Quantum Mechanics

Week 7

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Exercise Material

Webpage

Week 7

Review

Clicker-Questions

Midterm

Review of Last Week

- Any questions on last week's topics?
- Feedback on the previous session?

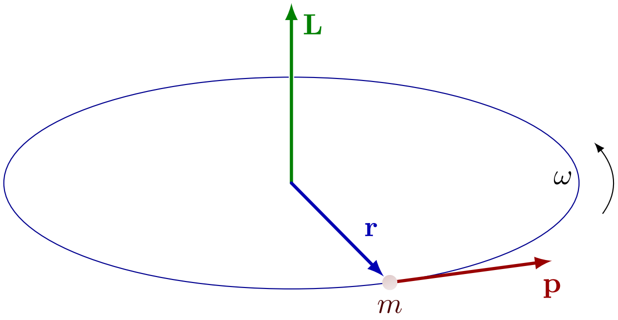
Review

Angular Momentum

Angular momentum in quantum mechanics is fundamental, revealing rotational symmetries of quantum systems.

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

- The angular momentum $\vec{L}$ is a three-dimensional vector represented by components $\vec{L}_x$, $\vec{L}_y$, and $\vec{L}_z$.
- It satisfies the squared magnitude relation: $L^2 = L_x^2 + L_y^2 + L_z^2$


$\mathbf{L}$
ω
$\mathbf{r}$
$\mathbf{p}$
m

Angular Momentum Operators

In quantum mechanics, the angular momentum operators are essential for describing rotational symmetries and are represented by $\hat{L}_x, \hat{L}_y$, and $\hat{L}_z$, corresponding to the axes of a three-dimensional Cartesian coordinate system.

$$
\begin{aligned}
[\hat{L}_x, \hat{L}_y] &= i\hbar\hat{L}_z, \\
[\hat{L}_y, \hat{L}_z] &= i\hbar\hat{L}_x, \\
[\hat{L}_z, \hat{L}_x] &= i\hbar\hat{L}_y.
\end{aligned}
$$

- Probabilistic measurements due to Uncertainty Principle.

Quantum Angular Momentum and Spherical

- Total angular momentum $\hat{L}^2$ commutes with its components $\hat{L}_x, \hat{L}_y, \hat{L}_z$.
- Commutation ensures simultaneous knowledge of $\hat{L}^2$ and one component $[\hat{L}^2, \hat{L}_{x,y,z}] = 0$ (typically $\hat{L}_z$)

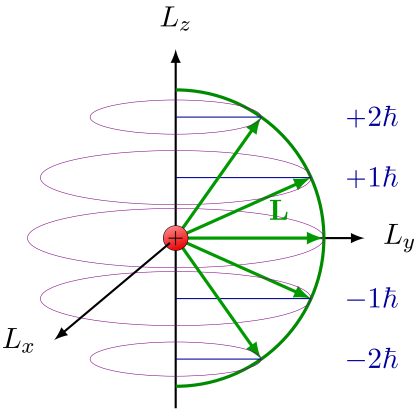
$$\boxed{\hat{L}^2 Y_\ell^{m_\ell} = \hbar^2 \ell(\ell+1) Y_\ell^{m_\ell} \rightarrow |\vec{L}| = \sqrt{\ell(\ell+1)}\hbar}$$

$$\boxed{\hat{L}_z Y_\ell^{m_\ell} = \hbar m_\ell Y_\ell^{m_\ell} \rightarrow L_z = m_\ell \hbar}$$

Spherical harmonics $Y_\ell^{m_\ell}$ are orthogonal eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$

Visualization of the Angular Momentum

- Quantum angular momentum, denoted as $|\vec{L}|$, is quantized.
- Magnitude $|\vec{L}|$ and L_z are precise; L_x and L_y are uncertain.
- The figure illustrates the probabilistic nature of L_x and L_y orientations.
- It depicts the cone of possible $\vec{L}$ directions with a well-defined L_z.
- L_z eigenvalue m_ℓ are the values between $-\ell, ..., \ell$
- Since $\ell < \sqrt{\ell(\ell+1)} \rightarrow |L_z| < |L|$, therefore $\vec{L}$ can never point along the z-axis.


L_z
$+2\hbar$
$+1\hbar$
$\mathbf{L}$
L_y
$-1\hbar$
L_x
$-2\hbar$

Spin: An Intrinsic Quantum Property

- Intrinsic to particles like electrons, hadrons, and atomic nuclei.
- Not a result of spatial movement but a fundamental quantum property.
- Elementary particles are point-like, making classical rotation inapplicable.
- Spin follows the same rules as Angular Momentum

Spin Operators and Quantization

The operators for spin follow eigenvalue equations:

$$\boxed{\hat{S}^2 f_s^{m_s} = \hbar^2 s(s+1) f_s^{m_s} \rightarrow |\vec{S}| = \sqrt{s(s+1)}\hbar}$$

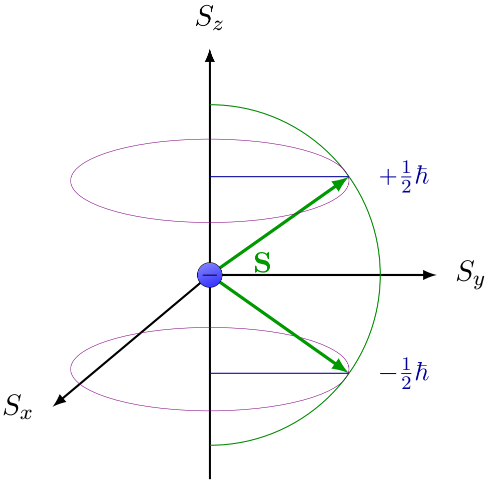
$$\boxed{\hat{S}_z f_s^{m_s} = \hbar m_s f_s^{m_s} \rightarrow S_z = m_s \hbar}$$

- S^2: total spin angular momentum.
- S_z: z-component of spin.

Spin is described by quantum numbers s and $m_s = -s, ..., s$. It is important to note that s can also be half-integers (Electron (Fermions) $s = \frac{1}{2}$, Photon (Bosons) $s = 1$)

Visualization of the Spin ($s = \frac{1}{2}$)

- Quantum Spin, denoted as $|\vec{S}|$, is quantized.
- Magnitude $|\vec{L}|$ and S_z are precise; S_x and S_y are uncertain.
- The figure illustrates the probabilistic nature of S_x and S_y orientations.
- It depicts the cone of possible $\vec{S}$ directions with a well-defined S_z.
- S_z eigenvalue m_s are the values between $-s, ..., s$
- Since $s < \sqrt{s(s+1)} \rightarrow |S_z| < |S|$, therefore $\vec{S}$ can never point along the z-axis.


S_z
$+\frac{1}{2}\hbar$
$\mathbf{S}$
S_y
$-\frac{1}{2}\hbar$
S_x

Spin Dirac Notation

- Eigenstates $f_s^{m_s}$ are not functions of r, θ, φ. There is no mathematical formulation for the eigenstates $\rightarrow$ Better to use Dirac Notation.

- Like in Angular Momentum, for standards we base our calculations in the commutation between $\hat{S}^2$ and $\hat{S}_z$.

For $s = \frac{1}{2}$ (e.g., electrons), the spin states in two dimensions are in the format $|s, m_s\rangle$:

$$\text{Spin-up state in z-basis: } f_{1/2}^{1/2} \equiv \left|\frac{1}{2}, \frac{1}{2}\right\rangle \equiv |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\text{Spin-down state in z-basis: } f_{1/2}^{-1/2} \equiv \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \equiv |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

A general spin state $|\chi\rangle$ of a spin-$\frac{1}{2}$ particle is in a superposition of both states:

$$|\chi\rangle = a\,|\uparrow\rangle + b\,|\downarrow\rangle,$$

Hermitian Operators for Spin $s = \frac{1}{2}$

Representation of 2x2 matrix operators in the basis $|\uparrow\rangle$ and $|\downarrow\rangle$ (z-Basis):

$$\hat{S}^2 \rightarrow \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \hat{S}_z \rightarrow \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\hat{S}_x \rightarrow \frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y \rightarrow \frac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Note: $|\uparrow\rangle$ and $|\downarrow\rangle$ are not eigenfunctions of $\hat{S}_x$ and $\hat{S}_y$ as they do not commute.

Raising and Lowering Operators for Spin

Spin raising and lowering operations also defined in the z-basis:

$$\hat{S}_+ \left|\downarrow\right\rangle = \hbar \left|\uparrow\right\rangle \quad \text{and} \quad \hat{S}_- \left|\uparrow\right\rangle = \hbar \left|\downarrow\right\rangle$$

$$\hat{S}_+ \to \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{S}_- \to \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Clicker-Questions

Midterm

- Next week midterm
- Covers Lectures 1-6
- Is a 20% bonus only
- All information on Moodle
- Practice with old Midterms
- Write down anything you might know, maximize points.