Quantum Mechanics

Week 4

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Contents

Pre-Reading Note

Dear Students,

Welcome to the course on Quantum Mechanics. As part of your learning resources, I will prepare a series of educational materials and sheets designed to complement the lectures.

Please note that these materials are **abridged versions** of the content from the textbook ***"Introduction to Quantum Mechanics By David J. Griffiths"***. They have been tailored to align with the class schedule and topics, providing you with concise summaries and key points for each topic covered.

It's important to understand that these sheets are **not standalone resources**. They are intended to be used in conjunction with the class material. For a deeper understanding and a more comprehensive view of each topic, I strongly encourage you to refer to the mentioned textbook.

The book provides detailed explanations, examples, and insights that go beyond the scope of our summaries. It will be an invaluable resource for you to solidify your understanding of Quantum Mechanics.

I cannot guarantee neither correctness nor completeness of the script. Please report any mistake directly to me.

Have fun with Quantum Mechanics!

Best regards,

Mark Benazet Castells

1 Harmonic Oscillator in Quantum Mechanics

1.1 Mathematical Formulation

In quantum mechanics, the harmonic oscillator is modeled by a particle in a quadratic potential well. The potential energy is a function of the particle's position x and is given by:

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2, \tag{1}$$

where k is the spring constant, m is the mass of the particle, and $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency of the oscillator. This potential represents a restoring force that increases linearly with displacement, a fundamental aspect of simple harmonic motion.

To determine the eigenstates of the harmonic oscillator, it is necessary to solve the Schrödinger equation with this potential.

1.2 Creation and Annihilation Operators

A powerful approach in quantum mechanics is the use of creation (raising) and annihilation (lowering) operators, which simplify the analysis of the harmonic oscillator:

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}\left(-i\hat{p} + m\omega x\right) \tag{2}$$

$$\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}\left(i\hat{p} + m\omega x\right), \tag{3}$$

where $\hat{p}$ is the momentum operator. These operators are fundamental in constructing the eigenstates of the harmonic oscillator and exploring its quantum properties.

It's noteworthy that there are two primary ways to solve the quantum harmonic oscillator problem: the analytic method, which involves directly solving the Schrödinger equation with the harmonic potential, and the algebraic method, which uses these creation and annihilation operators to construct the eigenstates.

1.3 Hamiltonian and Energy Levels

The Hamiltonian, representing the total energy (kinetic plus potential) of the system, is constructed using these operators:

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right) = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right). \tag{4}$$

In this model, the energy levels are quantized, and the ground state (lowest energy state) has an energy of $\frac{1}{2}\hbar\omega$, reflecting the zero-point energy characteristic of quantum systems.

The ground state wave function, ψ_0, for the quantum harmonic oscillator is a fundamental solution in quantum mechanics. This state represents the lowest energy state of the oscillator. The wave function is given by:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$$

where m is the mass of the particle, ω is the angular frequency of the oscillator, $\hbar$ is the reduced Planck constant, and x is the position. This Gaussian function reflects the probability amplitude for finding the particle at a position x.

The nth excited state of the system is obtained by applying the raising operator n times to the ground state wave function:

$$\psi_n = \frac{1}{\sqrt{n!}} (\hat{a}_+)^n \psi_0, \tag{5}$$

with the corresponding energy levels given by:

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega. \tag{6}$$

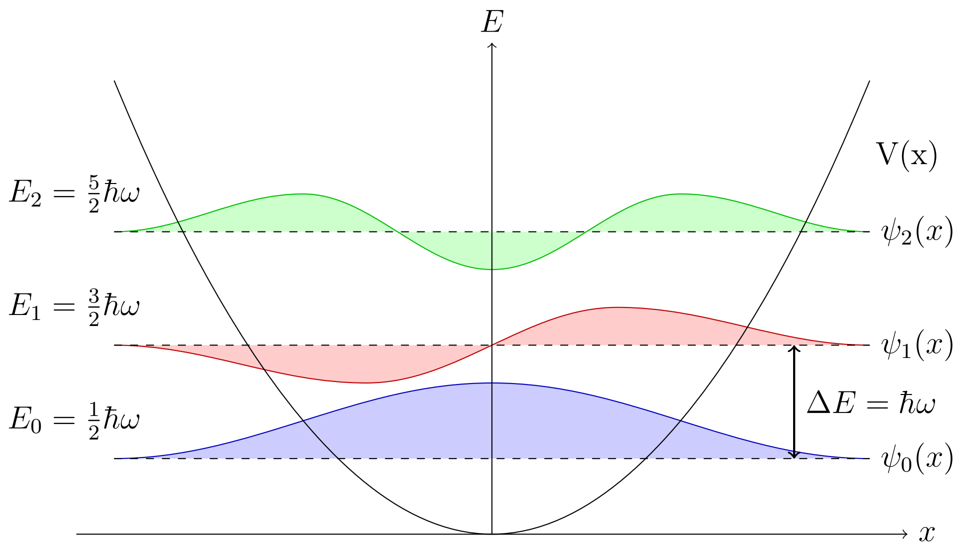

Figure 1: The first few energy states of a quantum harmonic oscillator.

1.4 Key Properties of the Quantum Harmonic Oscillator

The quantum harmonic oscillator exhibits several distinctive and important properties:

- **Quantized Energy Levels:** Unlike classical oscillators, the energy levels of a quantum harmonic oscillator are discrete and quantized. These energy levels are given by $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, where n is a non-negative integer representing the quantum number of the state.

- **Zero-Point Energy:** The ground state energy of the quantum harmonic oscillator ($n = 0$) is $\frac{1}{2}\hbar\omega$. This implies that even at absolute zero temperature, the oscillator possesses non-zero energy, a phenomenon absent in classical physics.

- **Wave Function Symmetry:** The wave functions, or the quantum states, of the harmonic oscillator exhibit symmetry. The ground state wave function is Gaussian, and higher state wave functions are Hermite-Gaussian. This symmetry leads to certain properties like the expectation value of position $\langle x \rangle$ being mostly zero. Furthermore, each subsequent wave function $\psi_{n+1}(x)$ has one more node than its predecessor $\psi_n(x)$.

- **Orthogonality:** The wave functions ψ_n are mutually orthogonal, satisfying the condition $\int_{-\infty}^{\infty} \psi_m^*(x)\psi_n(x)dx = \delta_{mn}$, where δ_{mn} is the Kronecker delta.

- **Completeness:** The set of wave functions ψ_n forms a complete basis for the space of square-integrable functions. Any square-integrable function can be expressed as a linear combination of these wave functions.

- **Existence Outside Potential Well:** The non-zero probability of finding a particle outside the potential well in quantum mechanics is due to the finiteness of the potential, not its shape. Whether the potential is x^2, x, or x^3, the key aspect is that it is not infinite like in an infinite square well. This finiteness allows the wave function to extend beyond the potential boundaries, enabling particle penetration into classically forbidden regions.

- **Classical Turning Points:** The quantum harmonic oscillator's behavior can be compared with its classical counterpart by examining the classical turning points, where the kinetic energy is zero.

- **Energy Gap:** The energy gap between successive levels in a quantum harmonic oscillator is fundamentally influenced by the angular frequency ω, which is in turn determined by the mass m and the spring constant k. Specifically, the angular frequency is given by $\omega = \sqrt{\frac{k}{m}}$. As a result, the energy gap between consecutive states, characterized by the expression $E_{n+1} - E_n = \hbar\omega$, increases with a stiffer spring (higher k) and decreases for a heavier mass (higher m).