Difference of convex functions in robust tube MPC

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Abstract— **We propose a robust tube-based Model Predictive Control (MPC) paradigm for nonlinear systems whose dynamics can be expressed as a difference of convex functions. The approach exploits the convexity properties of the system model to derive convex conditions that govern the evolution of robust tubes bounding predicted trajectories. These tubes allow an upper bound on a performance cost to be minimised subject to state and control constraints as a convex program, the solution of which can be used to update an estimate of the optimal state and control trajectories. This process is the basis of an iteration that solves a sequence of convex programs at each discrete time step. We show that the algorithm is recursively feasible, converges asymptotically to a fixed point of the iteration and ensures closed loop stability. The algorithm can be terminated after any number of iterations without affecting stability or constraint satisfaction. A case study is presented to illustrate an application of the algorithm.**

Keywords: nonlinear MPC, robust receding horizon control, convex optimisation, difference of convex functions

I. INTRODUCTION

Robust model predictive control (MPC) algorithms aim to provide performance and closed-loop stability guarantees in the presence of model or measurement uncertainty, while ensuring constraint satisfaction and tractable computation. Robust tube MPC refers to a collection of strategies that use information on the uncertainty affecting predictions to bound the future trajectories of the system [1], [2]. A tube consists of a sequence of sets containing predicted future system trajectories for all realisations of uncertainty.

Tube-based MPC has been successfully applied to robust stabilisation problems for linear systems [3]–[10]. The application of tube-based MPC to nonlinear systems poses a series of challenges: in particular online solution of nonconvex optimisation problems, safe approximation of uncertainty sets for states described by nonlinear dynamics, and the computational complexity associated with searching for general nonlinear feedback policies. An early implementation of robust MPC to nonlinear uncertain systems was proposed in [11] with a tube MPC framework that relies on the concept of constraint tightening as introduced in [12].

In some cases the special structure of the system dynamics can be exploited to good advantage to define a nonlinear tube MPC framework. In [13], a method is proposed for nonlinear systems with input-matched nonlinearities and nonlinear processes that take the form of piecewise affine maps. Feedback linearisation is used to cancel nonlinearities and a two-degree-of-freedom linear control law stabilises the system trajectory in a tube formed by robustly invariant sets. A method is proposed in [14] for constructing robust invariant sets for a class of Lipschitz nonlinear systems, relying on the computation of a quadratic Lyapunov function and an associated state feedback controller. In practice, this approach is limited since finding a global Lipschitz constant may result in a conservative controller with poor performance. Nonlinear system dynamics with a differential flatness property can also be advantageously reformulated to define a tube-based receding-horizon controller as in [15].

A common strategy in nonlinear tube-based MPC is to treat the nonlinearity in the dynamics as a bounded disturbance [16]–[19]. In [19], successive linear approximations around predicted trajectories are used to obtain a MPC law for nonlinear systems with differentiable dynamics. The proposed algorithm uses bounds on linearisation errors to construct an ellipsoidal tube and an associated control law. The controller achieves rejection of the linearisation error and stabilisation of the nonlinear system, and approach has guarantees of recursive feasibility and asymptotic stability.

These strategies rely for robustness on bounds on linearisation errors, which are either computed using fixed prior bounds on predicted future states, and are therefore conservative, or otherwise may present computational difficulties. In this work we show that tighter bounds can be obtained under certain convexity assumptions. We consider systems that can be expressed as a difference of convex functions [20], [21]. The convexity properties of these systems considerably simplify the problem of determining bounds on model states that are suitable for defining predicted state tubes. We derive convex conditions for the evolution of tubes bounding perturbations on future predicted trajectories. Analogously to successive linearisation approaches, these conditions form the basis of an iteration that solves a sequence of convex programs to minimise a bound on predicted cost subject to robust constraint satisfaction. The iteration is recursively feasible and convergent, and the resulting MPC algorithm is asymptotically stabilising.

The paper is organised as follows. Tube MPC based on successive linearisation is discussed in Section II, then Section III presents the proposed new MPC approach (DC-TMPC) based on the difference of convex functions. The convergence, stability and feasibility properties of the algorithm are discussed in Section IV. To illustrate the approach, we describe its application to the problem of regulating fluid in a pair of coupled tanks in Section V.

Notation: The value of a variable $x \in \mathbb{R}^{n_x}$ at the nth discrete time step is denoted $x[n]$, and a sequence predicted at discrete time n is denoted $\{x_k\}_{k=0}^{N-1} = \{x_0, \dots, x_{N-1}\}$, where x_k is the predicted value of $x[n+k]$. A symmetric matrix $P \in \mathbb{R}^{q \times q}$ is positive definite (resp. semidefinite) if

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$P \succ 0$ ($P \succeq 0$), and the weighted Euclidean norm is denoted $\|x\|_P := (x^\top P x)^{1/2}$. For $K \in \mathbb{R}^{p\times q}$ and sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^q$ we denote $\mathcal{A} \oplus \mathcal{B} := \{a + b : a \in \mathcal{A},\, b \in \mathcal{B}\}$, $K\mathcal{A} := \{Ka : a \in \mathcal{A}\}$, and $\|\mathcal{A}\|_P = \max_{a\in\mathcal{A}} \|a\|_P$. The kth element of a function $f : \mathbb{R}^p \to \mathbb{R}^q$ is denoted $[f]_k$ and f is called a DC function (or simply DC) if $f = g - h$, where $g, h : \mathbb{R}^p \to \mathbb{R}^q$ are such that $[g]_k, [h]_k$ are convex functions for $k = 1, \ldots, q$.

II. Successive linearisation tube MPC

Consider the following discrete-time dynamical system

$$x[n+1] = f(x[n], u[n]). \tag{1}$$

Here $x \in \mathcal{X}$, $u \in \mathcal{U}$ are state and control variables and $\mathcal{X} \subset \mathbb{R}^{n_x}$, $\mathcal{U} \subset \mathbb{R}^{n_u}$ are compact constraint sets. The goal is to regulate $(x[n], u[n])$ around a reference $(x^r[n], u^r[n])$ (where $x^r[n+1] = f(x^r[n], u^r[n])$ and $(x^r[n], u^r[n]) \in \mathcal{X} \times \mathcal{U}$), while minimising a quadratic cost. To simplify presentation we discuss here the case of a constant reference (x^r, u^r).

Successive linearisation MPC addresses the optimal control problem to be solved online at each discrete time step by considering perturbations of previously computed predicted trajectories of the system. This approach repeatedly linearises the model (1) around state and control trajectories, defined at time n by $\mathbf{x}^\circ = \{x^\circ_k\}_{k=0}^N$, $\mathbf{u}^\circ = \{u^\circ_k\}_{k=0}^{N-1}$, with $x^\circ_0 = x_0 = x[n]$ and $x^\circ_{k+1} = f(x^\circ_k, u^\circ_k)$ for $k = 0, \ldots, N-1$. In [19] the linearisation errors are treated as unknown bounded disturbances and a robustly stabilising two-degree of freedom controller is designed for the linearised dynamics by solving a sequence of convex optimisation problems.

Defining the predicted state and control perturbations as $s_k = x_k - x^\circ_k$ and $v_k = u_k - u^\circ_k$ respectively, we construct a tube (a sequence of sets) $\boldsymbol{\mathcal{S}} = \{\mathcal{S}_k\}_{k=0}^N$ so that $s_k \in \mathcal{S}_k$ for all k. The perturbed state satisfies $s_0 = 0$ and

$$s_{k+1} = A_k s_k + B_k v_k + g_k,$$

for $k = 0, \ldots, N-1$, where $A_k = \frac{\partial f}{\partial x}(x^\circ_k, u^\circ_k)$, $B_k = \frac{\partial f}{\partial u}(x^\circ_k, u^\circ_k)$, and g_k is the remainder term in the first-order Taylor expansion of f about (x°_k, u°_k). For the Jacobian linearisation to exist, the following assumption was implied

Assumption 1. *f is differentiable on $\mathcal{X} \times \mathcal{U}$.*

The predicted control perturbation sequence is parameterised with a feedforward term c_k and a linear feedback term $K_k s_k$, for $k = 0, \ldots, N-1$,

$$v_k = c_k + K_k s_k. \tag{2}$$

Defining $\Phi_k = A_k + B_k K_k$ and given bounds $g_k \in \mathcal{G}_k$ on linearisation errors, the sets $\mathcal{S}_k$ defining the tube cross-sections therefore satisfy $\mathcal{S}_0 = \{0\}$ and, for $k = 0, \ldots, N-1$,

$$\mathcal{S}_{k+1} \supseteq \Phi_k \mathcal{S}_k \oplus \{B_k c_k\} \oplus \mathcal{G}_k. \tag{3}$$

The tube $\boldsymbol{\mathcal{S}}$ and the feedforward sequence $\mathbf{c} = \{c_k\}_{k=0}^{N-1}$ are variables in an optimal tracking control problem with cost

$$J(\mathbf{c}, \boldsymbol{\mathcal{S}}, \mathbf{x}^\circ, \mathbf{u}^\circ) = \sum_{k=0}^{N-1} \big[\|\{x^\circ_k - x^r\} \oplus \mathcal{S}_k\|_Q^2 + \|\{u^\circ_k + c_k - u^r\} \oplus K_k \mathcal{S}_k\|_R^2 \big] + \|\{x^\circ_N - x^r\} \oplus \mathcal{S}_N\|_{\hat{Q}}^2 \tag{4}$$

where $Q \succ 0$, $R \succ 0$. The terminal weighting matrix $\hat{Q} \succ 0$ is chosen so that, for all $x \in \hat{\mathcal{X}}$,

$$\|x - x^r\|_{\hat{Q}}^2 \geq \|f(x, \hat{K}(x - x^r) + u^r) - x^r\|_{\hat{Q}}^2 + \|x - x^r\|_Q^2 + \|\hat{K}(x - x^r)\|_Q^2, \tag{5}$$

where the terminal set $\hat{\mathcal{X}}$ and feedback gain $\hat{K}$ satisfy

$$\hat{\mathcal{X}} \subseteq \mathcal{X}, \quad \hat{K}\hat{\mathcal{X}} \oplus \{u^r - \hat{K}x^r\} \subseteq \mathcal{U}. \tag{6}$$

The terminal set can be defined, for example, in terms of a scalar $\hat{\gamma}$:

$$\hat{\mathcal{X}} = \{x : \|x\|_{\hat{Q}}^2 \leq \hat{\gamma}\} \tag{7}$$

The terminal parameters $\hat{K}$, $\hat{Q}$ and $\hat{\gamma}$ can be computed simultaneously offline by solving a convex optimisation problem (see the Appendix). The gains $\{K_k\}_{k=0}^{N-1}$ in equation (2) can be computed, for example, using a dynamic programming recursion initialised with $P_N = \hat{Q}$, and, for $k = N-1, \ldots, 0$,

$$\begin{aligned}
\Delta_k &= B_k^\top P_{k+1} B_k + R \\
K_k &= -\Delta_k^{-1} B_k^\top P_{k+1} A_k \\
P_k &= Q + A_k^\top P_{k+1} A_k - A_k^\top P_{k+1} B_k \Delta_k^{-1} B_k^\top P_{k+1} A_k
\end{aligned} \tag{8}$$

The MPC optimisation at time n is initialised with a feasible predicted trajectory $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ of (1) and the following optimisation problem in variables $\mathbf{c}$, $\boldsymbol{\mathcal{S}}$ is solved sequentially

$$\begin{aligned}
\min_{\mathbf{c}, \boldsymbol{\mathcal{S}}} \;& J(\mathbf{c}, \boldsymbol{\mathcal{S}}, \mathbf{x}^\circ, \mathbf{u}^\circ) \text{ subject to, } \forall k \in \{0, \ldots, N-1\}, \\
& \Phi_k \mathcal{S}_k \oplus \{B_k c_k\} \oplus \mathcal{G}_k \subseteq \mathcal{S}_{k+1} \\
& \{x^\circ_k\} \oplus \mathcal{S}_k \subseteq \mathcal{X} \\
& \{u^\circ_k + c_k\} \oplus K_k \mathcal{S}_k \subseteq \mathcal{U} \\
& \|\{x^\circ_N - x^r\} \oplus \mathcal{S}_N\|_{\hat{Q}}^2 \leq \hat{\gamma} \\
& \mathcal{S}_0 = \{0\}.
\end{aligned} \tag{9}$$

The solution of (9) for $\mathbf{c}$ is used to update $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ by setting

$$s_0 \leftarrow 0, \tag{10a}$$

$$u^\circ_k \leftarrow u^\circ_k + c_k + K_k s_k, \tag{10b}$$

$$s_{k+1} \leftarrow f(x^\circ_k, u^\circ_k) - x^\circ_{k+1}, \tag{10c}$$

$$x^\circ_{k+1} \leftarrow f(x^\circ_k, u^\circ_k), \tag{10d}$$

for $k = 0, \ldots, N-1$, and the process (9)-(10) is repeated until $\|\mathbf{c}\|^2 := \sum_{k=0}^{N-1} \|c_k\|^2$ has converged to a sufficiently small value or the maximum number of iterations is reached. The control law is then implemented as

$$u[n] = u^\circ_0. \tag{11}$$

At time $n+1$ we set $x^\circ_0 \leftarrow x[n+1]$ and initialise a new iteration with $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ defined, for $k = 0, \ldots, N-1$, by

$$u^\circ_k \leftarrow u^\circ_{k+1}, \tag{12a}$$

$$x^\circ_{k+1} \leftarrow f(x^\circ_k, u^\circ_k), \tag{12b}$$

$$u^\circ_{N-1} \leftarrow \hat{K}(x^\circ_{N-1} - x^r) + u^r, \tag{12c}$$

$$x^\circ_N \leftarrow f(x^\circ_{N-1}, u^\circ_{N-1}). \tag{12d}$$

These definitions ensure recursive feasibility of the scheme. In addition, the optimal value of $J(\mathbf{c}, \boldsymbol{\mathcal{S}}, \mathbf{x}^\circ, \mathbf{u}^\circ)$ is non-increasing at successive iterations, implying that the iteration converges asymptotically: $\mathbf{c} \to 0$ and $\boldsymbol{\mathcal{S}} \to \{\{0\}, \ldots, \{0\}\}$, and also non-increasing at successive time steps $n = 0, 1, \ldots$, implying that $(x, u) = (x^r, u^r)$ is an asymptotically stable equilibrium (see e.g. [19], Theorems 7 and 8 for details).

Despite its attractive theoretical properties, the successive linearisation approach relies on bounds on the linearisation error $\mathcal{G}_k$, which are difficult to compute in general and might result in overly conservative bounds if no further assumptions are made on f. Moreover, the algorithm outlined in this section does not ensure convergence to a solution that satisfies the first order optimality conditions for the problem of minimizing $\sum_{k=0}^{N-1} [\|x_k - x^r\|_Q^2 + \|u_k - u^r\|_R^2] + \|x_N - x^r\|_{\hat{Q}}^2$ subject to $x_{k+1} = f(x_k, u_k)$, $x_k \in \mathcal{X}$ and $u_k \in \mathcal{U}$ since $\mathcal{G}_k$ may not be tight. To overcome these problems, we propose a method for systems (1) in which f is a DC function to obtain tight bounds on linearisation errors.

III. Successive convex programming tube MPC for DC systems

To derive a control algorithm with reduced computation and improved performance, we strengthen our assumptions on model (1) as follows.

Assumption 2. *f is DC on $\mathcal{X} \times \mathcal{U}$ and $\mathcal{X}, \mathcal{U}$ are convex sets.*

Under Assumption 2 the model (1) can be expressed

$$x[n+1] = f_1(x[n], u[n]) - f_2(x[n], u[n]), \tag{13}$$

where each component of the vector-valued functions f_1, f_2 is a convex function. Note that (13) includes systems in which f in (1) is either convex or concave. More generally, any twice continuously differentiable function is DC and any continuous function can be approximated arbitrarily closely by a DC function on a compact convex set [20], [21], which extends considerably the scope of the present method.

Given a predicted trajectory $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ of (1), the perturbed variables $s_k = x_k - x_k^\circ$, $v_k = u_k - u_k^\circ$ are governed by $s_0 = 0$ and, for $k = 0, \ldots, N-1$,

$$s_{k+1} = A_{1,k}s_k + B_{1,k}v_k + g_{1,k} - (A_{2,k}s_k + B_{2,k}v_k + g_{2,k}),$$

where for $i = 1, 2$, $A_{i,k} = \frac{\partial f_i}{\partial x}(x_k^\circ, u_k^\circ)$, $B_{i,k} = \frac{\partial f_i}{\partial u}(x_k^\circ, u_k^\circ)$, and $g_{i,k}$ represents the error in the Jacobian linear approximation of f_i around (x_k°, u_k°). Using the control law (2), defining $\Phi_{i,k} = A_{i,k} + B_{i,k}K_k$, $i = 1, 2$, and assuming bounds $g_{i,k} \in \mathcal{G}_{i,k}$, $i = 1, 2$, we obtain a set of constraints to be satisfied by the tube cross-sections $\mathcal{S}_k$ containing s_k:

$$\mathcal{S}_{k+1} \supseteq (\Phi_{1,k} - \Phi_{2,k})\mathcal{S}_k \oplus \{(B_{1,k} - B_{2,k})c_k\} \oplus \mathcal{G}_{1,k} \oplus -\mathcal{G}_{2,k}. \tag{14}$$

We now characterise the sets $\mathcal{S}_k$ and $\mathcal{G}_{i,k}$ to obtain a workable formulation of these constraints. Various parameterisations of the sets $\mathcal{S}_k$ are possible: polytopic, ellipsoidal, homothetic (e.g. [1]). In this work, for simplicity, we define a polytopic tube cross-section in terms of elementwise bounds:

$$\mathcal{S}_k = \{(s_1, \ldots, s_{n_x}) \in \mathbb{R}^{n_x} : \underline{s}_{k,j} \le s_j \le \overline{s}_{k,j},\ j = 1, \ldots, n_x\}.$$

Considering $g_{1,k}$, $g_{2,k}$ as functions of s_k and c_k:

$$g_{i,k} = f_i(x_k^\circ + s_k, u_k^\circ + v_k) - f_i(x_k^\circ, u_k^\circ) - \Phi_{i,k}s_k - B_{i,k}c_k,$$

the convexity of $f_i(x, u)$ for $(x, u) \in \mathcal{X} \times \mathcal{U}$ implies that $g_{i,k}(s, c)$ is convex for all (s, c) such that $(x^\circ + s, u^\circ + Ks + c)$ lies in $\mathcal{X} \times \mathcal{U}$. It follows that, for $i = 1, 2$,

$$\min_{s_k \in \mathcal{S}_k} g_{i,k}(s_k, c_k) = 0,$$
$$\max_{s_k \in \mathcal{S}_k} g_{i,k}(s_k, c_k) = \max_{s \in \mathcal{V}(\mathcal{S}_k)} g_{i,k}(s, c_k),$$

where $\mathcal{V}(\mathcal{S}_k)$ denotes the set of vertices of $\mathcal{S}_k$. The tube constraint (14) is therefore ensured by the convex inequalities

$$\underline{s}_{k+1} \le \min_{s \in \mathcal{V}(\mathcal{S}_k)} \{(\Phi_{1,k} - \Phi_{2,k})s + (B_{1,k} - B_{2,k})c_k - g_{2,k}(s)\},$$
$$\overline{s}_{k+1} \ge \max_{s \in \mathcal{V}(\mathcal{S}_k)} \{(\Phi_{1,k} - \Phi_{2,k})s + (B_{1,k} - B_{2,k})c_k + g_{1,k}(s)\}.$$

If no prior knowledge of their linear coefficients is exploited, each of these inequalities is equivalent to 2^{n_x} inequalities in $\mathbf{c}$ and the parameters $\{\underline{s}_k, \overline{s}_k\}_{k=0}^{N-1}$ that define $\boldsymbol{\mathcal{S}}$. Crucially, the resulting set of inequalities is convex, and sequences $\mathbf{c}$, $\boldsymbol{\mathcal{S}}$ can be computed for given $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ by solving the following convex optimisation problem

$$\begin{aligned}
&\min_{\mathbf{c}, \boldsymbol{\mathcal{S}}} \ J(\mathbf{c}, \boldsymbol{\mathcal{S}}, \mathbf{x}^\circ, \mathbf{u}^\circ) \\
&\quad \text{subject to, } \forall k \in \{0, \ldots, N-1\},\ \forall s \in \mathcal{V}(\mathcal{S}_k): \\
&\quad \underline{s}_{k+1} \le (\Phi_{1,k} - \Phi_{2,k})s + (B_{1,k} - B_{2,k})c_k - g_{2,k}(s) \\
&\quad \overline{s}_{k+1} \ge (\Phi_{1,k} - \Phi_{2,k})s + (B_{1,k} - B_{2,k})c_k + g_{1,k}(s) \\
&\quad x_k^\circ + s \subseteq \mathcal{X} \\
&\quad u_k^\circ + K_k s + c_k \subseteq \mathcal{U} \\
&\quad \|x_N^\circ + s - x^r\|_{\hat{Q}}^2 \le \hat{\gamma} \\
&\quad \underline{s}_0 = \overline{s}_0 = 0.
\end{aligned} \tag{15}$$

Problem (15) is solved repeatedly and $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ is updated according to (10a-d), until $\|\mathbf{c}\|$ is sufficiently small or the maximum number of iterations is reached. The control law is defined as (11), and at the subsequent discrete time instant we set $x_0^\circ \leftarrow x[n+1]$, and redefine $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ using (12a-d). The tube-based MPC strategy is summarised in Algorithm 1. We refer to this as DC-TMPC in what follows.

The DC-TMPC strategy requires (in line 7 of Algorithm 1) knowledge of a feasible predicted trajectory at time $n = 0$, namely a sequence $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ satisfying $(x_k^\circ, u_k^\circ) \in \mathcal{X} \times \mathcal{U}$, $k = 0, \ldots, N-1$ and $\|x_N^\circ - x^r\|_{\hat{Q}}^2 \le \hat{\gamma}$. An iterative approach to determining a feasible trajectory can be constructed by relaxing some of the constraints of (15) and performing successive optimisations to minimise constraint violations. For example, given a trajectory $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ of (1) that satisfies $(x_k^\circ, u_k^\circ) \in \mathcal{X} \times \mathcal{U}$, $k = 0, \ldots, N-1$, but not the terminal constraint $\|x_N^\circ - x^r\|_{\hat{Q}}^2 \le \hat{\gamma}$, a suitable iteration can be constructed by replacing $\hat{\gamma}$ in (15) with an optimisation variable γ and replacing the objective of (15) with γ. Repeatedly solving this convex program and updating $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ using (10a-d) will generate a convergent sequence of predicted trajectories with non-increasing values of $\|x_N^\circ - x^r\|_{\hat{Q}}^2$, which can be terminated when $\gamma \le \hat{\gamma}$ is achieved.

Algorithm 1: DC-TMPC algorithm.

1 **Initialization**
2 Compute $\hat{K}$, $\hat{Q}$, $\hat{\gamma}$ satisfying (5), (6) and (7)
3 **end**
4 **Online at times** $n = 0, 1, \ldots$
5 Set $x_0^\circ \leftarrow x[n]$
6 **if** $n = 0$ **then**
7 Find $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ such that $x_{k+1}^\circ = f(x_k^\circ, u_k^\circ)$, $(x_k^\circ, u_k^\circ) \in \mathcal{X} \times \mathcal{U}$ for $k = 0, \ldots, N-1$, and $\|x_N^\circ - x^r\|_{\hat{Q}}^2 \leq \hat{\gamma}$
8 **else**
9 Compute $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ using (12a-d)
10 **end**
11 Set $\|\mathbf{c}\| \leftarrow \infty$, $j \leftarrow 0$
12 **while** $\|\mathbf{c}\| >$ *tolerance* & $j <$ *maxIters* **do**
13 **for** $k \leftarrow N-1$ **to** 0 **do**
14 $A_{i,k} \leftarrow \frac{\partial f_i}{\partial x}(x_k^\circ, u_k^\circ), B_{i,k} \leftarrow \frac{\partial f_i}{\partial u}(x_k^\circ, u_k^\circ), i = 1, 2$
15 Compute K_k in (8) with $A_k = A_{1,k} - A_{2,k}$, $B_k = B_{1,k} - B_{2,k}$
16 Set $\Phi_{i,k} \leftarrow A_{i,k} + B_{i,k} K_k$
17 **end**
18 Solve (15) to find the optimal $\mathbf{c}$ and $\boldsymbol{\mathcal{S}}$
19 Update $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ using (10a-d)
20 Update the objective: $J^{j,n} \leftarrow J(\mathbf{c}, \boldsymbol{\mathcal{S}}, \mathbf{x}^\circ, \mathbf{u}^\circ)$
21 Update the iteration counter: $j \leftarrow j + 1$
22 **end**
23 Update the control input: $u[n] \leftarrow u_0^\circ$
24 **end**

IV. Feasibility, convergence and stability

This section establishes some important properties of DC-TMPC. We first show that $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ is a feasible predicted trajectory at every iteration of Algorithm 1 and at all times $n \geq 0$. This ensures firstly that problem (15) in line 18 is recursively feasible, and second that the while loop (lines 12-22) can be terminated after any number of iterations without compromising the stability of the MPC law, i.e. the approach allows *early termination* of the MPC optimisation.

Proposition 1 (Feasibility)**.** *Problem (15) in Algorithm 1 is feasible at all times $n \geq 0$ and all iterations $j \geq 0$.*

Proof. If $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ is a feasible trajectory, namely a trajectory of (1) satisfying $(x_k^\circ, u_k^\circ) \in \mathcal{X} \times \mathcal{U}$, $k = 0, \ldots, N-1$ and $\|x_N^\circ - x^r\|_{\hat{Q}}^2 \leq \hat{\gamma}$, then problem (15) is feasible since $(\mathbf{c}, \boldsymbol{\mathcal{S}}) = (0, \{\{0\}, \ldots, \{0\}\})$ trivially satisfies all constraints of (15). Furthermore, line 7 of Algorithm 1 requires that $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ is a feasible trajectory at $n = 0$, and thus ensures that (15) in line 18 is feasible for $n = 0$, $j = 0$. Therefore $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ computed in lines 9 and 19 of Algorithm 1 are necessarily feasible trajectories for all $n \geq 0$ and $j \geq 0$. $\square$

We next show that, at a given time step n, the optimal values of the objective function for problem (15) computed at successive iterations of Algorithm 1 are non-increasing.

Lemma 2. *The cost $J^{j,n}$ in line 20 of Algorithm 1 satisfies $J^{j+1,n} \leq J^{j,n}$ for all $j \geq 0$.*

Proof. This is a consequence of the recursive feasibility of problem (15) in line 18 of Algorithm 1 as demonstrated by Proposition 1. Specifically, $J(\mathbf{c}, \boldsymbol{\mathcal{S}}, \mathbf{x}^\circ, \mathbf{u}^\circ)$ is the maximum of a quadratic cost evaluated over the predicted tube, so the feasible suboptimal solution $(\mathbf{c}, \boldsymbol{\mathcal{S}}) = (0, \{\{0\}, \ldots, \{0\}\})$ at iteration $j+1$ must give a cost no greater than $J^{j,n}$. Therefore the optimal value at $j+1$ satisfies $J^{j+1,n} \leq J^{j,n}$. $\square$

Lemma 2 implies that $J^{j,n}$ converges to a limit as $j \to \infty$, and this is the basis of the following convergence result.

Proposition 3 (Convergence)**.** *The solution of (15) in Algorithm 1 at successive iterations at a given time step n satisfies $(\mathbf{c}, \boldsymbol{\mathcal{S}}) \to (0, \{\{0\}, \ldots, \{0\}\})$ as $j \to \infty$.*

Proof. From $J^{j,n} \geq 0$ and $J^{j+1,n} \leq J^{j,n}$ for all j, it follows that $J^{j,n}$ converges to a limit as $j \to \infty$. Since $Q \succ 0$ and $R \succ 0$, this implies that $\mathbf{c} \to 0$ and s_k, $k = 0, \ldots, N-1$, in the update (10a-d) in line 19 must also converge to zero. Furthermore, since $(\mathbf{c}, \boldsymbol{\mathcal{S}})$ minimizes the worst-case cost evaluated for the predicted tube, the optimal solutions of (15) must satisfy $\boldsymbol{\mathcal{S}} \to \{\{0\}, \ldots, \{0\}\}$ as $j \to \infty$. $\square$

We finally show that under the DC-TMPC law, $x = x^r$ is asymptotically stable.

Theorem 4 (Asymptotic stability)**.** *For the system (1) controlled by DC-TMPC, $x = x^r$ is an asymptotically stable equilibrium with region of attraction equal to the set of initial states $x[0]$ for which a feasible initial trajectory $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ exists in line 7 of Algorithm 1.*

Proof. Let $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ be the trajectory computed in line 9 of Algorithm 1 at $n+1$, for any initial state such that line 7 is feasible. Then, setting $(\mathbf{c}, \boldsymbol{\mathcal{S}}) = (0, \{\{0\}, \ldots, \{0\}\})$, and using $\mathcal{S}_0 = \{0\}$ and the property (5) of $\hat{Q}$, we obtain

$$J(\mathbf{c}, \boldsymbol{\mathcal{S}}, \mathbf{x}^\circ, \mathbf{u}^\circ) \leq J^{j_n,n} - \|x[n] - x^r\|_Q^2 + \|u[n] - u^r\|_R^2$$

where j_n is the final iteration count of Algorithm 1 at time n. The optimal solutions of (15) at time $n+1$ therefore give

$$J^{j_{n+1},n+1} \leq J^{0,n+1} \leq J^{j_n,n} - \|x[n] - x^r\|_Q^2 - \|u[n] - u^r\|_R^2,$$

Furthermore $J^{j,n}$ is a positive definite function of $x[n] - x^r$ since $Q \succ 0$, $R \succ 0$, and Lyapunov's direct method therefore implies that $x = x^r$ is asymptotically stable. $\square$

V. Case study

Consider the following nonlinear coupled tank model

$$x_1[n+1] = x_1[n] - \delta \frac{A_1}{A} \sqrt{2 g x_1[n]} + \delta \frac{k_p}{A} u[n],$$
$$x_2[n+1] = x_2[n] - \delta \frac{A_2}{A} \sqrt{2 g x_2[n]} + \delta \frac{A_1}{A} \sqrt{2 g x_1[n]},$$

where x_1, x_2, are the depths of liquid in a pair of connected tanks, A is the tank cross-section area, A_1, A_2 are the outflow orifices areas, g is acceleration due to gravity, and k_p is the pump gain. The problem at hand is to stabilise the system around a reference state $x_k^r = [(A_2/A_1)^2 h_r \; h_r]^\top$. The model can be written as a difference of convex functions

$$f = \begin{bmatrix} x_1[n] - \delta\frac{A_1}{A}\sqrt{2gx_1[n]} + \delta\frac{k_p}{A}u[n] \\ x_2[n] - \delta\frac{A_2}{A}\sqrt{2gx_2[n]} + \delta\frac{A_1}{A}\sqrt{2gx_1[n]} \end{bmatrix},$$

$$= \underbrace{\begin{bmatrix} x_1[n]-\delta\frac{A_1}{A}\sqrt{2gx_1[n]}+\delta\frac{k_p}{A}u[n] \\ x_2[n]-\delta\frac{A_2}{A}\sqrt{2gx_2[n]} \end{bmatrix}}_{f_1} - \underbrace{\begin{bmatrix} 0 \\ -\delta\frac{A_1}{A}\sqrt{2gx_1[n]} \end{bmatrix}}_{f_2}$$

Let $(\mathbf{x}_k^\circ, \mathbf{u}_k^\circ)$ be an N-step predicted trajectory evaluated at time n. Since the model depends linearly on the control input, the perturbations $s_k = x_k - x_k^\circ$ and $v_k = u_k - u_k^\circ$ under the predicted control law (2) are governed by the dynamics

$$s_{k+1} = (\Phi_{1,k} - \Phi_{2,k})s_k + (B_{1,k} - B_{2,k})c_k + g_{1,k}(s_k) - g_{2,k}(s_k)$$

where $\Phi_{1,k} = A_{1,k} + B_{1,k}K_k$, $\Phi_{2,k} = A_{2,k} + B_{2,k}K_k$ and

$$A_{1,k} = \begin{bmatrix} 1 - \frac{\delta A_1 g}{A\sqrt{2gx_{1,k}^\circ}} & 0 \\ 0 & 1 - \frac{\delta A_2 g}{A\sqrt{2gx_{2,k}^\circ}} \end{bmatrix},$$

$$A_{2,k} = \begin{bmatrix} 0 & 0 \\ -\frac{\delta A_1 g}{A\sqrt{2gx_{1,k}^\circ}} & 0 \end{bmatrix}, \quad B_{1,k} = \begin{bmatrix} \frac{\delta k_p}{A} \\ 0 \end{bmatrix}, \quad B_{2,k} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The feedback gains K_k are computed using the dynamic programming recursion (8) with $A_k = A_{1,k} - A_{2,k}$ and $B_k = B_{1,k} - B_{2,k}$ and state and input weights Q, R. The terminal parameters $\hat{K}$, $\hat{Q}$ and $\hat{\gamma}$ are obtained by solving the optimisation problem given in the Appendix with square terminal set bounds $\hat{\mathcal{X}} = \{x : |x - x^r| \leq \delta^x\}$, $\hat{\mathcal{U}} = \{u : |u - u^r| \leq \delta^u\}$. Denoting the bounds defining tube cross-sections $\mathcal{S}_k$ as $\underline{s}_k = [\underline{s}_{k,1} \;\; \underline{s}_{k,2}]^\top$, $\overline{s}_k = [\overline{s}_{k,1} \;\; \overline{s}_{k,2}]^\top$, the vertices of $\mathcal{S}_k$ are

$$V(\mathcal{S}_k) = \left\{ \begin{bmatrix} \underline{s}_{k,1} \\ \underline{s}_{k,2} \end{bmatrix}, \begin{bmatrix} \overline{s}_{k,1} \\ \underline{s}_{k,2} \end{bmatrix}, \begin{bmatrix} \underline{s}_{k,1} \\ \overline{s}_{k,2} \end{bmatrix}, \begin{bmatrix} \overline{s}_{k,1} \\ \overline{s}_{k,2} \end{bmatrix} \right\},$$

and the state and input constraint sets have the form $\mathcal{X} = \{x : \underline{h}[1 \; 1]^\top \leq x \leq \overline{h}[1 \; 1]^\top\}, \mathcal{U} = \{u : \underline{u} \leq u \leq \overline{u}\}$. Table I gives the model parameters.

To apply DC-TMPC to the coupled tank problem, Algorithm 1 is initialised by setting $\mathbf{u}^\circ$ to a constant voltage $u^r = 7.3\ V$, resulting in a feasible initial trajectory $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ for a horizon $N = 50$ and sampling interval $\delta = 1.4\,$s. Problem (15) is solved using the convex optimisation package CVX [22] with solver Mosek [23].

Figure 1 shows that the system successfully tracks the reference levels and the fluid level in tank 2 is stabilised around a height of 15 cm as required. Note that the overshoot for tank 1 is large, allowing the tanks to fill faster.

The influence of the input penalty R on the response is shown in Figure 2. For a large R, the response is slow with an energy efficient control law. By contrast, small values of R yield a more aggressive control with faster responses. Interestingly, the response for $R = 0.02$ makes the state and input inequality constraints active, which demonstrates the capabilities of the algorithm to generate a control command that does not violate constraints.

Convergence of the algorithm is demonstrated empirically in Figure 3 which shows the evolution of the first-iteration optimal objective $J^{0,n}$ as a function of the time step n. As expected, the objective decreases at each step.

The phase portrait in Figure 4 illustrates convergence of state trajectories and subsequent tightening of the state perturbation bounds for iterations $j = 1, \ldots, 5$ at time step $n = 0$. The sets $\mathcal{S}_k$ forming the cross sections of the tube are shown by black boxes. The sets become progressively tighter as the trajectory converges towards the optimum. The terminal set is represented by a red box, and we observe that all trajectories terminate within this set.

TABLE I

Coupled Tank Parameters [24].

Parameter	**Symbol**	**Value**	**Units**
Gravity acceleration	g	981	$\mathrm{cm\,s^{-2}}$
Pump gain	k_p	3.3	$\mathrm{cm^3 s^{-1} V^{-1}}$
Tank inside area	A	15.2	cm^2
Outflow orifice areas	A_1, A_2	0.13, 0.14	cm^2
Initial height	$x_1(0), x_2(0)$	0.2, 0.1	cm
Target height	h^r	15	cm
Target voltage	u^r	7.3	V
Input range	$[\underline{u}, \overline{u}]$	$[0, 24]$	V
State range	$[\underline{h}, \overline{h}]$	$[0.1, 30]$	cm
Terminal set size	δ^x, δ^u	$[1 \; 1]^\top, 1$	cm, V
Terminal cost	$\hat{Q}$	$\begin{bmatrix} 3.1 & 1.2 \\ 1.2 & 6.1 \end{bmatrix}$	cm^{-2}
Terminal gain	$\hat{K}$	$\begin{bmatrix} -0.8 & -0.5 \end{bmatrix}$	Vcm^{-1}
Terminal bound	$\hat{\gamma}$	2.8	–
State penalty	Q	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	cm^{-2}
Input penalty	R	0.1	V^{-2}
Horizon	N	50	–
Time step	δ	1.4	s
Max # of iterations	maxIters	5	–

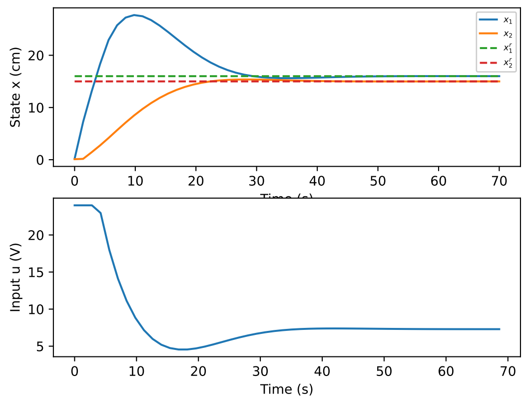

Fig. 1. State and input trajectories.

We now compare the convergence properties of DC-TMPC with the successive linearisation tube-based MPC algorithm in [19] (MPC-2011). As described in Section II, linearisation errors around predicted trajectories are treated as disturbances in MPC-2011. The approach uses state- and

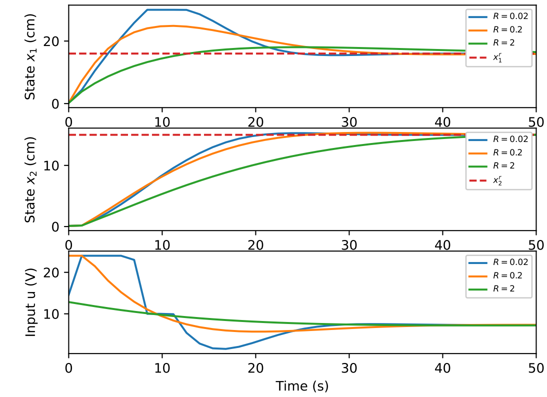

Fig. 2. Influence of input penalty R on the closed-loop response.

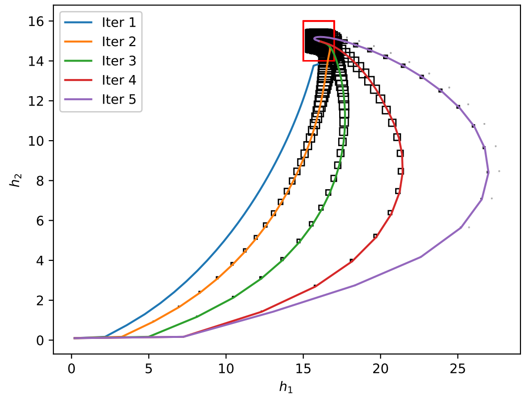

Fig. 4. Phase portrait at time step $n = 0$ with successive predicted state trajectories, associated bounds (black boxes) and terminal set (red box).

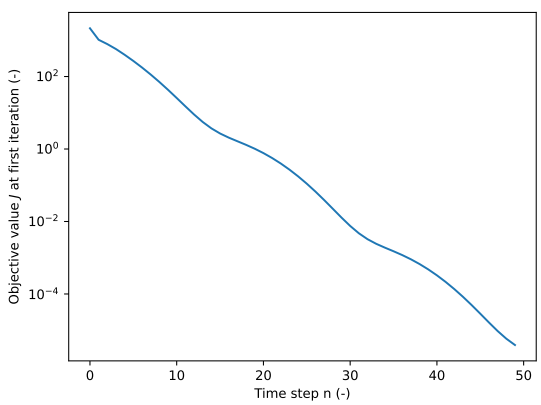

Fig. 3. Evolution of the objective value at first iteration, $J^{1,n}$.

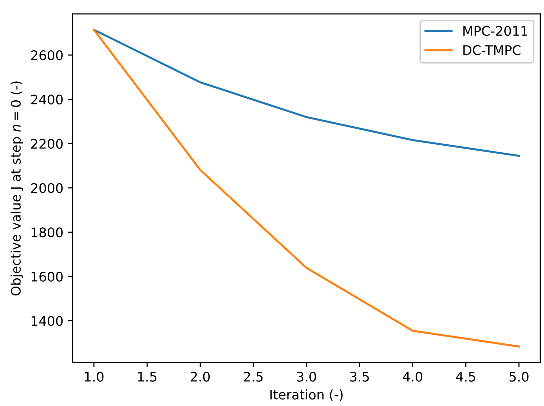

Fig. 5. Comparison of the objective value $J^{j,0}$ for both algorithms.

control-dependent bounds on these errors, but these are determined assuming a fixed operating region. Hence the approach is more conservative than DC-TMPC, which exploits the convex nature of the linearisation errors to find tighter bounds on the state perturbations. As a result, it is expected that DC-TMPC demonstrates faster convergence and a larger set of feasible initial conditions than MPC-2011. To demonstrate this, we apply the MPC-2011 algorithm to the coupled tanks model with the same parameters. For a given terminal set, the range of open loop input voltage allowable for initialising the algorithm with a feasible problem was found to be $[6.1, 9.3]$ V for DC-TMPC, while it was limited between $[7.2, 7.8]$ V for MPC-2011, showing a smaller feasible initial conditions set. This demonstrates the relative conservativeness of the state perturbation bounds in MPC-2011 over DC-TMPC, as expected. Finally, the faster convergence of DC-TMPC is shown in Figure 5, which compares the evolution of the objective value for both algorithms at the first time step, $J^{j,0}$, $j = 1, \dots, 5$. This achieves to demonstrate the superiority of DC-TMPC over the tube-based MPC-2011 algorithm with successive linearisations.

VI. Conclusion

This paper introduces DC-TMPC, a new method for robust nonlinear MPC applied to systems representable as a difference of convex functions. The method relies on successively approximating the system dynamics around predicted trajectories and exploits convexity in the linearisation errors to construct robust and non-conservative tubes containing the perturbed trajectories. Convergence, recursive feasibility and asymptotic stability of the proposed algorithm are demonstrated. The algorithm is applied to regulation of fluid levels in a coupled tank system.

Future work will include generalisation of the method to robustly stabilise the system in the presence of (additive) external disturbances, use of other parameterisations of the tube (e.g. ellipsoids or more general polytopes), and applications to problems such as robust trajectory generation for VTOL aircraft.

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Appendix

We summarise here a method for computing the terminal gain $\hat{K}$, terminal weighting matrix $\hat{Q}$, and terminal set bound $\hat{\gamma}$ by solving a semidefinite program (SDP). Given bounds $\bar{\mathcal{X}} = \{x : |x - x^r| \leq \delta^x\} \subseteq \mathcal{X}$, $\bar{\mathcal{U}} = \{u : |u - u^r| \leq \delta^u\} \subseteq \mathcal{U}$ on the state and control input within the terminal set, we assume that the nonlinear system dynamics can be represented in $\bar{\mathcal{X}} \times \bar{\mathcal{U}}$ using a set of linear models. The model approximation is assumed to satisfy, for all k,

$$f(x,u) - f(x^r,u^r) \in \mathrm{Co}\{A^{(i)}(x - x^r_k) + B^{(i)}(u - u^r_k), \; i = 1,\dots,m\}, \; \forall (x,u) \in \bar{\mathcal{X}} \times \bar{\mathcal{U}} \tag{16}$$

(where Co denotes the convex hull). In order that $\hat{Q}$ and $\hat{K}$ satisfy the inequality (5) we require, for all $x \in \bar{\mathcal{X}}$,

$$\|x - x^r\|^2_{\hat{Q}} \geq \big\|A^{(i)}(x - x^r) + B^{(i)}\hat{K}(x - x^r)\big\|^2_{\hat{Q}} + \|x - x^r\|^2_Q + \|\hat{K}(x - x^r)\|^2_R.$$

Since each term is quadratic in $x - x^r$, this condition is equivalent to a set of matrix inequalities, for $i = 1,\dots,m$,

$$\hat{Q} \succeq (A^{(i)} + B^{(i)}\hat{K})^\top \hat{Q}(A^{(i)} + B^{(i)}\hat{K}) + Q + \hat{K}^\top R\hat{K},$$

which can be expressed equivalently using Schur complements as LMIs in variables $S = \hat{Q}^{-1}$ and $Y = \hat{K}\hat{Q}^{-1}$:

$$\begin{bmatrix} S & (A^{(i)}S + B^{(i)}Y)^\top & S & Y \\ \star & S & 0 & 0 \\ \star & \star & Q^{-1} & 0 \\ \star & \star & \star & R^{-1} \end{bmatrix} \succeq 0, \; i = 1,\dots,m. \tag{17}$$

To ensure that the model approximation (16) remains valid we can exploit the positive invariance of the set $\hat{\mathcal{X}} = \{x : \|x - x^r\|_{\hat{Q}} \leq \hat{\gamma}\}$ for all $\hat{\gamma} > 0$, and impose the constraints

$$\{x : \|x - x^r\|^2_{\hat{Q}} \leq \hat{\gamma}\} \subseteq \bar{\mathcal{X}} \cap \{x : Kx \in \bar{\mathcal{U}}\}$$

which are equivalent to

$$\hat{\gamma}^{-1}[\delta^x]^2_i - [S]_{ii} \succeq 0, \qquad i = 1,\dots,n_x \tag{18}$$

$$\begin{bmatrix} \hat{\gamma}^{-1}[\delta^u]^2_i & [Y]_i \\ \star & S \end{bmatrix} \succeq 0, \qquad i = 1,\dots,n_u \tag{19}$$

To balance the requirements for good terminal controller performance and a large terminal set, we can minimise $\mathrm{tr}(\hat{Q}) + \alpha\hat{\gamma}^{-1}$ subject to the constraints (17), (18), (19) and

$$\begin{bmatrix} S & I \\ \star & \hat{Q} \end{bmatrix} \succ 0, \tag{20}$$

over variables $S = S^\top$, Y and $\hat{\gamma}^{-1}$, where α is a scalar constant that controls the trade-off between the competing objectives of minimising $\mathrm{tr}(\hat{Q})$ and minimising $\hat{\gamma}^{-1}$.