

# Lecture 8: Chaos

1-Dimensional discrete systems can be chaotic but continuous 1-D systems cannot. So we start with discrete maps:

$$x_{k+1} = F(x_k)$$

- A fixed point  $x^*$  satisfies

$$x^* = F(x^*)$$

- Linearise the Map around  $x^*$ :

$$x^* + w_{k+1} = F(x^*) + DF(x^*)w_k + \dots$$

Analyse the Jacobian

$$w_{k+1} = DF(x^*)w_k$$

# The unit circle

- The stability of  $x^*$  depends on whether the eigenvalues of the Jacobian  $DF(x^*)$  lie inside the unit circle (see e.g. Examples Sheet 1, q2)
- For the 1-D case this constraint reduces to
$$|DF(x^*)| < 1$$
- The same reasoning about stable, unstable and centre manifolds as in the case of continuous systems applies here.

# Cobweb diagrams

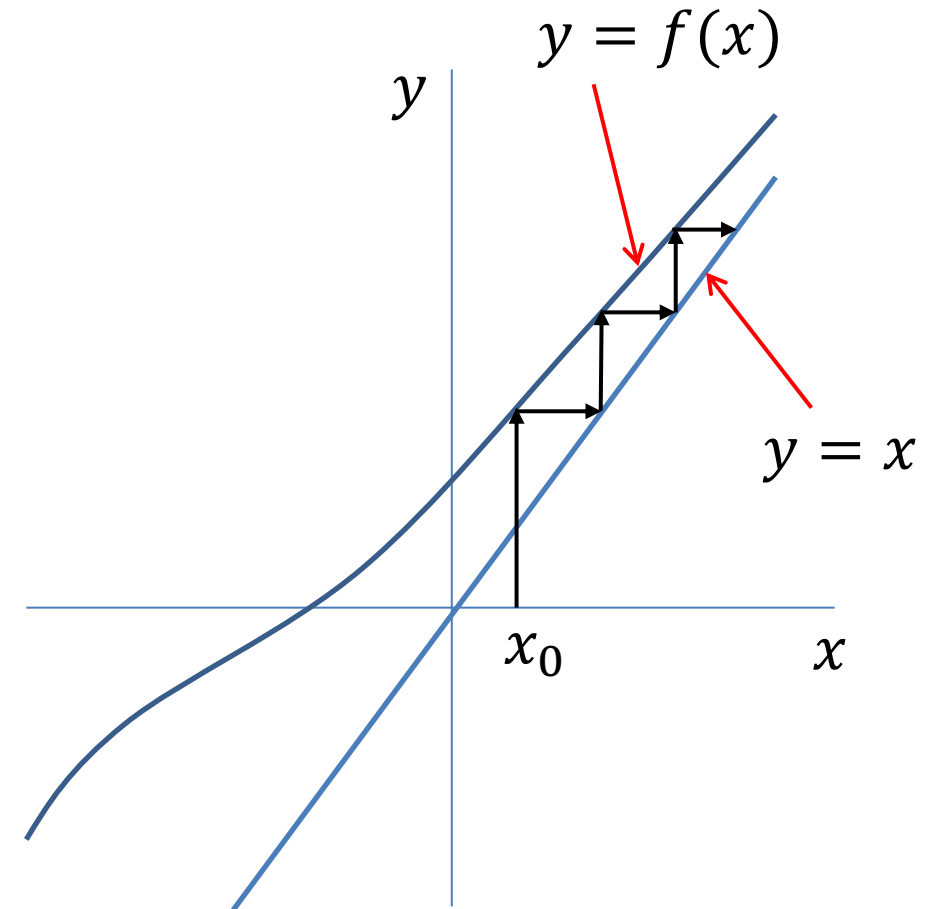
Consider the 1-D map

$$x_{k+1} = f(x_k)$$

To find  $x_1$  given  $x_0$ :

- plot  $x_0$  on the  $x$ -axis
- draw a vertical line to  $y = f(x)$
- transfer this value to  $x$  by drawing a horizontal line to  $y = x$ .

Repeat to find  $x_2, x_3 \dots$



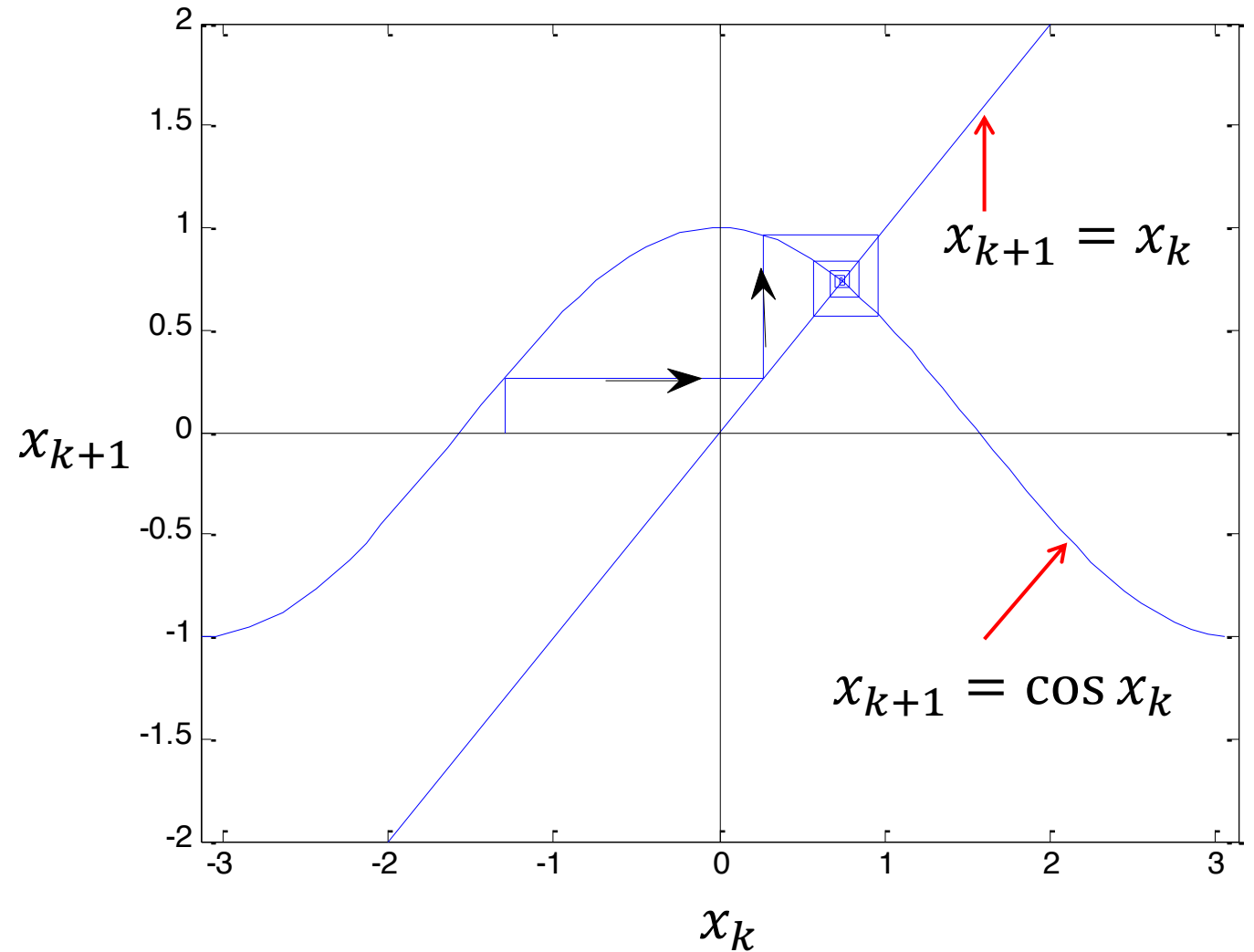
# Cosine map

Cosine map:

$$x_{k+1} \mapsto \cos(x_k)$$

has fixed point

$$x^* = 0.739$$



# Logistic map

- Model of e.g. population dynamics:

$$x_{k+1} = rx_k(1 - x_k)$$

Equilibria  $x^* = 0$  and  $x^* = (1 - 1/r)$

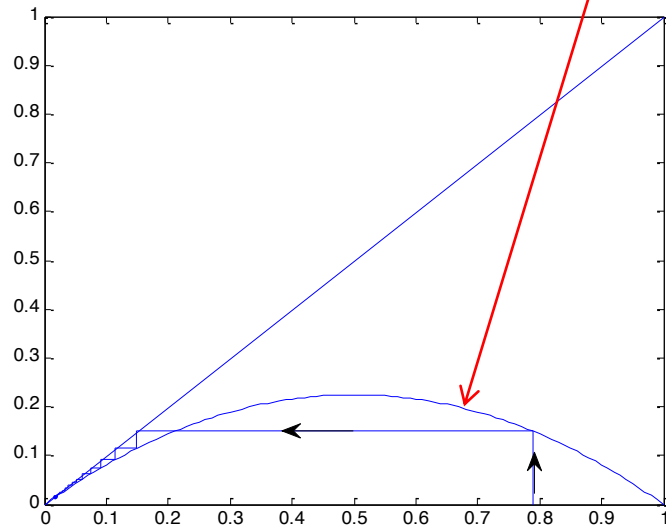
- Linearise:

$$w_{k+1} = r(1 - 2x^*)w_k$$

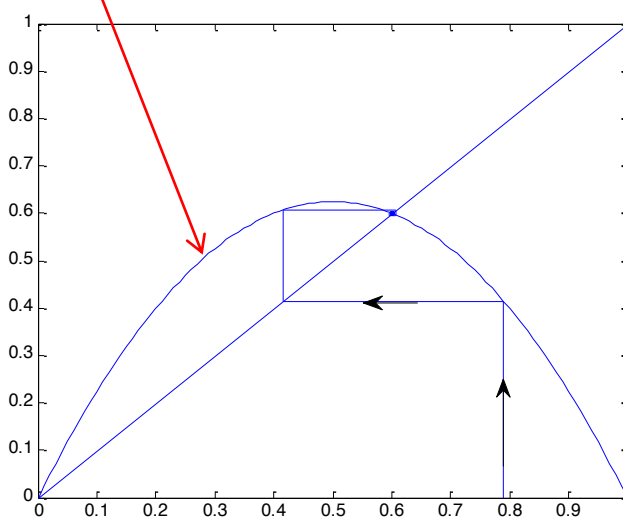
- If  $r < 1$ , then  $x^* = 0$  is stable and populations go extinct.
- If  $r > 1$ , then  $x^* = 0$  is unstable and populations grow – but what happens to the other equilibrium?

# Cobwebs for the logistic map

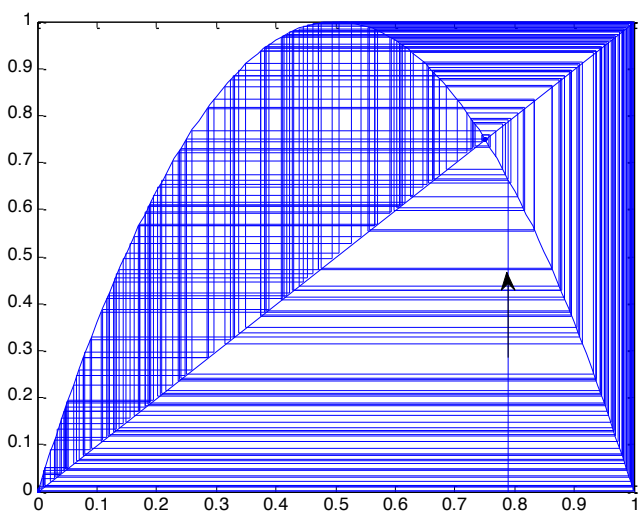
$$rx_k(1 - x_k)$$



$$r = 0.9$$



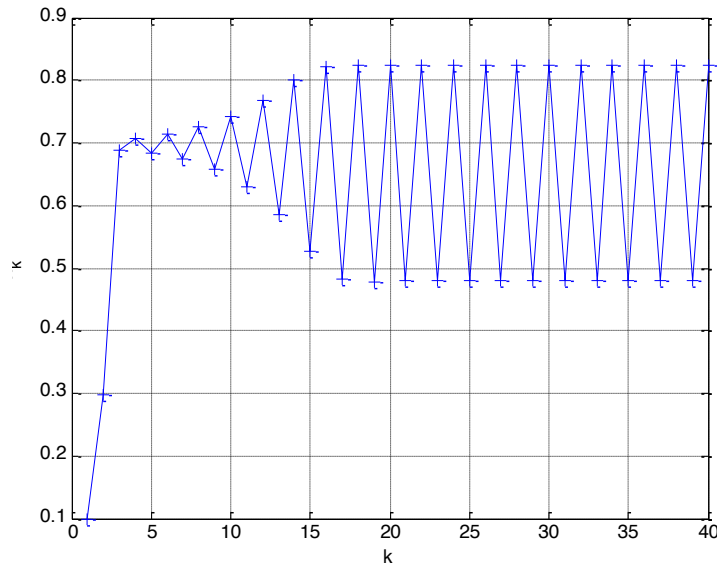
$$r = 2.5$$



$$r = 4.0$$

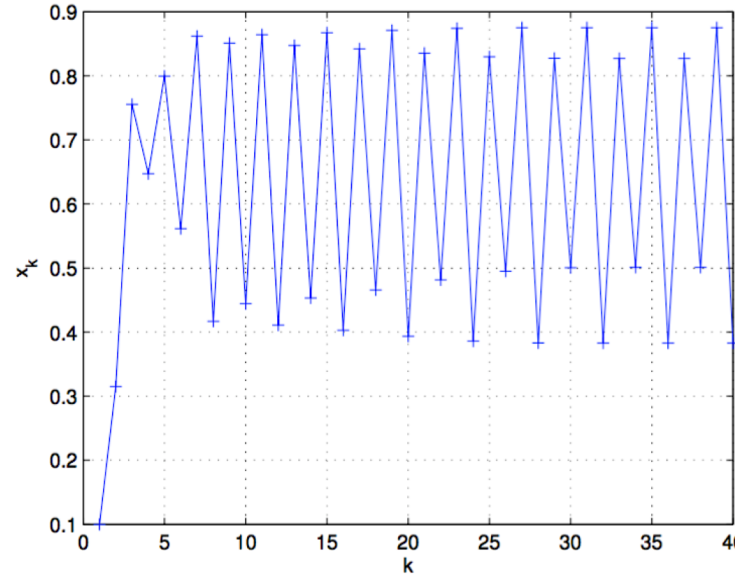
Solution becomes non-periodic for  $r = 3.59$

$r = 3.3$



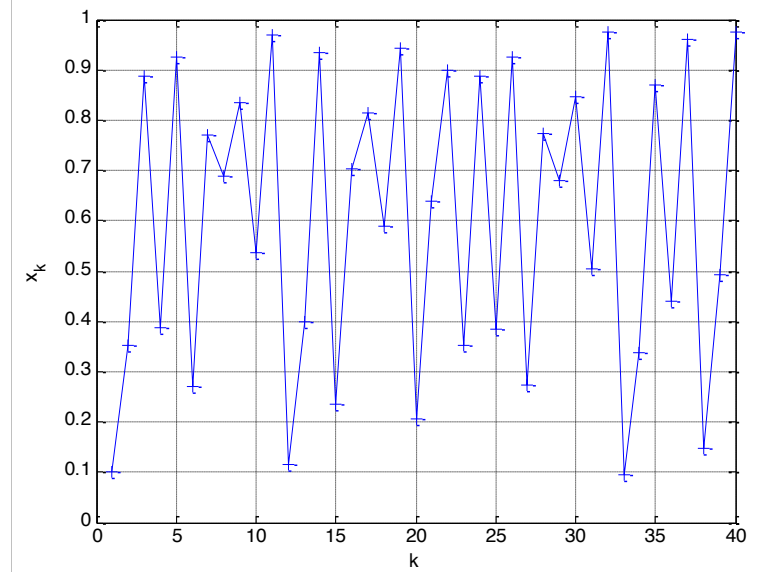
periodic

$r = 3.5$



double-periodic

$r = 3.9$



non-periodic

# Logistic map simulations

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# Behaviour of the logistic map

- For  $1 < r < 3$  the system is stable around  $(1 - 1/r)$
- For  $r > 3$  oscillations appear, with period doubling as  $r$  increases at well-defined values of  $r$ .
- The rate of doubling increases as  $r$  approaches  $3.5699\dots$  and  $x$  is then no longer periodic.
- As  $r$  increases, chaotic behaviour ensues with brief windows of periodic behaviour for certain ranges of  $r$ .



# Analysing periodicity

- Consider

$$x_{k+2} \mapsto f(f(x_k))$$

- The fixed points of this map are called 2-cycles (they have period 2), and are solutions of

$$r^2 x^* (1 - x^*) \{1 - r x^* (1 - x^*)\} = x^*.$$

The roots (see lecture notes) are stable for  $3 < r < 1 + \sqrt{6} = 3.4495$ .

- The next bifurcation is to a 4-cycle, then 8, etc. The changes in period occur as doublings with increasingly complex expressions for  $r$  and narrower stability ranges.

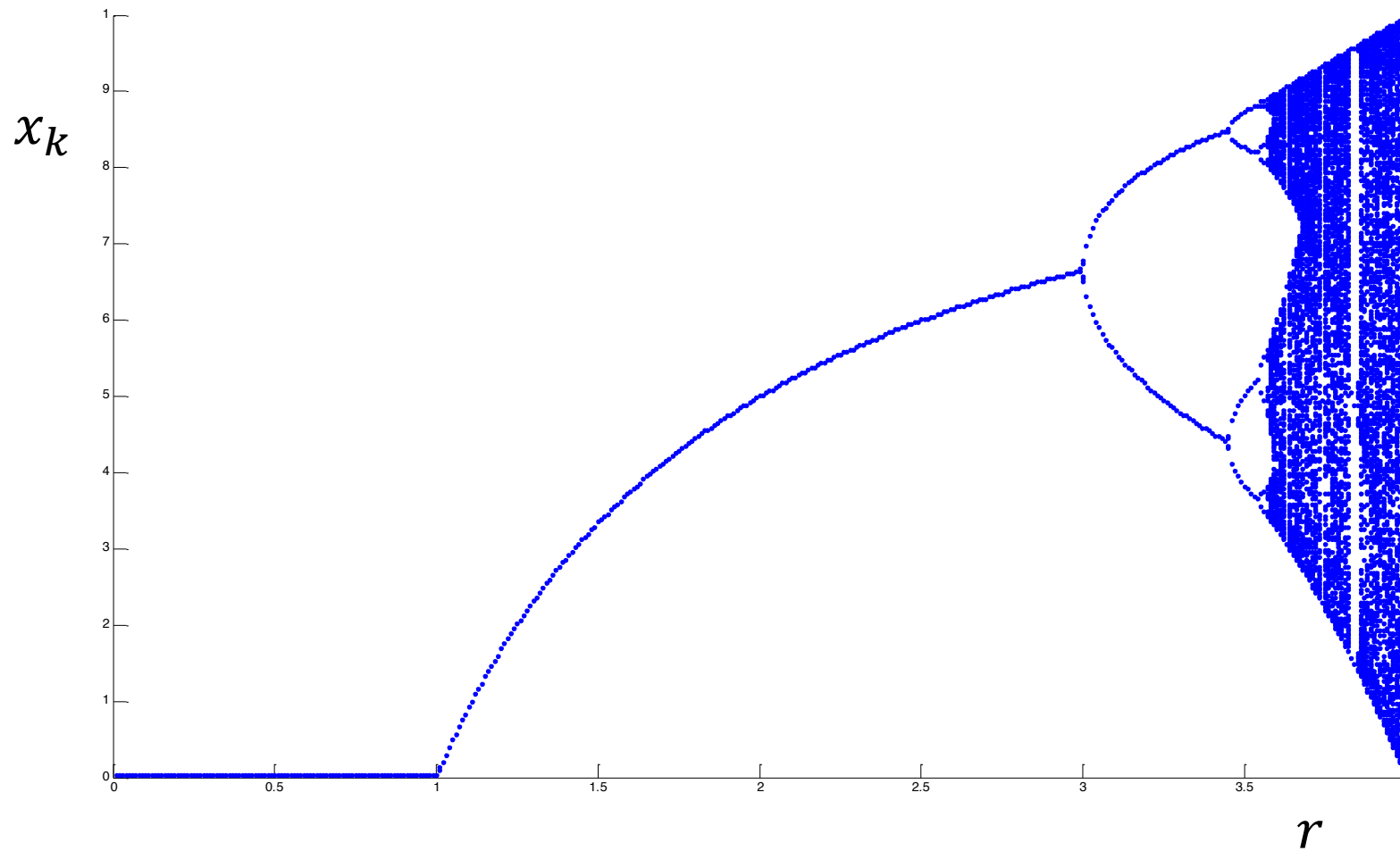
# Orbit diagrams

These are a way of displaying how a map changes with a parameter.

- Choose a value of the parameter  $r$  and a starting value for  $x$ .
- Run the iteration for, say, 300 iterations.
- Record the values of  $x$  for a further 300 or so iterations and plot them on the map.
- Change  $r$  and repeat.

Various plots are given in the lecture notes.

# The logistic map orbit



# Chaos

- The map illustrates dynamical chaos.
- Chaos is **aperiodic long-term** behaviour in a **deterministic** system that exhibits **sensitive dependence on initial conditions**.
  - Aperiodic long-term: These are trajectories that never settle down to fixed points or periodic orbits.
  - Deterministic: The trajectory is the solution of an equation with no noise – everything is certain and precise.
  - Sensitive to initial conditions: Points on trajectories that are near to each other diverge with time.

# Lyapunov exponent for maps

- Measures how fast solutions that are initially close diverge (sensitivity of initial conditions).
- Consider the effect on long-term behaviour of changing initial condition from  $x_0$  to  $x_0 + w_0$ :

$$\begin{array}{lll} x_1 + w_1 = f(x_0 + w_0) & \Rightarrow & w_1 = f(x_0 + w_0) - x_1 \\ x_2 + w_2 = f(f(x_0 + w_0)) & \Rightarrow & w_2 = f(f(x_0 + w_0)) - x_2 \\ \vdots & \vdots & \vdots \\ x_k + w_k = f(\cdots f(x_0 + w_0)) & \Rightarrow & w_k = f(\cdots f(x_0 + w_0)) - x_k \end{array}$$

**Definition:** The **Lyapunov exponent**,  $\lambda$ , is defined by

$$|w_k| = |w_0|e^{\lambda k}$$

# Lyapunov exponent for maps

Therefore

$$w_1 = f(x_0 + w_0) - x_1$$
$$\approx f(x_0) - x_1 + Df(x_0)w_0 = Df(x_0)w_0$$

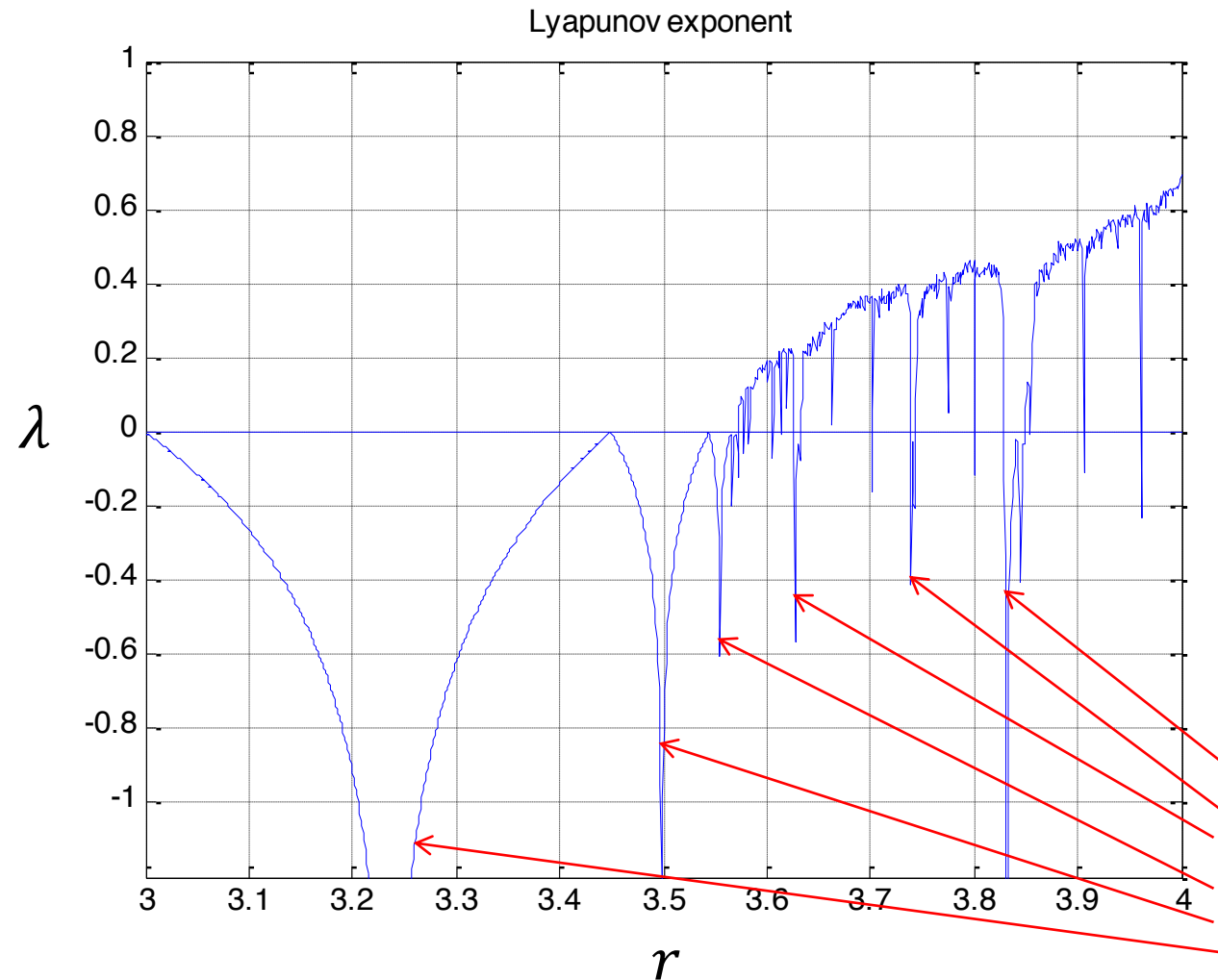
$$w_2 \approx Df(x_0)w_0$$
$$= Df(x_1)Df(x_0)w_0$$
$$\vdots$$

$$w_k = \prod_{i=1}^{k-1} Df(x_i) w_0$$

so

$$\lambda = \frac{1}{k} \ln \left| \frac{w_k}{w_0} \right| \approx \frac{1}{k} \sum_{i=0}^{k-1} Df(x_i) \quad (\text{accurate as } k \rightarrow \infty)$$

# Lyapunov exponents of the logistic map



Periodic  
solutions – dips  
in the exponent

# Chaos in Flows

- The **Lorenz Equations**: model of convection in the atmosphere

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz$$

$r$  = Rayleigh number,  $\sigma$  = Prandtl number,  $b$  is a positive constant.

- If  $r < 1$  the only equilibrium is the origin  $(0,0,0)$
- For  $r > 1$  two more equilibria appear via a pitchfork bifurcation:

$$(x^*, y^*, z^*) = \left( \sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1 \right)$$

$$(x^*, y^*, z^*) = \left( -\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1 \right)$$

- The equations are symmetric in  $x$  and  $y$ .



# Lorenz equations: volume contraction

- Compute  $\text{div } \mathbf{f}$  :

$$\begin{aligned}\nabla \cdot \mathbf{f} &= \frac{\partial \sigma(y - x)}{\partial x} + \frac{\partial (rx - y - xz)}{\partial y} + \frac{\partial (xy - bz)}{\partial z} \\ &= -(1 + \sigma + b) < 0\end{aligned}$$

Hence the integral of  $\nabla \cdot \mathbf{f}$  over any control volume is negative. So trajectories converge to a zero volume region of phase space.

- Is this zero volume solution a point or limit cycle? Neither!

# Lorenz equations: stability

- The solution cannot have unstable equilibrium points or unstable periodic orbits – such solutions imply expansion of the state space, not contraction.

Thus any fixed points must be stable or saddles, or if there are limit cycles they must be stable.

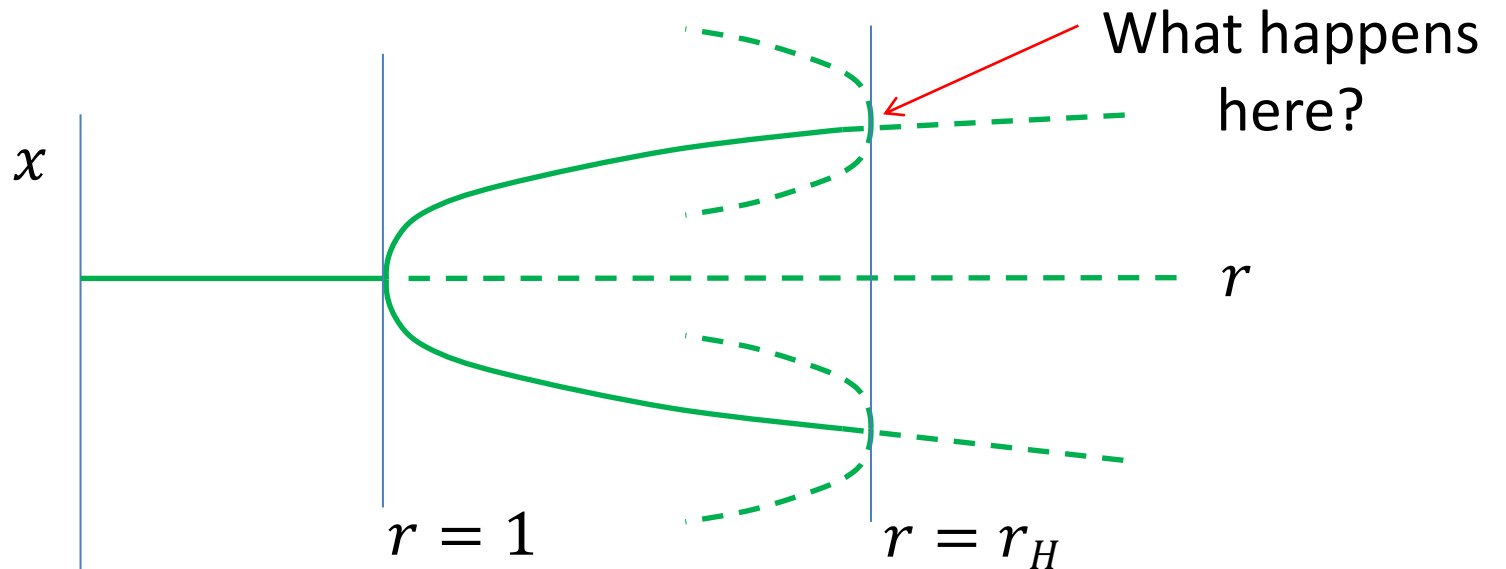
- Linearisation about the origin reveals a stable node for  $r < 1$  and a saddle  $r > 1$ .
- For  $r < 1$  the system is globally asymptotically stable (via Lyapunov) – there are no limit cycles and all trajectories fall into the origin.

# Lorenz equations: stability

- For  $1 < r < \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} = r_H$

the other two equilibrium points are stable and are surrounded by a saddle cycle (a type of unstable limit cycle). At  $r_H$  they undergo a Hopf bifurcation (the critical linearised eigenvalues are complex).

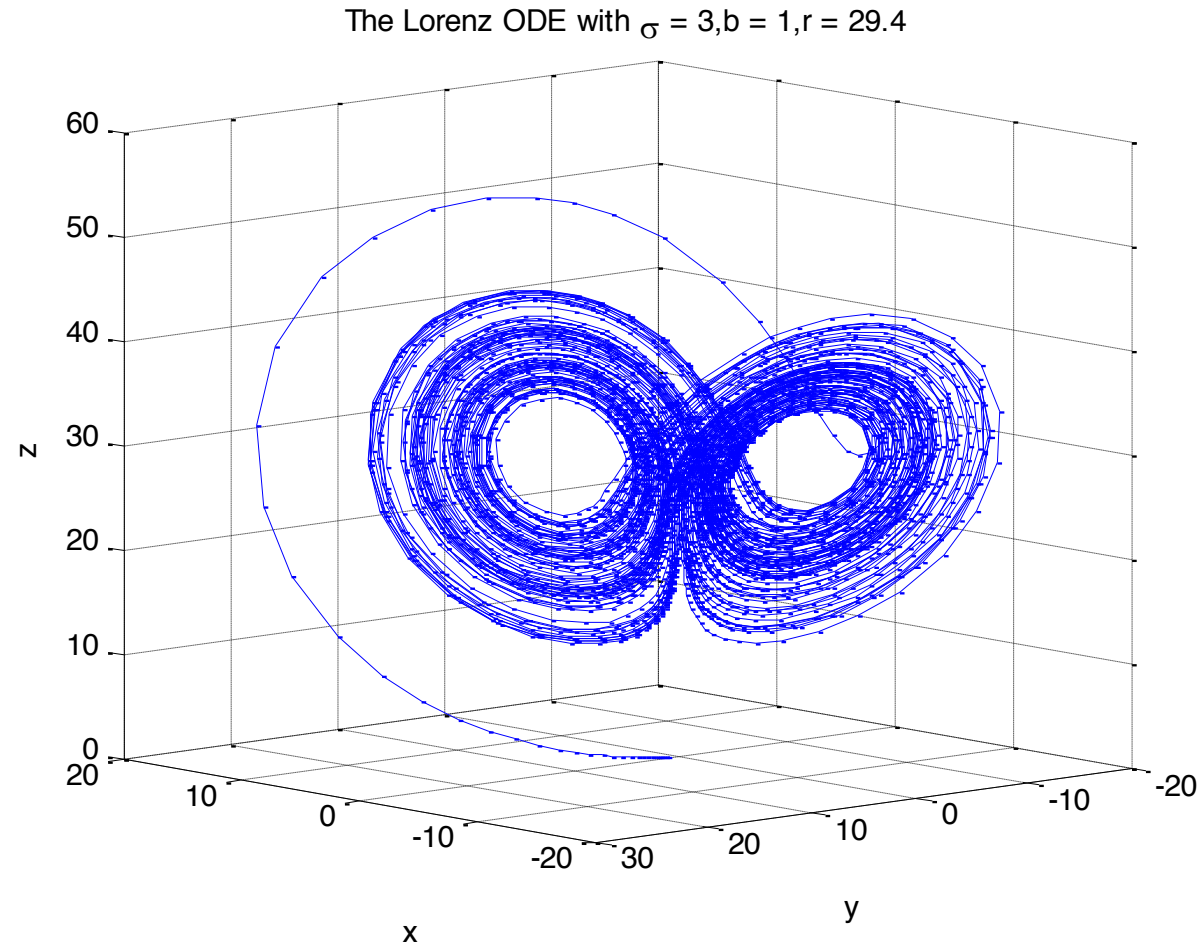
- For  $r = r_H$   
we have a subcritical saddle and for  $r > r_H$   
there are no other attractors nearby



# Clues to a Strange Attractor

- The volume is contracting.
  - There are no stable classical attractors.
  - There are no stable limit cycles for  $r > r_H$  (proved by Lorenz).
  - Trajectories cannot go to infinity.
- 
- There must be a zero volume object that attracts the trajectories – a **Strange Attractor**

# The Lorenz Butterfly



# Properties

- There is sensitivity to initial conditions.
- We can compute the rates of contraction along the principal axes of the Jacobian – these numbers are called the Lyapunov exponents (remember the Lyapunov exponent for maps). If the largest exponent is positive, phase space is stretching in that direction.
- The volume is contracting because the sum of the exponents is negative – but one or two may be positive and so the volume is growing in some direction(s) and shrinking in others.

# The Mandelbrot Set

- The map

$$z_{k+1} \mapsto z_k^2 + c$$

$z$  and  $c$  are complex. The point  $c$  is in the Mandelbrot set if this iteration remains bounded for all  $k$ . We can use colours to indicate the rate of divergence at  $c$

- The graph is a strange attractor for the map. The colours indicate how close the points shown are to the attractor.
- The set is a very complex object.

# The Mandelbrot Set

