04_diff_prog

September 11, 2023

1 Differentiable Programming 101

We study some initial examples of

- numerical differentiation
- symbolic differentiation
- automatic differentiation

2 Numerical Differentiation

Consider the sine function and its derivative,

$$f(x) = \sin(x), \quad f'(x) = \cos(x)$$

evaluated at the point $x = 0.1$.

[1]:
```python
import numpy as np
f = lambda x: np.sin(x)
x0 = 0.1
exact = np.cos(x0)
print("True derivative:", exact)
print("Forward Difference\tError\t\t\tCentral Difference\tError\n")
for i in range(10):
    h = 1/(10**i)
    f1 = (f(x0+h)-f(x0))/h
    f2 = (f(x0+h)-f(x0-h))/(2*h)
    e1 = np.abs(f1 - exact)
    e2 = np.abs(f2 - exact)
    print('%.5e\t\t%.5e\t\t%.5e\t\t%.5e'%(f1,e1,f2,e2))
```

```
True derivative: 0.9950041652780258
Forward Difference      Error                   Central Difference      Error

7.91374e-01             2.03630e-01             8.37267e-01
1.57737e-01
9.88359e-01             6.64502e-03             9.93347e-01
1.65751e-03
9.94488e-01             5.15746e-04             9.94988e-01
```

```
1.65833e-05
9.94954e-01              5.00825e-05              9.95004e-01
1.65834e-07
9.94999e-01              4.99333e-06              9.95004e-01
1.65828e-09
9.95004e-01              4.99183e-07              9.95004e-01
1.66720e-11
9.95004e-01              4.99136e-08              9.95004e-01
2.10021e-12
9.95004e-01              4.96341e-09              9.95004e-01
3.25943e-11
9.95004e-01              1.06184e-10              9.95004e-01
1.06184e-10
9.95004e-01              2.88174e-09              9.95004e-01
2.88174e-09
```

3 Symbolic Differentiation

Though very useful in simple cases, symbolic differentiation often leads to complex and redundant expressions. In addition, balckbox routines cannot be differentiated.

[2]:
```
from sympy import *
x = symbols('x')
#
diff(cos(x), x)
```

[2]:

$-\sin(x)$

[3]:
```
# a more complicated esxpression
def sigmoid(x):
  return 1 / (1 + exp(-x))

diff(sigmoid(x),x)
```

[3]:

$$\frac{e^{-x}}{\left(1+e^{-x}\right)^2}$$

Note that the derivative of

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

can be simply written as

$$\frac{d\sigma}{dx} = (1-\sigma(x))\sigma(x)$$

[4]:
```
# much more complicated
x,w1,w2,w3,b1,b2,b3 = symbols('x w1 w2 w3 b1 b2 b3')
y = w3*sigmoid(w2*sigmoid(w1*x + b1) + b2) + b3
diff(y, w1)
```

[4]:

$$\frac{w_2 w_3 x e^{-b_1 - w_1 x} e^{-b_2 - \frac{w_2}{e^{-b_1 - w_1 x} + 1}}}{\left(e^{-b_1 - w_1 x} + 1\right)^2 \left(e^{-b_2 - \frac{w_2}{e^{-b_1 - w_1 x} + 1}} + 1\right)^2}$$

[5]:
```
dydw1 = diff(y, w1)
print(dydw1)
```

```
w2*w3*x*exp(-b1 - w1*x)*exp(-b2 - w2/(exp(-b1 - w1*x) + 1))/((exp(-b1 - w1*x) +
1)**2*(exp(-b2 - w2/(exp(-b1 - w1*x) + 1)) + 1)**2)
```

4 Automatic Differentiation

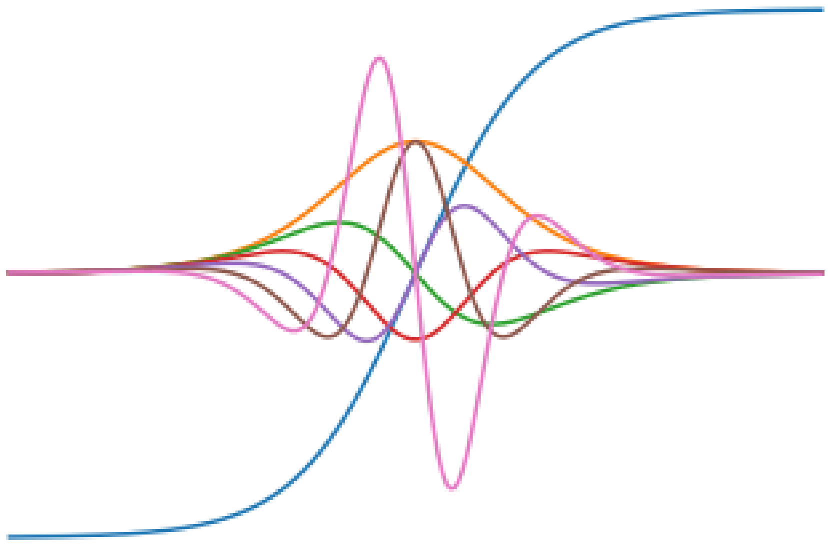
Here we show the simplicity and efficiency of `autograd` from `numpy`.

[6]:
```
import autograd.numpy as np
import matplotlib.pyplot as plt
from autograd import elementwise_grad as egrad  # for functions that vectorize␣
 ↪over inputs

# We could use np.tanh, but let's write our own as an example.
def tanh(x):
    return (1.0 - np.exp(-x))  / (1.0 + np.exp(-x))

x = np.linspace(-7, 7, 200)
plt.plot(x, tanh(x),
         x, egrad(tanh)(x),                                     # first  derivative
         x, egrad(egrad(tanh))(x),                              # second derivative
         x, egrad(egrad(egrad(tanh)))(x),                       # third  derivative
         x, egrad(egrad(egrad(egrad(tanh))))(x),                # fourth␣
 ↪derivative
         x, egrad(egrad(egrad(egrad(egrad(tanh)))))(x),         # fifth ␣
 ↪derivative
         x, egrad(egrad(egrad(egrad(egrad(egrad(tanh))))))(x))  # sixth ␣
 ↪derivative

plt.axis('off')
plt.savefig("tanh.png")
plt.show()
```



[7]:
```
from autograd import grad
grad_tanh = grad(tanh)              # Obtain its gradient function
gA = grad_tanh(1.0)                 # Evaluate the gradient at x = 1.0
gN = (tanh(1.01) - tanh(0.99)) / 0.02  # Compare to finite differences
print(gA, gN)
```

```
0.39322386648296376 0.3932226889551027
```

[ ]:
```
```