Problem Set 11

Problem 1

Suppose we put a delta-function bump in the center of the infinite square well:

$$\hat{H}' = \alpha\delta(x - a/2)$$

where α is a constant.

1. Find the first-order correction to the allowed energies. Explain why the energies are not perturbed for even n.

2. Find the first three nonzero terms in the expansion of the correction to the ground state wavefunction, i.e., $\psi_1^{(1)}$.

3. Find the second-order correction to the energies, $E_n^{(2)}$. Hint: you can sum the series explicitly, obtaining $-2m(\alpha/\pi\hbar n)^2$ for odd n.

Problem 2

Use a Gaussian trial function, $(\alpha/\pi)^{1/4}\exp[-\alpha x^2/2]$, to obtain the lowest upper bound on the ground state energy of the linear potential: $V(x) = C|x|$, where C is a constant.

Problem 3

Consider a quantum system with just three linearly independent states. Suppose the Hamiltonian, in matrix form, is

$$\hat{H} = V_0 \begin{pmatrix} 1-\varepsilon & 0 & 0 \\ 0 & 1 & \varepsilon \\ 0 & \varepsilon & 2 \end{pmatrix}$$

where V_0 is a constant, and ε is some small number ($\varepsilon \ll 1$).

1. Write down the eigenvectors and eigenvalues of the unperturbed Hamiltonian ($\varepsilon = 0$).

2. Solve for the exact eigenvalues of $\hat{H}$. Expand each of them as a power series in ε, up to second order.

3. Use first- and second-order nondegenerate perturbation theory to find the approximate eigenvalue for the state that grows out of the nondegenerate eigenvector of $\hat{H}^{(0)}$. Compare the exact result, from (a).

4. Use degenerate perturbation theory to find the first-order correction to the two initially degenerate eigenvalues. Compare the exact results.

Problem 4

Prove the following corollary to the variational principle:

If $\langle\psi|\hat{H}|\psi\rangle = 0,$ then $\hat{H} \geq E_{fe}$,
where ψ_{gs} is the ground state and E_{fe} is the energy of the first excited state.

Thus, if we can find a trial function that is orthogonal to the exact ground state, we can get an upper bound on the first excited state. In general, it is difficult to be sure that ψ is orthogonal to ψ_{gs}, since we do not typically know ψ_{gs}. However, if the potential $V(x)$ is an even function of x, then the ground state is likewise even, and hence any odd trial function will automatically meet the condition for the corollary.