Lecture 7: Local Bifurcations

- We have considered the shape (or topology) of the trajectories of solutions of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

- We now consider the structural stability of the topology of these trajectories near equilibrium points as the system parameters change

- Let $\boldsymbol{\mu}$ be a constant parameter and consider the solutions of

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}), \qquad \boldsymbol{\mu} \in \mathbb{R}^p, \mathbf{x} \in \mathbb{R}^{\boldsymbol{n}}$$

  $\boldsymbol{\mu}$ is called a bifurcation parameter or bifurcation vector

1-D Bifurcations

- The simplest case is a first order system with scalar parameter μ:

$$\dot{x} = f(x; \mu), \qquad x, \mu \in \mathbb{R}$$

- A bifurcation occurs when the number or type of the equilibria change as μ is changed, e.g. from stable to unstable.
- There are three types of bifurcation for this case:
  - Saddle-node
  - Transcritical
  - Pitchfork
- Bifurcations are usually analysed using 'normal forms', which are standardised equations that represent classes of equation.

Saddle-node bifurcation

- Normal form of a system that can have a saddle-node bifurcation:

$$\dot{x} = \mu - x^2$$

  $\mu > 0$: one stable equilibrium and one unstable equilibrium

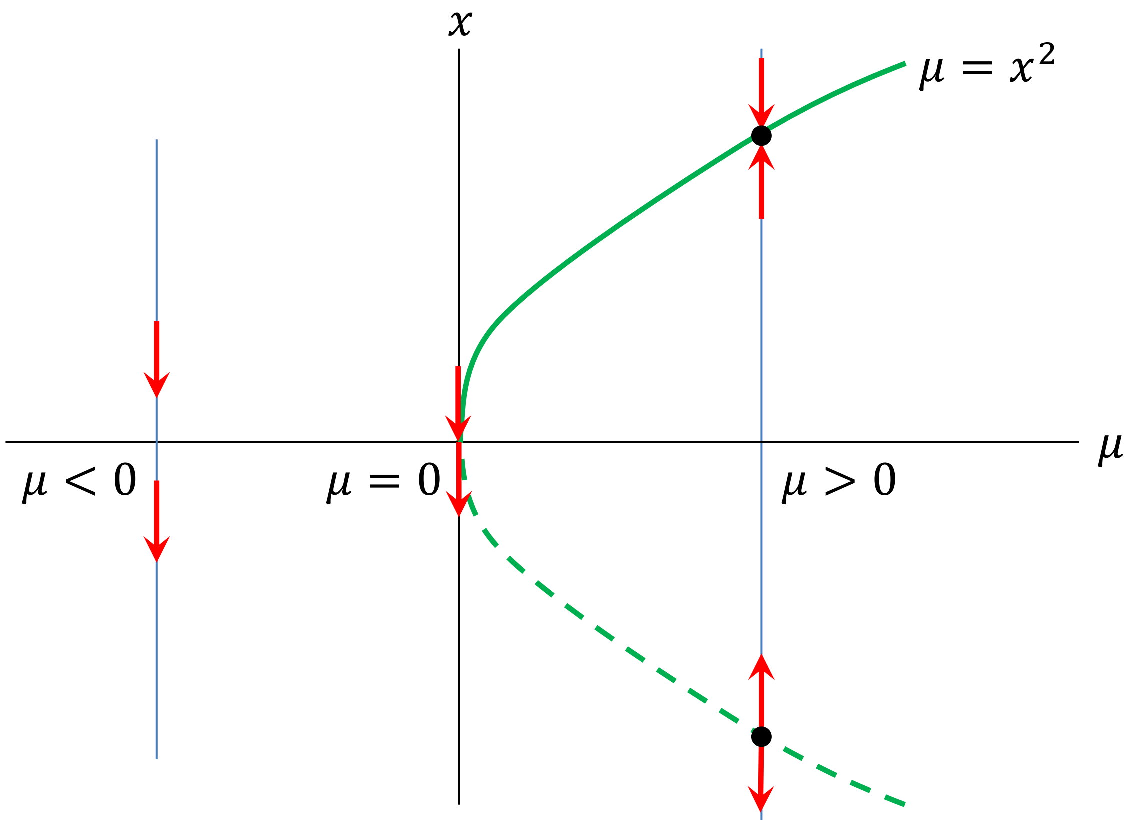
  $\mu = 0$: single equilibrium (called a saddle)

  $\mu < 0$: no equilibria

- A **bifurcation diagram** shows the position and type of equilibrium points on the vertical axis as μ varies along the horizontal axis.

- Solid line => stable equilibrium, dashed line => unstable equilibrium

Saddle-node bifurcation diagram



No equilibria Two equilibria

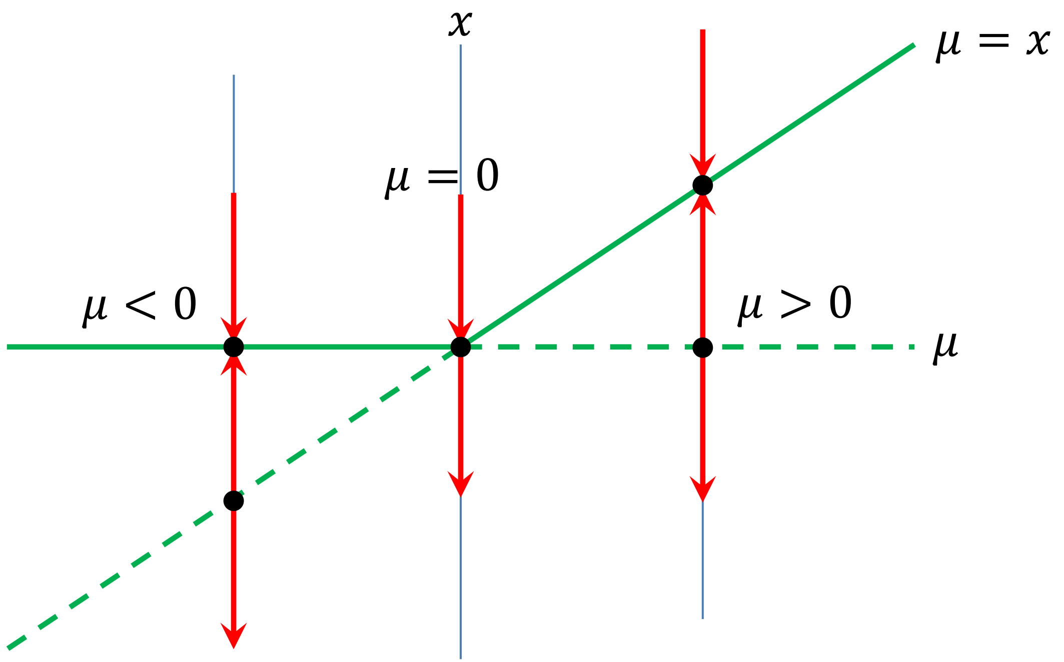
Transcritical bifurcation

Normal form:

$$\dot{x} = \mu x - x^2$$

Equilibria at $x = 0$ and $x = \mu$

Stability depends on μ; equilibria swap roles when $\mu = 0$ at a saddle.



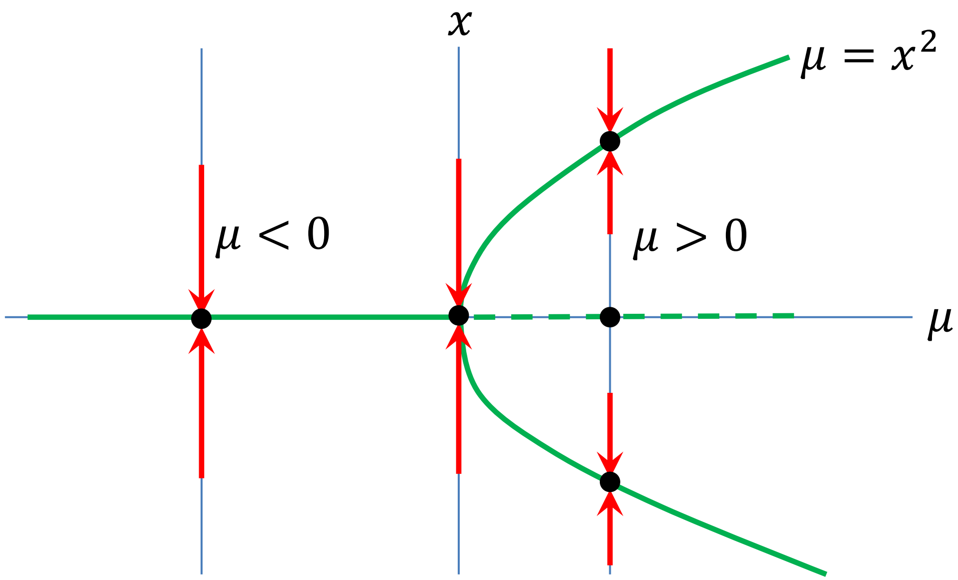
Pitchfork Bifurcation

Normal form:

$$\dot{x} = \mu x - x^3$$

$\mu > 0$: unstable equilibrium at $x = 0$ and stable equilibria at $x = \pm\mu^{1/2}$

$\mu < 0$: $x = 0$ is the only equilibrium and is stable.



Tangency conditions

- The locations, (x_0, μ_0), of bifurcation points are determined by the **tangency conditions**:

$$f(x_0, \mu_0) = 0$$
$$\left.\frac{\partial f(x, \mu_0)}{\partial x}\right|_{x_0} = 0$$

  (i.e. x_0 must be a non-hyperbolic equilibrium for $\mu = \mu_0$)

- If these are satisfied we have a candidate – but what sort of bifurcation is it?

Saddle-node bifurcation

$$f(x_0, \mu_0) = 0, \qquad \left.\frac{\partial f(x, \mu_0)}{\partial x}\right|_{x_0} = 0$$

$$\left.\frac{\partial f(x, \mu)}{\partial \mu}\right|_{x=x_0, \mu=\mu_0} \neq 0$$

($\Longrightarrow f$ is locally linear in μ)

$$\left.\frac{\partial^2 f(x, \mu)}{\partial x^2}\right|_{x=x_0, \mu=\mu_0} \neq 0$$

($\Longrightarrow f$ is locally quadratic in x)

Transcritical Bifurcation

$$f(x_0, \mu_0) = 0, \qquad \left.\frac{\partial f(x, \mu_0)}{\partial x}\right|_{x_0} = 0$$

$$\left.\frac{\partial f(x, \mu)}{\partial \mu}\right|_{x=x_0, \mu=\mu_0} = 0, \qquad \left.\frac{\partial^2 f(x, \mu)}{\partial x \partial \mu}\right|_{x=x_0, \mu=\mu_0} \neq 0$$

($\Longrightarrow$ f is locally bilinear in x, μ)

$$\left.\frac{\partial^2 f(x, \mu)}{\partial x^2}\right|_{x=x_0, \mu=\mu_0} \neq 0$$

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Pitchfork Bifurcation

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$$\left.\frac{\partial f(x, \mu)}{\partial \mu}\right|_{x=x_0, \mu=\mu_0} = 0, \qquad \left.\frac{\partial^2 f(x, \mu)}{\partial x \partial \mu}\right|_{x=x_0, \mu=\mu_0} \neq 0$$

($\Longrightarrow f$ is locally bilinear in x, μ)

$$\left.\frac{\partial^2 f(x, \mu)}{\partial x^2}\right|_{x=x_0, \mu=\mu_0} = 0, \qquad \left.\frac{\partial^3 f(x, \mu)}{\partial x^3}\right|_{x=x_0, \mu=\mu_0} \neq 0$$

($\Longrightarrow f$ is locally cubic in x)

Example

- Consider the system

$$\dot{x} = \mu \ln(x) + x - 1$$

- $f(x,\mu) = \mu \ln(x) + x - 1 = 0$ if and only if $x = 1$

  (N.B. $\frac{\partial f}{\partial x} = \frac{\mu}{x} + 1 > 0$ for all $x > 0$ and $f(x)$ undefined $x \leq 0$)

- For a bifurcation point we require $\left.\frac{\partial f(x,\mu)}{\partial x}\right|_{x_0=1} = \mu + 1 = 0$, i.e. $\mu = -1$
- At $(x_0, \mu_0) = (1, -1)$:

$$\frac{\partial f(x,\mu)}{\partial \mu} = \ln(x) = 0; \ \frac{\partial^2 f(x,\mu)}{\partial x^2} = -\frac{\mu}{x^2} = -1; \ \frac{\partial^2 f(x,\mu)}{\partial x \partial \mu} = \frac{1}{x} = 1$$

  so this is a transcritical bifurcation

2D Example

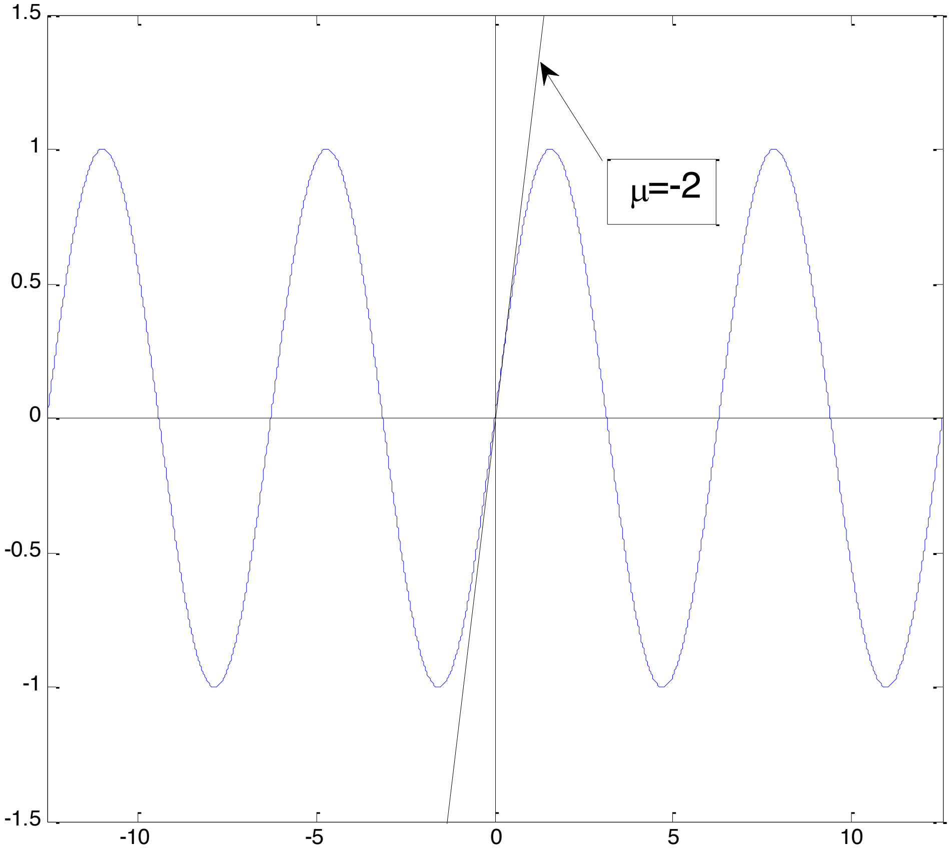
- Testing for the type of bifurcation for higher dimension phase spaces dependent on a single parameter is dealt with by Sotomayor’s Theorem (Section 4.2 in Perko) concerning systems with a single zero eigenvalue. Here we just consider a specific example.

- Question: Does the origin undergo a bifurcation for the following system

$$\dot{x} = \mu x + y + \sin x$$
$$\dot{y} = x - y$$

2D Example

The equilibrium points (x^*, y^*) satisfy $\dot{x} = 0, \dot{y} = 0$
so $x^* = y^*$ and $\sin x^* = -(\mu + 1)x^*$.



- If $\mu = -2, -(\mu + 1)x^*$ is tangential to $\sin x$.
- If $\mu < -2$ the line rotates anti-clockwise and there is only one solution.
- If $\mu > -2$ the line rotates clockwise and there are three solutions, then 5, then 7 and so on as μ increases; then the number of solutions decreases again for $\mu > -1$.
- Three of these solutions initially break apart at the origin at the first bifurcation.

2D Example

The Jacobian is

$$J = \begin{bmatrix} \mu + \cos x & 1 \\ 1 & -1 \end{bmatrix}$$

so at $x = 0$:

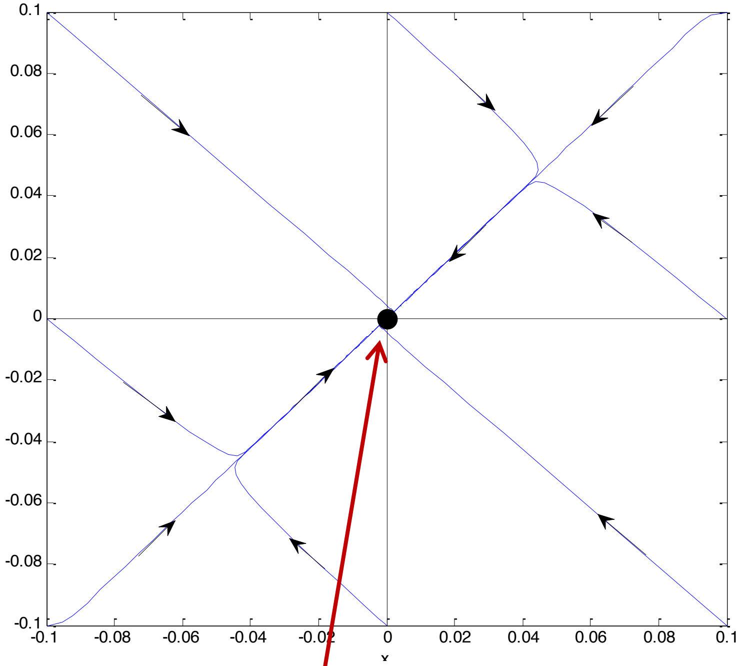
$$J = \begin{bmatrix} \mu + 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Eigenvalues, $\frac{\mu \mp \sqrt{(\mu+2)^2+4}}{2}$, are both negative if $\mu < -2$ and of opposite sign if $\mu > -2$ (the unstable direction is along the x-axis).

The previous slide demonstrates that there exist other equilibria (that are stable along the x-axis as well as the y-axis).

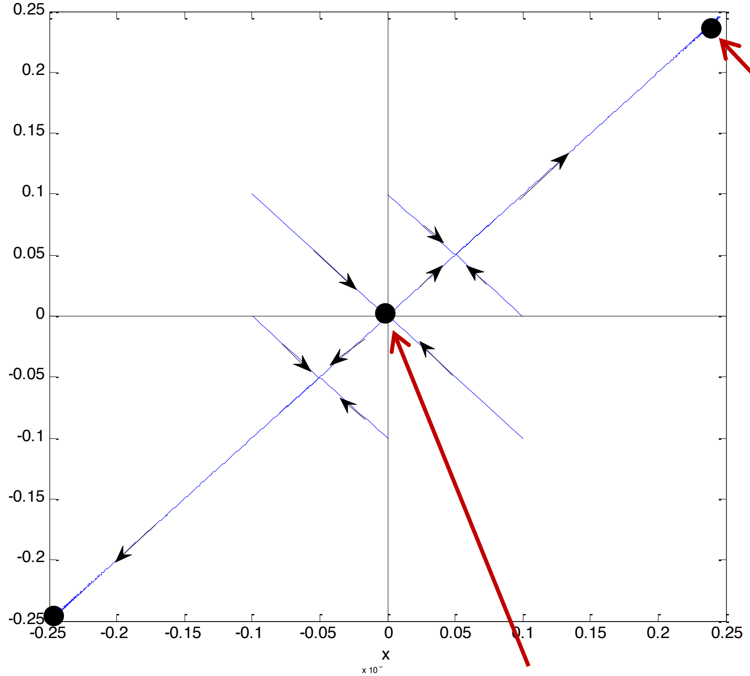
2D Example: phase plane plots

$\mu = -2.1$



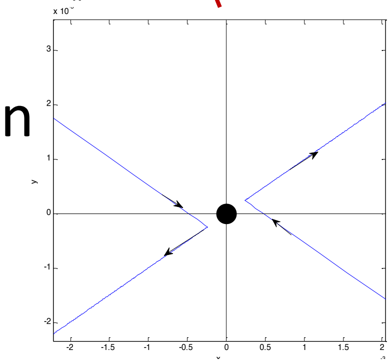
Only one equilibrium

$\mu = -1.99$

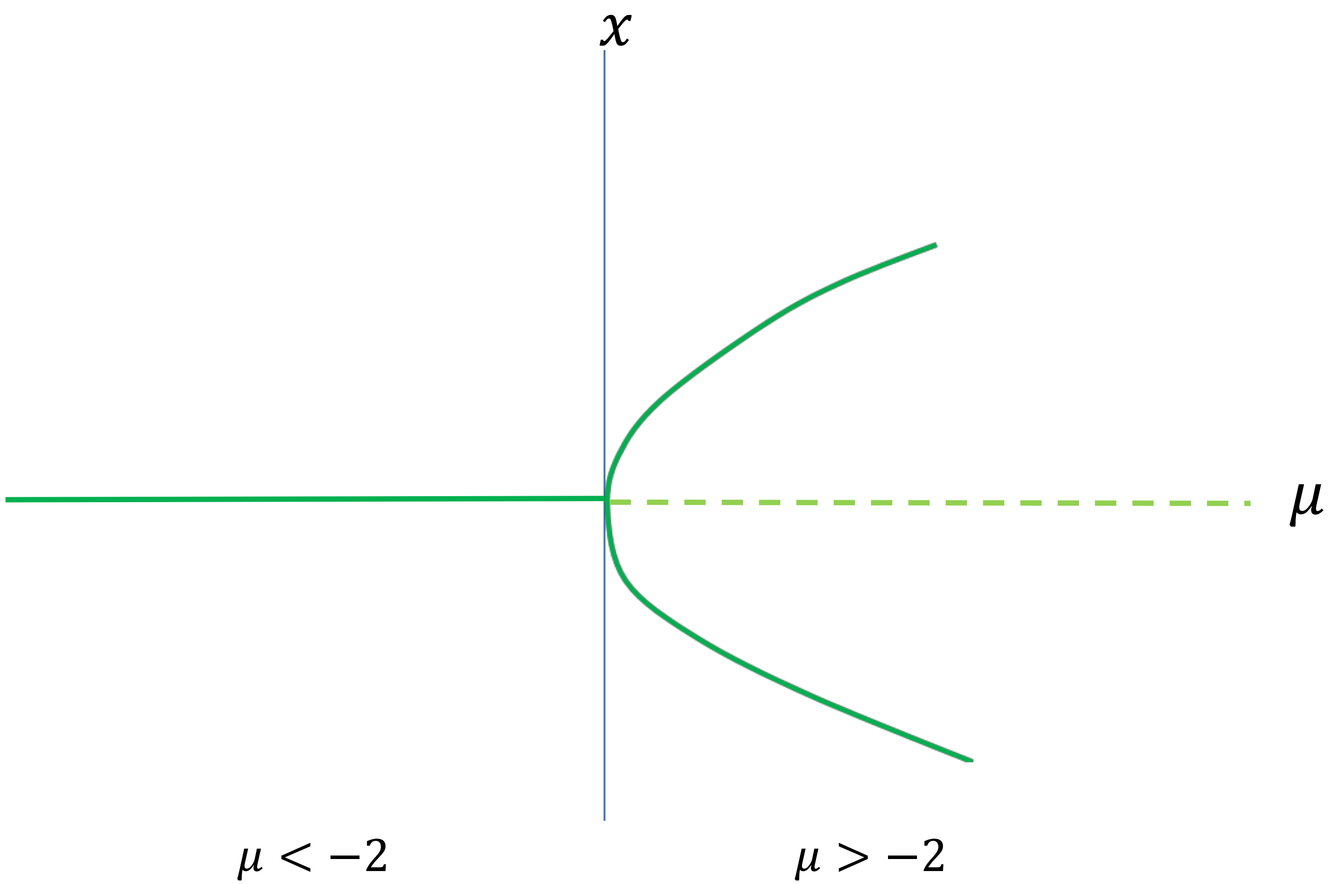


New equilibrium point

Exploded view of origin showing unstable equilibrium



Bifurcation diagram near the origin



Hopf Bifurcations

- The previous example dealt with a 2D system with a single eigenvalue that passes through zero. The example was found to have a pitchfork bifurcation.

- We now consider 2D systems in which the non-hyperbolic bifurcation point is a centre (with purely imaginary eigenvalues). We now have two eigenvalues that can change stability. This leads us to Hopf bifurcations.

Conditions for a Hopf Bifurcation

Consider the system

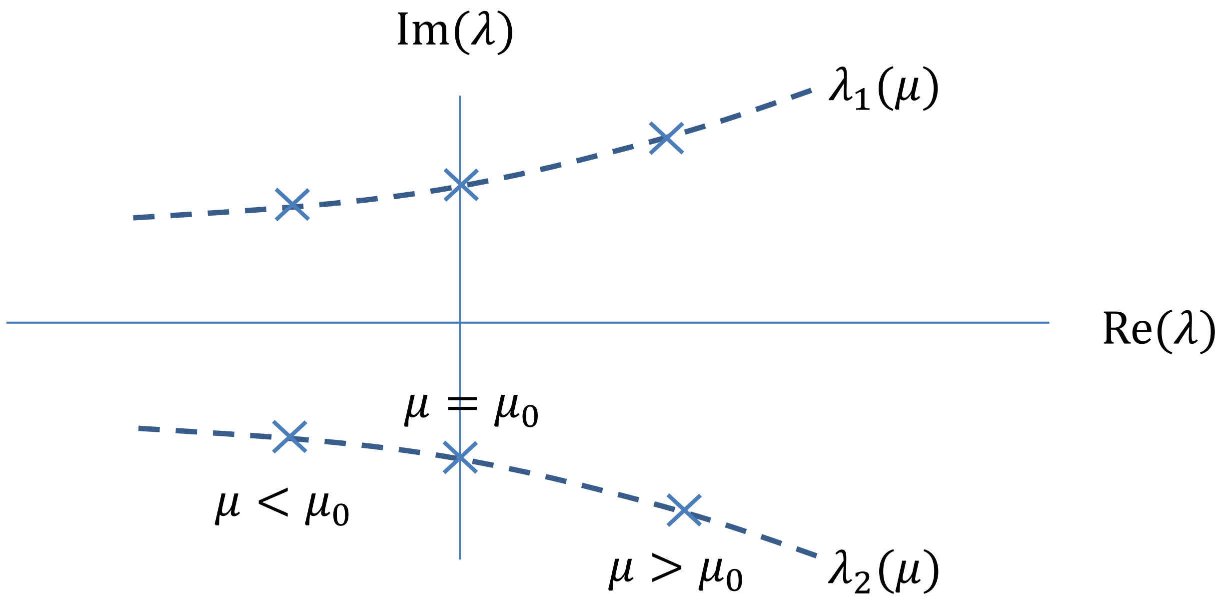
$$\dot{x} = f(x, y; \mu)$$
$$\dot{y} = g(x, y; \mu)$$

We have a Hopf bifurcation if

$$\lambda_{1,2}(\mu) = \alpha(\mu) \pm j\omega(\mu)$$

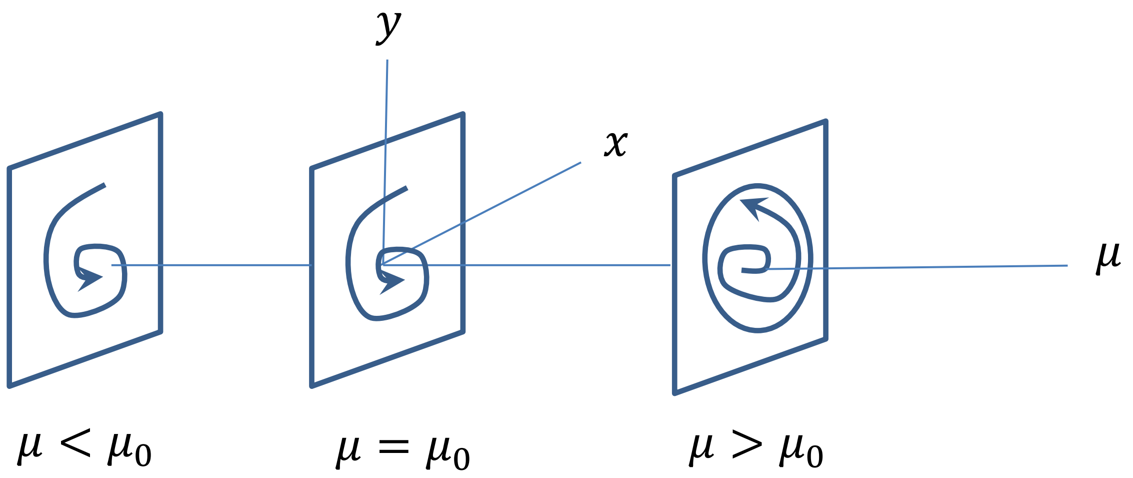
for $\mu_0 - \epsilon < \mu < \mu_0 + \epsilon$ for some $\epsilon > 0$ with:

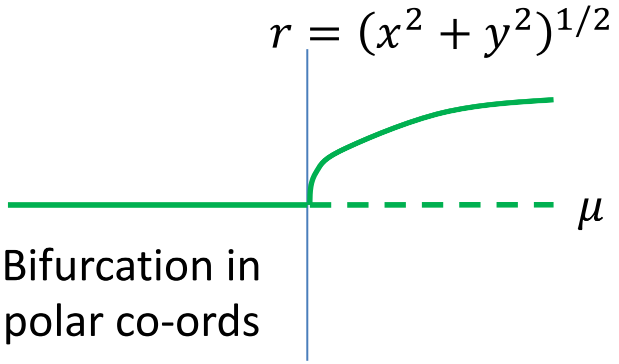
- $\alpha(\mu_0) = 0$
- $\alpha(\mu) < 0$ for $\mu < \mu_0$
- $\alpha(\mu) > 0$ for $\mu > \mu_0$



Supercritical Hopf Bifurcation

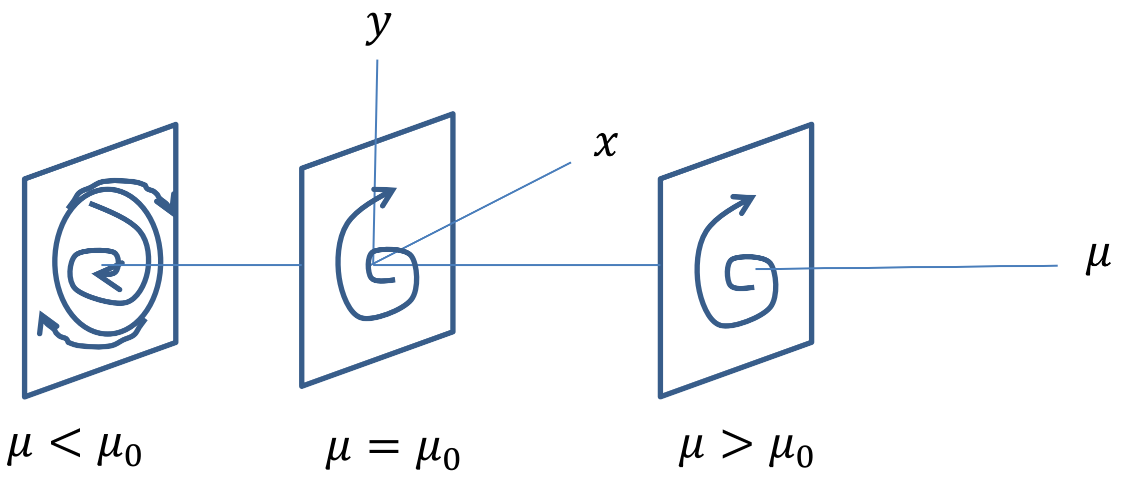
- For $\mu < \mu_0$ there is a stable spiral equilibrium point, which for $\mu > \mu_0$ becomes an unstable spiral with an enclosing stable limit cycle that expands with increasing μ.

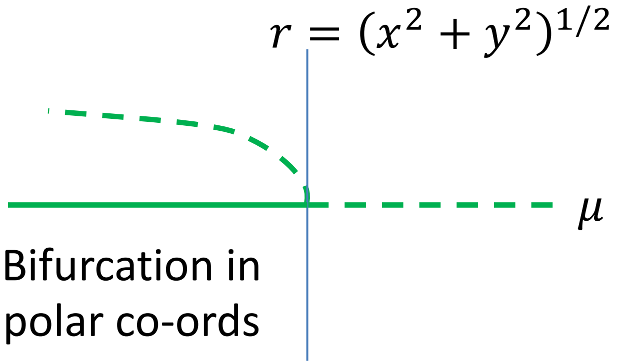




Subcritical Hopf Bifurcation

- For $\mu < \mu_0$ there is a stable spiral surrounded by an unstable limit cycle. As μ increases the unstable spiral becomes smaller and at $\mu = \mu_0$ the limit cycle collapses to a fixed point and for $\mu > \mu_0$ there is an unstable spiral.





Degenerate Hopf Bifurcation

- A stable spiral for $\mu < \mu_0$ becomes a nonlinear centre at $\mu = \mu_0$ (the orbit is not isolated and its ‘radius’ depends on an initial condition) and then an unstable spiral for $\mu > \mu_0$.

- Called **degenerate** because there is no limit cycle for $\mu < \mu_0$ or $\mu > \mu_0$.

Using Polar Co-ordinates

- Consider

$$\dot{x} = \mu x - y + xy^2$$
$$\dot{y} = x + \mu y + y^3$$

- $(x, y) = (0,0)$ is an equilibrium with Jacobian $\begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$.

- Jacobian has eigenvalues $\mu \pm j$.

  Therefore we suspect there is a Hopf bifurcation at $\mu = 0$.

  But what kind?

Using Polar Co-ordinates

$$0.5\frac{dr^2}{dt} = r\dot{r} = x\dot{x} + y\dot{y} = x(\mu x - y + xy^2) + y(x + \mu y + y^3)$$
$$= \mu r^2 + r^2 y^2$$

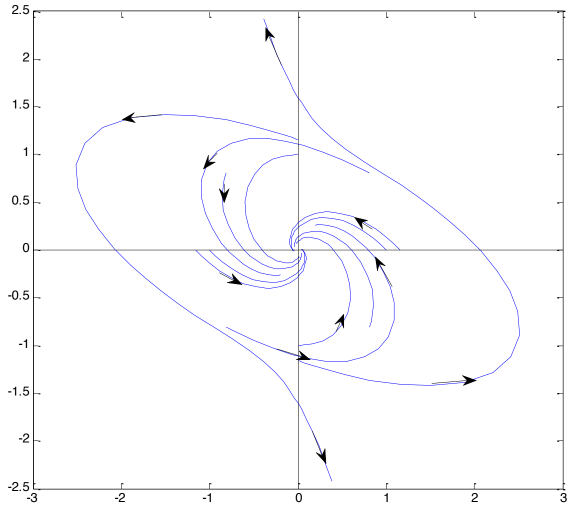
- If $\mu > 0$, then $\dot{r} = \mu r + r y^2 \geq \mu r$
  so $r(t)$ grows without limit, i.e. there is no limit cycle for $\mu > 0$.

- If $\mu = 0$, then $\dot{r} = r y^2 \geq 0$, so $r(t)$ grows with no nonlinear centre.
  Hence the bifurcation is not degenerate as there is no centre.

- If $\mu < 0$, then $\mu r + r y^2 < 0$ for small r, hence a stable spiral,
  confirming a subcritical Hopf bifurcation; expect an unstable limit cycle.

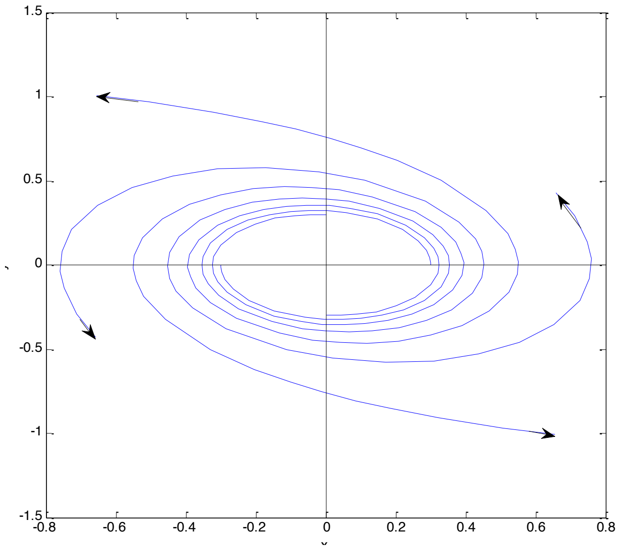
A sub-critical bifurcation

$\mu = -1$



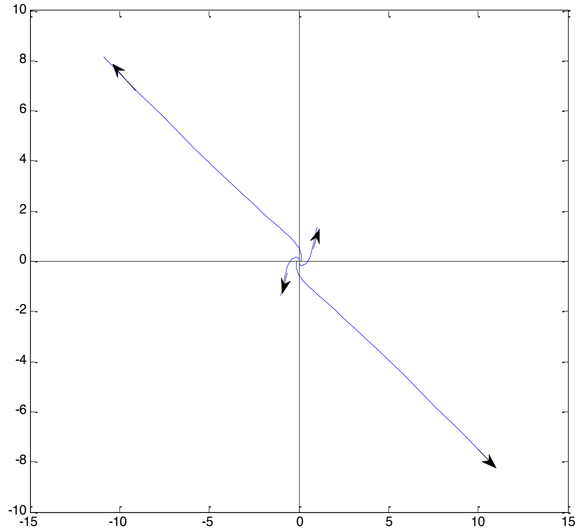
Stable spiral with unstable limit cycle

$\mu = 0$



Unstable spiral

$\mu = 1$



Unstable spiral