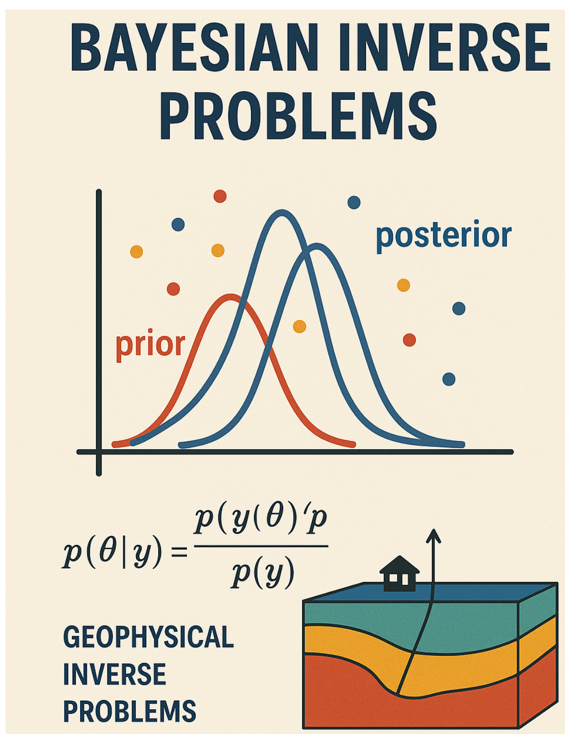
Bayesian Inverse Problems PRACTICAL-Solutions

Mark Asch - MAKUTU/2025



BIP - M. Asch - Practical for Lecture 02

Ex.1: Bayes - Priors and Posteriors

Suppose that the random variable $X \sim \mathcal{U}\,[0,1]$ and suppose that we know the conditional likelihood

$$Y|X = x \sim \mathcal{G}(x),$$

where the geometric law implies that

$$\mathrm{P}_{Y|X}(y|x) = x(1-x)^{y-1}$$

for $y = 1, 2, \ldots$ Find the posterior density of X given $Y = 2$.

SOLUTION:

- geometric law describes the probability that the first occurrence of success requires y independent (Bernouilli) trials, each with success probability x, sometimes written $\mathrm{P}(X = k) = (1-p)^{k-1}p.$

- according to Bayes' Law, the posterior

$$f_{X|Y}(x|2) = \frac{\mathrm{P}_{Y|X}(2|x) f_X(x)}{\mathrm{P}_Y(2)}$$

- but $Y|X = x \sim \mathcal{G}(x)$ so

$$\mathrm{P}_{Y|X}(2|x) = x(1-x)^{2-1} = x(1-x)$$

- the evidence (denominator), $\mathrm{P}_Y(2)$, can be calculated from the law of total probability

$$\begin{aligned}\mathrm{P}_Y(2) &= \int_{-\infty}^{\infty} \mathrm{P}_{Y|X}(2|x) f_X(x)\,\mathrm{d}x \\ &= \int_0^1 x(1-x)\,\mathrm{d}x \\ &= \frac{1}{6}\end{aligned}$$

- finally, from Bayes' formula

$$\begin{aligned}f_{X|Y}(x|2) &= \frac{x(1-x)\cdot 1}{1/6} \\ &= 6x(1-x), \quad 0 \le x \le 1.\end{aligned}$$

- Conclusions: the posterior is symmetric around $x = 0.5$ and concentrates the probability mass toward the middle of the interval, reflecting the information gained from observing $y = 2$.

- SEE: `Ex2.1-posterior.ipynb`

Ex. 2: MAP estimator

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

and suppose that

$$Y \mid X = x \sim \mathcal{G}(x)$$

is a geometric random variable. Find the MAP estimate of X given $Y = 3$.

SOLUTION:

- $Y|X = x \sim \mathcal{G}(x)$ so

$$\mathrm{P}_{Y|X}(y|x) = x(1-x)^{y-1}$$

for $y = 1, 2, \ldots$

- hence

$$\mathrm{P}_{Y|X}(3|x) = x(1-x)^{3-1} = x(1-x)^2$$

- we need to find the value of $x \in [0,1]$ that maximizes the numerator of Bayes' formula

$$\begin{aligned} \mathrm{P}_{Y|X}(y|x) f_X(x) &= x(1-x)^2 2x \\ &= 2x^2(1-x)^2 \end{aligned}$$

- differentiating thus expression, and equating to zero

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[2x^2(1-x)^2\right] &= 4x(1-x)^2 - 4x(1-x) \\ &= 4x(2x-1)(x-1) \\ &= 0 \quad \text{for } x = 0, 1/2, 1. \end{aligned}$$

- checking the sufficient condition for a maximum, we conclude that

$$\hat{x}_{\mathrm{MAP}} = \frac{1}{2}.$$

Ex. 3: MAP estimator of mean of IID random variables

Suppose that we are given a sequence $(x_1, \ldots, x_n)$ of IID $\mathcal{N}(\mu, \sigma_v^2)$ random variables and a prior distribution of the mean $g(\mu)$ is given by $\mathcal{N}(\mu_0, \sigma_m^2)$. Find the MAP estimate of μ.

SOLUTION:

- The function to be maximized is then given by

$$g(\mu)f(x \mid \mu) = \pi(\mu)L(\mu)$$

$$= \overbrace{\frac{1}{\sqrt{2\pi}\sigma_m} \exp\left(-\frac{1}{2}\left(\frac{\mu-\mu_0}{\sigma_m}\right)^2\right)}^{\pi(\mu)} \cdot \underbrace{\prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left(-\frac{1}{2}\left(\frac{x_j-\mu}{\sigma_v}\right)^2\right)}_{L(\mu)}$$

- Thanks to the IID property, this is equivalent to minimizing the following function of μ

$$\sum_{j=1}^{n} \left(\frac{x_j - \mu}{\sigma_v} \right)^2 + \left(\frac{\mu - \mu_0}{\sigma_m} \right)^2 .$$

- Then, the MAP estimator for μ is given by

$$\begin{aligned} \hat{\mu}_{\mathrm{MAP}} &= \frac{\sigma_m^2\, n}{\sigma_m^2\, n + \sigma_v^2} \left(\frac{1}{n} \sum_{j=1}^{n} x_j \right) + \frac{\sigma_v^2}{\sigma_m^2\, n + \sigma_v^2}\, \mu_0 \\ &= \frac{\sigma_m^2 \left(\sum_{j=1}^{n} x_j \right) + \sigma_v^2\, \mu_0}{\sigma_m^2\, n + \sigma_v^2}. \end{aligned}$$

which is a linear interpolation between the prior mean and the sample mean weighted by their respective covariances.

Ex. 4: MAP estimator of a noisy signal

Suppose that the signal $X \sim \mathcal{N}(0, \sigma_X^2)$ is transmitted over a communication channel. Assume that the received signal is given by

$$Y = X + \eta$$

where $\eta \sim \mathcal{N}(0, \sigma_\eta^2)$ is independent of X.

1. Find the ML estimate of X given the observation $Y = y$.

2. Find the MAP estimate of X given the observation $Y = y$.

3. Plot the prior, likelihood and posterior, as well as the ML and MAP estimates.

SOLUTION:

- Here we have the prior

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-x^2/2\sigma_X^2}.$$

- We also have $Y|X = x \sim \mathcal{N}(x, \sigma_\eta^2)$, so the likelihood

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_W} e^{-(y-x)^2/2\sigma_\eta^2}$$

- The ML estimate of X given $Y = y$ is the value of x that maximizes $f_{Y|X}(y|x)$. To do this, we need to minimize $(y-x)^2$ which is realized for $x = y$, so that

$$\hat{x}_{\mathrm{ML}} = y.$$

- The MAP estimate of X given $Y = y$ is the value of x that maximizes

$$f_{Y|X}(y|x) f_X(x) = C \exp\left\{ -\left[\frac{(y-x)^2}{2\sigma_\eta^2} + \frac{x^2}{2\sigma_X^2} \right] \right\},$$

where C is a constant that does not affect the maximization. This is equivalent to minimizing the quantity in the exponent,

$$\frac{(y-x)^2}{2\sigma_\eta^2} + \frac{x^2}{2\sigma_X^2}.$$

Differentiating, equating to zero, and solving, we obtain

$$\hat{x}_{\mathrm{MAP}} = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_\eta^2} y.$$

Ex. 5: CM (MMSE) estimator of a random variable

Let X be a continuous random variable with the following prior PDF

$$f_X(x) = 2x, \quad x \in [0,1] \,.$$

Suppose also that the likelihood is known,

$$f_{Y|X}(y|x) = \begin{cases} 2xy - x + 1 & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the CM estimate of X given the observation $Y = y$.

2. Plot the prior, likelihood and posterior, as well as the estimate.

SOLUTION:

- First we need to find the posterior density

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)}.$$

- The marginal can be computed as

$$\begin{aligned} f_Y(y) &= \int_0^1 f_{Y|X}(y|x)f_X(x)\,\mathrm{d}x && (1)\\ &= \int_0^1 (2xy - x + 1)2x\,\mathrm{d}x && (2)\\ &= \frac{4}{3}y + \frac{1}{3}, \quad \text{for } 0 \le y \le 1. && (3) \end{aligned}$$

- The posterior is then

$$f_{X|Y}(x|y) = \frac{6x(2xy - x + 1)}{4y + 1}, \quad \text{for } 0 \le x \le 1.$$

- The CM/MMSE estimate of X given the observation $Y = y$ is

$$
\begin{aligned}
\hat{x}_M &= \mathrm{E}[X|Y = y] \\
&= \int_0^1 x f_{X|Y}(x|y)\,\mathrm{d}x \\
&= \frac{1}{4y+1}\int_0^1 6x^2(2xy - x + 1)\,\mathrm{d}x \\
&= \frac{3y + 0.5}{4y+1}.
\end{aligned}
$$

Ex. 6: BCI for jointly normal RVs

Let X and Y be jointly normal random variables with $X \sim \mathcal{N}(0,1)$, $Y \sim \mathcal{N}(0,4)$ and covariance $\rho(X,Y) = 1/2$.

1. Find a 95% credible interval for X given that $Y = 2$ is observed.

2. Plot the prior, likelihood and posterior, as well as the interval estimate.

SOLUTION:

- We can show that if X and Y are jointly normal random variables with parameters μ_X,σ_X^2, μ_Y,σ_Y^2, and ρ, then, given $Y = y$ the variable X is normally distributed with posterior conditionals

$$\mathrm{E}[X|Y = y] = \mu_X + \rho\sigma_X \frac{y - \mu_Y}{\sigma_Y}$$
$$\mathrm{Var}(X|Y = y) = (1 - \rho^2)\sigma_X^2$$

- Therefore, $X|Y = 2$ is normal with

$$
\begin{aligned}
\mathrm{E}[X|Y = y] &= 0 + \frac{1}{2} \cdot \frac{2-1}{2} = \frac{1}{4}, \\
\mathrm{Var}(X|Y = y) &= \left(1 - \frac{1}{4}\right) \cdot 1 = \frac{3}{4}.
\end{aligned}
$$

- Here $\alpha = 0.05,$ so we need an interval $[a, b]$ for which

$$
\mathrm{P}(a \leq X \leq b|Y = 2) = 0.95
$$

- We usually choose a symmetric interval around the expected value $\mathrm{E}[X|Y = y] = 1/4.$ That is, we choose the interval in the form

$$
\left[\frac{1}{4} - c, \frac{1}{4} + c\right].
$$

This implies that

$$
\begin{aligned}
P\left(\frac{1}{4} - c \leq X \leq \frac{1}{4} + c | Y = 2\right) \\
= \Phi\left(\frac{c}{\sqrt{3/4}}\right) - \Phi\left(\frac{-c}{\sqrt{3/4}}\right) \\
= 2\Phi\left(\frac{c}{\sqrt{3/4}}\right) - 1 = 0.95.
\end{aligned}
$$

Solving for c, we find

$$c = \sqrt{3/4}\Phi^{-1}(0.975) \approx 1.70$$

Finally, the 95% credible interval for X is

$$\left[\frac{1}{4} - c, \frac{1}{4} + c\right] \approx [-1.45, 1.95].$$

Ex. 7: BIP Theory

Finite-Dimensional Examples: Scalar and vector cases -> Hints for solution

1. From the BIP theorem, we (see p.57 of `20_BIP.pdf`)

$$\pi^y(u) \propto \exp\left(-\frac{1}{2\delta^2}\|y - Au\|^2 - \frac{1}{2}|u|^2\right).$$

2. Try *completing the square* to show that the posterior mean and covariance are

$$\mu_\delta = \frac{A^\top y}{\delta^2 + \|A\|^2}, \quad \sigma_\delta^2 = \frac{\delta^2}{\delta^2 + \|A\|^2}.$$

3. Asymptotics - see p. 58

The vector-valued case (see pp. 59–61) is the matrix generalization of the above results. However, the asymptotics are more subtle...