Defence School of
Aeronautical Engineering

Aerosystems Engineer & Management
Training School

Academic Principles Organisation

Mathematics

BOOK 5
Algebraic Operations

WARNING

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KEY LEARNING POINTS

KLP	Description
MA3.2	Factorise 1st order algebraic expressions
MA3.6	Define Coefficient, variable, term, expression, equation and identity
MA3.7	Simplify algebraic expressions containing fractions, brackets, powers and roots.
MA3.8	Solve linear equations
MA3.10	Solve second degree (quadratic) equations using factorising and the formula method
MA3.11	Transpose a formula to change the subject.

NOTES:

ALGEBRA

ALPHANUMERIC CHARACTERS

1. Often, we are required to deal with certain quantities or measurements that are achieved through addition, subtraction, multiplication and division of numbers. Frequently this is accomplished without using definite numbers. For example, if we had to find the area of a rectangular plate we would multiply the its length by its breadth, this could be expressed in the form:

$$\text{Area of plate} = \text{length} \times \text{breadth}$$

2. This is a general statement, which applies to all rectangles as well as our plate. If symbols or letters are used a much shorter statement is obtained.

Using	A = the area of the rectangle
	L = the length of the rectangle
and	B = the breadth of the rectangle

The statement becomes:

$$A = L \times B$$

3. In this shortened statement we express the rule for finding the area of any rectangle: it is a general rule, and is called a *formula* or *algebraic equation*.

LETTERS AND SYMBOLS

4. When letters are used to represent numbers, any suitable choice may be made. In the previous equation A, L and B were employed but any letter could have been used if they were more suitable. As well as letters symbols can represent numbers; for instance, you may already be aware that the Greek letter ‘pi’ which has the symbol π is used to represent the constant number 3.141592654… or approximately 3.142 in many operations concerning circles. By common usage, however, certain letters or symbols are selected for a specific purpose.

ALGEBRAIC TERMS, EXPRESSIONS, EQUATIONS AND IDENTITIES.

5. An algebraic term is a simple algebraic statement that does not include a plus or minus sign. If the statement does include a plus or minus it becomes an algebraic expression, thus an expression can be made up of 2 or more terms. When an expression is made to equal another expression, or a number, it becomes an equation or in the case of something that is always equal regardless of the value of any of the variables, an identity.

Thus, for example:

$6mn^2$	is an algebraic term
$6mn^2+3pq$	is an algebraic expression
$2m+3=7$	is an equation (because it is only true when m = 2)
$m \times m = m^2$	is an identity (because it is true for all values of m)

Sometimes the symbol ≡ (identically equal to) is used instead of the normal equals sign for an identity, in situations where the meaning is not clear.

SUBSCRIPTS AND SUPERSCRIPTS USED WITH LETTERS AND SYMBOLS

6. On occasions letters may have a subscript after the letter, R_3 for example has a subscript 3. This is used when the same letter represents different numbers which are related to the same thing, the total resistance R_T may equal the sum of three resistances.

So, $R_T = R_1 + R_2 + R_3$.

7. Spoken as: "R, T equals R, 1 plus R, 2 plus R, 3." This must not be confused with superscript; R^3, which means R, cubed ($R \times R \times R$). Any letters that are in upper or lower case must stay as they are because upper case 'T' may represent one number while lower case 't' may represent a completely different one.

8. So 'T_1' is different to 'T', which is different to T^2, which is also different to 't'.

Example

Write an algebraic equation for the velocity of an aircraft in terms of the time and the displacement.

So, Velocity of aircraft equals displacement divided by time taken.
or Velocity of aircraft = displacement ÷ time taken

Equation $V = D \div T$ or $\left(V = \frac{D}{T}\right)$

Where: V = velocity
D = displacement
T = time

Note: It is important to indicate what the symbol or letter represents

EXERCISE 1

1. Classify the following as a term, expression, equation or identity:

   a. $x + x = 2x$

   b. $½mv^2$

   c. $u^2 + 2as$

   d. $2x+3=17$

2. Write an algebraic equation for the following:

   a. The total number of points scored in a rugby match equals the sum of the points for tries and conversions.

   b. The torque loading applied to a bolt is the product of the force applied and the length over which it is applied.

   c. The total price of an item equals the product of the net cost and the tax rate, plus the net cost.

LIKE TERMS

9. Like terms are numerical multiples of the same quantity.

   Thus 14x, 5x, and –2x, are like terms of 'x'

   or $7ab^2$, $-2ab^2$, and $+5ab^2$, are like terms of 'ab^2'

The numbers 7, -2 and 5 are called the numerical coefficient of the term and are the numbers that each term is multiplied by.

So $7ab^2 = 7\times ab^2$. And $-2ab^2 = -2 \times ab^2$.

If a letter or symbol is squared or cubed, x^2 or x^3 then compared to x they are not like terms.

So $2x$, $3x^2$, and $-6x^3$, are not like terms because x does not equal x^2 or x^3.

10. There are three laws which help us to simplify algebraic equations. These are:

   a. **Commutative Law**

$$a + b = b + a$$

$$a \times b = b \times a$$

This law does **not** apply to subtraction and division a – b does not = b – a.

   b. **Distributive Law**

$$(a + b)\, c = ac + bc$$

   c. **Associative Law**

$$(ab)c = a(bc)$$

$$(a + b) + c = a + (b + c)$$

ADDITION AND SUBTRACTION OF ALGEBRAIC TERMS

11. An expression consisting of a number of like terms can be reduced to a single term by adding and or subtracting the numerical coefficients. This is called simplifying and expression.

$7x - 5x + 3x = (7 - 5 + 3)\ x = 5x$

$3a^2 + 4a^2 = (3 + 4)\ a^2 = 7a^2$

$-7r - 4r = (-7 - 4)\ r = -11r$

$b^3 - 6b^3 = (1 - 6)\ b^3 = -5b^3$

12. Only like terms can be added or subtracted to simplify the expression. Thus $7a + 3b – 2s$ is an expression that cannot be simplified as all the terms are unlike. If an expression is made up of mixed terms some of which are alike the like terms can be added or subtracted to simplify the expression.

$5x + 7y - 4z - 3x + 8z – 2y + 3z$

$= (5 – 3)\ x + (7 – 2)\ y + (- 4 + 8 + 3)\ z$

$= 2x + 5y + 7z$

When an expression is simplified the order in which the terms are set out if they are different are to be alphabetical and or in descending orders of powers.

So $3b – 4c + 6a$ set out alphabetically would be.

$6a + 3b – 4c$

and $3x^2 + 4x^3 + 5x$ set out with descending order of powers would be,

$4x^3 + 3x^2 + 5x$

Example:

Simplify the following expression to its simplest form.

$3a – 4b + 7a^3 + b + 2a$

$= (3 + 2)\ a + (- 4 + 1)\ b + 7a^3$

$= 7a^3 + 5a – 3b$

EXERCISE 2

Write the following expressions in their simplest form:

a. $15b + 11b$

b. $15x – 3x + 7x$

c. $9a^2 – 4a^2 + 6a^2 + a^2$

d. $4xy + 3xy – 2xy – xy$

e. $4a – 5b + a + 6b + b$

f. $5x + 2y + 3z + x – y – 2z + 2x + 3z$

g. $3x^2 + 3xy^2 + 4xy^2 – x^2 – 2xy^2$

h. $3p + 4p^2 – p^3 + 5p^2 + 6p^3$

i. $x^2 – x^3 + 5x + 3x^2 + 4x^3$

j. $1.2a^3 – 3.4a^2 + 2.6a + 3.7a^2 + 3.6a – 2.8$

MULTIPLICATION SIGN

13. When using symbols multiplication signs are nearly always omitted, so $b \times h$ becomes bh. This obviously cannot be used with numbers, as you could not write 5×4 as 54. The multiplication sign can also be omitted when a symbol and number are being multiplied together, i.e. $3 \times p$ is written 3p.

14. The system may be extended to three or more quantities, hence $3 \times \pi \times R \times T$ is written $3\pi RT$. The order in which the symbols are written is unimportant to the value, as they are all numbers, which are being multiplied together. So, it could be written $\pi 3TR$ or $RT\pi 3$. It is usual; however, to write the number before symbols and symbols before letters, and the letters in alphabetical order, so it would normally be written $3\pi RT$.

DIVISION SIGN

15. The division sign $\div$ is used very rarely in algebra, as it is much more convenient to write the division as a fraction

$A \div B$ is written as a fraction $\frac{A}{B}$

and $\frac{Lm}{2\pi r}$ means $(Lm) \div (2\pi r)$

MULTIPLICATION AND DIVISION OF ALGEBRAIC QUANTITIES

16. The rules are precisely the same as those used with directed numbers, like signs are positive when multiplied or divided and unlike signs are negative.

Example

Multiplying

$(+a)(+b) = +(ab) = ab$

$(-a)(-b) = +(ab) = ab$

$(+a)(-b) = -(ab) = -ab$

$(-a)(+b) = -(ab) = -ab$

$(+5p)(+6q) = 5 \times 6 \times p \times q = 30pq$

$(+5p)(-2q) = -(5p)(2q) = -10pq$

Dividing

$(+a) \div (+b) = \frac{+a}{+b} = +\frac{a}{b} = \frac{a}{b}$

$(-a) \div (-b) = \frac{-a}{-b} = +\frac{a}{b} = \frac{a}{b}$

$(+a) \div (-b) = \frac{+a}{-b} = -\frac{a}{b}$

$(-a) \div (+b) = \frac{-a}{+b} = -\frac{a}{b}$

$-12a \div +3b = \frac{-12a}{+3b} = -\frac{4a}{b}$

SEQUENCE OF MIXED OPERATIONS ON ALGEBRAIC QUANTITIES

17. Since algebraic quantities contain symbols or letters, which represent numbers, the sequence of operation is exactly the same as with numbers. Remember the BODMAS rules (from work in book 1, Basic numeracy), which gives the initial letters of the correct sequence, i.e., Brackets, Order, Divide, Multiply, Add, and Subtract.

Thus

$$\begin{aligned} 2y^2 + (15y^4 - 3y^4) \div 4y^2 - y^2 &= 2y^2 + 12y^4 \div 4y^2 - y^2 \\ &= 2y^2 + 3y^2 - y^2 \\ &= 5y^2 - y^2 \\ &= 4y^2 \end{aligned}$$

EXERCISE 3

Simplify the following

1. $5x + 3x \times 4$
2. $4r – 4r \div 2$
3. $5y + ¼$ of $16y – 3y$
4. $5b – (20b^3 + 4b^3) \div 12b^2$
5. $(16ab^2 + 4ab^2) \div 2ab$

BRACKETS

18. Brackets are used for convenience in grouping terms together. When removing brackets each term within the bracket is multiplied by the quantity outside the bracket:

$4(a + b) = 4 \times a + 4 \times b = 4a + 4b$

$5(2y + 4z) = 5 \times 2y + 5 \times 4z = 10y + 20z$

$3(p – 2q) = 3 \times p - 3 \times 2q = 3p - 6q$

$a(p – q) = a \times p – a \times q = ap – aq$

$4x(3a + 4b) = 4x \times 3a + 4x \times 4b = 12ax + 16bx$

$3x(x + 4y) = 3x \times x + 3x \times 4y = 3x^2 + 12xy$

19. When a bracket has a minus sign outside it, the signs of all the terms inside the bracket are changed when the bracket is removed. The reasoning for this rule is the same as the previous section when multiplying positives and negatives:

$-2(3a – 4b) = (-2) \times 3a + (-2) \times (-4b) = -6a + 8b$

$-(p + q) = -p – q$

$-(a – b) = -a + b$

$-m(p + 3q) = -mp - 3mq$

20. When simplifying expressions containing brackets first multiply out the brackets and then add the like terms together:

$(4a + 4b) – (2a + 3b) = 4a + 4b – 2a – 3b = 2a + b$

$3(4x – 3y) – 2(3x + 5y) = 12x – 9y – 6x – 10y = 6x – 19y$

EXERCISE 4

Remove the brackets of the following:

1. $5(x + 3)$
2. $3(2a – 4b)$
3. $\frac{1}{2}(2a – 4b)$
4. $7(b – 5m)$
5. $-(x – 3y)$
6. $–(-5p – 6q)$
7. $-4(3a - 6)$
8. $3r(r + p)$
9. $–5x(3x + 4y)$
10. $a(x + y -z)$
11. $4a(ab + 4bc)$
12. $3x^2(2x^2 + 2y^2 – 3x)$
13. $–7t(2t^2 – t +5)$
14. $3abc(a + b - c)$

Remove the brackets and simplify the following:

15. $3(x + 1) + 2(x + 4)$
16. $3(x + 4) – (2x + 5)$
17. $5(2x – y) – 3(x + 2y)$
18. $– (4x + 4y – 3z) – 3(3x – 4y)$
19. $3(a – b) – 2(2a – 3b) + 4(a – 3b)$
20. $3x(x^2 + 7x – 1) – 2x(x^2 + 3) – 3(x^2 + 5)$

21. When multiplying brackets with more than 1 term, multiply every term in the first bracket individually with every term in the subsequent brackets and add the products. For example:

$$(a+b)(c+d) = ac + ad + bc + bd$$

$(a - 3)(a - 5) = a^2 – 3a – 5a + 15$ which simplifies to:

$$= a^2 – 8a + 15$$

$$(a + 2)^2 = (a + 2)(a + 2)$$

$$= a^2 + 4a + 4$$

Brackets with 2 terms are called binomial expressions, with 3 terms, trinomial etc.

Exercise 5

Find the product of the following binomial expressions:

1. $(m + n)(p - q)$
2. $(x + 4)(x + 5)$
3. $(2x + 4)(3x + 2)$
4. $(x - 2)(x + 7)$
5. $(2x + 6)(x - 1)$
6. $(3v + 2u)(2v - 3u)$
7. $(x + y)^2$
8. $(x^2 + y)(x^2 - y)$

NOTES

FACTORISATION OF QUADRATIC EXPRESSIONS

22. Factorisation is the reverse of multiplication of brackets.
Recall the method for finding the product of 2 binomial expressions is:

$$(a + b)(c + d) = ac + ad + bc + bd$$

Applying this method to binomials of the form (x + C), where C is a constant, gives expressions of the type shown below:

For example:

$$(x + 2)(x + 3) = x^2 + 2x + 3x + 6$$

And by adding the like terms:

$$= x^2 + 5x + 6$$

Expressions of this type are called Quadratic expressions; they contain a squared term, a linear term and a number.
It is possible, if given a quadratic expression, to decompose it into its constituent binomial factors. This process is called binomial factorisation and will be useful if we wish to go on to solve equations involving quadratics.

The general form of a quadratic expression is:

$ax^2 + bx + c$ where a, b and c are constants

this factorises to:

$$(x + \alpha)(x + \beta)$$

Our task is to find numbers α and β from a, b and c in the original expression.
Initially we will only consider expressions for which a = 1, then it can be seen that:

	If 1 = a
The product	$\alpha\beta = c$
And the sum	$\alpha + \beta = b$

so, the solution may be found as follows:

Example

Factorise: $x^2 + 5x + 4$

Here $\alpha\beta = 4$, so possible numbers for α and β are the factors of 4, i.e. (2,2) or (4,1) but we know that the sum of the factors, $\alpha + \beta = 5$ so 4 and 1 are the numbers required.

Thus: $x^2 + 5x + 4 = (x + 1)(x + 4)$

(As a check, if we multiply out the factors, we will obtain the original expression)

Example

Factorise, $x^2 - 7x - 8$

First list all the possible factors of -8

-8
4, -2
-4, 2
8, -1
-8, 1

Choose the 2 factors which add up to -7

i.e. -8 ,1

so, answer is:

$(x - 8)(x + 1)$ Multiply this out to check

Example

Factorise, $x^2 - 9x + 20$

Listing all possible factors of +20

Select the 2 which add up to -9

i.e. -5, -4

+20
20, 1
-20, -1
10, 2
-10, -2
5, 4
-5, -4

so, answer is:

$(x - 5)(x - 4)$ Multiply this out to check

Exercise 6

Factorise the following quadratic expressions:

1. $x^2 + 7x + 12$
2. $x^2 + 10x + 24$
3. $x^2 - 18x + 32$
4. $x^2 - 14x + 48$
5. $x^2 + 5x - 24$
6. $x^2 + 4x - 12$
7. $x^2 - 4x - 32$
8. $x^2 - 9$

PARTIAL FACTORISATION

23. Consider the following expression:

$$ac + ad + bc + bd$$

It can be seen that there is no common factor in all 4 terms, however "a" is common to the first 2 terms, and "b" is common in terms 3 and 4. So the expression can be partially factorised as follows:

$$ac + ad + bc + bd \quad = \quad a(c + d) + b(c + d)$$

If it can be arranged that the 2 brackets are the same, as in this case where they are both (c + d), then the partially common factors themselves may be bracketed, as follows:

$$a(c + d) + b(c + d) \quad = (a + b)(c + d)$$

The expression is now fully factorised. By multiplying out the 2 brackets it can be seen that the factorised version is equivalent to the original expression.

Example

Factorise as far as possible the expression:

$$12 + 3b + 4a + ab$$

There are no factors common to all 4 terms, so considering the first 2 terms only; the number 3 is common, i.e.

$$= 3(4 + b) + 4a + ab$$

If there is a factor common to terms 3 and 4 this too may be extracted,

$$= 3(4 + b) + a(4 + b)$$

Because the two brackets are the same, we can now group the partially common factors.

$= (3 + a)(4 + b)$ The expression is now fully factorised

The final step is dependent on arranging the 2 brackets to be the same; this may not always be possible, sometimes it may be necessary to re-arrange the original expression to achieve this.

Exercise 7

Factorise, as far as possible:

1. ac + 2bc – ad – 2 bd

2. xy + 2x + y + 2

3. 12 – 6a + 2b – ab

SOLVING QUADRATIC EQUATIONS

24. We know from our knowledge of expressions that the quadratic expressions in Exercise 6 cannot be solved because they are not equations. If an expression was formed into an equation a solution for the quadratic equation can be found. If we consider the quadratic equation:

$$x^2 + 4x + 3 = 0$$

When we factorise the above quadratic equation, we get:

$$(x+1)(x+3) = 0$$

We know for this equation to be true either one of the bracketed terms must have a value of zero to make the solution of the equation zero i.e.

$(x+1)\times 0 = 0$ $\quad (x+3) = 0$
or .
$0 \times (x+3) = 0$ $\quad (x+1) = 0$

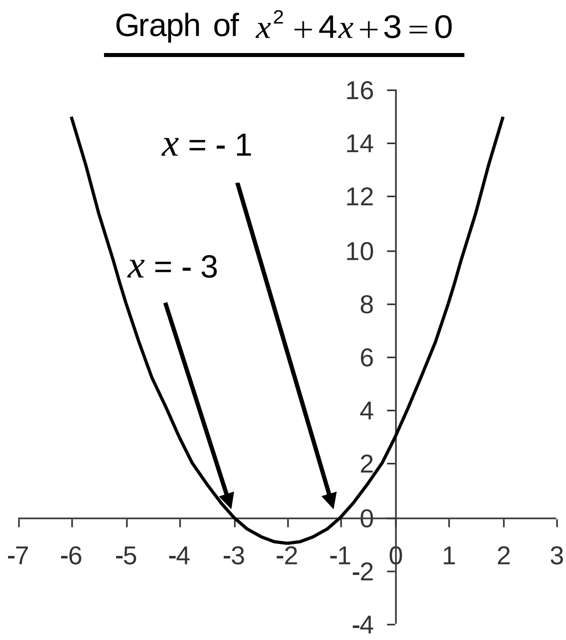
Therefore, we can state that either:

$(x+3) = 0$
or
$(x+1) = 0$

These two equations can be solved to find the value for x=0 by rearranging

$x = 0 - 3$
or
$x = 0 - 1$
giving
$x = -3$
and
$x = -1$

$x = -3$ and $x = -1$ are known as the roots of the equation (when the function crosses the x axis) and they are shown on the graph:



SOLVING QUADRATIC EQUATIONS USING THE QUADRATIC FORMULA

25. When we consider the quadratic equation

$x^2+4x+2=0$

We would look to factorise the equation to solve it however it is not clear of the factors. In this situation the quadratic formula can be used. The quadratic formula equates x too

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

And is true for the equation

$ax^2+bx+c=0$

To use the formula, we have to gain the values of a b and c respectively when mapped/compared against the above equation i.e.

$ax^2+bx+c=0$
$1x^2+4x+2=0$

We can now say

a=1
b=4
c=2

To find the values for x we would use the formula twice, gaining 2 values for x, this is due to the ± in front of the radical (root sign). We would create one value for x with a + value in front of the radical and one value for x with a – value in front of the radical as below: -

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-4 + \sqrt{4^2 - 4 \times 1 \times 2}}{2 \times 1}$$
$$x = -0.5858 \quad (4SF)$$

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-4 - \sqrt{4^2 - 4 \times 1 \times 2}}{2 \times 1}$$
$$x = -3.414 \quad (4SF)$$

Exercise 8

Solve the following equations using the factorisation method.

1.	$x^2 - x - 12=0$	2.	$x^2 + 3x - 10=0$
3.	$x^2 - 8x + 15=0$	4.	$x^2 - 10x + 9=0$

Exercise 9

Solve the following equations using the quadratic formula.

1.	$x^2 - 5x + 2=0$	2.	$x^2 - 4x + 1=0$
3.	$x^2 + 6x - 3=0$	4.	$x^2 + 7x - 2=0$

ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS

26. Consider the expression $\frac{a}{b}+\frac{c}{d}$ which is the addition of two fractional terms.

27. If we wish to express the sum of these algebraic fractions as one single fraction we must proceed as follows (this method is similar to the one used when adding or subtracting fractions containing numbers only).

28. First find the lowest common denominator as a rule it will be the product of the denominators in this case $b \times d$ which is bd. Each fraction is then expressed with bd as its denominator.

i.e., for the first fraction $\frac{a}{b}$ the lowest common denominator is bd, as we have multiplied the denominator by d, we must multiply the numerator by d, the fraction now becomes: $\frac{ad}{bd}$

For the second fraction, $\frac{c}{d}$ the lowest common denominator is bd, as we have multiplied the denominator by b, we must multiply the numerator by b, the fraction now becomes $\frac{cb}{bd}$

Now adding these fractions, we have:

$$\frac{ad}{bd}+\frac{cb}{bd}=\frac{ad+cb}{bd}$$

Examples

1. $\frac{1}{x}-\frac{1}{y} = \frac{y}{xy}-\frac{x}{xy}=\frac{y-x}{xy}$

2. $a-\frac{1}{b} = \frac{a}{1}-\frac{1}{b} = \frac{ab}{b}-\frac{1}{b} = \frac{ab-1}{b}$

3. $\frac{2}{b}+\frac{3}{c} = \frac{2c}{bc}+\frac{3b}{bc} = \frac{2c+3b}{bc}$

4. $\frac{1}{2b}-\frac{2}{3c} = \frac{3c}{6bc}-\frac{4b}{6bc} = \frac{3c-4b}{6bc}$

Exercise 10

Express the following as one fraction with a common denominator:

1. $\frac{1}{x}+\frac{1}{y}$

2. $1+\frac{1}{a}$

3. $\frac{m}{n}-1$

4. $\frac{b}{c}-c$

5. $\frac{a}{b}-\frac{c}{d}$

6. $\frac{a}{b}-\frac{1}{bc}$

7. $\frac{1}{xy}+\frac{2}{b}$

8. $\frac{3}{x}+\frac{x}{4}$

9. $\frac{4}{ab}+\frac{2}{cd}$

MULTIPLYING AND DIVIDING ALGEBRAIC FRACTIONS

29. The method used to multiply and divide algebraic fractions is the same as the one used for numerical fractions. If dividing the fraction after the division sign is inverted then multiplied as normal, cancelling down from top and bottom if possible, the multiplying numerators and denominators as shown.

$\frac{3x}{ab}\times\frac{2a}{xy}=\frac{3\cancel{x}}{\cancel{a}b}\times\frac{2\cancel{a}}{\cancel{x}y}$	Cancel between numerator and denominator
$=\frac{3}{b}\times\frac{2}{y}=\frac{6}{by}$	Multiply numerators and denominators to produce a fraction
$\frac{5x}{2ab}\div\frac{10x^2b}{3yz}=\frac{5x}{2ab}\times\frac{3yz}{10x^2b}$	Invert function after ÷ sign
$=\frac{^1\cancel{5}\cancel{x}}{2ab}\times\frac{3yz}{\cancel{10}_2\cancel{x}^2b}=\frac{3yz}{4ab^2x}$	Cancel between numerators and denominators and then multiply as before

Example:

Simplify the following:

1. $\frac{1}{x}\times\left(\frac{1}{x}+x\right)$

   $\frac{1}{x}\times\left(\frac{1}{x}+x\right)=\frac{1}{x^2}+\frac{^1\cancel{x}}{_1\cancel{x}}=\frac{1}{x^2}+1$

2. $\frac{ab}{cd}\times\frac{xy}{pq}$

   $\frac{ab}{cd}\times\frac{xy}{pq}=\frac{ab\times xy}{cd\times pq}=\frac{ab\,xy}{cd\,pq}$

3. $\frac{x+3}{x-3}\div\frac{3x}{x-3}$

   $\frac{x+3}{x-3}\div\frac{3x}{x-3}=\frac{x+3}{\cancel{x-3}_1}\times\frac{\cancel{x-3}^1}{3x}$

   $=\frac{x+3}{1}\times\frac{1}{3x}=\frac{x+3}{3x}$

4. $\frac{a}{b}\div\frac{1-6a}{2ab}$

   $\frac{a}{b}\div\frac{1-6a}{2ab}=\frac{a}{_1\cancel{b}}\times\frac{2a\cancel{b}^1}{1-6a}$

   $\frac{a}{1}\times\frac{2a}{1-6a}=\frac{2a^2}{1-6a}$

Exercise 11

Simplify the following:

1. $\frac{2}{a} \times \frac{b}{6}$

2. $\frac{3pq}{5} \times \frac{5p}{cq}$

3. $\frac{3x^2}{y} \times \frac{y^2}{x}$

4. $\frac{3d^3}{\pi r^2} \times \frac{\pi d}{4}$

5. $\frac{a+b}{a} \times \frac{a-b}{ab}$

6. $a \div \frac{1}{a}$

7. $\frac{3}{a} \div \frac{d}{a}$

8. $\frac{3x^2}{2a^2} \div \frac{3ax}{2}$

9. $\frac{ab}{c} \div \frac{ab}{c}$

10. $\frac{x-1}{x} \div \frac{3x}{x-1}$

SOLVING LINEAR EQUATIONS

30. When an equation has one unknown value contained within, x for instance, we may be asked to solve the equation for x i.e. Find the value of x. To solve for x or y or z whichever is the unknown value of the equation the equation needs to be rearranged with the unknown value alone on one side of the equals sign.

$$10 = a + 7$$

If we consider the equation above to isolate the *a* we must remove the +7 from the right-hand side (rhs) of the equation. We ask ourselves can we add, subtract, multiply or divide the rhs of the equation by a value that will make the + 7 go. When we consider the options, we will find that the best choice would be to subtract 7 from the rhs of the equation because from elementary maths we know:

$$7 - 7 = 0$$

However, to keep the balance of the equation as all equations are balanced around the equals sign we must also subtract 7 from the left-hand side (lhs) of the equation. When this technique is performed showing all steps we would get

$$\begin{aligned} 10-7 &= a+7-7 \\ 10-7 &= a+0 \\ 10-7 &= a \\ 3 &= a \end{aligned}$$

As 7-7 is equal to zero the seven is removed from the rhs of the equation but it then reappears on the lhs of the equation as it must be done to both sides of the equation. When we sum up the calculation we find our solution for *a*. To check the value is correct we could substitute the value back into the original equation and the equation should work out i.e. the lhs must equal the rhs

There are many different forms of equation that may arise and differing techniques may be required to solve them. When we consider the equation below:

$$10 = 5b$$

Again, in this case to isolate the *b* we must ask ourselves is there anything we can add, subtract, multiply or divide the rhs of the equation to get the 5 to go. When we consider the options the only thing possible in this case is to divide the rhs by 5 because from elementary maths we know:

$$\frac{5}{5} = 1$$

Keeping the balance and dividing both sides of the equation by 5 and looking at all the steps involved we would get:

$$\begin{aligned} \frac{10}{5} &= \frac{5b}{5} \\ \frac{10}{5} &= 1 \times b \\ \frac{10}{5} &= b \\ 2 &= b \end{aligned}$$

We notice that dividing through by 5 gives a value of 1 when cancelled with the 5 next to the b, and we know anything multiplied by 1 is itself which isolates the unknown value b and to keep the balance the lhs is divided by 5 also.

If we consider the equation below we can show how both of these techniques can be combined

$$2x + 3 = 6$$

When considering our options to solve for x we have 2 choices we could either divide the equation through by 2 first or subtract 3 from both of the equation. Either option can be chosen however if there are any separate terms that are not directly attached to the x it would be wise to move these terms away first making the calculation simpler, therefore the decision to subtract 3 from both sides of the equation is the best option. This would leave

$$2x+3-3=6-3$$
$$2x+0=3$$
$$2x=3$$

Our next step is to now divide both sides of the equation by 2 giving

$$\frac{2x}{2}=\frac{3}{2}$$
$$1x=1.5$$
$$x=1.5$$

The next technique to be explained shows how equations with differing fractional amounts can be solved. If we consider

$$\frac{x-4}{3}=\frac{x-2}{4}$$

When we examine the equation, we notice more than one value for x and also the fact that it is in fractional form. If we ask is there a value that the lhs of the equation could be multiplied by which would cancel out the 3 denominator our answer would be 3 however this would not work with the rhs of the equation where we can see that the rhs needs to be multiplied by a value of 4. If we use our knowledge of number fractions we can arrive at the fact that if we multiply both sides of the equation by the LCM (lowest common multiple) both the 3 and 4 will cancel with the LCM. If we consider the LCM of 3 and 4 we should arrive at an LCM of 12 the next step is to multiply both sides of the equation by the LCM of 12.

Note

- When adding or subtracting values we must ensure we do the same to each *side of the equation* to keep the balance.

- When multiplying or dividing values we must ensure we do the same to each *individual term in the equation* to keep the balance

$$\frac{12(x-4)}{3}=\frac{12(x-2)}{4}$$

Note the brackets are important

We can now cancel the 12 with the 3 and the 12 with the 4 as shown

$$4(x-4)=3(x-2)$$

As the equation is still in a factorised state our next step would be to expand the brackets

$$\begin{aligned}&4(x-4)=3(x-2)\\&4x-16=3x-6\end{aligned}$$

As we have 2 unknowns and they are on opposite sides of the equals sign we now need to rearrange to get the unknown values to one side of the equation and the known values to the other. This can be done by subtracting $3x$ from both sides of the equation and adding 16 to both sides of the equation giving

$$\begin{aligned}&4x-16+16-3x=3x-6+16-3x\\&4x-3x=-6+16\\&1x=10\\&x=10\end{aligned}$$

Exercise 12

Solve the following equations

1. $x + 2 = 7$
2. $t - 4 = 3$
3. $x - 8 = 12$
4. $3x = 9$
5. $\frac{y}{2} = 3$
6. $2x + 3 = 7$
7. $3x + 4 = -2$
8. $7x + 12 = 5$
9. $6x - 3x + 2x = 20$
10. $6m + 11 = 25 - m$
11. $5x - 10 = 3x + 2$
12. $0.3x = 1.8$
13. $1.2x - 0.8 = 0.8x + 1.2$
14. $5(m - 2) = 15$
15. $3(x - 1) - 4(2x + 3) = 14$
16. $\frac{x}{5} - \frac{x}{3} = 2$
17. $\frac{1}{3x} + \frac{1}{4x} = \frac{7}{20}$
18. $\frac{x+3}{4} - \frac{x-3}{5} = 2$
19. $\frac{2m-3}{4} = \frac{4-5m}{3}$
20. $\frac{x-2}{x-3} = 3$
21. $\frac{3}{x-2} = \frac{4}{x+4}$
22. $\frac{3}{2x+7} = \frac{5}{3(x-2)}$
23. $\frac{4p-1}{3} - \frac{3p-1}{2} = \frac{5-2p}{4}$
24. $\frac{3x-5}{4} - \frac{9-2x}{3} = 0$
25. $\frac{4x-5}{2} - \frac{2x-1}{6} = x$

EVALUATION OF FORMULAE AND EXPRESSIONS

Transposition of Formulae

31. Most formulae are remembered in a standard form, but for the purpose of solving a particular problem, it is often necessary to express a formula differently. This involves changing the **subject** of the formula and this process is called **transposition**.

Note: (1) *In the formula, $A = LB$, **A** is the subject.*
(2) *In the formula, $C = \pi d$, **C** is the subject.*
(3) *In the formula, $S = ut + ½at^2$, **S** is the subject.*

32. The basic rules of algebra apply equally to transposition of formulae as to solution of equations. This most important concept being that whatever we do to the left-hand side we must also do the right-hand side. For example:

a. If A = LB, transpose this formula to make **L** the subject.

Divide both sides by **B**: $\frac{A}{B} = L$

Reverse the formula: $L = \frac{A}{B}$

b. If $Y = \frac{X}{Z}$ transpose the formula to make X the subject.

Multiply both sides by **Z**: $YZ = X$

$\therefore$ $X = YZ$

c. If $a = \frac{b}{c}$ transpose the formula to make c the subject.

Multiply both sides by c: $ac = b$

Divide both sides by **a**: $c = \frac{b}{a}$

d. If $y = x + c$, transpose this formula to make x the subject.

Subtract **c** from both sides: $y – c = x$

$\therefore$ $x = y – c$

e. If $p = \frac{q - m}{r}$ transpose the formula to make q the subject.

Multiply both sides by **r**: $pr = q – m$

Add **m** to both sides: $pr + m = q$

$\therefore$ $q = pr + m$

33. If, after transposition, the subject is negative it can be made positive by multiplying both sides by –1. For example:

If $y - x = m$, transpose this formula to make **x** the subject.

Subtract y from both sides $-x = m - y$

Multiply both sides by –1: $x = y - m$

Examples

1. Transpose $x = \dfrac{y}{b}$ for y.

   Multiply both sides by b.

   Therefore $x \times b = \dfrac{y}{b} \times b$

   So $bx = y$

   Therefore $y = bx$

2. Transpose $m^3 = \dfrac{3x^2w}{p}$ for p.

   Multiple both sides by p.

   Therefore $m^3p = 3x^3w$

   Divide both sides by m^3

   Therefore $\dfrac{m^3p}{m^3} = \dfrac{3x^2w}{m^3}$

   Cancelling gives: $= \dfrac{3x^2w}{m^3}$

3. Transpose $\ell = a + (n - 1)d$ for n

   Subtract a from both sides.

   Therefore: $\ell - a = (n - 1)\ d$

   Divide both sides by d: $\dfrac{\ell - a}{d} = n - 1$

   Add 1 to both sides: $\dfrac{\ell - a}{d} + 1 = n$

   Rewritten as: $n = \dfrac{\ell - a}{d} + 1$

EXERCISE 13

Transpose the following to make the letter in the brackets the subject.

1. $Y = X + Z$ (X)
2. $a = b - c$ (b)
3. $p = q + s$ (s)
4. $I = m - n$ (n)
5. $Y = ZX$ (X)
6. $Sin\theta = \frac{O}{H}$ (O)
7. $Y = \frac{z}{X}$ (z)
8. $a = \frac{b}{c}$ (c)
9. $v = u + at$ (u)
10. $V = E - IR$ (I)
11. $v = u + at$ (t)
12. $p = \frac{mRT}{V}$ (T)
13. $s = ut + \frac{1}{2}at^2$ (u)
14. $\frac{1}{R} = \frac{1}{P}$ (P)
15. $R(u + s) = p$ (u)
16. $R_N = \frac{E}{I} - R_M$ (I)
17. $y = mx + c$ (x)
18. $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$ (P_2)
19. $Cos\theta = \frac{A}{H}$ (H)

TRANSPOSITION OF FORMULAE (POWERS AND ROOTS)

35. Consider the following examples:

a. If $\sqrt{X} = Y$, make X the subject:

Squaring both sides: $\left(\sqrt{X}\right)^2 = Y^2$

so $X = y^2$

b. If $x^2 = y$, make X the subject:

Square rooting both sides: $\sqrt{X^2} = \sqrt{Y}$

so $X = \sqrt{Y}$

c. If $a = b\sqrt{c}$, make c the subject:

Squaring both sides: $a^2 = \left(b\sqrt{c}\right)^2$

$a^2 = b^2\left(\sqrt{c}\right)^2$

$a^2 = b^2c$

Divide through b^2: $\frac{a^2}{b^2} = c$

Reverse: $c = \frac{a^2}{b^2}$

so: $c = \left(\frac{a}{b}\right)^2$

36. The square root of a given number, for example x, is the number r such that, $r^2 = x$. Every positive real number has 2 square roots, because if $r^2 = x$, then also $(-r)^2 = x$. For example:

a. The square roots of 4 are +2 and -2 because 2^2 and $(-2)^2 = 4$.

b. The square roots of 9 are +3 and -3 because 3^2 and $(-3)^2 = 9$.

Using the notation $\pm$, meaning “Plus or minus”:

c. The square root of 25 is ± 5 because $\pm 5^2 = 25$.

The positive square root is sometimes called the “Principal square root”

37. Instead of writing 'square root', to save time, we write $\sqrt{\ }$. This symbol is called the "Radical" and means "the positive square root of" (The number inside the radical sign is called the "Radicand")

For example:

a. The square roots of 16 can be written $\pm\sqrt{16}$.

b. $+\sqrt{36}$ and $-\sqrt{36}$ are both square roots of 36.

Although negative roots of non-negative real numbers always exist, variables used in engineering formulae often represent real life quantities which cannot by their nature take negative values. Examples of this would-be *M* for mass, *R* for resistance, *C* for capacitance etc. When transposing equations for such variables, where square roots have to be taken, in the vast majority of cases, because of the nature of the quantity being considered, we know that the correct solution is the positive one and the negative square root can be disregarded. From now on, we will assume that to be the case, i.e. that the equations we are dealing with are for variables that can only take positive values and the positive square root is the only one we will consider.

38. The cube root of a given number is such that, when it is cubed, the original number is again obtained. For example:

a. The cube root of 8 is 2, because $2^3 = 8$.

b. The cube root of 125 is 5 because $5^3 = 125$.

Instead of writing 'cube root' we write: $\sqrt[3]{\ }$

For example:

c. $\sqrt[3]{27}$ simply means the cube root of 27.

d. The cube root of 64 can be written $\sqrt[3]{64}$

39. It follows from the above the nth root of a given number is such that when it is raised to the power n, the original number is obtained. For example:

a. If $\sqrt[n]{x} = y$, then $x = y^n$

b. If $x = y^n$, then $\sqrt[n]{x} = y$

40. You should remember that, providing we add or subtract equal numbers or letters to both sides of equations or formulae by the same number or letter, the truth of the equation or formula is unaffected. We can now extend this concept to include powers and roots. For example:

$$x^2 = 9$$

Taking square roots of both sides $\sqrt{x^2} = \sqrt{9}$

$$\therefore \quad x = 3$$

Here we are assuming x is an engineering quantity that can only take positive values.

Exercise 14

Transpose the following formula to make the letter in brackets, the subject:
(You may assume that all variables can only take positive values)

1. $\sqrt{A} = d$ (A)
2. $p^2 = q$ (p)
3. $\sqrt[3]{X} = Y$ (X)
4. $a^3 = b$ (a)
5. $A = \pi r^2$ (r)
6. $(n - 1)^2 = t$ (n)
7. $p = q\sqrt{r}$ (r)
8. $Z = \sqrt{r^2 + x^2}$ (x)
9. $a = \sqrt{2bc}$ (c)
10. $X^2 = YZ^3$ (Z)
11. $y = kx^3$ (k)
12. $y = \dfrac{k}{\sqrt[3]{x}}$ (k)
13. $y = kx^3$ (x)
14. $y = \dfrac{k}{\sqrt[3]{x}}$ (x)

ANSWERS TO EXERCISES

Exercise 1

The answers to this exercise have to be in the same format, but may use different letters or symbols.

1. a. Identity (True for all values of x)

   b. Term (no + or -)

   c. Expression (no = sign)

   d. Equation (only true when $a = 7$)

2 a. $T_p = t + c$ where T_p = total points,

t = points for the tries,

and c = points for the conversions

b. $T = FL$ where T = torque, F = force, and L = length

c. $T = nx + n$ where T = total cost, n = net cost and x = tax

Exercise 2

1.	$26b$	2.	$19x$	3.	$12a^2$
4.	$4xy$	5.	$5a + 2b$	6.	$8x + y + 4z$
7.	$2x^2 + 5xy^2$	8.	$5p^3 + 9p^2 + 3p$	9.	$3x^3 + 4x^2 + 5x$
10.	$1.2a^3 + 0.3a^2 + 6.2a - 2.8$				

Exercise 3

1.	$17x$	2.	$2r$	3.	$6y$
4.	$3b$	5.	$10b$		

Exercise 4

1.	$5x + 15$	2.	$6a – 12b$	3.	$a – 2b$
4.	$7b – 35m$	5.	$3y – x$	6.	$5p + 6q$
7.	$24 – 12a$	8.	$3r^2 + 3rp$	9.	$-15x^2 – 20xy$
10.	$ax + ay – az$	11.	$4a^2b + 16abc$	12.	$6x^4 + 6x^2y^2 – 9x^3$
13.	$-14t^3 + 7t^2 – 35t$	14.	$3a^2bc + 3ab^2c – 3abc^2$		
15.	$5x + 11$	16.	$x + 7$	17.	$7x – 11y$
18.	$-13x + 8y + 3z$	19.	$3a – 9b$	20.	$x^3 + 18x^2 –9x - 15$

Exercise 5

1.	$mp – mq + np - nq$	2.	$x^2 + 9x + 20$	3.	$6x^2 +16x + 8$
4.	$x^2 + 5x – 14$	5.	$2x^2 + 4x – 6$	6.	$6v^2 – 5vu – 6u^2$
7.	$x^2 + 2xy + y^2$	8.	$x^4 – y^2$		

Exercise 6

1.	$(x + 3)(x + 4)$	6.	$(x - 2)(x + 6)$
2.	$(x + 6)(x + 4)$	7.	$(x - 8)(x + 4)$
3.	$(x - 16)(x - 2)$	8.	$(x + 3)(x - 3)$
4.	$(x - 6)(x - 8)$		
5.	$(x - 3)(x + 8)$		

Exercise 7

1. $c(a + 2b) – d(a + 2b)$

   $= (c – d)(a + 2b)$

3. $6(2 – a) + b (2 – a)$

   $= (6 + b)(2 – a)$

2. $x(y + 2) + 1(y + 2)$

   $= (x + 1)(y + 2)$

Exercise 8

1.	4 and -3	2.	-5 and 2
3.	5 and 3	4.	9 and 1

Exercise 9

1.	0.4384 and 4.562	2.	0.2679 and 3.732
3.	-6.464 and 0.4641	4.	0.2749 and -7.275

Exercise 10

1.	$\frac{y+x}{xy}$	2.	$\frac{a+1}{a}$	3.	$\frac{m-n}{n}$
4.	$\frac{b-c^2}{c}$	5.	$\frac{ad-cb}{bd}$	6.	$\frac{abc-b}{b^2c}$ or $\frac{ac-1}{bc}$
7.	$\frac{b+2xy}{xyb}$	8.	$\frac{12+x^2}{4x}$	9.	$\frac{4cd+2ab}{abcd}$

Exercise 11

1.	$\frac{b}{3a}$	2.	$\frac{3p^2}{c}$	3.	3xy
4.	$\frac{3d^4}{4r^2}$	5.	$\frac{a^2-b^2}{a^2b}$	6.	a^2
7.	$\frac{3}{d}$	8.	$\frac{x}{a^3}$	9.	1
10.	$\frac{x^2-2x+1}{3x^2}$				

Exercise 12

1.	5	2.	7	3.	20
4.	3	5.	6	6.	2
7.	-2	8.	-1	9.	4
10.	2	11.	6	12.	6
13.	5	14.	5	15.	-5.8
16.	-15	17.	1.66	18.	13
19.	0.9615384	20.	3.5	21.	20
22.	-53	23.	3.25	24.	3
25.	3.5				

Exercise 13 *in exercises 13 to 14 you may have an answer that looks slightly different, but is equivalent. Check with your tutor if you cannot see that your answers are the same as those given.*

1.	$x = y - z$	2.	$b = a + c$	3.	$s = p - q$
4.	$n = m - I$	5.	$x = \frac{y}{z}$	6.	$H \times Sin\theta = O$
7.	$z = yx$	8.	$c = \frac{b}{a}$	9.	$u = V - at$
10.	$I = \frac{E - V}{R}$	11.	$t = \frac{V - u}{a}$	12.	$T = \frac{PV}{mR}$
13.	$u = \frac{2S - at^2}{2t}$	14.	$P = R$	15.	$U = \frac{P}{R} - S$
16.	$I = \frac{E}{(R_N + R_M)}$	17.	$x = \frac{y - c}{m}$	18.	$P_2 = \frac{P_1V_1T_2}{T_1V_2}$
19.	$H = \frac{A}{Cos\theta}$				

Exercise 14

1.	$a = d^2$	2.	$p = \sqrt{q}$	3.	$X = Y^3$
4.	$a = \sqrt[3]{b}$	5.	$r = \sqrt{\frac{A}{\pi}}$	6.	$n = \sqrt{t} + 1$
7.	$r = \frac{p^2}{q^2}$	8.	$X = \sqrt{Z^2 - R^2}$	9.	$c = \frac{a^2}{2b}$
10.	$Z = \sqrt[3]{\frac{X^2}{Y}}$	11.	$k = \frac{y}{x^3}$	12.	$k = y\sqrt[3]{x}$
13.	$x = \sqrt[3]{\frac{y}{k}}$	14.	$x = \left(\frac{k}{y}\right)^3$		