Defence School of
Aeronautical Engineering

Aerosystems Engineer & Management Training School

Academic Principles Organisation

Mathematics

BOOK 3
Indices & Powers of Ten

KEY LEARNING POINTS

KLP	Description
MA3.3	Define the 6 laws of indices
MA3.4	Convert numbers to standard form and preferred standard form
MA3.5	Simplify and Solve arithmetic expressions in standard form and preferred standard form

Index form and Indices

1. Often, we are required to deal with numbers which are multiples of themselves,

For example:

$$100 = 10 \times 10$$

$$36 = 6 \times 6$$

$$125 = 5 \times 5 \times 5$$

$$64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

2. The problem here is that it can be difficult to read long strings of numbers presented in this way. Because this situation arises quite often, a special notation has been developed, called the Index notation. For example:

10×10	is written as	10^2
6×6	is written as	6^2
$5 \times 5 \times 5$	is written as	5^3
$2 \times 2 \times 2 \times 2 \times 2 \times 2$	is written as	2^6

Or more generally:

$a \times a \times a \ldots$ (n times)	is written as	a^n

3. The number such as “a” is called the base, and the number “n” is called the index (plural indices), which defines the amount of bases that need to be multiplied together.

4. There is a big advantage in using index notation, because the indices are easy to manipulate using a few simple *Rules of Indices*

The Rules of Indices

Rule 1	$a^m \times a^n$	=	a^{m+n}
Rule 2	$\dfrac{a^m}{a^n}$	=	a^{m-n}
Rule 3	$(a^m)^n$	=	$a^{m \times n}$
Rule 4	a^0	=	1
Rule 5	$\dfrac{1}{a^n}$	=	a^{-n}
Rule 6	$\sqrt[n]{a}$	=	$a^{\frac{1}{n}}$

<u>Rule 1</u>

$$\mathbf{a^m \times a^n = a^{m+n}}$$

5. To see how rule 1 works, consider the following:

$$10^2 \times 10^3 = (10 \times 10) \times (10 \times 10 \times 10) = 10^5$$

6. Therefore, the rule is that when two numbers, that have the same base, are multiplied together, we simply add the indices together.

Rule 2

$$\frac{a^m}{a^n} = a^{m-n}$$

7. Rule 2 works the opposite way to rule 1.

$$\frac{10^4}{10^2} \quad = \quad \frac{10 \times 10 \times \cancel{10} \times \cancel{10}}{\cancel{10} \times \cancel{10}} \quad = \quad 10^2 \quad = \quad 10^{4-2}$$

$$\frac{3^5}{3^2} \quad = \quad \frac{3 \times 3 \times 3 \times \cancel{3} \times \cancel{3}}{\cancel{3} \times \cancel{3}} \quad = \quad 3^3 \quad = \quad 3^{5-2}$$

8. Therefore, if two numbers with a common base are divided the index the denominator is taken away from the index of the numerator.

Rule 3

$$\mathbf{(a^m)^n = a^{m \times n}}$$

9. Consider the following:

$$\begin{aligned} (10^2)^3 &= 10^2 \times 10^2 \times 10^2 \\ &= (10 \times 10) \times (10 \times 10) \times (10 \times 10) \\ &= 10^6 \end{aligned}$$

10. Therefore, if a number in index form is raised to a power, we multiply the two indices.

Rule 4

$$\mathbf{a^0 = 1}$$

11. Rule 4 implies that if any base is raised to the power 0 the answer will be 1. This can be illustrated by looking at the following.

$\frac{10^2}{10^2} = 1$ but from the second rule, $\frac{10^2}{10^2} = 10^{2-2} = 10^0$

12. So, $10^0 = 1$ and as this can be repeated for any base, the rule is that

$$(\text{anything})^0 = 1$$

Rule 5

$$a^{-n} = \frac{1}{a^n}$$

13. Again, to illustrate this rule consider the following:

$$\frac{100}{10000} = \frac{1}{100} = \frac{1}{10^2} \qquad \text{But from rule 2, } \frac{10^2}{10^4} = 10^{2-4} = 10^{-2}$$

So $\frac{1}{10^2} = 10^{-2}$ and this can be repeated for any base or index.

14. If we consider the rule in fractional form we notice a simple pattern occurring

$$\frac{a^{-2}}{1} = \frac{1}{a^2} \qquad \textit{and} \qquad \frac{a^2}{1} = \frac{1}{a^{-2}}$$

15. When the base value raised to a power transfers from being on the top as numerator to being on the bottom as a denominator (or vice versa) the power reverses its sign.

Exercise 1

Use the first 5 rules of indices to simplify the following:

1. $6^3 \times 6^5$
2. $7^6 \times 7^7$
3. $a^3 \times a^7$
4. $a^4 \times a^5 \times a^9$
5. $\frac{10^6}{10^4}$
6. $a^5 \div a^3$
7. $\frac{a^{10}}{b^7}$
8. $a^{19} \div a^{18}$
9. $(6^2)^5$
10. $(a^3)^3$
11. 1°
12. $(a^2 \times a^3)^3$
13. $\frac{1}{x^4}$
14. $\frac{1}{a^4 \times a^2}$

Check your answers on page 13

16. The number $\sqrt{x}$ is referred to as “the positive square root of x”
It is defined as the positive number, which when multiplied by itself, equals x.

Thus, $\sqrt{9} = 3$, $\sqrt{36} = 6$, etc.

17. There are higher orders of roots, for example $\sqrt[3]{x}$ is the number when multiplied by itself 3 times, equals x, ($\sqrt[4]{x}$ 4 times etc.)

18. More generally, $\sqrt[n]{x}$ is the nth root of x (i.e. the number which when multiplied by itself n times equals x)

19. The sign $\sqrt{\ }$ is called the radical and this method of presentation of the root is called the radical form of the root. As you will see there is another form of presentation, called the index form.

20. Most importantly, If there is no number n shown with the radical, *we must assume that it is 2*. It is not normal to use the number 2 with the radical sign to signify a square root.

Rule 6

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

21. To illustrate this rule, consider the following:

$10 \times 10 = 100$ $\therefore\ 10 = \sqrt{100}$

$\sqrt{100} \times \sqrt{100} = 100$ writing the answer with an index we get

$\sqrt{100} \times \sqrt{100} = 100^1$

22. Our aim is to remove the square root sign and replace it with an index.

We can remove the square root sign and substitute the index x in its place

This would give us:

$100^x \times 100^x = 100^1$ By rule 1 we must add the indices, so taking the indices only

$x + x = 1$

$\frac{1}{2} + \frac{1}{2} = 1$

Therefore x must equal $\frac{1}{2}$ to satisfy the equation

Substituting this back into the original equation

$$100^{\frac{1}{2}} \times 100^{\frac{1}{2}} = 100^1$$

Therefore

$$\sqrt{100} = 100^{\frac{1}{2}}$$

23. So, the reciprocal of the power of the root becomes the new index of the base. (If the base already has an index, then combine this with the new fractional index, using rule 3)

24. Some additional examples are shown below:

$$\sqrt[3]{10} = 10^{\frac{1}{3}}$$

$$\sqrt{10} = 10^{\frac{1}{2}}$$

$$\sqrt[4]{5} = 5^{\frac{1}{4}}$$

$$\sqrt[4]{x^3} = (x^3)^{\frac{1}{4}} = x^{3\times\frac{1}{4}} = x^{\frac{3}{4}}$$

$$\sqrt[3]{10^{\frac{1}{2}}} = \left(10^{\frac{1}{2}}\right)^{\frac{1}{3}} = 10^{\frac{1}{2}\times\frac{1}{3}} = 10^{\frac{1}{6}}$$

$$\sqrt{5^{\frac{1}{4}}} = \left(5^{\frac{1}{4}}\right)^{\frac{1}{2}} = 5^{\frac{1}{4}\times\frac{1}{2}} = 5^{\frac{1}{8}}$$

$$\left(\sqrt[3]{a^2}\right)^{\frac{1}{2}} = \left((a^2)^{\frac{1}{3}}\right)^{\frac{1}{2}} = a^{2\times\frac{1}{3}\times\frac{1}{2}} = a^{\frac{2}{6}} = a^{\frac{1}{3}}$$

25. To simplify index problems, it will often be necessary to combine several of the laws during one operation, take it one step at a time understanding which law you are using and how. Try the following examples:

Exercise 1 (continued)

Use all 6 rules of indices to simplify the following:

15. $\sqrt{c^3}$

16. $\sqrt[3]{a^6}$

17. $\sqrt[3]{b^{\frac{1}{3}}}$

18. $\sqrt[3]{a^2} \times \sqrt{a^3}$

19. $\left(\sqrt[4]{a^8} \times a\right)^{\frac{1}{3}}$

20. $\dfrac{a^4 \times \sqrt{a^4}}{\sqrt[3]{a^2}}$

21. $\dfrac{\sqrt[3]{x^{\frac{1}{3}}}}{x^{\frac{1}{3}} \times \sqrt[3]{x^2}}$

22. $\dfrac{\sqrt{a^{-3}}}{a^{-3} \times \sqrt[3]{a^2}}$

Check your answers on page 13

Standard Form

26. This is a very useful method of representing both very large and very small numbers, that makes them easier to handle. The method uses powers of ten, which were used in the last section on indices. This method is known as **Standard Form**.

> ***Standard form** is sometimes called scientific notation*

27. To express any number in standard form, it is written as a number between 1 and 10 and multiplied by a power of 10 to reflect the magnitude of the number.

For example:

The number 200 can be thought of as 2 x 100, or using index notation, 2×10^2

Likewise, $20 = 2 \times 10 = 2 \times 10^1$

28. Using this system, we can now write any decimal number quite neatly, as the following examples show:

$234 = 2.34 \times 10^2$

$23.4 = 2.34 \times 10^1$

$2.34 = 2.34 \times 10^0$

$0.234 = 2.34 \times 10^{-1}$

$0.0234 = 2.34 \times 10^{-2}$

Remember any base raised to the power of zero equals one

29. Numbers written in this way are said to be in **standard form**. As you can see there is only 1 digit to the left of the decimal point and that digit must only take the value of either 1 through to 9.

<u>Examples</u>

1. Express 3500 in standard form.

Answer:

3500 is the same as 3.5 x 1000, or using the power of 10:

3500 = $\mathbf{3.5 \times 10^3}$ in standard form.

2. Express 0.004 in standard form.

Answer:

0.004 is the same as $4 \times \frac{1}{1000}$, or using the power of 10:

0.004 = $\mathbf{4 \times 10^{-3}}$ in standard form.

3. The planet Mercury is 58000000 kilometres from the Sun.

Express this distance in standard form.

Answer:

58000000 is the same as 5.8 x 10 000 000, and since:

10 000 000 = 10^7 Then:

58000000 = $\mathbf{5.8 \times 10^7}$ in standard form.

30. A quick method for converting numbers to standard form is to think of any number, say n, as being n x 10^0. Then, move the decimal point until the number is a decimal between 1 and 10

31. Each move of the decimal point must be accompanied by a change in the index of the 10 to preserve the value of the number, following the rule:

Move the decimal point to the left, increase the power of 10

Move the decimal point to the right, decrease the power of 10

e.g.

623 000 = 623 000 x 10^0 = 6.23 x 10^5

0.000 002 67 = 0.000 002 67 x 10^0 = 2.67 x 10^{-6}

Exercise 2

1. Express the following in standard form:

   a. 4600 b. 357 000 c. 0.00023

2. Light travels at 186 000 miles per second, express this value in standard form.

3. The top of Ben Nevis is 1343 metres above sea level, what is this height in standard form?

Check your answers on page 13

Arithmetic Using Standard form.

32. Expressing numbers in standard form also allows us to carry out basic arithmetic on very large and/or very small numbers, without having to deal with lots of zeros. For example, if you wanted to find the result of:

$550\,000 \times 0.000\,02$

33. If we did this calculation using long multiplication, it would be difficult to know where to put the decimal point; even some calculators may have trouble with it. But if we convert each number to standard form, the calculation becomes easier to handle. i.e. because

$550\,000\,000 = 5.5 \times 10^8$ and $0.000\,02 = 2 \times 10^{-5}$

So, we can say that:

$550\,000\,000 \times 0.000\,02 = 5.5 \times 10^8 \times 2 \times 10^{-5}$

This could be written as:

$(5.5 \times 2) \times (10^8 \times 10^{-5})$

Now:

$5.5 \times 2 = 11$ and $10^8 \times 10^{-5} = 10^3$

Remember the rules of indices

So:

$5.5 \times 2 \times 10^8 \times 10^{-5} = 11 \times 10^3$

Hence:

$550\,000\,000 \times 0.000\,02 = 11 \times 10^3 = 1.1 \times 10^4$

Exercise 3

Simplify the following leaving your answer in standard form:

Try to do these without using a calculator

1. $\dfrac{68\,000}{340}$

2. $400\,000 \times 0.0185$

3. $225\,000 \times 400\,000 \div 1500\,000$

Check your answers on page 13

Engineering Form (*Preferred Standard Form*)

34. There is another type of standard form known as Engineering Form, (sometimes called Preferred Standard Form), where the power of 10 is restricted to be a multiple of 3. This is very useful for engineering purposes because it is easy to convert from the power of 10 to the appropriate SI prefix (G, M, k, m, μ, n, p etc.) This means that the number accompanying the power of 10 can now take any value between 1 and <1000. Most scientific calculators have a function to convert answers to Engineering Form.

For example:

58000000 = 5.8×10^7 in standard form.

Or = 58×10^6 in engineering form.

0.000036 = 3.6×10^{-5} in standard form.

Or = 36×10^{-6} in engineering form.

<u>Exercise 4</u>

1. Express the following in engineering form:

   a. 4600 b. 357 000 c. 0.00023

2. Light travels at 186 000 miles per second, express this value in engineering form.

3. The top of Ben Nevis is 1343 metres above sea level, what is this height in engineering form?

Check your answers on page 13

Answers

Exercise 1

1. 6^8
2. 7^{13}
3. a^{10}
4. a^{18}
5. 10^2
6. a^2
7. $\dfrac{a^{10}}{b^7}$
8. a
9. 6^{10}
10. a^9
11. 1
12. a^{15}
13. x^{-4}
14. a^{-6}
15. $c^{\frac{3}{2}}$ *or* $c^{1.5}$
16. a^2
17. $b^{\frac{1}{9}}$
18. $a^{\frac{13}{6}}$ *or* $a^{2.166...}$
19. a
20. $a^{\frac{16}{3}}$ *or* $a^{5.33...}$
21. $x^{-\frac{8}{9}}$
22. $a^{\frac{5}{6}}$

Exercise 2

1.a. 4.6×10^3
1.b. 3.57×10^5
1.c. 2.3×10^{-4}

2. 1.86×10^5
3. 1.343×10^3

Exercise 3

1. $6.8 \times 10^4 \div 3.4 \times 10^2$
$= 2 \times 10^2$

2. $4 \times 10^5 \times 1.85 \times 10^{-2}$
$= 7.4 \times 10^3$

3. $2.25 \times 10^5 \times 4 \times 10^5 \div 1.5 \times 10^6$
$= 6 \times 10^4$

Exercise 4

1.a. 4.6×10^3

1.b. 357×10^3

1.c. 230×10^{-6}

2. 186×10^3

3. 1.343×10^3