{B}]$ to maximize the expected end-of-period value, where expectation is over the realization of ϵ :

$$\begin{aligned} V_1(s) &= \max_{B'} \left\{ F(\sigma^{-1}\Delta(s, B'))V^R(s, B') + \int_{\sigma^{-1}\Delta(s, B')}^1 [V^D(s, B') + \sigma\epsilon] dF(\epsilon) \right\} \\ &= \max_{B'} \left\{ V^D(s, B') + F(\sigma^{-1}\Delta(s, B'))\Delta(s, B') + \sigma \int_{\sigma^{-1}\Delta(s, B')}^1 \epsilon dF(\epsilon) \right\}. \end{aligned}$$

The second line indicates that the payoff conditional on B' is $V^D(s, B')$ plus the probability of repayment times the net gain from repayment plus $\sigma\epsilon$ in default states (that is, when $\epsilon > \sigma^{-1}\Delta(s, B')$). Using integration by parts, we can re-write this as:

$$V_1(s) = \max_{B'} \left\{ V^D(s, B') + \sigma - \sigma \int_{\sigma^{-1}\Delta(s, B')}^1 F(\epsilon) d\epsilon \right\}. \quad (4)$$

The right-hand side of (4) depends on the equilibrium value function through Δ . Hence, equation (4) implicitly defines an operator T , conditional on q , that maps V_0 to $V_1 = TV_0$. Given a price schedule, the solution to the government's problem V is the fixed point of this operator. It will be clear in what follows that the economics behind the self-fulfilling crises we study do not rely on the fact that the continuation value function is the same as the current value function. That is, the insights carry over to a finite horizon model or other non-stationary environments.

2.4 The Lenders' Problem

Given the small open economy and risk-neutral lenders assumptions, prices must satisfy a break-even condition. Specifically, let $\Delta(s, B') = V^R(s, B') - V^D(s, B')$ be part of an equilibrium, and let $\mathcal{B}(s)$ denote the government's equilibrium policy function for debt issuance. The price schedule $q : S \times (-\infty, \overline{B}] \rightarrow [0, R^{-1}]$ must equate the expected return on sovereign debt to the risk-free rate:

$$q(s, B') = \begin{cases} R^{-1} & \text{if } B' \leq 0, \\ R^{-1} \times F(\sigma^{-1}\Delta(s, B')) \times \mathbb{E}[F(\sigma^{-1}\Delta(s', \mathcal{B}(s')))|B'] & \text{if } B' \in (0, \overline{B}]. \end{cases} \quad (5)$$

The first line states that assets are always purchased at the risk-free price. In the second line, our timing implies that bondholders are vulnerable to two default decisions, one in the current period immediately after auction and one after next period's auction. Since next

assets), but the payments are lost to the government due to an un-modelled deadweight loss. As $\sigma \rightarrow 0$, this event never arises in equilibrium.

period's default probability depends on next period's debt choice, the pricing also depends on the government's borrowing policy function \mathcal{B} .

The fact that the break-even condition is imposed for all B' – even those that occur off equilibrium – is a perfection requirement: It ensures that if the government were to deviate from \mathcal{B} and issue a sub-optimal amount of debt, these bonds would be priced in a manner consistent with equilibrium behavior going forward.

2.5 Equilibrium

The definition of equilibrium is standard:

Definition 1. *An equilibrium consists of a price schedule q , a government value function V with associated policy \mathcal{B} such that: (i) V and \mathcal{B} solve the government's problem (4) given q ; and (ii) q satisfies the break-even condition (5) given \mathcal{B} .*

2.6 Static Multiplicity

For expositional purposes, it is useful to describe the multiplicity inherent in the environment by focusing on a single period, holding future (equilibrium) behavior constant. In particular, let $\{q, \mathcal{B}, V\}$ denote a “continuation equilibrium” that governs behavior from next period onwards. The multiplicity in the model is inherently “static” in the sense that we can hold continuation play fixed while supporting alternative prices and policies in the current period.

Fix an initial state, (Y, B) , and a choice of debt to be auctioned, $B' \in (0, \overline{B}]$. Given fixed (Y, B, B') , the only thing that varies are beliefs about government behavior at settlement. In particular, from (5) the key determinant of alternative equilibrium prices is the probability of default at settlement, $F(\sigma^{-1}\Delta(s, B'))$, which, in turn, depends on the equilibrium $\Delta = V^R - V^D$.

Beliefs are captured in our notation by the state variable ρ . For now, we suppress this state variable (for the current period) and construct alternative candidate outcomes for a given (Y, B, B') and a fixed continuation equilibrium (which implicitly includes a distribution over next period's beliefs). After establishing the type of equilibrium behavior that can be supported, we map particular outcomes to the belief state ρ .

Given a continuation equilibrium, we can define the reference price that satisfies (5) assuming the government will repay maturing bonds B with probability one at settlement. This replicates (for the current period) the Eaton-Gersovitz assumption that the government can commit to repayment of maturing bonds prior to the current period's auction. Hence,

we use the subscript EG to identify this reference price, which is defined by:

$$q_{EG}(B') \equiv R^{-1} \mathbb{E} [F(\Delta(s', \mathcal{B}(s')) | B')]. \quad (6)$$

The difference between (6) and (5) is we have set $F(\sigma^{-1} \Delta(s, B')) = 1$, reflecting the zero probability of default in the current period. This makes q_{EG} an upper bound on the equilibrium price schedule conditional on the continuation equilibrium. Given the i.i.d. assumptions and the fact that q_{EG} is determined by next period's default decisions, q_{EG} is a function only of B' . We are holding B' fixed for the present discussion, and hence for the remainder of the sub-section we subsume the argument of q_{EG} when convenient.

We can now rewrite the equilibrium condition (5) in terms of this reference price. Fixing (Y, B, B') , let $\tilde{q} \in [0, q_{EG}]$ denote a candidate equilibrium price for the newly auctioned bonds. The candidate price satisfies the break-even condition (5) if and only if:

$$\begin{aligned} \frac{\tilde{q}}{q_{EG}} &= \\ F \left(\frac{1}{\sigma} \left[u \left(Y - B + \left(\frac{\tilde{q}}{q_{EG}} \right) q_{EG} B' \right) - u \left(Y + \left(\frac{\tilde{q}}{q_{EG}} \right) q_{EG} B' \right) + \beta \mathbb{E} [V(s') | B'] - \beta \underline{V}^D \right] \right) \\ &\equiv \tilde{F} \left(\frac{\tilde{q}}{q_{EG}} \right), \end{aligned} \quad (7)$$

where the second line defines $\tilde{F} : [0, 1] \rightarrow [0, 1]$ for the given (Y, B, B') and q_{EG} .⁸ The left-hand side of expression (7) is the ratio of \tilde{q} to the price under within-period commitment to repay. The right hand side is the probability that

$$\sigma \epsilon \leq \tilde{\Delta} \equiv u(Y - B + \tilde{q} B') - u(Y + \tilde{q} B') + \beta \mathbb{E} [V(s') | B'] - \beta \underline{V}^D; \quad (8)$$

that is, $\sigma \epsilon$ is below the value of repayment minus the lowest default value.

Equation (7) characterizes how intra-period risk is priced. If $\epsilon \leq \sigma^{-1} \tilde{\Delta}$, the government does not default at settlement. This occurs with probability $F(\sigma^{-1} \tilde{\Delta})$. Given such a draw of ϵ , the bond is worth q_{EG} at settlement, where q_{EG} encapsulates the probability of default next period as well as the discount factor R^{-1} . If $\epsilon > \sigma^{-1} \tilde{\Delta}$, the government defaults on its maturing debt and the newly-auctioned bonds are worthless.

Our candidate \tilde{q} satisfies the conditions for equilibrium if and only if it satisfies equation (7); that is, if \tilde{q}/q_{EG} is a fixed point of \tilde{F} . As we have fixed B' and hence $q_{EG}(B')$, the static multiplicity then arises from the fact that there may be multiple roots \tilde{q} to (7).

We now discuss three possible classes of equilibrium prices. To guide the discussion,

⁸Specifically, $\tilde{F}(x) \equiv F(\sigma^{-1} [u(Y - B + x q_{EG} B') - u(Y + x q_{EG} B') + \beta \mathbb{E} [V(s') | B'] - \beta \underline{V}^D])$.

Figure 2: Within-Period Price Determination

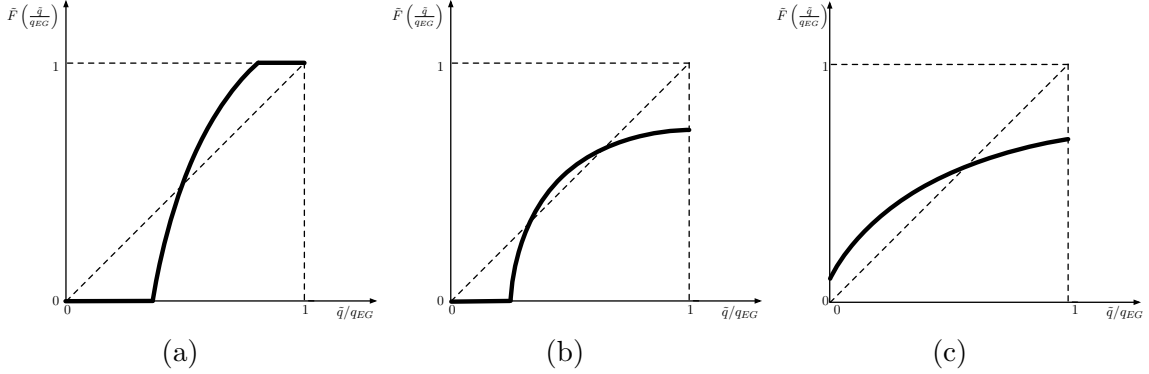


Figure 2 is a heuristic diagram depicting the fixed point problem $\tilde{q}/q_{EG} = \tilde{F}(\tilde{q}/q_{EG})$ of equation (7). In each panel of the figure, we let \tilde{q}/q_{EG} vary between zero and one on the horizontal axis and plot $\tilde{F}(\cdot)$. From (7), valid equilibrium prices consist of points where this function intersects the 45-degree line. \tilde{F} inherits the continuity of u and the concavity of u implies that \tilde{F} is strictly increasing when the government enters the period with debt, which is the case depicted.⁹ For $B \leq 0$, the government enters the period with assets; the proof of Proposition 2 establishes the government will never default in this case for small σ . We initially focus on panel (a), and then discuss the remaining two panels at the end of this section.

The “Eaton-Gersovitz” Price

One possibility is that $\tilde{q} = q_{EG}$ is an equilibrium; that is, there is zero intra-period risk of default. To establish when this is a valid equilibrium price, define the value of repayment at the Eaton-Gersovitz price by¹⁰

$$V_{EG}^R(Y, B, B') \equiv u(Y + q_{EG}(B')B' - B) + \beta \mathbb{E}[V(s')|B']. \quad (9)$$

Similarly, define the associated default value at price q_{EG} by $V_{EG}^D(Y, B') + \sigma\epsilon$, where:

$$V_{EG}^D(Y, B') \equiv u(Y + q_{EG}(B')B') + \beta \underline{V}^D. \quad (10)$$

⁹The proof of Proposition 4 establishes the concavity of \tilde{F} on $(0, 1)$ for a broad class of distributions F and utility functions.

¹⁰This is well defined if repayment is feasible (that is, $B \leq Y + q_{EG}(B')B'$); otherwise we set $V_{EG}^R(Y, B, B') = -\infty$.

The Eaton-Gersovitz price is a valid equilibrium price for (Y, B, B') if and only if

$$V_{EG}^R(Y, B, B') - V_{EG}^D(Y, B') \geq \sigma. \quad (11)$$

This implies that $\tilde{\Delta} \geq \sigma$ for $\tilde{\Delta}$ defined in equation (8) evaluated at $\tilde{q} = q_{EG}$. In panel (a) of Figure 2, q_{EG} is a valid equilibrium as $\tilde{F}(1) = 1$. In this scenario, by facing a high price at auction, the government's burden of repayment (both in terms of consumption today and debt going forward) is relatively low and the threat of default at settlement is therefore zero.

The Cole-Kehoe “Failed-Auction” Price

A second possibility is that $\tilde{q} = 0$ is an equilibrium. This is reminiscent of the canonical Cole-Kehoe rollover crisis, in which B' raises zero at auction and the government defaults at settlement with probability one.

Specifically, define¹¹

$$V_{CK}^R(Y, B, B') \equiv u(Y - B) + \beta \mathbb{E}[V(s')|B']. \quad (12)$$

Note that as $\tilde{q} = 0$, the value of repayment depends on B' only through the continuation value (as $B' \in s'$). For $\tilde{q} = 0$ to be an equilibrium price, (7) requires:

$$V_{CK}^R(Y, B, B') - u(Y) - \beta \underline{V}^D \leq 0.$$

In panel (a) of Figure 2, 0 is a valid equilibrium as $\tilde{F}(0) = 0$. In this scenario, by failing to raise any revenue at auction, the burden of repayment of maturing bonds is relatively high, leading to default with probability one at settlement, rationalizing the zero price.

Interior Prices

The third possibility is that there exists $\tilde{\epsilon} \in [0, 1]$ at which the government is indifferent to repayment and default. In particular, consider a $\tilde{\epsilon}$ such that:

$$\sigma \tilde{\epsilon} = u(Y - B + F(\tilde{\epsilon})q_{EG}B') - u(Y + F(\tilde{\epsilon})q_{EG}B') + \beta \mathbb{E}[V(s')|B'] - \beta \underline{V}^D. \quad (13)$$

In this case, the auction price is $\tilde{q} = F(\tilde{\epsilon})q_{EG}(B')$, and whether the government repays or defaults after the auction depends on the realization of $\epsilon \gtrless \tilde{\epsilon}$. The government strictly prefers to repay if $\epsilon < \tilde{\epsilon}$, default if $\epsilon > \tilde{\epsilon}$, and $\epsilon = \tilde{\epsilon}$ is the measure-zero point of indifference.

We refer to this case as an *interior* price:

¹¹As before, if $Y < B$, repayment is infeasible and we set $V_{CK}^R = -\infty$. The government always purchases assets at risk-free prices, and hence $V_{CK}^R(., ., B') = V_{EG}^R(., ., B')$ for $B' \leq 0$.

Definition 2. For a given (s, B') , an equilibrium price $q(s, B') = F(\tilde{\epsilon})q_{EG}(B')$ is **interior** if $\tilde{\epsilon} \in [0, 1]$ satisfies (13). The price is **strictly interior** if $\tilde{\epsilon} \in (0, 1)$.

At strictly interior prices, the government faces strictly positive but depressed (relative to q_{EG}) prices. At auction, the bonds are priced as if they are a standard “Eaton-Gersovitz” bond combined with a lottery on the realization of $\epsilon \geq \tilde{\epsilon}$.

Such an interior price is a generic feature of the static multiplicity inherent in the model. By this, we mean that if there are multiple prices that can be supported in equilibrium, then an interior price can also be supported. In particular, at a given (Y, B, B') , if the government prefers to repay when facing q_{EG} and to default when the price is zero, then there exists an interior equilibrium price:

Proposition 1. For a fixed (Y, B, B') , suppose that (i) $V_{EG}^R(Y, B, B') - V_{EG}^D(Y, B') \geq \sigma$, and (ii) $V_{CK}^R(Y, B, B') - V_{CK}^D(Y) \leq 0$, then there exists an interior equilibrium price. If the inequalities in (i) and (ii) are strict, then there exists a strictly interior equilibrium price.

In panel (a) of Figure 2, as $\tilde{F}(0) = 0$ and $\tilde{F}(1) = 1$, there must be a $x \in [0, 1]$ such that $\tilde{F}(x) = x$. As depicted, $x \in (0, 1)$, and hence constitutes a third distinct equilibrium price. There are also the knife-edge cases with only two valid prices, with either $\tilde{\epsilon} = 0$ or $\tilde{\epsilon} = 1$ satisfying equation (13).

Note that Proposition 1 does not require any restrictions on the distribution of ϵ other than compact support and a continuous *cdf*. The fact that Proposition 1 holds regardless of the size of σ is noteworthy given the empirical motivation. The time between an auction of new bonds and repayment of old may be quite short in practice. Even if the magnitude of high-frequency risk is arbitrarily small, the proposition establishes that the mere presence of such risks remains relevant for equilibrium price determination. One contribution of this paper is to incorporate such uncertainty into a model of self-fulfilling debt crises.

In this spirit, we sharpen the characterization of multiplicity using two approaches. One is place no restrictions on F and let $\sigma \rightarrow 0$, thereby studying the impact of arbitrarily small intra-period risk. An alternative view is that asset prices show significant volatility at high frequencies, particularly during crises, and therefore high-frequency risk may indeed be significant. Hence, the second approach is to place some structure on F without restricting σ to be arbitrarily small. We tackle each in turn.

Proposition 1 does not rule out that there may be more than one interior fixed point to equation (7). Moreover, the proposition provides sufficient conditions for an interior equilibrium, but it is silent on whether an interior price can be supported in other scenarios. The next proposition states that for small σ there is at most one interior price, and generically there exists either one equilibrium price or three:

Proposition 2. *For a given (Y, B, B') , there exists a $K > 0$ such that if $\sigma < K$, then there are only three possible equilibrium price configurations:*

- (i) *Zero is the only equilibrium price;*
- (ii) *$q_{EG}(B')$ is the only equilibrium price; or*
- (iii) *$\{0, q_{EG}(B'), F(\tilde{\epsilon})q_{EG}(B')\}$ can all be supported as equilibrium prices, where $\tilde{\epsilon} \in [0, 1]$ is the unique solution to equation (13).*

Proposition 1 gives sufficient conditions for an interior price. Proposition 2 states they are also necessary for small σ .

One can consider the limiting case of $\sigma = 0$ as a price that makes the government indifferent to default and repayment, which is supported as an equilibrium by the government randomizing over the default decision. That is, suppose the government's decision at settlement is to pick a probability of repayment: $p \in [0, 1]$. When facing a price \tilde{q} , the government's best response is:

$$\begin{aligned} p &= 1 \text{ if } u(Y - B + \tilde{q}B') + \beta\mathbb{E}[V(s')|B'] > u(Y + \tilde{q}B') + \beta\underline{V}^D; \\ p &= 0 \text{ if } u(Y - B + \tilde{q}B') + \beta\mathbb{E}[V(s')|B'] < u(Y + \tilde{q}B') + \beta\underline{V}^D; \text{ and} \\ p &\in [0, 1] \text{ if } u(Y - B + \tilde{q}B') + \beta\mathbb{E}[V(s')|B'] = u(Y + \tilde{q}B') + \beta\underline{V}^D. \end{aligned} \tag{14}$$

Similarly, the lenders' best response to an anticipated p at settlement is to bid $\tilde{q} = p \times q_{EG}$ at auction. A mixed-strategy equilibrium price is a pair (p, \tilde{q}) that satisfies $\tilde{q} = pq_{EG}$ and equation (14). Following the steps behind Proposition 1, it is straightforward to see that if $(p = 1, \tilde{q} = q_{EG})$ satisfies the first inequality in (14), and $(p = 0, \tilde{q} = 0)$ satisfies the second, then there is also a unique $p \in (0, 1)$ with an interior $\tilde{q} = pq_{EG} \in (0, q_{EG})$ that satisfies the third line of (14).

Reminiscent of Harsanyi (1973) purification, the mixed strategy price is the limit of the pure-strategy interior price as $\sigma \rightarrow 0$:¹²

Proposition 3. *Given (Y, B, B') with $B' > 0$, suppose $V_{EG}^R(Y, B, B') > V_{EG}^D(Y, B')$ and $V_{CK}^R(Y, B, B') < V_{CK}^D(Y)$. Let σ_n be a monotone decreasing sequence converging to zero. Then there exists an integer $N < \infty$ and a sequence $\tilde{\epsilon}_n$ satisfying (13) for each σ_n for $n > N$. Moreover, $p = \lim_{n \rightarrow \infty} F(\tilde{\epsilon}_n)$ and $\tilde{q} = \lim_{n \rightarrow \infty} F(\tilde{\epsilon}_n)q_{EG}(B')$ exist and satisfy (14).*

¹²Aguiar and Amador (2014b) introduced mixed strategies in a Cole-Kehoe model. That paper introduced them to rule out the repurchasing of long-term bonds in the event of a failed auction. See Section 2.10 for how buybacks are priced under pure strategies in our environment.

The preceding propositions characterized the static multiplicity for arbitrarily small intra-period risk. As noted above, an alternative view is that there is a non-negligible chance that significant information arrives between an auction and the next payment of debt. We now explore the scenario in which $\sigma \gg 0$. In particular, we show that there is a large class of utility functions and densities F for which there exist at most three possible equilibrium prices:

Proposition 4. *Suppose $Y > B > 0$ and $q_{EG}(B')B' > 0$. If $u''(c)$ is strictly increasing in $c \geq 0$ and $F'(\epsilon)$ is weakly decreasing in $\epsilon \in [0, 1]$, then there are at most three \tilde{q} that satisfy (7), with at most two interior. If $\tilde{q} = 0$ is not an equilibrium, then there is at most one equilibrium price that satisfies (7).*

Panels (b) and (c) in Figure 2 consider the scenario discussed in Proposition 4. In particular, in Panel (b), B' is such that there are three possible equilibrium prices, but $q_{EG}(B')$ is not among them. Rather, the highest possible price is associated with some risk of immediate default. A second interior price is associated with a greater chance of default within the period. Zero remains an equilibrium as well, which is associated with probability-one immediate default.

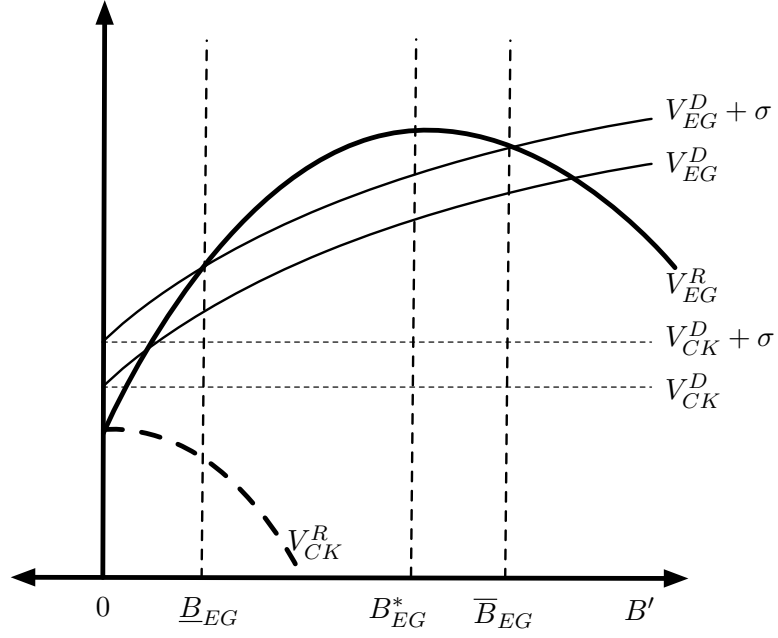
In Panel (c), B' is such that zero is no longer an equilibrium price. The only equilibrium price is an interior price. Again, this arises when σ is large enough that there is always a substantial risk (but not certainty) of default at settlement. In particular, such an outcome requires that the government strictly prefers to repay when $\epsilon = 0$ when facing a zero price and strictly prefers to default when $\epsilon = 1$ when facing the Eaton-Gersovitz price. From Proposition 2, this requires a σ bounded away from zero.

Not depicted in Figure 2 are the straightforward cases in which $\tilde{q} = 0$ or $\tilde{q} = q_{EG}(B')$ are the unique equilibrium prices. For the former, the \tilde{F} curve is below the 45-degree line for the entire domain. That is, even at the best possible price, the government strictly prefers to default. For the latter, even at a zero price the government strictly prefers to repay, implying the \tilde{F} curve lies above the 45-degree line over the entire domain $[0, 1]$.

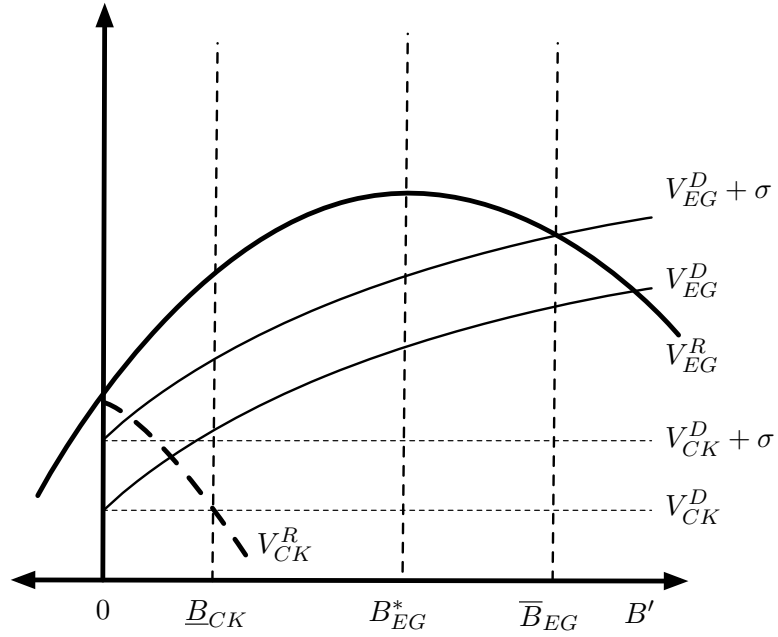
2.7 Regions of Multiplicity

We now discuss how the possibility of multiplicity varies with the states (Y, B) and choice B' . We continue focusing on static multiplicity by fixing a continuation value V and associated policy \mathcal{B} , and then analyzing alternative equilibrium behavior in the current period. For the formal analysis, there is no need to place additional restrictions on future equilibrium behavior (beyond those in the equilibrium definition); for clarity, the heuristic diagrams

Figure 3: Value Functions



(a) Canonical "Crisis Zone"



(b) Extended "Crisis Zone"

used in the exposition will assume that $\mathbb{E}[V(s')|B']$ and $q_{EG}(B')$ are continuous and weakly decreasing in B' .¹³

To get a better sense of when and how multiplicity arises in our environment, we use Figure 3. Both panels have the same set of curves, but are evaluated at different points in the state space (Y, B) . The solid hump-shaped line depicts $V_{EG}^R(Y, B, B')$ as we vary B' and fix (Y, B) . The non-monotonicity comes from the fact that as we increase B' , current consumption increases (subject to being on the upward part of the debt Laffer curve; that is, $d[q(B')B']/dB' > 0$). However, an increase in B' weakly reduces the continuation value. Once B' reaches the downward sloping part of the Laffer curve, V_{EG}^R unambiguously weakly declines in B' . The peak of V_{EG}^R indicates the optimal issuance policy given q_{EG} , which is denoted B_{EG}^* in the figure.¹⁴

We also include the value of repayment in the event of a failed auction, $V_{CK}^R(Y, B, B')$ defined in equation (12) (i.e., the value if $B' > 0$ were to be issued at a zero price). V_{CK}^R is depicted in Figure 3 by the downward sloping dashed line. For $B' > 0$, $V_{CK}^R(Y, B, \cdot)$ depends on B' only through continuation value, which we depict as decreasing in debt carried forward. As no revenue is raised at auction, V_{CK}^R is less than V_{EG}^R (on the domain such that $q_{EG}(B')B' > 0$). For $B' < 0$, V_{CK}^R tracks V_{EG}^R , as assets are always purchased at risk-free prices.

In the figure, we also include the upper and lower limits for the value of default given Y and the price schedule q_{EG} , which are the lines labeled V_{EG}^D and $V_{EG}^D + \sigma$ in the figure. The default values are increasing in B' as long as auction revenue is increasing; that is, as long as B' is on the upward sloping portion of the debt Laffer curve.

The horizontal dotted lines labeled V_{CK}^D and $V_{CK}^D + \sigma$ give the range of default values under a zero price. These are invariant to the choice of B' , as the price of issued bonds is always zero. The V_{EG}^D lines equal their V_{CK}^D counterparts when $B' = 0$, as auction revenue is zero regardless of price.

In Panel (a) we depict the canonical “Crisis Zone” studied by Cole-Kehoe. Specifically, V_{CK}^R is less than V_{CK}^D for all $B' \geq 0$. That is, if the government cannot raise a positive amount at auction, it strictly prefers to default regardless of the realization of ϵ . Thus, zero is an equilibrium price for all $B' > 0$.

On the other hand, if the government faced the q_{EG} schedule, it would issue $B' = B_{EG}^*$

¹³As a technical aside, the continuity of $V(s)$ in B is relatively straightforward to prove in the one-period-debt Eaton-Gersovitz model (see, for example, Auclert and Rognlie (2016) and Aguiar and Amador (2019)). The complication introduced by the Cole-Kehoe timing is that the price schedule q depends on both B and B' , and, depending on the equilibrium selection assumptions, q is potentially discontinuous in B and hence the budget set can shift discontinuously in the state variable B .

¹⁴We cannot state analytically that the function is continuous and single peaked. In all of our quantitative explorations, V_{EG}^R turns out to be single peaked.

and would repay at settlement with probability one. This is because $V_{EG}^R(Y, B, B_{EG}^*) > V_{EG}^D(Y, B_{EG}^*) + \sigma$. Hence, both default and repayment can be supported in equilibrium.

Due to the lack of commitment to repayment after the auction, the Eaton-Gersovitz price schedule may be valid on only a subset of the debt-issuance domain even under the “best” equilibrium beliefs. Specifically, $q_{EG}(B')$ is a valid equilibrium price schedule only if $V_{EG}^R(Y, B, B') \geq V_{EG}^D(Y, B') + \sigma$. Define:

$$\mathbb{B}_{EG}(Y, B) \equiv \{B' \in [0, \overline{B}] \mid V_{EG}^R(Y, B, B') \geq V_{EG}^D(Y, B') + \sigma\}. \quad (15)$$

On this domain, the Eaton-Gersovitz price is consistent with zero probability of default at settlement. As we are fixing (Y, B) in this discussion, we shall drop the arguments of \mathbb{B}_{EG} when convenient. If V_{EG}^R and V_{EG}^D are continuous in B' , \mathbb{B}_{EG} is a closed set, although it may be the empty set if default dominates repayment at all levels of debt issuances. When $V_{EG}^R(Y, B, \cdot) - V_{EG}^D(Y, \cdot)$ is single peaked and continuous, as it is depicted in Figure 3, we have $\mathbb{B}_{EG} = [\underline{B}_{EG}, \overline{B}_{EG}]$, where

$$\underline{B}_{EG}(Y, B) \equiv \min \mathbb{B}_{EG}(Y, B) \quad (16)$$

$$\overline{B}_{EG}(Y, B) \equiv \max \mathbb{B}_{EG}(Y, B). \quad (17)$$

Outside of \mathbb{B}_{EG} , there is a possibility that the government defaults even if it had auctioned debt at the Eaton-Gersovitz price, eliminating the Eaton-Gersovitz price schedule as a potential equilibrium outcome for such B' .

By Proposition 1, given that $q = 0$ is an equilibrium price for all $B' > 0$ in panel (a), an interior price can be supported in equilibrium for $B' \in \mathbb{B}_{EG}$. Specifically, the equilibria map to the example in Panel (a) of Figure 2. We do not depict the value when facing this third price schedule, although by construction it lies between V_{CK}^D and $V_{EG}^D + \sigma$.

Collecting results, in the scenario depicted in Panel (a) of Figure 3, there are three possible equilibrium price schedules (and combinations thereof): A zero price for all $B' > 0$, as well as both the Eaton-Gersovitz price and the interior price for $B' \in \mathbb{B}_{EG}$.

Following Cole and Kehoe (2000), the standard construction of a rollover crisis equilibrium is to allow a run to occur only if a zero price is valid for *all* $B' > 0$. That is, the government is unable to auction *any* amount of debt. This is the case depicted in panel (a). However, this is an overly narrow view of failed auctions. To explore this, we introduce and discuss panel (b).

Panel (b) is nearly identical to Panel (a); the lone difference is that the intercept of V_{CK}^R is greater than V_{CK}^D . The difference between panels (a) and (b) reflect different initial states (Y, B) , such that V_{EG}^R in panel (b) is shifted up relative to the default payoffs. Hence, even

at a zero price for new bond issuances, the government strictly prefers to repay maturing bonds at $\epsilon = 0$. This rules out zero as an equilibrium price in the neighborhood of $B' = 0$. Define:

$$\mathbb{B}_{CK}(Y, B) \equiv \{B' \in [0, \overline{B}] \mid V_{CK}^R(Y, B, B') \leq V_{CK}^D(Y)\}.$$

If $B' \in \mathbb{B}_{CK}$, then a zero price is supportable in equilibrium. The following states that \mathbb{B}_{CK} is non-empty if B is high enough or Y is low enough:

Proposition 5. *A necessary and sufficient condition for $\mathbb{B}_{CK}(Y, B) \neq \emptyset$ is*

$$u(Y - B) - u(Y) + \beta\sigma\mathbb{E}\epsilon \leq 0. \quad (18)$$

As $\beta\sigma\mathbb{E}\epsilon > 0$, the proposition says that a zero price can be supported for some $B' \in [0, \overline{B}]$ if B is high enough (and strictly positive) and Y is low enough (as $u'(Y - B) > u'(Y)$ when $B > 0$).

For a given (Y, B) such that (18) holds, let $\underline{B}_{CK} \equiv \inf \mathbb{B}_{CK}(Y, B)$. When $\mathbb{E}[V(s')|B']$ is continuous and weakly decreasing in B' , as depicted in Figure 3, $\mathbb{B}_{CK} = [\underline{B}_{CK}, \overline{B}]$. In Figure 3 Panel (b), a zero price is not an equilibrium for $B' < \underline{B}_{CK}$, but a failed auction is possible if the government were to auction $B' \geq \underline{B}_{CK}$. Hence, if creditors coordinate on the lowest possible price, the government faces a positive price for a non-trivial but *truncated* domain of debt issuances. We shall discuss this in more depth in the next section.

As V_{EG}^R lies above $V_{EG}^D + \sigma$ at $B' = 0$, the Eaton-Gersovitz price is valid for all $B' \leq \overline{B}_{EG}$; that is $\mathbb{B}_{EG} = [0, \overline{B}_{EG}]$. Thus, the distinction with Panel (a) is that in Panel (b) the Cole-Kehoe price schedule has a restricted domain, while the Eaton-Gersovitz price domain is no longer truncated on the left. As was the case in Panel (a), there are three supportable equilibrium price schedules over part of the domain in Panel (b). Specifically, for $B' \in \mathbb{B}_{CK} \cap \mathbb{B}_{EG}$, there is an interior equilibrium price.

2.8 Equilibrium Beliefs and Prices

We now discuss how we map the possible equilibrium prices to the creditor belief sunspot ρ . In particular, we consider three possible belief regimes, each roughly corresponding to one of the three types of prices discussed above. We index beliefs by the exogenous state variable $\rho \in \{P, C, O\}$. The element “P” denotes pessimistic (or “worst-case”) beliefs, “O” denotes optimistic (or “best-case”) beliefs, and “C” denotes “concerned.” Note that these beliefs differ in regard to the imminent repayment of maturing debt; beliefs over future outcomes are held fixed (although it is straightforward to introduce persistence in belief regimes).

For all beliefs, $q(s, B') = R^{-1}$ for $B' \leq 0$. For expositional simplicity, we omit this domain from the characterizations below. It is also useful to define $q_I(Y, B, B')$ as the largest interior price, when an interior price exists:¹⁵

$$q_I(Y, B, B') = \max_{\epsilon \in [0,1]} \{F(\epsilon)q_{EG}(B') \text{ s.t. Equation (13) holds}\}.$$

Recall that the above analysis established conditions under which the interior price, when it exists, is unique in which case the max operator just returns this price.

If $\rho = O$, then creditors coordinate on the highest possible equilibrium price for state (Y, B, B') . In particular, for $B' > 0$:¹⁶

$$q(Y, B, \rho = O, B') = \begin{cases} q_{EG} & \text{if } B' \in \mathbb{B}_{EG}(Y, B) \\ \max\{0, q_I(Y, B, B')\} & \text{otherwise.} \end{cases} \quad (19)$$

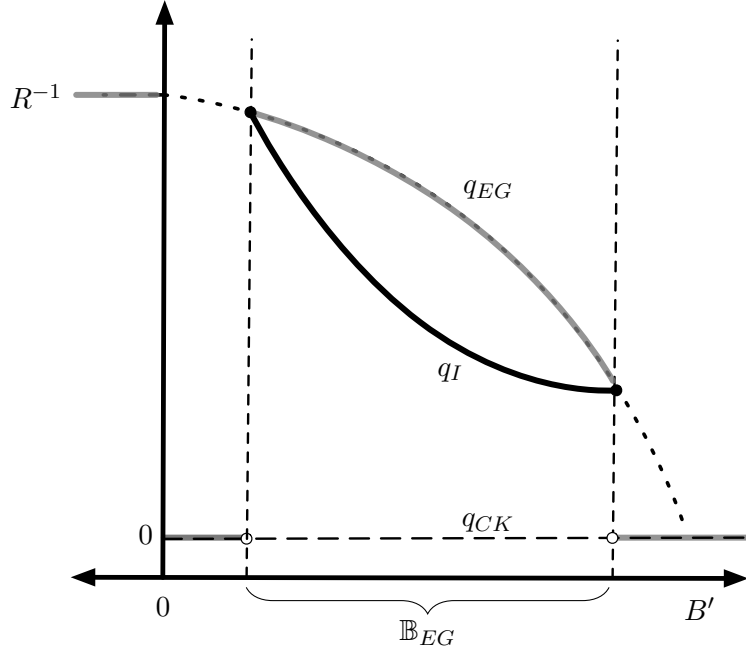
That is, if $q_{EG}(B')$ is supportable as an equilibrium price, then $q(Y, B, O, B') = q_{EG}(B')$. Otherwise, creditors coordinate on the highest interior price, if one exists, and, if no interior price exists, creditors coordinate on zero. From Proposition 2, we know that as $\sigma \rightarrow 0$, there is no interior price if $q_{EG}(B')$ is not supportable, and hence this last line is unambiguously zero.

Panels (a) and (b) of Figure 4 depict the price selection associated with the scenarios of the respective panels of Figure 3. Figure 4 assumes $\sigma \rightarrow 0$ for expositional simplicity. The downward sloping dotted line is $q_{EG}(B')$. The shaded portion of that line depicts the optimistic price schedule when $B' \in \mathbb{B}_{EG}$. Recall that for Figure 3 Panel (a), $\underline{B}_{EG} > 0$, and hence there is an interval in the neighborhood above $B' = 0$ in which the Eaton-Gersovitz price is not sustainable in equilibrium. As $\sigma \rightarrow 0$, Proposition 2 states that the only sustainable price on this domain is zero. For $B' \in \mathbb{B}_{EG}$, the optimistic price schedule is q_{EG} . For $B' > \bar{B}_{EG}$, the only sustainable price is again zero for small σ . In Panel (b) of Figure 3, we have $\underline{B}_{EG} = 0$, and hence the optimistic price schedule tracks q_{EG} for all $B' \leq \bar{B}_{EG}$ in Panel (b) of Figure 4.

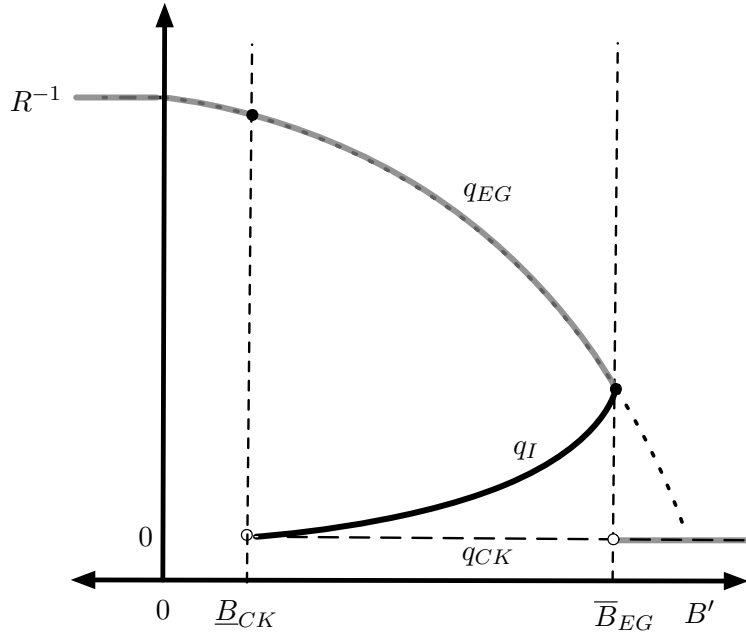
¹⁵Note that solutions to equation (13) are the zeros of a continuous function, and hence constitute a closed set. Thus, the maximal element is well defined.

¹⁶Implicit in this definition is the fact that the second case selects zero if no interior price exists.

Figure 4: Price Schedules



(a) Price Schedule Associated with Figure 3 Panel (a)



(b) Price Schedule Associated with Figure 3 Panel (b)

If $\rho = P$, then creditors coordinate on the lowest possible equilibrium price:

$$q(Y, B, \rho = P, B') = \begin{cases} 0 & \text{if } B' > \underline{B}_{CK}(Y, B) \\ q_I(Y, B, B') & \text{if } B' \leq \underline{B}_{CK} \text{ and } q_I(Y, B, B') > 0 \\ q_{EG}(B') & \text{otherwise.} \end{cases} \quad (20)$$

If a failed auction is possible at (Y, B, B') , then when $\rho = P$ creditors coordinate on the zero price. The one exception is for $B' = \underline{B}_{CK} \equiv \inf \mathbb{B}_{CK}(Y, B)$. We set $q(s, \underline{B}_{CK})$ equal to a strictly positive price (either $q_I > 0$ or q_{EG}); otherwise, the domain on which prices are strictly positive is open as $B' \uparrow \underline{B}_{CK}$, and the optimal debt issuance policy may not be well defined. As we shall see, the last two lines of the definition imply that the pessimistic price schedule coincides with the concerned price schedule for $B' \leq \underline{B}_{CK}$.

In Panel (a) of Figure 4, the pessimistic price schedule is the horizontal dashed line at zero. Recall that in Figure 3 Panel (a) we depicted the canonical Cole-Kehoe Crisis Zone with $\underline{B}_{CK} = 0$. Hence, a zero price is supportable for all $B' > 0$. In panel (b), however, $\underline{B}_{CK} > 0$. Hence, the pessimistic beliefs track q_{EG} for a non-trivial portion of the positive domain. As depicted, our selection ensures that q is upper semi-continuous at \underline{B}_{CK} , and hence the zero price applies only for $B' > \underline{B}_{CK}$.

If $\rho = C$, creditors focus on the within-period uncertainty and coordinate on the largest interior price, if one exists (and otherwise coordinate on the optimistic price). In particular,

$$q(Y, B, \rho = C, B') = \begin{cases} q_I(Y, B, B') & \text{if } q_I(Y, B, B') > 0 \\ q(Y, B, \rho = O, B') & \text{otherwise,} \end{cases} \quad (21)$$

where as with $\rho = P$, we restrict $q_I > 0$ to ensure a well-behaved budget set. Hence, for concerned beliefs, creditors coordinate on the price associated with the highest supportable indifference thresholds. The coordination on the highest root is chosen so that if the boundary case of $\tilde{\epsilon} = 1$ satisfies (13), then q_{EG} is the equilibrium price. We will return to this feature when we discuss the government's debt issuance policy below. As established in Proposition 2, if σ is small, then there is at most one interior price, and this additional level of selection is without loss.

In Panel (a) of Figure 4, we have a well defined interior price on \mathbb{B}_{EG} . As the government is indifferent to default when facing q_{EG} at the boundaries of this domain, the interior price coincides with q_{EG} at \underline{B}_{EG} and \overline{B}_{EG} . On the interior of \mathbb{B}_{EG} , there is a strictly interior price.

In Panel (b), an interior price is only sustainable on $[\underline{B}_{CK}, \overline{B}_{EG}]$. At \underline{B}_{CK} , the government is indifferent to default at the zero price. Hence, q_I approaches zero as $B' \downarrow \underline{B}_{CK}$. As before,

q_I approaches q_{EG} as $B' \uparrow \bar{B}_{EG}$.

The fact that $q_I(Y, B, \cdot)$ is non-monotonic in B' reflects the two sources of risk. As the government auctions more debt it relaxes the burden on repaying maturing bonds. For $B > 0$, concavity of u implies that more auction revenue increases $u(Y - B + qB')$ more than $u(Y + qB')$, raising the likelihood of repayment at settlement. However, an increase in B' may lower the continuation value and increase the probability of default. The net effect is ambiguous. In Panel (a), we depict the latter effect dominating and q_I is always decreasing. In Panel (b), the former effect dominates and the interior price is increasing in B' .

The non-monotonicity reflects that the interior price captures elements of both a failed auction and the standard Eaton-Gersovitz concerns about future default. That is, creditors worry whether the government is raising enough auction revenue to repay maturing debt in the current period as well as about whether the government will default in the next period. The pessimistic beliefs put all the weight on the former, generating a zero price. The optimistic beliefs put all the weight on the latter, generating the Eaton-Gersovitz price. And the interior price is a mixture of the two.

The price schedule depends on maturing debt B as well as new issuances, and the comparative static with respect to B is also not necessarily monotone. While perhaps counter-intuitive that more legacy debt is not always a negative for current spreads, it is the combination of two intuitive forces. On the one hand, a lower B reduces the scope for negative outcomes (either q_{CK} or q_I), favoring the Eaton-Gersovitz price. This may generate a higher price conditional on $\rho \neq 0$, as the only sustainable price becomes q_{EG} as B falls. However, if an interior price exists, it may *increase* as B declines. This reflects that the point of indifference between repayment and default occurs at a lower auction price the smaller is outstanding debt. That is, governments with low amounts of legacy debt are less prone to a crisis, but *conditional* on having a crisis, it will involve a relatively severe spike in spreads to make default a credible threat.¹⁷

Note that under any of the beliefs discussed above, the government may face a discontinuous price schedule. Clearly, the government will never issue B' just to the right of a discontinuous drop in the price schedule when the continuation value is decreasing in B' . Such a debt issuance raises less revenue and depresses the continuation value. The next subsection discusses optimal debt issuance and formally confirms that intuition. Moreover,

¹⁷More precisely, the comparative statics are such that the interior price increases in B for small σ . Recall that this is the case depicted in Panel (a) of Figure 2, and an increase in B shifts the increasing part of the \tilde{F} curve to the right, increasing the point of intersection with the 45-degree line. For large σ , there may be multiple interior prices with differing comparative statics (panel (b) of Figure 2), or a unique interior price that is the maximal price sustainable with any beliefs (panel (c)) that has the opposite comparative static to the small σ case.

we establish that there are conditions under which the government will never issue debt at *any* part of the domain with a strictly interior price. That is, the government will optimally choose a debt level that zeros out intra-period default risk.

2.9 Debt Policy Functions

We now discuss the government's optimal debt issuance policy given an equilibrium price schedule and continuation value. We begin with perhaps the most surprising insight. Namely, under certain general conditions, in equilibrium the government will never issue debt at a strictly interior price. Rather, it opts to issue a larger amount of debt in order to eliminate intra-period risk. We begin this subsection by stating and discussing this result, and then turn to the more familiar cases of debt issuance under pessimistic and optimistic beliefs. The analysis assumes $\sigma > 0$, and we will note how things change at the limit $\sigma = 0$.

We begin with the following result:

Proposition 6. *For a given $s \in S$ with $\rho = C$, consider two possible debt issuances $\{B'_1, B'_2\}$ with $q(s, B'_i)$ interior $i = 1, 2$. If (i) $\frac{q(s, B'_1)}{q_{EG}(B'_1)} \leq \frac{q(s, B'_2)}{q_{EG}(B'_2)}$, and (ii) $q(s, B'_1)B'_1 \leq q(s, B'_2)B'_2$, then B'_2 weakly dominates B'_1 as a policy choice, and strictly dominates if either of the two inequalities is strict.*

The first condition says that the intra-period risk of B'_2 is less than that of B'_1 . In particular, if $\tilde{\epsilon}_i$, $i = 1, 2$, are the respective thresholds from equation (13), then $F(\tilde{\epsilon}_2) \geq F(\tilde{\epsilon}_1)$, making it less likely the government defaults at settlement after issuing B'_2 . The second condition says that B'_2 raises more auction revenue than B'_1 .

Proposition 6 states that, when choosing over the interior price schedule, the government favors less intra-period risk and more auction revenue, all else equal. Applying this result to Figures 3 and 4, the government strictly prefers to issue at \bar{B}_{EG} than any debt level that fetches an interior price and raises less revenue:

Corollary 1. *For a given $s \in S$ with $\rho = C$, suppose there exists $\bar{B}_{EG} = \max \mathbb{B}_{EG}$ such that $q(s, \bar{B}_{EG}) = q_I(s, \bar{B}_{EG}) = q_{EG}(\bar{B}_{EG})$. Then \bar{B}_{EG} dominates any B' as a debt issuance such that $q(s, B') < q_{EG}(B')$ and $q(s, B')B' \leq q_{EG}(\bar{B}_{EG})\bar{B}_{EG}$.*

At \bar{B}_{EG} , the auction occurs at the Eaton-Gersovitz price, but at a debt level that is elevated relative to B_{EG}^* . By issuing additional debt, the government can eliminate the risk of not repaying maturing debt at settlement, but at the cost of raising the probability of default next period.

Proposition 6 implies that the government will never issue debt on the interior of \mathbb{B}_{EG} , unless \bar{B}_{EG} is on the wrong side of the debt Laffer curve. On the interior, the government

receives a low price due to concern about intra-period risk. As the government is indifferent at the margin to such risk, it has an incentive to choose debt that minimizes intra-period risk. The proposition says that the government will issue debt such that there is zero intra-period risk.¹⁸ That is, it will only issue debt at the EG price in equilibrium. Hence, the primary impact of concerned beliefs is not to deliver an interior price *on the equilibrium path*, but rather to push the government to issue an alternative level of debt. In particular, it may lead the government to issue more debt than it would under the optimistic beliefs. We shall return to this point in the next subsection when we talk about alternative types of debt crises. This result will also be echoed in the policy functions of the quantitative model in Section 4; in particular, see Figure 6.

For the case depicted in panel (a) of Figure 3, the proposition states that the government will issue at $\bar{B}_{EG} > B_{EG}^*$. In particular, the result states that the points on the interior of $(\underline{B}_{EG}, \bar{B}_{EG})$ are dominated by the right end point. Given that V_{EG}^D is strictly increasing for $B' \leq \bar{B}_{EG}$, \bar{B}_{EG} also dominates \underline{B}_{EG} . The proposition leaves open the possibility that the government may issue $B' > \bar{B}_{EG}$, if that raises more revenue at auction. In that case, the government is attracted by the extra revenue, which it keeps in the case of default. However, Proposition 2 states that as $\sigma \rightarrow 0$, the points outside of \mathbb{B}_{EG} have a price that approaches zero, and hence these alternatives are also dominated.

For the case of panel (b) of Figure 3, the government will either issue \underline{B}_{CK} or \bar{B}_{EG} , assuming $B' > \bar{B}_{EG}$ does not raise additional revenue. Issuing \underline{B}_{CK} yields the Eaton-Gersovitz price, as even at a zero price the government does not default. It is in general ambiguous whether the repayment value at \underline{B}_{CK} is greater or less than the repayment value at \bar{B}_{EG} . The proposition establishes that issuances on the interior of $(\underline{B}_{CK}, \bar{B}_{EG})$ are dominated by \bar{B}_{EG} . In the case of panel (b), therefore, the government may over-issue or under-issue relative to B_{EG}^* when facing concerned beliefs. The common implication is that the government's debt issuance yields the Eaton-Gersovitz price in equilibrium, but the quantity of issuance is distorted relative to B_{EG}^* .

For large σ , there may be an interior price at $B' > \bar{B}_{EG}$ which raises enough additional revenue that it warrants risking default at settlement (and keeping the revenue) rather than issuing \bar{B}_{EG} and securing the EG price. In our simulations of the quantitative model presented in Section 4, issuance at $B' > \bar{B}_{EG}$ under concerned beliefs does not happen in equilibrium.

When $\sigma = 0$, interior prices are supported with mixed strategies. In this case, the

¹⁸This intuition uses the fact that fixing B' , the government always prefers a better price in exchange for a lower probability of intra-period default. It leaves out the impact on the continuation value from increasing B' . The proof establishes that, under concerned beliefs, the former effect dominates the latter.

government is indifferent over a range of issuances at interior prices. However, the proposition states that only \bar{B}_{EG} is valid for σ in the neighborhood of zero.

One final point. The incentive to “over-issue” to avoid intra-period default risk exists if revenue is increasing along the concerned price schedule all the way to \bar{B}_{EG} . This can be true even if \bar{B}_{EG} is on the downward sloping portion of the *EG revenue curve*. Indeed, in the simulations reported later in the paper, this turns out to be the case. Thus, in our framework, sovereign’s behavior might well put the economy on the wrong side of the revenue Laffer curve drawn for the EG price. Of course, the reason why the sovereign is on the wrong side of the EG Laffer curve is that it is facing the concerned, not the EG, price schedule, and for concerned price schedule it is on the upward portion of the corresponding revenue curve.

We now turn to debt issuance under the more familiar optimistic and pessimistic beliefs. Under optimistic beliefs, the government faces the best supportable price schedule. For $\sigma \rightarrow 0$, this implies the government will optimize over $B'(-\infty, 0] \cup \mathbb{B}_{EG}$ conditional on the $q_{EG}(B')$ price schedule. This is similar to the standard problem under intra-period commitment (conditional on future equilibrium play), with the only difference introduced by the current timing being that not all B' are consistent with guaranteed repayment at settlement. That is, \mathbb{B}_{EG} does not necessarily include all B' such that $q_{EG}(B') > 0$, as the government has the option to default immediately after the auction. For large σ , the government facing optimistic creditors may choose to issue debt at an interior price; in particular, for large σ there are debt levels which cannot sustain q_{EG} but have a strictly positive price.

As in the standard model, if Y is low enough and B high enough, the government defaults with probability one. In that case, the value of default dominates any achievable repayment value. This can be considered a “fundamental” default in the sense that default occurs regardless of creditor beliefs.

Under pessimistic beliefs, the government faces a price of zero for $B > \underline{B}_{CK}$. If $\underline{B}_{CK} = 0$, we are in the standard Cole-Kehoe crisis scenario. That is, the government cannot issue any amount of debt at a positive price. It therefore has the option of purchasing assets and repaying B or defaulting with probability one. If $\underline{B}_{CK} > 0$, there is a range of debt that carries a strictly positive price. As $\sigma \rightarrow 0$, this price becomes the Eaton-Gersovitz price. Issuing a small amount of debt at this price may dominate outright default (or purchasing assets). If so, then the equilibrium outcome of pessimistic beliefs is not default but truncated debt issuance.

The interesting commonality between the three belief regimes as $\sigma \rightarrow 0$ is that all imply the government either issues at Eaton-Gersovitz prices or defaults with probability one. The

impact of creditor beliefs is on what level of issuances (if any) the government undertakes at auction. Observed equilibrium prices will differ across beliefs due to the differences in debt issuances, but the observed equilibrium prices will all be points on the best possible price schedule.

2.10 Long-Term Debt and Buybacks

So far we have assumed one-period bonds. In this subsection, we briefly indicate what changes if the debt is of longer duration. The new margin introduced by long duration debt is that the sovereign can save by buying back existing debt. This raises the issue as to what price this buyback can occur and whether the buyback price is subject to belief shocks.

We assume that the sovereign borrows by issuing long-term non-contingent bonds. These bonds pay a coupon every period up to and including the period of maturity, which, without loss of generality, we normalize to r^* per unit of face value, where r^* is the (constant) international risk-free rate. With this normalization, a risk-free bond will have an equilibrium price of one. For tractability, we consider a bond with random maturity, as in Leland (1994).¹⁹ In particular, each bond matures next period with a constant hazard rate $\lambda \in [0, 1]$. We let the unit of a bond be infinitesimally small, and let maturity be i.i.d. across individual bonds, such that with probability one a fraction λ of any non-degenerate portfolio of bonds matures each period. The constant hazard of maturity implies that all bonds are symmetric before the realization of maturity at the start of the period, regardless of when they were purchased. Note as well the expected maturity of a bond is $1/\lambda$ periods, and so $\lambda = 0$ is a console and $\lambda = 1$ is one-period debt. While we vary λ across quantitative exercises, within any specific environment there is only one maturity traded. With these conventions, a portfolio of sovereign bonds of measure x receives a payment (absent default) of $(r^* + \lambda)x$, and has a continuation face value of $(1 - \lambda)x$.

If the government honors its obligations at settlement, its payoff is:

$$V^R(s, B') = u(Y - (r^* + \lambda)B + q(s, B')[B' - (1 - \lambda)B]) + \beta \mathbb{E}[V(s')|s, B'], \quad (22)$$

If the government defaults, its payoff is

$$V^D(s) \equiv u(Y + q(s, B')[B' - (1 - \lambda)B]) + \beta \mathbb{E}V^D(s') + \sigma\epsilon, \quad (23)$$

When $B' - (1 - \lambda)B > 0$, the government is adding to its (non-maturing) debt stock.

¹⁹See also Hatchondo and Martinez (2009), Chatterjee and Eyigungor (2012) and Arellano and Ramarayanan (2012).

Here the situation is similar to the one-period debt case analyzed previously. The pricing of the new debt will depend on the investors beliefs about the likelihood of default at settlement on the maturing obligations (term $-(r^* + \lambda)B$ in (22)).

More interestingly, when $B' - (1 - \lambda)B < 0$ the government is buying back some its debt rather than issuing more debt. Since the buyback takes place at the time of the auction, any funds spent on the buyback are sunk. This is why they are deducted from both the repayment consumption level and the default consumption level.

The analog of our interior price condition with one period debt (13) is now given by

$$\begin{aligned} \sigma \tilde{\epsilon} = & u(Y - (r^* + \lambda)B + \tilde{q} \times [B' - (1 - \lambda)B]) - u(Y + \tilde{q} \times [B' - (1 - \lambda)B]) \\ & + \beta \mathbb{E}[V(s')|s, B'] - \beta \mathbb{E}V^D(s'). \end{aligned} \quad (24)$$

where $\tilde{q} = F(\tilde{\epsilon})q_{EG}$. Note that the right hand side of (24) is decreasing in q if there are buybacks ($B' < (1 - \lambda)B$), hence the right-hand side is decreasing in $\tilde{\epsilon}$. Therefore, with buybacks, there can be only one equilibrium price for a fixed (s, B') . This leads to the following result.

Proposition 7. *Given s and B' , if $B' < (1 - \lambda)B$, the equilibrium price $q(s, B')$ is unique.*

Even though the equilibrium buyback price is unique, it could still reflect positive intra-period default risk because the sovereign still has to service outstanding debt (the term $-(r^* + \lambda)B$ in (22)) which it may choose not to do at settlement. However, the next proposition asserts that if by reducing its buybacks the sovereign can reduce both intra-period risk *and* the amount being paid out to lenders, it is optimal for it to do so. This effectively means that when buybacks occur, they occur at EG prices.

Proposition 8. *Consider two possible buybacks $\{B'_1, B'_2\}$ with $q(s, B'_i)$ interior $i = 1, 2$, $B'_i < (1 - \lambda)B$ and $B'_1 > B'_2$. If (i) $\frac{q(s, B'_1)}{q_{EG}(B'_1)} \geq \frac{q(s, B'_2)}{q_{EG}(B'_2)}$ and (ii) $q(s, B'_1)[B'_1 - (1 - \lambda)B] \geq q(s, B'_2)[B'_2 - (1 - \lambda)B]$ then B'_1 weakly dominates B'_2 and does so strictly if either of these two inequalities is strict.*

3 A Unified Approach to Crises

In the data, debt crises take a variety of forms. The Mexico 1994-5 crisis, as argued by Cole and Kehoe (1996), had the hallmarks of a classic run. A large literature has explored “sudden stops” in which a country’s borrowing is severely reduced, but not eliminated. On the other hand, some countries increase debt during a crisis; for example, see the case-study of Italy 2008-12 in Bocola and Dovis (2016). There are also plausibly “fundamental”

defaults in which beliefs do not matter, that is a country's high debt and deep recession generate a default as the only equilibrium outcome. In many, if not all, sovereign debt crises, a third party such as the IMF or the EU has played a role, although the size of third-party participation varies across events. Finally, crises often cluster in time, so that more than one country is subject to high spreads over the same period. In this section, we show that the framework outlined above is a useful lens through which to view this varied set of experiences. In particular, the model admits a very rich menu of different types crises, some of which end in immediate default and some of which do not involve default but nevertheless affect equilibrium prices and debt positions. The results established in the previous section allow us to pinpoint what combination of state variables and creditor beliefs give rise to the alternative scenarios.

3.1 Cole-Kehoe Rollover Crisis

We begin with the canonical Cole-Kehoe crisis. Consider Panel (a) of Figure 3. A zero price is an equilibrium price schedule for all $B' > 0$. Hence, if creditors are pessimistic, the government cannot issue bonds and then defaults at settlement with probability one. For this to be a valid equilibrium outcome, Y must be sufficiently low and B sufficiently high such that $\max_{B' \leq 0} V_{EG}^R(Y, B, B') \leq V_{EG}^D(Y, 0)$; that is, the best the government can do purchasing assets or issuing zero new bonds is dominated by default. In Figure 3 Panel (a), $q_{EG}(B')$ is also a valid equilibrium price on part of the domain. In particular, under optimistic beliefs, the government will issue B_{EG}^* and repay with certainty at settlement. Hence, for this part of the state space of (Y, B) , the realization of a sunspot determines whether the government defaults or repays.

3.2 Self-Fulfilling Sudden Stop

Our expanded notion of crisis outcomes now includes the contemporaneous effects of the threat of a rollover crisis over a strict sub-set of the debt domain. This can lead to an alternative “forced deleveraging” or a “sudden stop” crisis. Consider Panel (b) of Figure 3. A zero price is an outcome here only if $B' \geq \underline{B}_{CK}(Y, B) > 0$, and to avoid this outcome, the government will optimally choose a lower borrowing level. That is, even under pessimistic beliefs, the government can issue some debt at strictly positive prices. However, it cannot issue to the level of the optimistic beliefs optimum, B_{EG}^* .

For expositional simplicity, let $\sigma \rightarrow 0$, in which case the pessimistic price schedule is $q_{EG}(B')$ for $B' \leq \underline{B}_{CK}$ in Panel (b) of Figure 3, and zero for $B' > \underline{B}_{CK}$. The optimal response for the government is to issue $\arg\max_{B' \leq \underline{B}_{CK}} V_{EG}^R(Y, B, B')$. In Panel (b) of Figure

3 the value obtained by issuing debt on this restricted domain dominates default. Thus, pessimistic beliefs do not necessarily generate an immediate default, but may instead force the government to issue a relatively small amount of new debt and cut back on consumption (relative to the optimistic benchmark). The individual lender's beliefs are supported in equilibrium due to the concern that if the government issues more debt than \underline{B}_{CK} , other creditors will fail to bid at auction and the government will not raise enough at the auction to warrant repaying maturing debt. For this to be an equilibrium outcome, (Y, B) must be more favorable than in the canonical rollover crisis, but at the same time, Y must be low enough and B high enough that $\underline{B}_{CK}(Y, B)j < \bar{B}_{EG}$. Just as in the rollover crisis, whether or not the government is forced to delever depends upon the realization of the sunspot variable.

Interestingly, the forced deleveraging choice may also be the equilibrium outcome under concerned beliefs. From Proposition 6 and the subsequent discussion, we know that as $\sigma \rightarrow 0$, the government will either issue at \bar{B}_{EG} or the pessimistic-beliefs optimal issuances. Which it chooses depends on whether $V_{EG}^R(Y, B, \underline{B}_{CK}) \gtrless V_{EG}^R(Y, B, \bar{B}_{EG})$.

3.3 Overborrowing and Slow Motion Crises

Under concerned beliefs, Proposition 6 states conditions under which \bar{B}_{EG} dominates issuing at an interior price. As noted above, this implies that the government either issues \bar{B}_{EG} or delevers to $B' \leq \underline{B}_{CK}$. If \bar{B}_{EG} is preferable and $\bar{B}_{EG} > B_{EG}^*$, then concerned beliefs lead to “over borrowing” relative to optimistic beliefs. While the price obtained is the Eaton-Gersovitz price, it will imply high spreads due to the greater level of debt issuance.

The concerned beliefs induce over borrowing because the creditors are worried that if the government were to issue less debt, it would not have adequate revenue to repay maturing debt at settlement without pushing consumption down to the point it prefers to default. This concern is reflected by the interior equilibrium price schedule. In this case, current over borrowing can lead to a sort of slow motion crisis in which the government over issues today, raising the possibility of future default. This is reminiscent of the dynamics of Lorenzoni and Werning (2013)'s slow-moving crises, but in our case the multiplicity is entirely static. In particular, conditional on issuing \bar{B}_{EG} , the equilibrium price is unique. However, the off-equilibrium beliefs about intra-period risk lead the government to issue more debt than is optimal under optimistic beliefs.

3.4 Uncertain Bailouts

Another interpretation of the intra-period risk is uncertainty over the size of a potential bailout from a third party, like the IMF or the ECB. Suppose at the time of settlement, the

government seeks assistance from the third party. The third party commits to transfer $\sigma\varepsilon$ conditional on repayment, with ε drawn from a distribution with support $[0, 1]$. The value of repayment conditional on the draw of ε is:

$$u(Y - B + q(s, B')B' + \sigma\varepsilon) + \beta\mathbb{E}[V(s')|B'].$$

The value of default is $V^D(s, B')$ given in equation (2). The government repays as long as:

$$u(y - B + q(s, B')B' - \sigma\varepsilon) \geq u(Y + q(s, B')B') + \beta\underline{V} - \beta\mathbb{E}[V(s')|B'].$$

Rearranging, we have repayment if and only if:

$$\varepsilon \geq \frac{1}{\sigma} \cdot \underbrace{u(Y + q(s, B')B') + \beta\underline{V} - \beta\mathbb{E}[V(s')|B']}_{\equiv \hat{\Delta}(s, B')} - Y + B - q(s, B')B',$$

where u^{-1} is the inverse of the utility function. This expression takes the same form (with a reversed inequality) as the benchmark model's expression (3). Hence, we can recast the interior price crises as uncertainty over whether (and by how much) the government will be bailed out. Thus, the possibility of a bailout introduces room for coordination on an intermediate event between a failed auction and the EG outcome, with the same implications for prices and debt issuances as in the benchmark model.

3.5 Intra-period Risk and High Spreads

Proposition 6 included a condition that issuing \overline{B}_{EG} raised as much revenue as the alternative debt issuances which fetched an interior price. If \overline{B}_{EG} is on the wrong side of the Debt Laffer Curve, then \overline{B}_{EG} may no longer be the optimal choice when $\rho = C$. In this case, it may be optimal to issue $B' < \overline{B}_{EG}$ at a strictly interior price. Lenders then face intra-period risk and prices are depressed relative to optimistic beliefs. Hence, the crisis is associated with high spreads combined with an interior probability of default at settlement.²⁰

3.6 Fundamental Defaults

The model also admits a standard “fundamental” default in which the unique equilibrium at a given (Y, B) is default. This occurs when the debt level is sufficiently high and the income level sufficiently low that repayment is dominated by default regardless of creditor

²⁰Note that if σ is large, then it may be the case that even optimistic beliefs carry with them intra-period risk.

beliefs.

3.7 Contagious Crises

Finally, we note that many sovereign debt crises involve several countries simultaneously, typically referred to as contagion. Conceptually it is not difficult to imagine how to extend the model to include such possibilities. First, the ϵ shock itself may be correlated across countries. If the shock is interpreted as the legal or political ramifications of default, it is plausible that the severity of default punishment is not independent across similar countries. The fact that external debt is typically issued in a handful of financial centers, such as New York and London, make correlated punishments a natural assumption. Hence, if lenders are concerned about the failure of enforcement (a high ϵ in our notation) in one country, there is a reason to have the same concern regarding other debtors.

The bailout interpretation in Section 3.4 also naturally lends itself to cross-country correlation, given that the same third party may be present in multiple countries. The fact that the EU bailout of Greece involved creditors taking losses may have fostered uncertainty about the size of a potential bailout for other indebted euro countries.

Moreover, if the same pool of lenders are operating in many markets, a coordination failure in one market may lead to a similar failure in others. In terms of our notation, the belief variable ρ may not be independently distributed across economies.

4 Quantitative Implications

In this section, we explore how the insights from the preceding analysis manifest in standard versions of quantitative sovereign debt models. There are two primary goals to this exercise. One is to establish that variation in creditor beliefs have a significant impact in reasonably calibrated models. We do this both for short-term (one period) quantitative debt models of Aguiar and Gopinath (2006) and Arellano (2008) and for long-term debt models of Hatchondo and Martinez (2009), Chatterjee and Eyigungor (2012) and Arellano and Ramanarayanan (2012). A second goal is to assess how the magnitude of the intra-period risk, that is, σ , affects key simulated moments. The preceding section proved a number of results for the case of $\sigma \rightarrow 0$, and flagged how these results may change for $\sigma \gg 0$. Hence, whether the magnitude of σ is important is a quantitative question, one we answer in this section.

One key insight from the analytical model is that, when confronted with concerned beliefs, the government adjusts debt issuances in order to avoid intra-period risk. This section verifies that this result holds true in a richer, quantitative models. In particular, the government

prefers to redirect concerned beliefs into expenditure volatility rather than pay the price of compensating lenders for intra-period risk. The fact that debt issuances are more extreme under concerned beliefs also leads to more spread volatility, but the spreads are reflections of future default risk rather than intra-period settlement risk.

4.1 Set-Up

As in the analytical model, the government faces a stochastic endowment and trades a one-period non-contingent bond with risk-neutral lenders. Following Aguiar and Gopinath (2006, 2007), the growth rate of endowment is the sum of two terms: the first is an AR(1) process, \hat{g}_t ; the second is an MA(1), η_t . This can be interpreted as an environment with stochastic trend growth as well as an i.i.d. transitory shock:

$$\ln y_t - \ln y_{t-1} = g_t + \eta_t$$

where

$$\begin{aligned} g_t &= (1 - \rho_g)\mu_g + \rho_g g_{t-1} + \sigma_g \epsilon_t \\ \eta_t &= \sigma_s [s_t - s_{t-1}] \end{aligned}$$

and ϵ_t and s_t are i.i.d. draws from a standard Normal distribution.

Our benchmark uses Mexican quarterly real GDP data to calibrate the endowment process. We also consider an alternative calibration using Argentine data. The benchmark is chosen as Mexico is a relatively typical emerging market (see Aguiar and Gopinath (2007)). The Argentine calibration is common in the sovereign debt literature and also represents a relatively volatile economy compared with other emerging and developed economies. The parameters for the benchmark are: $\sigma_s = .0033$, $\mu_g = .0034$, $\rho_g = .445$, and $\sigma_g = .011$.

Default involves a proportional loss of endowment as well as temporary exclusion from financial markets. Exclusion ends with a Poisson hazard rate, after which the government is in good credit standing with zero debt. Following Aguiar and Gopinath (2006), we set the quarterly Poisson re-entry probability in the default state to 12.5%. The calibration of the proportional endowment loss parameter is described below.

We make standard parameter choices for preferences and asset markets. Specifically, the quarterly risk-free rate is 1%; the sovereign has separable power-utility preferences with $\text{CRRA} = 2.0$. We also follow Aguiar and Gopinath (2006) in setting $\beta = 0.8$.

4.2 Belief Regimes

Our benchmark exercise involves positive but minimal intra-period risk, which is designed to capture the limiting behavior as σ approaches zero. To that end, we assume ϵ is uniform on $[0, 1]$ and $\sigma = 0.0001$. For the “large σ ” alternative, we increase σ by a factor of 50; that is, in the alternative $\sigma = 0.005$.

Our benchmark model lets the belief state ρ transit between the optimistic and the concerned regimes. To cleanly highlight the mechanics of the model, we restrict the benchmark to two belief regimes and set the probability of the pessimistic belief regime to zero. We continue to assume ρ is i.i.d. over time, and use a moment-matching procedure to compute the probability $\rho = C$. Specifically, we select the probability $\rho = C$ and the proportional default cost d to match two key empirical moments for Mexico: average Debt/(Quarterly GDP) of 65.6%, and the standard deviation of sovereign spreads of 2.5%.²¹

The parameters that match our moments are proportional endowment loss in default of $d = 17.55\%$ and a concerned belief arrival rate of 2.25%. That is, lenders become concerned about intra-period risk once every forty-four quarters.

In addition to the benchmark, we consider four alternative models using the Mexico calibration. These are comparative static type exercises that hold everything constant and vary one key element of the environment.

Alternative I restricts beliefs to the optimistic regime with probability one. That is, the bond-price schedule is always the best that can be supported in equilibrium. This is reminiscent of the standard calibrations of the Eaton-Gersovitz model, with the one caveat that the lack of intra-period commitment may restrict the domain on which the Eaton-Gersovitz price can be supported.

Alternative II has beliefs fluctuating between optimistic and pessimistic: $\rho \in \{O, P\}$. As in the benchmark, we continue to assume the negative belief regime arrives with probability 0.0225, but now the negative regime coordinates on the worse possible price schedule.

Alternative III uses the benchmark belief process but multiplies σ by a factor of 50 to $\sigma = 0.005$. Thus, Alternative III is designed to highlight the quantitative role of significant intra-period risk.

Alternative IV increases the concerned regime’s arrival rate to 5%. At the same time we reduce the endowment loss in default to 12.5%. We describe below the expositional purpose of this alternative specification.

²¹The debt-to-GDP ratio is the mean external debt of Mexico between 2002Q1 through 2014Q3. The sovereign spread is the average EMBI for Mexico over this period.

4.3 Simulated Moments

Key simulated moments for the benchmark model and the four alternatives are presented in Table 1.

The benchmark model is designed to deliver an average debt-to-GDP (quarterly) of 65.6%. The default frequency is low relative to the data; namely, a default occurs once every two hundred years. This is a well-known feature of one-period debt models that do not have implicit “debt forgiveness” in low endowment states via a non-linear default cost. Interestingly, despite this relative low frequency of default, spreads are quite volatile. This reflects that when the concerned regime arrives, the government decides to issue a significant amount of debt at high yields, as discussed in Corollary 1. We discuss this feature in more detail below.

Table 1: Simulated Moments

Model		Mean Debt-to-GDP (Quarterly)	Default Frequency (Annualized)	StDev($r - r^*$) (Annualized)	$\frac{\text{StDev}(\ln c)}{\text{StDev}(\ln y)}$
BM		65.6%	0.5%	2.5%	1.20
Alt I	$\rho = O$	66.1%	0.1%	0.1%	1.15
Alt II	$\rho \in \{O, P\}$	30.2%	0.0%	0.0%	1.05
Alt III	$\sigma = 0.05$	64.8%	0.4%	2.5%	1.20
Alt IV	$\left\{ \begin{array}{l} \Pr(\rho = C) = 5\% \\ d=.125 \end{array} \right.$	38.9%	0.1%	2.3%	1.23

Alternative I explores the same environment, but without shifts in beliefs. This alternative model generates slightly higher debt levels, reduces the default frequency by a factor of five, and cuts spread volatility by a factor of 25. Comparing between the benchmark and this alternative indicates that the presence of the concerned regime, albeit rare in the benchmark, is pivotal in generating default and spread volatility. The comparison also indicates that the presence of the concerned regime generates additional expenditure volatility.

Alternative II replaces the concerned belief regime with the pessimistic regime. This enhances the severity of shifts in beliefs. The endogenous response of the government is to borrow much less; indeed, the average debt level is less than half of the benchmark value. The avoidance of leverage is sufficient to eliminate default on the equilibrium path. Thus, the threat of a full-blown failed auction is sufficiently severe that the government forgoes sovereign risk all together. This does not mean, however, that model dynamics are unaffected by rollover risk. Net exports become markedly more countercyclical under this

specification,²² indicating more frequent rapid capital outflows and sudden stops.

Alternative III increases the magnitude of intra-period risk relative to the negligible value of the benchmark model. The simulated moments are essentially unchanged by this increase, save for a modest deleveraging that takes place in response to the heightened level of overall risk. The similarity between the benchmark and Alternative III reflects that the government prefers to adjust its debt choices rather than face the additional spread due to intra-period risk (see Proposition 6 and Corollary 1).

Alternative IV increases the frequency of the concerned regime from the benchmark's 2.25% to 5.0%. At the same time, we lower the proportional cost of default to 12.5%. This alternative is useful to understand how the model behaves when belief-driven crises arrive with greater frequency. The response is to maintain a lower debt level on average, induce a lower default rate, all while also continuing to generate spread and consumption volatility. We shall discuss the economics behind this behavior once we have explored the benchmark in greater detail.

4.4 Price Schedules and Policy Functions

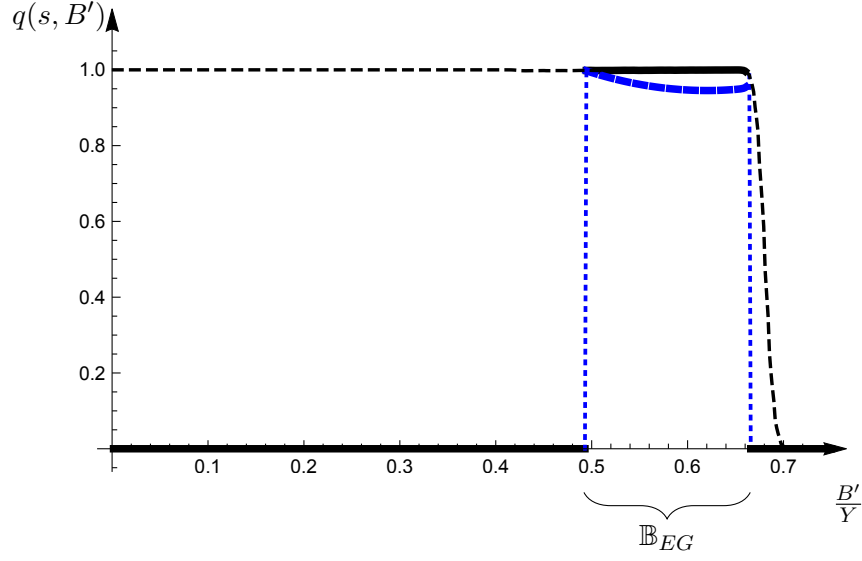
Figure 5 depicts the equilibrium price schedules the government faces at auction in the benchmark model. In particular, in Panel (a), we set endowment growth and the initial debt-to-GDP to their respective ergodic means, and trace out prices as we vary the future state B' . In Panel (b) we depict the price schedule at the same endowment state but for a lower level of initial debt; namely, $B/Y = 41.8\%$.

In each panel, the thin dashed black line depicts the reference price schedule q_{EG} introduced in the analytical model; that is, this is the break-even price of a bond if creditors were certain of repayment at the current period's settlement. The thick solid black line is the optimistic price schedule and the thick blue dashed line is the concerned price schedule.

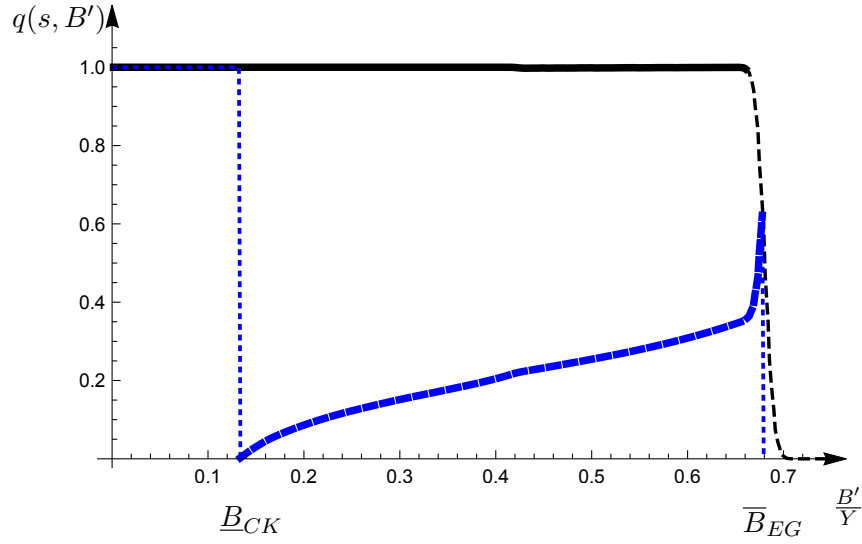
Figure 5 is the quantitative model's version of Figure 4. The economics are also the same as in the analytical model. Namely, in Panel (a), we are in the canonical Cole-Kehoe Crisis Zone and zero is always an equilibrium price. If the benchmark incorporated pessimistic beliefs in the current period, the associated price schedule would be zero for all $B' > 0$. For very low levels of future debt, q_{EG} cannot be supported in equilibrium and the equilibrium price is zero regardless of beliefs. This reflects that insufficient revenue at today's auction makes repayment at the current period's settlement non-credible. There is a subset of the debt domain, which we again label \mathbb{B}_{EG} , on which the Eaton-Gersovitz price schedule is supportable. On this domain, an interior price schedule is also a valid equilibrium outcome,

²²Specifically, the correlation of nx/y and y is -0.10 in the benchmark and drops to -0.26 in Alternative II.

Figure 5: Price Schedules: Evaluated at Mean Endowment Growth



(a) Initial State: $B/Y = 65.6\%$



(b) Initial State: $B/Y = 41.8\%$

and creditor beliefs determine which schedule is realized at auction. Note that \mathbb{B}_{EG} includes the ergodic mean and surrounding debt levels. For larger values of future debt, prices fall to zero again, regardless of beliefs, as the future debt burden is great enough to trigger immediate default.

In Panel (b), for debt issuances less than \underline{B}_{CK} , the only possible equilibrium price is the Eaton-Gersovitz price. This reflects that legacy debt is at such a low level, the government is willing to repay even without auctioning new debt. If creditors coordinated on pessimistic beliefs, the price schedule would be q_{EG} for $B'/Y \leq \underline{B}_{CK}$ and zero otherwise. For $B'/Y \in [\underline{B}_{CK}, \overline{B}_{EG}]$, three prices are consistent with equilibrium; namely, zero, the Eaton-Gersovitz price, and an interior price. For $B'/Y > \overline{B}_{EG}$, the only supportable price is zero.

Figure 6 depicts the government's policy function. The horizontal axis is the initial debt state; the solid black line is the optimal issuance policy in response to the optimistic price schedule; and the dashed blue line is the optimal issuance policy in response to the concerned price schedule. Each policy function is evaluated at the mean endowment state. The diagonal line from the origin is the 45-degree line. Debt levels to the right of the depicted lines involve default with probability one, and therefore do not have well-defined policies.

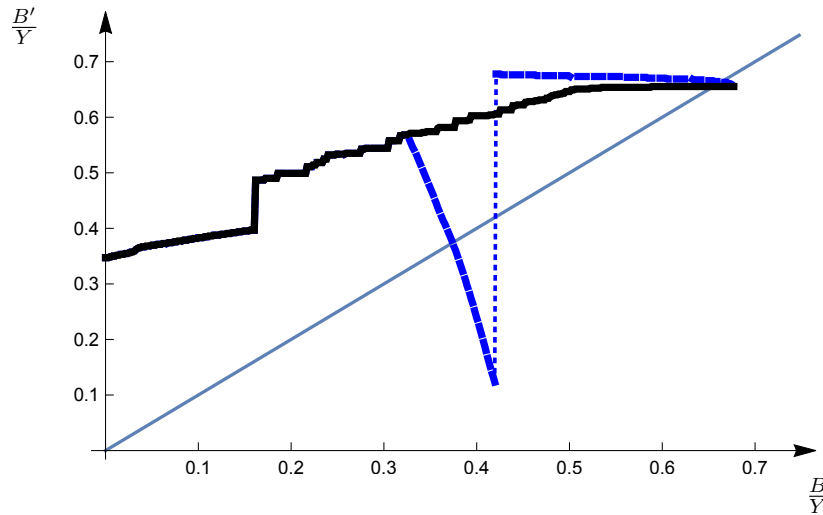


Figure 6: Debt-Issuance Policy Functions

The policy response to the optimistic price schedule has a familiar upward slope, before leveling out near the ergodic mean. Hence, if the government were to consistently face opti-

mistic beliefs, it would quickly issue debt until close to the ergodic mean. After building up its debt level, endowment shocks induce small variation in debt levels until a low endowment realization occurs in the presence of relatively high debt, at which point the government defaults. This is the familiar debt dynamics from the standard quantitative Eaton-Gersovitz models.

The policy response to the concerned price schedule is markedly different and tracks the intuition provided by Proposition 6, Corollary 1, and the subsequent discussion in that section. At low initial debt levels and low issuances, the concerned price schedule overlaps with the optimistic schedule (see Figure 5b). Hence, the policies overlap on this domain in Figure 6. For initial debt levels between 33 and 42 percent of output, the concerned regime forces the government to truncate its debt issuances. Such low debt levels secure the Eaton-Gersovitz price and avoid intra-period risk. For the same reason, but with the opposite outcome, the government over issues (relative to the optimistic policy) for debt levels beyond 42 percent of the endowment. Again, the government avoids intra-period risk, but on this domain finds it preferable to issue additional debt.

In regard to the price schedule depicted in Figure 5a, the government is issuing at the point the concerned schedule meets up with the Eaton-Gersovitz schedule at the upper boundary of \mathbb{B}_{EG} . This reflects the forces captured in Proposition 6. The fact that the Eaton-Gersovitz price schedule is sharply declining while the concerned price schedule is flat generates the marked differences in policy. When facing optimistic prices, the steeply downward sloping price schedule in the neighborhood of \bar{B}_{EG} discourages the government from issuing such a high level of debt. In fact, \bar{B}_{EG} is on the downward sloping part of the optimistic-price debt Laffer curve.

Conversely, under concerned beliefs, the government faces a much flatter (or even upward sloping) price schedule near \bar{B}_{EG} . This is because at lower levels of debt issuances, creditors are worried about repayment at settlement. Hence, \bar{B}_{EG} is on the upward part of the concerned-price debt Laffer curve. From Corollary 1, this implies that \bar{B}_{EG} is the optimal policy. Nevertheless, the high levels of debt carried into next period pose a significant risk for creditors, and spreads are relatively high. In the Eaton-Gersovitz model, future default rates are closely tied to debt levels, generating a familiar sharply declining price schedule once default becomes an outcome. This is obviously present in the concerned price schedule as well, but is offset by a different force. Under concerned beliefs, *intra-period* default risk is very sensitive to debt issuances, with relatively *low* levels of issuance generating higher risk due to the concern about whether the government has raised enough to repay old debt at settlement. This countervailing force, which is not present under Eaton-Gersovitz timing, generates the flatter price schedule and induces the government to issue more debt.

While the concerned price schedule is flatter in the neighborhood of \overline{B}_{EG} , it is not high. That is, future default risk is significant, and this is reflected in spreads. In the benchmark simulation, the average spread is 0.6%, reflecting the overall low default rate. However, conditional on the concerned regime, the average spread is 16%, more than 25 times the unconditional mean. Moreover, 23 percent of the defaults coincide with the arrival of the concerned regime (relative to the 2.25 percent unconditional arrival rate of concerned beliefs). However, nearly half (46%) of all defaults are immediately preceded by the concerned regime. Thus, the concerned regime may lead directly to default, but otherwise frequently leads the government to over borrow and increase the risk default in the next period.

The quantitative model indicates that it is possible to have the rich variety of outcomes that we discussed in Section 3, including “excessive” debt issuance and sudden stops, depending on the current state.²³

Although not part of the benchmark belief process, if the government were to face pessimistic prices, the optimal response would track the concerned policy, including the low debt issuance portion, until 0.42, but not include the over-issuance segment. This is because pessimistic prices are zero for $B' > \overline{B}_{CK}$. Hence, pessimistic beliefs lead the government to either sharply reduces debt issuances, or, if initial debt is above 44 percent of income, simply default. In the simulation of Alternative II reported in Table 1, we see the threat of being forced to choose between a sharp reduction in debt and default is sufficient for the government to avoid these outcomes in equilibrium. In particular, the government’s mean debt level is half that of the benchmark in the presence of pessimistic belief-driven crises. This is sufficient to eliminate default altogether as well as sharp deleveraging episodes (the latter evidenced by the relative low consumption volatility).

An interesting contrast to both the benchmark and the pessimistic alternative is Alternative IV. In this alternative, the concerned beliefs occur with more than double the rate in the benchmark. That, on its own, would amplify the gap between the benchmark and Alternative I. Specifically, defaults and volatility increase relative to the benchmark. To highlight richer set of possibilities, Alternative IV combines the more frequent concerned crises with a lower default cost (reducing the latter from 17.6% to 12.5%). This reduces the amount of debt that can be sustained in equilibrium; hence, the ergodic mean is a little more than half of the benchmark value. Interestingly, the spread and consumption volatility are similar to

²³ As a caveat in mapping Proposition 6 to the quantitative implementation, the intersection point of the two price schedules at the boundary of \mathbb{B}_{EG} may not coincide with a choice on the quantitative debt grid. This is clearer in Figure 5b, where the two curves fail to intersect at \overline{B}_{EG} . However, typically the two curves are fairly close and thus the level of intra-period risk in the simulated equilibrium is negligible. In the benchmark model, this is responsible for a small fraction of the defaults (the default frequency is about 0.4% without them in the benchmark model), but they affect neither the average debt-to-GDP nor the spread volatility nor any of the other key features of our model.

the benchmark model, but the default frequency is only a fifth of the benchmark value (0.1% versus 0.5%). This demonstrates that volatility can coexist without frequent defaults. The volatility is generated because the arrival of the concerned regime triggers a sharp deleveraging. Recalling the benchmark policy function in Figure 6, the sharp deleveraging portion of the domain is closer to the ergodic mean in Alternative IV. Hence, the simulation features frequent “sudden stops,” but few outright defaults.

For expositional reasons, we have restricted attention to at most two belief regimes in the support of ρ for the benchmark and alternatives. It is straightforward to envision richer Markov processes for beliefs including non-i.i.d. processes. This would conceivably generate the full mix of crises in one simulation, including standard defaults, rollover defaults in the pessimistic regime, over-borrowing in response to concerned beliefs, and sudden stops.

4.5 Long-Term Bonds

In this section, we examine the case of long-term bonds. We first show that rollover crises can occur with long-term bonds, provided the resources the government has access to repay maturing obligations is not all of output. Thus issuance dynamics change in the presence of concerned or pessimistic beliefs even if debt is long-term. Second, we explore at what prices buybacks could occur in equilibrium and whether or not they are undertaken.

To calibrate the maturity for the long-term debt model, we target the average maturity of Mexico. Specifically, we use the average ratio of debt service to the face value of debt for all public and publicly guaranteed debt in Mexico from 1980-2015. This implies a quarterly $\lambda = 4.5\%$, or an average maturity that is roughly five years. Over this time period, Mexican government tax revenue averaged 10.7% of total GDP. To match the average debt-to-revenue burden of Mexico, we target a debt to endowment ratio of 6.1. This is a more realistic target when considering roll-over crises, as adjusting tax revenue as a fraction of GDP is difficult to do immediately in response to a failed auction. Adjusting the default cost to $d = .35$ achieves the indebtedness target. Otherwise, the calibration is the same as Section 4, with the one exception being we raise the volatility of the i.i.d. component of the endowment in order to achieve convergence.²⁴

We assume the same structure on beliefs as the benchmark model, i.e., a 97.75% chance of O and a 2.25% chance of C , and consider as well an alternative in which beliefs switch instead between O and P with the same probabilities. The results can be found in Table 2 as well as Figure 7, both of which make clear that divergent beliefs matter substantially in

²⁴Chatterjee and Eyigungor (2012) note that this component is important in attaining convergence in the long-term debt model, and we corroborate this here. In particular, we set $\sigma_s = .03$, which allows us to achieve convergence at a tolerance of $1e - 4$.

the long-term debt case at empirical maturities.

Table 2: Long-Term Debt Model Moments

	Benchmark Beliefs	O/P Alternative
Def. Freq.	6.8%	0.0%
Avg. Spread	7.4%	0.0%
Ratio of Debt-to-Tax Revenue	6.1	6.3
σ_c/σ_y	1.57	2.30

From Table 2 one can see that our benchmark with concerned beliefs generates a fairly large number of defaults, while the alternative optimistic/pessimistic case generates none (just as in our one-period debt model). This is because the government “overborrows” in the benchmark beliefs in response to concerned beliefs while deleveraging in the pessimistic case. Overborrowing is even more potent in generating defaults than it is in the one-period-debt model. Further, as we would expect, consumption volatility is higher in the O/P case because crises always induce sharp reductions in debt. Interestingly, though, debt levels are higher in the O/P case. This is a product of debt dilution: Investors in the O/P case know that with some probability a P shock will occur tomorrow, and the sovereign will be induced to delever, which raises the price of tomorrow’s bonds and thus their payoffs. Consequently, they are willing to extend more credit today.

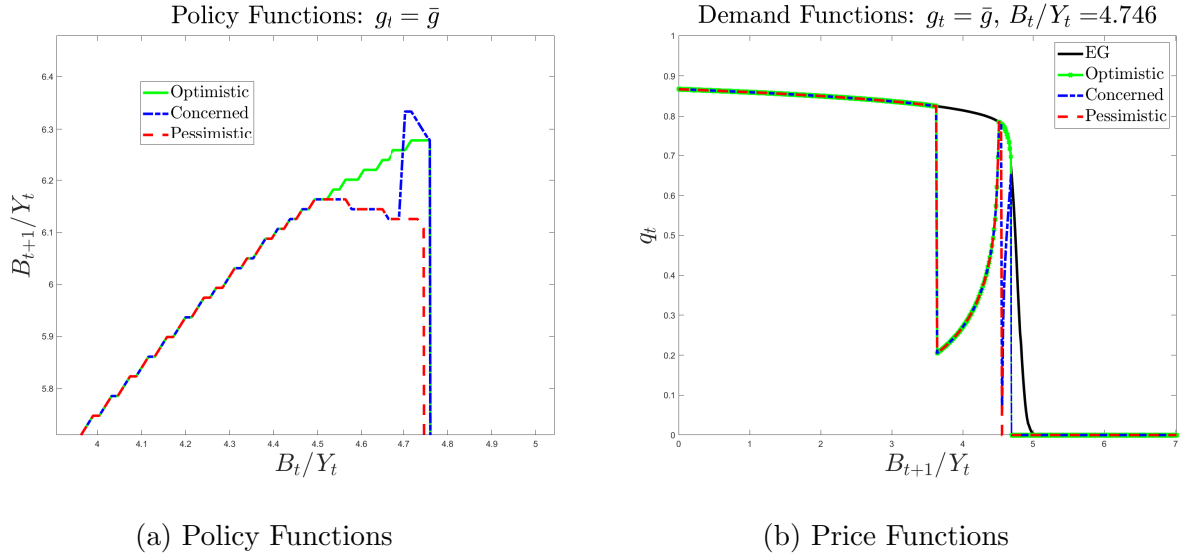


Figure 7: Long-Term Debt

Another feature worth noting here is the shape of the pricing schedule in the buyback region of Figure 7b. In accordance with Proposition 7, the equilibrium price is indeed unique.

It is also non-monotonic. For low levels of buybacks, there is an interior price, but as the buybacks get larger, the price ratchets up discontinuously to the EG price. This is a result of the changing continuation value implied by a buyback. As buybacks get larger, the threshold $\tilde{\epsilon}$ that makes the government indifferent to repayment or default initially is declining. Once the buybacks are large enough, the continuation value is high enough that there is no interior $\tilde{\epsilon}$, and the risk of intra-period default goes to zero. A final thing to note is that buybacks never occur on the equilibrium path in either belief regime.

5 Conclusion

In this paper, we show that the possibility of uncertainty between an auction and the next payment opens the door to a rich taxonomy of crises. In particular, we extend the nature of self-fulfilling crises to include bond issuances at fire-sale prices during a rollover crisis. This was motivated by the fact that crises in practice are often associated with positive issuances at abnormally high spreads. If creditors coordinate on such short-term uncertainty, then the government has an incentive to completely eliminate intra-period risk by adjusting the amount of debt auctioned. In the one-period bond model, introducing such self-fulfilling fire sales generates a five-fold increase in default probability and a 25-fold increase in spread volatility. The nature of desperate deals in the model and the associated equilibrium behavior provide a novel lens to interpret the interest rate spikes and debt dynamics observed in recent sovereign debt crises.

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Appendix A Proofs

A.1 Proof of Proposition 1

Proof. Let us define $H : [0, 1] \rightarrow \mathbb{R}$ by

$$H(\epsilon) \equiv u(Y - B + F(\epsilon)q_{EG}(B')B') - u(Y + F(\epsilon)q_{EG}(B')B') - \sigma\epsilon + \beta\mathbb{E}[V(s')|B'] - \beta\underline{V}^D. \quad (25)$$

Equation (13) is satisfied at $\tilde{\epsilon}$ if and only if $H(\tilde{\epsilon}) = 0$. Note that ϵ only enters H via the functions u and F , both of which are continuous. Hence, H is continuous. Premise (i) states that $H(1) \geq 0$. Premise (ii) states that $H(0) \leq 0$. By continuity there is at least one $\epsilon \in [0, 1]$ such that $H(\epsilon) = 0$. If $H(1) > 0$ and $H(0) < 0$, then the ϵ that solves $H(\epsilon) = 0$ must be strictly interior. \square

A.2 Proof of Proposition 2

Proof. For a given (Y, B, B') , let q_{EG} denote $q_{EG}(B')$. If $q_{EG} = 0$, the only equilibrium price is zero. If $B' \leq 0$, the only equilibrium price is $q_{EG} = R^{-1}$, as assets always trade at the risk-free price. Thus the proposition is trivially true for $B' \leq 0$ and $q_{EG} = 0$. Henceforward, assume $q_{EG}B' > 0$.

Define the function $g : [0, 1] \rightarrow \mathbb{R}$ by:

$$g(\epsilon) \equiv u(Y - B + F(\epsilon)q_{EG}B') - u(Y + F(\epsilon)q_{EG}B') + \beta\mathbb{E}[V(s')|B'] - \beta\underline{V}^D. \quad (26)$$

Then $H : [0, 1] \rightarrow \mathbb{R}$ defined in equation (25) in the proof of Proposition 1 can be written as:

$$H(\epsilon) \equiv g(\epsilon) - \sigma\epsilon. \quad (27)$$

Zero is an equilibrium if and only if $H(0) \leq 0$; q_{EG} is an equilibrium price if and only if $H(1) \geq 0$; and $\tilde{\epsilon}$ solve equation (13) if and only if $H(\tilde{\epsilon}) = 0$. We consider the cases of $B < 0$, $B = 0$, and $B > 0$ in turn:

If $B < 0$: If $B < 0$, then $g(\epsilon) > 0$ for all $\epsilon \in [0, 1]$. This uses the fact that $\beta\mathbb{E}[V(s')|B'] - \beta\underline{V}^D \geq 0$. This inequality follows from the fact that $V(s) \geq \max_{B' \leq \bar{B}} \mathbb{E}[V^D(s, B') + \sigma\epsilon] \geq \underline{V}^D$. As g is continuous on the compact domain $[0, 1]$, it achieves a minimum $\underline{g} \equiv \min_{\epsilon \in [0, 1]} g(\epsilon) > 0$. If $\sigma < \underline{g}$, then $H(\epsilon) = g(\epsilon) - \sigma\epsilon \geq \underline{g} - \sigma\epsilon > \sigma(1 - \epsilon) \geq 0$. Hence, $H(\epsilon) > 0$ for all ϵ , and $q_{EG}(B')$ is the only possible equilibrium price. Thus, the proposition holds for $B < 0$ by setting $K = \underline{g} > 0$.

If $B = 0$: If $B = 0$, then $g(\epsilon) = \beta\mathbb{E}[V(s')|B'] - \beta\underline{V}^D$, which is independent of ϵ . If $g = 0$, then $H(\epsilon) = -\sigma\epsilon$, and the only possible equilibrium price is 0 for any $\sigma > 0$. If $g > 0$, then, letting $K = g > 0$, for $\sigma < K$, we have $H(\epsilon) = K - \sigma\epsilon > 0$. In this case, $H(\epsilon) \geq H(1) > 0$ and the only possible price is q_{EG} . Thus, the proposition holds for $B = 0$.

If $B > 0$: If $B > 0$, then concavity of u implies that $g(\epsilon)$ is strictly increasing in ϵ . In particular,

$$g'(\epsilon) = [u'(Y - B + F(\epsilon)q_{EG}B') - u'(Y + F(\epsilon)q_{EG}B')] F'(\epsilon)q_{EG}B' > 0,$$

where the last inequality uses the fact that $F'(\epsilon) \geq \alpha > 0$ and $q_{EG}B' > 0$. Let

$$K \equiv \min_{\epsilon \in [0, 1]} [u'(Y - B + F(\epsilon)q_{EG}B') - u'(Y + F(\epsilon)q_{EG}B')] \underline{f}q_{EG}B' > 0,$$

where the minimum exists as u' is a continuous function and is strictly positive given that $g'(\epsilon) > 0$ for all $\epsilon \in [0, 1]$. Then $H'(\epsilon) = g'(\epsilon) - \sigma \geq K - \sigma$. If $\sigma < K$, then $H'(\epsilon) > 0$. Hence there is at most one $\tilde{\epsilon}$ such that $H(\tilde{\epsilon}) = 0$. If $H(0) \leq 0 \leq H(1)$, then $\{0, q_{EG}, F(\tilde{\epsilon})q_{EG}\}$ are all equilibrium prices. If $H(1) > H(0) > 0$, then only q_{EG} is an equilibrium price. If $H(0) < H(1) < 0$, then only zero is an equilibrium price. Thus, the proposition holds for $B > 0$. \square

A.3 Proof of Proposition 3

Proof. First, note that $B \neq 0$. To see this, when $B = 0$, we have

$$V_{EG}^R(Y, 0, B') - V_{EG}^D(Y) = \beta (\mathbb{E}V(s') - \underline{V}^D) = V_{CK}^R(Y, 0, B') - V_{CK}^D(Y),$$

which is inconsistent with the inequalities in the proposition's premise. Hence, we take $B' \neq 0$ in what follows. As (Y, B, B') is fixed throughout, we drop these arguments from the notation below. The premise implies $V_{EG}^R > V_{CK}^R$, and hence it must be the case that $q_{EG} > 0$.

Define $h : [0, 1] \rightarrow \mathbb{R}$ by:

$$h(x) \equiv u(Y - B + xq_{EG}B') - u(Y + xq_{EG}B').$$

h represents the net current period flow utility from repayment over default when the price is xq_{EG} . By strict concavity of u and $q_{EG}B' > 0$, we have $h'(x) \gtrless 0$ when $B \gtrless 0$. The inequalities in the premise imply:

$$h(1) + \beta (\mathbb{E}V(s') - \underline{V}^D) > 0 > h(0) + \beta (\mathbb{E}V(s') - \underline{V}^D).$$

By continuity of h , there exists a $p \in [0, 1]$ such that $h(p) + \beta (\mathbb{E}V(s') - \underline{V}^D) = 0$. At price $\tilde{q} = pq_{EG}$, the government is indifferent to defaulting or repaying when $\sigma = 0$. At price \tilde{q} and if the government randomizes by defaulting with probability $1 - p$, the lenders break even. Hence, (\tilde{q}, p) is a mixed strategy equilibrium.

Let $K > 0$ be defined as in Proposition 2. Let $\bar{\sigma} \equiv V_{EG}^R - V_{EG}^D$, which is strictly positive by the proposition's premise. Let N be defined by:

$$N \equiv \inf\{n \geq 0 \mid \sigma_n < \min\{\bar{\sigma}, K\}\}.$$

As the sequence σ_n is monotonically decreasing, for $n > N$, we have

$$V_{EG}^R - V_{EG}^D - \sigma_n > V_{EG}^R - V_{EG}^D - \bar{\sigma} = 0,$$

and hence q_{EG} is supportable as an equilibrium price. As $V_{CK}^R < V_{CK}^D$, zero is also an equilibrium price. For each $n > N$ and associated σ_n , by Proposition 2, there exists a unique ϵ_n such that equation (13) holds. Define $p_n \equiv F(\epsilon_n)$. From (13) and the fact that $\tilde{\epsilon}_n \leq 1$, we have

$$0 \leq h(p_n) - h(p) = \sigma_n \tilde{\epsilon}_n \leq \sigma_n.$$

Hence, as $\sigma_n \rightarrow 0$, $h(p_n) - h(p) \rightarrow 0$. To establish convergence in the arguments of h , define

$$\kappa \equiv \min_{x \in [0, 1]} |h'(x)|.$$

As h' is continuous and h' is either strictly positive or negative for all $x \in [0, 1]$ given $B \neq 0$, $\kappa > 0$ is well defined. Note that by definition

$$h(p_n) - h(p) \geq \kappa |p_n - p|.$$

Hence, $h(p_n) - h(p) \rightarrow 0$ implies $|p_n - p| \rightarrow 0$. □

A.4 Proof of Proposition 4

Proof. Using H defined by (25) in the proof of Proposition 1, we have $\tilde{q} = 0$ satisfies (7) if $H(0) \leq 0$; $\tilde{q} = q_{EG}(B')$ if $H(1) \geq 0$; and $\tilde{q} \in (0, q_{EG}(B'))$ if $H(\tilde{\epsilon}) = 0$ for some $\tilde{\epsilon} \in (0, 1)$. Define

$$h(x) \equiv u(Y - B + x) - u(Y + x). \tag{28}$$

We have

$$H'(\epsilon) = h'(F(\epsilon)q_{EG}(B')B')F'(\epsilon)q_{EG}(B')B' - \sigma. \quad (29)$$

By definition of h ,

$$\begin{aligned} h'(x) &= u'(Y - B + x) - u'(Y + x) \\ h''(x) &= u''(Y - B + x) - u''(Y + x). \end{aligned}$$

As $B > 0$ and u strictly concave, we have $h'(x) > 0$ for $x \geq 0$. If u'' is strictly increasing, then $h''(x) < 0$. Hence, as ϵ increases, $h'(F(\epsilon)q_{EG}(B')B')$ strictly decreases and $F'(\epsilon)$ weakly decreases. As both are positive, their product decreases and H' is strictly decreasing. This establishes that there are at most two roots to H ; that is, there are at most two interior prices that satisfy (13). If $H(0) \leq 0$, then $\tilde{q} = 0$ satisfies (7), and there are at most three possible equilibrium prices, two of which being interior. If $H(0) > 0$, then $\tilde{q} = 0$ is not an equilibrium. As $H(0) > 0$ and H' strictly decreasing, there is at most one interior solution $\tilde{\epsilon}$ to (13), with $H'(\tilde{\epsilon}) < 0$. If there is such a $\tilde{\epsilon} < 1$, then $H(1) < 0$ and $\tilde{q} = F(\tilde{\epsilon})q_{EG}(B')$ is the only equilibrium price. Otherwise, $H(\epsilon) \geq 0$ for all $\epsilon \in [0, 1]$, and $\tilde{q} = q_{EG}(B')$ is the only equilibrium price. \square

A.5 Proof of Proposition 5

Proof. We first establish that $\mathbb{E}[V(s')|B'] \geq \underline{V}^D + \sigma\mathbb{E}\epsilon$ for all B' . To see this, it is always feasible to issue zero new debt and default with probability one on maturing debt:

$$\begin{aligned} V(s) &\geq u(Y) + \sigma\mathbb{E}\epsilon + \beta\underline{V}^D \\ &= \sigma\mathbb{E}\epsilon + u(Y) - \mathbb{E}u(Y') + \underline{V}^D, \end{aligned}$$

where the second line uses the fact that $\underline{V}^D \equiv \mathbb{E}u(Y')/(1-\beta)$. Taking expectation over s' for any $B' \leq \bar{B}$, we have

$$\mathbb{E}[V(s')|B'] \geq \underline{V}^D + \sigma\mathbb{E}\epsilon. \quad (30)$$

From the definitions of V_{CK}^R and V_{CK}^D , we can re-write \mathbb{B}_{CK} as:

$$\mathbb{B}_{CK}(Y, B) \equiv \{B' \in [0, \bar{B}] \mid u(Y - B) - u(Y) + \beta\mathbb{E}[V(s')|B'] - \beta\underline{V}^D \leq 0\}. \quad (31)$$

Equation (30) implies that if $B' \in \mathbb{B}_{CK}(Y, B)$, then

$$\begin{aligned} 0 &\geq u(Y - B) - u(Y) + \beta\mathbb{E}[V(s')|B'] - \beta\underline{V}^D \\ &\geq u(Y - B) - u(Y) + \beta\sigma\mathbb{E}\epsilon. \end{aligned}$$

That is, (18) is a necessary condition for $\mathbb{B}_{CK} \neq \emptyset$. To show that (18) is also a sufficient condition, suppose (18) satisfied. For B' larger than the natural borrowing limit (which we have assumed is strictly less than \bar{B}), repayment is infeasible for any endowment realization and default occurs with probability one. Hence, there exists a $\tilde{B} \in (0, \bar{B})$ such that $\mathbb{E}[V(s')|B' \geq \tilde{B}] = \sigma\mathbb{E}\epsilon + \underline{V}^D$. Therefore, $[\tilde{B}, \bar{B}] \subset \mathbb{B}_{CK}(Y, B)$. As $[\tilde{B}, \bar{B}] \neq \emptyset$ when (18) holds, \mathbb{B}_{CK} is not empty. \square

A.6 Proof of Proposition 6

Proof. If $q(s, B'_i)$ is interior for $i = 1, 2$, there exist respective $\tilde{\epsilon}_i$ that satisfy (13). As $q(s, B'_i)/q_{EG}(B'_i) = F(\tilde{\epsilon}_i)$, and F is strictly increasing on its support, we have $\tilde{\epsilon}_1 \leq \tilde{\epsilon}_2$. By (13), we also have:

$$V^R(s, B'_i) = V^D(s, B'_i) + \sigma\tilde{\epsilon}_i,$$

for $i = 1, 2$. Using this, the expected payoff from B'_i at the time of auction can be written:

$$\begin{aligned} & \mathbb{E} \max \{V^R(s, B'_i), V^D(s, B'_i) + \sigma \tilde{\epsilon}\} \\ &= \mathbb{E} \max \{V^D(s, B'_i) + \sigma \tilde{\epsilon}_i, V^D(s, B'_i) + \sigma \epsilon\} \\ &= V^D(s, B'_i) + \sigma \mathbb{E} \max \{\tilde{\epsilon}_i, \epsilon\}. \end{aligned}$$

As $q(s, B'_1)B'_1 \leq q(s, B'_2)B'_2$, we have $V^D(s, B'_1) \leq V^D(s, B'_2)$. This, plus the fact that $\tilde{\epsilon}_1 \leq \tilde{\epsilon}_2$ implies that B'_2 weakly dominates B'_1 as a debt choice. If either $q(s, B'_1)B'_1 < q(s, B'_2)B'_2$ or $\tilde{\epsilon}_1 < \tilde{\epsilon}_2$, the preference for B'_2 is strict. \square

A.7 Proof of Proposition 7

Proof. Fix s and $B' < (1 - \lambda)B$ and let q_{EG} denote $q_{EG}(s, B')$. For $\tilde{q} \in [0, q_{EG}]$, define $\hat{F}(\tilde{q})$ by:

$$\begin{aligned} & \hat{F}(\tilde{q}) \equiv \\ & F\left(\frac{1}{\sigma} \left[u(Y - (r^* + \lambda)B + \tilde{q} \times [B' - (1 - \lambda)B]) - u(Y + \tilde{q} \times [B' - (1 - \lambda)B]) + \beta \mathbb{E}[V(s')|s, B'] - \beta \mathbb{E}V^D(s') \right] \right). \end{aligned}$$

A $\tilde{q} \in [0, q_{EG}]$ is an equilibrium price if and only if $\tilde{q}/q_{EG} = \hat{F}(\tilde{q})$. Note that \hat{F} maps $[0, q_{EG}]$ into $[0, 1]$, is weakly decreasing (as $B' < (1 - \lambda)B$), and continuous. Hence, there exists one and only one \tilde{q} that satisfies the equilibrium condition. \square

A.8 Proof of Proposition 8

Proof. Since $q(s, B'_i)$ are interior equilibrium prices for $i = 1, 2$, there exist thresholds $\tilde{\epsilon}_i \in [0, 1]$ such that:

$$V^R(s, B'_i) = V^D(s, B'_i) + \sigma \tilde{\epsilon}_i, \text{ for } i = 1, 2.$$

Conditional on auctioning B'_i , the expected payoff is:

$$\begin{aligned} & \int_0^1 \max \{V^R(s, B'_i), V^D(s, B'_i) + \sigma \epsilon\} dF(\epsilon) \\ &= \int_0^1 \max \{V^D(s, B'_i) + \sigma \tilde{\epsilon}_i, V^D(s, B'_i) + \sigma \epsilon\} dF(\epsilon) \\ &= V^D(s, B'_i) + \sigma \int_0^1 \max \{\tilde{\epsilon}_i, \epsilon\} dF(\epsilon). \end{aligned}$$

From condition (i) in the proposition statement, we have $\tilde{\epsilon}_1 \geq \tilde{\epsilon}_2$, with strict inequality if (i) is strict. As the default value is increasing in net auction revenue, from premise (ii) in the proposition, we have $V^D(s, B'_1) \geq V^D(s, B'_2)$, with strict inequality if (ii) is strict. Thus, the expected value from auctioning B'_1 is weakly greater than B'_2 , and strictly if either (i) or (ii) is strict. \square

Proof. Since $q(s, B'_i)$ are interior equilibrium prices for $i = 1, 2$, there exist thresholds $\tilde{\epsilon}_i \in [0, 1]$ such that:

$$V^R(s, B'_i) = V^D(s, B'_i) + \sigma \tilde{\epsilon}_i, \text{ for } i = 1, 2.$$

Conditional on auctioning B'_i , the expected payoff is:

$$\begin{aligned}
& \int_0^1 \max \{ V^R(s, B'_i), V^D(s, B'_i) + \sigma \epsilon \} dF(\epsilon) \\
&= \int_0^1 \max \{ V^D(s, B'_i) + \sigma \tilde{\epsilon}_i, V^D(s, B'_i) + \sigma \epsilon \} dF(\epsilon) \\
&= V^D(s, B'_i) + \sigma \int_0^1 \max \{ \tilde{\epsilon}_i, \epsilon \} dF(\epsilon).
\end{aligned}$$

From condition (i) in the proposition statement, we have $\tilde{\epsilon}_1 \geq \tilde{\epsilon}_2$, with strict inequality if (i) is strict. As the default value is increasing in net auction revenue, from premise (ii) in the proposition, we have $V^D(s, B'_1) \geq V^D(s, B'_2)$, with strict inequality if (ii) is strict. Thus, the expected value from auctioning B'_1 is weakly greater than B'_2 , and strictly if either (i) or (ii) is strict. □