Problem Set 7

Problem 1

A **hydrogenic atom** consists of a single electron orbiting a nucleus with Z protons. For example, $Z = 1$ for hydrogen itself, $Z = 2$ for helium with one electron removed, $Z = 3$ for lithium with two electrons removed, etc. Determine the Bohr energies $E_n(Z)$, the binding energy $E_1(Z)$, and the Bohr radius $a(Z)$ for a hydrogenic atom. Hint: there is nothing new to calculate here. The potential is still spherically symmetric, but the strength of the potential is increased by a factor of Z.

Problem 2

1. Starting with the **canonical commutation** relations for the Cartesian components of the operators $\hat{\vec{r}}$ and $\hat{\vec{p}}$, determine the following:

$$[\hat{L}_z, \hat{x}], \quad [\hat{L}_z, \hat{y}], \quad [\hat{L}_z, \hat{z}], \quad [\hat{L}_z, \hat{p}_x], \quad [\hat{L}_z, \hat{p}_y], \quad [\hat{L}_z, \hat{p}_z].$$

2. Evaluate the commutators: $[\hat{L}_z, \hat{r}^2]$ and $[\hat{L}_z, \hat{p}^2]$.

3. Show that the Hamiltonian $\hat{H}$ commutes with all three components of $\hat{L}$ if the potential is spherically symmetric. Thus, $\hat{H}, \hat{L}^2$, and $\hat{L}_z$ all commute. What is the significance of this?

Problem 3

Two particles of mass m are attached to the ends of a massless rigid rod of length a. The system is free to rotate in three dimensions about the center, but this center point cannot move.

1. Show that the allowed energies of this rigid rotor are:

$$E_n = \frac{\hbar^2 n(n+1)}{ma^2}, \quad \text{for } n = 0, 1, 2 \dots$$

   Hint: First express the classical energy in terms of the total angular momentum.

2. What are the normalized eigenfunctions for this system? What is the degeneracy of the nth energy level?

Problem 4

An electron is in the spin state:

$$|\chi\rangle = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

1. Determine the normalization constant, A.

2. Find the expectation values of $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$.

3. Find the standard deviations in $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$.

4. Confirm that your results are consistent with the generalized uncertainty principle for $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$.

$$\sigma_{\hat{A}}^2 \sigma_{\hat{B}}^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2 .$$

Problem 5

We introduced a specific spin 1/2 basis in lecture, i.e.,

$$\left|\frac{1}{2},+\frac{1}{2}\right\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \begin{pmatrix}0\\1\end{pmatrix},$$

where the ket notation represents $|s, m_s\rangle$.

1. In this basis, determine the eigenvalues and eigenvectors for $\hat{S}_y$.

2. If you measure S_y on the general spin state,

$$|\chi\rangle = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix},$$

what values would you get and with what probabilities? Check that the probabilities add to one. Note: a and b do not have to be real.

3. If you measure S_y^2 on the same general state, what values would you get and with what probabilities?