Mathematical Presentation of Value at Risk (VaR) and Conditional Value at Risk (CVaR)

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ML-PREP 2025

Abstract

In this short note, we provide a comprehensive mathematical presentation of Value at Risk and Conditional Value at Risk, along with Python implementations and applications to environmental risk analysis.

Value at Risk (VaR)

Mathematical Definition

Value at Risk (VaR) is a statistical measure that quantifies the maximum potential loss of a portfolio over a specific time horizon at a given confidence level.

Let's denote:

- X as a random variable representing the profit/loss of a portfolio,
- $\alpha \in (0, 1)$ as the confidence level (typically 95%, 99% or 99.9%)

The Value at Risk at confidence level α is defined as

$$\mathrm{VaR}_\alpha(X) = -\inf\{x \in \mathbb{R} : P(X \leq x) \geq 1 - \alpha\}.$$

In simpler terms, VaR is the threshold value such that the probability of the loss exceeding this value is at most $1 - \alpha$.

Properties of VaR

1. VaR is not sub-additive: $\mathrm{VaR}_\alpha(X + Y) \leq \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y)$ does not always hold.
2. VaR only considers the probability of losses exceeding a threshold, not the magnitude of those extreme losses.
3. VaR is not a coherent risk measure because it lacks sub-additivity.

Conditional Value at Risk (CVaR)

Mathematical Definition

Conditional Value at Risk (CVaR), also known as Expected Shortfall (ES), addresses some limitations of VaR by measuring the expected loss given that the loss exceeds the VaR threshold. It is defined as

$$\mathrm{CVaR}_\alpha(X) = -\mathbb{E}[X | X \leq -\mathrm{VaR}_\alpha(X)].$$

For continuous distributions, CVaR can be expressed as

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{1-\alpha} \int_0^{1-\alpha} \mathrm{VaR}_\gamma(X)\, \mathrm{d}\gamma.$$

Properties of CVaR

1. CVaR is a coherent risk measure (satisfies monotonicity, sub-additivity, homogeneity, and translation invariance).
2. CVaR accounts for the severity of losses beyond the VaR threshold.
3. CVaR is always greater than or equal to VaR: $\mathrm{CVaR}_\alpha(X) \geq \mathrm{VaR}_\alpha(X)$.

Methods for Computing VaR and CVaR

1. Historical Method

The historical method uses historical data to estimate VaR and CVaR without making assumptions about the distribution of returns.

Process:

1. Collect historical returns data.
2. Sort returns in ascending order.
3. For VaR at confidence level α, find the return at the $(1 - \alpha)$ quantile.
4. For CVaR, calculate the average of all returns below the VaR threshold.

Advantages:

- No distribution assumptions required.
- Captures the actual empirical distribution of returns.
- Preserves fat tails and skewness present in the data.

Limitations:

- Highly dependent on the specific historical period used.
- May not account for future scenarios not present in historical data.

- Requires sufficient historical data for reliable estimates.

2. Parametric Method

The parametric method assumes returns follow a specific probability distribution (usually normal).

Process:

1. Collect historical returns data.
2. Calculate mean (μ) and standard deviation (σ) of returns.
3. For VaR at confidence level α,

$$\mathrm{VaR}_\alpha = -(\mu + \sigma \cdot z_\alpha),$$

   where z_α is the α-quantile of the standard normal distribution.

4. For CVaR at confidence level α,

$$\mathrm{CVaR}_\alpha = -\left(\mu + \sigma \cdot \frac{\phi(z_\alpha)}{\alpha}\right),$$

   where $\phi(z)$ is the standard normal probability density function, $\mathcal{N}(0,1)$.

Advantages:

- Computationally efficient.
- Requires less historical data.
- Easy to implement and explain.

Limitations:

- Normal distribution assumption may not hold for financial returns, or environmental events (non stationary).
- Underestimates risk for fat-tailed distributions.
- Poor performance during market stress periods.

3. Monte Carlo Method

The Monte Carlo method simulates a large number of possible future scenarios based on assumptions about the return distribution.

Process:

1. Specify a stochastic process for the returns.
2. Generate a large number of random scenarios.
3. Calculate the portfolio value for each scenario.

4. Determine VaR as the appropriate percentile of the simulated distribution.
5. Calculate CVaR as the average of losses exceeding VaR.

Advantages:

- Flexible approach that can incorporate various distributional assumptions.
- Can model complex portfolios and risk factors.
- Able to incorporate correlation structures and time-varying volatility.

Limitations:

- Computationally intensive.
- Results depend on the quality of the simulation model.
- Parameter calibration can be challenging.

Applications to Environmental Risk Analysis

1. Flood Damage Cost Assessment

In flood risk management, VaR and CVaR can quantify the financial impact of flooding events:

- VaR (95%): The flood damage cost that will not be exceeded with 95% probability in a given year.
- CVaR (95%): The expected damage cost in the worst 5% of flood scenarios.

This helps in:

- Setting appropriate flood insurance premiums.
- Planning emergency response budgets.
- Evaluating cost-effectiveness of flood protection infrastructure.

2. Pollution Emission Control

For industrial facilities monitoring emissions:

- VaR (95%): The emission level that will not be exceeded with 95% probability.
- CVaR (95%): The expected emission level during the worst 5% of operating conditions.

Applications include:

- Designing appropriate pollution control equipment.
- Setting compliance thresholds and penalties.
- Evaluating environmental impact of industrial activities.

3. Drought Risk Management

For water resource management:

- VaR (95%): The water shortage level that will not be exceeded with 95% probability.
- CVaR (95%): The expected water shortage during the worst 5% of drought scenarios.

This helps in:

- Planning water conservation measures.
- Designing reservoir operations.
- Developing drought contingency plans.

Comparison of VaR and CVaR in Environmental Risk Analysis

Aspect	VaR	CVaR
Risk Information	Threshold value only	Expected loss in tail
Extreme Events	May underestimate	Better captures extreme events
Regulatory Use	Traditional measure	Increasingly adopted
Decision Making	Simple threshold	Better for risk-averse decisions
Coherence	Not coherent	Coherent risk measure

Conclusion

VaR and CVaR provide valuable quantitative frameworks for risk assessment and management in both financial and environmental contexts. While VaR offers a straightforward threshold that is easily communicated, CVaR provides a more complete picture of tail risk that is particularly valuable when managing risks associated with extreme environmental events.

The Python implementation provided, demonstrates how these measures can be calculated using different methodologies and applied to environmental risk scenarios, offering a valuable toolkit for risk managers, environmental scientists, and policymakers.