Quantum Mechanics

Week 5

Mark Benazet Castells

March 27, 2024

Exercise Material

Webpage

Week 3

Recap

Review

Exercises

Review of Last Week

- Any questions on last week's topics?
- Feedback on the previous session?

Recap

Quantum Mechanics Postulates

- Wave functions $\Psi(x,t)$ represent physical states in the **Hilbert Space**.

- $\Psi(x,t)$ must be *square-integrable* and *normalized*:

$$\boxed{\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 1.}$$

- If the Wave function lays in the Hilbert Space, we can represent it as a vector:

$$\Psi = \sum_{n=1}^{\infty} c_n f_n(\vec{r}) \rightarrow \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_{\psi_n} = |\Psi\rangle \quad \textbf{"ket"}$$

- Observable properties are extracted using **linear Hermitian operators**.

- Hermitian operators ensure real expectation values:

$$\boxed{\langle \Psi | \hat{Q} \Psi \rangle = \langle \hat{Q} \Psi | \Psi \rangle}$$

Determinate States

An eigenvalue equation for an operator $\hat{Q}$ and a state Ψ is expressed as:

$$\boxed{\hat{Q}\Psi = q\Psi,}$$

indicating that if a system is in state Ψ, any measurement of $\hat{Q}$ will consistently yield the eigenvalue q.

- States Ψ solving the eigenvalue equation are called **determinate states**.
- The set of all possible eigenvalues q of $\hat{Q}$ forms its **spectrum**.
- Eigenfunctions with the same eigenvalue are termed **degenerate states**.

Generalized Interpretation

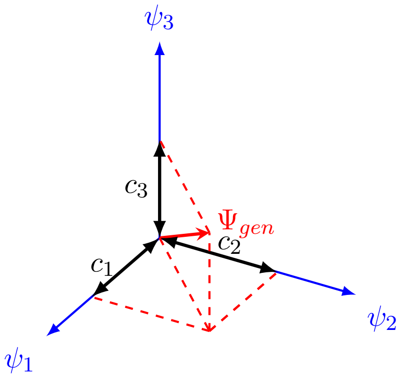
When Ψ is not an eigenfunction of $\hat{Q}$, a measurement yields an eigenvalue q_n with probability $|c_n|^2$:

$$\boxed{\langle Q \rangle = \sum_n |c_n|^2 q_n,}$$

where $\langle Q \rangle$ is the expected value of $\hat{Q}$.

To find the coefficients c_n, project Ψ onto the eigenfunctions ψ_n:

$$\boxed{c_n = \langle \psi_n | \Psi \rangle.}$$



Given $\vec{\Psi}_{gen} = c_1\vec{e}_{\psi_1} + c_2\vec{e}_{\psi_2} + c_3\vec{e}_{\psi_3}$, we define:

$$|\Psi\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}_{\psi_n}$$

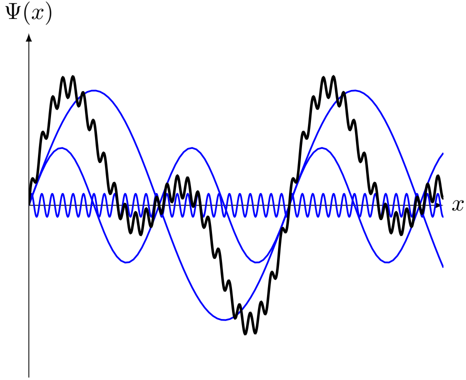
Where, $c_n = \langle\psi_n|\Psi\rangle$, encapsulating the system's state in terms of basis vectors $\hat{\psi}_n$.

Review

Generalized Uncertainty Principle

$[\hat{A}, \hat{B}] = 0$

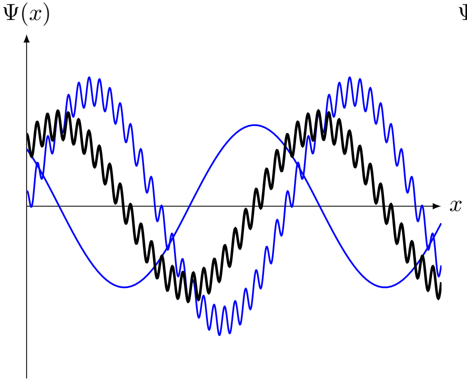
If $\hat{A}$ and $\hat{B}$ are compatible, precise measurements of these observables are possible as they share eigenfunctions, implying $\sigma_A \sigma_B = 0$. Thus, both can be measured simultaneously without state interference.

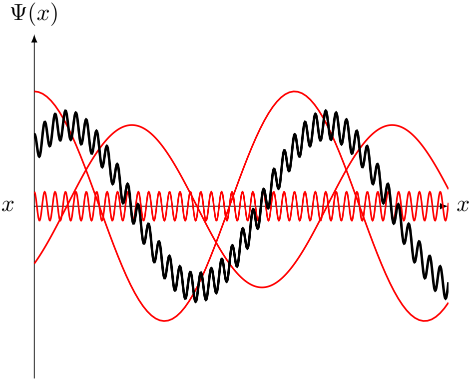


Where Ψ is the Wave function and ψ_n are the eigenfunctions of $\hat{A}$ and $\hat{B}$.

$[\hat{A}, \hat{B}] \neq 0$

If $\hat{A}$ and $\hat{B}$ are incompatible, precise measurement of these observables is impossible due to differing eigenfunctions, implying $\sigma_A \sigma_B \neq 0$. Hence, simultaneous measurement is unfeasible as it alters the state.





Where Ψ is the wave function, ψ_A are the eigenfunctions of operator $\hat{A}$, and ψ_B are the eigenfunctions of operator $\hat{B}$.

Therefore, we introduce the **generalized uncertainty principl**e for non-commuting operators:

$$\boxed{\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2}$$

For $\hat{A} = \hat{x}$, $\hat{B} = \hat{p}$, $[\hat{x}, \hat{p}] = i\hbar$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{\hbar}{2} \right)^2$$

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

Continuous Eigenfunctions

Position Operator

To illustrate eigenfunctions of the position operator $\hat{x}$:

$$\hat{x}g(x) = x'g(x),$$

where $\hat{x}$ is the operator, $g(x)$ the eigenfunction, and x' the eigenvalue. The Dirac delta function, $\delta(x - x')$, uniquely satisfies:

$$\boxed{\hat{x}\delta(x - x') = x'\delta(x - x').}$$

- It is not in the Hilbert Space as it is not Square-integrable.
- It ensures orthonormality—position eigenstates are mutually orthogonal. Dirac delta instead of Kronecker delta.
- Completeness is achieved as any spatial function can be expressed through these eigenstates.

Momentum Operator

For the momentum operator $\hat{p}$, we seek a function $f_{p'}$ that satisfies:

$$\hat{p} f_{p'} = p' f_{p'},$$

where p' is the eigenvalue. The solution is:

$$\boxed{f_{p'}(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ip'x}{\hbar}\right),}$$

- It is not in the Hilbert Space as it is a free-particle.
- It ensures orthonormality. Dirac delta instead of Kronecker delta.
- Completeness is achieved as any spatial function can be expressed through these eigenstates.

Momentum eigenfunctions correspond to waves with wavelength $\lambda = \frac{h}{p'}$.

T/F L6

Midterm

Exercises

Exercise 2

Remark: Use the properties from the ZF

Exercise 3

Exercise 4

Really useful to practice how to do such problems.

Questions?

THANK YOU!

The Dirac Delta Function

The Dirac delta function, denoted $\delta(x)$, is defined as:

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ \infty & \text{if } x = 0, \end{cases}$$

with the integral property:

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

Furthermore, it has the sifting property where for any function $f(x)$: $f(x) \cdot \delta(x-a) \to f(a) \cdot \delta(x-a)$, But in practice, it really works in the integral form:

$$\int_{-\infty}^{\infty} f(x) \cdot \delta(x - a)\, dx = f(a).$$