Problem Set 8

Problem 1

For typical two-particle quantum mechanical systems, the interaction potential $V(\vec{r}_1, \vec{r}_2)$ only depends on the vector $\vec{r} = \vec{r}_1 - \vec{r}_2$. In that case, the Schrödinger equation separates if we change variables from $\vec{r}_1, \vec{r}_2$ to $\vec{r}$ and $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1 + m_2)$, where $\vec{R}$ is the center of mass.

1. Show that $\vec{r}_1 = \vec{R} + \frac{m_r}{m_1}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_r}{m_2}\vec{r}$, $\vec{\nabla}_1 = \vec{\nabla}_r + \frac{m_r}{m_2}\vec{\nabla}_R$ and $\vec{\nabla}_2 = -\vec{\nabla}_r + \frac{m_r}{m_1}\vec{\nabla}_R$, where m_1 and m_2 are the masses of particles 1 and 2 and

$$m_r = \frac{m_1 m_2}{m_1 + m_2}$$

is the **reduced mass** of the system.

2. Show that the T.I.S.E. becomes

$$\left[\frac{-\hbar^2}{2(m_1+m_2)}\nabla_R^2 - \frac{\hbar^2}{2m_r}\nabla_r^2 + V(\vec{r})\right]\psi = E\psi$$

3. Separate the variables, letting $\psi(\vec{R}, \vec{r}) = \psi_R(\vec{R})\psi_r(\vec{r})$. Note that ψ_R satisfies the one-particle Schrödinger equation, with total mass $(m_1 + m_2)$ in place of m, zero potential, and energy E_R, while ψ_r satisfies the one-particle Schrödinger equation, with reduced mass m_r in place of m, potential $V(\vec{r})$, and energy E_r. The total energy is $E = E_r + E_R$. This shows that the center of mass moves like a free particle, and the relative motion (i.e. the motion of particle 2 with respect to particle 1) is the same as if we had a single particle with the reduced mass subject to the potential V. Exactly the same decomposition occurs in classical mechanics, where a two-body problem is reduced to an equivalent one-body problem.

4. Using the above, determine the percentage error (to two significant digits) in our calculated binding energy of the hydrogen atom where previously we just used the electron mass instead of the reduced mass.

Problem 2

In lecture, we discussed a two-particle system where each particle was sitting in a one-particle state (neglecting spin), $\psi_a(\vec{r})$ or $\psi_b(\vec{r})$. If the two particles are indistinguishable, we must construct the two wavefunctions (neglecting spin):

$$\psi_\pm = C[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)]$$

1. If ψ_a and ψ_b are orthogonal, and both normalized, what is the constant C?

2. If $\psi_a = \psi_b$ and it is normalized, what is the constant C? Is this case possible for bosons, fermions, or both?

Problem 3

Neglecting spin, imagine that you are given two noninteracting particles, each of mass m, in an infinite square well. If one particle is in the state $\psi_{n'}$ and the other is in $\psi_{n''}$ where $n' \neq n''$, calculate $\langle (x_1 - x_2)^2 \rangle$ for the following cases:

1. The two particles are distinguishable.

2. The two particles are indistinguishable and the wavefunction is symmetric with respect to exchange of the two particles.

3. The two particles are indistinguishable and the wavefunction is antisymmetric with respect to exchange of the two particles.

Note: the solution to problem #1 from problem set #2 will be helpful to do this problem.