Defence School of
Aeronautical Engineering

Aerosystems Engineer & Management
Training School

Academic Principles Organisation

MATHEMATICS

BOOK 7

Pythagoras' Theorem & Trigonometry

WARNING

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KEY LEARNING POINTS

KLP	Description
MA5.1	Construct a right-angled triangle from given data.
MA5.2	Apply Pythagoras' theorem.
MA5.3	Define sine, cosine and tangent ratios for angles 0° - 90°
MA5.4	Determine angles and sides of a right-angle triangle.

PYTHAGORAS' THEORUM & TRIGONOMETRY

1. The conventional way to label a triangle is as shown in figure 1, the angles or corners are identified with capital letters and the sides are labelled with the lower-case letter of the angle opposite

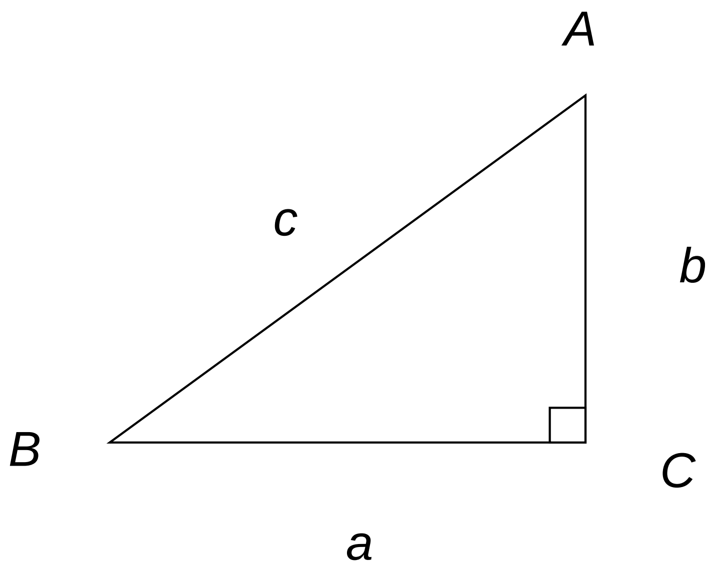


Figure 1

The Theorem of Pythagoras

2. Consider the right-angled triangle below:

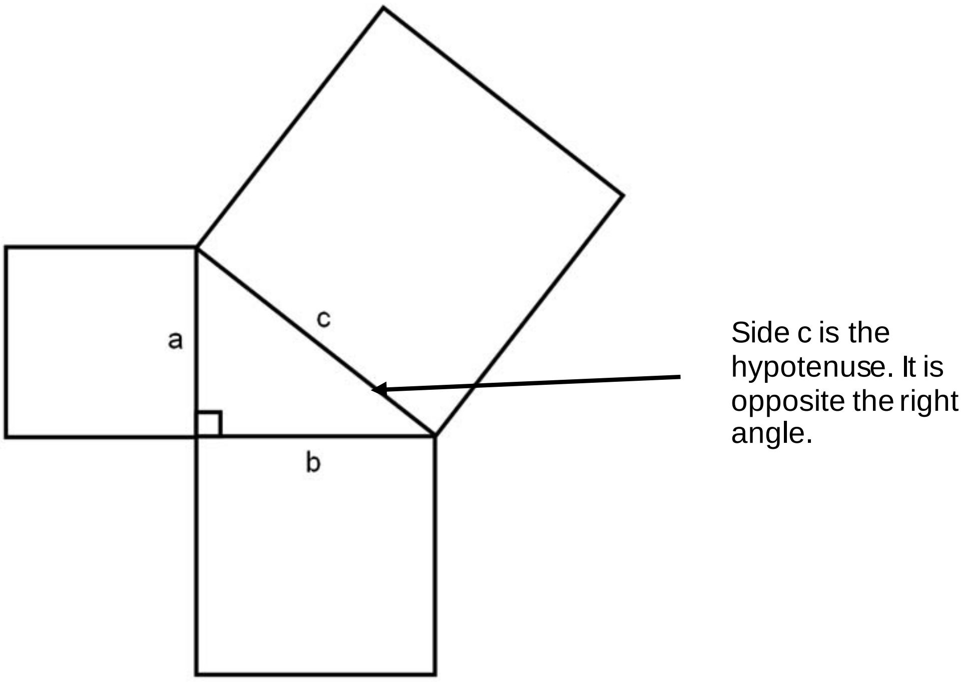


Figure 2

Theorem of Pythagoras

In a right-angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

3. This means that the area of the square drawn on side c in the diagram, is equal to the sum of the areas of the squares drawn on sides a and b.

4. If we write this as a formula:

the area of the square, side **a**, will be $\mathbf{a} \times \mathbf{a}$

the area of the square, side **b**, will be $\mathbf{b} \times \mathbf{b}$

the area of the square, side **c**, will be $\mathbf{c} \times \mathbf{c}$

so, we have:

$$c \times c = a \times a + b \times b$$

5. We normally write this as:

$$c^2 = a^2 + b^2$$

Or $$c = \sqrt{a^2 + b^2}$$

6. This is the theorem of Pythagoras and applies to right-angled triangles only.

Now consider these examples:

Example 1

7. Using the theorem of Pythagoras

$$c^2 = a^2 + b^2$$

8. Given a right-angled triangle has side *a* = 4 cm and side *b* = 3 cm. Calculate the length of side *c*.

Substituting for a and b:

$$c^2 = a^2 + b^2$$

$$c^2 = 16 + 9$$

$$c^2 = 25$$

$$c = \sqrt{25}$$

$$\therefore\ c \quad = \ 5cm$$

9. This is a common triangle, used in the building trade to construct a right-angle. It is referred to as the 3:4:5 right-angled triangle or builder's triangle.

Example 2

10. Use the theorem of Pythagoras to calculate the hypotenuse of a right-angled triangle whose other two sides measure 3 ft and 7 ft.

The hypotenuse is always the longest side. Using the formula:

$$c^2 = a^2 + b^2$$

$$c^2 = 3^2 + 7^2$$

$$c^2 = 58$$

11. Continue by pressing the square root button on your calculator followed by 58 then the equals button

$$c = \sqrt{58} = 7.616$$

To one decimal place, the hypotenuse is **7.6 ft**.

Try this simple exercise:

Exercise 1

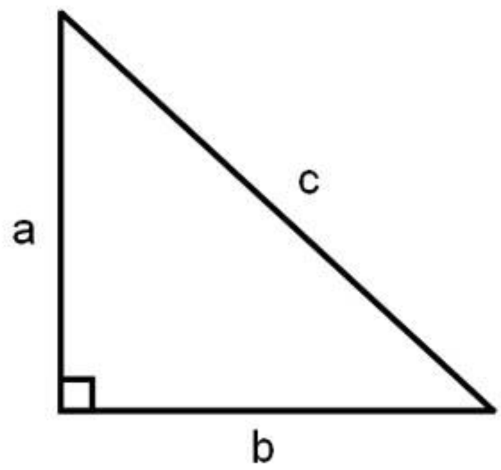


Figure 3

12. Given the triangle above, calculate the size of the missing side in each case. Answer to one decimal place, where appropriate.

	a	b	c
1.	3 cm	?	5 cm
2.	8 cm	?	10 cm
3.	?	12 cm	13 cm
4.	7 m	?	25 m
5.	24 m	?	26 m
6.	8 mm	?	15 mm
7.	?	10 m	14 m
8.	4 cm	?	7 cm
9.	?	12 cm	16 cm
10.	6 m	?	15 m

Check your answers with those on page 8.

13. Now try this exercise. Remember, the hypotenuse is the longest side, and it is always the side opposite the right angle.

Exercise 2

The following diagrams are not drawn to scale. Give answers correct to one decimal place where necessary. Use the theorem of Pythagoras in each case.

11. Find the side marked x.

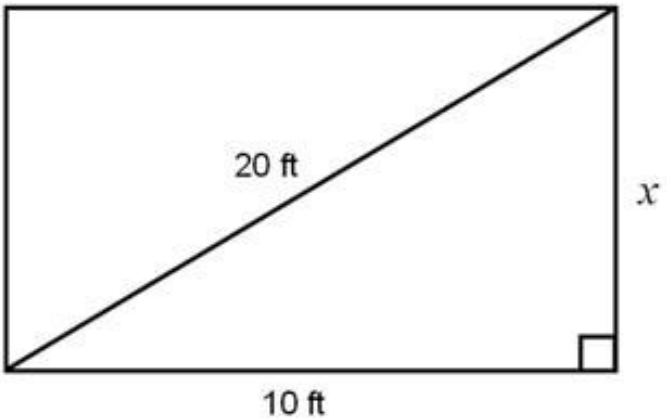


Figure 4

12. The diagram below shows a section of a trellis, made up of six struts. Calculate:

    a. the length of BC.

    b. the total length of wood required to build this section of trellis.

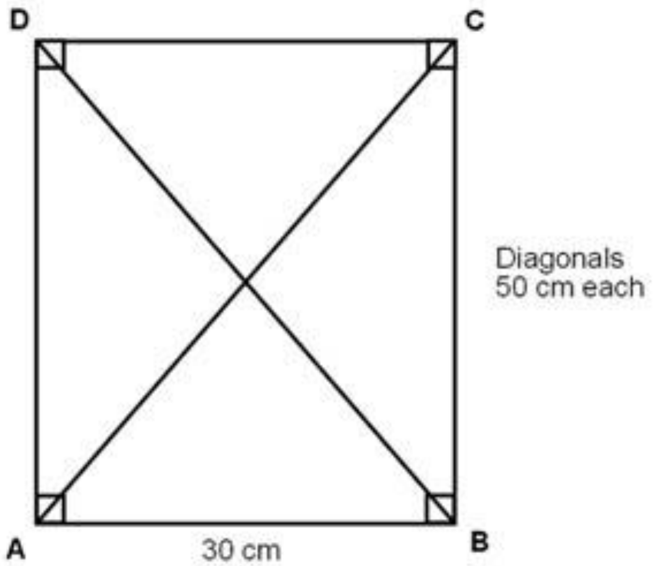


Figure 5

Exercise 2 is continued on the next page.

13. The sides of a field measures 500 m x 300 m.

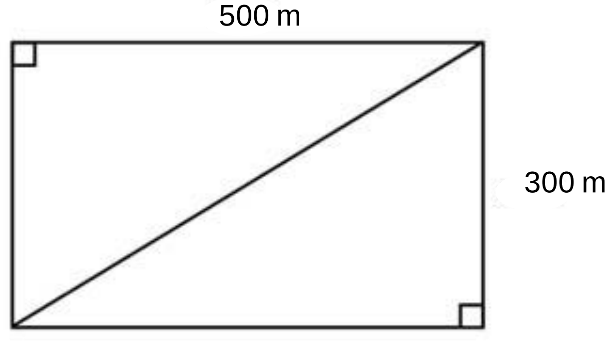


Figure 6

Calculate the length of a diagonal line drawn as shown.

14. If a ladder 12 ft long is rested against a vertical wall as shown, with the foot of the ladder 4 ft from the base of the wall, how far up the wall will the ladder reach?

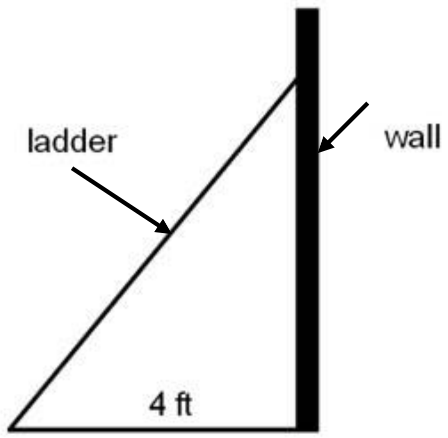


Figure 7

Check your answers with those on page 8

Answers

Exercise 1

1. 4cm
2. 6cm
3. 5cm
4. 24m
5. 10m
6. 12.7mm
7. 9.8m
8. 5.7cm
9. 10.6cm
10. 13.7m

Exercise 2

11. 17.3ft
12. a. 40cm
    b. 240cm
13. 583.1m
14. 11.3ft

TRIGONOMETRY

14. Trigonometry is the study of the relationships between the angles and the sides in triangles.

15. The ratios used in this unit are only applicable to right-angled triangles. Note that the diagrams in this unit are not drawn to scale, so you may not be able to find any sides or angles by measurements.

16. We shall name the sides of the right-angled triangle first.

Consider these triangles:

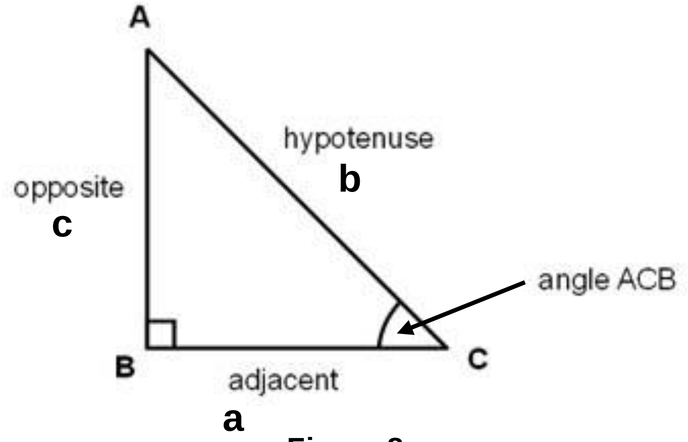


Figure 8

17. The **hypotenuse** is always the longest side. The longest side is the side opposite the right angle. In this case, the **hypotenuse** is side **b or AC**.

18. The side opposite the angle given, or required, is called the **opposite** side. In the case of the triangle above, angle **C** or **ACB** is **opposite** side **C** or **AB**.

19. The other side is adjacent to the angle given, or required, and is called the **adjacent** side. Side **a** or **BC** is **adjacent** to angle **C** or **ACB** in figure 7.

20. In the triangle below, the angle given, or required, is the angle **A** or **BAC**.

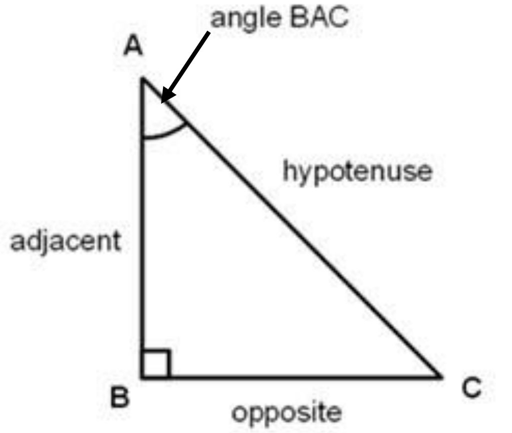


Figure 9

The sides are marked relative to the angle **A (**or **BAC)** this time. Sometimes we label the angle with a letter like x, often we use a Greek letter for example θ or α

Try this short exercise:

Exercise 3

Name the sides relative to the marked angle in each case.

15.

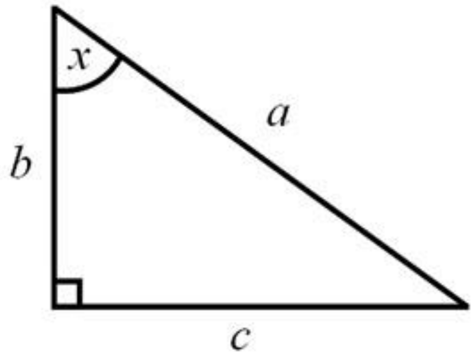


Figure 10

16.

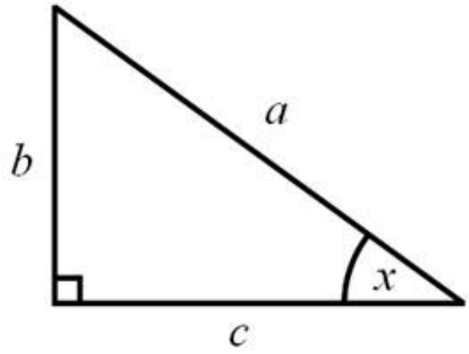


Figure 11

17.

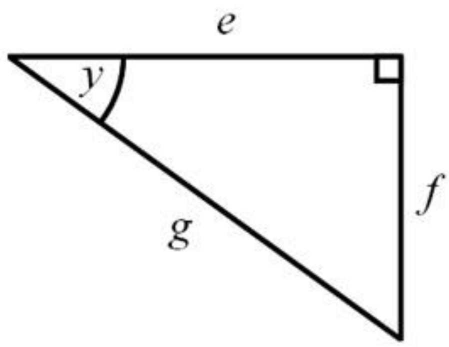


Figure 12

18.

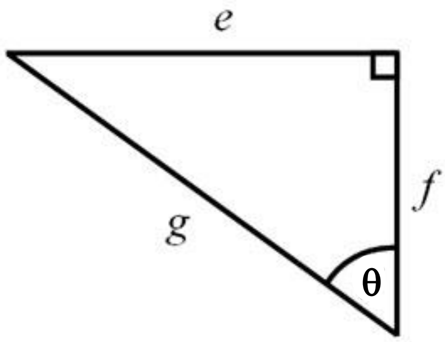


Figure 13

Check your answers with those on page 20.

TRIGONOMETRY RATIOS

21. There are three trigonometry ratios.

22. These ratios are sine (**sin**), cosine (**cos**) and tangent (**tan**).

23. The trigonometry ratios are given below:

$$\sin\theta = \frac{opposite}{hypotenuse} \qquad S = \frac{O}{H}$$

$$\cos\theta = \frac{adjacent}{hypotenuse} \qquad C = \frac{A}{H}$$

$$\tan\theta = \frac{opposite}{adjacent} \qquad T = \frac{O}{A}$$

24. You may have heard these ratios referred to as:

SOH – CAH – TOA

25. This should help you to remember the ratios.

26. Incidentally if we use these labels instead of **ABC** for angles and **abc** for side lengths Pythagoras becomes $H^2 = O^2 + A^2$ and $H = \sqrt{O^2 + A^2}$

THE SINE RATIO

Study the following examples.

Example 3

27. Using the following triangle, find the size of the marked side.

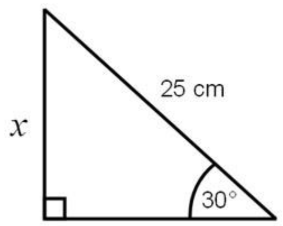


Figure 14

The marked side, x, is opposite the angle 30°.

The hypotenuse the other length given and is 25 cm.

The two sides involved are therefore the opposite side and the hypotenuse. O and H.

28. The letters O and H are together in SOH, so we can say that:

$$\sin\theta = \frac{O}{H} \quad \therefore \quad H\sin\theta = O$$

Substituting in the values given

$25 \times \sin 30° = x$

The length of x is **12.5 cm**.

Example 4

29. Find the value of y to one decimal place.

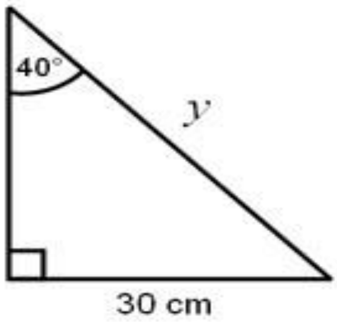


Figure 15

30. y is opposite the right angle, so y is the hypotenuse. It also quite clearly the longest side which is always the hypotenuse

31. The side opposite the 40° angle is 30 cm.

32. We need to use the side O and H.

33. This is then SOH and we use the sin of angle 40° as follows:

$$\sin \boldsymbol{\theta} = \frac{O}{H} \quad \therefore \quad H \sin \boldsymbol{\theta} = O \qquad and \qquad H = \frac{O}{\sin \boldsymbol{\theta}}$$

34. Substitute in the Values given

$$y = \frac{30}{\sin 40^0} = 46.672$$

To one decimal place y is **46.7 cm**.

35. As this hypotenuse of the triangle and is therefore the longest side of the triangle check to see if the answer is sensible.

Now try the exercise on the next page.

Exercise 4

Find the side marked x in each case, giving your answer correct to one decimal place.

19.

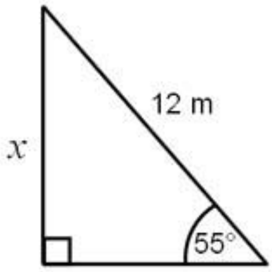


Figure 16

20.

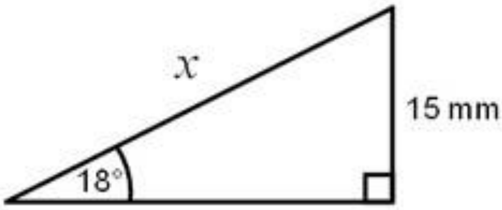


Figure 17

21.

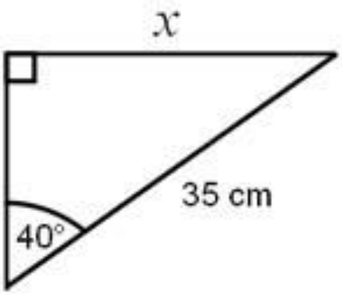


Figure 18

22.

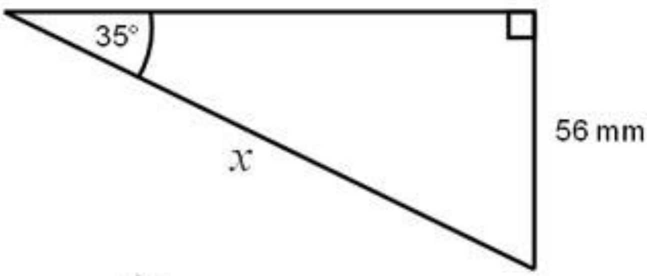


Figure 19

23.

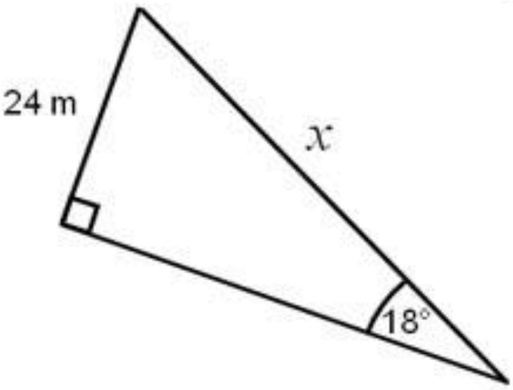


Figure 20

Check your answers with those on page 20

Notes:

THE COSINE RATIO

Study these examples.

Example 5

36. Find the value of y in the following diagram. Give the answer to two decimal places.

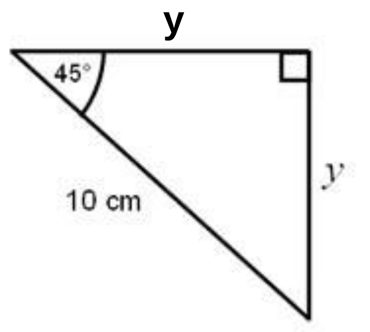


Figure 21

37. The hypotenuse is the side opposite the right angle and is equal to 10 cm.

38. y is the side adjacent to the angle of 45°.

39. We are going to use the sides A and H.

SOH-CAH-TOA

A and H are both in CAH.

40. We shall use the cosine of the angle.

$$\cos\boldsymbol{\theta} = \frac{A}{H} \quad \therefore \quad H\cos\boldsymbol{\theta} = A$$

41. Substitute in the values given:

$$y = 10 \times \cos 45°$$

42. The length of y is **7.07 cm** to two decimal places.

Example 6

43. Find the value of z, to one decimal place.

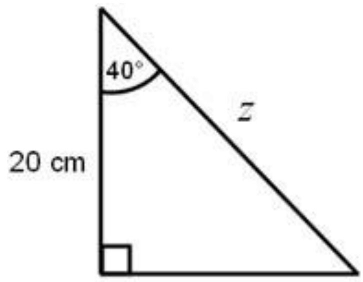


Figure 22

44. As before check to find which sides are involved.

45. z is the side opposite the right angle, so it is the hypotenuse.

46. The 20cm side is adjacent to the angle of 40°.

We need to use A and H, CAH.

$$\cos\boldsymbol{\theta} = \frac{A}{H} \quad \therefore \; H\cos\boldsymbol{\theta} = A \qquad \textit{and} \qquad H = \frac{A}{\cos\boldsymbol{\theta}}$$

47. Substitute in the values from the question

$$z \times \cos 40° = 20$$

48. Divide both sides by cos 40°:

$$z = \frac{20}{\cos 40°}$$

49. The length of z is **26.1 cm** to one decimal place.

Try this exercise:

Exercise 5

50. Find the side marked x in the following diagrams, giving your answers correct to one decimal place.

24.

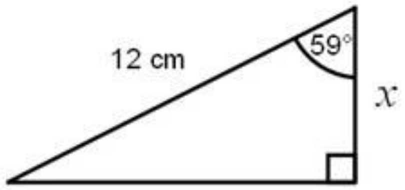


Figure 23

25.

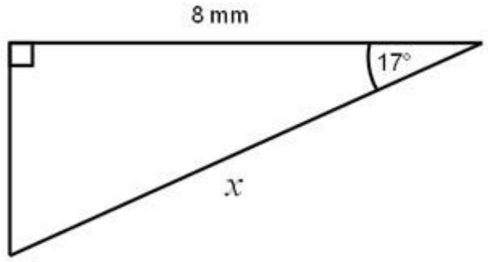


Figure 24

26.

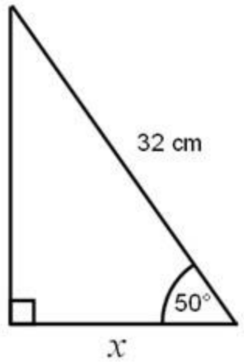


Figure 25

27.

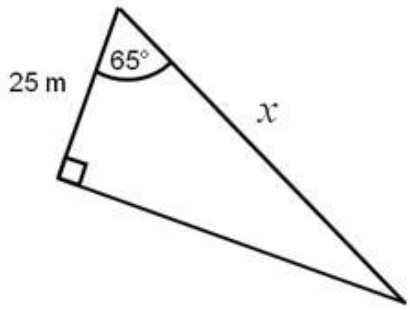


Figure 26

28.

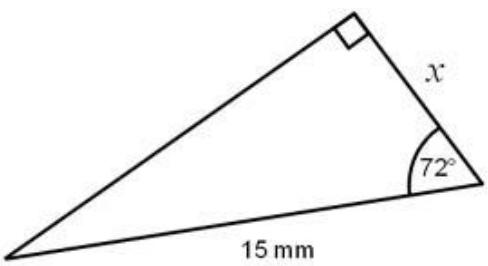


Figure 27

Check your answers with those on page 20 before studying the next examples.

THE TANGENT RATIO

Example 5

51. Find the value of p to one decimal place.

52. *p is opposite the 25° angle.*

53. The 15cm side is adjacent to the 25° angle.

We need to use O and A, TOA.

$$\tan\boldsymbol{\theta} = \frac{O}{A} \therefore \quad A\tan\boldsymbol{\theta} = O$$

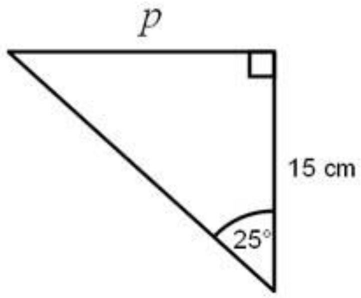


Figure 28

Substitute in the values from the question

$$p = 15 \times \tan 25° = 6.995$$

The length of p is **7.0 cm** to one decimal place.

Example 6

54. Find the value of q to one decimal place.

55. The 14cm side is opposite the angle 25°.

56. q is adjacent to the 25° angle.

O and A. Use TOA.

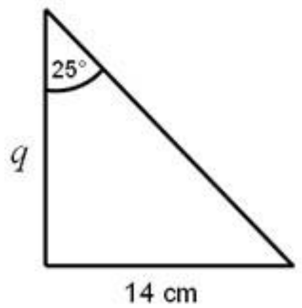


Figure 29

$$\tan\boldsymbol{\theta} = \frac{O}{A}$$

$$\therefore \quad A\tan\boldsymbol{\theta} = O \qquad and \qquad A = \frac{O}{\tan\boldsymbol{\theta}}$$

57. Substitute in the values given

$$q = \frac{14}{\tan 25} = 30.023$$

The length of q is **30.0 cm** to one decimal place.

Try the exercise on the next page.

Exercise 6

58. Find the value of x in each case, giving your answers correct to two decimal places for this exercise

29.

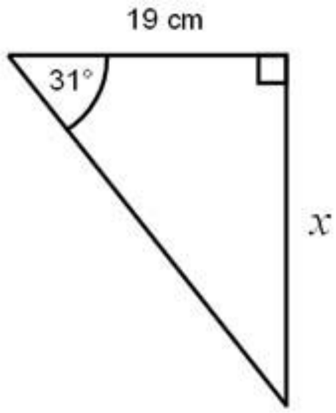


Figure 30

30.

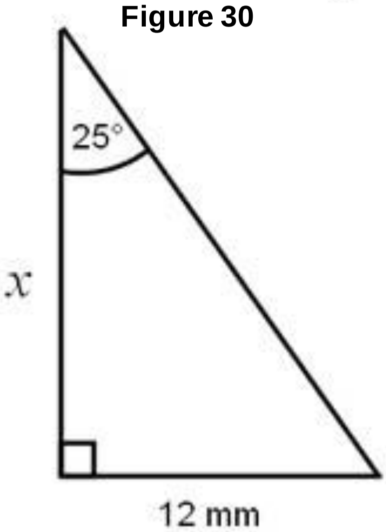


Figure 31

31.

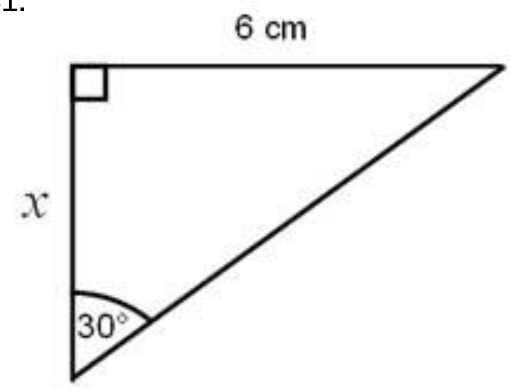


Figure 32

32.

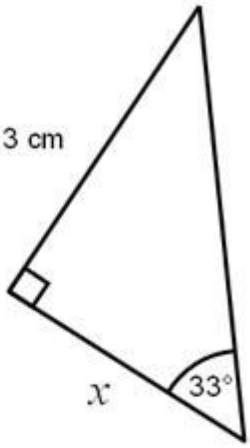


Figure 33

33.

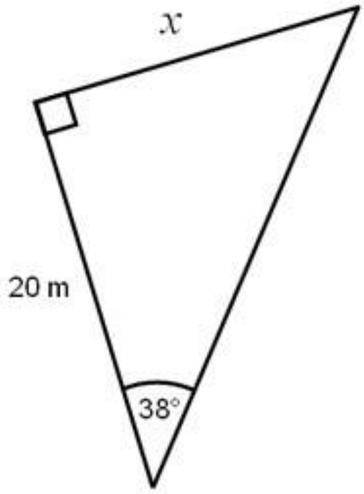


Figure 34

34.

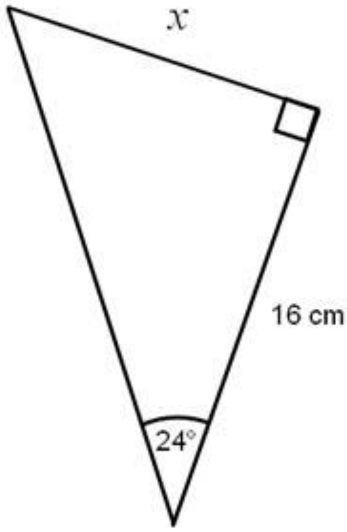


Figure 35

Check your answers with those on page 20.

Try this miscellaneous exercise:

Exercise 7

59. Find the length of the side marked x in the following exercise, giving your answers correct to two decimal places. You will need to decide whether to use sin, cos or tan.

35.

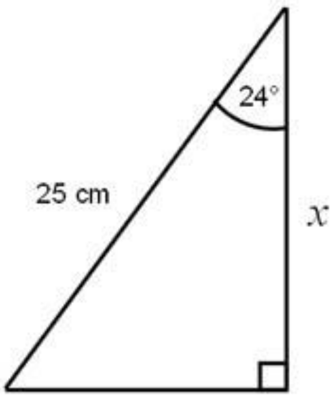


Figure 36

36.

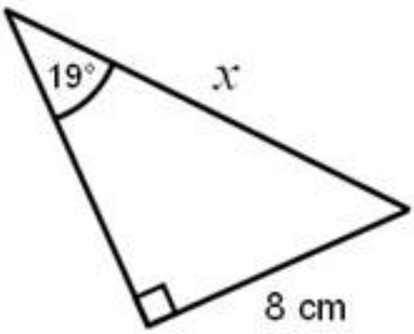


Figure 37

37.

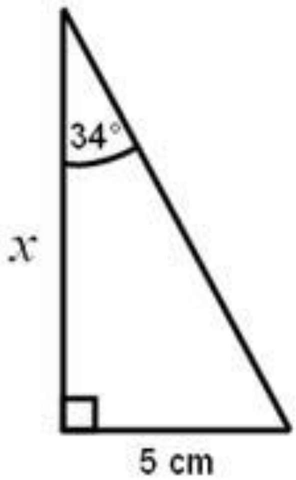


Figure 38

38.

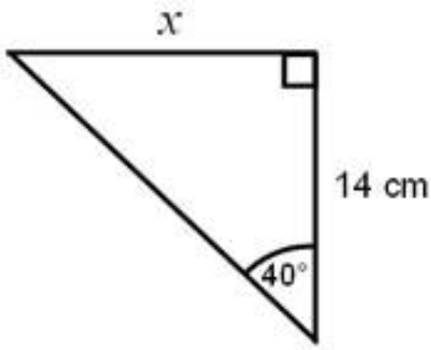


Figure 39

39.

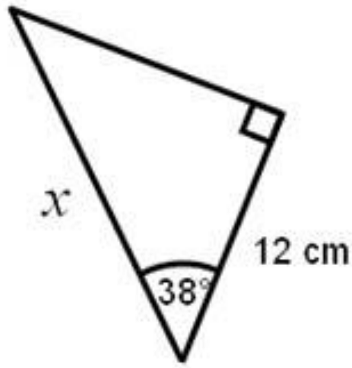


Figure 40

40.

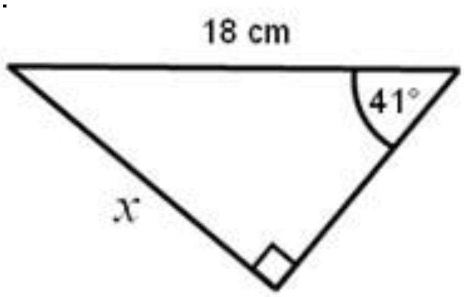


Figure 41

Check your answers with those over the page.

Answers

Exercise 3

15. a = hypotenuse
    b = adjacent
    c = opposite

16. a = hypotenuse
    b = opposite
    c = adjacent

17. e = adjacent
    f = opposite
    g = hypotenuse

18. e = opposite
    f = adjacent
    g = hypotenuse

Exercise 4

19. 9.8 m
20. 48.5 mm
21. 22.5 cm
22. 97.6 mm
24. 77.7 m

Exercise 5

24. 6.2 cm
25. 8.4 mm
26. 20.6 cm
27. 59.2 m
28. 4.6 mm

Exercise 6

29. 11.42 cm
30. 25.73 mm
31. 10.39 cm
32. 4.62 cm
33. 15.63 m
34. 7.12 cm

Exercise 7

35. 22.84 cm
36. 24.57 cm
37. 7.41 cm
38. 11.75 cm
39. 15.23 cm
40. 11.81 cm

INVERSE TRIGONOMETRICAL RATIOS

60. The ratios used in this unit are only applicable to right-angled triangles. Note that the diagrams in this unit are not drawn to scale, so you may not find any sides or angles by measurement.

61. Sometimes you will be given two sides of a right-angled triangle and asked to calculate an angle.

Consider the following examples:

Example 7

62. Given that sin $A = 0.5$ find the value of angle A.

If $\sin A = 0.5$,

Then $A = \sin^{-1} 0.5$ = 30°

Example 8

63. Find the value of x to the nearest degree.

64. The hypotenuse is 9 cm.

65. The 7cm side is opposite the required angle.

Sides O and H are involved. **SOH**.

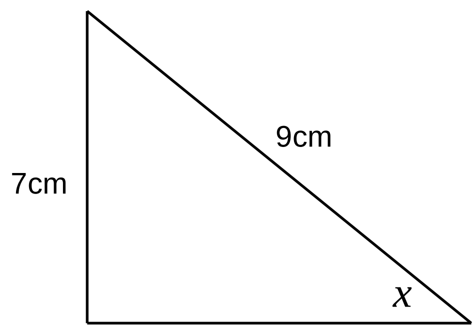


Figure 42

66. We need to use the sin of angle x.

$$\sin x = \frac{O}{H}$$

$$\sin x = \frac{7}{9}$$

$$\therefore x = \sin^{-1}\left(\frac{7}{9}\right) = 51°$$

Angle θ is equal to **51°**

Example 9

67. Find the value of θ.

68. The hypotenuse is 12 cm.

69. The side adjacent to the required angle is 6 cm.

A and H. **CAH**.

70. Use the cos of the angle.

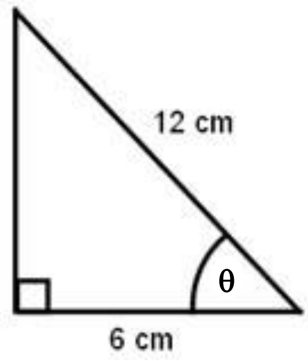


Figure 43

$$\cos \boldsymbol{\theta} = \frac{A}{H} \qquad \cos \boldsymbol{\theta} = \frac{6}{12}$$

$$\therefore \theta = \cos^{-1}\left(\frac{6}{12}\right) = 60^\circ$$

Example 10

71. Find the value of θ to the nearest degree.

72. The side opposite θ is 10 cm.

73. The side adjacent to θ is 6 cm.

O and A. **TOA**.

74. Use the tan of the angle.

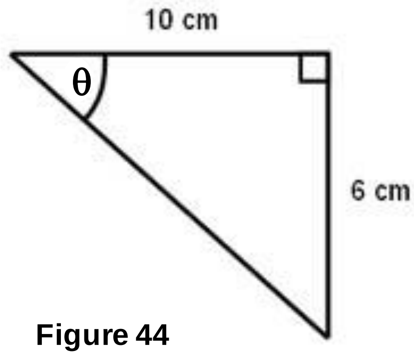


Figure 44

$$\tan \boldsymbol{\theta} = \frac{O}{A}$$

$$\tan \theta = \frac{6}{10}$$

$$\therefore \theta = \tan^{-1}\left(\frac{6}{10}\right) = 31^\circ \quad \text{(2SF)}$$

Try the exercise on the next page.

Try this exercise:

Exercise 8

Give all your answers to the nearest degree.

41. Find angle x

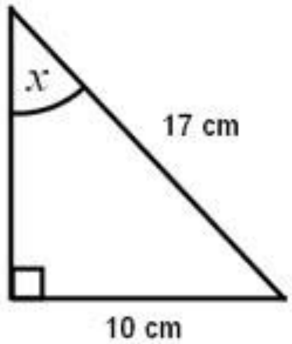


Figure 45

42. Find angle y.

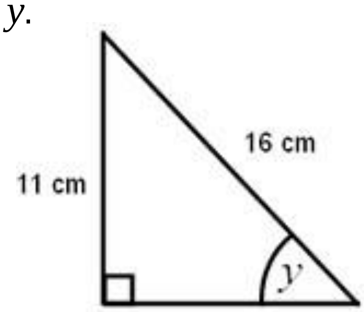


Figure 46

43. Find angle θ.

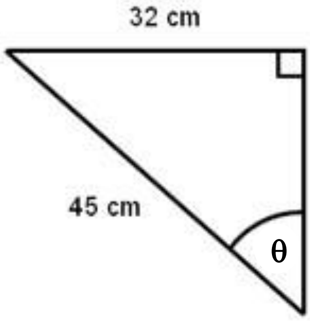


Figure 47

44. Find angle g.

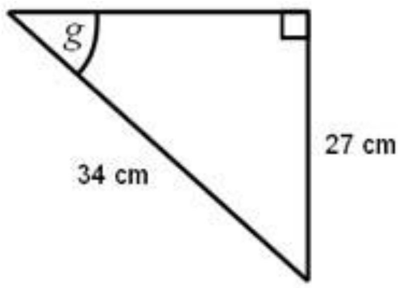


Figure 48

Check your answers on page 28.

Try this exercise:

Exercise 9

Give all answers in this exercise correct to the nearest degree.

45. Find angle z

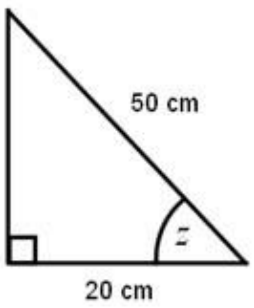


Figure 49

46. Find angle θ.

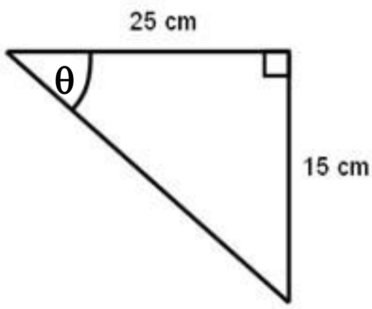


Figure 50

Check your answers on page 28.

Composite Figures

75. Trigonometry questions may involve more than one triangle. Each triangle needs to be considered independently when deciding which side is the hypotenuse, the opposite and the adjacent.

Try the following exercise:

Exercise 10

1.

Using the diagram shown:

a) find the length of AC.

b) calculate angle BCA to the nearest degree.

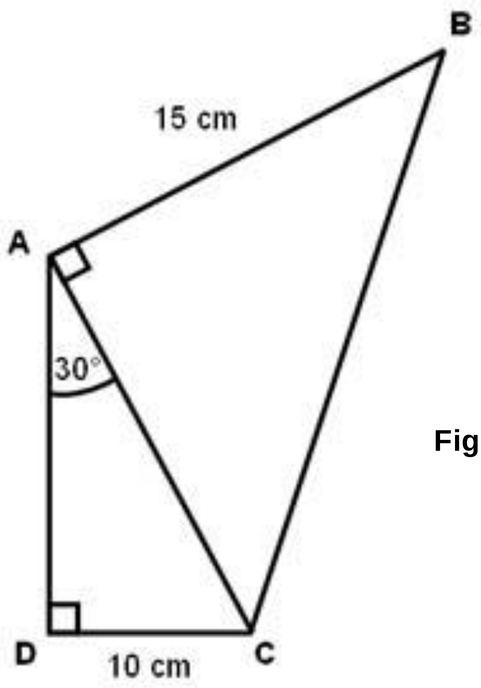


Figure 51

2. The diagram below is of a motorway system linking four towns marked A, B, C and D. The distance between A and B is 20 miles, angle DAB = 25° and angle DCB = 60°. Calculate correct to one decimal place:

a) the distance AD.

b) the distance BD.

c) the distance DC.

d) the total distance AC.

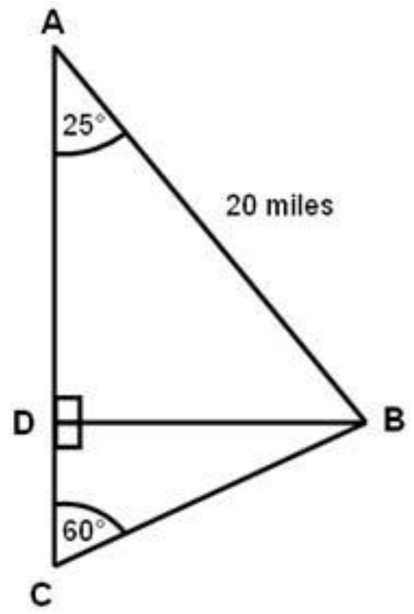


Figure 52

3. The diagram below shows two pylons marked AB and CD, where AB is of height 105 m and CD is of height 75 m. Angle CED is 35° and AED is a straight line. Find, correct to the nearest whole number:

   a) the length of CE.

   b) the size of angle AEB.

   c) the length of BE.

   d) the size of angle ECB.

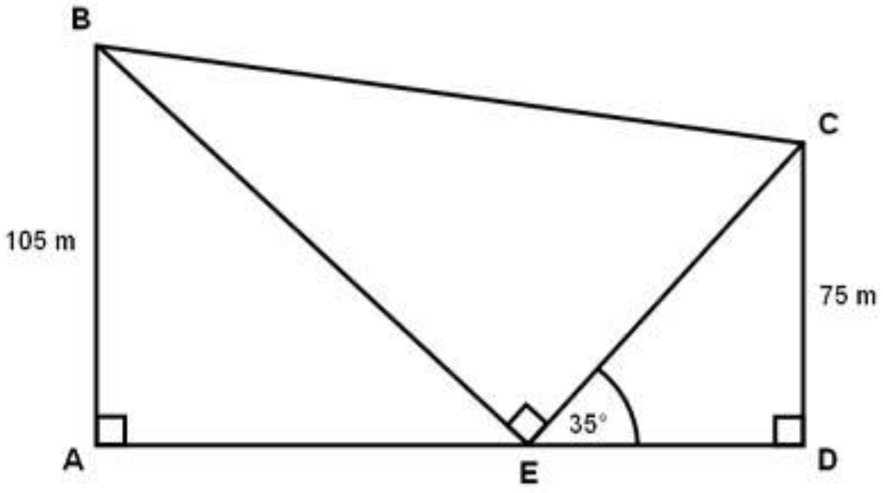


Figure 53

Answers on page 28

Answers

Exercise 8

41. 36°
42. 43°
43. 45°
44. 53°

Exercise 9

45. 66°
46. 31°

Exercise 10

1. a) 20 cm
   b) 37°
2. a) 18.1 miles
   b) 8.5 miles
   c) 4.9 miles
   d) 23.0 miles
3. a) 131 m
   b) 55°
   c) 128 m
   d) 44°

Curves Sketching and the Periodic Nature of the Sine and Cosine Function.

Sketch the sine and cosine wave forms over one complete cycle.
State the periodic properties of the sine and cosine.

Exercise 11

- Complete the table below which covers the range from 0° to 360°.
- Evaluate to two decimal places. (The table is in 15° intervals.)
- Select suitable scales to produce, on the graph paper given, a graph which fills as much of the paper as practicable.
- Using the line of best fit, the graph should be a smooth curve.
- Plot the sine and cosine graphs on the same axis.
- What is similar about the two graphs?
- What is different about them?

θ[°]	0	15	30	45	60	75	90	105	120	135	150	165	180
sinθ													
cosθ													
θ[°]	**195**	**210**	**225**	**240**	**255**	**270**	**285**	**300**	**315**	**330**	**345**	**360**	
sinθ													
cosθ													

For the sine function on your graph, note that over the range of angles, 0° to 360°, the waveform starts and finishes at the same value.

What would happen if you were to extend the table from 360° to 720° , what would the graph look like?

Cosine Function

For the cosine function on your graph, note that over the range of angles 0° to 360°, the waveform starts and finishes at the same value.

What would happen if you were to extend the table from 360° to 720°, what would the graph look like?

Comparing the sine and cosine waveforms it can be seen that there is a relationship between the two.
Because they are the same shape they are both said to be *sinusoidal* functions
In our example the cosine waveform can be said to be 90° ahead (leading) the sine wave.

A waveform can be said to be leading or lagging, depending on which waveform is chosen to be the reference.
The phase shift can be added or subtracted to the waveform, then for any angle θ :

$$\sin\theta = \cos(\theta - 90°) \quad \text{and} \quad \cos\theta = \sin(\theta + 90°)$$

This can be expressed in alternative form where the angles are not in degrees but in radians. We will not be using this unit of measurement on the maths course but you may come across it in other subjects

$$\sin\omega t = \cos\left(\omega t - \frac{\pi}{2}\right) \quad \text{and} \quad \cos\omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$$

Multiples of Sin and Cos functions

Consider the function:

$y = A\cos\theta$ where A is a constant

Exercise 12

Take the example when A = 2

Try sketching y = 2 sinθ for values of θ from 0° to 360°, use the table below.

θ[°]	0	15	30	45	60	75	90	105	120	135	150	165	180
2sinθ													
θ°	195	210	225	240	255	270	285	300	315	330	345	360	
2sinθ													

What, do you conclude, is the effect of constant A in the function?

Consider the function:

$$y = \cos(n\theta) \qquad \text{where n is a constant}$$

Exercise 13

Take the example when n = 2

Try sketching y = sin 2θ for values of θ from 0° to 360°, use the table below.

θ°	0	15	30	45	60	75	90	105	120	135	150	165	180
Sin2θ													
θ°	195	210	225	240	255	270	285	300	315	330	345	360	
Sin2θ													

What, do you conclude, is the effect of constant n in the function?

Thus, we have seen that any sinusoidal function can be represented by the formula:

$$y = A\sin(n\theta \pm \phi) \qquad \text{where } A, n \text{ and } \phi \text{ are constants}$$

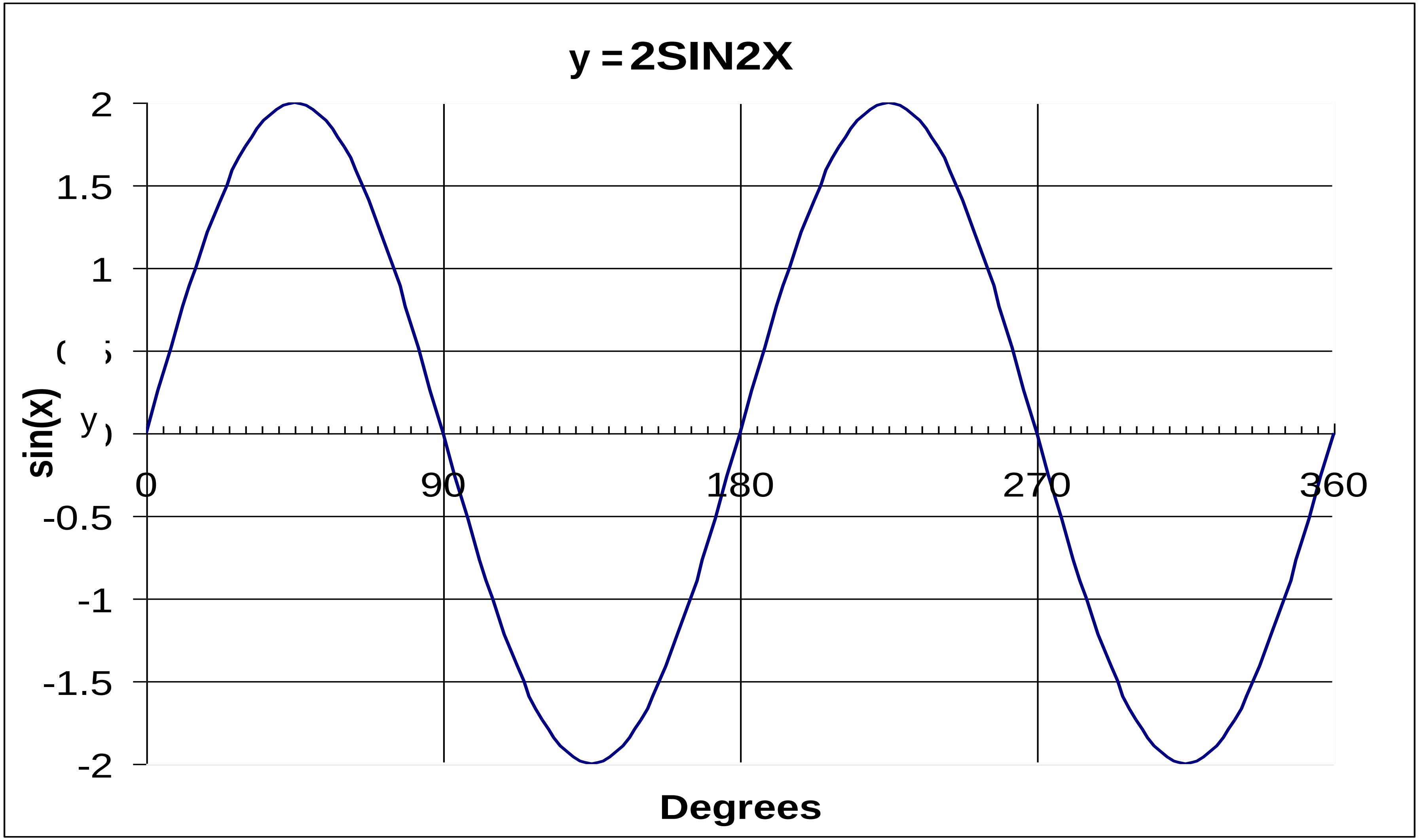
Exercise 14

State the effect of each of the constants A, n and ϕ in the sinusoidal function equation.

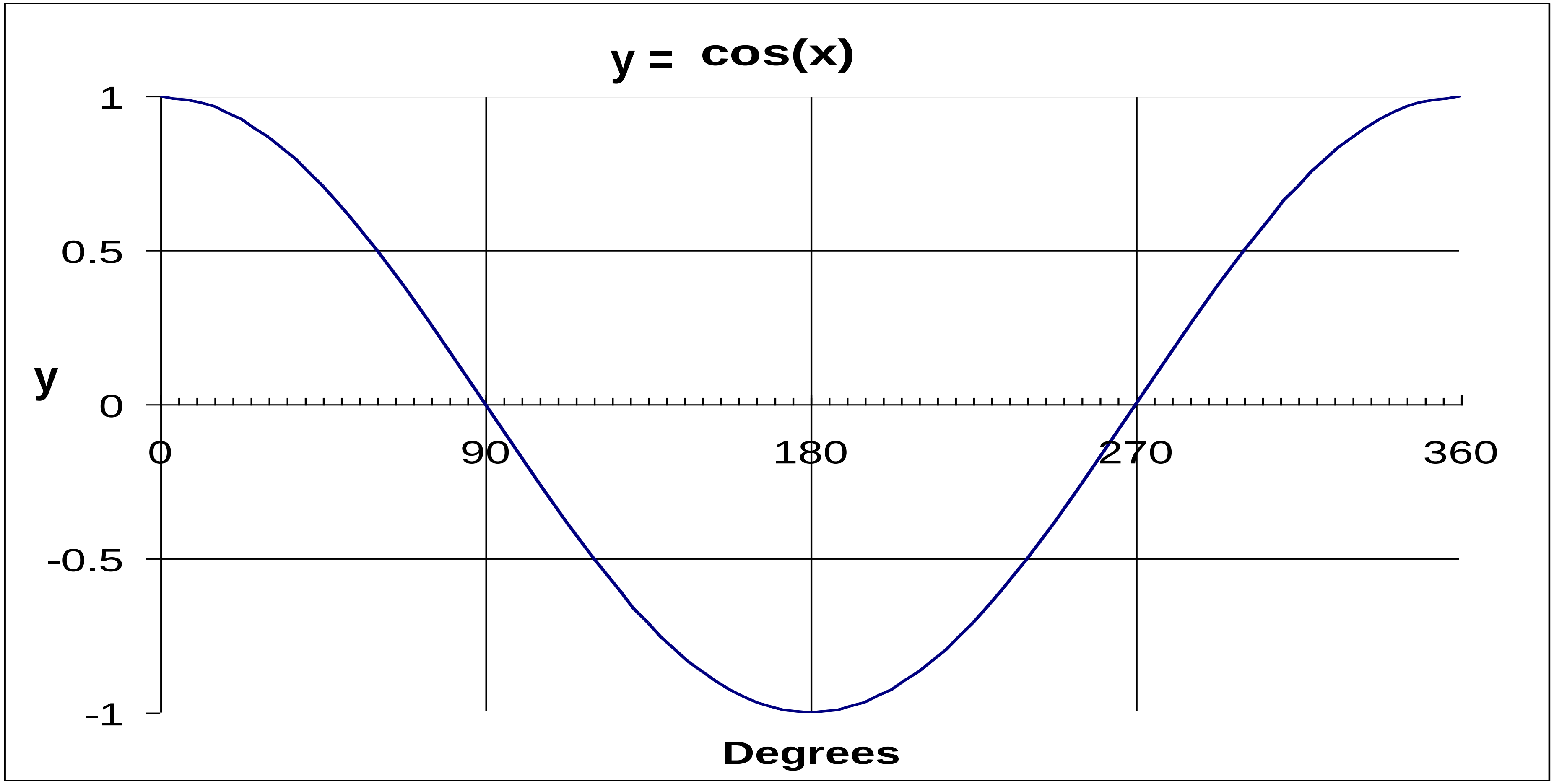
Exercise 15

Answer the questions on the following pages. These questions are similar to examination style questions.

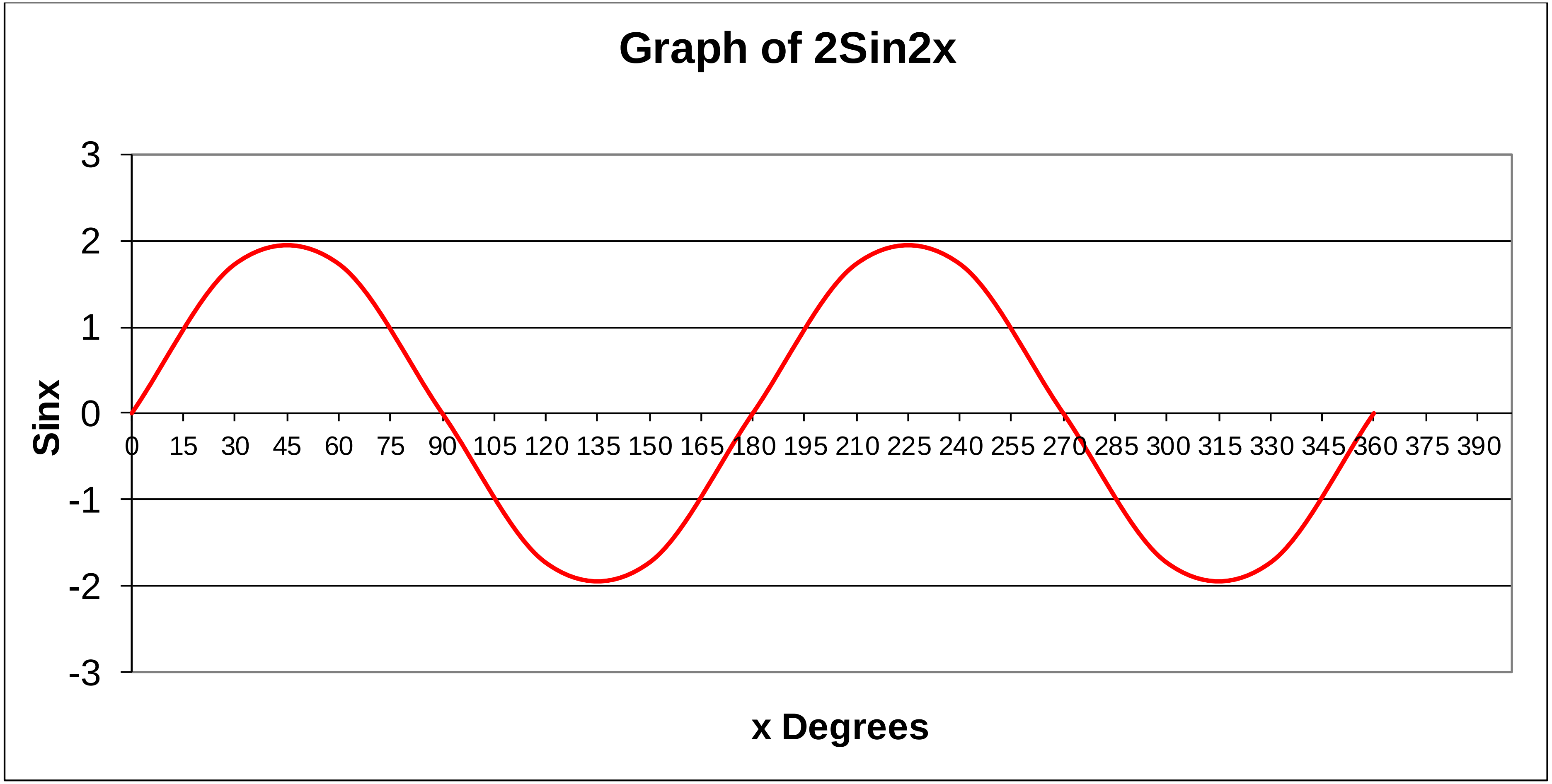
Check your answers on page 36



1. Using the graph above, at which angles will the sinusoidal waveform have at amplitude of - 0.5?
2. Using the graph above, at which angles will the sinusoidal waveform have at amplitude of 1.5?



3. Using the graph above, at which angles will the cosine waveform have at amplitude of - 0.5?
4. Using the graph above, at which angles will the cosine waveform have at amplitude of 0.5?



Graph of y = 2 sin 2x

5. Using the graph above, at which angles will 2 sin2x have a value of 1.72?
6. Using the graph above, what is the value of 2 sin2x at 165°
7. At what other angles on this graph will 2 sin2x have the same value as at 165°?

1.72

Exercise 16

1. Plot the graph of $y = 3\cos 2\theta$ over the range 0° to 360°. Find all the solutions when $y = 1.2$.

2. Plot the graph of $y = 2\sin 3\theta$ over the range 0° to 360°. Find all the solutions when $y = -1.6$

Periodic Properties of Trigonometric Waveforms

You will have noticed from the graphs in Figure 1 and 2 that the plotted waveforms repeat after a 360° cycle. This is the periodic property of the waveforms.

SUMMARY

- Both the SINE and COSINE waves repeat continuously over 360° cycles.

- COSINE waves can be described by the equation:

$$\cos\theta = \sin(\theta + 90°)$$

where θ is any angle.

- SINE waves can be expressed by the equation:

$$\sin\theta = \cos(\theta - 90°)$$

where θ is any angle.

Answers

Exercise 14

A = Amplitude
n = Frequency
ϕ = Phase

Exercise15

1. 97°, 173°, 277°, 353°
2. 24°, 66°, 204°, 246°
3. 120°, 240°
4. 60°,300°
5. 30°, 60°, 210°, 240°
6. -1
7. 105°, 285°, 345°

(answers given correct to nearest degree)

Exercise16

1. 33°, 47°, 213°, 327°
2. 78°, 102°, 198°, 222°, 318°, 342°

(answers given correct to nearest degree)

Notes