Tariff Wars and Net Foreign Assets*

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Abstract

This paper examines whether and how international financial claims accumulated during a period of relatively free trade can be settled once a trade war erupts. We identify the conditions under which net claims can be honored even under balanced trade – or, in the extreme, autarky. The analysis also reveals a potential equilibrium multiplicity in which a trade war impoverishes one country while enriching its trading partner, with the identity of the winner and loser determined by a sunspot. The key adjustment mechanism is the tariff-induced shift in the terms of trade (and the corresponding real exchange rate), which revalues the relative gross asset positions to ensure that any residual trade (if any) is consistent with the inherited financial claims.

1 Introduction

The two eras of globalization between the two world wars and after the 1970s witnessed large declines in trade costs and substantial increases in cross-border asset positions. Both eras eventually ran afoul of pent up protectionist sentiment, exemplified in the US by the Smoot-Hawley and Trump tariffs for the two eras, respectively. The question we study is whether and how the stock of international financial claims accumulated during a period of relatively free trade can be paid once a trade war breaks out. We provide conditions under which net claims can be honored even if trade is balanced or, in the limit, completely shut down. The analysis also uncovers an interesting potential multiplicity in which a trade war immiserizes one country and benefits its

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trading partner, with the winner and loser determined by a sunspot. The key channel through which all these phenomona are supported in equilibrium is by an appropriate shift in the equilibrium terms of trade (and associated real exchange rate) that revalues the relative gross asset positions to ensure that whatever trade occurs (if any) is consistent with the inherited financial claims.

The basic logic can be seen very clearly in a standard two-country (Home and Foreign), two-good static exchange economy that can be analyzed using a standard Edgeworth box. The initial endowment point is translated by inherited gross claims (positive or negative) denominated in each good to be payable in each country. (Hence, there are four potential assets). Consider the net value of these positions. Within each country, claims on either good can be aggregated using domestic prices. Across borders, the aggregated claims use international relative prices.

Suppose the trade war is severe enough that autarky (i.e., no trade in either good) is the resulting equilibrium. How is this supported in equilibrium? First, domestic prices in each country are determined by relative endowments, which can be used to aggregate net claims in each country. The question then becomes is there an international relative price that zeros out these claims. This is possible if the countries' external gross claims on each other have the same sign. For example, if both country's have a strictly positive gross claim, then there is a unique strictly positive and finite relative price that equates these claims. The same holds if both countries have strictly negative gross positions in the other country. High enough tariffs then ensure that there is no scope for goods arbitrage even if this international relative price differs from either domestic relative price. Further restrictions on the asset positions can ensure that autarky is the *only* equilibrium for high enough tariffs.

Autarky is not the only balanced-trade outcome, however: for moderate tariff levels there exists a continuum of bilateral tariff combinations that deliver balanced but strictly positive trade while satisfying contractual financial obligations. For example, suppose Home is a net debtor at free trade prices and Foreign imposes tariffs. With cross-border asset positions of the same sign, there is a Home tariff that implements the correct international terms of trade that zeros out its net debt in equilibrium. Balanced trade then becomes consistent with both budget sets.

We also discuss the competing effects that sign the welfare consequences of a trade war. The net debtor at free trade prices gains (and the creditor loses) during a trade war due to the revaluation of assets. On the other hand, if countries hold positive claims on the other country's export good, then the change in the terms of trade favor the exports of the creditor (as the debtor's portfolio is long in that good). If both countries have short positions in the other country's export, then the terms of trade move in favor of the debtor's export. Finally, both suffer from the loss of the gains from trade. Hence, the consequences for both country's welfare are ambiguous, depending on reduction on the gains from trade as well as the explicit and implicit shifts in wealth due to the changes in the terms of trade.

In practice, many countries are indebted in a foreign currency (say, the US dollar for many emerging and developing economies). We map this to our framework by considering a country, say Home, that owes a claim denominated in its imported good (under free trade). We provide bounds on Foreign's tariffs that expose Home to a "debt dilemma." In particular, the conditions admit an equilibrium terms of trade that increases the real burden of the debt to the point that Home must export *both* goods to Foreign in order to satisfy its budget constraint. Thus, Home is immiserized at the expense of Foreign by the real burden of its liability. This is reminiscent of an emerging market trying to service its dollar debt when confronted with a US policy that generates a sharp appreciation of the dollar.

Home's debt dilemma arises when it has a short position in its imported good and Foreign's tariff are in a non-empty interval. These conditions do not preclude that Foreign also has a liability in its imported good and Home has tariffs in the relevant interval. Thus both countries may simultaneously face the potential for a debt dilemma. It then becomes indeterminate which (if any, as autarky may still be an outcome) of the countries find themselves immiserized by the real appreciation of their liability.

With the baseline results in hand, we show how the benchmark analysis can be extended to many goods. The gross claims can be aggregated within country at domestic relative prices, and then, as in the benchmark, there is a single international relative price that equate the cross-border claims. From this point, it is straightforward to follow the Arrow-Debreu tradition and think of distinct goods as indexed by time and states of nature. This allows us to speak to trade wars in dynamic environments. From this perspective, the domestic interest rate is used to aggregate claims within a country, and then the exchange rate is pinned down by the need to revalue cross-border claims.

Our questions and the associated answers echo the Transfer Problem famously debated by Keynes (1929) and Ohlin (1929) in the aftermath of the first world war. That debate turned on whether a transfer by one country to the another affected the terms-of-trade. With identical homothetic preferences, the answer is no, due to the fact that the distribution of wealth does not influence relative prices in such an environment. However, when the marginal dollar is spent differently, such transfers do affect the terms-of-trade. Our analysis is closest to the treatment in Samuelson (1952, 1954) with an environment in which preferences are identical and homothetic, but where there are trade frictions. Samuelson shows that when tariffs are present, the required transfer has an adverse impact on the payer's terms-of-trade. Our question is the mirror image – if, under free trade, one country needs to make a net transfer to the other, can the disruption to trade generate a terms of trade that nullifies the need to make the payment.[1]

[1]Epifani and Gancia (2017) study global imbalances in a trade model and discuss how the price indexes react

Prompted by the recent policy events, there are several new papers on the interactions between tariffs and trade deficits.[2] Pujolas and Rossbach (2024) study the welfare effects of trade wars in an Armington model with trade imbalances, where countries have exogenous international net (but not gross) positions. Ignatenko et al. (2025) quantify the welfare effects of tariffs in a model with trade imbalances where net foreign asset positions are given but trade deficits endogenously respond to policy. Costinot and Werning (2025) analyze a dynamic model with fixed terms-of-trade and study the effects of a permanent increase in tariffs on the trade deficit by affecting the incentives of domestic households to save and consume. Obstfeld (2025) surveys the arguments for the determination of the U.S. trade deficit and argues that trade policies are not the main driver. In his discussion of this paper, Perri (2025) uses an international macro model to analyze the effects of tariff shocks, and their persistence, on the trade deficit, emphasizing also that the response of the economy to other shocks may be hampered by the presence of tariffs. The papers cited above do not study valuation effects, the main goal of our work.

Itskhoki and Mukhin (2025), in a related and contemporaneous paper to ours, study the optimal unilateral tariff in the presence of valuation effects.[3] Our focus is also on valuation effects but we study instead the economy response to a trade war where bilateral tariffs are imposed.[4]

The role of valuation effects for understanding the behavior of the net foreign asset positions goes back to the pioneering work of Lane and Milesi-Ferretti (2001), Tille (2003), and Gourinchas and Rey (2007). This initial work focused in a large part on the behavior of the dollar exchange rate to account for changes in asset valuations. Recent work by Atkeson, Heathcote, and Perri (2025) highlights that even if the deterioration of the US dollar accounted for the surprising stability of the US NFA position in the early 2000s (even in the presence of very large trade deficits), it is crucial to consider the spectacular performance of the US stock markets relative to foreign ones to account for the US NFA deterioration after 2008. Our paper shows that valuation effects necessary arise to rebalance the net foreign asset positions in the presence of a trade war, shedding light into the recent persistent deterioration of the US dollar as a response of US trade policy announcements.

The paper is structured as follows. Section 2 sets up the environment. Section 3 discusses the free-trade benchmark. Section 4 provides the main results characterizing when a trade war leads to autarky or to balanced trade.Section 5 studies a novel multiplicity that arises when countries

differently in a trade model with increasing returns a la Krugman.

[2]There is previous work on global imbalances, trade deficits and trade frictions. See for example, Razin and Svensson (1983), Dekle, Eaton, and Kortum (2007), Fitzgerald (2012), Reyes-Heroles (2017), Kehoe, Ruhl, and Steinberg (2018), Alessandria, Bai, and Woo (2024), among others.

[3]There is a classic and very general theory on optimal tariffs emphasizing the terms-of-trade effects, but that does not consider valuation effects. See Dixit and Norman (1980) and Dixit (1985) for comprehensive summaries.

[4]Recent papers, starting with Bianchi and Coulibaly (2025), have studied the optimal monetary response to tariffs. See also Werning, Lorenzoni, and Guerrieri (2025) and Auclert, Rognlie, and Straub (2025).

have liabilities in the other country's goods. Section 6 discusses the case with more general asset structures. Section 7 extends the model to arbitrary number of goods, time and uncertainty. Section 8 concludes. All proofs are collected in the Appendix.

2 Environment

The environment is static and consists of two countries, Home and Foreign, and two goods, A and B. Letting a star denote Foreign, let endowments of A and B be denoted $Y = (Y_A, Y_B)$ and $Y^* = (Y_A^*, Y_B^*)$, with global endowments denoted by $\overline{Y}_A \equiv Y_A + Y_A^*$ and $\overline{Y}_B \equiv Y_B + Y_B^*$.

Preferences are assumed to be identical for Home and Foreign and characterized by a homothetic utility function $u(x_A, x_B)$ over non-negative consumption of good A, x_A, and good B, x_B. By homotheticity, we can define the Marginal Rate of Substitution (MRS) between B and A as a function of the ratio $x = x_B/x_A$:

$$g(x) \equiv \frac{u_B(1,x)}{u_A(1,x)},$$

where we make the standard assumption $g'(x) < 0$. In addition, we make the following Inada assumptions: $\lim_{x\to 0} g(x) = \infty$ and $\lim_{x\to\infty} g(x) = 0$. Let h denote the inverse of g, which is well defined given the strict monotonicity of MRS:

$$h(x) \equiv g^{-1}(x).$$

Since preferences are identical, there are gains from trade only if endowments differ. We assume all endowments are positive, and we label the goods so that Home has a relative abundance in good A.

Assumption 1. *The endowments are strictly positive $Y_A, Y_B, Y_A^*, Y_B^* > 0$. And Home is relatively abundant in good A:*

$$\frac{Y_B^*}{Y_A^*} > \frac{\overline{Y}_B}{\overline{Y}_A} > \frac{Y_B}{Y_A}.$$

We could set one price to one to serve as numeraire, but for expositional clarity we assume all prices are denominated in some world currency. Let the price of good A at Home in terms of this world currency be p_A, and let the price of good B be p_B. Let (p_A^*, p_B^*) denote the corresponding prices of good A and B, respectively, in Foreign. There are no physical costs of trade, but each country may levy a non-negative tariff on imports. Specifically, let $\tau \geq 0$ denote Home's tariff and $\tau^* \geq 0$ Foreign's tariff. Goods arbitrage requires that

$$\frac{p_A}{p_A^*} = \begin{cases} (1+\tau) & \text{for } c_A > Y_A \\ 1/(1+\tau^*) & \text{for } c_A < Y_A \end{cases}$$

with $p_A/p_A^* \in [1/(1+\tau^*), (1+\tau)]$ if $c_A = Y_A$,

$$\frac{p_B}{p_B^*} = \begin{cases} (1+\tau) & \text{for } c_B > Y_B \\ 1/(1+\tau^*) & \text{for } c_B < Y_B \end{cases}$$

with $p_B/p_B^* \in [1/(1+\tau^*), (1+\tau)]$ if $c_B = Y_B$.

Asset Markets: The countries begin the period with gross claims on each other denominated in the two goods. It is important to be precise about *where* the claims are settled; that is, whether the goods are paid before or after import taxes are levied. For an asset that promises a unit of good A, for example, there are two possibilities. One is that payment is made in Home, and thus carries a payout of p_A in the world numeraire; a second is that payment is made in Foreign, and thus carries a payout of p_A^*.[5] For reference, Table 1 collects the asset nomenclature we use throughout.

Table 1: Asset Nomenclature

Notation	Definition	Value in Global Numeraire
a_A	Home's asset (liability if negative) denominated in A to be paid by Foreign in Foreign	$p_A^* a_A$
a_B	Home's asset (liability if negative) denominated in B to be paid by Foreign in Foreign	$p_B^* a_B$
a_A^*	Foreign's asset (liability if negative) denominated in A to paid by Home in Home	$p_A a_A^*$
a_B^*	Foreign's asset (liability if negative) denominated in B to paid by Home in Home	$p_B a_B^*$

Let a_k denote Home's claims on Foreign (that is, Foreign's liabilities) in good $k \in \{A, B\}$ that

[5]There is also a third possibility: that payment is made in what Dixit and Norman (1980) refer to as "border" prices, which is the price after any export taxes are levied but before any import tariffs. For example, if assets are settled in a global financial center outside both Home and Foreign's customs area. We do not pursue this alternative as the analysis is similar to what we present in Section 4.4 under Assumption 4. As in the case of Section 4.4, the advantage of introducing a global financial center is that the valuations of the assets are not directly affected by the level of tariffs or export taxes given a terms of trade, and isolates the role of the terms of trade in balancing trade when tariffs are high.

are paid in Foreign. Let a_k^* denote Foreign's claims on Home (that is, Home's liabilities) in good k to be paid in Home. With this notation, the budget constraints faced by Home and Foreign households are:

$$\begin{aligned} p_A(c_A + a_A^*) + p_B(c_B + a_B^*) &= p_A Y_A + p_A^* a_A + p_B Y_B + p_B^* a_B + T \\ p_A^*(c_A^* + a_A) + p_B^*(c_B^* + a_B) &= p_A^* Y_A^* + p_A a_A^* + p_B^* Y_B + p_B a_B^* + T^*, \end{aligned} \tag{1}$$

where T and T^* are government transfers in Home and Foreign, respectively. Households' optimality implies that

$$g\left(\frac{c_B}{c_A}\right) = \frac{p_B}{p_A} \quad \text{and} \quad g\left(\frac{c_B^*}{c_A^*}\right) = \frac{p_B^*}{p_A^*}.$$

Finally, the governments' budget constraints are

$$\begin{aligned} T &= \tau p_A^* \max\{(c_A - Y_A), 0\} + \tau p_B^* \max\{(c_B - Y_B), 0\}, \\ T^* &= \tau^* p_A \max\{(c_A^* - Y_A^*), 0\} + \tau^* p_B \max\{(c_B^* - Y_B^*), 0\}. \end{aligned} \tag{2}$$

The definition of equilibrium is standard:

Definition 1. *Given a pair of tariffs, (τ, τ^*), we define an* equilibrium *as a vector $\{c_A, c_B, c_A^*, c_B^*, p_A, p_B, p_A^*, p_B^*\}$ such that: (i) households optimize subject to their budget constraints; (ii) prices satisfy goods arbitrage; (iii) the government budget constraints hold, and (iv) the aggregate resource constraints are satisfied: $c_A + c_A^* = \overline{Y}_A$ and $c_B + c_B^* = \overline{Y}_B$.*

3 Free Trade Equilibrium

We first discuss the simple benchmark of free trade (that is, when $\tau = \tau^* = 0$). Prices are equalized in this case. Let $\rho^{FT} \equiv \frac{p_B}{p_A} = \frac{p_B^*}{p_A^*}$ under free trade. Household optimality implies:

$$\rho^{FT} = g(c_B^{FT}/c_A^{FT}) = g(c_B^{*FT}/c_A^{*FT}) = g(\overline{Y}_B/\overline{Y}_A), \tag{3}$$

where the last equality follows from the resource constraint and the fact that $c_B/c_A = c_B^*/c_A^*$. Under free trade, the budget constraint of Home becomes

$$c_A^{FT} + \rho^{FT} c_B^{FT} = Y_A + a_A - a_A^* + \rho^{FT}(Y_B + a_B - a_B^*),$$

which together with (3) pins down the unique free trade equilibrium allocation. Note that price equalization implies that asset payouts are equalized across countries; that is, only net positions $a_k - a_k^*, k \in \{A, B\}$ matter.

It will be useful to characterize the equilibrium using an Edgeworth Box. This is depicted in Figure 1. The dimensions of the box are $\overline{Y}_A \times \overline{Y}_B$. Home's consumption of A and B is depicted from the southwest corner, with A on the horizontal axis and B on the vertical axis. Foreign consumption is the mirror image, with its origin in the northeast corner.

Given identical, homothetic preferences, the indifference curves are tangent along the diagonal. The diagonal thus outlines the efficient contract curve. Homotheticity also implies that ρ^{FT} is independent of initial endowments, and is equal to $\rho^{FT} \equiv g\left(\frac{\overline{Y}_B}{\overline{Y}_A}\right)$.

The endowment point is depicted by the point Y. The point $F \equiv (Y_A + a_A - a_A^*, Y_B + a_B - a_B^*)$ depicts the endowment adjusted by initial asset positions. As depicted, we have $a_A < a_A^*$ and $a_B > a_B^*$, which means that Home is short good A and long good B. This places F to the northwest of Y. The line through point F depicts (c_A, c_B) such that Home's budget set is satisfied at relative prices ρ^{FT}. The free-trade equilibrium is anchored where this budget line intersects the diagonal, E^{FT}. As depicted, E^{FT} is to the left and above Y, indicating that Home imports good B and exports good A. This is our benchmark scenario:[6]

Assumption 2 (Two-way Trades). *Under free trade, $c_A^{FT} < Y_A$ and $c_B^{FT} > Y_B$.*

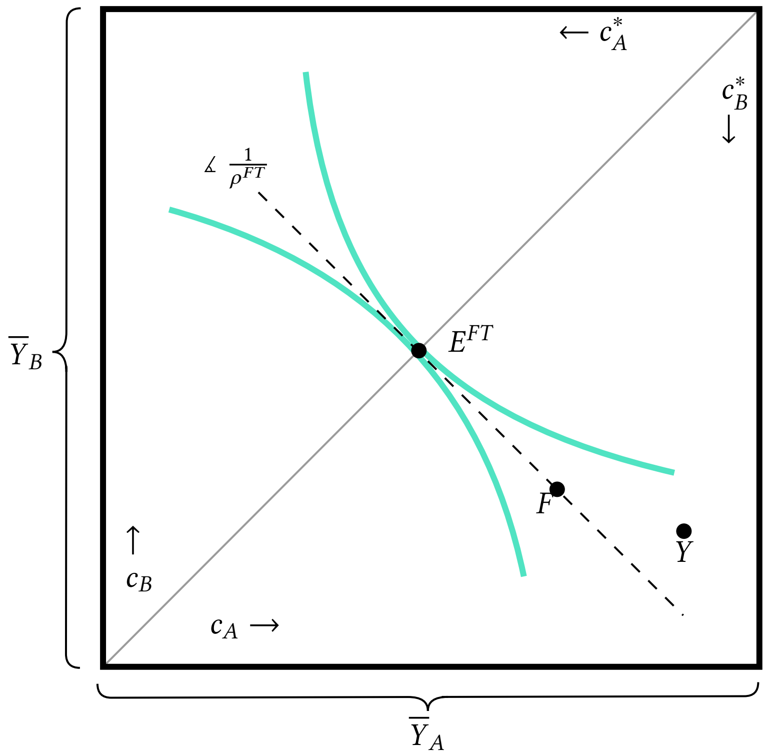
Whether Home is a net debtor or creditor depends on the relative position of its endowment point (Y) and its financial position (F) given the world price ρ^{FT}. Note that in the figure, given that the point Y is to the right of Home's budget line, Home is a net debtor: Home's expenditure is strictly lower than its total endowment income ($c_A^{FT} + \rho^{FT} c_B^{FT} < Y_A + \rho^{FT} Y_B$). In what follows we will proceed with this case (and the converse is obtained by switching labels):

Assumption 3 (Home is a Net Debtor). *Under free trade, $c_A^{FT} + \rho^{FT} c_B^{FT} < Y_A + \rho^{FT} Y_B$.*

Depicting equilibrium for the alternative cases in which F is to the northeast, southeast, and southwest of Y is straightforward. In each case, the location of Y with respect to where the line extending through F with slope $-1/\rho^{FT}$ intersects the diagonal pins down the pattern of trade.

[6] We are postponing the cases when country A (or B) is so indebted that under free trade it exports both goods. We take this up in Section 5.

Figure 1: Free Trade Equilibrium



4 Equilibria with Tariffs: A Simple Asset Structure

We now analyze the economy with tariffs. We begin characterizing tariff equilibria by considering environments in which each country has an external position (which could be negative) only in the other country's abundant good and zero claim on the other country's scarce good. Note that this is the relevant case under complete specialization, where $Y_B = Y_A^* = 0$.

Assumption 4. *Each country has zero external position in the other country's scarce good. That is,* $a_A = a_B^* = 0$.

This implies that any claims in good A are settled in Home and any claims in good B are settled in Foreign. This assumption is not necessary for what follows (and we review results for the more general case in Section 6), but it allows for the characterization of equilibrium to closely parallel the approach used for free trade. In particular, for equilibria in which Home does not import A and Foreign does not import B, tariffs affect the relative value of assets only through the terms of trade. To see this, suppose that Home exports A and imports B. In this case, goods arbitrage requires $p_A^* = (1 + \tau^*)p_A$ and $p_B = (1 + \tau)p_B^*$. Substituting this into Home's budget constraint (1), and using (2) to eliminate T, we obtain after some rearranging:

$$c_A + a_A^* + \rho c_B + (1+\tau)\rho a_B^* = Y_A + \rho Y_B + (1+\tau^*)a_A + \rho a_B,$$

where $\rho \equiv p_B^*/p_A$ is the international terms-of-trade. In general, as we shall discuss later, tariffs affect the country's budget set both through the terms-of-trade and directly. Assumption 4 eliminates the latter effect in cases in which the direction of trade mirrors the free trade equilibrium.

We proceed with the characterization of equilibrium by considering different asset configurations that satisfy Assumption 4. Subsection 4.1 assumes that each country weakly owes claims on its abundant good and Subsection 4.2 assumes that the product of gross positions is strictly positive, whether each individual position is positive or negative. We postpone to Section 5 the case in which countries are debtors in their scarce good.

4.1 Liabilities in the Abundant Good

Our first case is when each country's liabilities (if any) are denominated in their abundant good:

Assumption 5. *Each country's liabilities (if any) are denominated in its abundant good:*

$$a_B \geq 0 \text{ and } a_A^* \geq 0.$$

The equilibrium has the following features:[7]

Lemma 1. *Suppose Assumptions 1 through 5 hold. Then any tariff equilibrium is characterized by $(c_A, c_B, c_A^*, c_B^*, \rho)$ such that:*

(i) Home (weakly) exports good A and (weakly) imports good B: $c_A \leq Y_A$ and $c_B \geq Y_B$;

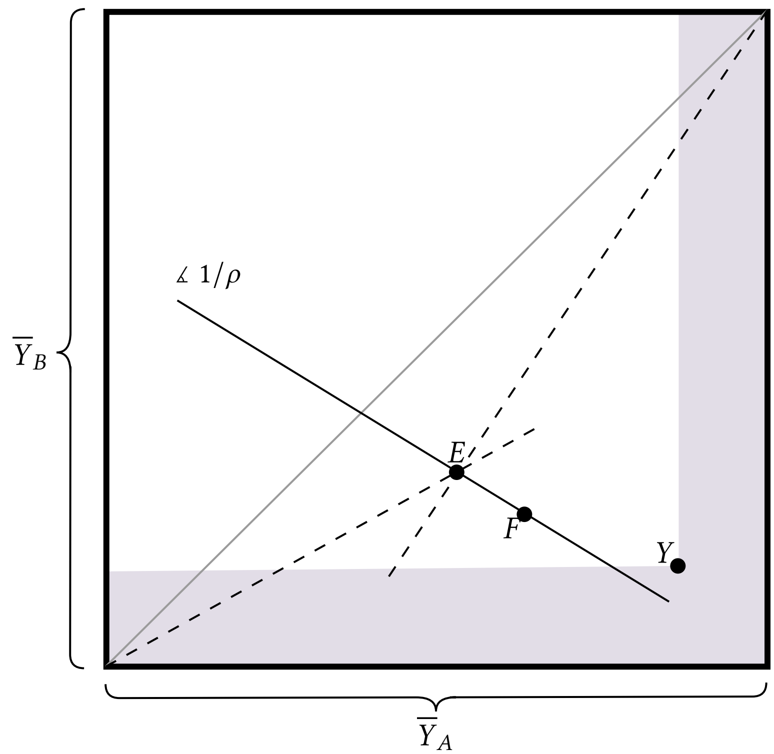
(ii) Households optimize and goods arbitrage holds:

$$g(c_B/c_A) = \alpha\rho \quad \text{and} \quad g(c_B^*/c_A^*) = \rho/\alpha^*, \tag{4}$$

where $\alpha, 1/\alpha^* \in [1/(1+\tau^*), 1+\tau]$ and $\alpha = 1+\tau$ if $c_B > Y_B$ and $\alpha^* = (1+\tau^*)$ if $c_A < Y_A$;

[7] We do not explore at this point whether the equilibrium is unique for a given set of tariff rates. We will see that if the tariffs are high enough the equilibrium is unique. For discussions of mutiplicity in an exchange economy see Kehoe (1998) and the more recent take by Toda and Walsh (2017).

Figure 2: Tariff War Equilibrium: Simple Assets Case



(iii) And budget sets and the resource constraints are satisfied:

$$\begin{aligned} c_A + \rho c_B &= Y_A + \rho Y_B - a_A^* + \rho a_B \\ c_A^* + \rho c_B^* &= Y_A^* + \rho Y_B^* + a_A^* - \rho a_B, \end{aligned} \tag{5}$$

and $c_A + c_A^* = \overline{Y}_A$, $c_B + c_B^* = \overline{Y}_B$.

The corresponding equilibrium prices are any (p_A, p_B, p_A^*, p_B^*) *such that*

$$\rho \equiv p_B^*/p_A, \quad p_B = \alpha p_B^*, \quad \text{and} \quad p_A^* = \alpha^* p_A.$$

To unpack this lemma, we shall refer to Figure 2, which displays the associated Edgeworth box. As before, $Y = (Y_A, Y_B)$ is the endowment point and $F = (Y_A - a_A^*, Y_B + a_B)$ is the endowment translated by the asset positions. Assumption 4 guarantees that F is not in the shaded region.

The result in part (i) of the lemma shows that the presence of tariffs *cannot reverse* the pattern of trade, which is intuitive given that tariffs are levied on whichever good(s) is imported. If trade were reversed then import tariffs would fall on the relative abundant good, reducing the incentive to import them compared to the free trade equilibrium.[8] This implies that the equilibrium cannot be in the interior of the shaded region of Figure 2. It also implies that we can focus on the ratio of $p_B^*/p_A = \rho$ as the terms-of-trade in any equilibrium.

If trade is strictly positive in both goods, we have $p_B/p_A = (1+\tau)\rho$ and $p_B^*/p_A^* = \rho/(1+\tau^*)$. This implies that the households' first-order conditions are:

$$\left(\frac{1}{1+\tau}\right) g\left(\frac{c_B}{c_A}\right) = \rho = (1+\tau^*) g\left(\frac{\overline{Y}_B - c_B}{\overline{Y}_A - c_A}\right).$$

In the Edgeworth box of Figure 2, the ray from Home's origin traces out $c_B/c_A = h\left((1+\tau)\rho\right)$, where recall that $h(x)$ is the inverse of g. Similarly, the ray from Foreign's origin contains the locus $c_B^*/c_A^* = h(\rho/(1+\tau^*))$. Consistent with tariffs distorting Home's consumption toward good A and Foreign toward B, we depict the rays below the free-trade diagonal.

The feasibility conditions in equation (5) are obtained by substituting T and T^* in the respective household budget sets using the governments' budget constraints. As discussed above, from (5) we can see that in any equilibrium the terms of trade, $\rho = p_B^*/p_A$, plays the dual role of being the international relative price of the goods, as well as being the relative price of the international claims. Equation (5) states that feasible consumption allocations for Home are those along the line with slope $-1/\rho$ passing through the point F. Using the resource constraint, Foreign's allocations viewed from the opposite origin lie along the same line.

The equilibrium ρ is such that these rays intersect along the budget line, depicted as point E. This allocation satisfies both country's household optimality as well as the resource constraints. As drawn, E is to the northwest of F. This means $c_B > Y_B + a_B$, and so Home's imports of B are greater than its asset. Similarly, $c_A < Y_A - a_A^*$, and Home's exports of A, $Y_A - c_A$, are greater in magnitude than its debt a_A. As $(1+\tau)(1+\tau^*)$ increases, the equilibrium is pushed further to the southeast, away from the free trade contract curve. If the equilibrium point lies below F, then $c_B < Y_B + a_B$ and Home's imports of B are less than its assets a_B. Similarly, if it lies to the right of F, then $c_A > Y_A - a_A^\star$, and Home's exports of A are less than its debt.

If trade is not strictly positive, the equilibrium will lie along the border of the shaded region. Such boundary points are supported in equilibrium by the appropriate α or α^* in (4). That is, if a good is not actively traded, the respective MRS and equilibrium prices can take values within the specified intervals spanned by the tariff wedges. The budget line continues to be defined by the international terms of trade, ρ.

[8] However, a note of caution is required as Section 5 shows that this is not necessarily true without Assumption 4.

4.2 Tariff Wars and Balanced Trade Equilibria

The preceding subsection discussed the possibility of zero trade in one of the goods. This begs the question of whether autarky is a possibility in the presence of gross asset positions. More generally, are there equilibria with tariffs in which trade is balanced, even though countries may have claims on each other that make one or the other a net debtor under free trade. In this subsection, we explore such equilibria.

For some of the results, we will appeal to the following modification of Assumption 5:

Assumption 6. *Gross positions are strictly positive and of the same sign:*

$$a_B \times a_A^* > 0.$$

Note that if $a_B > 0$, this is a special case of Assumption 5. However, the assumption also allows for the case where both $a_B < 0$ and $a_A^* < 0$, which is that both countries have liabilities in their scarce good.

Under this asset structure, for large enough tariffs Autarky is an equilibrium:

Proposition 1 (Autarky is an Equilibrium). *Suppose that Assumptions 1 - 4 and 6 hold. There exists $\underline{\tau}, \underline{\tau}^*$ such that for any $\tau \geq \underline{\tau}$ and $\tau^* \geq \underline{\tau}^*$, the autarkic allocation ($c_A = Y_A, c_B = Y_B, c_A^* = Y_A^*, c_B^* = Y_B^*$) is an equilibrium. In the autarkic equilibrium, the terms of trade for Home (the debtor country), $\hat{\rho} = p_B^*/p_A$, are:*

$$\hat{\rho} = \frac{a_A^*}{a_B}.$$

For autarky to be an equilibrium, domestic relative prices must be such that household optimality is satisfied at domestic endowments:

$$p_B/p_A = g(Y_B/Y_A) \equiv \rho_{Aut}, \quad \text{and} \quad p_B^*/p_A^* = g(Y_B^*/Y_A^*) \equiv \rho_{Aut}^*. \tag{6}$$

The budget constraint for Home requires that

$$p_B^* a_B = p_A a_A^*$$

That is, *if Autarky is an equilibrium, there is no trade in goods, and the net foreign asset positions of each country must equal zero* at equilibrium prices. It follows that the terms of trade, $\hat{\rho}$, that are necessary to balance the foreign asset positions must be

$$\hat{\rho} \equiv \frac{a_A^*}{a_B}.$$

This shows the need for Assumption 6, which ensures a strictly positive relative price.

Finally, equilibrium requires that zero trade is consistent with goods market arbitrage. From (6), the ratio of prices across countries satisfies:

$$\frac{p_A^*}{p_A} = \frac{p_B^*}{p_A}\frac{p_A^*}{p_B^*} = \frac{\hat{\rho}}{\rho_{Aut}^*}, \quad \text{and} \quad \frac{p_B^*}{p_B} = \frac{p_B^*}{p_A}\frac{p_A}{p_B} = \frac{\hat{\rho}}{\rho_{Aut}}.$$

Goods market arbitrage requires:

$$\frac{p_A}{p_A^*} \in \left[\frac{1}{1+\tau^*}, 1+\tau\right], \quad \text{and} \quad \frac{p_B}{p_B^*} \in \left[\frac{1}{1+\tau^*}, 1+\tau\right].$$

Given $\hat{\rho}, \rho_{Aut}^*, \rho_{Aut}$, this holds for (τ, τ^*) sufficiently high. (We are left to check Foreign's budget constraint, but this one holds by Walras' law.)

As noted above, Assumption 6 is not a special case of Assumption 5. However, when both assumptions are satisfied, we can leverage Lemma 1 to show that Autarky is the unique outcome for large enough tariffs:

Proposition 2 (Autarky is the Only Equilibrium). *Suppose that Assumptions 1 through 6 hold. Then there exists $\underline{\tau}, \underline{\tau}^*$ such that for any $\tau \geq \underline{\tau}$ and $\tau^* \geq \underline{\tau}^*$, the autarkic allocation $(c_A = Y_A, c_B = Y_B, c_A^* = Y_A^*, c_B^* = Y_B^*)$ is the* unique equilibrium. *In the autarkic equilibrium, the terms of trade for Home (the debtor country), $\hat{\rho} = p_B^*/p_A$, are worse than under free trade:*

$$\hat{\rho} = \frac{a_A^*}{a_B} > \rho^{FT}.$$

The result rests on the fact that as both tariffs rise, the relative value of gross claims forces both countries into autarky. To see the logic, note that as tariffs become very large, an equilibrium allocation cannot remain in the area where $c_A < Y_A$ and $c_B^* < Y_B^*$ (the interior of the non-shaded region in Figure 2). In particular, if there is trade in both goods, household optimality requires $c_B/c_A = h((1+\tau)\rho)$, which implies that as τ increases, $c_B/c_A \to 0$, which is inconsistent with $c_B > Y_B$. A similar argument implies that $c_B^*/c_A^* = h(\rho/(1+\tau^*)) \to \infty$ as τ^* becomes large.

Lemma 1 then implies that the only potential alternative to autarky is one-way trade in a single good: either $c_A < Y_A$ and $c_B = Y_B$, or $c_A = Y_A$ and $c_B > Y_B$. Under the first potential alternative scenario, Home's budget constraint becomes:

$$c_A - Y_A + \rho \underbrace{(c_B - Y_B)}_{=0} = \rho a_B - a_A^* = (1+\tau^*) g\left(\frac{c_B^*}{c_A^*}\right) a_B - a_A^*,$$

where $\rho = p_B^*/p_A$ and the last equality follows from the premise that Foreign is importing good A and therefore its marginal rate of substitution equals $p_B^*/p_A^* = \rho/(1+\tau^*)$. The premise that $c_A^* > Y_A^*$ and $c_B^* = Y_B^*$ also implies

$$\frac{\rho}{1+\tau^*} = g\left(\frac{c_B^*}{c_A^*}\right) > g\left(\frac{Y_B^*}{Y_A^*}\right) \equiv \rho_{Aut}^*.$$

Combining these two expressions and realling that $a_B > 0$ by Assumptions 5 and 6, we have

$$c_A - Y_A > (1+\tau^*)\rho_{Aut}^* a_B - a_A^*.$$

As τ^* becomes large, $c_A - Y_A$ becomes positive, violating the premise that Home exports good A. The reason is that a higher tariff in Foreign makes Home relatively wealthier, and appreciates its claim on Foreign goods. A similar argument in reverse shows that $c_A = Y_A$ with $c_B > Y_B$ cannot happen for sufficiently high τ. Thus the unique equilibrium outcome when tariffs are sufficiently high is autarky.

It is also straightforward to see that if Home is a net debtor under free trade, its terms of trade needs to depreciate in a trade war that leads to autarky. To see this, Home's debtor status implies

$$\rho^{FT} a_B - a_A^* < 0.$$

Using $a_B > 0$, this implies $\rho^{FT} < a_A^*/a_B \equiv \hat{\rho}$. The premise that $a_B > 0$ and $a_A^* > 0$ implies that Home is long Foreign's export (under free trade) and short its own. Hence, an improvement in the relative international price of Foreign's export rebalances financial claims in Home's favor. If $a_B < 0$ and $a_A^* < 0$ (which is also consistent with Assumption 6), then the terms of trade would shift in the favor of Home's export.

The shift in the terms of trade is necessary for balanced trade to be consistent with initial asset positions (under Assumption 5). Interestingly, this can be used to support balanced-trade equilibria that are not autarkic, which we explore next.

We first state a result and then provide a diagrammatic derivation. Let us define

$$\underline{\theta} \equiv \frac{\overline{Y}_B - h(\hat{\rho})\left(Y_A - \hat{\rho} Y_B^*\right)}{h(\hat{\rho})\hat{\rho}\overline{Y}_A + Y_A^* - \hat{\rho} Y_B}.$$

We have the following result:

Proposition 3 (Balanced Trade Equilibria)**.** *Suppose that Assumptions 1 through 6 hold. Suppose also that $g(Y_B^*/Y_A^*) < \hat{\rho} < g(Y_B/Y_A)$. Let*

$$T^* \equiv [\mathbb{I}_{\{\underline{\theta}>0\}}(\hat{\rho}/g(\underline{\theta}) - 1), \hat{\rho}/g(Y_B^*/Y_A^*) - 1).$$

Then

(i) T^* is non-empty.

(ii) For any $\tau^* \in T^*$ there exists a $\tau \geq 0$ such that there exists a tariff equilibrium with active trade such that trade is balanced:

$$Y_A - c_A = -\hat{\rho}(Y_B - c_B),$$

and Home exports good A.

We already know that if tariffs at Home and Foreign are high enough, then autarky emerges as the sole equilibrium. In this case, trade must necessarily be balanced since no trade occurs. The proposition states that balanced trade is attainable away from autarky if Foreign tariffs are in the specified non-empty set.

Balanced trade occurs when $Y_A - c_A + \rho(Y_B - c_B) = 0$. From the budget constraint, this requires that net foreign assets positions are zero:

$$\rho a_B = a_A^*.$$

Therefore, the terms-of-trade must be as in the autarky case, $\rho = \hat{\rho} \equiv a_A^*/a_B$.

The construction of the set of tariffs consistent with balanced and nonzero trade proceeds as in the characterization of the tariff-war equilibrium of Figure 2, but in reverse. Specifically, we know *a priori* the required terms-of-trade and seek the associated tariffs. Balanced trade is thus associated with the budget line through F with slope $-1/\hat{\rho}$, as shown in Figure 3. Recall that $F = (Y_A - a_A^*, Y_B + a_B)$; the fact that the slope of the budget line is $-a_B/a_A^*$ therefore implies it contains the endowment point $Y = (Y_A, Y_B)$, as well. That is, the value of the net foreign asset positions are zero along this line.

The requirement that $g(Y_B^*/Y_A^*) < \hat{\rho} < g(Y_B/Y_A)$ guarantees that in the absence of tariffs, when facing a hypothetical world price of $\hat{\rho}$, Home will be an exporter of good A, while Foreign will be an exporter of good B. A tariff equilibrium cannot reverse the pattern of trade, so the equilibrium must remain to the north-west of the endowment point.

For a given foreign tariff rate τ^*, we obtain the slope of the ray from Foreign's origin:

$$\frac{c_B^*}{c_A^*} = h\left(\hat{\rho}/(1+\tau^*)\right).$$

Where this intersects the budget line is the balanced-trade equilibrium candidate associated with τ^*. Given the c_B/c_A associated with this intersection, we can recover the necessary Home tariff from Home's optimality condition:

$$g\left(\frac{c_B}{c_A}\right) = (1+\tau)\hat{\rho}.$$

The condition in Proposition that requires $\tau^* \geq \mathbb{1}_{\{\underline{\theta}>0\}}(\hat{\rho}/g(\underline{\theta}) - 1)$ ensures that this allocation is consistent with a non-negative Home tariff.

Proceeding in this way, for each candidate τ^* large enough, we obtain a unique τ that balances trade. The proof of Proposition 3 derives this formally. Note that as tariffs increase, the equilibrium allocation approaches the autarkic outcome along the budget line. The condition that $g(Y_B^*/Y_A^*) < \hat{\rho}$ guarantees that indeed the autarky limit is reached when $\tau^* = \hat{\rho}/g(Y_B^*/Y_A^*) - 1$.

Figure 3 shows a tariff war that achieves balanced trade. The dashed line is the budget line at free-trade prices, ρ^{FT}. As we did in Figure 2, we depict the case of Home being a net debtor at free-trade prices, and hence the free-trade budget line is to the left of the endowment point. The solid line represents the new budget line at prices $\hat{\rho} > \rho^{FT}$. Let point E represents the point on this new budget line associated with a zero tariff at Home (which will contain the ray from Home's origin with slope $h(\hat{\rho})$). At this point, an equilibrium with balanced trade is obtained when Foreign imposes the tariff that equates its marginal rate of substitution at point E's allocation to $\hat{\rho}/(1+\tau^*)$.

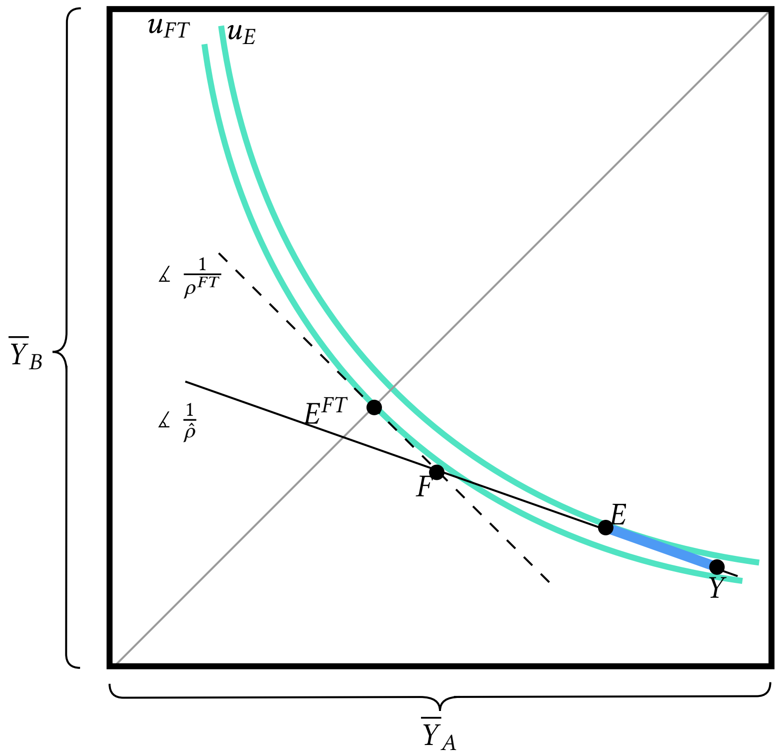
As we increase Home and Foreign's tariff, the associated rays from the respective origins rotate away from the diagonal, tracing out the set of balanced trade equilibria (depicted by the segment connecting E to Y). As both tariffs increase, we eventually converge to the autarkic allocation.

4.3 Welfare Consequences

Figure 3 can also speak to the welfare consequences of a trade war. Recall that free-trade equilibrium occurs where the original budget line with slope $1/\rho^{FT}$ intersects the diagonal, shown by point E^{FT} in the figure. Home and Foreign's indifference curves are tangent at this point.

There are three effects of the trade war on each country's welfare. Recall that E is associated

Figure 3: Tariff War Equilibrium With Balanced Trade



Note: Home is a net debtor, and its terms-of-trade worsen.

with zero Home tariff, and hence Home's indifference curve is tangent to the trade-war budget line at that point. As we move from E to Y, Home's welfare decreases. Similarly, at E, the steepness of Foreign's indifference curve is $(1 + \tau^*)/\hat{\rho} \geq 1/\hat{\rho}$.[9] Hence, as we move along the budget line with slope $-1/\hat{\rho}$ from E to Y, Foreign's welfare falls. This reflects that the terms of trade are given by $\hat{\rho}$, and the distortion in consumption generated by the tariffs is increasing as we move from E towards Y. That is, the movement from E to Y involves increases in τ and τ^* without any further change in the international terms of trade.

As Foreign's indifference curve is tangent to the free trade budget line at E^{FT}, and $\rho^{FT} < \hat{\rho}$ (for $a_B > 0$), then any point to the right of F (which includes Autarky) represents a lower welfare for Foreign. As Foreign is a net creditor under free trade, the loss reflects the decline in the value of its net assets (on top of the loss from the distortion to trade), and the associated shrinkage in its budget set. There is a potential positive for Foreign, which is the improvement in its terms of trade. For the points to the right of F, this is dominated by the other two factors. However, it may be the case that point E lies to the left of point F, opening the door to a potential welfare gain relative to free trade. Such gains are supported by the improved terms of trade dominating the asset and trade-distortion effects.

[9]Specifically, $-dc_B^*/dc_A^* = -1/g(c_B^*/c_A^*) = (1 + \tau^*)/\hat{\rho} \geq 1/\hat{\rho}$.

Home is a net debtor under free trade, and it is more intuitive that a trade-war can benefit it. Figure 3 is drawn in such way that even autarky is an improvement for Home when compared to the free trade outcome: as depicted, the free-trade indifference curve for Home u^{FT} lies below the autarky allocation Y. However, such an outcome is not a necessity: it can well be the case that all balanced trade outcomes generate a welfare loss in Home.

If Home is short its abundant good ($a_B < 0$) and Assumption 6 (but not Assumption 5) is satisfied, then the terms of trade move in its favor (see Appendix A). The fact that Home finds its debt position zeroed out and its terms of trade improved in a trade war suggests that Home should construct such a portfolio ex ante. However, in Section 5, we show that such positions make it vulnerable to an immiserizing debt crisis.

While we do not consider the possibility of default, one could extend the analysis to limited commitment with Autarky as the outside option. In that scenario, the respective indifference curves through point Y denote the bounds on welfare. The portion of the line segement $E - Y$ that lies between the two curves contain the set of tariff war equilibria that are consistent with no default.

One perspective on these welfare consequences is to evaluate the real debt burden faced by a country. Homotheticity of preferences imply that utility is a monotonic transformation of some composite commodity $C(x_A, x_B)$, where C is a constant returns to scale aggregator. This aggregator is common across country's due to symmetric preferences. Let $\mathcal{P}$ be the associated ideal price index, which inherits the constant-returns-to-scale property from C. In particular, Home and Foreign price indices are given by:

$$
\begin{aligned}
P &\equiv \mathcal{P}(p_A, p_B) = p_A \mathcal{P}(1, p_B/p_A) \\
P^* &\equiv \mathcal{P}(p_A^*, p_B^*) = p_A^* \mathcal{P}(1, p_B^*/p_A^*).
\end{aligned}
$$

With this notation, we can revisit the welfare consequences of the trade war in Figure 3, recalling that $\rho = \hat{\rho} = a_A^*/a_B$. In particular, at point E, the real value of Home's claim on Foreign is

$$
\frac{p_B^* a_B}{P} = \frac{p_B^* a_B}{p_A \mathcal{P}(1, \hat{\rho})} = \frac{\hat{\rho} a_B}{\mathcal{P}(1, \hat{\rho})} = \frac{a_B}{\mathcal{P}(1/\hat{\rho}, 1)} > \frac{a_B}{\mathcal{P}(1/\rho^{FT}, 1)},
$$

where recall that at E we have $\tau = 0$ and $\hat{\rho} > \rho^{FT}$. Hence, the trade war increase the real value of Home's claim on Foreign relative to free trade. Similarly, the real value of its liability falls:

$$
\frac{p_A a_A^*}{P} = \frac{a_A^*}{\mathcal{P}(1, \hat{\rho})} < \frac{a_A^*}{\mathcal{P}(1, \rho^{FT})}.
$$

Thus, for the debtor country, the trade war raises the real value of its assets and depresses the real value of its liabilities.

This discussion highlights the competing effects operating in the model: the welfare consequences of the valuation effects, the terms of trade movements in relation to free trade, and the adverse effects of tariff distortions as a trade war intensifies.

4.4 Tariff Wars: No Gross Positions

Propositions 1, 2, and 3 study the case where $a_B \times a_A^* > 0$ and show how in an equilibrium with high tariffs, the positions rebalance to zero due to changes in international prices. However, if this condition is not satisfied, then trade cannot be balanced in a trade war as there is no strictly positive terms-of-trade that can rebalance the net foreign assets. Then what happens in this case when tariffs are high enough?

Suppose $a_B = 0$ and $a_A^* > 0$. This is a natural case where Home only has an initial liability in units of its abundant good. This maps into a common case in the international macro literature in which the net asset position is pinned down by a gross position in a single asset.

Note that the conditions in Lemma 1 applies, and we know that the pattern of trade cannot be reversed in any tariff equilibrium. So, let us conjecture then that when tariffs are high enough, the equilibrium is one where Home exports a_A^* units of its abundant good to Foreign, and imports nothing. In that case, we know that

$$c_A = Y_A - a_A^*, \quad c_B = Y_B, \quad c_A^* = Y_A^* + a_A^*, \quad \text{and} \quad c_B^* = Y_B^*.$$

This consumption allocation satisfies part (i) and part (iii) of Lemma 1. We are just left to check part (ii). For this, the following must hold:

$$g(c_B^*/c_A^*) = \rho/(1+\tau^*) \quad \text{and} \quad g(c_B/c_A)/\rho = \alpha \in [1/(1+\tau^*), 1+\tau].$$

The first term above determines ρ, and to satisfy the second, it suffices that

$$\frac{g(c_B/c_A)}{(1+\tau^*)g(c_B^*/c_A^*)} \in \left[\frac{1}{1+\tau^*}, 1+\tau\right].$$

If $g(c_B/c_A) > g(c_B^*/c_A^*)$, then good arbitrage for good B will be satisfied for sufficiently high τ. We can obtain a simple condition for $g(c_B/c_A) > g(c_B^*/c_A^*)$. This occurs if:

$$\frac{Y_B}{Y_A - a_A^*} < \frac{Y_B^*}{Y_A^* + a_A^*}. \tag{7}$$

It is possible to show that this inequality follows from Assumptions 1 and 2.[10] Hence, the conjecture we have proposed satisfies all of the conditions in Lemma 1, and it is an equilibrium. We summarize this in the following Lemma:

Lemma 2. *Suppose Assumptions 1, 2, 3, and 4 hold. Suppose that $a_B = 0$ and $a_A^* > 0$. Then the allocation ($c_A = Y_A - a_A^*, c_B = Y_B, c_A^* = Y_A^* + a_A^*, c_B^* = Y_B^*$) constitute an equilibrium for a home tariff τ high enough. In such an equilibrium, the terms-of-trade, ρ, is given by:*

$$\rho = (1 + \tau^*) g(c_B^*/c_A^*).$$

A curious reader may ask how Lemma 2 and Proposition 2 can both hold, in particular, if we start from an initial $a_B > 0$, and $a_A* > 0$ and then take the limit as $a_B \to 0$. For any $a_B > 0$, Proposition 2 tells us that autarky is the unique equilibrium for high enough tariffs. But Lemma 2 tells us that when $a_B = 0$, the equilibrium is no longer autarky, but rather one where $c_A < Y_a$. The resolution is by noticing that as a_B goes to zero, the terms-of-trade that sustain autarky in Proposition 2, $\hat{\rho}$, goes to infinity, and thus, the required tariff rates must also go to infinity.

5 Exchange Rates and the (Self-Fulfilling) Debt Dilemma

In practice, countries are often indebted in a foreign currency. How does that map into our framework, and what are the implications for a trade war? We address these questions in this section. In particular, we establish a surprising indeterminacy due to the presence of tariffs that can lead to large shifts in relative welfare based on a sunspot.

Recall that the preceding analysis rested on Assumption 5 that $a_A^* \geq 0$ and $a_B \geq 0$; that is, each country's liabilities are denominated in its abundant good. What happens if $a_B < 0$ or $a_A^* < 0$? That is, Home must repay a liability denominated in its scarce good (to be delivered in Foreign), or vice versa for a Foreign liability and its scarce good. We begin with the first case:

Proposition 4 (a)**.** ***Home's Debt Dilemma:*** *Suppose Assumptions 1, 2, and 4 hold. If $a_B < 0$*

[10] Basically, the assumption that good A is abundant at Home and that in a free trade equilibrium, Home consumes a level of good B higher than its endowment, guarantees that starting at an endowment point $(Y_A - a_A^*, Y_B)$, Home most import the Foreign abundant good under free trade. Thus Home remains "abundant" in good A, which is condition (7).

and τ^* satisfies

$$0 < \frac{(c_A^{FT} + \rho^{FT} c_B^{FT}) - (1/h(\rho^{FT}) + \rho^{FT}) Y_B}{-\rho^{FT} a_B} < \tau^* < \frac{c_A^{FT} + \rho^{FT} c_B^{FT}}{-\rho^{FT} a_B}$$

Then there exists a tariff equilibrium where $c_A < Y_A$; $c_B < Y_B$; $g(c_B/c_A) = \rho^{FT}$; and

$$c_A - c_A^{FT} + \rho^{FT}(c_B - c_B^{FT}) = \tau^* \rho^{FT} a_B < 0.$$

Note that in this equilibrium, Home exports *both* goods. We can construct such an equilibrium via guess and verify. Goods arbitrage when $c_A < Y_A$ and $c_B < Y_B$ implies that $p_A^* = (1+\tau^*)p_A$ and $p_B^* = (1+\tau^*)p_B$. Hence, relative prices are the same across countries and Foreign and Home households' equate their marginal rates of substitution. That is, their indifference curves are tangent and the conjectured equilibrium allocation lies along the diagonal. We depict our candidate equilibrium as point E_1 in Figure 4.

Home's budget constraint is:

$$\begin{aligned} p_A c_A + p_B c_B &= p_A(Y_A - a_A^*) + p_B(Y_B + (1+\tau^*)a_B) \\ \Rightarrow c_A + \rho^{FT} c_B &= Y_A - a_A^* + \rho^{FT}(Y_B + (1+\tau^*)a_B). \end{aligned}$$

This is the free-trade budget line translated down by $\tau^* a_B < 0$. Define $F_1 \equiv (Y_A - a_A^*, Y_B + (1+\tau^*)a_B)$ as the translated financial position of Home in our conjectured equilibrium. Point E_1 in the figure is the intersection of a line with slope $-1/\rho^{FT}$ from F_1 to the diagonal.

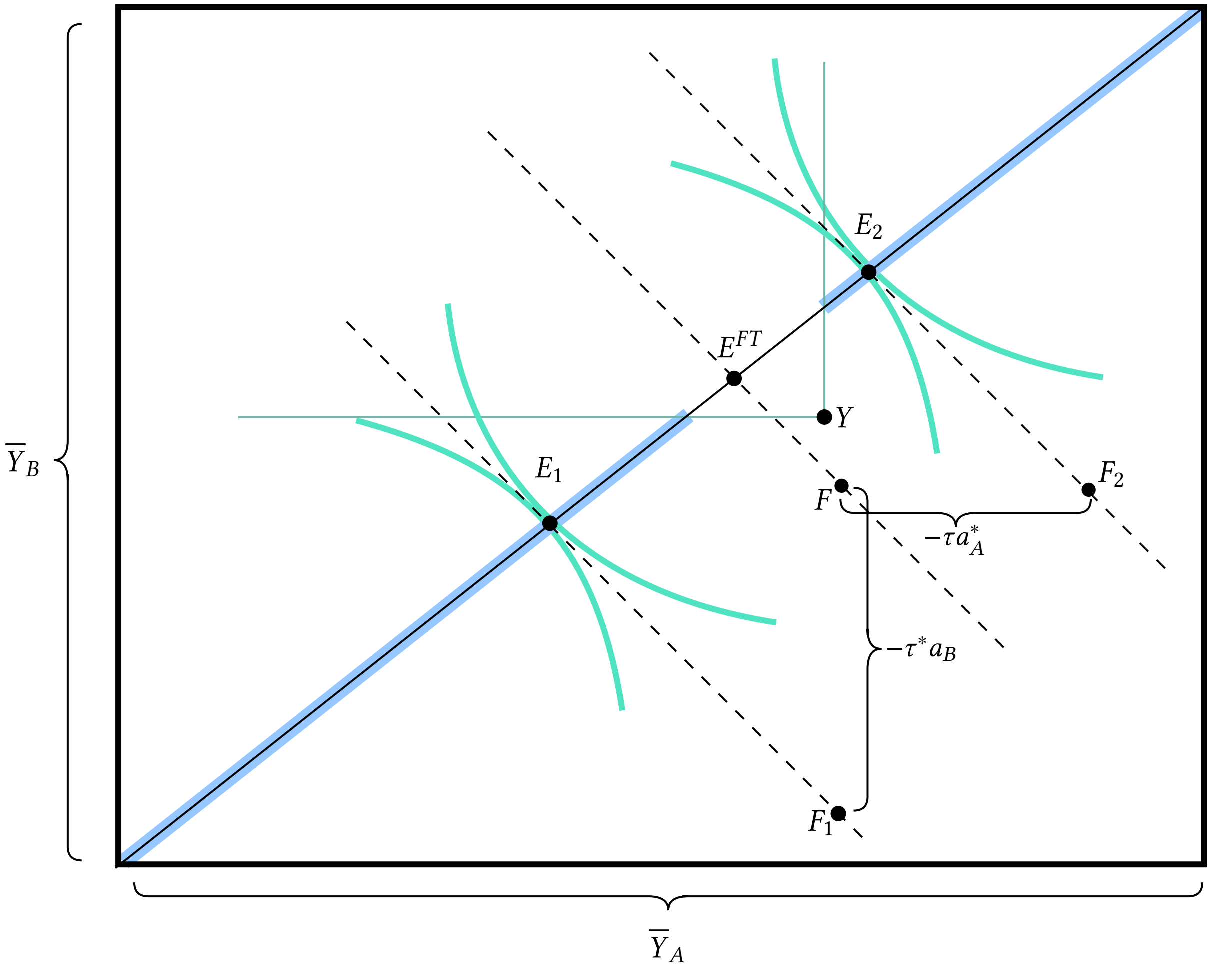
The last thing to check is that this intersection is indeed in the region such that $0 < c_A < Y_A$ and $0 < c_B < Y_B$. As we have moved along the diagonal from free trade to the left, we automatically have $c_A < c_A^{FT} < Y_A$, where the latter inequality is Assumption 2. The lower bound on τ^* in the proposition ensure that F_1 is far enough to the left of E^{FT} such that $c_B < Y_B$. The upper bound on τ^* ensures that it is still feasible for Home to pay the additional $\tau^* \rho^{FT} a_B$ relative to free trade; that is, that consumption is not forced to be negative to satisfy the budget set.

As E_1 is on the diagonal and strictly below the free trade equilibrium, it is unambigously the case that Home suffers in welfare terms from the foreign tariff relative to free trade and Foreign gains. For an economic interpretation, define the real exchange rate by

$$Q \equiv \frac{P^*}{P},$$

where an increase in Q is a Foreign appreciation and a Home depreciation. Under free trade,

Figure 4: Tariff War: Multiplicity with Short Selling



Note: Multiplicity of equilibria when $a_B < 0$ and $a_A^* < 0$. E_1 and E_2 represent two tariff equilibria for the same tariff rates (τ, τ^*). The shaded light blue segments in the contract curve represent allocations that can now be attained as an equilibrium outcome for some tariff combinations of (τ, τ^*).

$\mathcal{Q} = 1$. At E_1, we have

$$\mathcal{Q} = \frac{(1+\tau^*)\mathcal{P}(1, \rho^{FT})}{\mathcal{P}(1, \rho^{FT})} = 1 + \tau^* > 1.$$

As τ^* increases, Foreign's real exchange rate appreciates. As $a_B < 0$, as τ^* increases the burden of delivering good B to Foreign (in Foreign) increases for Home. It is "as if" Home must pay a liability in an appreciated foreign currency due to the foreign tariff. This immiserization forces Home to export both goods in order to service its debt.

A symmetric result holds for Foreign:

Proposition 4 (b). ***Foreign's Debt Dilemma:*** *Suppose assumptions 1, 2, and 4 hold. If $a_A^* < 0$ and τ satisfies*

$$0 < \frac{(c_A^{*FT} + \rho^{FT} c_B^{*FT}) - (1 + h(\rho^{FT})\rho^{FT}) Y_A^*}{-a_A^*} < \tau < \frac{c_A^{*FT} + \rho^{FT} c_B^{*FT}}{-a_A^*}$$

Then there exists a tariff equilibrium where $c_A^* < Y_A^*$, $c_B^* < Y_B^*$, $g(c_B^*/c_A^*) = \rho^{FT}$ and

$$c_A^* - c_A^{*FT} + \rho^{FT}(c_B^* - c_B^{*FT}) = \tau a_A^*.$$

This scenario is depicted by point E_2 in Figure 4. F_2 is the position $(Y_A - (1+\tau)a_A^*, Y_B + a_B)$, which is the free trade position translated to the right by τa_A^*. As Home's tariff increases, Foreign's liability in good A (delivered in Home) becomes more of a burden, and it must turn to exporting both goods. This pushes the equilibrium toward Foreign's origin, reducing Foreign's welfare and increasing Home's.

Importantly, note that τ plays no role in the condition for E_1 and τ^* plays no role in the condition for E_2. This is because both equilibria feature only one country with imports, and the other country's tariff is not relevant. This implies that conditions for both equilibria can be simultaneously satisfied. That is, either scenario can be an equilibrium *for a fixed set of tariffs τ and τ^**:[11]

Corollary 1. *Suppose Assumptions (1), (2), and (4) hold and $a_B < 0$ and $a_A^* < 0$. If τ and τ^* satisfy the inequalities in Proposition 4 Parts (a) and (b), then there exist (at least) two equilibria, one featuring Home exporting both goods and the other featuring Foreign exporting both goods.*

It is worth emphasizing that these equilibria are dramatically different. In one of them, Home is poor, and consumes strictly less than its endowment of both goods. In the other one, Foreign is poor, and consumes strictly less than its endowment of both goods. Which country has a "debt dilemma" is thus completely dependent on the coordination of beliefs (i.e., a sunspot). The multiplicity arises from the revaluation of net foreign asset positions due to the direction of trade. Under Assumption 4, these equilibria do not emerge when $a_B \geq 0$ and $a_A^* \geq 0$ and tariffs are high enough. The reason is that in such cases, higher tariffs in Foreign make the Home wealthier (rather than poorer) when trade flows from Home to Foreign, invalidating the equilibrium conjec-

[11]Note that if $a_B < 0$ and $a_A^* < 0$, then Assumption 6 is satisfied. If the upper thresholds in Proposition 4 (a) and (b) exceed $\underline{\tau}$ and $\underline{\tau}^*$ in Proposition 1, then autarky is a third equilibrium.

ture that Home is poor enough. However, when a country is indebted in a “foreign currency” (i.e., a liability in its scarce good to be settled in the other country) it is vulnerable to a self-fulfilling welfare loss if tariffs are appropriately high.

6 General Asset Structures

In the previous sections, we impose Assumption 4, which assumes that contracts denominated in a good are settled in the country for which that good is abundant. In the Appendix, we have versions of all the previous lemmas and propositions that apply under more general asset structures enumerated in Table 1, and show that the insights we have discussed carry through. We briefly discuss these results next.

In general, for the existence of an autarkic equilibrium, what is necessary is just that[12]

$$(a_A + \rho^*_{Aut} a_B) \times (a^*_A + \rho_{Aut} a^*_B) > 0.$$

That is, using the autarkic relative prices, the value of Home claims in foreign goods $a_A + \rho^*_{Aut} a_B$ must have the same sign as the value of Foreign claims in home goods $a^*_A + \rho_{Aut} a^*_B$.

The corresponding terms of trade that emerge in an autarkic equilibrium is such that the value of these claims exactly cancel out:

$$\hat{\rho} = \frac{p^*_B}{p_A} = \frac{a^*_A + \rho_{Aut} a^*_B}{a_A/\rho^*_{Aut} + a_B}.$$

For the uniqueness of the autarkic equilibrium when tariffs rates are large enough we need that [13]

$$\begin{aligned} a_A + \rho^{FT} a_B > 0, &\quad \text{and} \quad a^*_A + \rho^{FT} a^*_B > 0. \\ a_A + \rho^*_{Aut} a_B > 0, &\quad \text{and} \quad a^*_A + \rho_{Aut} a^*_B > 0. \end{aligned}$$

which holds immediately if all the asset positions on individual goods are strictly positive.

Note that in the free trade equilibrium, Home was a net debtor, and thus

$$(a_A + \rho^{FT} a_B) - (a^*_A + \rho^{FT} a^*_B) < 0 \Rightarrow \rho^{FT} < \frac{a^*_A + \rho^{FT} a^*_B}{a_A/\rho^{FT} + a_B}.$$

Whether the terms of trade worsen for the debtor country in a trade war (that is, $\hat{\rho} > \hat{\rho}^{FT}$) depends

[12]See Proposition 6 in the Appendix.
[13]See Proposition 7 in the Appendix.

on the composition of its foreign assets and liabilities and how far are the free trade relative prices to the domestic autarky prices. The important point is that the changes in the terms of trade must rebalance the net positions to zero.

7 Arbitrary Number of Goods, Time, and Uncertainty

We have done all of our analysis in an environment with only two goods. In this section we discuss how the results generalize to a world with an arbitrary number of goods.

Let Home and Foreign have endowments of N goods, denoted by vectors $\boldsymbol{y} = \{Y_1, \ldots, Y_N\}$ and $\boldsymbol{y}^* = \{Y_1^*, \ldots, Y_N^*\}$.[14] We assume that the endowments are all strictly positive $Y_i > 0$, $Y_i^* > 0$ for all $i \in \{1, \ldots, N\}$. The utility functions of the corresponding representative households are $u(\boldsymbol{c})$ and $u^*(\boldsymbol{c}^*)$, where $\boldsymbol{c} = \{c_1, \ldots, c_N\}$ and $\boldsymbol{c}^* = \{c_1^*, \ldots, c_N^*\}$ are the consumption vectors. Let $\boldsymbol{p} = \{p_1, \ldots, p_N\}$ and $\boldsymbol{p}^* = \{p_1^*, \ldots, p_N^*\}$ denote the domestic price vectors at Home and at Foreign, all quoted in a numeraire unit. We let $\boldsymbol{a} = \{a_1, \ldots, a_N\}$ denote Home ownership of assets in Foreign, denominated in each of the goods; and similarly we denote by $\boldsymbol{a}^* = \{a_1^*, \ldots, a_N^*\}$ Foreign ownership of assets denominated in goods at Home.

Home households maximize utility subject to their budget constraint:

$$\max_{\boldsymbol{c}} u(\boldsymbol{c}) \quad \text{subject to} \quad \boldsymbol{p} \cdot \boldsymbol{c} \leq \boldsymbol{p} \cdot \boldsymbol{y} + \boldsymbol{p}^* \cdot \boldsymbol{a} - \boldsymbol{p} \cdot \boldsymbol{a}^* + T.$$

where T is Home's tariff revenue, and the operator $\cdot$ denotes the dot product. Similarly, the Foreign households' problem is

$$\max_{\boldsymbol{c}^*} u^*(\boldsymbol{c}^*) \quad \text{subject to} \quad \boldsymbol{p}^* \cdot \boldsymbol{c}^* \leq \boldsymbol{p}^* \cdot \boldsymbol{y}^* + \boldsymbol{p} \cdot \boldsymbol{a}^* - \boldsymbol{p}^* \cdot \boldsymbol{a} + T^*.$$

where T^* is Foreign's tariff revenue.

We will maintain the assumption that tariffs are set uniformly across all imported goods, and let $\tau \geq 0$ and $\tau^* \geq 0$ denote the Home and Foreign tariffs as before. International good arbitrage requires that for all $i \in \{1, \ldots, N\}$:

$$\frac{p_i}{p_i^*} = \begin{cases} (1+\tau) & \text{for } c_i > Y_i \\ 1/(1+\tau^*) & \text{for } c_i < Y_i \end{cases}$$
$$\text{with } p_i/p_i^* \in [1/(1+\tau^*), (1+\tau)] \text{ if } c_i = Y_i.$$

[14] Following Dixit and Norman (1980), it should not be too hard to extend this analysis to include production, factors, and intermediate goods.

The governments' budget constraints are:

$$T = \tau \sum_{i=1}^{N} p_i^* \max\{c_i - y_i, 0\}, \qquad \text{and} \qquad T^* = \tau^* \sum_{i=1}^{N} p_i \max\{c_i^* - y_i^*, 0\}.$$

Finally, the resource constraints are:

$$\boldsymbol{c} + \boldsymbol{c}^* = \boldsymbol{y} + \boldsymbol{y}^*.$$

With all of the above, we can define a competitive equilibrium for a given set of tariffs rates, $\{\tau, \tau^*\}$, as standard: given prices, households optimizes; the government budget constraint holds, the resource constraint holds, and the prices are consistent with international goods arbitrage.[15]

Denote by $\boldsymbol{\rho}_{aut}$ and $\boldsymbol{\rho}^*_{aut}$ the vectors of relative prices with respect to good 1 consistent with autarky for both Home and Foreign. That is, $\boldsymbol{\rho}_{aut} = \{1, \rho_2, \ldots, \rho_N\}$ and $\boldsymbol{\rho}^*_{aut} = \{1, \rho_2^*, \ldots, \rho_N^*\}$ such that:

$$\rho_i = \frac{u_i(\boldsymbol{y})}{u_1(\boldsymbol{y})}, \qquad \text{and} \qquad \rho_i^* = \frac{u_i^*(\boldsymbol{y}^*)}{u_1(\boldsymbol{y}^*)} \qquad \text{for all } i \in \{2, \cdots, N\}.$$

where u_i and u_i^* denote the partial derivatives of u and u^* with respect to good i.

Then, we have the following result.

Proposition 5. *Suppose that $\boldsymbol{a}$ and $\boldsymbol{a}^*$ are such that*

$$(\boldsymbol{\rho}_{aut} \cdot \boldsymbol{a}^*) \times (\boldsymbol{\rho}^*_{aut} \cdot \boldsymbol{a}) > 0.$$

Then there exists $\underline{\tau}$ and $\underline{\tau}^*$ such that for all tariffs $\tau \geq \underline{\tau}$ and $\tau^* \geq \underline{\tau}^*$, the autarky allocation $\boldsymbol{c} = \boldsymbol{y}$ and $\boldsymbol{c}^* = \boldsymbol{y}^*$ is an equilibrium.

Proof. In the text. □

The proof of this proposition follows the same argument for Proposition 1. Given that the consumption allocations must equal the endowments, we know that equilibrium Home and Foreign

[15] The main difference of this setup with the classic analysis in Dixit and Norman (1980) is the presence of p^* in Home's budget constraint (and of p in Foreign's budget constraint), reflecting that Home (and Foreign) has ownership of claims *that are not located within its physical boundaries*. In the classic analysis, the levels of p and p^* do not matter for equilibrium determination, only relative prices across goods within the locations are necessary. This is Lerner symmetry: there is an equivalence in the choices of export and import taxes. In our environment, with p and p^* in the budget constraints, this no longer holds, and Lerner symmetry breaks. This important break of Lerner symmetry in the presence of cross-country asset positions was first noted and emphasized in Itskhoki and Mukhin (2025).

domestic prices must be:

$$\boldsymbol{p} = p_1 \times \boldsymbol{\rho}_{aut}$$
$$\boldsymbol{p}^* = p_1^* \times \boldsymbol{\rho}^*_{aut}.$$

for some p_1 and p_1^*. Note that given endowments, the ratio p_1^*/p_1 will be proportional to the real exchange rate.

In an autarkic equilibrium, it must be that the net foreign asset positions of each country zeros out; there is no trade; and no tariff revenue, $T = T^* = 0$. Hence the budget constraints of both Home and Foreign hold with equality for $\boldsymbol{c} = \boldsymbol{y}$ and $\boldsymbol{c}^* = \boldsymbol{y}^*$. For the net foreign asset positions to zero out we need that:

$$\boldsymbol{p} \cdot \boldsymbol{a}^* = \boldsymbol{p}^* \cdot \boldsymbol{a} \quad \Rightarrow \quad p_1 \boldsymbol{\rho}_{aut} a^* = p_1^* \boldsymbol{\rho}^*_{aut} \boldsymbol{a},$$

which will hold for a particular ratio:

$$\frac{p_1^*}{p_1} = \frac{\boldsymbol{\rho}_{aut} \cdot \boldsymbol{a}^*}{\boldsymbol{\rho}^*_{aut} \cdot \boldsymbol{a}} > 0,$$

where the inequality follows from the assumption in the proposition. In this way, asset positions pin down the real exchange rate in the absence of trade. Given that the optimality condition holds by the construction of $\boldsymbol{p}$ and $\boldsymbol{p}^*$, it follows the autarkic consumption solves the households' problem at Home and Foreign. Clearly the autarky allocation satisfy the resource constraints.

The only equilibrium condition left to check is international goods arbitrage. For that, we need to ensure that the relative prices for each good across countries are consistent with no trade:

$$\frac{p_i}{p_i^*} = \frac{p_1}{p_1^*} \frac{u_i(\boldsymbol{y})}{u_1(\boldsymbol{y})} \frac{u_1^*(\boldsymbol{y}^*)}{u_i^*(\boldsymbol{y}^*)} \in \left[\frac{1}{1+\tau^*}, 1+\tau\right] \text{ for all } i \in \{1, \ldots, N\}.$$

The following lower bounds on tariffs ensure this is true for all goods:

$$\tau \geq \underline{\tau} = \left(\frac{\boldsymbol{\rho}^*_{aut} \cdot \boldsymbol{a}}{\boldsymbol{\rho}_{aut} \cdot \boldsymbol{a}^*}\right) \max_{i \in \{1,\ldots,N\}} \left\{\frac{u_i(\boldsymbol{y})}{u_1(\boldsymbol{y})} \frac{u_1^*(\boldsymbol{y}^*)}{u_i^*(\boldsymbol{y}^*)}\right\},$$
$$\tau^* \geq \underline{\tau}^* = \left(\frac{\boldsymbol{\rho}_{aut} \cdot \boldsymbol{a}^*}{\boldsymbol{\rho}^*_{aut} \cdot \boldsymbol{a}}\right) \max_{i \in \{1,\ldots,N\}} \left\{\frac{u_i^*(\boldsymbol{y}^*)}{u_1^*(\boldsymbol{y}^*)} \frac{u_1(\boldsymbol{y})}{u_i(\boldsymbol{y})}\right\}.$$

If tariffs satisfy these bounds, then the autarky allocation is an equilibrium.

Just as Dixit and Norman (1980) highlight: once the analysis has been extended to an arbitrary number of goods, it is straightforward to also introduce intertemporal trade (and uncertainty) into the analysis. This just requires re-interpreting different goods as different times or states, and intra-temporal prices as inter-temporal ones, as in the classic Arrow-Debreu analysis.

The result of Proposition 5 then tells us that, in a dynamic environment, an increase in bilateral tariffs that is high enough makes autarky an equilibrium as long as the total value of the asset positions of each country have the same sign *when evaluated at the respective autarky prices.* In this equilibrium, the net foreign asset positions must rebalance to zero, and the domestic intertemporal prices of each country converge to their respective intertemporal prices in autarky. That is, the real exchange rate plays the role of re-balancing the asset positions, while the domestic interest rates align with their autarkic values. This result provides an interpretation of why the recent trade war affected the dollar exchange rate but had no major discernible impact on interest rates. In a severe enough trade war, the exchange rate and the interest rates are not connected but rather are balancing distinct equilibrium forces.

8 Conclusion

Using a simple two-country, two-good model with international asset positions, we have characterized the equilibria that arise when countries impose tariffs in a trade war. We have shown that when countries hold foreign assets denominated in the foreign abundant good, then for high enough tariffs, autarky is the unique equilibrium. In this case, the terms of trade for the ex-ante debtor country worsen in a trade war. When countries hold more general asset positions, then multiple equilibria can arise for the same tariff rates. In such cases, the direction of trade affects the valuation of foreign assets and liabilities, leading to different possible equilibrium outcomes. We have shown how the analysis generalizes to multiple goods, trade across time and uncertainty.

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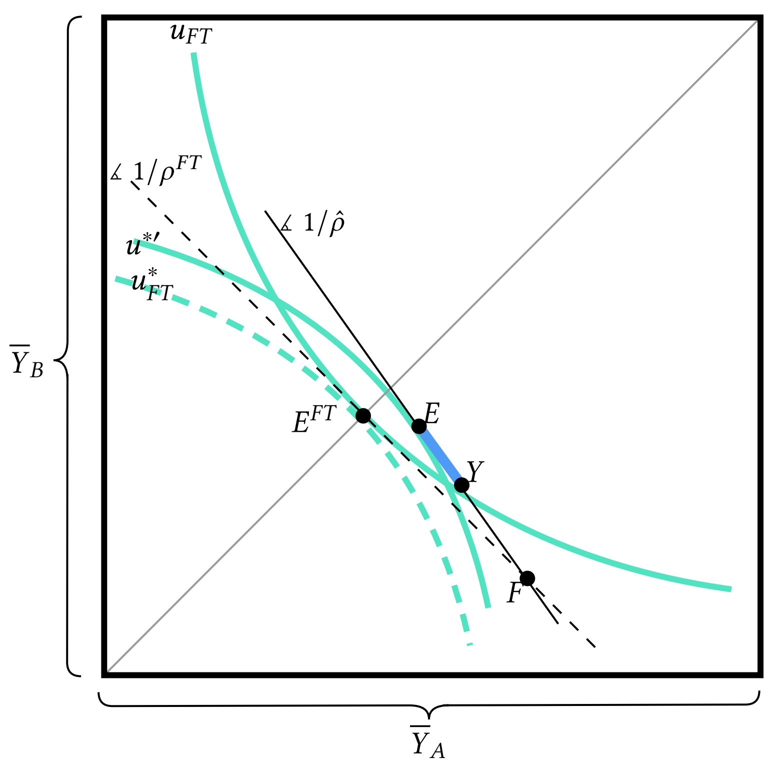
A The other case: Home is a net debtor, and its terms-of-trade improve

Figure 5 shows the case where Home starts as a net debtor, but its terms of trade improve once a trade-war with balanced trade happens. The point E is now determined by the foreign zero tariff rate, represented by the tangency line of Foreign's indifference curve ($u^{*\prime}$) with the budget line given by $\hat{\rho}$. The points from E to Y represent points that are attained in a tariff war where trade is balanced. As we move from E to Y, tariffs increase, eventually reaching the autarky equilibrium. Note that in this case, Foreign is always worse off. It started as a creditor, and the tariff war does not improve its terms-of-trade. Whether Home is better of or not depends on the location of the point E and the endowment Y. As drawn in the graph, Home benefits. Although we can construct this case, it is worth noting that it does not satisfy Assumption 5.

Figure 5: Tariff War Equilibrium With Balanced Trade



Note: Home is a net debtor, and its terms-of-trade improve.

B Proofs

We will proceed to state and prove the results for the general case, without imposing Assumption 4. The statements in the paper arise naturally under that additional restriction.

B.1 Proof of Proposition 1

Recall $\rho_{Aut} = g(Y_B/Y_A)$ and $\rho^*_{Aut} = g(Y^*_B/Y^*_A)$ are the autarkic relative prices at Home and Foreign respectively.

We will prove the following more general version of Proposition 1:

Proposition 6 (Autarky is an Equilibrium). *There exists $\underline{\tau}, \underline{\tau}^*$ such that for any $\tau \geq \underline{\tau}$ and $\tau^* \geq \underline{\tau}^*$, the autarkic allocation $(c_A = Y_A, c_B = Y_B, c^*_A = Y^*_A, c^*_B = Y^*_B)$ is an equilibrium if*

$$(a_A + \rho^*_{Aut} a_B) \times (a^*_A + \rho_{Aut} a^*_B) > 0.$$

In the autarkic equilibrium, the terms of trade for Home (the debtor country), $\hat{\rho} = p^*_B/p_A$, are:

$$\hat{\rho} = \frac{a^*_A + \rho_{Aut} a^*_B}{\frac{a_A}{\rho^*_{Aut}} + a_B}.$$

Proof. The proof follows the text. The budget constraint for Home under the autarkic allocation requires that:

$$p^*_A a_A + p^*_B a_B = p_A a^*_A + p_B a^*_B.$$

We also know from household optimality that

$$p_B/p_A = \rho_{Aut}, \quad \text{and} \quad p^*_B/p^*_A = \rho^*_{Aut}.$$

Plugging back into the budget constraint we get:

$$p^*_B \left(\frac{a_A}{\rho^*_{Aut}} + a_B \right) = p_A (a^*_A + \rho_{Aut} a^*_B).$$

And thus

$$\hat{\rho} = \frac{p^*_B}{p_A} = \frac{a^*_A + \rho_{Aut} a^*_B}{\frac{a_A}{\rho^*_{Aut}} + a_B}, > 0$$

where the inequality is guaranteed by the condition in the Proposition.

It follows then that the prices of the same good across countries are:

$$\frac{p^*_A}{p_A} = \frac{\hat{\rho}}{\rho^*_{Aut}}, \quad \text{and} \quad \frac{p^*_B}{p_B} = \frac{\hat{\rho}}{\rho_{Aut}}$$

The final step is the good arbitrage conditions. For the autarky allocation, we need that:

$$p_A/p_A^* \in [1/(1+\tau^*), 1+\tau],$$
$$p_B/p_B^* \in [1/(1+\tau^*), 1+\tau],$$

So, it suffices to set $\underline{\tau} = \max\{p_A/p_A^*, p_B/p_B^*, 1\} - 1$ and $\underline{\tau}^* = \max\{p_A^*/p_A, p_B^*/p_B, 1\} - 1$. And thus for any (τ, τ^*) with $\tau \geq \underline{\tau}$ and $\tau^* \geq \underline{\tau}^*$, good arbitrage holds. □

This proposition implies Proposition 1 under Assumption 4 (and also under the less important assumptions 1, 2, and 3).

B.2 Proof of Lemma 1

We will state and prove a more general version of Lemma 1. For that, we will use the following version of Assumptions 4 and 5:

Assumption 7. *Under free trade, each country has a non-negative claim in the foreign goods. That is,*

$$a_A + \rho^{FT} a_B \geq 0, \quad \textit{and} \quad a_A^* + \rho^{FT} a_B^* \geq 0.$$

Note that Assumptions 4 and 5 imply Assumption 7. We have this generalized version of Lemma 1:

Lemma 3. *Suppose Assumptions 1, 2, 3, and 7 hold. Then any tariff equilibrium is characterized by $(c_A, c_B, c_A^*, c_B^*, \rho)$ such that:*

(i) Home (weakly) exports good A and (weakly) imports good B: $c_A \leq Y_A$ and $c_B \geq Y_B$;

(ii) Households optimize and goods arbitrage holds:

$$g(c_B/c_A) = \alpha\rho \quad \textit{and} \quad g(c_B^*/c_A^*) = \rho/\alpha^*,$$

where $\alpha, 1/\alpha^* \in [1/(1+\tau^*), 1+\tau]$ and $\alpha = 1+\tau$ if $c_B > Y_B$ and $\alpha^* = (1+\tau^*)$ if $c_A < Y_A$;

(iii) And budget sets and the resource constraints are satisfied:

$$c_A + \rho c_B = Y_A + \rho Y_B - (a_A^* + \alpha\rho a_B^*) + \alpha^* a_A + \rho a_B$$
$$c_A^* + \rho c_B^* = Y_A + \rho Y_B + (a_A^* + \alpha\rho a_B^*) - \alpha^* a_A + \rho a_B,$$

and $c_A + c_A^* = \overline{Y}_A$, $c_B + c_B^* = \overline{Y}_B$.

The corresponding equilibrium prices are any (p_A, p_B, p_A^*, p_B^*) *such that*

$$\rho \equiv p_B^*/p_A, \quad p_B = \alpha p_B^*, \quad \textit{and} \quad p_A^* = \alpha^* p_A.$$

Proof. We start by ruling out some cases.

Case 1: $c_A > Y_A$ **and** $c_B \leq Y_B$**.** This case implies that

$$\frac{c_B}{c_A} < \frac{Y_B}{Y_A} < \frac{\overline{Y}_B}{\overline{Y}_A} < \frac{Y_B^*}{Y_A^*} < \frac{c_B^*}{c_A^*}.$$

This implies that

$$\frac{p_B}{p_A} = g(c_B/c_A) > g(c_B^*/c_A^*) = \frac{p_B^*}{p_A^*}.$$

Goods arbitrage implies that $p_A = (1+\tau)p_A^*$. Using this and the above, we have that

$$\frac{p_B}{p_B^*} > (1+\tau).$$

But this is a contradiction of the good B's arbitrage, which requires $p_B/p_B^* \in [1/(1+\tau^*), 1+\tau]$.

Case 2: $c_A > Y_A$ **and** $c_B > Y_B$**.** In this case, $p_A = (1+\tau)p_A^*$ and $p_B = (1+\tau)p_B^*$. The households optimality condition are then:

$$g(c_B/c_A) = g(c_B^*/c_A^*) = g(\overline{Y}_B/\overline{Y}_A) = \rho^{FT}.$$

Consider now the budget constraint for Foreign:

$$p_A^*(c_A^* + a_A) + p_B^*(c_B^* + a_B) = p_A^* Y_A^* + p_B^* Y_B^* + p_A a_A^* + p_B a_B^*$$

where we used that $T^* = 0$ given that $c_A > Y_A$ and $c_B > Y_B$. Dividing by p_B^*, we get

$$c_A^* + \rho^{FT} c_B^* = Y_A^* + \rho^{FT} Y_B^* + (a_A^* + \rho^{FT} a_B^*) - (a_A + \rho^{FT} a_B) + \tau(a_A^* + \rho^{FT} a_B^*)$$

But note that $Y_A^* + \rho^{FT} Y_B^* + (a_A^* + \rho^{FT} a_B^*) - (a_A + \rho^{FT} a_B)$ is the total wealth available for consumption under free trade for Foreign.

In this conjectured equilibrium, Foreign has weakly higher wealth as $\tau(a_A^* + \rho^{FT} a_B^*) \geq 0$ by Assumption 7 and that $\tau \geq 0$. Given that Foreign households face the same prices as under free trade but have more resources, they cannot choose a level of consumption c_A^*, c_B^* that is lower than the free trade allocation

$c_A^{*,FT}, c_B^{*,FT}$. It follows then that $c_A^* > c_A^{*,FT}$. But $c_A^{*,FT} > Y_A^*$, generating a contradiction.

The contradictions in cases 1 and 2 imply that $c_A \leq Y_A$. A symmetric argument, using that $\tau^* \geq 0$ and Assumption 7 implies that $c_B^* \leq Y_B^*$.

This shows the necessity of part (i). Letting $\alpha = p_B/p_B^*$ and $\alpha^* = p_A^*/p_A$, we obtain the necessity of part (ii) immediately from households optimality and good arbitrage.

Finally, for the necessity of part (iii), let's start from Home's budget constraint:

$$p_A(c_A - Y_A) + p_B(c_B - Y_B) + p_A a_A^* + p_B a_B^* = p_A^* a_A + p_B^* a_B + T$$

where T is

$$T = \tau p_B^* \max\{(c_B - Y_B), 0\}$$

and where we have used that $\max\{(c_A - Y_A), 0\} = 0$ from part (i). Dividing by p_A we obtain:

$$(c_A - Y_A) + \frac{p_B}{p_A}(c_B - Y_B) + a_A^* + \frac{p_B}{p_A} a_B^* = \frac{p_A^*}{p_A} a_A + \frac{p_B^*}{p_A} a_B + \tau \frac{p_B^*}{p_A} \max\{(c_B - Y_B), 0\}$$

Using the definition of ρ, α, and α^*, we can rewrite this as

$$(c_A - Y_A) + \frac{p_B}{p_B^*}\rho(c_B - Y_B) + a_A^* + \alpha\rho a_B^* = \alpha^* a_A + \rho a_B + \tau\rho \max\{(c_B - Y_B), 0\}$$

Note that if $c_B = Y_B$, this becomes:

$$(c_A - Y_A) + a_A^* + \alpha\rho a_B^* = \alpha^* a_A + \rho a_B$$

And if $c_B > Y_B$, we have that $\frac{p_B}{p_B^*} = 1 + \tau$, and the budget constraints becomes:

$$(c_A - Y_A) + \rho(c_B - Y_B) + a_A^* + \alpha\rho a_B^* = \alpha^* a_A + \rho a_B$$

Hence, a single budget constraint encompases both cases:

$$(c_A - Y_A) + \rho(c_B - Y_B) + a_A^* + \alpha\rho a_B^* = \alpha^* a_A + \rho a_B$$

A symmetric argument shows the budget constraint for Foreign.

We have shown that a tariff equilibrium must satisfy conditions (i), (ii) and (iii). The reverse is straightforward: we can construct an equilibrium if conditions (i), (ii), and (iii) are satisfied. □

B.3 Proof of Proposition 2

First, we will strengthen a bit Assumption 7:

Assumption 8. *Under free trade, each country has a non-negative claim in the foreign goods. That is,*

$$a_A + \rho^{FT} a_B > 0, \quad \text{and} \quad a_A^* + \rho^{FT} a_B^* > 0.$$

Note that if Assumption 4 holds, and $a_B \times a_A^* > 0$, then Assumption 8 holds. We now state and proof a more general version of Proposition 2:

Proposition 7 (Autarky is the Only Equilibrium)**.** *Suppose that Assumptions 1, 2, 3, and 8 hold. There exists $\underline{\tau}, \underline{\tau}^*$ such that for any $\tau \geq \underline{\tau}$ and $\tau^* \geq \underline{\tau}^*$, the autarkic allocation ($c_A = Y_A, c_B = Y_B, c_A^* = Y_A^*, c_B^* = Y_B^*$) is the* unique equilibrium *if*

$$a_A + \rho_{Aut}^* a_B > 0, \quad \text{and} \quad a_A^* + \rho_{Aut} a_B^* > 0.$$

In the autarkic equilibrium, the terms of trade for Home (the debtor country), $\hat{\rho} = p_B^*/p_A$, are worse than under free trade:

$$\hat{\rho} = \frac{a_A^* + \rho_{Aut} a_B^*}{\frac{a_A}{\rho_{Aut}^*} + a_B}.$$

Proof. We already know by Proposition 6 that Autarky is an equilibrium when the tariff rates are high enough. The condition on Proposition 7 imply that Lemma 3 applies. Our approach is to rule all possible cases but autarky.

Case 1: $c_A < Y_A$ and $c_B = Y_B$. We start by noticing that

$$g(c_B^*/c_A^*) = \frac{\rho}{1+\tau^*}, \quad \text{and} \quad g(c_B/c_A) = \alpha\rho,$$

for $\alpha \in [1/(1+\tau^*), 1+\tau]$. The budget constraint for Home implies that:

$$\begin{aligned} c_A - Y_A &= (1+\tau^*)a_A - a_A^* + \rho(a_B - \alpha a_B^*) \\ &= (1+\tau^*)(a_A + g(c_B^*/c_A^*)a_B) - (a_A^* + g(c_B/c_A)a_B^*) \end{aligned}$$

Given $c_A < Y_A$ and $c_B = Y_B$, we have that $c_B^*/c_A^* < Y_B^*/Y_A^*$ and $c_B/c_A > Y_B/Y_A$, and thus

$$g(c_B^*/c_A^*) > g(Y_B^*/Y_A^*) = \rho_{Aut}^*, \quad \text{and} \quad g(c_B/c_A) < g(Y_B/Y_A) = \rho_{Aut}.$$

Now from household's optimality we have that

$$g(c_B^*/c_A^*) = \frac{1}{\alpha(1+\tau^*)} g(c_B/c_A).$$

Note that $\alpha \in [1/(1+\tau^*), 1+\tau]$, and thus $\alpha(1+\tau^*) \geq 1$. This implies that

$$c_B^*/c_A^* \geq c_B/c_A \Rightarrow Y_B^*/c_A^* \geq Y_B/(\overline{Y}_A - c_A^*) \Rightarrow Y_B^*\overline{Y}_A - Y_B^*c_A^* \geq Y_B c_A^* \Rightarrow Y_B^*\overline{Y}_A \geq \overline{Y}_B c_A^*.$$

and thus given that $c_B = Y_B$,

$$c_B^*/c_A^* \geq \overline{Y}_B/\overline{Y}_A.$$

A similar argument then shows that

$$c_B/c_A \leq \overline{Y}_B/\overline{Y}_A.$$

Taken together, we have then that

$$\rho_{Aut}^* = g(Y_B^*/Y_A^*) < g(c_B^*/c_A^*) \leq g(\overline{Y}_B/\overline{Y}_A) = \rho^{FT} \leq g(c_B/c_A) < g(Y_B/Y_A) = \rho_{Aut}.$$

Now, if a_B is positive, we have that

$$a_A + g(c_B^*/c_A^*)a_B \geq a_A + \rho_{Aut}^* a_B.$$

If a_B is negative, then,

$$a_A + g(c_B^*/c_A^*)a_B \geq a_A + \rho_{FT} a_B.$$

Together, this means that

$$a_A + g(c_B^*/c_A^*)a_B \geq \min\{a_A + \rho_{FT} a_B, a_A + \rho_{Aut}^* a_B\} > 0.$$

where the inequality follows from Assumption 8 and the condition in the Proposition.
Similarly, we can show that

$$a_A^* + g(c_B/c_A)a_B^* \leq \max\{a_A^* + \rho_{FT} a_B^*, a_A^* + \rho_{Aut} a_B^*\}.$$

Returning to the budget constraint, we have then that:

$$c_A - Y_A \geq (1+\tau^*) \underbrace{\min\{a_A + \rho_{FT} a_B, a_A + \rho_{Aut}^* a_B\}}_{>0} - \max\{a_A^* + \rho_{FT} a_B^*, a_A^* + \rho_{Aut} a_B^*\}.$$

It follows that $c_A > Y_A$ for τ^* large enough, generating a constradiction and ruling Case 1 out.

Case 2: $c_A = Y_A$ and $c_B > Y_B$. This case can be ruled out with a symmetric argument to the one used for Case 1, and showing that for τ sufficiently large, it cannot be that $c_B^* < Y_B^*$.

Case 3: $c_A < Y_A$ **and** $c_B > Y_B$**.** From optimality and goods arbitrage we have that

$$g(c_B/c_A) = (1+\tau)\rho, \quad \text{and} \quad g(c_B^*/c_A^*) = \rho/(1+\tau^*).$$

It follows then that

$$g(c_B/c_A) = (1+\tau)(1+\tau^*)g(c_B^*/c_A^*).$$

Now, $c_A = Y_A$ and $c_B > Y_B$ imply that

$$c_B/c_A > Y_B/Y_A \Rightarrow g(c_B/c_A) < g(Y_B/Y_A),$$

and

$$c_B^*/c_A^* < Y_B^*/Y_A^* \Rightarrow g(c_B^*/c_A^*) > g(Y_B^*/Y_A^*).$$

Taken together we have that

$$g(Y_B/Y_A) > g(c_B/c_A) = (1+\tau)(1+\tau^*)g(c_B^*/c_A^*) > (1+\tau)(1+\tau^*)g(Y_B^*/Y_A^*).$$

If that tariffs are sufficiently high so that

$$(1+\tau)(1+\tau^*) > g(Y_B/Y_A)/g(Y_B^*/Y_A^*),$$

then we have a contradiction. Thus, this case can also be ruled out for tariff rates high enough.

According to Lemma 3, the only case left if autarky. We know by Proposition 6 that autarky is an equilibrium, which then completes the proof. □

A final thing to show with respect to Proposition 2 is that the terms of trade deteriorate for Home under Assumptions 4 and 5. To see this, just note that $\hat{\rho} = a_A^*/a_B > \rho^{FT}$ given that home is a debtor under free trade: $\rho^{FT} a_B - a_A^* < 0$.

B.4 Proof of Proposition 3

Proof. Let $t_1 = \mathbb{I}\left(\underline{\theta} > 0\right)\left(\frac{\hat{\rho}}{g(\underline{\theta})} - 1\right)$ and $t_2 = \frac{\hat{\rho}}{g(Y_B^*/Y_A^*)} - 1$.

The first thing to check is that $t_1 < t_2$. Note that $t_2 > 0$ by the assumption that $g(Y_B^*/Y_A^*) < \hat{\rho}$. So if $\theta \leq 0$, it follows that $t_1 < t_2$. For $\theta > 0$, doing some algebra, we get that

$$\underline{\theta} < \frac{Y_B^*}{Y_A^*} \Leftrightarrow Y_B(Y_A^* + \hat{\rho}Y_B^*) < g^{-1}(\hat{\rho})Y_A(Y_A^* + \hat{\rho}Y_B^*) \Leftrightarrow Y_B < g^{-1}(\hat{\rho})Y_A$$

which holds by $g(Y_B/Y_A) > \hat{\rho}$.

Consider a $\tau^* \in [t_1, t_2)$, and let us construct a tariff equibrium for some $\tau \geq 0$. Given that there is trade in equilibrium, $g(c_B^*/c_A^*) = \hat{\rho}/(1+\tau^*)$. Define

$$\theta^* \equiv g^{-1}\left(\frac{\hat{\rho}}{1+\tau^*}\right) = \frac{c_B^*}{c_A^*} < \frac{Y_B^*}{Y_A^*}, \tag{8}$$

where the last equality follows from condition (i): $g(Y_B^*/Y_A^*) < \hat{\rho}/(1+\tau^*)$. Using the fact that $a_A + \hat{\rho} a_B = 0$ by definition of $\hat{\rho}$, the Foreign budget constraint at world prices implies

$$c_A^* = \frac{Y_A^* + \hat{\rho} Y_B^*}{1 + \hat{\rho}\theta^*},$$

where we have used $c_B^* = \theta^* c_A^*$. The proposed equilibrium allocation for Foreign is thus:

$$(c_A^*, c_B^*) = \left(\frac{Y_A^* + \hat{\rho} Y_B^*}{1+\hat{\rho}\theta^*}, \frac{\theta^*\left(Y_A^* + \hat{\rho} Y_B^*\right)}{1+\hat{\rho}\theta^*}\right). \tag{9}$$

Note that $c_A^* > 0$ and $c_B^* > 0$. From the resource conditions, we can obtain the implied consumption allocation for Home:

$$\begin{aligned} c_A &= \overline{Y}_A - c_A^* = \frac{Y_A + \hat{\rho}(\theta^*\overline{Y}_A - Y_B^*)}{1+\hat{\rho}\theta^*} = Y_A - \frac{\hat{\rho}(Y_B^* - \theta^* Y_A^*)}{1+\hat{\rho}\theta^*} \\ c_B &= \overline{Y}_B - c_B^* = \frac{\overline{Y}_B - \theta^* Y_A^* + \hat{\rho}\theta^* Y_B}{1+\hat{\rho}\theta^*} = Y_B + \frac{Y_B^* - \theta^* Y_A^*}{1+\hat{\rho}\theta^*}. \end{aligned} \tag{10}$$

As $Y_B^* > \theta^* Y_A^*$ from (8), we have $c_B > Y_B > 0$ and $c_A < Y_A$ as required. That is, Home exports good A. To ensure that $c_A \geq 0$, we require

$$0 \leq Y_A + \hat{\rho}(\theta^*\overline{Y}_A - Y_B^*) \iff \frac{\hat{\rho} Y_B^* - Y_A}{\hat{\rho}\overline{Y}_A} \leq \theta^* = g^{-1}\left(\frac{\hat{\rho}}{1+\tau^*}\right).$$

If $\hat{\rho} Y_B^* \leq Y_A$, then this condition is satisfied immediately. Otherwise, the following condition ensures $c_A \geq 0$:

$$\tau^* \geq \frac{\hat{\rho}}{g\left(\frac{\hat{\rho} Y_B^* - Y_A}{\hat{\rho}\overline{Y}_A}\right)} - 1. \tag{11}$$

Next, we need to construct a tariff τ such that the allocation in (10) is optimal for Home. Consider a candidate τ given by

$$\tau = \frac{g\left(c_B/c_A\right)}{\hat{\rho}} - 1,$$

where c_A and c_B are given by (10). with this tariff rate, the proposed allocation would be optimal for

Home but we need to verify $\tau \geq 0$:

$$\tau \geq 0 \iff (1+\tau)\hat{\rho} \geq \hat{\rho} \iff \frac{c_B}{c_A} = g^{-1}((1+\tau)\hat{\rho}) \leq g^{-1}(\hat{\rho}).$$

Substituting for c_B/c_A using (10), we require

$$g^{-1}(\hat{\rho}) \geq \frac{\overline{Y}_B + \hat{\rho}\theta^* Y_B - \theta^* Y_A^*}{Y_A + \hat{\rho}(\theta^* \overline{Y}_A - Y_B^*)}$$

$$\iff$$

$$\left(g^{-1}(\hat{\rho})\hat{\rho}\overline{Y}_A + Y_A^* - \hat{\rho}Y_B\right)\theta^* \geq \overline{Y}_B - g^{-1}(\hat{\rho})\left(Y_A - \hat{\rho}Y_B^*\right). \tag{12}$$

Note that the left hand side coefficient is strictly positive:

$$g^{-1}(\hat{\rho})\hat{\rho}\overline{Y}_A + Y_A^* - \hat{\rho}Y_B > (Y_B/Y_A)\hat{\rho}\overline{Y}_A + Y_A^* - \hat{\rho}Y_B = \hat{\rho}Y_B(\overline{Y}_A/Y_A - 1) + Y_A^* > 0$$

where the first inequality follows from $g^{-1}(\hat{\rho}) > Y_B/Y_A$ (a premise in the lemma).
Dividing through (12) by this coefficient, we obtain the following condition as requirement for the Home tariff to be non-negative:

$$\theta^* \geq \frac{\overline{Y}_B - g^{-1}(\hat{\rho})\left(Y_A - \hat{\rho}Y_B^*\right)}{g^{-1}(\hat{\rho})\hat{\rho}\overline{Y}_A + Y_A^* - \hat{\rho}Y_B} \equiv \underline{\theta}. \tag{13}$$

If $\overline{Y}_B - g^{-1}(\hat{\rho})\left(Y_A - \hat{\rho}Y_B^*\right) \leq 0$, and thus $\underline{\theta} \leq 0$, then $\theta^* \geq \underline{\theta}$ and the condition holds.
Otherwise, if $\underline{\theta} > 0$, using $g(\theta^*) = \hat{\rho}/(1+\tau^*)$, the condition is equivalent to

$$\tau^* \geq \frac{\hat{\rho}}{g(\underline{\theta})} - 1. \tag{14}$$

And thus $\tau^\star \in T^*$ implies that the Home tariff is non-negative.
Finally, let us come back to the restriction we needed for $c_A \geq 0$, (11). Note that,

$$\frac{\hat{\rho}Y_B^* - Y_A}{\hat{\rho}\overline{Y}_A} < \underline{\theta}.$$

Thus if $\underline{\theta} \leq 0$, we have that $\hat{\rho}Y_B^* - Y_A < 0$. And if $\underline{\theta} > 0$, we have that

$$\frac{\hat{\rho}}{g(\underline{\theta})} - 1 > \frac{\hat{\rho}}{g\left(\frac{\hat{\rho}Y_B^* - Y_A}{\hat{\rho}\overline{Y}_A}\right)} - 1$$

and (11) is implied by (14). And htus $\tau^\star \in T^*$ is sufficient to guarantee $c_A \geq 0$.
Summarizing this: for any $\tau^* \geq [t_1, t_2)$ condition (13) holds, and thus there exists a $\tau \geq 0$ such that the allocation in (9) and (10) is a balanced trade equilibrium. In this equilibrium, Home exports good A.

□

B.5 Proof of Proposition 4

We now state and prove a general version of Proposition 4 that does not require Assumption 4:

Proposition 8. *Suppose that Assumptions 1 and 2. Then*

(i) If $a_A + \rho^{FT} a_B < 0$ and τ^* satisfies

$$0 < \frac{(c_A^{FT} + \rho^{FT} c_B^{FT}) - (1/h(\rho^{FT}) + \rho^{FT})Y_B}{-(a_A + \rho^{FT} a_B)} < \tau^* < \frac{c_A^{FT} + \rho^{FT} c_B^{FT}}{-(a_A + \rho^{FT} a_B)},$$

then there exists a tariff equilibrium where $c_A < Y_A$, $c_B < Y_B$, $g(c_B/c_A) = \rho^{FT}$ and

$$c_A - c_A^{FT} + \rho^{FT}(c_B - c_B^{FT}) = \tau^*(a_A + \rho^{FT} a_B).$$

(ii) If $a_A^* + \rho^{FT} a_B^* < 0$ and τ satisfies

$$0 < \frac{(c_A^{*FT} + \rho^{FT} c_B^{*FT}) - (1 + \rho^{FT} h(\rho^{FT}))Y_A^*}{-(a_A^* + \rho^{FT} a_B^*)} < \tau < \frac{c_A^{*FT} + \rho^{FT} c_B^{*FT}}{-(a_A^* + \rho^{FT} a_B^*)},$$

then there exists a tariff equilibrium where $c_A^* < Y_A^*$, $c_B^* < Y_B^*$, $g(c_B^*/c_A^*) = \rho^{FT}$ and

$$c_A^* - c_A^{*FT} + \rho^{FT}(c_B^* - c_B^{*FT}) = \tau(a_A^* + \rho^{FT} a_B^*).$$

Proof. We will prove the part (i) of the proposition. Part (ii) follows by symmetry.
Suppose that τ^* satisfies the condition in part (i).
Let Home prices be such that $p_B/p_A = \rho^{FT}$, and Foreign prices be such that $p_B^*/p_A^* = \rho^{FT}$. Let $p_A = p_A^*(1+\tau^*)$ and $p_B = p_B^*(1+\tau^*)$. Note that with these prices, goods arbitrage holds.
Let c_A and c_B be such that $g(c_B/c_A) = \rho^{FT}$ and Home budget constraint holds with equality. This guarantees that Home's household optimality conditions hold. Home's budget constraint can be rewritten as:

$$c_A + \rho^{FT} c_B = Y_A + \rho^{FT} Y_B + p_A^*/p_A(a_A + \rho^{FT} a_B) - (a_A^* + \rho^{FT} a_B^*).$$

Using the fact that $p_A^*/p_A = (1+\tau^*)$, we can rewrite this as

$$c_A + \rho^{FT} c_B = Y_A + \rho^{FT} Y_B + (1+\tau^*)(a_A + \rho^{FT} a_B) - (a_A^* + \rho^{FT} a_B^*).$$

And sing the fact that $c_A^{FT} + \rho^{FT} c_B^{FT} = Y_A + \rho^{FT} Y_B + (a_A + \rho^{FT} a_B) - (a_A^* + \rho^{FT} a_B^*)$, we can rewrite this once more as

$$c_A + \rho^{FT} c_B = c_A^{FT} + \rho^{FT} c_B^{FT} + \tau^*(a_A + \rho^{FT} a_B).$$

Note that $\tau^* < (c_A^{FT} + \rho^{FT} c_B^{FT})/(-(a_A + \rho^{FT} a_B))$ by the condition in the Proposition. This implies that $c_A + \rho^{FT} c_B > 0$. Hence c_A and c_B are non-negative.

Next, we need to check that $c_A < Y_A$ and $c_B < Y_B$. Note that $c_A = c_B/h(\rho^{FT})$. Thus, from the budget constraint, we have that

$$c_B(1/h(\rho^{FT}) + \rho^{FT}) = c_A^{FT} + \rho^{FT} c_B^{FT} + \tau^*(a_A + \rho^{FT} a_B).$$

For $c_B < Y_B$, it suffices to show that:

$$Y_B(1/h(\rho^{FT}) + \rho^{FT}) > c_A^{FT} + \rho^{FT} c_B^{FT} + \tau^*(a_A + \rho^{FT} a_B).$$

But this is the condition in Proposition that imposes $\frac{(c_A^{FT}+\rho^{FT}c_B^{FT})-(1/h(\rho^{FT})+\rho^{FT})Y_B}{-(a_A+\rho^{FT}a_B)} < \tau^*$.

For $c_A < Y_A$, recall from Assumptions 1 and 2 that $c_A^{FT} < Y_A$ and $c_B^{FT} > Y_B$. Given that we have shown that $c_B < Y_B$, it follows that $c_B < c_B^{FT}$. Then, $c_A < c_A^{FT} < Y_A$, by homotheticity.

Using the resource constraints, we can obtain c_A^* and c_B^*. Note that $c_A^* = \overline{Y}_A - c_A > 0$ and $c_B^* = \overline{Y}_B - c_B > 0$. Foreign optimality condition holds as $g(c_B^*/c_A^*) = g(c_B/c_A) = g(\overline{Y}_B/\overline{Y}_A) = \rho^{FT}$. We need to check that Foreign's budget constraint holds with equality. This follows from Walras law.

We have thus construct an equilibrium with the properties in part (i) of the Proposition.

A final thing to check for part (i) follows by noticing that

$$Y_B(1/h(\rho^{FT}) + \rho^{FT}) < c_A^{FT} + \rho^{FT} c_B^{FT} = c_B(1/h(\rho^{FT}) + \rho^{FT})$$

given that $c_B^{FT} > Y_B$. This implies that indeed

$$0 < \frac{(c_A^{FT} + \rho^{FT} c_B^{FT}) - (1/h(\rho^{FT}) + \rho^{FT})Y_B}{-(a_A + \rho^{FT} a_B)}.$$

Part (ii) follows by symmetric argument. □

B.6 Proof of Lemma 2

Proof. The proof follows from Lemma 1, which can be applied as Assumptions 5 holds as $a_B \geq 0$ and $a_A^* \geq 0$.

We just need to show that inequality (7) follows from Assumptions 1 and 2.

Note that under free trade, Home's budget constraint is:

$$c_A^{FT} + \rho^{FT} c_B^{FT} = Y_A + \rho^{FT} Y_B - a_A^*.$$

From the household first order condition under free trade:

$$g(c_B/c_A) = \rho^{FT} \Rightarrow c_B^{FT} = h(\rho^{FT})c_A^{FT}.$$

Using this in the budget constraint leads to:

$$(1/h(\rho^{FT}) + \rho^{FT})c_B^{FT} = Y_A - a_A^* + \rho^{FT} Y_B$$

Assumption 2 requires that $c_B^{FT} > Y_B$. Using this in the above equation, we get that:

$$(1/h(\rho^{FT}) + \rho^{FT})Y_B^{FT} < Y_A - a_A^* + \rho^{FT} Y_B \Rightarrow \frac{Y_B}{Y_A - a_A^*} < h(\rho^{FT}) = \frac{\overline{Y}_B}{\overline{Y}_A}.$$

A symmetric argument for Foreign shows that

$$h(\rho^{FT}) = \frac{\overline{Y}_B}{\overline{Y}_A} < \frac{Y_B^*}{Y_B^* + a_A^*}.$$

Taken together these last two imply condition (7). The rest of the proof follows from argument in the body of the text. □