Die geniale Zusammenfassung

Control Systems 2

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General Math

$\angle\left(\frac{N}{D}\right) = \angle(N) - \angle(D)$ $\qquad$ $\angle(e^{jx}) = x$

$\angle(a+b)^n = n\cdot\angle(a+b)$ $\qquad$ $\angle(z) = \arctan\left(\frac{Im(z)}{Re(z)}\right)$

$\left|A\cdot\frac{N}{D}\right| = |A|\frac{|N|}{|D|}$ $\qquad$ $\frac{1}{2a}\left(-b\pm\sqrt{b^2-4ac}\right)$

$|e^{-j\omega T}| = |\cos(\omega T) - j\cdot\sin(\omega T)| = 1$

$\left|\frac{(a+jb)^x}{(c+jd)^y}\right| = \frac{(\sqrt{a^2+b^2})^x}{(\sqrt{c^2+d^2})^y}$

$\angle\left(\frac{(a+jb)^x}{(c+jd)^y}\right) = x\angle(a+jb) - y\angle(c+jd)$

$= x\cdot arctan\left(\frac{b}{a}\right) - y\cdot arctan\left(\frac{d}{c}\right)$ (in rad rechnen!)

$A = TDT^{-1}, D = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \lambda_n\end{pmatrix},\ e^{At} = Te^DT^{-1} = T\begin{pmatrix}e^{\lambda_1} & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & e^{\lambda_n}\end{pmatrix}T^{-1}$

2x2 inverse: $A^{-1} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{\det(A)}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

3x3 inverse: $\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}^{-1} = \frac{1}{\det(A)}\cdot\begin{pmatrix}ei-fh & ch-bi & bf-ce\\ fg-di & ai-cg & cd-af\\ dh-eg & bg-ah & ae-bd\end{pmatrix}$

Omega from frequency: $\omega = 2\pi\cdot f$

Push-through rule: $C_1(\mathbb{I} - C_2C_1)^{-1} = (\mathbb{I} - C_1C_2)^{-1}C_1$

Moore-Penrose pseudo inverse:
Used to invert a non-square matrix:
For a **tall** system ($l > m$), $\text{rank}(P) = m$:
Left pseudo-inverse: $\qquad A^\dagger = (A^*A)^{-1}A^*, A^\dagger A = \mathbb{I}_m$
For a **fat** system ($l < m$), $\text{rank}(P) = l$:
Right pseudo-inverse: $\qquad A^\dagger = A^*(AA^*)^{-1}, AA^\dagger = \mathbb{I}_l$
Hermitian Transpose A^*:
A^* means A^T plus complex conjugate *:
$A = \begin{bmatrix}i & 1\\ 2 & 3-i\end{bmatrix} \Rightarrow A^* = \begin{bmatrix}-i & 2\\ 1 & 3+i\end{bmatrix}$

Best of CS I

$A = \left.\frac{\partial f}{\partial x}\right|_{x_{eq},u_{eq}}$ $\quad$ $B = \left.\frac{\partial f}{\partial u}\right|_{x_{eq},u_{eq}}$ $\quad$ $C = \left.\frac{\partial g}{\partial x}\right|_{x_{eq},u_{eq}}$ $\quad$ $D = \left.\frac{\partial f}{\partial u}\right|_{x_{eq},u_{eq}}$

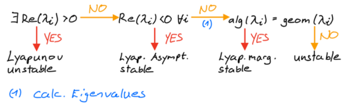
Stability

Stability analyses the behaviour of the system near the equilibrium point.

Poles			
stable	$Re(p_i) < 0$	bigger $\lvert Re(p_i)\rvert$ → faster response	
unstable	$Re(p_i) > 0$	bigger $\lvert Im(p_i)\rvert$ → higher freq. oscill.	
Zeros			
overshoot	$Re(z_i) < 0$	Miminumphase	Stronger if near origin
undershoot	$Re(z_i) > 0$	Non-Minimumphase	

Lyapunov Stability

Lyapunov stability: An LTI system is Lyapunov st. if for every bounded I.C. and zero input, the state remains bounded.
det $(A - \mathbb{I}\lambda)$:



BIBO

A System is BIBO stable if for all bounded inputs, the output remains bounded.
For LTI-Systems: Asymptotic stability ⟷ BIBO stability
+ controllability
+ observability

Nyquist Criterion

N = Z - P
N: number of clockwise encirclements of -1
Z: number of zeroes (c.l. poles)
P: number of poles (unstable c.l. poles)

Bode Plot

Shows gain and phase response of LTI systems for different frequencies. Useful for quantitative arguments and control-system design; may be misleading in determining closed-loop stability (e.g. for open-loop unstable system); gives steady state answer.

Decibel	$\lvert F(j\omega)\rvert_{dB} = 20\cdot\log_{10}(\lvert F(j\omega)\rvert)$
Backward	$\lvert F(j\omega)\rvert = 10^{\frac{\lvert F(j\omega)\rvert_{dB}}{20}}$
Stability	If T.F. is o.l. stable, verify that $\angle L(j\omega) >$ 180° when $\lvert L(j\omega)\rvert_{dB} = 0\ dB = 1$
Gain Margin	$\lvert L(j\omega)\rvert$ at $\angle(L(j\omega)) = -180°$
Phase Margin	Phase distance to $-180°$ when $\lvert L(j\omega)\rvert_{dB} = 0\ dB = 1$
Crossover	ω_c where $\lvert G(j\omega)\rvert = 1 = 0\ dB$ $\qquad \omega_c \approx \omega_b$
Bandwidth	ω_b where $\lvert T(j\omega)\rvert = \frac{1}{\sqrt{2}}$

$L(j\omega)$	0.01	0.1	0.5	0.32	1	10	20	100
$\lvert L(j\omega)\rvert_{dB}$	-40	-20	-6	10	0	20	26.02	40

Draw Bode plot:
1. Rewrite TF in proper form
2. Separate TF in its constituent parts
3. Draw Bode Plot for each part
4. Add plots from 3.

Term	Magnitude	Phase
K (konstant)	$20log_{10}(\lvert K\rvert)$	$K > 0: 0°$ $K < 0: \pm 180°$
Pole at Origin $\frac{1}{s}$ (integrator)	$-20dB/dec$ passing through $0dB$ at $\omega = 1$	$-90°$
Zero at Origin **s** (differentiator)	$+20dB/dec$ passing through $0dB$ at $\omega = 1$	$+90°$
Real Pole $\frac{1}{\frac{s}{\omega_0}+1}$	- Draw low frequency asymptote at 0 dB - Draw high frequency asymptote at -20 dB/decade - Connect lines at ω_0.	- Draw low frequency asymptote at 0° - Draw high frequency asymptote at -90° - Connect with a straight line from $0.1\cdot\omega_0$ to $10\cdot\omega_0$
Real Zero $\frac{s}{\omega_0}+1$	Same than real Pole, but asymptote +20dB/dec	Same than real pole but asymptote +90°
Underdamped Pole $\frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right)+1}$	$\begin{cases}0dB & \omega<\omega_0\\ -40dB/dec & else\end{cases}$ Peak at ω_0 with amplitude -20log (2ζ)	$\begin{cases}0° & \omega<\omega_0\\ -180° & else\end{cases}$ connected from $\omega_0\cdot 10^{-\zeta}$ to $\omega_0\cdot 10^{\zeta}$
Underdamped Zero $\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right)+1$	$\begin{cases}0dB & \omega<\omega_0\\ +40dB/dec & else\end{cases}$ Dip at ω_0 w/ amplitude 20log (2ζ)	$\begin{cases}0° & \omega<\omega_0\\ 180° & else\end{cases}$ connected from $\omega_0\cdot 10^{-\zeta}$ to $\omega_0\cdot 10^{\zeta}$
Time Delay e^{-sT}	No change in magnitude	Phase drops linearly $\phi = -\omega T$ (radians) $\phi = -\omega T\cdot {}^{180°}/_{\pi}$ (deg)

Note: zero is of the form $\frac{s}{-\omega_0} + 1$: Magnitude remains, phase is $-90°$. Analogously for Poles ($+$ instead of $-90°$)

Positive Definiteness (PD)

Test 1: A real, symmetric Matrix is (semi) PD IFF it has only (non-negative) positive eigenvalues
Test 2: Sylvester's Criterion: A symmetric matrix is PD if all its **leading** (i.e. on the main diagonal) principal minors are positive.
It is semi PD if all its principal minors are positive
[BILD: example TODO]

Controllability/Reachability

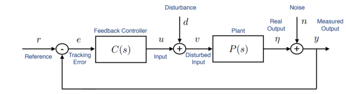
They are structural properties of a system: same in any representation / unchangeable by any control strategy
Controllable if $\mathcal{C}$ has full rank:
$\mathcal{C} = [B\ AB\ A^2B\cdots A^{n-1}B];\ A\in\mathbb{R}^{n\times n}$ $\qquad \det(\mathcal{C}) \neq 0$
Observable if $\text{rank}(\mathcal{O}) = \text{rank}\begin{bmatrix}C\\ \vdots\\ CA^{n-1}\end{bmatrix} = n$
Reachable IFF $\text{rank}(\mathcal{R}) = \text{rank}[B\quad AB\quad\cdots\quad A^{n-1}B] = n$
Controllability and Reachability are only different in discrete time.

Popov-Belevitch-Hautus Test (PBH):
Rule for detectability applies for observability if $n\ \forall\lambda\ \in C$
Rule for stability applies for detectability if $n\ \forall\lambda\ \in C$
Detectable: All unstable states are observable.
An eigenvalue λ is detectable if $\text{rank}\begin{bmatrix}\lambda\cdot\mathbb{I} - A\\ C\end{bmatrix} = n$
A system is detectable if this holds $\forall\lambda\in\mathbb{C}^+$

Stabilizable if all unstable states are controllable/reachable
To see if a λ is stabilizable, check if $\text{rank}[(\lambda\mathbb{I} - A)\quad B] = n$
A system is stabilizable if the above holds $\forall\lambda\in\mathbb{C}^+$
$(A,B)\ controllable \Leftrightarrow (A^T,B^T)\ observable$
$(A,C)\ observable \Leftrightarrow (A^T,C^T)\ controllablee$

Loop Transfer Functs. & The Gangs

We have a standard Loop with:



Then the relations between the in- and outputs are:

$$\begin{bmatrix}y\\ \eta\\ v\\ u\\ e\end{bmatrix} = \begin{bmatrix}TF & PS & S\\ TF & PS & -T\\ CFS & PS & -T\\ CFS & -T & -CS\\ FS & -PS & -S\end{bmatrix}\cdot\begin{bmatrix}r\\ d\\ n\end{bmatrix}$$

With P as the plant, C as the controller and
Sensitivity: $S = \frac{1}{1+PC}$ $\quad$ Load sens.: $PS = \frac{P}{1+PC}$ $\quad$ $TF = \frac{PCF}{1+PC}$
Compl. Sens.: $T = \frac{PC}{1+PC}$ $\quad$ Noise sens.: $CS = \frac{C}{1+PC}$ $\quad$ $CF = \frac{CF}{1+PC}$
And the tracking error $\varepsilon = (1-T)r - PSd + Tn$
The **Gang of Four** is when $F = 1$; S+T=1

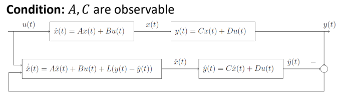
State Feedback

We have a system and don't know the states X. The closed loop dynamics are determined by $A_{cl} = A - BK \Rightarrow$ use K to place poles of CL TF. Works if A, B are reachable.
Pole placement by hand:
- Decide on pole locations and create p_{des}
- Parametrise $K = [k_1\quad k_2], \dim(K) = \dim(B^T)$
- Compute $p(\lambda) = \det(\lambda\mathbb{I} - A_{cl}) = \det(\lambda\mathbb{I} - (A - BK))$
- Compare coefficients of $p_{desired}$ and $p(\lambda)$

Pole placement with Ackermann's formula:
- $K = [0\quad\cdots\quad 0\quad 1]\mathcal{R}^{-1}\cdot p_{des}(A)$
  (→ if $p_{\text{des}} = (s+3)^2 \Rightarrow (A + 3\mathbb{I})^2$)
- $p_{des}(A)$ is the desired CL. Characteristics, evaluated at s=A
  → if all EV are real, no overshoot is possible
  → real parts of poles - given by decrease function of error
- $\mathcal{R}$ is the reachability matrix (must have full rank for $\mathcal{R}^{-1}$)

State Estimation / Luenberger Obs.

Idea: use the difference of real output y and model output $\hat{y}$ to estimate state $\hat{x}$. Control and estimation do not interact with each other – can be designed independently
Condition: A, C are observable



- Set $\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u + L(y - \hat{y})$ $\qquad L(y-\hat{y}) = LC(x - \hat{x})$
- Define error $\hat{e} = x - \hat{x} \Rightarrow \dot{\hat{e}} = \dot{x} - \dot{\hat{x}} = \cdots = (A - LC)\hat{e}$
- L is the observer gain matrix

Finding L like in *State Feedback*:
- $\dot{\hat{e}} = (A - LC)\hat{e}, L = [l_1\quad\cdots\quad l_n]^{T}$
- Set p_{des} with the desired poles
- $\det(\lambda\mathbb{I} - (A - LC)) = p_{des}$ and compare coefficients

Or with Ackermann:
- $L = p_{des}(A)\cdot\mathcal{O}^{-1}\cdot[0\quad\cdots\quad 0\quad 1]^T$
- If $C = B^T = [1\ 1\ 1] \Longrightarrow LC = K$

Separation Principle

Idea: Control $(A - BK)$ and estimation $(A - LC)$ do not interact and can thus be designed independently.

$$\tilde{x} = \begin{bmatrix}x\\ e\end{bmatrix};\ \dot{\tilde{x}} = \begin{bmatrix}\dot{x}\\ \dot{e}\end{bmatrix} = \underbrace{\begin{bmatrix}A - BK & BK\\ 0 & A - LC\end{bmatrix}}_{A_{cl}}\cdot\tilde{x}$$

$$\sigma(A_{cl}) = \sigma(A - BK)\cup\sigma(A - LC)$$

Rule of thumb: Observer 5 to 10x faster than controller.
Eigenvalues of A-BK ~ 0,1 of A-LC

Reachability / Observability Decomp.

Reachability Decomposition:
distinguish reachable from unreachable system
Take $z = \begin{bmatrix}z_1\\ z_2\end{bmatrix} = Tx$
with s reachable states z_1 and $(n-s)$ non-reach. states z_2
- Find $T^{-1} = [\underbrace{v_1\quad\cdots\quad v_s}_{\text{Image Basis of }\mathcal{R}} \mid \underbrace{v_{s+1}\quad\cdots\quad v_n}_{\text{Completion}}]$
- Image Basis: Bring Matrix into pivot form: Vectors with a pivot (the first two) are the base:
  $\begin{bmatrix}* & * & *\\ 0 & * & *\\ 0 & 0 & 0\end{bmatrix}$
- Completion: linearly independent row vectors
- Set $\tilde{A} = TAT^{-1}, \tilde{B} = TB$ and $\tilde{C} = CT^{-1}$
- $\Rightarrow \tilde{A} = \begin{bmatrix}* & *\\ 0 & *\end{bmatrix}, \tilde{B} = \begin{bmatrix}1\\ 0\end{bmatrix}$ are decoupled

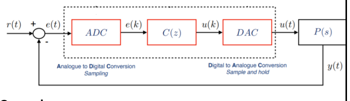
Reachable canonical Form:
It has the form:

$$\tilde{A} = \begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & 0 & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}\end{bmatrix} \text{ and } \tilde{B} = \begin{bmatrix}0\\ \vdots\\ 1\end{bmatrix}$$

The a_i can be found by comparing the coefficients of $\det(\lambda\mathbb{I} - A) = \det(\lambda\mathbb{I} - \tilde{A})$

Observability Decomposition:
Take $z = \begin{bmatrix}z_1\\ z_2\end{bmatrix} = Tx$
with r observable states z_1 and $(n-r)$ non-observable state z_2
- Find $T^{-1} = [\underbrace{v_1\quad\cdots\quad v_r}_{\text{Completion}} \mid \underbrace{v_{r+1}\quad\cdots\quad v_n}_{\text{Basis of kernel}}]$
- Completion: linearly independent row vectors
- Kernel: Solutions x of $\mathcal{O}\cdot x = 0$ of the observability matrix
- Set $\tilde{A} = TAT^{-1}, \tilde{B} = TB$ and $\tilde{C} = CT^{-1}$
- $\mathcal{O}_{new} = T^{-1}\mathcal{O}T$

Discrete Time Control



General:
Sampling period: T_s $\qquad$ Sampling frequency $f_s = \frac{1}{T_s}$
ADC: $e(k) = e(t_k) = e(k\cdot T_s)\quad k\in N$
$t_k = t_{k-1} + kT_s$
DAC: hold → $u(t) = u^d[k]$ $\qquad k\cdot T_s \leq t < (k-1)T_s$

Aliasing

Aliasing occurs when the sampling frequency is too low.
Solution (theory) $\underbrace{f_s}_{\text{sampling freq.}} > 2\cdot\underbrace{f_b}_{\text{highest freq of system}} := f_{\text{Nyquist}}$
Solution (practice) $f_s \geq 5\cdot f_{\text{Nyquist}}$
Analogously: $\omega_s \geq 10\omega_c$
Other solution: Anti-aliasing-filter (AAF): Low-pass filter, attenuates high frequencies of analogue signal entering controller; introduces phase lag
Example: Divide largest component by 2π and compare to the given sampling frequency 3Hz (or T_s=1/3)

$$\cos\left(\frac{\pi}{3}t\right) \to {}^{\frac{\pi}{3}}/_{2\pi} = \frac{1}{6}Hz;\ 3Hz > 2\cdot\frac{1}{6}Hz: no\ aliasing$$
$$\cos(4\pi t) \to {}^{4\pi}/_{2\pi} = 2Hz;\ 3Hz < 2\cdot 2\ Hz: aliasing$$

Time Delays (TD)

Time delays arise in the process of discretisation.
In frequency space: $y(s) = e^{-Ts}\cdot U(s)$ $\qquad$ with a delay T
Properties:
Does not influence magnitude, introduces phase lag
$\underbrace{\phi_{m,T}}_{\text{Phase margin w/ delay}} = \underbrace{\phi_m}_{\text{Phase m. w/o delay}} - \omega_c T$
Reduction of phase margin can lead to instability
$|e^{-j\omega T}| = 1$ $\qquad$ Delay does not affect magnitude
$\angle(e^{-j\omega T}) = \omega t$ $\qquad$ The higher ω the bigger the lag
Time delays in control design:
1. Ignore TD, design controller, check ϕ and increase phase margin if needed
2. Approximation of TD with **Padé Approximation:** (Taylor)

- $e^{-Ts} \simeq \frac{e^{-Ts}}{e^{Ts}} \simeq \frac{2/T - s}{2/T + s}$
- Introduces pole and non-minimum phase zero (limits control bandwith)

Controller implementation

```
%% 1. initialization (compute these values only once)
% sampling period
Ts = 0.05;
% initial error
e_sum = e0;
% controller coefficients  k_p = kp;
k_i = ki;
k_d = kd;
%% 2. loop every Ts seconds
% read ADC
e = adc();
% calculate proportional term
u_p = k_p*e;
% calculate integral term
e_sum = e_sum + e*Ts;
u_i = k_i*e_sum;
% calculate derivative term
e_diff = (e - e_last)/Ts;
u_d = k_d*e_diff;
% compute control u(k)
u = u_p + u_i + u_d;
% write to output
dac(u);
% registry shift
e_last = e;
% repeat until end of operations
```

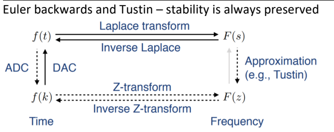
Emulation

Computers cannot do limits (integration and derivatives) → emulate a Control:
1. Design a continuous time controller
2. Choose sampling rate small enough (see "Aliasing")
3. If needed, design an anti-aliasing filter
4. Modify controller to account for phase lag (due to delay)
5. Emulate controller using Euler / Tustin
   a. Substitute s (from below) into the TF → get Tf(z)
6. Analyse open loop stability

Condition: no aliasing
Numerical integration:
Use: $\boldsymbol{p}f(t) = \dot{f}(t)$ $\qquad \boldsymbol{q}^{\pm 1}f(kT_s) = f(kT_s \pm T_s)$

Euler Forward	Euler Backwards	Tustin (trapezoids)	Exact
$p \simeq \frac{q-1}{T_s}$	$p \simeq \frac{q-1}{qT_s}$	$p \simeq \frac{2}{T_s}\frac{q-1}{q+1}$	
$s \simeq \frac{z-1}{T_s}$	$s \simeq \frac{z-1}{zT_s}$	$s \simeq \frac{2}{T_s}\frac{z-1}{z+1}$	$s = \frac{1}{T_s}\ln(z)$
$\lvert z\rvert \simeq \lvert 1 + T_s s\rvert$	$\lvert z\rvert \simeq \frac{1}{\lvert 1 - T_s s\rvert}$	$\lvert z\rvert \simeq \frac{\lvert 1 + T_s s\rvert}{\lvert 1 - T_s s\rvert}$	$z = e^{s\cdot T_s}$

Euler backwards and Tustin – stability is always preserved



With:

	Time	Frequency
Continuous	p	s
Discrete	q	z

Example: Discretise $y(t) = \frac{d}{dx}x(t)$ using Euler Backwards:

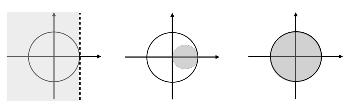
$$\dot{x} = px = \left(\frac{q-1}{qT_s}\right)x = \left(\frac{1-\frac{1}{q}}{T_s}\right)x = \frac{x(kT_s) - x(kT_s - T_s)}{T_s} =$$
$$= \frac{x(kT_s) - x((k-1)T_s)}{T_s} = y(kT_s)$$

***Z*-Transform:**
- Calculate TF in frequency domain in continuous Time (Laplace Transform), then substitute $s = \cdots$

Definition: $\qquad F(z) = \sum_{k=-\infty}^{\infty} f[k]z^{-k}$
Linearity: $\qquad Z(\alpha f[k] + \beta g[k]) = \alpha F(z) + \beta G(z)$
Time-shift: $\qquad Z(f[k - k_0]) = z^{-k_0}F(z))$

Pole mapping:
- Euler back. And Tustin: If a continuous time pole is stable, the discrete pole is (always!) also stable
- Euler forward: A stable continuous pole can become an unstable discrete time pole (calculate to check)

Shaded area: where continuous-time stable poles are mapped
Stable poles lie inside the unit circle.



Euler forward $\qquad$ Euler backwards $\qquad$ Tustin (trapezoids)

Transforming A,B,C,D Matrices:
- Set $\dot{x} = px$ (using the appropriate p)
- Solve so that we have
- $x[k+1] = \underbrace{(*)}_{A_{d,f}}x[k] + \underbrace{(*)}_{B_{d,f}}u[k]$
- And $y[k] = \underbrace{(*)}_{C_{d,f}}x[k] + \underbrace{(*)}_{D_{d,f}}u[k]$

Exact discretisation
$A_d = e^{AT_s} = e^A$ $\qquad C_d = C;$
$B_d = \int_0^{T_s} e^{A\lambda}d\lambda B = A^{-1}(e^{AT_s} - \mathbb{I})B$ $\qquad D_d = D$
Example:
$A = \begin{bmatrix}1 & 2\\ 0 & -3\end{bmatrix}; EVal: 1;\ -3; EVec: \begin{bmatrix}0\\ 1\end{bmatrix}; \begin{bmatrix}-1\\ 2\end{bmatrix};\ e^A = Te^DT^{-1}$

MIMO Control

Transfer function → Transfer Matrices with n states, m inputs, l outputs
$x\in\mathbb{R}^n,\quad u\in\mathbb{R}^m,\quad y\in\mathbb{R}^l$
$A\in\mathbb{R}^{n\times n},\quad B\in\mathbb{R}^{n\times m},\quad C\in\mathbb{R}^{l\times n},\quad D\in\mathbb{R}^{l\times m}$
$P(s) = C(s\mathbb{I} - A)^{-1}B + D,\qquad P\in\mathbb{R}^{l\times m}$
Where P is the transfer matrix.
$G_1G_2 \neq G_2G_1$ Do not commute - General rule: start from output and proceed against signal flow towards input

	Input	Output
Loop TF	$L_I(s) = C(s)P(s)$	$L_O(s) = P(s)C(s)$
Sensitivity fun.	$S_I(s) = (\mathbb{I} + L_I(s))^{-1}$	$S_O(s) = (\mathbb{I} + L_O(s))^{-1}$
Complimentary Sensitivity fun.	$T_I(s) = (\mathbb{I} + L_I(s))^{-1}L_I(s)$	$T_O(s) = (\mathbb{I} + L_O(s))^{-1}L_O(s)$

- Fully controllable if: #outputs = #normal rank TF
- 1 input affect more than 1 output - interactive system
- 1 input affects 1 output – decoupled (SISO in parallel)

Avoid complexity and use SISO controller. • Decentralized Control, directly use SISO. • Decoupled Control, diagonalize system and use SISO control. + less effort - suboptimal results	Multivariable control, optimizing cost function. • LQR control (H_2) • H-∞ control + optimal results - modelling effort

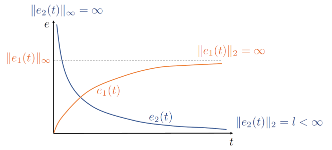
Push-through rule: $G_1(\mathbb{I} - G_2G_1)^{-1} = (\mathbb{I} - G_1G_2)^{-1}G_1$

Finding Poles and Zeros

Recipe (finding poles and zeros)
1) Find all minors (det. of square submatrices)
2) Find pole polynomial $\pi(s)$ where $\pi(s)$ is the least common denominator of all minors (KGV der Teiler). The different denom. of the TFs must be contained in the least common denom.
   Ex: TFs: $\frac{1}{s}, \frac{1}{(s-1)^2}, \frac{s^2-s+2}{(s+2)} \to s(s+2)(s-1)^2$
3) MIMO poles are root of $\pi(s)$
4) Normalize max. minors with $\pi(s)$, that means that the denominator of the max. minors is the pole polynomial.
   MINO zeros are root of greatest divisor of numerators of normalized max. minors.
   Ex: 2nd order: $\frac{s^2(s-1)(s+2)}{s(s+2)(s-1)^2}, \frac{s(s+1)(s-1)(s+2)}{s(s+2)(s-1)^2}, \frac{s(s^2-s+2)}{s(s+2)(s-1)^2} \to \boldsymbol{s}$
   Terms that appear in all numerators (here only s zero@0)
- #Poles gives the order of internal description of syst.

Norms

Definition of a norm: (2-Norm: 'Energy', ∞-Norm: 'Peak')



- Non-negativity: $\| x \| \geq 0$
- Positive definiteness: $\| x \| = 0 \iff x = 0$
- Homogeneity: $\| \alpha x \| = |\alpha| \, \| x \|, \forall \alpha \in \mathbb{C}$
- Triangular inequality:
  $\| x + y \| \leq \| x \| + \| y \|, \quad \forall x, y \in \mathcal{X}$

Vector norms:

- P-th norm: $\| x \|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}, \; p = 1,2,..$
- Two norm: $\| x \|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$
- Infinity norm: $\| x \|_\infty = \| x \|_{max} = \max |x_i|$

Matrix norms:

- Euclidian norm: (tr = spur)
  $$\| G \|_F = \sqrt{\sum_{i,j} |g_{ij}|^2} = \sqrt{tr(G^* G)}$$
- Max element norm: $\| G \|_{max} = \max |g_{ij}|$

Induced matrix norms:
(quantify maximum gain (amplification) of an output vector for any possible input direction at a given frequency).
They all satisfy multiplicative property:
$\| G_2 G_1 \|_{i,p} \leq \| G_2 \|_{i,p} \cdot \| G_1 \|_{i,p}$

- Induced p-norm
  $$\| G \|_{i,p} = \max_{w \neq 0} \frac{\| G\omega \|_p}{\| \omega \|_p} = \max_{\|\omega\|_p = 1} \| G\omega \|_p, \quad p = 1,2, \ldots$$
- Induced 2-norm:
  $$\| G \|_{i,2} = \sqrt{\rho(G^* G)} = \sqrt{|\lambda_{max}(G^* G)|} = \overline{\sigma}(G)$$
  where σ is the largest singular value of G (see SVD).
  Induced norm of A is maximum amplification u $\to$ y
- Induced infinity norm:
  $$\| G \|_{i,\infty} = \max\left(\sum_j |g_{ij}|\right)$$
  $\to$ Sum over row vectors

Signal (temporal) norms:

- p-th norm: $\| e(t) \|_p = \left(\int_{-\infty}^{\infty} \sum_{i=1}^n |e_i(\tau)|^p d\tau\right)^{\frac{1}{p}}$
- Two norm ("energy" of signal):
  $$\| e(t) \|_2 = \sqrt{\int_{-\infty}^{\infty} \sum_{i=1}^n |e_i(\tau)|^2 d\tau}$$
- Infinity norm: peak value in time:
  $$\| e(t) \|_\infty = \sup_\tau (\max_i |e_i(\tau)|)$$
  where sup is the least upper bound.

System norms:

- $\mathcal{H}_2$ Infinity norm: (not an induced norm) energy of system
  $$\| G(s) \|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i,j=1}^n |g_{ij}(j\omega)|^2 d\omega}$$
- $\mathcal{H}_\infty$ Infinity norm: peak of maximum singular value:
  $$\| G(s) \|_\infty = \sup_\omega \| G(j\omega) \|_{i,2} = \sup_\omega \overline{\sigma}(G(j\omega))$$

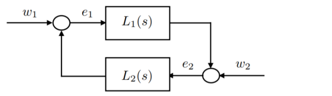
MIMO Stability

BIBO Stability: A system is BIBO stable, if there is a finite constant such that the output norm is bounded by the input norm:
$$\| y \|_\infty \leq k \, \| u \|_\infty$$
Closed loop transfer function has poles all in the open left-hand plane (all real parts of the roots of the characteristic polynomial are strictly negative). E.g. if all eigenvalues of A are $\lambda < 0$.

Internal stability:
Bounded input at any place, for any IC, all states remain bounded for all future time.
$$\begin{bmatrix} Y(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} (\mathbb{I} + PC)^{-1}PC & (\mathbb{I} + PC)^{-1}P \\ (\mathbb{I} - CP)^{-1}C & -(\mathbb{I} + CP)^{-1}CP \end{bmatrix} \begin{bmatrix} R(s) \\ D(s) \end{bmatrix}$$
Or $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \end{bmatrix} \cdot \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$

$\Rightarrow$ Each internal TF stable ($Re(\pi) < 0$) $\iff$ internal stability.
Internal stability $\Rightarrow$ external stability
Internal stability ($\Leftarrow$) external stability (only if observable & controllable, so no pole/zero cancellations)

Small gain theorem: Provides a sufficient condition for closed loop stability. Let $L_1(s), L_2(s)$ be stable, rational and proper. Then the closed loop interconnection (TF) is stable if $\|L_1\| \cdot \|L_2\| < 1$. The norm can be any norm satisfying the multiplicative property (e.g. infinity norm).



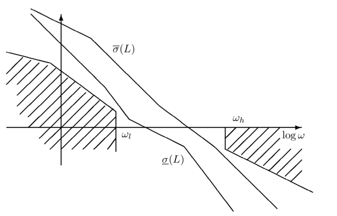
Small gain theorem for internal stability (BIBO)
$$\begin{bmatrix} e_1 \\ e_1 \end{bmatrix} = \begin{bmatrix} (\mathbb{I} - L_2 L_1)^{-1} & (\mathbb{I} - L_2 L_1)^{-1} L_2 \\ (\mathbb{I} - L_1 L_2)^{-1} L_1 & (\mathbb{I} - L_1 L_2)^{-1} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$$
w_2 captures effect of noise, w_1 effect of disturbance/input

MIMO Performance and Singular Values

Loop shaping of MIMO systems is possible with singular values, but way too tedious, thus e.g. LQR control is much easier.

Disturbance attenuation (low ω)	Noise rejection/robustness (high ω)
$\underline{\sigma}(C) \gg 1$	$\overline{\sigma}(C) \leq M$
$\underline{\sigma}(CP) \gg 1$	$\overline{\sigma}(CP) \ll 1$
$\underline{\sigma}(PC) \gg 1$	$\overline{\sigma}(C) \ll 1$

Inverse: $\overline{\sigma}(A^{-1}) = \frac{1}{\underline{\sigma}(A)}$
Fan's Theorem: $\sigma_i(A) - \overline{\sigma}(B) \leq \sigma_i(A + B) \leq \sigma_i(A) + \overline{\sigma}(B)$
$\underline{\sigma}(A) - 1 \leq \underline{\sigma}(\mathbb{I} + A) \leq \underline{\sigma}(A) + 1$
Product: $\overline{\sigma}(AB) \leq \overline{\sigma}(A)\overline{\sigma}(B)$ and $\underline{\sigma}(AB) \geq \underline{\sigma}(A)\underline{\sigma}(B)$
Determinant: $det(X + AB) = det(X + B^{-1}XA)$



Bandwidth: defined as a region for MIMO system. If a single number is required, the worst-case scenario is provided.

Good performance

$\eta = T_O r - S_O P d - T_O n$
$\varepsilon = r - \eta = S_O r - S_O P d + T_O n$
$v = C S_O r - S_I d - C S_O n$
$u = S_I C r - T_I d - S_I C n$

- Make "small":

Performance criteria	Transfer function	Low frequency (r,d)	High frequency (n)	Transfer function
Good tracking	$d \to \varepsilon$	$S_O(s)P(s)$	$T_O(s)$	$n \to \varepsilon$
Good tracking	$r \to \varepsilon$	$S_O(s)$		
Disturbance attenuation (input)	$d \to v$	$S_I(s)$		
Disturbance attenuation (output)	$d \to \eta$	$S_O(s)P(s)$		
Noise rejection (input)			$C(s)S_O(s)$	$n \to v$
Noise rejection (output)			$T_O(s)$	$n \to \eta$
Robustness to additive model uncertainty			$T_O(s), T_I(s)$	$\Delta \to \det(I + PC)$
Input saturation			$C(s)$	

Weighted Performance
Performance constraints may be encoded in frequency dependent weighting matrices W_s, W_T, W_C:
Low frequency disturbance rejection: $\|W_s S\| < 1$
High frequency noise attenuation: $\|W_T T\| < 1$
Control effort: $\|W_C CS\| < 1$

Singular Value Decomposition (SVD)

SVD informs on strong and weak input / output directions of a MIMO plant. A singular value is sort of the gain of the plant.
$$G = U \Sigma V^*$$
where
$\boldsymbol{U} = [u_1, \ldots, u_l] \in \mathbb{C}^{l \times l}$ (orthonormal; **output directions**),
$\boldsymbol{V} = [v_1, \ldots, v_m] \in \mathbb{C}^{m \times m}$ (orthogonal $[V \cdot V^T = \mathbb{I}]$ and unit length, **input directions**)
$$\boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_p \end{bmatrix},$$
$\sigma_1 \geq \cdots \geq \sigma_j \geq 0, \; j = \min(l, m)$ and
$\sigma_i = \sigma_i(G) = \sqrt{\lambda_i(G^* G)}$

Input-Output bounds: $\underline{\sigma}(G(j\omega)) \leq \frac{\|y\|_2}{\|u\|_2} \leq \overline{\sigma}(G(j\omega))$

Condition number: $\kappa(G) = \frac{\overline{\sigma}(G)}{\underline{\sigma}(G)} \geq 1$

measurement how directional plant is. Big condition number means strongly directional plant -> ill conditioned system, more difficult to control. Is the ration of the max. and min. singular values.

Recipe SVD:
1) Find eigenvalues of $G^* G$
2) Singular value: $\sigma = \sqrt{Eigenvalues(G^* G)}$.
3) Build Σ with σ_1 a diagonal matrix in descending order, Note: $\dim(\Sigma) = \dim(G)$.
4) Compute right singular vectors v_i (**input dir.,** normalized eigenvectors of $G^* G$). Make matrix unitary (vectors orthonormal) by using Gram-Schmidt if necessary
5) *Shortcut:* $u_i = \frac{1}{\sigma_i} \cdot G v_i$ for $\sigma_i \neq 0$
6) Compute left singular vectors u_i (**output dir.,** normalized eigenvectors of GG^*).

Example: Find max. amplification of SVD: largest SV in first column – use first column of V and U and calculate magnitude and phase
$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1.5 \end{bmatrix}; V = \begin{bmatrix} \sqrt{\frac{3}{8}} + \sqrt{\frac{3}{8}} j & \sqrt{2}/2 \\ 1/2 & \sqrt{2}j/2 \end{bmatrix}; U = \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{3}j}{6} & 0 \\ \frac{1}{\sqrt{3}} + \frac{j}{\sqrt{3}} & 1 \end{bmatrix}$$
$$|V_{max}| = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}; \angle V_{max} = \begin{bmatrix} 0.78 \\ 0 \end{bmatrix}; |U_{max}| = \begin{bmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \end{bmatrix}; \angle U_{max} = \begin{bmatrix} 0.52 \\ 0.78 \end{bmatrix}$$
$$u(t) = \left[|V_{max}| \cdot \cos(\omega_0 t + \angle V_{max})\right]$$
$$y_\infty(t) = 3 \cdot \left[|U_{max}| \cdot \cos(\omega_0 t + \angle U_{max})\right]$$

Decentralized control:

Useful when MIMO system has low degree of interaction between inputs and outputs. Needs same number of inputs and outputs.

- Tall system ($l > m$): Has more outputs than inputs and not all outputs are affected by inputs.
- Fat system ($l < m$): Has more inputs than outputs

Relative Gain Array (RGA)
Allows to quantify "degree of coupling" of a MIMO system:
$$\Lambda(P) = RGA(P) = P \times (P^\dagger)^T$$
where $\times$ indicates the element-by-element multiplication (Hadamard od Shur product) and $\dagger$ denotes the pseudo-inverse (Moore-Penrose).
$$\lambda_{ij} = \frac{gain \; from \; u_j \; to \; y_i \; with \; all \; other \; loops \; open}{gain \; from \; u_j \; to \; y_i with \; all \; other \; loops \; closed}$$

$\lambda_{ij} = 1$	Open & closed loop gains are equal. Interact. doesn't affect channel (Pick for pairing)
$\lambda_{ij} = 0$	Open loop gain is zero. I.e. no effect of j-th input to i-th output. (don't pick for pairing)
$0 < \lambda_{ij} < 1$	Closing the loop increases channel gain. Pair only if close to 1.
$\lambda_{ij} > 1$	Closing loop decr. Channel gain. The higher, the more interaction. Pair only if close to 1.
$\lambda_{ij} < 1$	Closed and open loop gains have opposite signs (do not pick for pairing)

RGA properties:

- RGA elements are dimensionless – units of inputs and outputs do not matter
- Rows and columns of RGA add up to 1.
- RGA is invariant to input/output scaling
- RGA is property of the system and does not depend on the chosen controller

For 2x2:

- $[RGA]_{11} = [PGA]_{22} = \frac{P_{11}P_{22}}{P_{11}P_{22} - P_{12}P_{21}}$
- $[RGA]_{12} = 1 - [RGA]_{11}$
- $\lambda_{11} = 1/(1 - \frac{P_{12}P_{21}}{P_{11}P_{22}})$, $\Lambda(\text{P}) = \begin{pmatrix} \lambda_{11} & 1 - \lambda_{11} \\ 1 - \lambda_{11} & \lambda_{11} \end{pmatrix}$

For $\boldsymbol{n \times n}$

- If $[RGA] = \begin{pmatrix} \approx 1 & \approx 0 \\ \approx 0 & \approx 1 \end{pmatrix}$, SISO control is possible!

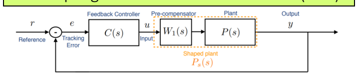
$$P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \; or \; \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \Rightarrow RGA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$P = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix} \; or \; \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \Rightarrow RGA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Frequency Dependency RGA:

- If $RG(s = 0)$ has positive diagonal entries, SISO control possible.
- If $RGA(j\omega) = \mathbb{I}$ at $\omega_c \pm$ one decade, one can ignore cross couplings and do SISO control.

RGA number:
$\| \Lambda(P) - I \|_{sum}$ where $\| A \|_{sum} = \sum |a_{ij}|$
A low RGA number indicates diagonal dominance and hence good pairing.

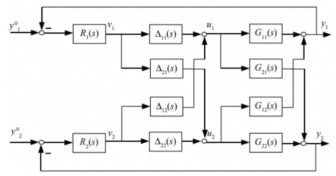
Decoupling and internal model control (IMC)



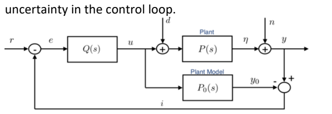
Objective: find compensator(s) such that shaped plant is diagonal at a specific frequency. Assume plant is square (if not, use pseudo-inverse in place of inverse).
Dynamic decoupling: Diagonal shaped plant at all freq.
Steady-state decoupling: Diagonal shaped plant only at steady state.
Internally stable control loop
Q-Parameterization: define $Q = C(s)\left(\mathbb{I} + P(s)C(s)\right)^{-1}$
If $C(s)$ is proper, $Q(s)$ must be proper
$P(s)$ stable, then closed loop system is stable if and only if $Q(s)$ is stable.
Implications:

- finding Q $\iff$ finding C
- If Q is stable, internal stability guaranteed, even if C is unstable
- Through Q-Param. Relation between S(s) and C(s) is linear.

$C = Q(\mathbb{I} - PQ)^{-1}$ is the parametrization of all internally stabilizing controllers for any stable TF matrix Q (given stable P).
Forward Decoupling:
Idea: Put a Δ decoupling block in front of the plant:



Internal model control:
Internal model approach: feedback only mismatch between model prediction and actual measured output, i.e., uncertainty in the control loop.



$$Y = (\mathbb{I} - PQ)N + P(\mathbb{I} - QP)D + PQR + PQ(P - P_0)U$$

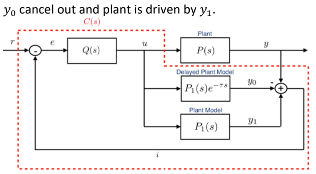
- If $P = P_0$ and $Q = P^{-1}$, then $Y = R$
- But if P is strictly proper, $Q = P^{-1}$ not realizable.
  Solution: add sufficient number of stable poles to Q:
  $$Q(s) = \frac{1}{(s + \alpha)^r} P^{-1}(s), \quad \alpha > 0, \quad r \in \mathbb{N}$$
- Problem: If P has RHP-zeros, Q has unstable poles
- Solution: Invert only the minimum phase component. Make the non-minimum phase zeros, minimum phase. Amplitude will stay the same, phase will change.
- Problem: time delays cannot be inverted (would generate future predicting).
- Solution: Ignore delay when inverting plant to determine Q, but considering when calculating C (or with Pade-approx.).

Additive Uncertainty

Simply add an uncertainty $W_1 \Delta W_2$ to our plant. This gives:
$$L = P_{unc} C = (P + W_1 \Delta W_2)C$$
The Nyquist criterion is fulfilled if
$$\underbrace{|W_1 \Delta W_2 \cdot C|}_{\text{r of uncert. circle}} < \underbrace{|1 + L|}_{\text{dist. to (-1, 0) point}}$$

Smith predictor

Used as dead time compensator for huge delays. Idea: have an internal model for delayed and non-delayed plant. y and y_0 cancel out and plant is driven by y_1.



+ very fast, helps to control dead time systems

- if $P \neq P_0$ or $\tau_0 \neq \tau \Longrightarrow$ Performance can be really bad.
- inherent time delay T cannot be eliminated (just adverse effects)

MIMO Controller Synthesis

No meaningful phase information for MIMO systems – we need systematic methods. There are two main families:
$\mathcal{H}_2$ methods
optimisation problem in time domain; minimise the mean value of a suitable error signal over all frequencies
$\mathcal{H}_\infty$ methods
optimisation problem in frequency domain; minimise worst possible value of a suitable error signal over all frequencies

Linear Quadratic Regulator (LQR) $\to$ $\mathcal{H}_2$

Goal is to find the best balance between the performance, represented by R and the energy needed, represented by Q. Always a trade-off, so best combination is needed!
Find a control input $u(t), t \in [0, \infty]$ such that
$J_{LQR} = \int_0^\infty \| z(t) \|_2^2 + \rho \, \| u(t) \|_2^2 \, dt, \; \rho \in \mathbb{R}^+$ is minimized.
$$J_{LQR} = \int_0^\infty u^T R u + x^T Q x + 2x^T N u \, dt$$
$u^T R u - error$
$x^T Q x - given \; energy$
where $u \in \mathbb{R}^{m \times 1}, \; z \in \mathbb{R}^{k \times 1}, \; x \in \mathbb{R}^{n \times 1}$, (N in most cases 0)
$R = F^T \bar{Q} F + \rho \bar{R}, \quad Q = E^T \bar{Q} E, \quad N = E^T \bar{Q} F,$
$K = R^{-1}(N + PB)^T$ and R, Q are symmetric and positive definite to solve ARE
Solution exists if:

- The system (A, B) is stabilizable (all unstable p. control.)
- The pair $(\tilde{A}, \tilde{Q}) = (A - BR^{-1}N^T, Q - NR^{-1}N^T)$ is detectable.

Then: $u_{LQR}(t) = -R^{-1}(N + PB)^T x(t) = -Kx(t)$ where P is the real, symmetric, positive definite solution to the Riccati equation:
$$ARE: \quad A^T P + PA - (N + PB)R^{-1}(N^T + B^T P) + Q = 0$$
Goal is to find P!

Hamiltonian Method (only solves linear problems)
$H = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^* \end{pmatrix}$ now $(P \; -\mathbb{I})H\begin{pmatrix} \mathbb{I} \\ P \end{pmatrix} = 0 \to ARE$
a) Build Hamiltonian H
b) Find eigenvalues of H
c) Find eigenvectors of stable eigenvalues ($Re(\lambda) < 0$)
d) Write $\begin{pmatrix} x_1^{n \times n} \\ x_2^{n \times n} \end{pmatrix} = (v_1 \ldots v_i) \Longrightarrow P = X_2 X_1^{-1}$

Direct Method
a) Parametrize P using its properties (symmetric, pos. def.)
b) Plug P into ARE and solve nonlinear problems to find $P_1, P_2, \ldots$ exclude solutions that lead to non-positive definiteness.

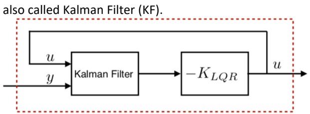
Dynamics of closed-loop system (see State Feedback):
$(A - BK) = \pi_{cl} =$ *closed loop poles*
Weighting Matrices (only relative size matters!)

R > Q	R < Q
High cost on input	Low cost on input
Expensive control	Aggressive control
$\Longrightarrow$ conservative controller	$\Longrightarrow$ fast response controller

Changing J by a constant has no influence on bandwith
Example: infinite-horizon LQR controller
Cost function: $\int_0^\infty 64x(t)^2 + u(t)^2 dt \quad$ Q=64 $\quad$ R=1
1st order sys: $\dot{x}(t) = 3x(t) + 0.5u(t) \quad$ A=3 $\quad$ B=0.5
$$CARE: 0 = P \cdot BR^{-1}B^T \cdot P - PA - A^T P - Q$$
$$= \frac{1}{4}P^2 - 6P - 64$$
P=32 (has to be positive: -8 not a sol) $\; K = R^{-1}B^T P = 0.5 \cdot 32 = 16$

Linear Quadratic Gaussian (LQG)

LQG is the union of LQR and a Linear Quadratic Estimator (LQE),
also called Kalman Filter (KF).



LQG

Kalman Filter: Estimate state of system by combining measurements from different sources that ware subject to noise. If all noises are Gaussian and zero mean, the KF minimizes the mean square error of the estimated variables.

- Easy and efficient to implement.
- Optimal for LTI systems with Gaussian zero mean noise
  - Works incredibly well in practice

LQR guarantees stability of closed loop system.
+ high level approach $\to$ tune performance
- LQR & KF robust $\to$ LQG no robustness guarantees