Tariff Wars and Net Foreign Assets*

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Abstract

This note addresses whether and when a trade war that imposes balanced trade (or even zero trade) can be consistent with initial non-zero net foreign asset positions. Using a bilateral trade model, we exploit insights from the classic literature on the Transfer Problem to characterize when gross asset or liability positions and tariff policies generate an endogenous terms-of-trade effect that ensures the value of assets and liabilities balance. As long as gross positions denominated in different goods differ in sign, there exists a continuum of bilateral tariff policies that ensure balanced trade and that satisfy the contractual financial obligations. If the new terms-of-trade do not reverse the initial direction of trade, balanced trade is consistent with non-zero exports and imports. In general, high enough bilateral tariffs lead to an autarkic outcome where no trade occurs and the net foreign asset positions rebalance to zero.

1 Introduction

In April 2025, the Trump administration proposed a set of tariffs that had the purported goal of balancing trade for the United States, not only in the aggregate but bilaterally. The US has a large negative net foreign asset position, which is a combination of large gross positions denominated in different currencies (or goods for real assets such as equities). Most countries also have large gross and net positions (Lane and Milesi-Ferretti, 2001; Gourinchas and Rey, 2014). This raises the question of how such contracted international net liabilities can be paid if trade is balanced. We address this question in a static two-country two-good exchange economy in which countries

*The views expressed here are entirely those of the authors. They do not necessarily represent the views of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

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start with non-zero gross claims on each other that must be paid in the current period. Financial claims are real and can be denominated in either of the two goods. We characterize the set of initial assets and tariff policies under which the endogenous terms of trade adjust to ensure that any net positions are zeroed out, consistent with a balanced trade equilibrium. As long as gross positions are such that a country is long in one good and short the other, there are bilateral tariffs large enough that lead to autarky and revalue the zero net foreign asset positions to zero. If the terms-of-trade that zero out the value of the net foreign asset positions are consistent with the original pattern of trade, there exists a continuum of tariff pairs imposed in the respective countries that ensure balanced and non-zero trade and that satisfies the original contractual financial obligations.

Our question and the associated answer echoes the Transfer Problem famously debated by Keynes (1929) and Ohlin (1929) in the aftermath of the first world war. That debate turned on whether a transfer by one country to the another affected the terms-of-trade. With identical homothetic preferences, the answer is no, due to the fact that the distribution of wealth does not influence relative prices in such an environment. However, when the marginal dollar is spent differently, such transfers do affect the terms-of-trade. Our analysis is closest to the treatment in Samuelson (1952, 1954) with an environment in which preferences are identical and homothetic, but where there are trade frictions. Samuelson shows that when tariffs are present, the required transfer has an adverse impact on the payer's terms-of-trade. We leverage this insight to explore under what patterns of claims and tariffs do the terms-of-trade endogenously adjust to zero out any net foreign asset position.

This result naturally raises questions of how a potential trade war affects the willingness to take such international positions ex ante, and what effects this has on equilibrium interest rates or asset prices. Moreover, even ex-post, the welfare consequences are not immediate from the change in the value of net foreign assets. We briefly discuss these questions and other extensions in the conclusion.

Prompted by the recent policy events, there are several new papers on the interactions between tariffs and trade deficits. Pujolas and Rossbach (2024) study the welfare effects of trade wars in an Armington model with trade imbalances, where countries have exogenous international net (but not gross) positions. Ignatenko et al. (2025) quantify the welfare effects of tariffs in a model with trade imbalances where net foreign asset positions are given but trade deficits endogenously respond to policy. Costinot and Werning (2025) analyze a dynamic model with fixed terms-of-trade and study the effects of a permanent increase in tariffs on the trade deficit by affecting the incentives of domestic households to save and consume. In a very related and contemporaneous paper, Itskhoki and Mukhin (2025) argue that absent valuation effects, tariffs do not affect the long-run trade deficit. Our note focuses precisely on valuation effects when countries have gross positions in potentially different assets.[1]

2 Environment

The environment is static and consists of two countries, Home and Foreign, and two goods, A and B. Letting a star denote Foreign, let endowments of A and B be denoted $Y = (Y_A, Y_B)$ and $Y^* = (Y_A^*, Y_B^*)$, with global endowments denoted by $\overline{Y}_A \equiv Y_A + Y_A^*$ and $\overline{Y}_B \equiv Y_B + Y_B^*$.

Preferences are assumed to be identical for Home and Foreign and characterized by a homothetic utility function $u(x_a, x_b)$ over consumption of good A, x_A, and good B, x_B. By homotheticity, we can define the Marginal Rate of Substitution (MRS) between B and A as a function of the ratio $x = x_B/x_A$:

$$g(x) \equiv \frac{u_B(1,x)}{u_A(1,x)}, \tag{1}$$

where we make the standard assumption $g'(x) < 0$. In addition, we make the following Inada assumptions: $\lim_{x\to 0} g(x) = \infty$ and $\lim_{x\to\infty} g(x) = 0$.

We assume that Home starts with gross foreign asset positions $a = (a_A, a_B)$, where $a_A \in (-\overline{Y}_A, \overline{Y}_A)$ represents Home's claim on Foreign in units of good A and $a_B \in (-\overline{Y}_B, \overline{Y}_B)$ are claims in units of good B. Anticipating asset market clearing, Foreign claims are $a^* = (-a_A, -a_B)$.

We shall take as our benchmark a scenario in which Home exports good A and imports good B. Anticipating this, we will impose that Home is endowed with relatively more of good A:

$$\frac{Y_B^*}{Y_A^*} > \frac{\overline{Y}_B}{\overline{Y}_A} > \frac{Y_B}{Y_A}. \tag{C1}$$

Let Home consumers face a price of one for good A and a price of p_B for good B. Foreign consumers face respective prices (p_A^*, p_B^*). There is a consumption tax on good B at Home, $\tau \in (-1, \infty)$, and a consumption tax on good A at Foreign, $\tau^* \in (-1, \infty)$. Below we discuss conditions for the equivalence between taxes and tariffs. Trade is otherwise costless and prices must satisfy the following arbitrage conditions:

$$p_B = (1+\tau)p_B^*$$
$$p_A^* = 1 + \tau^*.$$

[1]There is previous work on global imbalances, trade deficits and trade frictions. See for example, Razin and Svensson (1983), Dekle, Eaton, and Kortum (2007), Fitzgerald (2012), Reyes-Heroles (2017), Timothy J. Kehoe, Ruhl, and Steinberg (2018), Alessandria, Bai, and Woo (2024), among others.

Define the international relative price of B to be:

$$\rho \equiv \frac{p_B^*}{p_A} = p_B^*. \tag{2}$$

We then have $p_B = (1+\tau)\rho$ and $p_B^*/p_A^* = \rho/(1+\tau^*)$. Using this convention, the representative Home and Foreign consumers face respective budget sets are:

$$\begin{aligned} c_A + (1+\tau)\rho c_B &= (Y_A + a_A) + \rho(Y_B + a_B) + T \\ (1+\tau^*)c_A^* + \rho c_B^* &= (Y_A^* - a_A) + \rho(Y_B^* - a_B) + T^*, \end{aligned} \tag{3}$$

where T and T^* is a government transfer in Home and Foreign, respectively, in units of A. The governments' budget sets are:

$$\begin{aligned} T &= \tau \rho c_B \\ T^* &= \tau^* c_A^*. \end{aligned} \tag{4}$$

We are now ready to define an equilibrium.

Definition 1. *Given taxes $\{\tau, \tau^*\}$ and initial endowments and assets, a* tax equilibrium *is an allocation $\{c_A, c_B, c_A^*, c_B^*\}$ and a relative ρ such that (i) households optimize subject to (3); (ii) the budget sets of the governments (4) hold; and (iii) aggregate resource constraints are satisfied: $c_A + c_A^* = \overline{Y}_A$ and $c_B + c_B^* = \overline{Y}_B$.*

Household optimization implies:

$$\begin{aligned} g\left(\frac{c_B}{c_A}\right) &= p_B = (1+\tau)\rho \\ g\left(\frac{c_B^*}{c_A^*}\right) &= \frac{p_B^*}{p_A^*} = \frac{\rho}{1+\tau^*}. \end{aligned} \tag{5}$$

Substituting the resource conditions into (5), substituting the tariff's revenue into the budget constraints (3), and using Walras Law to drop Foreign's budget constraint, we can simplify the characterization of an equilibrium by solving for three values, $\{c_A, c_B, \rho\}$ that satisfy the following three relationships:

$$\begin{aligned} \left(\frac{1}{1+\tau}\right) g\left(\frac{c_B}{c_A}\right) = \rho &= (1+\tau^*) g\left(\frac{\overline{Y}_B - c_B}{\overline{Y}_A - c_A}\right) \\ c_A + \rho c_B &= Y_A + a_A + \rho(Y_B + a_B). \end{aligned} \tag{6}$$

An equilibrium exists as long as initial asset positions are within a well defined set. However, the presence of taxes can induce multiple equilibria even in this symmetric world.[2]

Lemma 1. *Define* $\rho^{FT} \equiv g(\overline{Y}_B/\overline{Y}_A)$. *For any* τ, τ^*, *if*

$$
\begin{aligned}
Y_A + a_A + \rho(Y_B + a_B) &> 0 \\
Y_A^* - a_A + \rho(Y_B^* - a_B) &> 0
\end{aligned}
\tag{7}
$$

for $\rho \in \left\{\rho^{FT}, \frac{\rho^{FT}}{1+\tau}, (1+\tau^*)\rho^{FT}\right\}$, *then an equilibrium exists with* ρ *in between* $\rho^{FT}/(1+\tau)$ *and* $(1+\tau^*)\rho^{FT}$.

Equivalence to Tariffs

Let us briefly discuss the mapping of the above economy with taxes to a situation with tariffs. Suppose that each country imposes a weakly positive tariff on its imports. That is, Home imposes a tariff $t \geq 0$ on Home's imported good and Foreign imposes a tariff t^* on Home's export.

Let the pre-tariff world price of A be normalized to 1, and the pre-tariff world price of B be normalized to ρ. Goods arbitrage at home now becomes

$$
\begin{aligned}
&p_A = \begin{cases} (1+t) & \text{for } c_A > Y_A \\ 1 & \text{for } c_A < Y_A \end{cases} \\
&\text{with } p_A \in [1, (1+t)] \text{ if } c_A = Y_A, \\
&p_B = \begin{cases} (1+t)\rho & \text{for } c_B > Y_B \\ \rho & \text{for } c_B < Y_B \end{cases} \\
&\text{with } p_B \in [\rho, (1+t)\rho] \text{ if } c_B = Y_B.
\end{aligned}
\tag{8}
$$

[2]For an early result showing the existence of multiple equilibria in an Edgeworth box exchange economy, see Timothy J Kehoe (1998). For more recent work, see Toda and Walsh (2017).

Similarly, in Foreign:

$$
\begin{aligned}
p_A^* &= \begin{cases} (1+t^*) & \text{for } c_A^* > Y_A^* \\ 1 & \text{for } c_A^* < Y_A^*, \end{cases} \\
&\text{with } p_A^* \in [1, (1+t^*)] \text{ if } c_A^* = Y_A^*, \\
p_B^* &= \begin{cases} (1+t^*)\rho & \text{for } c_B^* > Y_B^* \\ \rho & \text{for } c_B^* < Y_B^*, \end{cases} \\
&\text{with } p_B^* \in [\rho, (1+t^*)\rho] \text{ if } c_B^* = Y_B^*.
\end{aligned} \tag{9}
$$

The budget constraints faced by Home and Foreign households are:

$$
\begin{aligned}
p_A c_A - Y_A + p_B(c_B - Y_B) &= a_A + \rho a_B + T \\
p_A^*(c_A^* - Y_A^*) + p_B^*(c_B^* - Y_B^*) &= -(a_A + \rho a_B) + p_A^* T^*,
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
T &= t\rho \max\{(c_B - Y_B), 0\} + t \max\{(c_A - Y_A), 0\} \\
T^* &= t^*\rho \max\{(c_B^* - Y_B^*), 0\} + t^* \max\{(c_A^* - Y_A^*), 0\}.
\end{aligned} \tag{11}
$$

We define a *tariff equilibrium* as expected: a vector $\{c_A, c_B, c_A^*, c_B^*, \rho, p_A, p_B, p_A^*, p_B^*\}$ such that households optimize subject to their budget constraints (10), with prices now given by (8) and (9), the government budget constraints (11) holds, and the aggregate resource constraints are satisfied: $c_A + c_A^* = \overline{Y}_A$ and $c_B + c_B^* = \overline{Y}_B$.

To see the mapping to the tax equilibrium, note that in a tariff equilibrium

$$
g\left(\frac{c_B}{c_A}\right) = \begin{cases} (1+t)\rho & \text{if } c_A < Y_A, c_B > Y_B \\ \frac{\rho}{1+t} & \text{if } c_A > Y_A, c_B < Y_B, \\ \rho & \text{if } c_A > Y_A, c_B > Y_B \text{ or } c_A < Y_A, c_B < Y_B, \end{cases} \tag{12}
$$

and

$$
\begin{aligned}
g\left(\frac{c_B}{c_A}\right) &\in \left[\frac{\rho}{1+t}, (1+t)\rho\right], && \text{if } c_A = Y_A, c_B = Y_B \\
g\left(\frac{c_B}{c_A}\right) &\in \left[\rho, (1+t)\rho\right], && \text{if } c_A < Y_A, c_B = Y_B \text{ or } c_A = Y_A, c_B > Y_B \\
g\left(\frac{c_B}{c_A}\right) &\in \left[\frac{\rho}{1+t}, \rho\right], && \text{if } c_A = Y_A, c_B < Y_B \text{ or } c_A > Y_A, c_B = Y_B,
\end{aligned} \tag{13}
$$

and similarly for Foreign. The budget constraints of the households become:

$$\begin{aligned} c_A + \rho c_B &= Y_A + a_A + \rho(Y_B + a_B) \\ c_A^* + \rho c_B^* &= Y_A^* - a_A + \rho(Y_B^* - a_B) \end{aligned} \tag{14}$$

after substituting in for the tariff revenue. Note that these budget constraints are the same as in the tax equilibrium, as expected.

Lemma 2. *If $\{c_A, c_B, c_A^*, c_B^*\}$ and ρ satisfy (12), (13) (as well their respective versions for Foreign), (14), and the aggregate resource constraints hold, then $\{c_A, c_B, c_A^*, c_B^*\}$ and ρ constitute a tariff equilibrium with tariffs (t, t^*).*

It follows that we can always find a domestic tax $\tau > -1$ such that Home's relative consumption is the same as with the tariff for a given terms of trade ρ. The same argument applies to B. As a result, it is immediate that there is always a tax equilibrium that generates the same allocation as a tariff equilibrium:

Lemma 3. *Suppose an allocation $\{c_A, c_B, c_A^*, c_B^*\}$ and terms-of-trade ρ is part of a tariff equilibrium. Then there exists an identical tax equilibrium with $\tau = \rho/g(c_B/c_A) - 1$ and $\tau^* = g(c_B^*, c_A^*)/\rho - 1$.*

The other direction is a bit more subtle as we are requiring that tariffs be non-negative and uniform across imports if both goods are imported. When an equilibrium with domestic taxes is such that each country exports one good and imports the other, and the taxes distort consumption away from imports, then it is also an equilibrium with nonnegative tariffs. Another result is that if Home does not import or export a good in a tariff equilibrium, then increasing tariffs does not change the equilibrium allocation:[3]

Lemma 4. *Suppose that for (t, t^*), there is a tariff equilibrium with either $c_A = Y_A$ or $c_B = Y_B$, then the same allocation with the same terms of trade constitutes a tariff equilibrium for any tariffs $(t', t^{*\prime})$ with $t' \geq t$ and $t^{*\prime} \geq t^*$.*

To see the economics behind this lemma, suppose Home imports B and consumes its endow-

[3] This result is immediate in a world without financial positions (that is, $a_A = a_B = 0$). In that case, $c_A = Y_A$ (or $c_B = Y_B$) implies that each country is in autarky. It is easy to see then that an increase in import tariff rates does not impact the equilibrium.

ment of A. Let ρ denote the terms-of-trade at the initial tariff rates, and from the above we have $p_B/p_A \in [\rho, (1+t)\rho]$. As t increases, the same terms-of-trade and the same domestic relative price is still consistent with equilibrium. Specifically, as the domestic price of B increases, the domestic price of A keeps pace. This increase in the price of A generates a growing gap between the domestic and world price of A; however, the higher tariff makes it unprofitable to import A despite the increase in the local price, keeping $c_A = Y_A$.

3 Equilibrium Characterization

We first characterize a Free Trade Equilibrium for which $\tau = \tau^* = 0$, before turning to the case with taxes. Given Lemma 4, we conduct the analysis with taxes to avoid the more cumbersome notation involving tariffs. When the tax equilibrium involves a trade reversal, we flag that such equilibria may not be consistent with tariffs as they involve an import subsidy.

3.1 Free Trade Equilibrium

It will be useful to characterize the equilibrium using an Edgeworth Box. This is depicted in Figure 1. The dimensions of the box are $\overline{Y}_A \times \overline{Y}_B$. Home's consumption of A and B is depicted from the southwest corner, with A on the horizontal axis and B on the vertical axis. Foreign consumption is the mirror image, with its origin in the northeast corner.

Given identical, homothetic preferences, the indifference curves are tangent along the diagonal. The diagonal thus outlines the efficient contract curve. Homotheticity also implies that ρ^{FT} is independent of initial endowments, and is equal to $\rho^{FT} \equiv g\left(\frac{\overline{Y}_B}{\overline{Y}_A}\right)$.

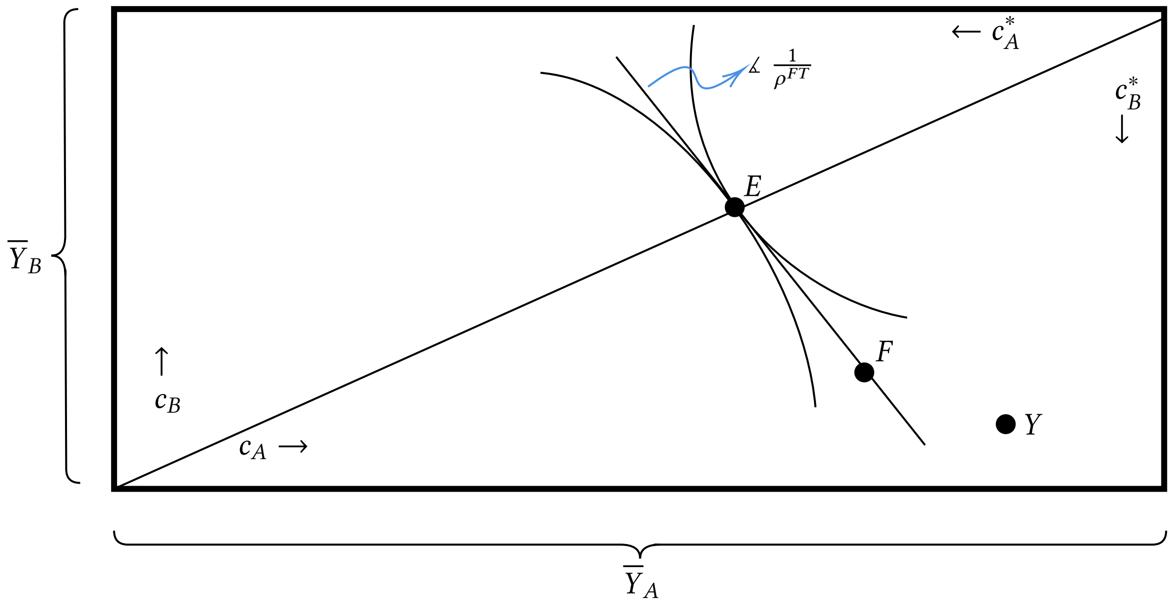
The endowment point is depicted by the point Y, using condition (C1). The point $F \equiv (Y_A + a_A, Y_B + a_B)$ depicts the endowment adjusted by initial asset positions. As depicted, we have $a_A < 0$ and $a_B > 0$, which means that Home is short on good A and long on good B. This places F to the northwest of Y. The line through point F depicts (c_A, c_B) such that Home's budget set is satisfied at relative prices ρ^{FT}:

$$c_A + \rho^{FT} c_B = Y_A + a_A + \rho^{FT}(Y_B + a_B).$$

The free-trade equilibrium is anchored where this budget line intersects the diagonal, E^{FT}. As depicted, E^{FT} is to the left and above Y, indicating that Home imports good B and exports good A. This is our benchmark scenario, and we will discuss alternatives below.

Whether Home is a net debtor or creditor depends on the relative position of its endowment point (Y) and its financial position (F) given the world price ρ. Note that in the figure, given that the point Y is to the left of Home's budget line, Home is a net debtor: Home's expenditure is strictly lower than its total income ($Y_A + \rho Y_B$).

Figure 1: Free Trade Equilibrium



3.2 The Tariff-War Equilibrium

Now suppose that taxes are set to $\{\tau, \tau^*\}$, with $\tau, \tau^* \geq 0$ with at least one tariff being strictly positive. We proceed in the following manner to characterize the equilibrium.

From Home's optimality condition, we have

$$\frac{c_B}{c_A} = h\left((1+\tau)\rho'\right),$$

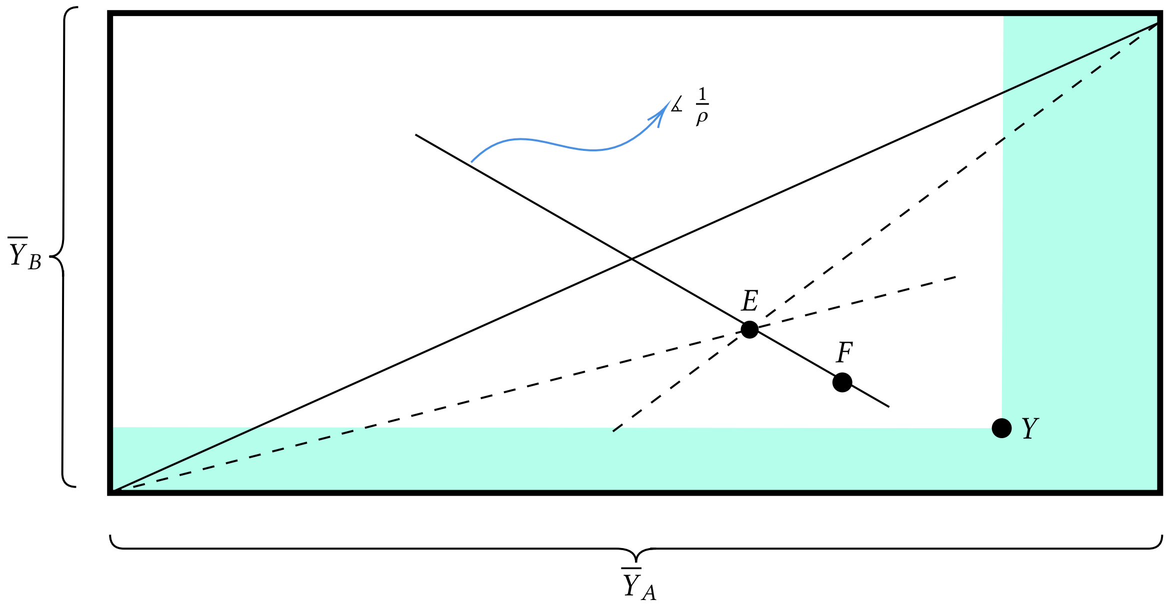
where $h(x)$ is the inverse of g (which is well defined given that $g' < 0$). Thus, our candidate ρ' is associated with a ray from Home's origin in the Edgeworth box, which we depict with a dashed line in Figure 2. Consistent with tariffs distorting Home's consumption toward good A and Foreign toward B, we depict this ray below the free-trade diagonal. That is, our candidate is such that $(1+\tau)\rho' > \rho^{FT} > \rho'/(1+\tau^*)$, so that the Home's relative price of B is higher than it would be under free trade and the Foreign's is lower.

Similarly, Foreign's relative consumption is given by

$$\frac{c_B^*}{c_A^*} = \frac{\overline{Y}_A - c_A}{\overline{Y}_B - c_B} = h\left(\rho'/(1+\tau^*)\right).$$

We depict this as a dashed ray from Foreign's origin. Where these rays intersect, point E, is an allocation that satisfies the first equalities in (6). Drawing a line through this point with slope $-\rho'$, we check if it contains the point F. If it does, the budget condition in (6) is satisfied and ρ' is a valid conjecture. If not, this is not a feasible allocation for one of the countries. As depicted in

Figure 2: Tariff War Equilibrium



the figure, it is feasible for Foreign, as F is on the proposed budget line.

As $g' < 0$, if the candidate ρ' decreases, then $\frac{c_B}{c_A}$ increases, and the ray from the Home origin steepens, rotating *toward* the diagonal. As ρ' decreases, we have that $\frac{c_B^*}{c_A^*}$ increases as well, and so the ray from the Foreign origin rotates *away* from the diagonal. This shifts the intersection toward the northeast, increasing Home's consumption and decreasing Foreign's. Moreover, the budget line flattens with a lower ρ'. We can thus find the candidate $\rho' = \rho$ that satisfies the budget set (as formally established by Lemma 1).

As drawn, E is to the northwest of F. This means $c_B > Y_B + a_B$, and so Home's imports of B are greater than its asset. Similarly, $c_A < Y_A + a_A$, and Home's exports of A, $Y_A - c_A$, are greater in magnitude than its debt $-a_A$. As $(1+\tau)(1+\tau^*)$ increases, the equilibrium is pushed further to the southeast, away from the free trade contract curve. If the equilibrium point lies below F, then $c_B < Y_B + a_B$ and Home's imports of B are less than its assets a_B. Similarly, if it lies to the right of F, then $c_A > Y_A + a_A$, and Home's exports of A are less than its debt.

There is the question of what happens as we follow the budget line further to the southeast, extending into the shaded area in Figure 2. For those points, either Home starts importing A and exporting B, or Foreign starts importing B and exporting A. These allocations are indeed an equilibrium with appropriate consumption taxes, but they may not be under our tariff instruments (as the pattern of trade has been reversed). If they arise because one of the countries subsidizes its imports, it will not be achievable with tariffs. The fact that tariffs are always non-negative, and adversely affect imports, rules out such tariff equilibria.

4 Balanced-Trade Tariff Wars

Is it possible that a set of tariffs can lead to balanced trade? And, if so, under what initial asset positions? And are such tariffs unique? In this section, we answer this set of questions. We first state the result and then provide a diagrammatic derivation. We leave the formal proof to the appendix.

Let us define

$$\underline{\theta} \equiv \frac{\overline{Y}_B - g^{-1}(\hat{\rho})\left(Y_A - \hat{\rho} Y_B^*\right)}{g^{-1}(\hat{\rho})\hat{\rho}\overline{Y}_A + Y_A^* - \hat{\rho} Y_B}.$$

Proposition 1 (Balanced Trade Equilibria)**.** *Suppose that Condition C1 holds, and initial asset positions satisfy $a_A \times a_B < 0$. Let $\hat{\rho} \equiv -a_A/a_B$.*

(i) For any pair $(t, t^*) \in [\underline{t}, \infty) \times [\underline{t}^*, \infty)$ where

$$\underline{t} \equiv \max\left\{\hat{\rho}/g\left(Y_B/Y_A\right), g\left(Y_B/Y_A\right)/\hat{\rho}\right\} - 1$$
$$\underline{t}^* \equiv \max\left\{\hat{\rho}/g\left(Y_B^*/Y_A^*\right), g\left(Y_B^*/Y_A^*\right)/\hat{\rho}\right\} - 1$$

there exists a tariff equilibrium that features no trade, $c_A = Y_A, c_B = Y_B$, and where $\hat{\rho}$ are the terms of trade.

(ii) Suppose that $g(Y_B^*/Y_A^*) < \hat{\rho} < g(Y_B/Y_A)$. Let $T^* \equiv [\mathbb{I}_{\{\underline{\theta}>0\}}(\hat{\rho}/g(\underline{\theta})-1), \hat{\rho}/g(Y_B^*/Y_A^*)-1)$. T^* is non-empty and for any $t^* \in T^*$ there exists a $t \geq 0$ such that $\hat{\rho}$ is a tariff equilibrium terms-of-trade, and trade is balanced:

$$Y_A - c_A = -\hat{\rho}(Y_B - c_B).$$

In any such equilibrium Home exports good A.

(iii) There is no tariff equilibrium with balanced trade where Home imports good A.

Balanced trade occurs when $Y_A - c_A + \rho(Y_B - c_B) = 0$. From the budget constraint, this requires that net foreign assets positions are zero:

$$a_A + \rho a_B = 0.$$

Therefore, the terms-of-trade must be $\rho = \hat{\rho} \equiv -a_A/a_B$. We immediately see why $a_A \times a_B < 0$ is necessary for a well-defined balanced-trade relative price (unless we include the uninteresting case of $a_A = a_B = 0$). Moreover, unless we are in the knife-edge case in which $\rho^{FT} = \hat{\rho}$, that is, in which free trade leads to balanced trade, we also need nonzero tariffs.

Part (i) of the Proposition shows that if *both Home and Foreign tariffs* are sufficiently high, there is an equilibrium with no trade at all. In this equilibrium, the terms of trade, $\hat{\rho}$, are exactly such that the net foreign asset position of each country is zero.

Part (ii) proceeds to show that autarky is not necessarily the only equilibrium outcome with balanced trade. The construction of the set of tariffs consistent with balanced and nonzero trade proceeds as in the characterization of the tariff-war equilibrium above, but in reverse. Specifically, we know *a priori* the required terms-of-trade and seek the associated tariffs. Balanced trade is thus associated with the budget line through F with slope $-\hat{\rho}$, as shown in Figure 3.

The requirement that $g(Y_B^*/Y_A^*) < \hat{\rho} < g(Y_B/Y_A)$ guarantees that in the absence of tariffs, when facing a hypothetical world price of $\hat{\rho}$, Home will be an exporter of good A, while Foreign will be an exporter of good B. A tariff equilibrium cannot reverse the pattern of trade, so the equilibrium must remain to the north-west of the endowment point.

For a given foreign tariff rate t^*, we obtain the slope of the ray from Foreign's origin:

$$\frac{c_B^*}{c_A^*} = h\left(\hat{\rho}/(1+t^*)\right).$$

Where this intersects the budget line is the balanced-trade equilibrium candidate associated with t^*. Given the c_B/c_A associated with this intersection, we can recover the necessary Home tariff from Home's optimality condition:

$$g\left(\frac{c_B}{c_A}\right) = (1+t)\hat{\rho}.$$

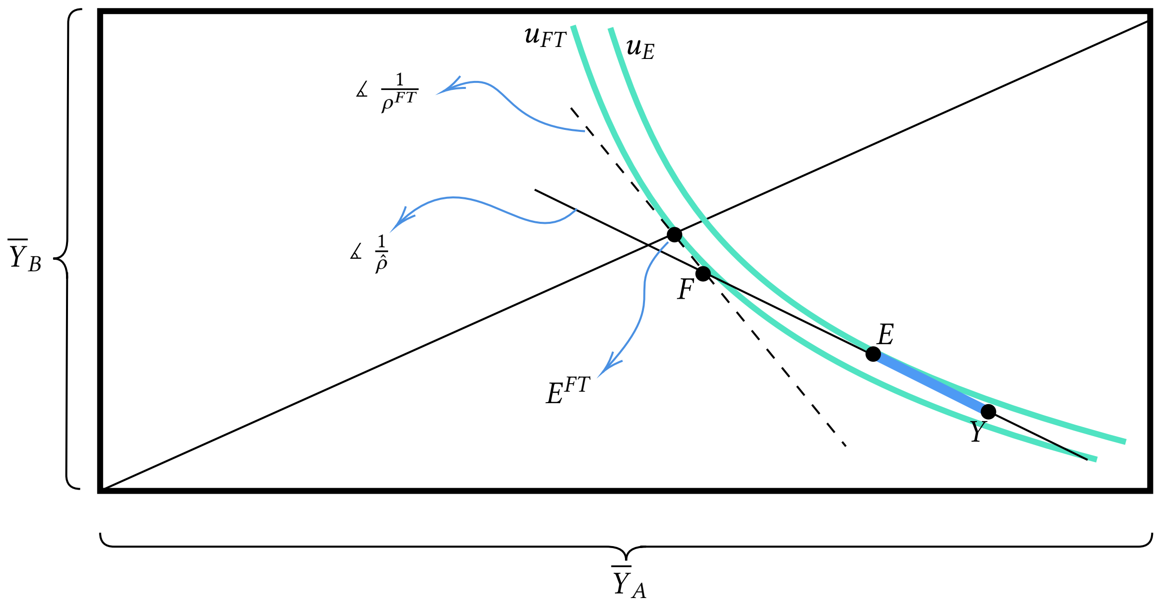
The condition in Proposition that requires $t^* \geq \mathbb{I}_{\{\underline{\theta}>0\}}(\hat{\rho}/g(\underline{\theta}) - 1)$ ensures that this allocation is consistent with a non-negative Home tariff.

Proceeding in this way, for each candidate t^* large enough, we obtain a unique t that balances trade. The proof of Proposition 1 derives this formally. Note that as tariffs increase, the equilibrium allocation approaches the autarkic outcome along the budget line. The condition that $g(Y_B^*/Y_A^*) < \hat{\rho}$ guarantees that indeed the autarky limit is reached when $t^* = \hat{\rho}/g(Y_B^*/Y_A^*) - 1$.

Part (iii) finally shows that the pattern of trade cannot be reverse in any balanced trade equilibrium, and thus the above discussion cover all the possible tariff equilibrium cases.

Figure 3 shows a tariff war that achieves balanced trade. The dashed line is the budget line at free-trade prices, ρ^{FT}. Note that in the way we have drawn this case, Home is a net debtor at free-trade prices: Home's endowment point lies in a higher budget line than its endowment point once adjusted by its net foreign asset position. The solid line represents the new budget

Figure 3: Tariff War Equilibrium With Balanced Trade



line at prices $\hat{\rho}$. In this case, both F and Y lie on the same budget line, hence the net foreign asset positions have fallen to zero. Point E represents a point in this new budget line associated with a zero tariff at Home. At this point, a tariff equilibrium with balanced trade is obtained with a positive tariff in Foreign. All points on the shaded blue line, connecting point E with point Y, are balanced trade equilibria. As we move towards Y on this line, tariffs are increasing in both Home and Foreign, eventually converging to the autarkic allocation.

The Figure can also speak to the welfare consequences of the trade war. Recall that free-trade equilibrium occurs where the original budget line with slope ρ^{FT} intersects the diagonal, shown by point E^{FT} in the Figure. Home and Foreign welfare is strictly decreasing as we move from point E towards point Y, that is, as tariffs increase. As depicted, Home's indifference curve is tangent to the budget line with $P_B/P_A = \hat{\rho}$ at point E; that is, $t = 0$. All other points on the segment E-Y are on lower Home indifference curves. The reason is that the terms of trade are given by $\hat{\rho}$, and the distortion in consumption generated by the tariffs is increasing as we move from E towards Y. We also know that autarky is inside the original budget set of Foreign under free trade prices (as it is a net creditor at those prices), so Foreign will be strictly worse off at Y when compared to E^{FT}. Foreign could potentially benefit from a trade war that is not intense enough to reach autarky, but only if the terms of trade move in its favor (in the Figure, the terms of trade do improve for Foreign but Foreign is always strictly worse off). Home was a a net debtor, and it is more intuitive that a trade-war can benefit it. The Figure is drawn in such way that even autarky is an improvement for Home when compared to the free trade outcome: the indifference curve for Home in the autarky allocation (denoted by u_E) is higher than at the free trade equilibrium, u_{FT}. However, such an outcome is not a necessity: it can well be the case that all balanced trade outcomes generate a welfare loss in Home. This discussion highlights the competing effects operating in the model: the welfare consequences of the valuation effects, the terms of trade movements in relation to free trade, and the adverse effects of tariff distortions as a trade war intensifies.

5 Conclusion

In this paper, we characterized the set of policies and gross asset positions such that balanced trade is consistent with financial obligations. Using a standard two-by-two environment, we showed that as long as gross positions differ in sign, there exists a set of bilateral tariffs that balanced trade and zeros out any net financial position vis-a-vis the other country. The welfare consequences are ambiguous, and the valuation effects can potentially offset or amplify the loss of the gains from trade.

The analysis considers only the ex-post scenario of paying off inherited liabilities during a trade war. This has obvious implications for the ex ante question of international asset prices and interest rates. The possibility of a future trade war will inevitably distort ex ante lending, as lenders will be wary of seeing their net claims devalued during a subsequent trade war. One way to avoid this in the environment considered here is for portfolios to be either long in both goods or short in both goods. That is, ruling out leveraged claims on assets. This distortion away from such portfolios would have important consequences in richer models where competing gross positions play a role, say for the diversification of risk or location of multinational activity.

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A Proofs

Proof of Lemma 1

Proof. For the free trade case, we will guess and verify that ρ^{FT} is an equilibrium price. Setting $c_B = c_A(\overline{Y}_B/\overline{Y}_A)$ satisfies the two conditions in the first line of (6). Then

$$c_A = \frac{Y_A + a_A + \rho^{FT}(Y_B + a_B)}{1 + \rho^{FT}\overline{Y}_B/\overline{Y}_A}$$

satisfies the resource condition. The associated $c_A^* = \overline{Y}_A - c_A$ is

$$c_A^* = \frac{Y_A^* - a_A + \rho^{FT}(Y_B^* - a_B)}{1 + \rho^{FT}\overline{Y}_B/\overline{Y}_A}.$$

Condition (7) ensures that $c_A > 0$ and $c_A^* > 0$, verifying the conjectured price supports an equilibrium. A similar allocation can be supported if $(1+\tau)(1+\tau^*) = 1$, but $\tau, \tau^* \neq 0$. In this case, optimality implies $g(c_B/c_A^*) = g(c_B^*/c_A^*) = g(\overline{Y}_B/\overline{Y}_A)$. The terms-of-trade can then be obtained from $(1+\tau)\rho = \rho/(1+\tau^*) = g(\overline{Y}_B/\overline{Y}_A)$.

Moving to the case when $(1+\tau)(1+\tau^*) \neq 1$, define $h \equiv g^{-1}$ to be the inverse of g. From Home's optimality, we have $c_B = h(\rho(1+\tau))c_A$. Substituting this into the second line of (6), we have

$$c_A = \Phi_1(\rho) \equiv \frac{Y_A + a_A + \rho\,(Y_B + a_B)}{1 + \rho h(\rho(1+\tau))}. \tag{15}$$

Similarly, Foreign optimization implies $c_B^* = (\overline{Y}_B - c_B) = h(\rho/(1+\tau^*))c_A^* = h(\rho/(1+\tau^*))(\overline{Y}_A - c_A)$. Using $c_B = h(\rho(1+\tau))c_A$, we can substitute and re-arrange to obtain

$$c_A = \Phi_2(\rho) \equiv \frac{\overline{Y}_A h(\rho/(1+\tau^*)) - \overline{Y}_B}{h(\rho/(1+\tau^*)) - h(\rho(1+\tau))}. \tag{16}$$

We can evaluate these expressions at the end points of $\boldsymbol{P}$. Letting $\underline{\rho} \equiv \frac{\rho^{FT}}{1+\tau}$, we have:

$$\Phi_1\left(\underline{\rho}\right) = \frac{Y_A + a_A + \underline{\rho}(Y_B + a_B)}{1 + \rho^{FT}h(\rho^{FT})}.$$

Using $h(\rho^{FT}) = \overline{Y}_B/\overline{Y}_A$ and factoring out $\overline{Y}_A$, we have

$$\begin{aligned}\Phi_1\left(\underline{\rho}\right) &= \left(\frac{Y_A + a_A + \frac{\rho^{FT}}{1+\tau}(Y_B + a_B)}{\overline{Y}_A + \rho^{FT}\overline{Y}_B}\right)\overline{Y}_A \\ &\leq \left(\frac{Y_A + a_A + \frac{\rho^{FT}}{1+\tau}(Y_B + a_B)}{\overline{Y}_A + \frac{\rho^{FT}}{1+\tau}\overline{Y}_B}\right)\overline{Y}_A \\ &= \left(1 - \frac{Y_A^* - a_A + \frac{\rho^{FT}}{1+\tau}(Y_B^* - a_B)}{\overline{Y}_A + \frac{\rho^{FT}}{1+\tau}\overline{Y}_B}\right)\overline{Y}_A < \overline{Y}_A\end{aligned}$$

where the second line follows as the numerator is strictly positive by the premise on asset positions and both $\overline{Y}_B$ and $\overline{Y}_A$ are strictly positive and the third also uses the premise on assets. For $\Phi_2\left(\underline{\rho}\right)$,

$$\Phi_2\left(\underline{\rho}\right) = \frac{\overline{Y}_A h\left(\frac{\rho^{FT}}{(1+\tau)(1+\tau^*)}\right) - \overline{Y}_B}{h\left(\frac{\rho^{FT}}{(1+\tau)(1+\tau^*)}\right) - \overline{Y}_B/\overline{Y}_A} = \overline{Y}_A.$$

Hence, $\Phi_1\left(\underline{\rho}\right) < \Phi_2\left(\underline{\rho}\right)$.
Turning to the other endpoint, let $\overline{\rho} \equiv (1+\tau^*)\rho^{FT}$. Then

$$\Phi_1\left(\overline{\rho}\right) = \frac{Y_A + a_A + \overline{\rho}\left(Y_B + a_B\right)}{1 + \overline{\rho}h(\rho^{FT}(1+\tau)(1+\tau^*))} > 0,$$

where the inequality follows from the premise on assets. We also have the numerator of

$$\Phi_2\left(\overline{\rho}\right) = \frac{\overline{Y}_A h(\rho^{FT}) - \overline{Y}_B}{h(\overline{\rho}/(1+\tau^*)) - h(\overline{\rho}(1+\tau))} = 0,$$

as $h(\rho^{FT}) = \overline{Y}_B/\overline{Y}_A$ and the denominator is nonzero.
At $\rho = \rho^{FT}/(1+\tau)$, we have $\Phi_1(\rho) > 0$ and $\Phi_2(\rho) = 0$. At $\rho = (1+\tau^*)\rho^{FT}$, we have $\Phi_1(\rho) > \overline{Y}_A$, $\Phi_2(\rho) = \overline{Y}_A$. Hence, $\Phi_1\left(\overline{\rho}\right) > \Phi_2\left(\overline{\rho}\right)$. By continuity, there is a point in $\boldsymbol{P}$ at which $\Phi_1\left(\rho\right) = \Phi_2\left(\rho\right)$. This defines a price ρ with an associated allocation that satisfies the equilibrium conditions. □

Proof of Lemma 2

Proof. Suppose that we have an allocation that satisfies the conditions in the lemma. Using the tariff revenue in (11), the households budget constraints (10) hold given that (14) hold. And the aggregate resource constraints hold by the premise.
The households optimality conditions are

$$g(c_B/c_A) = p_B/p_A \text{ and } g(c_B^*/c_A^*) = p_B^*/p_A^*$$

If $c_A \neq Y_A$, then p_A is determined by (8). In that case, we let $p_B = g(c_B/c_A) * p_A$. This value of p_B

satisfies (8) given that (12)-(13) hold. If $c_B \neq y_B$, then (8) determines p_B, and p_A is chosen such that $p_A = p_B/g(c_B/c_A)$. And this value of p_A satisfies (8) given that (12)-(13) hold. In a similar manner, we proceed to show the same for the Foreign prices when $c_A^* \neq Y_A^*$ or $c_B^* \neq Y_B^*$. The last case to consider is then when $c_A = Y_A$, $c_B = Y_B$, $c_A^* = Y_A^*$, and $c_B^* = Y_B^*$. In that case, we can pick p_A to be any value in $[1, (1+t)]$ and let $p_B = g(c_B/c_A)p_A$. Given that $g(c_B/c_A) \in [\rho/(1+t), (1+t)\rho]$ in this case, it follows that p_B satisfies (8). And we can do the same for Foreign. Hence, we have constructed prices that are consistent with goods arbitrage and support the household optimality conditions. Given that the households budget constraints hold, and the resource constraints hold, we have then a tariff equilibrium. □

Proof of Lemma 3

Proof. Since the budget sets are identical at world prices, and the conditions in (6) are satisfied by construction, we only need to check that $\tau > -1$ and $\tau^* > -1$. As $g(c_B/c_A) \leq (1+t)\rho$, $\tau \geq -t/(1+t) > -1$. Similarly, as $g(c_B^*/c_A^*) \geq \rho/(1+t^*)$, $\tau \geq -t^*/(1+t^*) > -1$. □

Proof of Lemma 4

Proof. First, we argue that the same ρ and the same consumption allocation that is a tariff equilibrium for tariffs t, t^* is also an equilibrium for tariff $t', t^{*\prime}$. For the Home household, their optimization condition is given by (13). This condition holds for $t' \geq t$ if it holds for t, as $\rho \geq 0$. The same applies to the equivalent conditions for Foreign given that $t^{*\prime} \geq t^*$. The budget constraints of households in both Home and Foreign, (14), hold as neither ρ nor the consumption allocation have changed. The resource constraints also hold. And thus, we have satisfied all of the conditions that are needed for a tariff equilibrium. □

Proof of Proposition 1

We first narrow the scope for possible equilibria:

Lemma 5. *In any tariff equilibrium with balanced trade we have* $g\left(\frac{Y_B^*}{Y_A^*}\right) \leq \hat{\rho}(1+t^*)$.

Proof. Recall that in any tariff equilibrium, $g(c_B^*/c_A^*) \leq (1+t^*)\hat{\rho}$. To generate a contradiction of the lemma, suppose $g\left(\frac{Y_B^*}{Y_A^*}\right) > \hat{\rho}(1+t^*)$. This implies,

$$g\left(\frac{Y_B^*}{Y_A^*}\right) > (1+t^*)\hat{\rho} \geq g\left(\frac{c_B^*}{c_A^*}\right) \iff \frac{Y_B^*}{Y_A^*} < \frac{c_B^*}{c_A^*}. \tag{17}$$

Foreign's budget set in balanced trade, $c_A^* + \hat{\rho}c_B^* = Y_A^* + \hat{\rho}Y_B^*$, implies that if $c_B^* \leq Y_B^*$ then $c_A^* \geq Y_A^*$. Hence, for (17) to hold requires $c_B^* > Y_B^*$ and $c_A^* < Y_A^*$ and Foreign is exporting A and Home is exporting

B. Equilibrium requires $g\left(c_B^*/c_A^*\right) = (1+t^*)\hat{\rho}$ and $g\left(c_B/c_A\right) = \hat{\rho}/(1+t)$. As $t, t^* \geq 0$, we have

$$g\left(\frac{Y_B^*}{Y_A^*}\right) > g\left(\frac{c_B^*}{c_A^*}\right) = (1+t^*)\hat{\rho} \geq \frac{\hat{\rho}}{1+t} = g\left(\frac{c_B}{c_A}\right) > g\left(\frac{Y_B}{Y_A}\right)$$
$$\Rightarrow \frac{Y_B^*}{Y_A^*} < \frac{Y_B}{Y_A},$$

where the first and last inequalities in the top line use the fact that Home is exporting B and importing A and vice versa for Foreign. Hence, we have a contradiction of Condition (C1). Thus, the condition in the lemma is a necessary condition for a tariff equilibrium with balanced trade. □

The proof of the above lemma established the following corollary:

Corollary 1. *In a tariff equilibrium with balanced trade, either there is zero trade (Autarky) or Home exports A and imports B.*

We next show conditions for Autarky to be an equilibrium:

Lemma 6. *If t^* is such that:*

(i) $t^* \geq \frac{\hat{\rho}}{g\left(Y_B^*/Y_A^*\right)} - 1$; *and*

(ii) $t^* \geq \frac{g\left(Y_B^*/Y_A^*\right)}{\hat{\rho}} - 1$

then there exists a $t \geq 0$ such that Autarky is an equilibrium.

Proof. Note that the two conditions together imply

$$g\left(\frac{Y_B^*}{Y_A^*}\right) \in \left[\frac{\hat{\rho}}{1+t^*}, (1+t^*)\hat{\rho}\right], \tag{18}$$

This condition implies that $c_A^* = Y_A^*$ and $c_B^* = Y_B^*$ is consistent with Foreign optimality. The balanced-trade budget constraint is also satisfied in autarky. The remaining equilibrium condition is Home optimality:

$$g\left(\frac{Y_B}{Y_A}\right) \in \left[\frac{\hat{\rho}}{1+t}, (1+t)\hat{\rho}\right].$$

This is satisfied for any Home tariff such that

$$1+t \geq \max\left\{\frac{\hat{\rho}}{g\left(Y_B/Y_A\right)}, \frac{g\left(Y_B/Y_A\right)}{\hat{\rho}}\right\}.$$

Note that the lower bound on $1+t$ is weakly greater than one. Thus, if t^* satisfies the conditions in the lemma, there always exists a $t \geq 0$ such that Autarky is an equilibrium. $\square$

Given Lemmas 5 and 6, we only need to consider the remaining scenario: $g(Y_B^*/Y_A^*) < \hat{\rho}/(1+t^*)$. In this case, Autarky is not consistent with Foreign optimization. By Corollary 1, a balanced trade equilibrium in this case must involve Foreign exporting B and importing A. We now establish necessary and sufficient conditions for such an equilibrium.

Before stating the conditions, we define the following object:

$$\underline{\theta} \equiv \frac{\overline{Y}_B - g^{-1}(\hat{\rho})\left(Y_A - \hat{\rho}Y_B^*\right)}{g^{-1}(\hat{\rho})\hat{\rho}\overline{Y}_A + Y_A^* - \hat{\rho}Y_B}.$$

Lemma 7. *Suppose that $g(Y_B^*/Y_A^*) < \hat{\rho} < g(Y_B/Y_A)$. Let $t_1 = \mathbb{I}\left(\underline{\theta} > 0\right)\left(\frac{\hat{\rho}}{g(\underline{\theta})} - 1\right)$ and $t_2 = \frac{\hat{\rho}}{g(Y_B^*/Y_A^*)} - 1$. Then $t_1 < t_2$ and for any $t^* \in [t_1, t_2)$ there exists a $t \geq 0$ such that a tariff equilibrium with balanced and non-zero trade exists. In such equilibrium, Home exports good A.*

Proof. The condition that $t^* < t_2$ rules out Autarky as an equilibrium, as Foreign's optimality would be violated. Hence, $g(c_B^*/c_A^*) = \hat{\rho}/(1+t^*)$. Define

$$\theta^* \equiv g^{-1}\left(\frac{\hat{\rho}}{1+t^*}\right) = \frac{c_B^*}{c_A^*} < \frac{Y_B^*}{Y_A^*}, \tag{19}$$

where the last equality follows from condition (i): $g(Y_B^*/Y_A^*) < \hat{\rho}/(1+t^*)$. Using the fact that $a_A + \hat{\rho}a_B = 0$ by definition of $\hat{\rho}$, the Foreign budget constraint at world prices implies

$$c_A^* = \frac{Y_A^* + \hat{\rho}Y_B^*}{1+\hat{\rho}\theta^*},$$

where we have used $c_B^* = \theta^* c_A^*$. The proposed equilibrium allocation for Foreign is thus:

$$(c_A^*, c_B^*) = \left(\frac{Y_A^* + \hat{\rho}Y_B^*}{1+\hat{\rho}\theta^*}, \frac{\theta^*\left(Y_A^* + \hat{\rho}Y_B^*\right)}{1+\hat{\rho}\theta^*}\right). \tag{20}$$

Note that $c_A^* > 0$ and $c_B^* > 0$. From the resource conditions, we have

$$\begin{aligned} c_A &= \overline{Y}_A - c_A^* = \frac{Y_A + \hat{\rho}(\theta^*\overline{Y}_A - Y_B^*)}{1+\hat{\rho}\theta^*} = Y_A - \frac{\hat{\rho}(Y_B^* - \theta^* Y_A^*)}{1+\hat{\rho}\theta^*} \\ c_B &= \overline{Y}_B - c_B^* = \frac{\overline{Y}_B - \theta^* Y_A^* + \hat{\rho}\theta^* Y_B}{1+\hat{\rho}\theta^*} = Y_B + \frac{Y_B^* - \theta^* Y_A^*}{1+\hat{\rho}\theta^*}. \end{aligned} \tag{21}$$

As $Y_B^* > \theta^* Y_A^*$ from (19), we have $c_B > Y_B > 0$ and $c_A < Y_A$. That is, Home exports good A.
To ensure that $c_A \geq 0$, we require

$$0 \leq Y_A + \hat{\rho}(\theta^* \overline{Y}_A - Y_B^*) \iff \frac{\hat{\rho} Y_B^* - Y_A}{\hat{\rho} \overline{Y}_A} \leq \theta^* = g^{-1}\left(\frac{\hat{\rho}}{1+t^*}\right).$$

If $\hat{\rho} Y_B^* \leq Y_A$, then this condition is satisfied immediately. Otherwise, the following condition ensures $c_A \geq 0$:

$$t^* \geq \frac{\hat{\rho}}{g\left(\frac{\hat{\rho} Y_B^* - Y_A}{\hat{\rho} \overline{Y}_A}\right)} - 1. \tag{22}$$

This is condition (ii) of the lemma. Finally, we need to verify the allocation in (21) is optimal for Home. Consider a candidate t given by

$$t = \frac{g\left(c_B/c_A\right)}{\hat{\rho}} - 1,$$

where c_A and c_B are given by (21). By construction, the proposed allocation is optimal for Home given t. We need to verify $t \geq 0$:

$$t \geq 0 \iff (1+t)\hat{\rho} \geq \hat{\rho} \iff \frac{c_B}{c_A} = g^{-1}((1+t)\hat{\rho}) \leq g^{-1}(\hat{\rho}).$$

Substituting for c_B/c_A using (21), we require

$$g^{-1}(\hat{\rho}) \geq \frac{\overline{Y}_B + \hat{\rho}\theta^* Y_B - \theta^* Y_A^*}{Y_A + \hat{\rho}(\theta^* \overline{Y}_A - Y_B^*)}$$

$$\iff$$

$$\theta^* \left(g^{-1}(\hat{\rho})\hat{\rho}\overline{Y}_A + Y_A^* - \hat{\rho} Y_B\right) \geq \overline{Y}_B - g^{-1}(\hat{\rho})\left(Y_A - \hat{\rho} Y_B^*\right).$$

Note that

$$g^{-1}(\hat{\rho})\hat{\rho}\overline{Y}_A + Y_A^* - \hat{\rho} Y_B > (Y_B/Y_A)\hat{\rho}\overline{Y}_A + Y_A^* - \hat{\rho} Y_B = \hat{\rho} Y_B(\overline{Y}_A/Y_A - 1) + Y_A^* > 0$$

where the first inequality follows from $g^{-1}(\hat{\rho}) > Y_B/Y_A$ (a premise in the lemma).
Dividing through, we require

$$\theta^* \geq \frac{\overline{Y}_B - g^{-1}(\hat{\rho})\left(Y_A - \hat{\rho} Y_B^*\right)}{g^{-1}(\hat{\rho})\hat{\rho}\overline{Y}_A + Y_A^* - \hat{\rho} Y_B} \equiv \underline{\theta}. \tag{23}$$

If $\overline{Y}_B - g^{-1}(\hat{\rho})\left(Y_A - \hat{\rho} Y_B^*\right) \leq 0$, and thus $\underline{\theta} \leq 0$, then $\theta^* \geq \underline{\theta}$. Otherwise, using $g(\theta^*) = \hat{\rho}/(1+t^*)$, this is equivalent to

$$t^* \geq \frac{\hat{\rho}}{g(\underline{\theta})} - 1. \tag{24}$$

Now, we note that

$$\frac{\hat{\rho} Y_B^* - Y_A}{\hat{\rho} \overline{Y}_A} < \underline{\theta}.$$

Thus if $\underline{\theta} \leq 0$, we have that $\hat{\rho} Y_B^* - Y_A < 0$. And if $\underline{\theta} > 0$, we have that

$$\frac{\hat{\rho}}{g(\underline{\theta})} - 1 > \frac{\hat{\rho}}{g\left(\frac{\hat{\rho} Y_B^* - Y_A}{\hat{\rho} \overline{Y}_A}\right)} - 1$$

and (22) is implied by (24).

Summarizing this: for any $t^* \geq [t_1, t_2)$ (as defined in the lemma) condition (23) holds, and thus there exists a $t \geq 0$ such that the allocation in (20) and (21) is a balanced trade equilibrium. In this equilibrium, Home exports good A.

The last thing to check is that $t_1 < t_2$. Note that $t_2 > 0$ by the assumption that $g(Y_B^*/Y_A^*) < \hat{\rho}$. So if $\theta \leq 0$, it follows that $t_1 < t_2$. For $\theta > 0$, doing some algebra, we get that

$$\underline{\theta} < \frac{Y_B^*}{Y_A^*} \Leftrightarrow Y_B(Y_A^* + \hat{\rho} Y_B^*) < g^{-1}(\hat{\rho}) Y_A (Y_A^* + \hat{\rho} Y_B^*) \Leftrightarrow Y_B < g^{-1}(\hat{\rho}) Y_A$$

which holds by $g(Y_B/Y_A) > \hat{\rho}$. □

We are now ready to prove Proposition 1:

Proof. Lemma 6 guarantees part (i). Part (ii) is guaranteed by Lemma 7. Corollary 1 guarantees part (iii). □