

# Adaptive Model Predictive Control: Robustness and Parameter Estimation

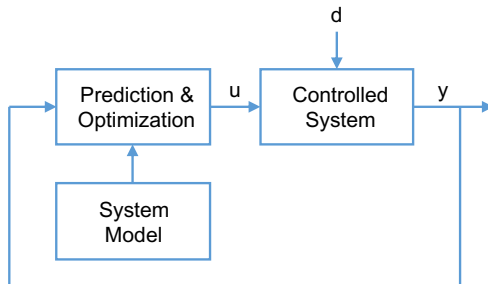
Mark Cannon

Joint work with Matthias Lorenzen, University of Stuttgart  
and Xiaonan Lu, University of Oxford



# Motivation

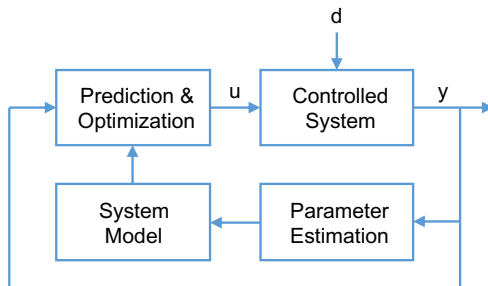
Robust MPC paradigm:



- ▶ MPC requires adequate models of the system, uncertainty, disturbances
- ▶ Amount of uncertainty in the model crucially affects performance
- ▶ Large effort (time & money) spent on model identification offline

# Motivation

Adaptive MPC paradigm:



- ▷ Identify model online
- ▷ Require: robust constraint satisfaction  
closed loop stability & performance guarantees  
parameter convergence

# Motivation

An idea with a long history: e.g. self-tuning control, DMC, GPC ...

[Clarke, Tuffs, Mohtadi, 1987]

Revisited with new tools:

- Set membership estimation

[Bai, Cho, Tempo, 1998]

- Robust tube MPC

[Langsson, Chrysoschoos, Rakovic, Mayne, 2004]

- Dual adaptive/predictive control

[Lee & Lee, 2009]

# Motivation

## Recent work on MPC with model adaptation

- Focus on online learning & identification:

- Persistency of Excitation constraints

[Marafioti, Bitmead, Hovd, 2014]

- Kalman filter-based parameter estimation with covariance matrix in cost

[Heirung, Ydstie, Foss, 2017]

- Gaussian process regression, particle filtering

[Klenske, Zeilinger, Scholkopf, Hennig, 2016]

[Bayard & Schumitzky, 2010]

- Focus on robust constraint satisfaction and performance:

- Constraints based on prior uncertainty set, online update of cost only

[Aswani, Gonzalez, Sastry, Tomlin, 2013]

- Set-based identification, stable FIR plant model

[Tanaskovic, Fagiano, Smith, Morari, 2014]

# Motivation

This talk considers how to

- ensure robust constraint satisfaction;
- update constraints & costs online via set-membership & point estimates;
- enforce parameter convergence via persistency of excitation conditions.

Outline:

- 1 Set membership parameter estimation
- 2 Polytopic tube robust MPC
- 3 Parameter convergence and time-varying parameters

# Parameter set estimate

Plant model with unknown parameter vector  $\theta^*$  and disturbance  $w$ :

$$x_{k+1} = A(\theta^*)x_k + B(\theta^*)u_k + w_k$$

Assume the model is affine in  $\theta^*$  (assumed constant)

$$x_{k+1} = D_k \theta^* + d_k + w_k \quad \left\{ \begin{array}{l} D_k = D(x_k, u_k) \\ d_k = A_0 x_k + B_0 u_k \end{array} \right.$$

with stochastic disturbance  $w_k \in \mathcal{W}$  a.s.,  $\mathcal{W}$  known, compact, polytopic

If  $x_k, x_{k-1}, u_{k-1}$  are known, then  $\theta^*$  must lie in the “unfalsified set”:

$$\Delta_k = \{\theta : x_k = D_{k-1}\theta + d_{k-1} + w, w \in \mathcal{W}\}$$

Hence update the parameter set estimate  $\Theta_k$  via

$$\Theta_k = \Theta_{k-1} \cap \Delta_k$$

# Parameter set estimate

- ▶ If  $\Theta_0$  is a compact polytope, then  $\Theta_k$  is a compact polytope for all  $k > 0$   
But the update  $\Theta_k = \Theta_{k-1} \cap \Delta_k$  has potentially unbounded complexity!

- ▶ Instead, use fixed complexity sets defined for given  $H_\Theta$  by

$$\Theta_k = \{\theta : H_\Theta \theta \leq h_k\}$$

and update  $\Theta_k$  by solving a set of linear programs:

$$[h_k]_i = \max_{w \in \mathcal{W}, \theta \in \Theta_{k-1}} [H_\Theta]_i \theta \quad \text{s.t.} \quad x_k = D_{k-1} \theta + d_{k-1} + w$$

Then

$$\Theta_{k-1} \cap \Delta_k \subseteq \Theta_k \subseteq \Theta_{k-1}$$

since

- ▶  $[H_\Theta]_i \theta \leq [h_k]_i$  for all  $\theta \in \Delta_k \cap \Theta_{k-1}$  implies  $\Theta_{k-1} \cap \Delta_k \subseteq \Theta_k$
- ▶  $[h_k]_i \leq \max_{\theta \in \Theta_{k-1}} [H_\Theta]_i \theta = [h_{k-1}]_i$  implies  $\Theta_k \subseteq \Theta_{k-1}$



## Parameter point estimate

To ensure closed loop  $l^2$  stability, we define the MPC cost in terms of a point estimate  $\hat{\theta}_k$  of  $\theta^*$ , computed using a LMS filter

Given a parameter estimate  $\hat{\theta}_k$ , let  $\hat{x}_{1|k} = A(\hat{\theta}_k)x_k + B(\hat{\theta}_k)u_k$   
Then for a given parameter update gain  $\mu > 0$  satisfying

$$1/\mu > \sup_{(x,u) \in \mathcal{Z}} \|D(x,u)\|^2$$

the point estimate  $\hat{\theta}_k$  is defined

$$\begin{aligned}\tilde{\theta}_k &= \hat{\theta}_{k-1} + \mu D^\top(x_{k-1}, u_{k-1})(x_k - \hat{x}_{1|k-1}) \\ \hat{\theta}_k &= \Pi_{\Theta_k}(\tilde{\theta}_k)\end{aligned}$$

where  $\Pi_{\Theta_k}$  is the Euclidean projection onto  $\Theta_k$

Here  $\mathcal{Z}$  is the joint state and control constraint set (assumed bounded)  
and the point estimate update becomes simply a projection onto  $\Theta_k$  if  $\mu \rightarrow 0$

# Parameter point estimate

The closed loop  $l^2$  gain property is based on the following result

## Lemma (Point estimate)

*If  $\sup_{k \in \mathbb{N}} \|x_k\| < \infty$  and  $\sup_{k \in \mathbb{N}} \|u_k\| < \infty$ , then  $\theta_k \in \Theta_k$  for all  $k$  and*

$$\sup_{T \in \mathbb{N}, w_k \in \mathcal{W}, \hat{\theta}_0 \in \Theta_0} \frac{\sum_{k=0}^T \|\tilde{x}_{1|k}\|^2}{\frac{1}{\mu} \|\hat{\theta}_0 - \theta^*\|^2 + \sum_{k=0}^T \|w_k\|^2} \leq 1$$

*where  $\tilde{x}_{1|k} = A(\theta^*)x_k + B(\theta^*)u_k - \hat{x}_{1|k}$  is the 1-step prediction error*

# Control Problem

Consider robust regulation of the system

$$x_{k+1} = A(\theta)x_k + B(\theta)u_k + w_k$$

with  $\theta \in \Theta_k$ ,  $w_k \in \mathcal{W}$ , subject to the state and control constraints

$$Fx_k + Gu_k \leq \mathbf{1} = [1 \ \cdots \ 1]^\top$$

## Assumption (Robust stabilizability)

*There exists a set  $\mathcal{X} = \{x : Vx \leq \mathbf{1}\}$  and feedback gain  $K$  such that  $\mathcal{X}$  is  $\lambda$ -contractive for some  $\lambda \in [0, 1)$ , i.e.*

$$V\Phi(\theta)x \leq \lambda\mathbf{1}, \quad \text{for all } x \in \mathcal{X}, \theta \in \Theta_0.$$

*where  $\Phi(\theta) = A(\theta) + B(\theta)K$ .*

# Control Problem

State and control input sequences predicted at time  $k$ :  $u_{i|k}, x_{i|k}$ ,  $i = 0, 1, \dots$  are expressed in terms of decision variables  $\mathbf{v} = (v_{0|k}, \dots, v_{N|k})$ :

$$u_{i|k} = \begin{cases} Kx_{i|k} + v_{i|k} & i = 0, 1, \dots \\ Kx_{i|k} \end{cases}$$

The regulation cost is defined in terms of point estimate  $\hat{\theta}_k$ :

$$J_N(x_k, \hat{\theta}_k, \mathbf{v}_k) = \sum_{i=0}^{N-1} \left( \|\hat{x}_{i|k}\|_Q^2 + \|\hat{u}_{i|k}\|_R^2 \right) + \|\hat{x}_{N|k}\|_P^2$$

where  $\hat{x}_{i|k}$ ,  $\hat{u}_{i|k}$  are defined by

$$\begin{aligned} \hat{x}_{i+1|k} &= A(\hat{\theta}_k)\hat{x}_{i|k} + B(\hat{\theta}_k)\hat{u}_{i|k} \\ \hat{u}_{i|k} &= K\hat{x}_{i|k} + v_{i|k} \end{aligned}$$

and  $P \succeq \Phi^\top(\theta)P\Phi(\theta) + Q + K^\top RK$  for all  $\theta \in \Theta_0$

# Tube MPC

A sequence of sets (a “tube”) is constructed to bound the predicted state  $x_{i|k}$ , with  $i$ th cross section,  $\mathcal{X}_{i|k}$ :

$$\mathcal{X}_{i|k} = \{x : Vx \leq \alpha_{i|k}\}$$

where  $V$  is determined offline and  $\alpha_{i|k}$  are online decision variables

- Ⓐ For robust satisfaction of  $x_{i|k} \in \mathcal{X}_{i|k}$ , we require

$$V\Phi(\theta)x + VB(\theta)v_{i|k} + \bar{w} \leq \alpha_{i+1|k} \quad \text{for all } x \in \mathcal{X}_{i|k}, \theta \in \Theta_k$$

where  $[\bar{w}]_i = \max_{w \in \mathcal{W}} [V]_i w$

- Ⓑ For robust satisfaction of  $Fx_{i|k} + Gu_{i|k} \leq \mathbf{1}$ , we require

$$(F + GK)x + Gv_{i|k} \leq \mathbf{1} \quad \text{for all } x \in \mathcal{X}_{i|k}$$

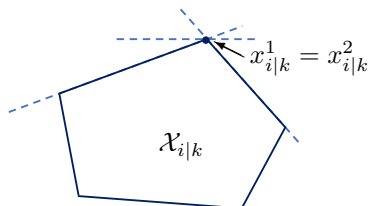
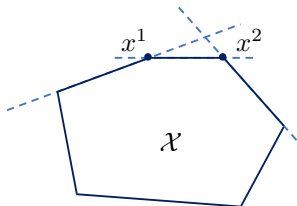
Condition (A) is bilinear in  $x$  and  $\theta$ , but it can be expressed in terms of linear inequalities using a vertex representation of either  $\mathcal{X}_{i|k}$  or  $\Theta_k$

# Tube MPC

We generate the vertex representation:

$$\mathcal{X}_{i|k} = \text{co}\{x_{i|k}^1, \dots, x_{i|k}^m\}$$

using the property that  $\{x : [V]_r x \leq [\alpha_{i|k}]_r\}$  is a supporting hyperplane of  $\mathcal{X}_{i|k}$  for each  $r$ :



Hence each vertex  $x_{i|k}^j$  is given by the intersection of hyperplanes corresponding to a fixed set of rows of  $V$ , and

$$x_{i|k}^j = U^j \alpha_{i|k}$$

for some  $U^j$ , determined offline from the vertices of  $\mathcal{X} = \{x : Vx \leq \mathbf{1}\}$

# Tube MPC

In terms of both hyperplane and the vertex descriptions of  $\mathcal{X}_{i|k}$ , the robust tube constraints become

- Ⓐ  $V\Phi(\theta)U^j\alpha_{i|k} + VB(\theta)v_{i|k} + \bar{w} \leq \alpha_{i+1|k}$  for all  $\theta \in \Theta_k$ ,  $j = 1, \dots, m$
- Ⓑ  $(F + GK)U^j\alpha_{i|k} + Gv_{i|k} \leq \mathbf{1}$ ,  $j = 1, \dots, m$

Now condition (B) is linear and (A) can be equivalently written as linear constraints using

## Lemma (Polyhedral set inclusion)

Let  $\mathcal{P}_i = \{x : F_i x \leq f_i\} \subset \mathbb{R}^n$  for  $i = 1, 2$ . Then  $\mathcal{P}_1 \subseteq \mathcal{P}_2$  iff

$$\exists \Lambda \geq 0 \text{ such that } \Lambda F_1 = F_2 \text{ and } \Lambda f_1 \leq f_2$$

# Robust MPC online optimization problem

Summary of constraints in the online MPC optimization at time  $k$ :

$$Vx_k \leq \alpha_{0|k}$$

$$\Lambda_{i|k}^j H_\Theta = VD(U^j \alpha_{i|k}, KU^j \alpha_{i|k} + v_{i|k})$$

$$\Lambda_{i|k}^j h_k \leq \alpha_{i+1|k} - Vd(u^j \alpha_{i|k}, KU^j \alpha_{i|k} + v_{i|k}) - \bar{w}$$

$$\Lambda_{i|k}^j \geq 0$$

$$(F + GK)U^j \alpha_{i|k} + Gv_{i|k} \leq \mathbf{1}$$

$$\Lambda_{N|k}^j H_\Theta = VD(U^j \alpha_{N|k}, KU^j \alpha_{N|k})$$

$$\Lambda_{N|k}^j h_k \leq \alpha_{N|k} - Vd(u^j \alpha_{N|k}, KU^j \alpha_{N|k}) - \bar{w}$$

$$\Lambda_{N|k}^j \geq 0$$

$$(F + GK)U^j \alpha_{N|k} \leq \mathbf{1}$$

for  $i = 0, \dots, N-1, j = 1, \dots, m$

Let  $\mathcal{D}(x_k, \Theta_k)$  be the feasible set for the decision variables  $\mathbf{v}_k, \boldsymbol{\alpha}_k, \boldsymbol{\Lambda}_k$



# Robust adaptive MPC algorithm

Offline: Choose  $\Theta_0$ ,  $\mathcal{X}$ , feedback gain  $K$ , and compute  $P$

Online, at each time  $k = 1, 2, \dots$ :

1 Given  $x_k$ , update the set  $(\Theta_k)$  and point  $(\hat{\theta}_k)$  parameter estimates

2 Compute the solution  $(\mathbf{v}_k^*, \boldsymbol{\alpha}_k^*, \boldsymbol{\Lambda}_k^*)$  of the QP

$$\begin{aligned} \min_{\mathbf{v}_k, \boldsymbol{\alpha}_k, \boldsymbol{\Lambda}_k} \quad & J(x_k, \hat{\theta}_k, \mathbf{v}_k) \\ \text{subject to} \quad & (\mathbf{v}_k, \boldsymbol{\alpha}_k, \boldsymbol{\Lambda}_k) \in \mathcal{D}(x_k, \Theta_k) \end{aligned}$$

3 Apply the control law  $u_k^* = Kx_k + v_{0|k}^*$

# Robust adaptive MPC algorithm

## Theorem (Closed loop properties)

*If  $\theta^* \in \Theta_0$  and  $\mathcal{D}(x_0, \Theta_0) \neq \emptyset$ , then for all  $k > 0$ :*

- ❶  $\theta^* \in \Theta_k$
- ❷  $D(x_k, \Theta_k) \neq \emptyset$
- ❸  $Fx_k + Gu_k \leq \mathbf{1}$

*and the closed loop system is finite-gain  $l^2$ -stable, i.e. there exist constants  $c_0, c_1, c_2 > 0$  such that for all  $T$ :*

$$\sum_{k=0}^T \|x_k\|^2 \leq c_0 \|x_0\|^2 + c_1 \|\hat{\theta}_0 - \theta^*\|^2 + c_2 \sum_{k=0}^T \|w_k\|^2$$

# A numerical example

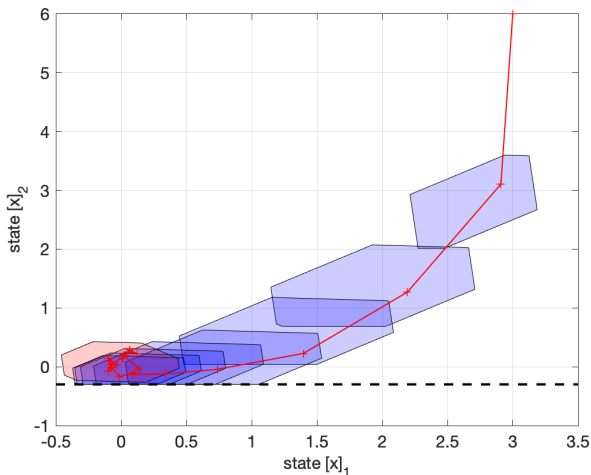
Second-order linear system with

$$(A(\theta), B(\theta)) = (A_0, B_0) + \sum_{i=1}^3 (A_i, B_i)\theta_i$$

$$A_0 = \begin{bmatrix} 0.5 & 0.2 \\ -0.1 & 0.6 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.042 & 0 \\ 0.072 & 0.03 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.015 & 0.019 \\ 0.009 & 0.035 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ -0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.0397 \\ 0.059 \end{bmatrix}.$$

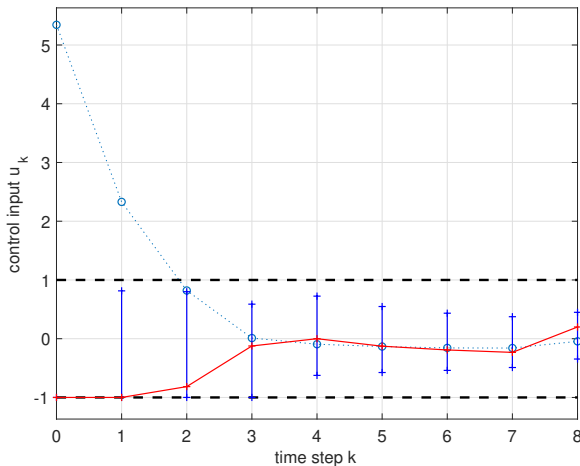
- ▶ true parameter  $\theta^* = [0.8 \ 0.2 \ -0.5]^\top$ , initial set  $\Theta_0 = \{\theta : \|\theta\|_\infty \leq 1\}$ .
- ▶ disturbance uniformly distributed on  $\mathcal{W} = \{w \in \mathbb{R}^2 : \|w\|_\infty \leq 0.1\}$ ,  $w_k$
- ▶ state and input constraints:  $[x]_2 \geq -0.3$  and  $u_k \leq 1$ .

## A numerical example: constraint satisfaction



**Figure:** Realized closed-loop trajectory from initial condition  $x_0 = [3 \ 6]^T$  (red line), predicted state tube at time  $k = 0$  (tube cross-sections: blue, terminal set: pink)

# A numerical example: constraint satisfaction



**Figure:** Realized closed-loop trajectory from initial condition  $x_0 = [3 \ 6]^\top$  (red line), predicted control tube at time  $k = 0$  (tube cross-sections: blue)

# Persistent excitation

- ▷ A regressor  $\Psi_k$  is persistently exciting (PE) if

$$\beta_1^2 \mathbb{I} \preceq \frac{1}{l} \sum_{k=k_0+1}^{k_0+l} \Psi_k \Psi_k^\top \preceq \beta_2^2 \mathbb{I}$$

for some  $\beta_1, \beta_2, l > 0$  and all  $k_0$  (Narendra, 1987).

- ▷ Define the diameter of  $\Theta$  as  $\text{dia}(\Theta) = \sup_{\theta_1, \theta_2 \in \Theta} \|\theta_1 - \theta_2\|$

## Convergence of set membership parameter estimate

If the noise bound  $w \in \mathcal{W}$  is tight and the regressor  $D_k$  is persistently exciting, then  $\text{dia}(\Theta_k) \rightarrow 0$  with probability one [Bai, Cho, Tempo, 1998].

- ▷  $\mathcal{W}$  is a tight noise bound if the support of the probability distribution of  $w$  is equal to  $\mathcal{W}$

# Persistent excitation

Regressor:  $\Psi_k = D_k^\top = [A_1 x_k + B_1 u_k \quad \cdots \quad A_p x_k + B_p u_k]^\top$

Consider the PE condition evaluated over a window that includes  $n$  past time-steps plus current time:

$$\sum_{k=-n}^{k=0} D_k^\top D_k \succeq \beta_1^2 \mathbb{I}$$

This is nonconvex in  $u_0 = Kx_k + v_0|_k$ , but we can linearise to obtain a convex condition. Thus, let  $u_0 = u_0^* + \delta u$ , so that

$$\begin{aligned} D_0^\top D_0 &\succeq D(x_0, u_0^*)^\top D(x_0, u_0^*) + D^\top(x_0, u_0^*) [B_1 \delta u \quad \cdots \quad B_p \delta u] \\ &\quad + [B_1 \delta u \quad \cdots \quad B_p \delta u]^\top D(x_0, u_0^*) \end{aligned}$$

Therefore a sufficient condition for  $\sum_{k=-n}^{k=0} D_k^\top D_k \succeq \beta_1^2 \mathbb{I}$  is an LMI in  $\delta u$ :

$$\begin{aligned} \text{LMI}(\delta u) : \sum_{k=-n}^{k=-1} D_k^\top D_k &+ D(x_0, u_0^*)^\top D(x_0, u_0^*) \\ &+ D(x_0, u_0^*)^\top [B_1 \delta u \quad \cdots \quad B_p \delta u] + [B_1 \delta u \quad \cdots \quad B_p \delta u]^\top D(x_0, u_0^*) \succeq \beta_1^2 \mathbb{I} \end{aligned}$$

# Robust adaptive MPC algorithm with PE constraint

Offline: Choose  $\Theta_0$ ,  $\mathcal{X}$ ,  $\beta_1$ , feedback gain  $K$ , and compute  $P$

Online, at each time  $k = 1, 2, \dots$ :

- 1 Given  $x_k$ , update the set  $(\Theta_k)$  and point  $(\hat{\theta}_k)$  parameter estimates
- 2 Compute the solution  $(\mathbf{v}_k^*, \boldsymbol{\alpha}_k^*, \boldsymbol{\Lambda}_k^*)$  of the QP

$$\begin{aligned} \min_{\mathbf{v}_k, \boldsymbol{\alpha}_k, \boldsymbol{\Lambda}_k} J(x_k, \hat{\theta}_k, \mathbf{v}_k) \\ \text{subject to } (\mathbf{v}_k, \boldsymbol{\alpha}_k, \boldsymbol{\Lambda}_k) \in \mathcal{D}(x_k, \Theta_k) \end{aligned}$$

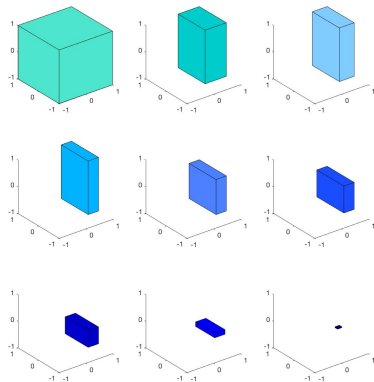
- 3 If  $\sum_{k=-n}^{k=-1} D_k^\top D_k + D(x_0, u_0^*)^\top D(x_0, u_0^*) \not\leq \beta_1 \mathbb{I}$ :

- (a) Re-run the MPC optimization with  $v_{0|k} = v_{0|k}^* + \delta u$  and  $\text{LMI}(\delta u)$  as additional constraints
- (b) If a feasible solution exists, set  $v_{0|k}^* \leftarrow v_{0|k}^* + \delta u^*$

- 4 Apply the control law  $u_k^* = Kx_k + v_{0|k}^*$



## A numerical example: parameter set



**Figure:** Parameter set  $\Theta_k$  at time steps  $k \in \{0, 1, 2; 10, 25, 50; 100, 500, 5000\}$

$\Theta$ set	Volume (%)	Cost*
$\Theta_0$	100	62.22
$\Theta_1$	26.1	61.13
$\Theta_2$	18.3	61.03
$\Theta_{10}$	12.7	60.96
$\Theta_{25}$	8.3	60.93
$\Theta_{50}$	6.3	60.77
$\Theta_{100}$	3.4	59.45
$\Theta_{500}$	0.7	57.94
$\Theta_{5000}$	0.0089	53.95
$\theta^*$	-	52.70

**Table:** Volume of  $\Theta_k$  as  $\Theta_k/\Theta_0 \times 100\%$ ; Cost\* with same initial  $x_0$  and constraints

# Time-varying parameters

## Assumption (time-varying parameters)

*There exists a constant  $r_\theta$  such that the parameter vector  $\theta_k^*$  satisfies  $\theta_k^* \in \Theta_0$  for all  $k$  and  $\|\theta_{k+1}^* - \theta_k^*\| \leq r_\theta$*

Define the dilation operator:

$$R_i(\Theta) = \{\theta : H_\Theta \theta \leq h + i r_\theta \mathbf{1}\}$$

Then the parameter set update can be expressed

$$\Theta_k = R_1(\Theta_{k-1} \cap \Delta_k) \cap \Theta_0$$

and  $\Theta_k$  is replaced in the tube MPC constraints by

$$\Theta_{i|k} = R_i(\Theta_k) \cap \Theta_0$$

# Robust adaptive MPC algorithm with time-varying parameters

Parameter estimate bounds and recursive feasibility properties are unchanged:

## Theorem (Closed loop properties)

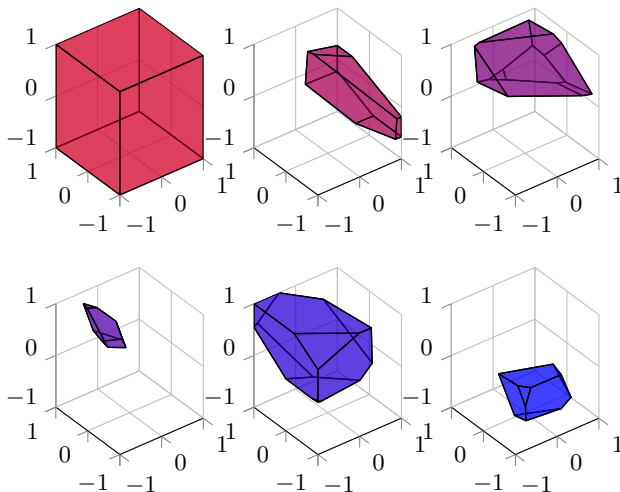
*If  $\theta^* \in \Theta_0$  and  $\mathcal{D}(x_0, \Theta_0) \neq \emptyset$ , then for all  $k > 0$ :*

- ①  $\theta^* \in \Theta_k$
- ②  $D(x_k, \Theta_k) \neq \emptyset$
- ③  $Fx_k + Gu_k \leq \mathbf{1}$

But the LMS filter has an additional tracking error, which invalidates the  $l^2$  properties, i.e. “certainty equivalence” no longer applies

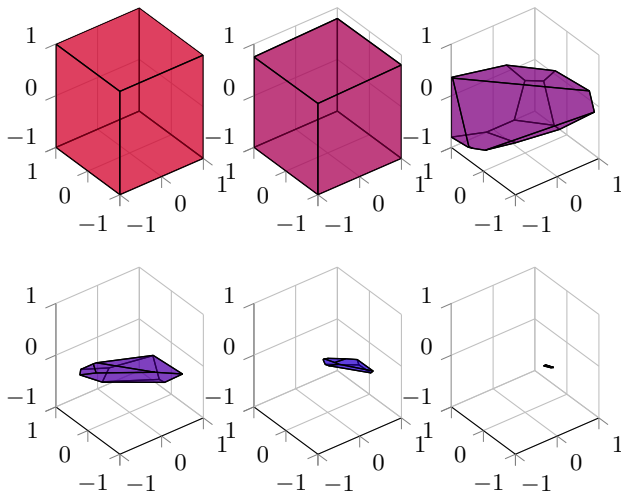
However other performance measures can be used in this context, such as the min-max approach of [Lorenzen, Allgöwer, Cannon, 2017]

## A numerical example: time-varying parameters



**Figure:** Parameter set  $\Theta_k$  at times  $k \in \{0, 100, 200, 300, 400, 500\}$  for the time-varying system with  $r_\theta = 0.01$

## A numerical example: time-varying parameters



**Figure:** Parameter set  $\Theta_k$  at times  $k \in \{0, 5, 25, 70, 120, 500\}$  for the non-time-varying case for comparison

# Conclusions & Outlook

## Conclusions:

- Adaptive robust MPC with closed loop guarantees is computationally tractable
- Set-membership parameter estimation and LMS point estimates are obvious choices for MPC cost functions and robust constraints
- Nonconvex PE conditions can be relaxed to convex sufficient conditions

## Future work

- How to ensure recursive feasibility with PE constraints?
- Are PE conditions better handled by adding terms to the MPC cost (similar to MPC-based dual control)?
- How can we relax the requirement of prior knowledge of a robustly stabilizing local feedback law?

## References:

- M. Lorenzen, M. Cannon, & F. Allgöwer, “Robust MPC with recursive model update”  
Automatica (to appear)
- X. Lu, M. Cannon, “Robust adaptive tube model predictive control” ACC19 (submitted)

# Recursive Feasibility

## Proposition (Recursive Feasibility)

*The online MPC optimisation is feasible at all times  $t \geq 1$  if the optimisation is feasible at  $t = 0$ .*

Proof: Define the optimal solution at time  $t - 1$  as  $(\mathbf{v}_{t-1}, \boldsymbol{\alpha}_{t-1}, \boldsymbol{\sigma}_{t-1}, \boldsymbol{\tau}_{t-1})$  and the suboptimal solution at time  $t$  as  $(\tilde{\mathbf{v}}_t, \tilde{\boldsymbol{\alpha}}_t, \tilde{\boldsymbol{\sigma}}_t, \tilde{\boldsymbol{\tau}}_t)$ .

- ① update equation and constraints are satisfied for  $k \in \mathbb{N}_{0,N-2}$  due to the feasibility at time  $t - 1$  and  $\Theta_t \subseteq \Theta_{t-1}$
- ② update equation and constraints are satisfied for  $k = N - 1$  because of the terminal constraint
- ③ bound on cost  $(\sigma_k, \tau_k)$  are satisfied for  $k \in \mathbb{N}_{0,N-1}$  because of definition
- ④ initial constraint is satisfied because  $Vx_t \leq \alpha_{1|t-1}$
- ⑤ terminal constraints are satisfied because of the feasibility at time  $t - 1$  and the definition of  $\tilde{\boldsymbol{\alpha}}_{N|t}$  implies  $\tilde{\boldsymbol{\alpha}}_{N|t} \leq \boldsymbol{\alpha}_{N|t-1}$

# Asymptotic bounds on state and control

## Proposition (Asymptotic bounds on state and control)

*If the MPC optimization is feasible at  $t = 0$ , then the state and control input,  $x_t$  and  $u_t$ , satisfy the asymptotic bounds:*

$$\limsup_{t \rightarrow \infty} \max\{Qx_t\} \leq \bar{s}_\infty(\lambda_\infty)$$

$$\limsup_{t \rightarrow \infty} \max\{Ru_t\} \leq \bar{t}_\infty(\lambda_\infty)$$

where  $\lambda_\infty = \lim_{t \rightarrow \infty} \lambda_t$ .

Proof: Consider the rate of decrease in cost

$$\begin{aligned} J_{t-1}^o - J_t^o &\geq \left[ \max\{Qx_{t-1}\} - \bar{s}_\infty(\lambda_{t-1}) \right]_{\geq 0} + \left[ \max\{Ru_{t-1}\} - \bar{t}_\infty(\lambda_{t-1}) \right]_{\geq 0} \\ &\quad - (\max\{H_Q \mathbf{1}\} + \max\{H_R \mathbf{1}\}) \left( \frac{1}{(1-\lambda_{t-1})^2} - \frac{1}{(1-\lambda_t)^2} \right) \max\{\bar{w}\} \end{aligned}$$