# Quantum Mechanics

Week 6

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# Contents

# Pre-Reading Note

Dear Students,

Welcome to the course on Quantum Mechanics. As part of your learning resources, I will prepare a series of educational materials and sheets designed to complement the lectures.

Please note that these materials are **abridged versions** of the content from the textbook ***"Introduction to Quantum Mechanics By David J. Griffiths"***. They have been tailored to align with the class schedule and topics, providing you with concise summaries and key points for each topic covered.

It's important to understand that these sheets are **not standalone resources**. They are intended to be used in conjunction with the class material. For a deeper understanding and a more comprehensive view of each topic, I strongly encourage you to refer to the mentioned textbook.

The book provides detailed explanations, examples, and insights that go beyond the scope of our summaries. It will be an invaluable resource for you to solidify your understanding of Quantum Mechanics.

I cannot guarantee neither correctness nor completeness of the script. Please report any mistake directly to me.

Have fun with Quantum Mechanics!

Best regards,

*Mark Benazet Castells*

# 1 Quantum Measurement

Quantum measurement is fundamentally tied to the concept of observables, represented by operators in Hilbert Space, and the probabilistic nature of quantum mechanics.

## 1.1 Observables and Projection Operators

An observable in quantum mechanics is associated with a set of orthogonal projection operators, which represent the eigenfunctions or eigenstates of the observable in Hilbert Space. Mathematically, an observable $\hat{O}$ can be expressed as a sum of projection operators $P_i$, each corresponding to an eigenvalue $\lambda_i$:

$$\hat{O} = \sum_i \lambda_i P_i, \quad \text{where} \quad P_i = |\psi_i\rangle \langle\psi_i| \,.$$

Here, $|\psi_i\rangle$ are the orthogonal eigenstates of $\hat{O}$.

## 1.2 Probabilistic Nature of Quantum Measurement

The first measurement of a quantum system is inherently probabilistic (if we are not starting in an eigenstate). This randomness is a fundamental aspect of quantum mechanics, not just due to experimental limitations. The probability of obtaining a particular eigenvalue $\lambda_i$ when measuring an observable $\hat{O}$ on a state $|\Psi\rangle$ is given by the Born rule:

$$P(\lambda_i) = \langle\Psi| P_i |\Psi\rangle \,.$$

Let's confirm our understanding with a brief **sanity check**:
Representing the state $|\Psi\rangle$ as:

$$|\Psi\rangle = \sum_n c_n |\psi_n\rangle \,,$$

where $c_n$ are coefficients, and applying the projection operator $P_i = |\psi_i\rangle \langle\psi_i|$, we get:

$$P(\lambda_i) = \left(\sum_m c_m^* \langle\psi_m|\right) |\psi_i\rangle \langle\psi_i| \left(\sum_n c_n |\psi_n\rangle\right) .$$

Using eigenstate orthogonality, $\langle\psi_m|\psi_n\rangle = \delta_{mn}$, simplifies to:

$$P(\lambda_i) = c_i^* c_i.$$

Given $|\psi_i\rangle$ are normalized ($\langle\psi_i| |\psi_i\rangle = 1$), we arrive at:

$$P(\lambda_i) = |c_i|^2,$$

matching our expected quantum probability.

**Remark:** The detailed explanation (and the projection operators) provided here is for a deeper understanding and can be considered supplementary. In the context of this course, you may accept these concepts as foundational without delving into the intensive mathematical details.

## 1.3 Ensembles in Quantum Measurement

An ensemble in quantum mechanics refers to a large collection of identically prepared systems. Measurements on ensembles allow for the statistical interpretation of quantum mechanics, providing insight into the distribution of outcomes over many identical experiments.

## 1.4 Commutation Relations and Measurement

The commutation relation between two operators $\hat{A}$ and $\hat{B}$ has profound implications for quantum measurements:

**Commuting Operators $[\hat{A}, \hat{B}] = 0$:**

- Commuting operators share a common eigenbasis, indicating compatible observables.
- Simultaneous measurement of both observables is possible with arbitrary precision.
- The order of measurements does not affect the outcome: $\hat{A}\hat{B}\left|\psi_n\right\rangle = \hat{B}\hat{A}\left|\psi_n\right\rangle$.

**Non-Commuting Operators $[\hat{A}, \hat{B}] \neq 0$:**

- Non-commuting operators do not share a common eigenbasis, implying independent observables.
- Simultaneous precise measurement of both is fundamentally impossible.
- The order of measurements affects the outcome: $\hat{A}\hat{B}\left|\psi_n\right\rangle \neq \hat{B}\hat{A}\left|\psi_n\right\rangle$.
- The Uncertainty Principle arises, linking the uncertainties in measurements of these observables.

# 2 Generalized Uncertainty Principle

The Uncertainty Principle is a fundamental concept in quantum mechanics that arises from the probabilistic nature of quantum measurement, particularly when dealing with non-commuting observables.

## 2.1 Mathematical Formulation

For any pair of observables $A$ and $B$, the Uncertainty Principle is expressed as:

$$\sigma_A^2 \cdot \sigma_B^2 \geq \left( \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2 \tag{1}$$

In this equation, $\sigma_A$ and $\sigma_B$ are the standard deviations (measures of uncertainty) for observables $A$ and $B$. The term $\langle [\hat{A}, \hat{B}] \rangle$ represents the expected value of the commutator of the corresponding operators.

## 2.2 Understanding the Implications

This principle highlights a fundamental limit to the precision of simultaneous measurements of certain pairs of observables. It implies that as the measurement precision of one observable increases, the precision of the other decreases, provided the observables do not commute. This is not just an experimental limitation but a basic feature of quantum systems.

### Example

A classic example of the Uncertainty Principle is the relationship between position ($x$) and momentum ($p$). Their standard deviations, $\sigma_x$ and $\sigma_p$, obey the relation:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2},$$

where $\hbar$ is the reduced Planck constant. This relation signifies that one cannot simultaneously measure the position and momentum of a particle with arbitrary precision.

# 3 Kronecker Delta and Dirac Delta Function

## 3.1 Kronecker Delta

The Kronecker delta, denoted as $\delta_{ij}$, is a function of two variables (often integers) that is 1 if the variables are equal, and 0 otherwise. It is defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

In quantum mechanics, the Kronecker delta is used to express orthogonality and normalization conditions, especially in discrete systems.

## 3.2 Dirac Delta Function

The Dirac delta function, denoted as $\delta(x)$, is a generalized function or distribution. It is not a function in the traditional sense but is defined by its behavior under integration:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a).$$

This property indicates that the Dirac delta function is zero everywhere except at $x = a$, where it is infinitely high such that its total integral is 1. In quantum mechanics, the Dirac delta function is used to represent continuous normalization conditions, like in the case of wavefunctions in position space.

**Remark:** While the Kronecker delta is used in discrete contexts (like summing over indices in matrix operations), the Dirac delta function is essential in continuous contexts, often seen in integrals over space or time.