Defence School of
Aeronautical Engineering

Aerosystems Engineer & Management Training School

Academic Principles Organisation

MATHEMATICS

BOOK 10

Graphs & Simultaneous Equations

GRAPHS AND SIMULTANEOUS EQUATIONS

KEY LEARNING POINTS

KLP	Description
MA4.1	Define and use coordinate systems
MA4.2	Draw a fully labelled graph from Cartesian coordinates
MA4.3	Interpolate linear graphs to determine x and y coordinates
MA4.4	Extrapolate linear graphs to determine x and y coordinates
MA4.5	Determine the values *y*, *m*, *x*, *and c* from linear graphs and formulate the equation of the line
MA4.6	Solve using the graphical method a pair of simultaneous equations
MA4.7	Draw graphical representations of sine and cosine waveforms

GRAPHS AND SIMULTANEOUS EQUATIONS

1. Service publications and reports make extensive use of pictorial illustrations to assist the reader in gaining a greater depth of understanding of their content. The most common form of illustration that is used is the **graph**.

AXES

2. The first step in drawing a graph is to draw two lines at right angles to each other, as shown below. These lines are called the **axes of reference**. The vertical axis is usually called the **y-axis** and the horizontal one is the **x-axis**. The point where the two axes cross is called the **origin** of the graph.

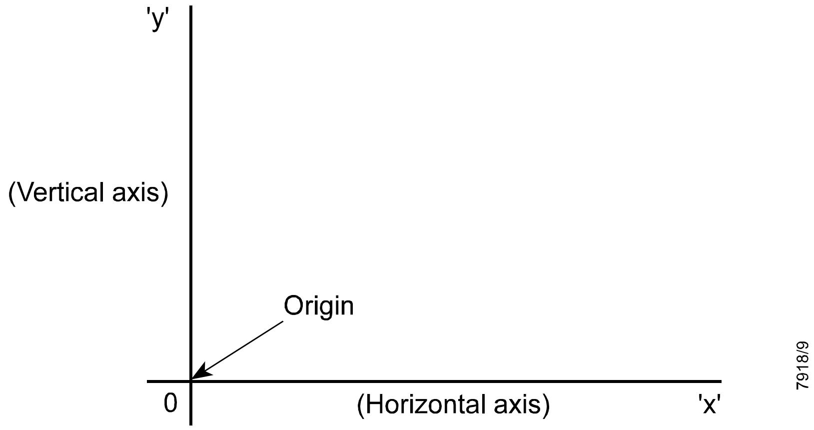


Figure 1

CO-ORDINATES (CARTESIAN/RECTANGULAR)

3. We use an ordered pair of **co-ordinates** to identify points on a graph (the system is the same as that used in map reading to locate or identify a particular point on the ground). In the graph below values of **y** are to be plotted against values of **x**. The point **P** has been plotted so that x = 4 and y = 6. The values of 4 and 6 are the **co-ordinates** of **P** and are written in brackets separated by a comma i.e. (4, 6). The **x** value is always written **first** and the **y** value last.

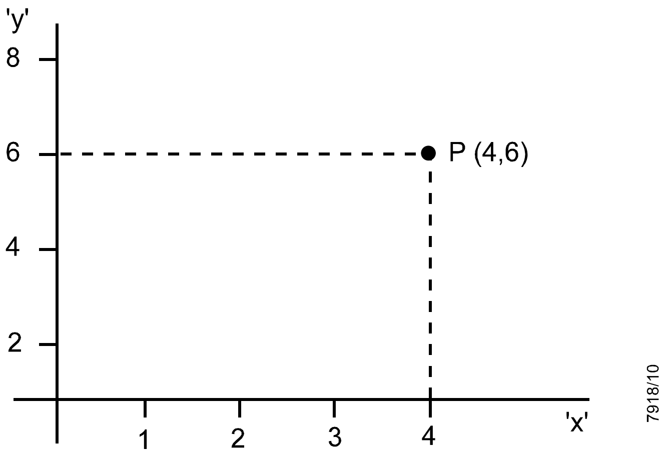


Figure 2

Example:

Give the co-ordinates of the point **B** and **C** in the graph below.

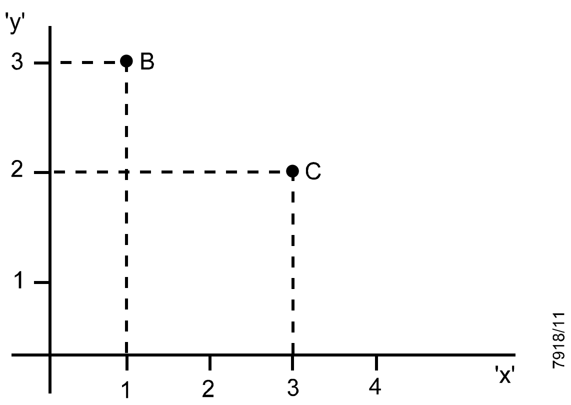


Figure 3

Solution:

a. **B** is 1 along the x-axis and 3 up the y-axis.

$\therefore$ **B** is at (1, 3).

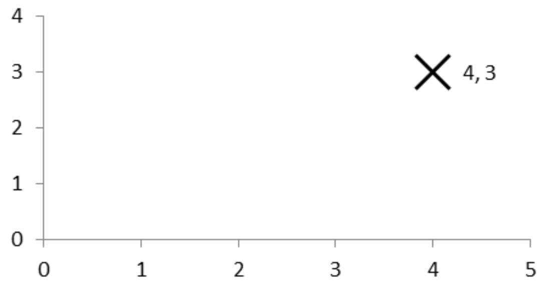
b. **C** is 3 along the x-axis and 2 up the axis

$\therefore$ **C** is at (3,2)

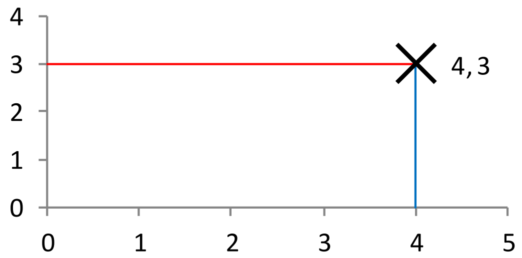
CO-ORDINATES (POLAR)

4. There is another method that can be used to identify a specific point on a graph this being the polar method. The polar method utilises the length of a line and its elevation to identify a specific point. It will be used for drawing vectors in the science unit of the course, phasors in the electrics unit of the course (1236) and drift calculations in the flight navigation phase of the course (1236).

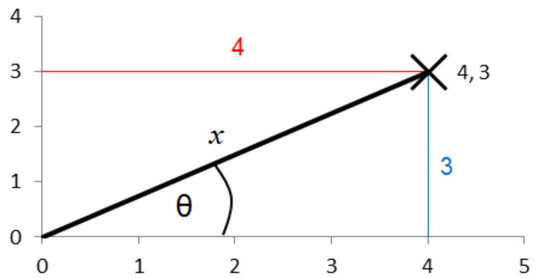
5. If we consider a point on a graph defined by its Cartesian co-ordinate, we could also consider that point in its polar form. The graph below defines the Cartesian point (4,3)



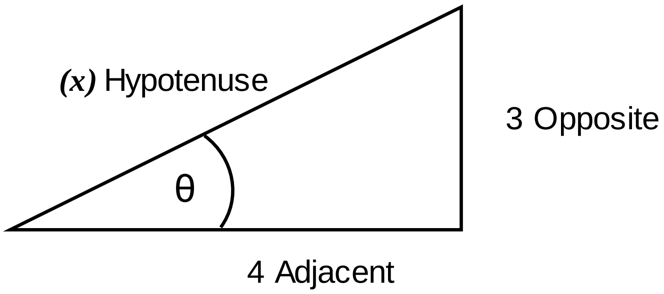
If we put the lines in place to define this point we get.



The polar co-ordinate for this point would be its length and elevation from the origin (0,0) of the graph. If a triangle is constructed about the point 4,3



We would get the triangle below:-



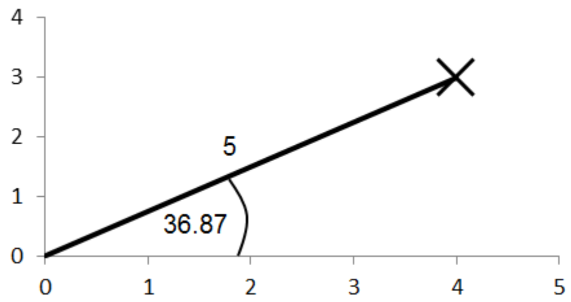
6. Using trigonometrical ratios the magnitude of θ and x can be ascertained.

$$\theta = \tan^{-1}\frac{Opposite}{Adjacent} \qquad \theta = \tan^{-1}\frac{3}{4} \qquad \theta = 36.86°$$

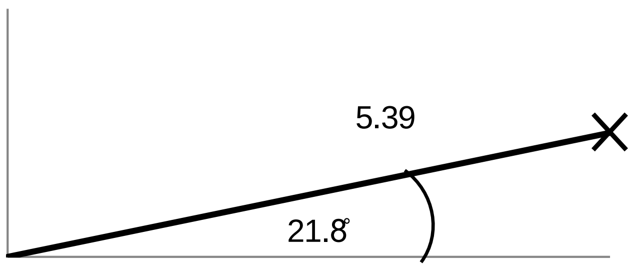
$$\sin\theta = \frac{Opposite}{Hypotenuse} \qquad Hypotenuse\ (x) = \frac{Opposite}{\sin\theta} \qquad Hypotenuse(x) = \frac{3}{\sin 36.86}$$

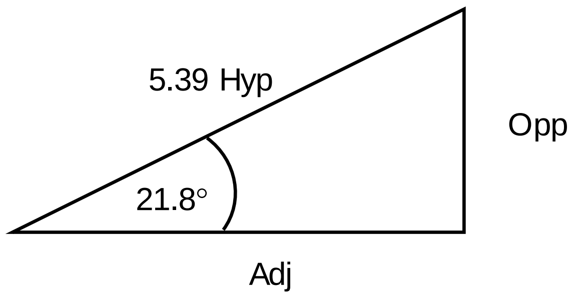
$$Hypotenuse\ (x) = 5$$

7. With this information we can define the polar coordinate as 5 @ 36.87° and show its graphical representation as below:



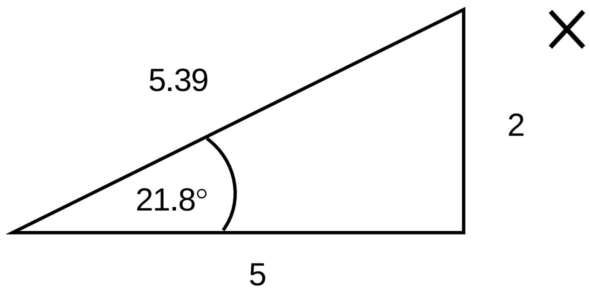
8. Similarly, if we were presented with a polar coordinate we may require to change it back into its corresponding Cartesian coordinate this process is the reverse. 5.39 @ 21.8° is shown below using trigonometric ratios we can revert it back to a Cartesian coordinate





$\sin\theta = \dfrac{Opp}{Hyp}$ $\qquad Hyp \times \sin\theta = Opp$ $\qquad 5.39 \times \sin 21.8 = Opp$ $\qquad 2 = Opp$

$\cos\theta = \dfrac{Adj}{Hyp}$ $\qquad Hyp \times \cos\theta = Opp$ $\qquad 5.39 \times \cos 21.8 = Adj$ $\qquad 5 = Adj$



9. Translating the trigonometry gives a Cartesian coordinate of (5,2)

EXERCISE 1

Change the following Cartesian co-ordinates in to polar coordinates

1. 2,3
2. 5,7
3. 8,3
4. 2.5,9.5
5. 0.25,0.75

Change the following polar co-ordinates in to Cartesian coordinates

6. 7.81@50.19°
7. 8.86@16.39°
8. 7.62@66.8°
9. 5@20°
10. 7.5@35°

Answers on page 21

10. It is essential you understand the principle behind changing coordinates however most modern scientific calculators have a function to change them for you. Ask your lecturer to show you the calculator method.

GRAPHS OF EXPERIMENTAL DATA

11. Every graph shows a relationship between two sets of numbers. Graphs are commonly used to illustrate the results of an experiment or test. When drawing graphs of experimental data, the quantity that is being varied by the operator is called the **independent variable** and is usually put on the horizontal axis. The quantity that is measured is called the **response** or **dependent variable** and is usually put on the vertical axis.

Example:

12. The following table gives the amount of stretch measured in a control cable run under a range of increasing loads. Plot a graph to represent this information.

Load (kg)	10	20	30	40	50
Stretch (mm)	3.5	7	10.5	14	17.5

Solution:

13. Here the stretch that occurs in the cable is dependent upon the load that is being applied, so the load will appear on the horizontal axis as it is the independent variable.

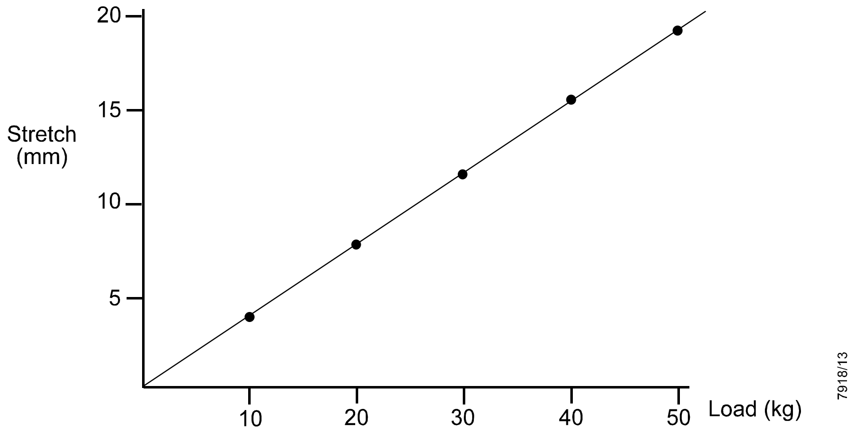


Figure 5

NEGATIVE VALUES

14. When experimental data includes negative values, the straightforward L axes arrangement can no longer be used. The four possible arrangements commonly used are as follow:

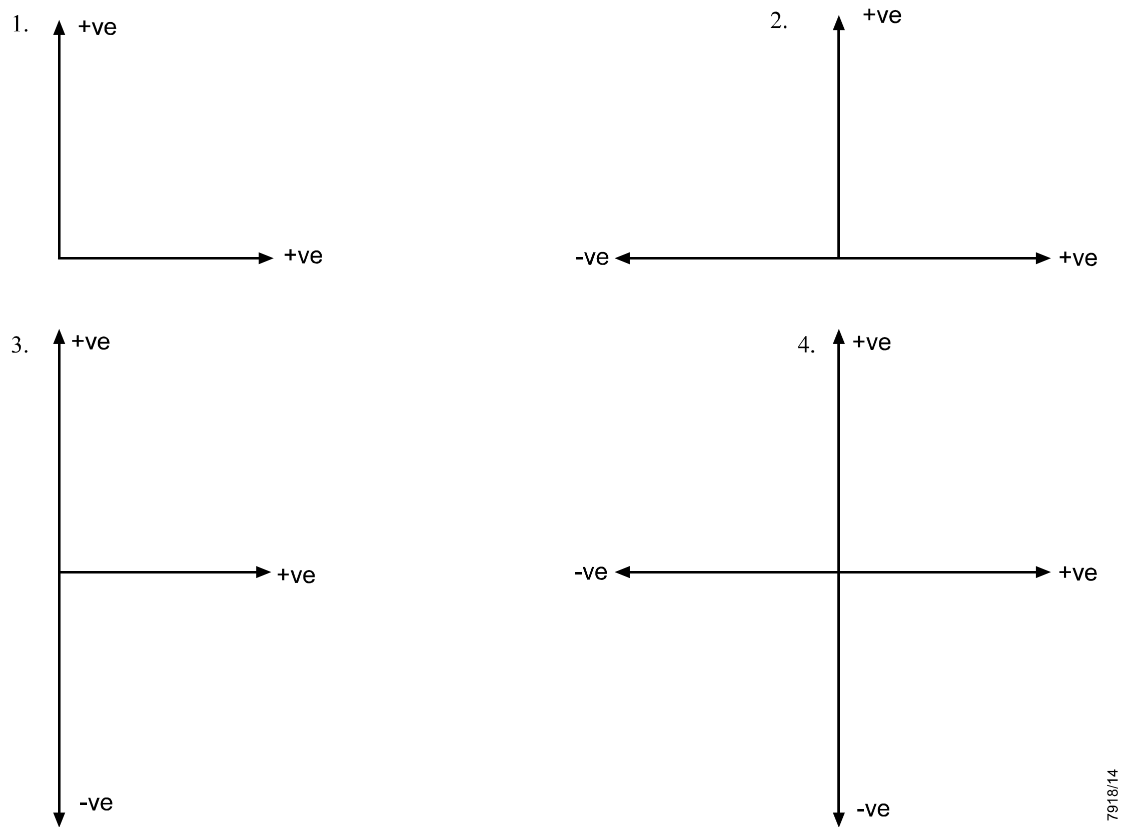


Figure 6

EXERCISE 2

The values below give the relationship between force applied and resultant acceleration, in an experiment carried out on a trolley rolling along an inclined plane.

Force in Newtons	-3	-2	-1	0	1	2	3
Acceleration in m/s^2	-5	-2	1	4	7	10	13

Display these results in a suitable graph.

LINE OF BEST FIT

15. All the graphs that you will plot will be **line graphs**, where the line is either straight or a smooth curve. However, when we use a line graph to plot actual **results** that have been obtained in a trial, it is **very** rare for the points to be exactly on a straight line or curve, and we have to draw the **line of best fit** that appears to fit the results. The difference between an individual data points and the line is called **the residual,** the line of best fit is technically the line which minimises the square of the residuals, this can be found by using a computer programme (for example excel) or by eye. Thus, the line of best fit will tend to average out any errors the results.

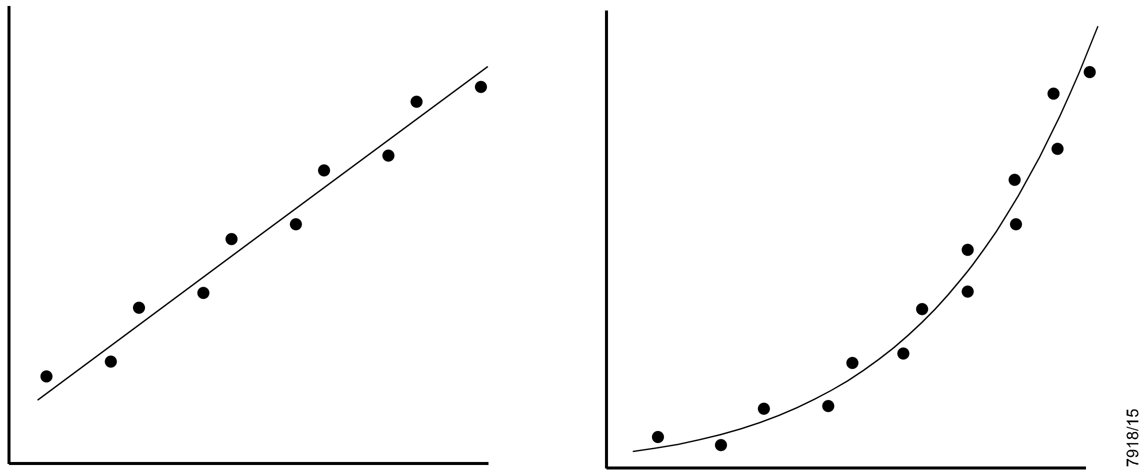


Figure 7

16. It is more difficult to plot the line of best fit for a curve than for a straight line, for this reason, variables which follow a square law are sometimes plotted on logarithmic graph paper, which has the effect of reducing the curve to a straight line.

INTERPOLATION

17. When readings are taken from a graph from between the plotted points making use of the best line, this is known as **interpolation** of the graph. When the best line method is used, all subsequent readings must come from the line, even if they coincide with the observed values. This has the effect of averaging out errors in the observations

EXTRAPOLATION

18. When readings are calculated from points not plotted on the graph, this is known as **extrapolation**.

19. When plotting a graph, you must always try to ensure that the following points are covered.

a. Draw the graph as large as the paper will conveniently allow.

b. Label each axis clearly.

c. Indicate the scale of each axis.

EXERCISE 3

1. On graph paper plot the points **A** (3, 1), **B** (7, 5), **C** (1, 3) and **D** (5,7). Join up the sides to make the rectangle **ACDB**. What are the co-ordinates of the join of the diagonals **AD** and **BC**?

2. The table below shows the comparative values of two temperature scales, Celsius (°C) and Fahrenheit (°F). Draw a line graph to represent this data and determine the following:

   a. The equivalent temperature (°F) at 6°C.

   b. The equivalent temperature (°F) at 18°C.

°C	0	3	9	12	15
°F	32	37	48	54	59

3. The following table shows how the resistance of a certain wire varies with temperature:

Temperature: T (°C)	0	10	30	50	70	90
Resistance: R (ohms)	150	156	168	180	192	204

Plot a graph of R against T and from the graph determine:

   a. The temperature when the resistance is 190 ohms.

   b. The resistance when the temperature is 100°C.

4. The following table gives the results of an experiment in which the current was measured for various values of applied potential difference:

Potential Difference PD (volts)	1	2	3	5	6	7
Current 1 (mA)	6.3	8	9.3	12.6	14	16

Plot the results and determine from the graph:

   a. The current when the PD is 4 volts

   b. The PD required to give a current of 8.5 mA.

5. The following table shows the relationship between the increase in internal diameter of a washer and temperature.

Increase in mm	0	.0016	.0032	.0048	.0064	.0080	.0096	.0112	.0128
Temperature in °C	-40	-30	-20	-10	0	10	20	30	40

Plot a graph to show these results and calculate the:

   a. increase at 15°C

   b. temperature required for a .0072 mm increase.

Answers on page 21

GRAPHS OF FUNCTIONS

20. A function is a mapping of one variable to another, following a **rule** (or **law**), for example

$$y = 5x + 3$$

21. Often, we are required to draw graphs of functions, the only functions will concern ourselves with on this course are functions where the variables (x and y for example) Have order 1. Such functions produce straight lines when plotted.

For example:

The function $y = 5x + 3$ will produce a straight line when plotted, because both x and y have order (index) = 1

The function $s = 3t^2 + 2$ will ***not*** produce a straight line when plotted, because the variable t has order 2 (i.e. not equal to 1)

The function $y = \frac{2}{x} + 3$ will ***not*** produce a straight line when plotted, because the variable x has order (index) = -1 (i.e. not equal to 1)

22. To produce a graph of given function, for example $y = 5x + 3$, start by compiling a table of values, to cover the range required, say from x = -3 to +3

x	-3	0	3
$5x$	-15	0	15
+3	+3	+3	+3
y	-12	+3	18

23. Because we know that the graph will be a straight line, it is theoretically possible to get away with calculating and plotting only 2 points, but it is good practice to plot at least 3 points, to act as a check on our arithmetic.

24. Now on the graph paper, construct suitable axes, plot the points and join up the points with a straight line. If the points do not line up, then something is wrong and we should check our calculations.

EXERCISE 4

1. Choosing suitable axes, draw the graph of the function $y = 3x$, covering values of x from -3 to +3

2. On the same axes, include the graph of the function $y = -x + 4$

3. For which values of x and y are both equations satisfied simultaneously?

Answer on page 21

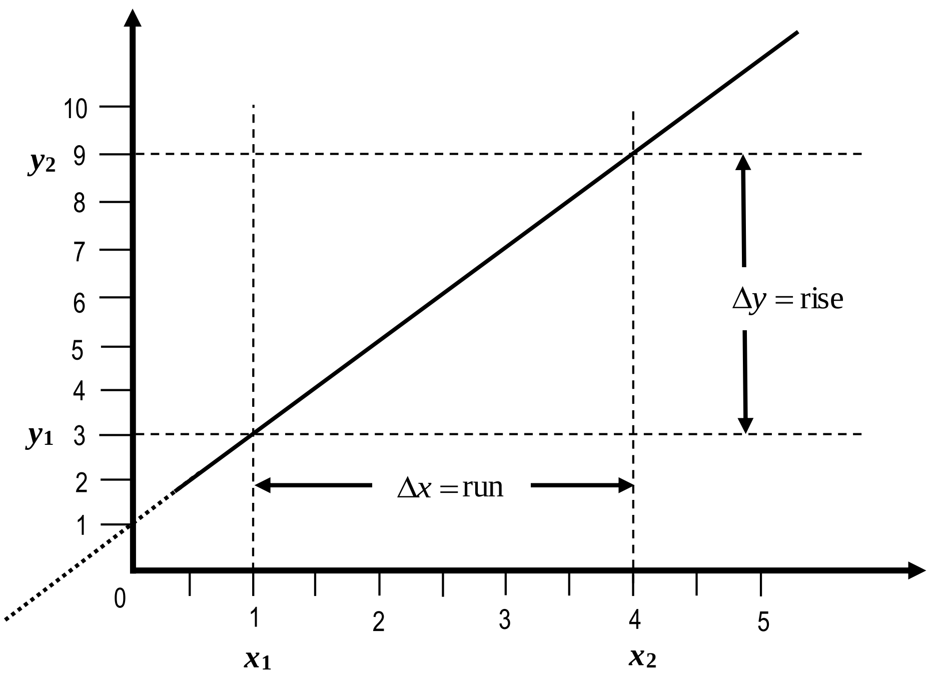
FUNCTIONS FROM GRAPHS

25. If we have a graph of a function, it is possible to use it to find the equation of the rule. (Sometimes referred to as the **Law** of the graph).

26. Providing that the graph is a straight line, (in this course we will only consider straight line graphs) the equation will be of the form $y = mx + c$, where y and x are the two variables and m and c are two constants. Our task is to find the constants m and c from the graph.

27. The constant m is the slope, or **gradient** of the line. This is the increase in the value of y divided by the corresponding increase in the value of x, (i.e. **rise** over **run**) between any two points on the line.

Consider the following graph:



The Greek letter Δ (delta) *is read as "change in" some books use the lower case* δ *meaning "small change in"*

Figure 8

28. First, choose any two convenient points on the graph. Here we have chosen the points (1,3) and (4,9). Then the gradient is:

$$\text{gradient} = m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-3}{4-1} = \frac{6}{3} = 2$$

Thus, we have determined that the value of m in the formula is 2

Note that the rise ($y_2 - y_1$) and run ($x_2 - x_1$) is measured in terms of the scale on the axes and not the number of squares on the graph paper.

Note also that if the graph slopes **upwards** (i.e. bottom left to top right, as in figure 8) then the value of m is positive, but if the slope is **downwards**, then m is negative. (If the line is horizontal, then $m = 0$ and if vertical, m is indeterminate)

29. The constant c is the value of y when $x = 0$, this is the y **intercept** of the line and can simply be read from the graph. In the case of the example shown in Fig 8, the value of the y intercept is 1 and so $c = 1$

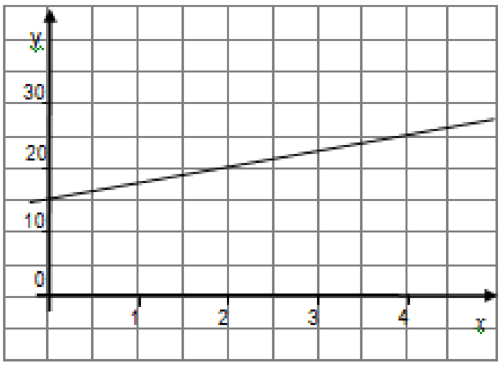
Thus, the full equation of the line on the graph shown in fig 8 is:

$$y = 2x + 1$$

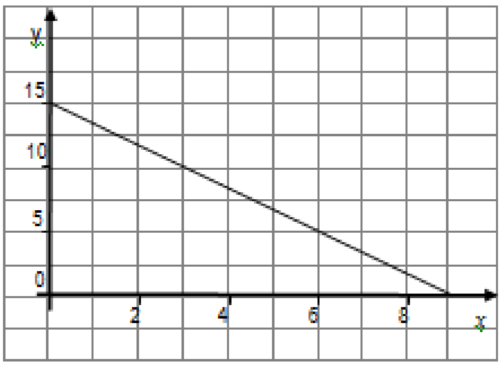
EXERCISE 5

For the four straight line graphs shown below, find the equations of the lines.

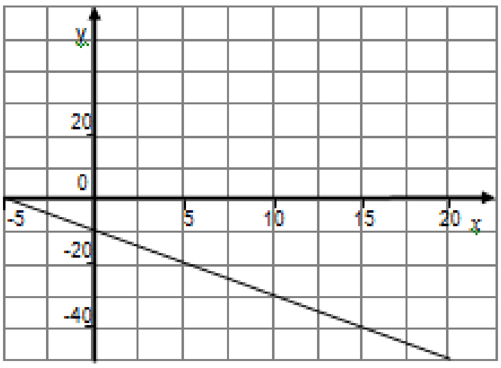
Graph 1



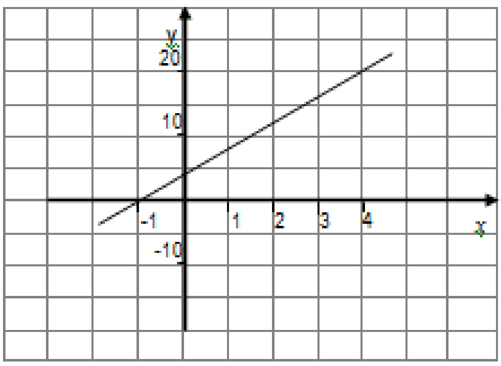
Graph 2



Graph 3



Graph 4



Answers on page 21

EQUATIONS WITHOUT GRAPHS

30. Provided we are told that the graph of a function is a straight line, it is possible to find the equation of the line, given two points on the line, without drawing the graph.

31. For example, consider the straight line which passes through the Cartesian points (1,5) and (4,14). We know, because the line is straight, that the equation will be of the form $y = mx + c$

Using the formula for the gradient:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{14 - 5}{4 - 1} = \frac{9}{3} = 3$$

So far, we know that the equation is:

$$y = 3x + c$$

32. To find the value of c, substitute a value for x and a value for y that we know is true, i.e. from one of the points, say (1,5).

$$5 = 3 \times 1 + c$$

33. This gives an equation with just one unknown which can be solved in the usual way:

$$5 = 3 \times 1 + c$$

$$\therefore 5 - 3 \times 1 = c$$
$$\therefore c = 2$$

34. So, the equation of the straight line which passes through the points (1,5) and (4,14) is:

$$y = 3x + 2$$

EXERCISE 6

Find the equations of the straight lines which pass through the following pairs of Cartesian points.

1. (0,4) and (2,2)
2. (-1, -1) and (5,3)
3. (3,5) and (5,3)
4. (1,4) and (6,9)
5. (7,1) and (½, -½)

Answers on page 22

NOTES

SIMULTANEOUS EQUATIONS

35. A function such as $y = 0.8x + 1$ can be thought of as an equation with two unknowns, x and y. As such, there are an infinite number of possible values (solutions) for x and y, all of which lie on the line drawn in figure 8. (page 8)

36. If, however we consider a second equation, which is **linearly independent** from (i.e. not a multiple of) the first, then there is just one unique solution for the situation of both equations being true **at the same time** (i.e. simultaneously). This unique solution is the point on the graph where the two lines, representing the two equations cross.

37. Recall exercise 3 where we plotted graphs of the functions

$y = -x + 4$ and on the same axis

$y = 3x$

38. The point on the axis where the two lines cross is the point (1,3), thus x = 1 and y = 3 is the only solution that fits both equations simultaneously. This may be verified by substitution in the original equations.

EXERCISE 7

39. Use the graphical method to find the solution to the following pair of simultaneous equations:

$$y = 6x - 12$$
$$y = -2x + 18$$

Answer on page 22

40. Whilst it is always possible to solve simultaneous equations graphically, the process is time consuming and sometimes inexact. It may be more convenient to solve the equations using algebra to eliminate one of the variables. For example,

If we know that $y = -x + 4$

And also, that $y = 3x$

41. Because the two ys are the same number, then we can say that

$3x = -x + 4$

42. This is now a simple equation with just one unknown and can be solved using normal methods, i.e.

43. The problem is now half-solved; we still have to find the value of y that fits the equations. This can be done by substituting $x = 1$ into any of the two equations to find the corresponding value of y.

44. The second equation looks more convenient

$y = 3 \times 1 = 3$

45. We now have the full solution, so we can say:

The equations

$$y = -x + 4$$
$$y = 3x$$

46. Are simultaneously satisfied, when $x = 1$ and $y = 3$

This result can be verified by substitution in the second equation.

EXERCISE 8

Find simultaneous solutions to the following pairs of equations:

1. $$y = -2x - 3$$
$$y = x + 5$$

2. $$y = 6x - 12$$
$$y = 18 - 2x$$

3. $$y = \frac{3x}{2} + 5$$
$$y = -4x + 14$$

4. $$y = 2x + 1$$
$$y = 10 - x$$

5. $$y = 2x - 3$$
$$y = \frac{5}{2}x - 8$$

6. $$y = 9 - x$$
$$y = x + 3$$

Answers on page 22

IMPLICIT PRESENTATION OF SIMULTANEOUS EQUATIONS

47. An equation such as $y = 6x - 12$ is called an **explicit** equation, because the two unknowns, x and y are presented at either side of the equals sign. Often, because of the way that simultaneous equation type problems arise in practical situations, the equations themselves are presented in what is called the **implicit** form, with both unknowns at the same side of the equals sign. Commonly, when the equations are presented implicitly, because there is no reference to a graphical axis, the variables are signified by letters like A and B instead of x and y.

48. As an example, consider the following question, which reflects a real-life situation:

Q. *A bag contains 7 bolts and 4 nuts and weighs 610g, another bag contains 6 bolts and 3 nuts and weighs 510g. Write a pair of simultaneous equations reflecting this information.*

Answer:
Let the weight of a bolt be A g and the weight of a nut be B g. Then:

7A + 4B = 610
And 6A + 3B = 510 are the two simultaneous equations.

49. To solve the equations, it is always possible to convert them to the explicit form and proceed as before, but this can result in some awkward numbers.

A more common approach is to eliminate one of the variables as follows:

Call the equations (i) and (ii) as shown

7A + 4B = 610 …………………. (i)
6A + 3B = 510 …………………. (ii)

50. Firstly, decide which variable to eliminate, say B. Our task is to get the coefficient (multiplying number) of B to be the same in both equations. For this to happen we must multiply equation (i) by 3 throughout and equation (ii) by 4 throughout.

21A + 12B = 1830 ……………. (i)
24A + 12B = 2040 ……………. (ii)

51. Now if we subtract the LHS of equation (i) from the LHS of (ii) and the Corresponding RHS of (i) from (ii) the variable B will be eliminated as follows:

24A – 21A + 12B – 12B = 2040 – 1830

So 3A = 210

And A = 70

52. Note that at this stage, if the variables to be eliminated have *opposite* signs, then the two equations must be *added*, rather than subtracted.

53. Now that we have found the value of A we can substitute this in one of the original equations and solve for B, as follows:

Taking equation (ii)

$$6(70) + 3B = 510$$

$$\therefore \quad 420 + 3B = 510$$

$$\therefore \quad 3B = 510 - 420$$

$$\text{And} \quad B = 30$$

So, the bolts weigh 70g and the nuts weigh 30g.
(This can be verified by substitution)

54. Consider the following further example:

Solve the simultaneous equations

$3A + 2B = 12$ …………………. (i)
$4A - \ B = 5$ …………………. (ii)

We have decided to eliminate B, so multiplying (ii) by 2

$3A + 2B = 12$ …………………. (i)
$8A - 2B = 10$ …………………. (ii)

Because the B's have opposite signs in the 2 equations, we must add (i) to (ii)

$3A + 8A + 2B – 2B = 10 + 12$

therefore $\quad 11A = 22$

and $\quad A = 2$

substituting for $A = 2$ in (ii)

$4(2) - \ B = 5$

$8 – 5 = B = 3$

So, the solution is $A = 2$ and $B = 3$

This can be verified by substitution.

EXERCISE 9

Find simultaneous solutions to the following pairs of equations:

1. $3A + 2B = 12$
   $2A + 3B = 13$

2. $A + B = 9$
   $A - B = 3$

3. $A - B = 5$
   $A + 2B = -3$

4. $m + n = 12$
   $m - n = 3$

5. $p - q = 18$
   $2p - q = 6$

6. $A + 3B = 7$
   $2A - 2B = 6$

7. $4a - 6b = -2.5$
   $7a - 5b = -0.25$

8. $3a + b = 7$
   $a = 11 - b$

9. $2a + b = 9$
   $a - b = 0$

10. $7I_1 + 6I_2 = 115$
    $8I_1 + 5I_2 = 111$

11. Little Johnny Smith goes to the school tuck shop on two separate days in one week. On Monday he buys 3 gobstoppers and 2 sherbet fountains and this cost him 37p. On Tuesday, he has only 15p, but this is just enough to buy 1 gobstopper and 1 sherbet fountain. How much is a gobstopper and how much is a sherbet fountain.

12. It is not possible to find a determinate solution for a and b in the simultaneous equations

$4a + b = 14$ and
$8a + 2b = 28$ (Try and see)

Why is this?

Answers on page 22

ANSWERS TO EXERCISES

Exercise 1

1. 3.61@56.31°
2. 8.60@54.46°
3. 8.54@20.56°
4. 9.82@75.26°
5. 0.79@71.57°
6. 5,6
7. 8.5,2.5
8. 3,7
9. 4.7,1.71
10. 6.14,4.3

Exercise 3

1. (4,4)

2. a. 42.8
   b. 64.4

3. a. 67
   b. 210

3. a. 11
   b. 2.4

5. a. 0.0088
   b. 5°C

Exercise 4

3. (1,3)

Exercise 5

1. $y = 2\frac{1}{2}x + 15$

2. $y = -1\frac{2}{3}x + 15$

3. $y = -2x - 10$

4. $y = 4x + 4$

Exercise 6

1. $y = -x + 4$
2. $y = \frac{2}{3}x - \frac{1}{3}$
3. $y = -x + 8$
4. $y = x + 3$
5. $y = \frac{3}{13}x - \frac{8}{13}$

Exercise 7

$x = 3.75$

$y = 10.5$

Exercise 8

1. $x = -2\frac{2}{3}, y = 2\frac{1}{3}$
2. $x = 3\frac{3}{4}, y = 10\frac{1}{2}$
3. $x = 1.64, y = 7.45$
4. $x = 3, y = 7$
5. $x = 10, y = 17$
6. $x = 3, y = 6$

Exercise 9

1.	A = 2,	B = 3
2.	A = 6,	B = 3
3.	A = $2\frac{1}{3}$,	B = $-2\frac{2}{3}$
4.	m = 7.5,	n = 4.5
5.	p = -12,	q = -30
6.	A = 4,	B = 1
7.	a = 0.5,	B = 0.75
8.	a = -2,	B = 13
9.	a = 3,	B = 3
10.	I_1 = 7,	I_2 = 11

11. Gobstoppers are 7p and Sherbets are 8p

12. The two equations are not linearly independent (one is a multiple of the other) see para. 18.

Notes