

Organisation

C21 Model Predictive Control

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4 lectures

Michaelmas Term 2018



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- ▷ 4 lectures: week 3 Monday 3-4 pm LR2
 Thursday 3-4 pm LR2

- week 4 Monday 3-4 pm LR2
 Thursday 3-4 pm LR2

- ▷ 1 class:
week 6 Thursday 2-3 pm or 3-4 pm or 4-5 pm LR6

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Course outline

Lecture 1

Introduction

1. Introduction to MPC and constrained control
2. Prediction and optimization
3. Closed loop properties
4. Disturbances and integral action
5. Robust tube MPC

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Books

- ① B. Kouvaritakis and M. Cannon, *Model Predictive Control: Classical, Robust and Stochastic*, Springer 2015

Recommended reading: Chapters 1, 2 & 3

- ② J.B. Rawlings and D.Q. Mayne, *Model Predictive Control: Theory and Design*. Nob Hill Publishing, 2009

- ③ J.M. Maciejowski, *Predictive control with constraints*. Prentice Hall, 2002

Recommended reading: Chapters 1–3, 6 & 8

Introduction

Classical controller design:

- ① Determine plant model
- ② Design controller (e.g. PID)
- ③ Apply controller

discard model

$$x_{k+1} = f(x_k, u_k)$$



Model predictive control (MPC):

- ① Use model to predict system behaviour
- ② Choose optimal trajectory
- ③ Repeat procedure (feedback)



Overview of MPC

Model predictive control strategy:

- ① Prediction
- ② Online optimization
- ③ Receding horizon implementation

- ① Prediction

* Plant model: $x_{k+1} = f(x_k, u_k)$

* Simulate forward in time (over a prediction horizon of N steps)

$$\text{input sequence: } \mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ u_{1|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix} \xrightarrow{\text{defines}} \text{state sequence: } \mathbf{x}_k = \begin{bmatrix} x_{1|k} \\ x_{2|k} \\ \vdots \\ x_{N|k} \end{bmatrix}$$

Notation:

$(u_{i|k}, x_{i|k})$ = predicted i steps ahead | evaluated at time k

$x_{0|k} = x_k$

Overview of MPC

- ② Optimization

* Predicted quality criterion/cost:

$$J(x_k, \mathbf{u}_k) = \sum_{i=0}^N l_i(x_{i|k}, u_{i|k})$$

$l_i(x, u)$: stage cost

* Solve numerically to determine optimal input sequence:

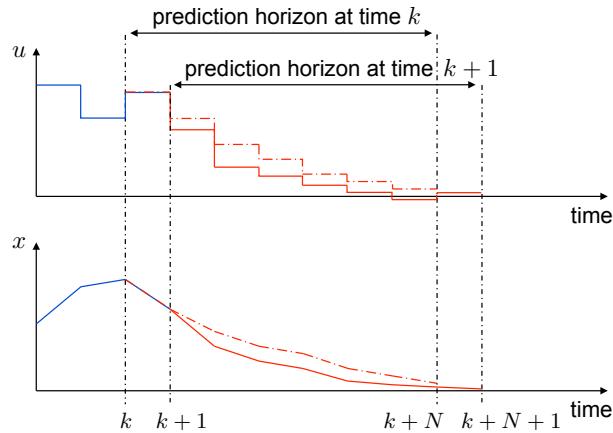
$$\begin{aligned} \mathbf{u}_k^* &= \arg \min_{\mathbf{u}_k} J(x_k, \mathbf{u}_k) \\ &= (u_{0|k}^*, \dots, u_{N-1|k}^*) \end{aligned}$$

- ③ Implementation

* Use first element of \mathbf{u}_k^* \Rightarrow actual plant input $u_k = u_{0|k}^*$

* Repeat optimization at each sampling instant $k = 0, 1, \dots$

Overview of MPC



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Overview of MPC

Optimization is repeated online at each sampling instant $k = 0, 1, \dots$



receding horizon:

$$\mathbf{u}_k = (u_{0|k}, \dots, u_{N-1|k})$$

$$\mathbf{x}_k = (x_{1|k}, \dots, x_{N|k})$$

$$\mathbf{u}_{k+1} = (u_{0|k+1}, \dots, u_{N-1|k+1}) \quad \mathbf{x}_{k+1} = (x_{1|k+1}, \dots, x_{N|k+1})$$

\vdots

\vdots

- * provides feedback since $\mathbf{u}_k^*, \mathbf{x}_k^*$ are functions of x_k
 - so reduces effects of model error and measurement noise
- * and compensates for finite number of free variables in predictions
 - so improves closed-loop performance

Example

Plant model:

$$x_{k+1} = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u_k$$

$$y_k = [-1 \ 1] x_k$$

Cost:

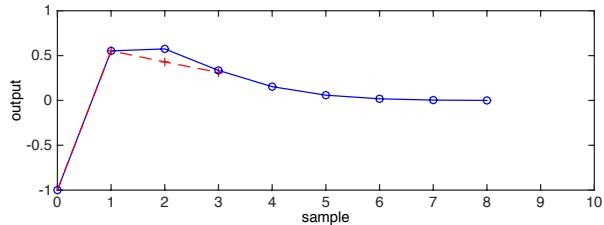
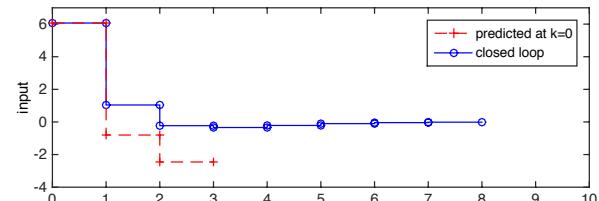
$$\sum_{i=0}^{N-1} (y_i^2 + u_{i|k}^2) + y_{N|k}^2$$

Prediction horizon: $N = 3$

Free variables in predictions: $\mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ u_{1|k} \\ u_{2|k} \end{bmatrix}$

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Example



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Motivation for MPC

Advantages

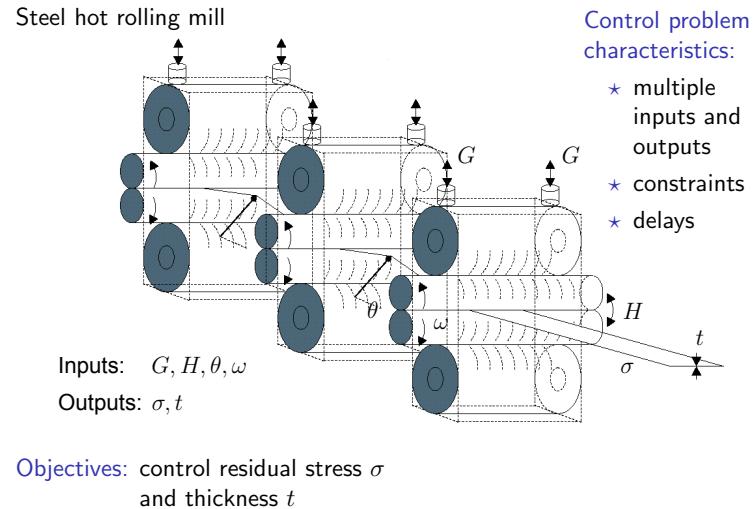
- ▷ Flexible plant model
 - e.g. multivariable
linear or nonlinear
deterministic, stochastic or fuzzy
- ▷ Handles constraints on control inputs and states
 - e.g. actuator limits
safety, environmental and economic constraints
- ▷ Approximately optimal control

Disadvantages

- ▷ Requires online optimization
 - e.g. large computation for nonlinear and uncertain systems

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Applications: Process Control



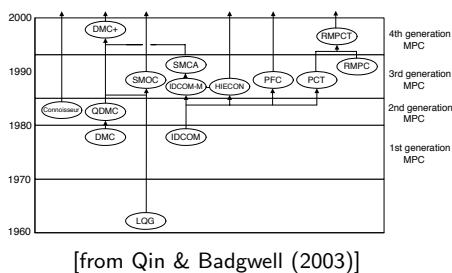
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Historical development

Control strategy reinvented several times

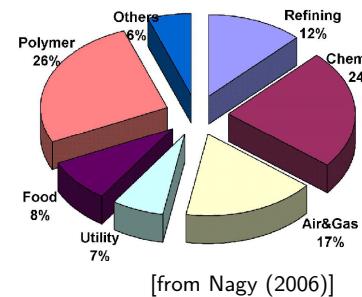
LQ(G) optimal control	1950's–1980's
industrial process control	1980's
constrained nonlinear control	1990's–today

Development of commercial MPC algorithms:



Applications: Chemical Process Control

- Applications of predictive control to more than 4,500 different chemical processes (based on a 2006 survey)
- MPC applications in the chemical industry



Typical control problems:

- * nonlinear dynamics
- * slow sampling rates
- * non-quadratic utility functions

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Applications: Electromechanical systems

Variable-pitch wind turbines



Prediction model

Linear plant model: $x_{k+1} = Ax_k + Bu_k$

▷ Predicted \mathbf{x}_k depends **linearly** on \mathbf{u}_k [details in Lecture 2]

▷ Therefore the cost is **quadratic** in \mathbf{u}_k $\mathbf{u}_k^\top H \mathbf{u}_k + 2f^\top \mathbf{u}_k + g(x_k)$
and constraints are **linear** $A_c \mathbf{u}_k \leq b(x_k)$

▷ Online optimization:

$$\min_{\mathbf{u}} \mathbf{u}^\top H \mathbf{u} + 2f^\top \mathbf{u} \quad \text{s.t.} \quad A_c \mathbf{u} \leq b_c$$

is a convex Quadratic Program (QP),
which is reliably and efficiently solvable

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Applications: Electromechanical systems

Predictive swing-up and balancing controllers



Autonomous racing for remote controlled cars



Prediction model

Nonlinear plant model: $x_{k+1} = f(x_k, u_k)$

▷ Predicted \mathbf{x}_k depends **nonlinearly** on \mathbf{u}_k

▷ In general the cost is **nonconvex** in \mathbf{u}_k : $J(x_k, \mathbf{u}_k)$
and the constraints are **nonconvex**: $g_c(x_k, \mathbf{u}_k) \leq 0$

▷ Online optimization:

$$\min_{\mathbf{u}} J(x_k, \mathbf{u}) \quad \text{s.t.} \quad g_c(x_k, \mathbf{u}) \leq 0$$

is nonconvex
may have local minima
may not be solvable efficiently or reliably

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Prediction model

Discrete time prediction model

- ▷ Predictions optimized periodically at $t = 0, T, 2T, \dots$
- ▷ Usually $T = T_s$ = sampling interval of model
- ▷ But $T = nT_s$ for $n > 1$ is also possible, e.g. if $T_s <$ time required to perform online optimization

$n =$ integer so a time-shifted version of the optimal input sequence at time k can be implemented at time $k + 1$
(allows a guarantee of stability – [Lecture 3])

e.g. if $n = 1$, then $\mathbf{u}_{k+1} = (\mathbf{u}_{1|k}, \dots, \mathbf{u}_{N-1|k}, \mathbf{u}_{N|k})$ is possible,
where $(\mathbf{u}_{0|k}, \mathbf{u}_{1|k}, \dots, \mathbf{u}_{N-1|k}) = \mathbf{u}_k^*$

Constraints

Constraints are present in almost all control problems

- ▷ Input constraints, e.g. box constraints:

$$\underline{u} \leq u_k \leq \bar{u} \quad (\text{absolute})$$

$$\Delta u \leq u_k - u_{k-1} \leq \bar{\Delta u} \quad (\text{rate})$$

- * typically active during transients, e.g. valve saturation or d.c. motor saturation

- ▷ State constraints, e.g. box constraints

$$\underline{x} \leq x_k \leq \bar{x} \quad (\text{linear})$$

- * can be active during transients, e.g. aircraft stall speed
- * and in steady state, e.g. economic constraints

Prediction model

Continuous time prediction model

- ▷ Predicted $u(t)$ need not be piecewise constant,
e.g. 1st order hold gives continuous, piecewise linear $u(t)$
or $u(t) =$ polynomial in t (piecewise quadratic, cubic etc)
- ▷ Continuous time prediction model can be integrated online,
which is useful for nonlinear continuous time systems
- ▷ This course: discrete-time model and $T = T_s$ assumed

Constraints

Classify constraints as either **hard** or **soft**

- ▷ Hard constraints must be satisfied at all times,
if this is not possible, then the problem is **infeasible**

- ▷ Soft constraints can be violated to avoid infeasibility

- ▷ Strategies for handling soft constraints:

- * impose (hard) constraints on the probability of violating each soft constraint
- * or remove active constraints until the problem becomes feasible

- ▷ This course: only hard constraints are considered

Constraint handling

Suboptimal methods for handling input constraints:

- ➊ Saturate the unconstrained control law
constraints are then usually ignored in controller design
- ➋ “De-tune” the unconstrained control law
increase the penalty on u in the performance objective
- ➌ Use an anti-windup strategy
to put limits on the state of a dynamic controller
(typically the integral term of a PI or PID controller)

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Constraint handling

De-tuning of optimal control law:

$$K_{LQ} = \text{optimal gain for LQ cost } J^\infty = \sum_{k=0}^{\infty} (\|x_k\|_Q^2 + \|u_k\|_R^2)$$

Increase R until $u = K_{LQ}x$ satisfies constraints for all initial conditions

Example:

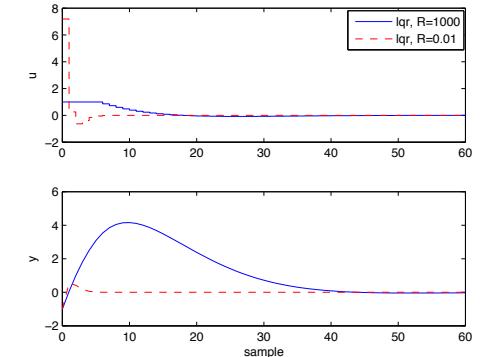
$$(A, B, C) \text{ as before}$$

$$10^{-2} \leq R \leq 10^3$$



settling time increased
from 6 to 40

- ★ $y(t) \rightarrow 0$ slowly
- ★ stability can be ensured



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Constraint handling

Effects of **input saturation**, $\underline{u} \leq u_k \leq \bar{u}$

unconstrained control law: $u = u^0$

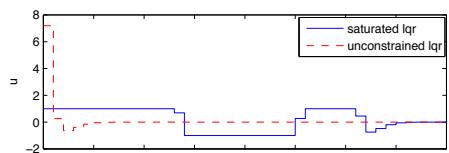
saturated control law: $u = \max\{\min\{u^0, \bar{u}\}, \underline{u}\}$

Example:

$$(A, B, C) \text{ as before}$$

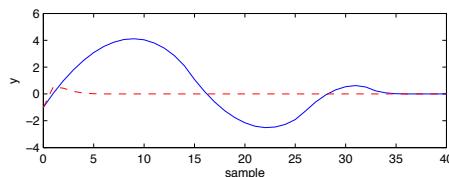
$$\underline{u} = -1, \bar{u} = 1$$

$$u^0 = K_{LQ}x$$



Input saturation causes

- ★ poor performance
- ★ possible instability
(since the open-loop system is unstable)

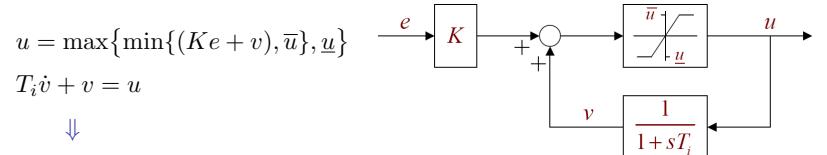


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Constraint handling

Anti-windup prevents instability of the controller while the input is saturated

Many possible approaches, e.g. anti-windup PI controller:



$$\underline{u} \leq u \leq \bar{u} \implies u = K \left(e + \frac{1}{T_i} \int^t e dt \right)$$

$$u = \underline{u} \text{ or } \bar{u} \implies v(t) \rightarrow \underline{u} \text{ or } \bar{u} \text{ exponentially}$$

Strategy is suboptimal and may not prevent instability

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Constraint handling

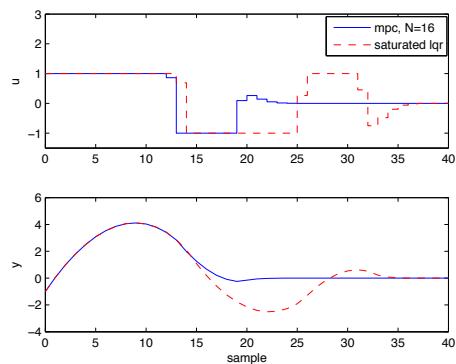
Anti-windup is based in the past behaviour of the system, whereas MPC optimizes future performance

Example:

(A, B, C) as before

MPC vs saturated LQ
(both based on cost J^∞):

- * settling time reduced to 20 by MPC
- * stability is guaranteed with MPC



Summary

- ▷ Predict performance using plant model
 - e.g. linear or nonlinear, discrete or continuous time
- ▷ Optimize future (open loop) control sequence
 - computationally much easier than optimizing over feedback laws
- ▷ Implement first sample, then repeat optimization
 - provides feedback to reduce effect of uncertainty
- ▷ Comparison of common methods of handling constraints:
 - saturation, de-tuning, anti-windup, MPC

Prediction and optimization

- Input and state predictions
- Unconstrained finite horizon optimal control
- Infinite prediction horizons and connection with LQ optimal control
- Incorporating constraints
- Quadratic programming

Review of MPC strategy

At each sampling instant:

- ➊ Use a model to **predict** system behaviour over a finite future horizon
- ➋ Compute a control sequence by solving an **online optimization** problem
- ➌ Apply the **first element** of optimal control sequence as control input



Advantages

- ★ flexible plant model
- ★ constraints taken into account
- ★ optimal performance

Disadvantage

- ★ online optimization required

Prediction equations

Linear time-invariant model:

$$x_{i+1|k} = Ax_{i|k} + Bu_{i|k}$$

assume x_k is measured at $k = 0, 1, \dots$

$$x_{0|k} = x_k$$

$$x_{1|k} = Ax_k + Bu_{0|k}$$

\vdots

$$x_{N|k} = A^N x_k + A^{N-1} B u_{0|k} + A^{N-2} B u_{1|k} + \dots + B u_{N-1|k}$$

\Downarrow

$$\mathbf{x}_k = \mathcal{M}x_k + \mathcal{C}\mathbf{u}_k$$

$$\mathcal{C} = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \quad \mathcal{M} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}$$

or

$$x_{i|k} = A^i x_k + \mathcal{C}_i \mathbf{u}_k, \quad \mathcal{C}_i = i\text{th row of } \mathcal{C}$$

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Prediction equations

Linear time-invariant model:

$$x_{k+1} = Ax_k + Bu_k$$

assume x_k is measured at $k = 0, 1, \dots$

$$\text{Predictions: } \mathbf{x}_k = \begin{bmatrix} x_{1|k} \\ \vdots \\ x_{N|k} \end{bmatrix} \quad \mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix}$$

$$\text{Quadratic cost: } J(x_k, \mathbf{u}_k) = \sum_{i=0}^{N-1} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) + \|x_{N|k}\|_P^2$$

$$(\|x\|_Q^2 = x^\top Q x, \quad \|u\|_R^2 = u^\top R u \\ P = \text{terminal weighting matrix})$$

Prediction equations

Predicted cost:

$$\begin{aligned} J_k &= \sum_{i=0}^{N-1} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) + \|x_{N|k}\|_P^2 \\ &= x_k^\top Q x_k + \mathbf{x}_k^\top \mathbf{Q} \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R} \mathbf{u}_k \quad \left\{ \begin{array}{l} \mathbf{Q} = \text{diag}\{Q, \dots, Q, P\} \\ \mathbf{R} = \text{diag}\{R, \dots, R, R\} \end{array} \right. \end{aligned}$$

\Downarrow

$$J_k = \mathbf{u}_k^\top H \mathbf{u}_k + 2x_k^\top F^\top \mathbf{u}_k + x_k^\top G x_k$$

where

$$H = \mathcal{C}^\top \mathbf{Q} \mathcal{C} + \mathbf{R} \quad \leftarrow \mathbf{u} \times \mathbf{u} \text{ terms}$$

$$F = \mathcal{C}^\top \mathbf{Q} \mathcal{M} \quad \leftarrow \mathbf{u} \times x \text{ terms}$$

$$G = \mathcal{M}^\top \mathbf{Q} \mathcal{M} + Q \quad \leftarrow x \times x \text{ terms}$$

time-invariant model $\Rightarrow H, F, G$ can be computed offline

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Prediction equations – example

Plant model: $x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k$

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

Prediction horizon $N = 4$: $\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.079 & 0 & 0 & 0 \\ 0.157 & 0 & 0 & 0 \\ 0.075 & 0.079 & 0 & 0 \\ 0.323 & 0.157 & 0 & 0 \\ 0.071 & 0.075 & 0.079 & 0 \\ 0.497 & 0.323 & 0.157 & 0 \\ 0.068 & 0.071 & 0.075 & 0.079 \end{bmatrix}$

Cost matrices $Q = C^\top C$, $R = 0.01$, and $P = Q$:

$$H = \begin{bmatrix} 0.271 & 0.122 & 0.016 & -0.034 \\ 0.122 & 0.086 & 0.014 & -0.020 \\ 0.016 & 0.014 & 0.023 & -0.007 \\ -0.034 & -0.020 & -0.007 & 0.016 \end{bmatrix} \quad F = \begin{bmatrix} 0.977 & 4.925 \\ 0.383 & 2.174 \\ 0.016 & 0.219 \\ -0.115 & -0.618 \end{bmatrix}$$

$$G = \begin{bmatrix} 7.589 & 22.78 \\ 22.78 & 103.7 \end{bmatrix}$$

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Prediction equations: LTV model

Aside: Linear time-varying model: $x_{k+1} = A_k x_k + B_k u_k$

assume x_k is measured at $k = 0, 1, \dots$

Predictions: $x_{0|k} = x_k$

$$x_{1|k} = A_k x_k + B_k u_{0|k}$$

$$x_{2|k} = A_{k+1} A_k x_k + A_{k+1} B_k u_{0|k} + B_{k+1} u_{1|k}$$

\vdots

\Downarrow

$$x_{i|k} = \prod_{j=i-1}^0 A_{k+j} x_k + \mathcal{C}_i(k) u_k, \quad i = 0, \dots, N$$

$$\mathcal{C}_i(k) = \begin{bmatrix} \prod_{j=i-1}^1 A_{k+j} B_k & \prod_{j=i-1}^2 A_{k+j} B_{k+1} & \cdots & B_{k+i-1} & 0 & \cdots & 0 \end{bmatrix}$$

\Downarrow

$H(k)$, $F(k)$, $G(k)$ depend on k and must be computed online

Unconstrained optimization

Minimize cost:

$$\mathbf{u}^* = \arg \min_{\mathbf{u}} J, \quad J = \mathbf{u}^\top H \mathbf{u} + 2x^\top F^\top \mathbf{u} + x^\top Gx$$

differentiate w.r.t. \mathbf{u} : $\nabla_{\mathbf{u}} J = 2H\mathbf{u} + 2Fx = 0$

$$\Downarrow$$

$$\mathbf{u} = -H^{-1}Fx$$

$= \mathbf{u}^*$ if H is positive definite i.e. if $H \succ 0$

Here $H = \mathcal{C}^\top Q \mathcal{C} + R \succ 0$ if: $\begin{cases} R \succ 0 \& Q, P \succeq 0 & \text{or} \\ R \succeq 0 \& Q, P \succ 0 \& \mathcal{C} \text{ is full-rank} \end{cases}$

\Updownarrow
(A, B) controllable

Receding horizon controller is linear state feedback:

$$u_k = -[I \ 0 \ \cdots \ 0] H^{-1}Fx_k$$

is the closed loop response optimal? is it even stable?

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Example

Model: A, B, C as before, cost: $J_k = \sum_{i=0}^{N-1} (y_{i|k}^2 + 0.01u_{i|k}^2) + y_{N|k}^2$

► For $N = 4$: $\mathbf{u}_k^* = -H^{-1}Fx_k = \begin{bmatrix} -4.36 & -18.7 \\ 1.64 & 1.24 \\ 1.41 & 3.00 \\ 0.59 & 1.83 \end{bmatrix} x_k$

$$u_k = [-4.36 \ -18.7] x_k$$

► For general N : $u_k = Lx_k$

	$N = 4$	$N = 3$	$N = 2$	$N = 1$
L	$[-4.36 \ -18.69]$	$[-3.80 \ -16.98]$	$[1.22 \ -3.95]$	$[5.35 \ 5.10]$

$\lambda(A + BL)$

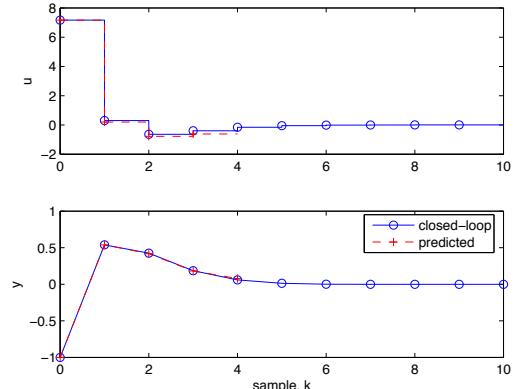
	$N = 4$	$N = 3$	$N = 2$	$N = 1$
$[-4.36 \ -18.69]$	$[-3.80 \ -16.98]$	$[1.22 \ -3.95]$	$[5.35 \ 5.10]$	
$0.29 \pm 0.17j$	$0.36 \pm 0.22j$	$1.36, 0.38$	$2.15, 0.30$	
stable	stable	unstable	unstable	

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Example

Horizon: $N = 4$, $x_0 = (0.5, -0.5)$



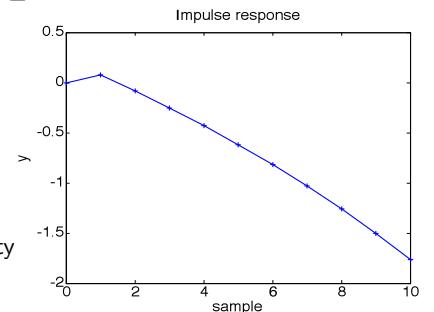
Receding horizon control

Why is this example unstable for $N \leq 2$?

System is non-minimum phase

↓
impulse response changes sign

↓
hence short horizon causes instability



Solution:

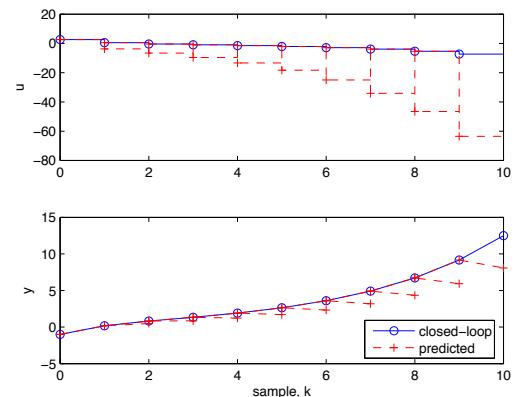
- ★ use an **infinite** horizon cost
- ★ but keep a **finite** number of optimization variables in predictions

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Example

Horizon: $N = 2$, $x_0 = (0.5, -0.5)$

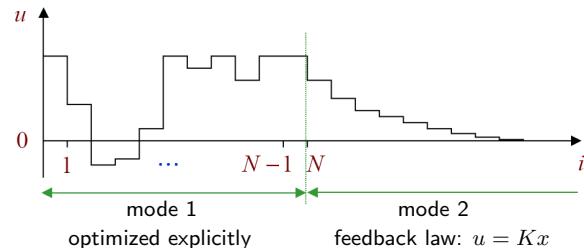


Observation: predicted and closed loop responses are different for small N

Dual mode predictions

An infinite prediction horizon is possible with **dual mode** predictions:

$$u_{i|k} = \begin{cases} \text{optimization variables} & i = 0, \dots, N-1, \text{ mode 1} \\ Kx_{i|k} & i = N, N+1, \dots, \text{ mode 2} \end{cases}$$



Feedback gain K : stabilizing and determined offline

unconstrained LQ optimal for $\sum_{i=0}^{\infty} (\|x_i\|_Q^2 + \|u_i\|_R^2)$
(usually)

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Infinite horizon cost

If the predicted input sequence is

$$\{u_{0|k}, \dots, u_{N-1|k}, Kx_{N|k}, K\Phi x_{N|k}, \dots\}$$

then

$$\sum_{i=0}^{\infty} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) = \sum_{i=0}^{N-1} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) + \|x_{N|k}\|_P^2$$

where

$$P - (A + BK)^T P (A + BK) = Q + K^T R K$$

Lyapunov matrix equation (discrete time)

Note:

- * if $Q + K^T R K \succ 0$, then the solution P is unique and $P \succ 0$
- * matlab: $P = \text{dlyap}(\text{Phi}', \text{RHS});$
 $\text{Phi} = A + B * K;$ $\text{RHS} = Q + K' * R * K;$
- * P is the steady state Riccati equation solution if K is LQ-optimal

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Connection with LQ optimal control

Predicted cost:

$$J(x_k, (u_{0|k}, \dots, u_{N-1|k})) = \sum_{i=0}^{N-1} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) + \|x_{N|k}\|_P^2$$

where $P - (A + BK)^T P (A + BK) = Q + K^T R K$
and $K = \text{LQ-optimal}$

$$u_{0|k}^* = Kx_k \text{ where } \mathbf{u}_k^* = \arg \min_{\mathbf{u}} J(x_k, \mathbf{u}) = (u_{0|k}^*, \dots, u_{N-1|k}^*)$$

$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$

The Bellman principle of optimality implies:

$$\{u_{0|k}, u_{1,k}, \dots\} \text{ optimal} \iff \begin{cases} \{u_{0|k}, \dots, u_{N-1|k}\} \text{ optimal} \\ \text{and } K \text{ LQ-optimal} \end{cases}$$

$\Uparrow \quad \Uparrow \quad \Uparrow \quad \Uparrow \quad \Uparrow \quad \Uparrow$

Infinite horizon cost

Proof that the predicted cost over the mode 2 horizon is $\|x_{N|k}\|_P^2$:

Let $J^\infty(x_0) = \sum_{i=0}^{\infty} (\|x_i\|_Q^2 + \|u_i\|_R^2)$, with $u_i = Kx_i$, $x_{i+1} = \Phi x_i \forall i$,

– then $J^\infty(x_0) = \sum_{i=0}^{\infty} (\|\Phi^i x_0\|_Q^2 + \|K\Phi^i x_0\|_R^2)$
 $= x_0^\top \underbrace{\left[\sum_{i=0}^{\infty} (\Phi^i)^\top (Q + K^T R K) \Phi^i \right]}_{=S} x_0$

– but $\Phi^\top S \Phi = \sum_{i=1}^{\infty} (\Phi^i)^\top (Q + K^T R K) \Phi^i$
 $= S - (Q + K^T R K)$

so if $\Phi = A + BK$, then $S = P$ and $J^\infty(x_{N|k}) = \|x_{N|k}\|_P^2$

Connection with LQ optimal control – example

► Model parameters (A, B, C) as before

LQ optimal gain for $Q = C^\top C$, $R = 0.01$: $K = \begin{bmatrix} -4.36 & -18.74 \\ 3.92 & 4.83 \end{bmatrix}$
Lyapunov equation solution: $P = \begin{bmatrix} 3.92 & 4.83 \\ 13.86 & \end{bmatrix}$

► Cost matrices for $N = 4$:

$$H = \begin{bmatrix} 1.44 & 0.98 & 0.59 & 0.26 \\ & 0.72 & 0.44 & 0.20 \\ & & 0.30 & 0.14 \\ & & & 0.096 \end{bmatrix} \quad F = \begin{bmatrix} 3.67 & 23.9 \\ 2.37 & 16.2 \\ 1.36 & 9.50 \\ 0.556 & 4.18 \end{bmatrix} \quad G = \begin{bmatrix} 13.8 & 66.7 \\ & 413 \end{bmatrix}$$

► Predictive control law: $u_k = -[1 \ 0 \ 0] H^{-1} F x_k$
 $= [-4.35 \ -18.74] x_k$

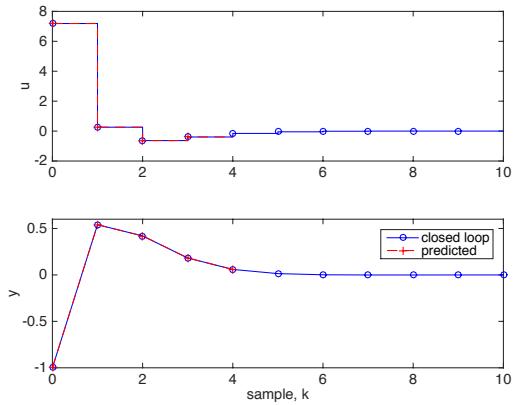
(identical to the LQ optimal controller)

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Connection with LQ optimal control – example

- Response for $N = 4$, $x_0 = (0.5, -0.5)$



Infinite horizon cost
no constraints } \implies identical predicted and closed loop responses

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Dual mode predictions

Aside: Pre-stabilized dual mode predictions with better numerical stability

- Vectorized form: $\mathbf{x}_k = \mathcal{M}x_k + \mathcal{C}\mathbf{c}_k$

$$\mathbf{x}_k := \begin{bmatrix} x_{1|k} \\ \vdots \\ x_{N|k} \end{bmatrix}, \quad \mathbf{c}_k := \begin{bmatrix} c_{0|k} \\ \vdots \\ c_{N-1|k} \end{bmatrix}$$

$$\mathcal{M} = \begin{bmatrix} \Phi & & & & \\ \Phi^2 & & & & \\ \vdots & & & & \\ \Phi^N & & & & \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} B & 0 & \cdots & 0 \\ \Phi B & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi^{N-1}B & \Phi^{N-2}B & \cdots & B \end{bmatrix}$$

- Cost: $J(x_k, (u_{0|k}, \dots, u_{N-1|k})) = J(x_k, \mathbf{c}_k)$

2 - 18

Dual mode predictions

Aside: Pre-stabilized dual mode predictions with better numerical stability

- Inputs

$$\begin{array}{ll} \text{mode 1} & u_{i|k} = Kx_{i|k} + c_{i|k}, \quad i = 0, 1, \dots, N-1 \\ \text{mode 2} & u_{i|k} = Kx_{i|k}, \quad i = N, N+1, \dots \end{array}$$

- Dynamics

$$\begin{array}{ll} \text{mode 1} & x_{i+1|k} = \Phi x_{i|k} + B c_{i|k}, \quad i = 0, 1, \dots, N-1 \\ \text{mode 2} & x_{i+1|k} = \Phi x_{i|k}, \quad i = N, N+1, \dots \end{array}$$

where $(c_{0|k}, \dots, c_{N-1|k})$ are optimization variables

Input and state constraints

Infinite horizon unconstrained MPC = LQ optimal control

but MPC can also handle constraints

Consider constraints applied to mode 1 predictions:

- ★ input constraints: $\underline{u} \leq u_{i|k} \leq \bar{u}, \quad i = 0, \dots, N-1$

$$\Leftrightarrow \begin{bmatrix} I \\ -I \end{bmatrix} \mathbf{u}_k \leq \begin{bmatrix} \bar{\mathbf{u}} \\ -\underline{\mathbf{u}} \end{bmatrix} \quad \text{where} \quad \bar{\mathbf{u}} = [\bar{u}^\top \ \dots \ \bar{u}^\top]^\top$$

$$\underline{\mathbf{u}} = [\underline{u}^\top \ \dots \ \underline{u}^\top]^\top$$

- ★ state constraints: $\underline{x} \leq x_{i|k} \leq \bar{x}, \quad i = 1, \dots, N$

$$\Leftrightarrow \begin{bmatrix} \mathcal{C}_i \\ -\mathcal{C}_i \end{bmatrix} \mathbf{u}_k \leq \begin{bmatrix} \bar{x} \\ -\underline{x} \end{bmatrix} + \begin{bmatrix} -A^i \\ A^i \end{bmatrix} x_k, \quad i = 1, \dots, N$$

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Input and state constraints

Constraints on mode 1 predictions can be expressed

$$A_c \mathbf{u}_k \leq b_c + B_c x_k$$

where A_c, B_c, b_c can be computed offline since model is time-invariant

The online optimization is a quadratic program (QP):

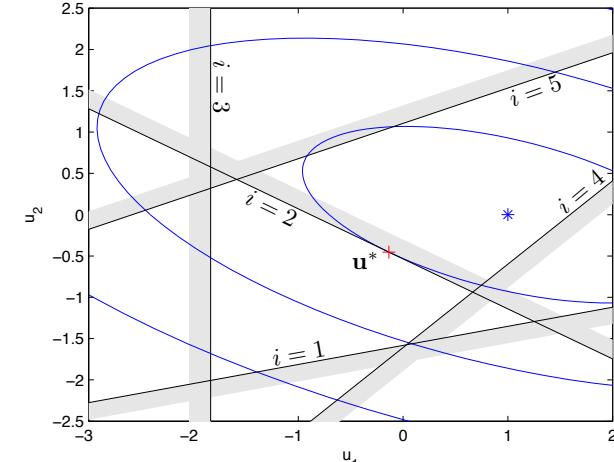
$$\begin{aligned} & \text{minimize}_{\mathbf{u}} \quad \mathbf{u}^\top H \mathbf{u} + 2x_k^\top F^\top \mathbf{u} \\ & \text{subject to} \quad A_c \mathbf{u} \leq b_c + B_c x_k \end{aligned}$$

which is a convex optimization problem with a unique solution if

$$H = \mathcal{C}^\top \mathcal{Q} \mathcal{C} + \mathbf{R} \text{ is positive definite}$$

2 - 20

Active constraints – example



A QP problem with 5 inequality constraints
active set at solution: $\mathcal{I} = \{2\}$

2 - 22

QP solvers: (a) Active set

Consider the QP: $\mathbf{u}^* = \arg \min_{\mathbf{u}} \mathbf{u}^\top H \mathbf{u} + 2f^\top \mathbf{u}$
subject to $A\mathbf{u} \leq b$

and let (A_i, b_i) = i th row/element of (A, b)

- ▷ Individual constraints are **active** or **inactive**

active	inactive
$A_i \mathbf{u}^* = b_i, \forall i \in \mathcal{I}$	$A_i \mathbf{u}^* \leq b_i, \forall i \notin \mathcal{I}$
b_i affects solution	b_i does not affect solution

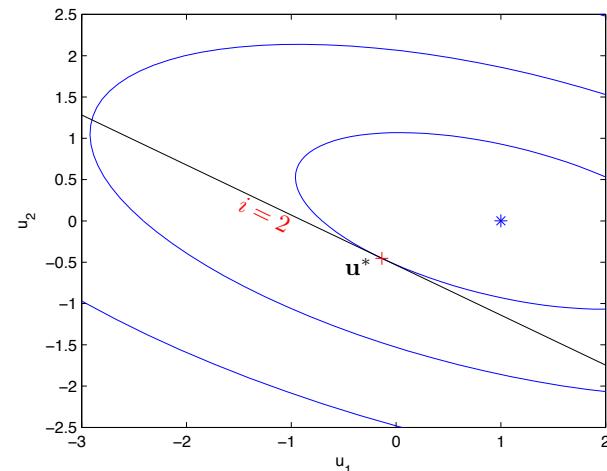
- ▷ Equality constraint problem: $\mathbf{u}^* = \arg \min_{\mathbf{u}} \mathbf{u}^\top H \mathbf{u} + 2f^\top \mathbf{u}$
subject to $A_i \mathbf{u} = b_i, \forall i \in \mathcal{I}$

- ▷ Solve QP by searching for \mathcal{I}

- * one equality constraint problem solved at each iteration
- * optimality conditions (**KKT conditions**) identify solution

2 - 21

Active constraints – example



An equivalent equality constraint problem

2 - 22

QP solvers: (a) Active set

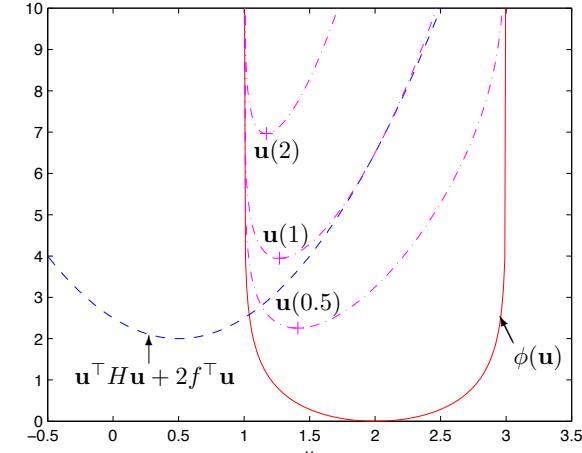
- ▷ Computation:
 $O(N^3 n_u^3)$ additions/multiplications per iteration (conservative estimate)
upper bound on number of iterations is exponential in problem size
- ▷ At each iteration choose trial active set using: cost gradient
constraint sensitivities

↓

number of iterations needed is often small in practice
- ▷ In MPC $\mathbf{u}_k^* = \mathbf{u}^*(x_k)$ and $\mathcal{I}_k = \mathcal{I}(x_k)$
hence initialize solver at time k using the solution computed at $k-1$

2 - 23

Interior point method – example



$\mathbf{u}(\mu) \rightarrow \mathbf{u}^* = 1$ as $\mu \rightarrow \infty$
but $\min_{\mathbf{u}} \mu(\mathbf{u}^T H \mathbf{u} + 2f^T \mathbf{u}) + \phi(\mathbf{u})$ becomes ill-conditioned as $\mu \rightarrow \infty$

2 - 25

QP solvers: (b) Interior point

- ▷ Solve an unconstrained problem at each iteration:

$$\mathbf{u}(\mu) = \min_{\mathbf{u}} \mu(\mathbf{u}^T H \mathbf{u} + 2f^T \mathbf{u}) + \phi(\mathbf{u})$$
where
 $\phi(\mathbf{u})$ = barrier function ($\phi \rightarrow \infty$ at constraints)
 $\mathbf{u} \rightarrow \mathbf{u}^*$ as $\mu \rightarrow \infty$
- Increase μ until $\phi(\mathbf{u}^*) > 1/\epsilon$ (ϵ = user-defined tolerance)
- ▷ # operations per iterations is constant, e.g. $O(N^3 n_u^3)$
iterations for given ϵ is polynomial in problem size

↓

Computational advantages for large-scale problems
e.g. # variables $> 10^2$, # constraints $> 10^3$
- ▷ No general method for initializing at solution estimate

2 - 24

QP solvers: (c) Multiparametric

$$\begin{aligned} \text{Let } \mathbf{u}^*(\mathbf{x}) &= \arg \min_{\mathbf{u}} \mathbf{u}^T H \mathbf{u} + 2\mathbf{x}^T F^T \mathbf{u} \\ \text{subject to } A\mathbf{u} &\leq b + B\mathbf{x} \end{aligned}$$

then:

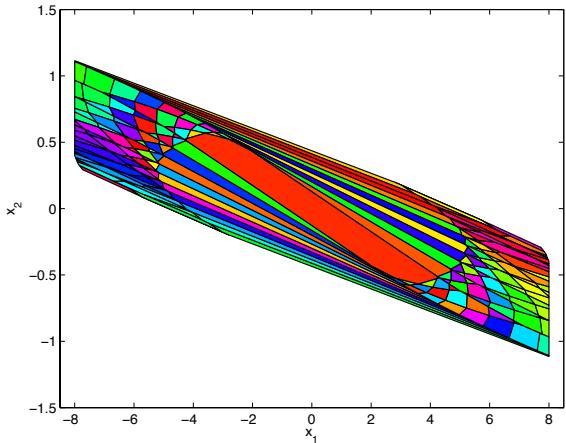
- * \mathbf{u}^* is a continuous function of \mathbf{x}
- * $\mathbf{u}^*(\mathbf{x}) = K_j \mathbf{x} + k_j$ for all \mathbf{x} in a polytopic set \mathcal{X}_j

▷ In principle each K_j, k_j and \mathcal{X}_j can be determined offline

▷ But number of sets \mathcal{X}_j is usually large (depends exponentially on problem size)
so online determination of j such that $x_k \in \mathcal{X}_j$ is difficult

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Multiparametric QP – example



Model: (A, B, C) as before,
cost: $Q = C^T C$, $R = 1$, horizon: $N = 10$
constraints: $-1 \leq u \leq 1$, $-\mathbf{1} \leq x/8 \leq \mathbf{1}$

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Lecture 3

Closed loop properties of MPC

3 - 1

Summary

- ▷ Predicted control inputs: $\mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix}$
- and states: $\mathbf{x}_k = \begin{bmatrix} x_{1|k} \\ \vdots \\ x_{N|k} \end{bmatrix} = \mathcal{M}x_k + \mathcal{C}\mathbf{u}_k$
- ▷ Predicted cost: $J(x_k, \mathbf{u}_k) = \sum_{i=0}^{N-1} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) + \|x_{N|k}\|_P^2$
 $= \mathbf{u}_k^\top H \mathbf{u}_k + 2x_k^\top F^\top \mathbf{u}_k + x_k^\top G x_k$
- ▷ Online optimization of cost subject to linear state and input constraints is a QP problem:

$$\underset{\mathbf{u}}{\text{minimize}} \quad \mathbf{u}^\top H \mathbf{u} + 2x_k^\top F^\top \mathbf{u}$$

$$\text{subject to} \quad A_c \mathbf{u} \leq b_c + B_c x_k$$

Closed loop properties of MPC

- Review: infinite horizon cost
- Infinite horizon predictive control with constraints
- Closed loop stability
- Constraint-checking horizon
- Connection with constrained optimal control

3 - 2

Review: infinite horizon cost

Short prediction horizons cause poor performance and instability, so

- * use an infinite horizon cost: $J(x_k, \mathbf{u}_k) = \sum_{i=0}^{\infty} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2)$

- * keep optimization finite-dimensional by using **dual mode predictions**:

$$u_{i|k} = \begin{cases} \text{optimization variables} & i = 0, \dots, N-1, \text{ mode 1} \\ Kx_{i|k} & i = N, N+1, \dots, \text{ mode 2} \end{cases}$$

mode 1: $\mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix}$ \mathbf{u}_k optimized online

mode 2: $u_{i|k} = Kx_{i|k}$ K chosen offline

3 - 3

Review: MPC online optimization

- ▷ Unconstrained optimization: $\nabla_{\mathbf{u}} J(x, \mathbf{u}^*) = 2H\mathbf{u}^* + 2Fx = 0$, so

$$\mathbf{u}^*(x) = -H^{-1}Fx$$

$$\implies \text{linear controller: } u_k = K_{\text{MPC}}x_k$$

K_{MPC} = LQ-optimal if K = LQ-optimal (in mode 2)

- ▷ Constrained optimization:

$$\mathbf{u}^*(x) = \arg \min_{\mathbf{u}} \mathbf{u}^\top H\mathbf{u} + 2x^\top F^\top \mathbf{u}$$

subject to $A_c \mathbf{u} \leq b_c + B_c x$

$$\implies \text{nonlinear controller: } u_k = K_{\text{MPC}}(x_k)$$

3 - 5

Review: infinite horizon cost

- ▷ Cost for mode 2: $\sum_{i=N}^{\infty} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) = \|x_{N|k}\|_P^2$

P is the solution of the **Lyapunov equation**

$$P - (A + BK)^\top P(A + BK) = Q + K^\top R K$$

- ▷ Infinite horizon cost:

$$\begin{aligned} J(x_k, \mathbf{u}_k) &= \sum_{i=0}^{N-1} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) + \|x_{N|k}\|_P^2 \\ &= \mathbf{u}_k^\top H\mathbf{u}_k + 2x_k^\top F^\top \mathbf{u}_k + x_k^\top Gx_k \end{aligned}$$

3 - 4

Constrained MPC – example

- ▷ Plant model: $x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k$

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$\text{Constraints: } -1 \leq u_k \leq 1$$

- ▷ MPC optimization (constraints applied only to mode 1 predictions):

$$\begin{aligned} &\underset{\mathbf{u}}{\text{minimize}} \quad \sum_{i=0}^{N-1} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) + \|x_{N|k}\|_P^2 \\ &\text{subject to} \quad -1 \leq u_{i|k} \leq 1, \quad i = 0, \dots, N-1 \end{aligned}$$

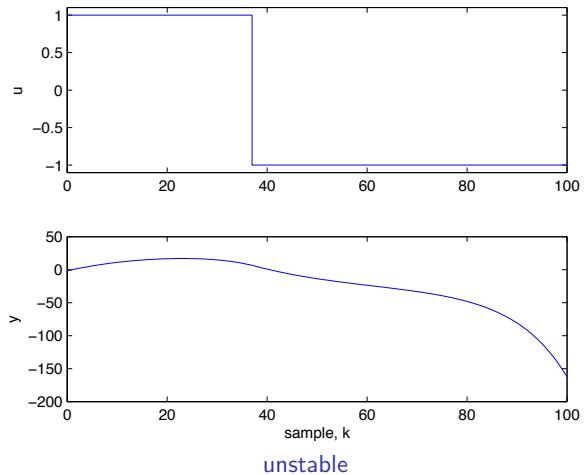
$$Q = C^\top C, \quad R = 0.01, \quad N = 2$$

... performance? stability?

3 - 6

Constrained MPC – example

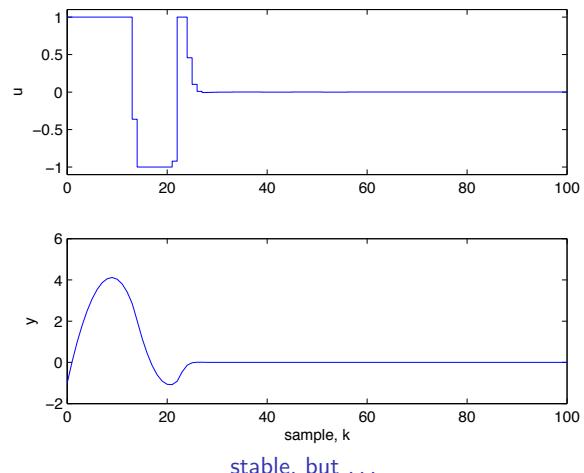
Closed loop response for $x_0 = (0.8, -0.8)$



3 - 7

Constrained MPC – example

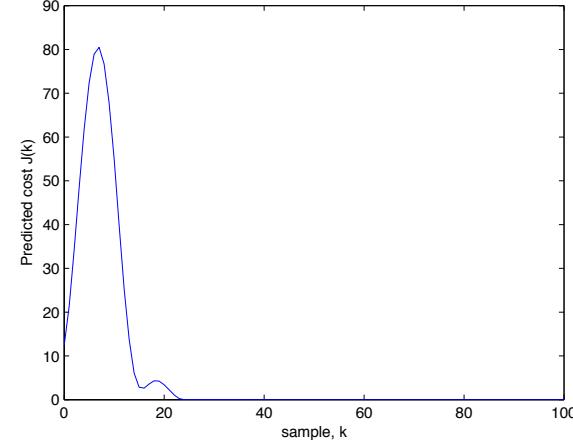
Closed loop response for $x_0 = (0.5, -0.5)$



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Constrained MPC – example

Optimal predicted cost $x_0 = (0.5, -0.5)$



... increasing $J_k \implies$ closed loop response does not follow predicted trajectory

3 - 9

Stability analysis

How can we guarantee the closed loop stability of MPC?

- (a). Show that a Lyapunov function exists demonstrating stability
- (b). Ensure that optimization feasible is at each time $k = 0, 1, \dots$

▷ For Lyapunov stability analysis:

- * consider first the unconstrained problem
- * use predicted cost as a trial Lyapunov function

▷ Guarantee feasibility of the MPC optimization recursively

by ensuring that feasibility at time k implies feasibility at $k + 1$

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Discrete time Lyapunov stability

Consider the system $x_{k+1} = f(x_k)$, with $f(0) = 0$

- ▷ Definition: $x = 0$ is a **stable** equilibrium point if
for all $R > 0$ there exists r such that
 $\|x_0\| < r \implies \|x_k\| < R$ for all k
- ▷ If continuously differentiable $V(x)$ exists with
 - (i). $V(x)$ is positive definite and
 - (ii). $V(x_{k+1}) - V(x_k) \leq 0$
 then $x = 0$ is a stable equilibrium point

Lyapunov stability

Trial Lyapunov function:

$$J^*(x_k) = J(x_k, \mathbf{u}_k^*)$$

$$\text{where } J(x_k, \mathbf{u}_k) = \sum_{i=0}^{\infty} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2)$$

* $J^*(x)$ is positive definite if:

(a). $R \succeq 0$ and $Q \succ 0$, or

(b). $R \succ 0$ and $Q \succeq 0$ and (A, Q) is observable

since then $J^*(x_k) \geq 0$ and $J^*(x_k) = 0$ if and only if $x_k = 0$

* $J^*(x)$ is continuously differentiable

... from analysis of MPC optimization as a multiparametric QP

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Discrete time Lyapunov stability

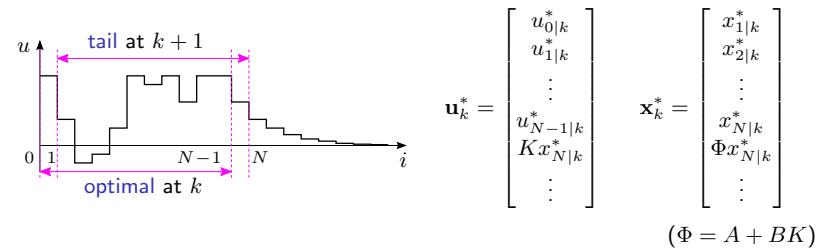
Consider the system $x_{k+1} = f(x_k)$, with $f(0) = 0$

- ▷ Definition: $x = 0$ is an **asymptotically stable** equilibrium point if
 - (i). $x = 0$ is stable and
 - (ii). r exists such that $\|x_0\| < r \implies \lim_{k \rightarrow \infty} x_k = 0$
- ▷ If continuously differentiable $V(x)$ exists with
 - (i). $V(x)$ is positive definite and
 - (ii). $V(x_{k+1}) - V(x_k) < 0$ whenever $x_k \neq 0$
 then $x = 0$ is an asymptotically stable equilibrium point

Lyapunov stability

Construct a bound on $J^*(x_{k+1}) - J^*(x_k)$ using the “tail” of the optimal prediction at time k

Optimal predicted sequences at time k :

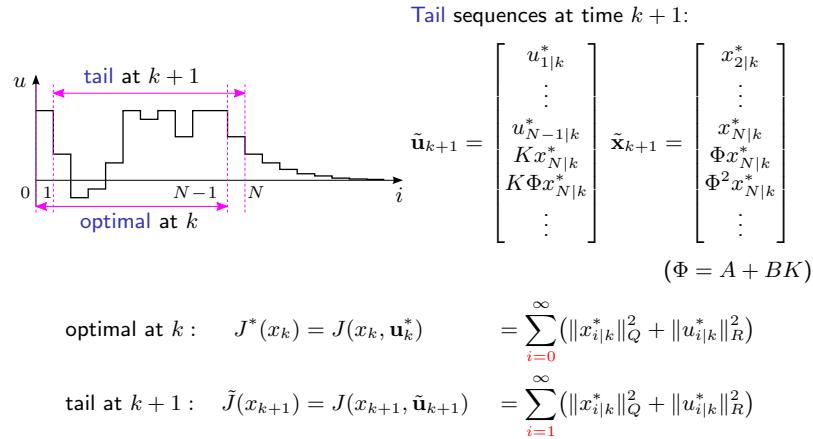


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Lyapunov stability

Construct a bound on $J^*(x_{k+1}) - J^*(x_k)$ using the “tail” of the optimal prediction at time k



Lyapunov stability

The bound $J^*(x_{k+1}) - J^*(x_k) \leq -\|x_k\|_Q^2 - \|u_k\|_R^2$ implies:

- ➊ the closed loop cost cannot exceed the initial predicted cost, since summing both sides over all $k \geq 0$ gives

$$\sum_{k=0}^{\infty} (\|x_k\|_Q^2 + \|u_k\|_R^2) \leq J^*(x_0)$$

- ➋ $x = 0$ is asymptotically stable

- * if $R \succeq 0$ and $Q \succ 0$, this follows from Lyapunov's direct method
- * if $R \succ 0$, $Q \succeq 0$ and (A, Q) observable, this follows from:
 - (a). stability of $x = 0$ \Leftarrow Lyapunov's direct method
 - (b). $\lim_{k \rightarrow \infty} (\|x_k\|_Q^2 + \|u_k\|_R^2) = 0 \Leftarrow \sum_{k=0}^{\infty} (\|x_k\|_Q^2 + \|u_k\|_R^2) < \infty$

Lyapunov stability

Construct a bound on $J^*(x_{k+1}) - J^*(x_k)$ using the “tail” of the optimal prediction at time k

Predicted cost for the tail:

$$\tilde{J}(x_{k+1}) = J^*(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2$$

but $\tilde{\mathbf{u}}_{k+1}$ is suboptimal at time $k + 1$, so

$$J^*(x_{k+1}) \leq \tilde{J}(x_{k+1})$$

Therefore

$$J^*(x_{k+1}) \leq J^*(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2$$

Stability analysis

How can we guarantee the closed loop stability of MPC?

- (b). Ensure that optimization feasible is at each time $k = 0, 1, \dots$

- ▷ Guarantee feasibility of the MPC optimization recursively by ensuring that feasibility at time $k \implies$ feasibility at $k + 1$

Terminal constraint

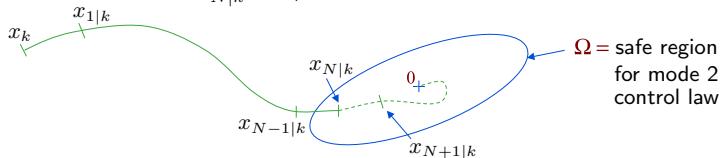
The basic idea



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Terminal constraint

Terminal constraint: $x_{N|k} \in \Omega$, where $\Omega = \text{terminal set}$



Choose Ω so that:

- (a). $x \in \Omega \implies \begin{cases} \underline{u} \leq Kx \leq \bar{u} \\ \underline{x} \leq x \leq \bar{x} \end{cases}$
- (b). $x \in \Omega \implies (A + BK)x \in \Omega$

then Ω is invariant for the mode 2 dynamics and constraints, so

$$x_{N|k} \in \Omega \implies \begin{cases} \underline{u} \leq u_{i|k} \leq \bar{u} \\ \underline{x} \leq x_{i|k} \leq \bar{x} \end{cases} \text{ for } i = N, N+1, \dots$$

i.e. constraints are satisfied over the infinite mode 2 prediction horizon

Stability of constrained MPC

Prototype MPC algorithm

At each time $k = 0, 1, \dots$

(i). solve $\mathbf{u}_k^* = \arg \min_{\mathbf{u}_k} J(x_k, \mathbf{u}_k)$

$$\text{s.t. } \underline{u} \leq u_{i|k} \leq \bar{u}, \quad i = 0, \dots, N-1$$

$$\underline{x} \leq x_{i|k} \leq \bar{x}, \quad i = 1, \dots, N$$

$$x_{N|k} \in \Omega$$

(ii). apply $u_k = u_{0|k}^*$ to the system

Asymptotically stabilizes $x = 0$ with region of attraction \mathcal{F}_N ,

$$\mathcal{F}_N = \left\{ x_0 : \exists \{u_0, \dots, u_{N-1}\} \text{ such that } \begin{array}{l} \underline{u} \leq u_i \leq \bar{u}, \quad i = 0, \dots, N-1 \\ \underline{x} \leq x_i \leq \bar{x}, \quad i = 1, \dots, N \\ x_N \in \Omega \end{array} \right\}$$

= the set of all feasible initial conditions for N -step horizon and terminal set Ω

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Terminal constraints

Make Ω as large as possible so that the feasible set \mathcal{F}_N is maximized, i.e.

$$\Omega = \mathcal{X}_\infty = \lim_{j \rightarrow \infty} \mathcal{X}_j$$

where

- * $\mathcal{X}_j = \text{initial conditions for which constraints are satisfied for } j \text{ steps}$ with $u = Kx$

$$= \left\{ x : \begin{array}{l} \underline{u} \leq K(A + BK)^i x \leq \bar{u} \\ \underline{x} \leq (A + BK)^i x \leq \bar{x} \end{array} \quad i = 0, \dots, j \right\}$$

- * $\mathcal{X}_\infty = \mathcal{X}_\nu$ for some finite ν if $|\text{eig}(A + BK)| < 1$

⇓

$x \in \mathcal{X}_\infty$ if constraints are satisfied on a finite constraint checking horizon

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Terminal constraints – Example

Plant model:

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k$$

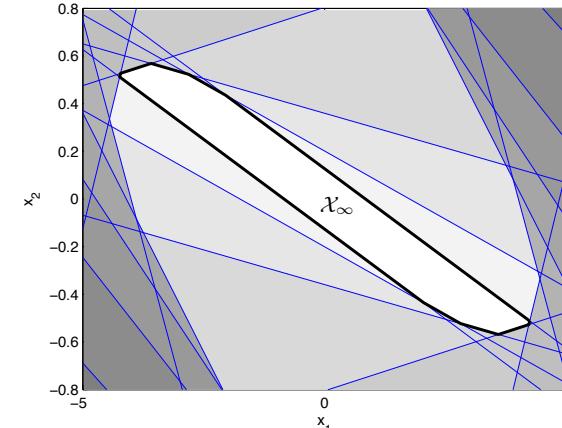
$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} \quad C = [-1 \quad 1]$$

input constraints: $-1 \leq u_k \leq 1$

mode 2 feedback law: $K = [-1.19 \quad -7.88]$
 $= K_{LQ}$ for $Q = C^\top C, R = 1$

Terminal constraints – example

Constraints: $-1 \leq u \leq 1$



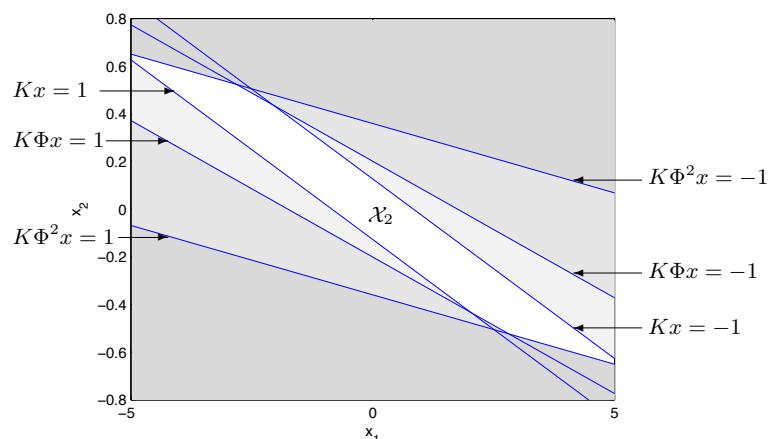
$\mathcal{X}_4 = \mathcal{X}_5 = \dots = \mathcal{X}_j$ for all $j > 4$ so $\mathcal{X}_4 = \mathcal{X}_\infty$

3 - 22

3 - 23

Terminal constraints – example

Constraints: $-1 \leq u \leq 1$



Terminal constraints – example

In this example \mathcal{X}_∞ is determined in a finite number of steps because

- ① $(A + BK)$ is strictly stable, and
- ② $((A + BK), K)$ is observable

$$\text{①} \Rightarrow \left\{ \begin{array}{l} \text{shortest distance of hyperplane} \\ K(A + BK)^i x \leq 1 \text{ from origin} \end{array} \right\} = \frac{1}{\|K(A + BK)^i\|_2} \rightarrow \infty \text{ as } i \rightarrow \infty$$

- ② $\Rightarrow \mathcal{X}_\infty$ is bounded because $x_0 \notin \mathcal{X}_\infty$ if x_0 is sufficiently large

Here $\{x : -1 \leq K(A + BK)^i x \leq 1\}$ contains \mathcal{X}_4 for $i > 4$



$$\mathcal{X}_\infty = \mathcal{X}_4$$

constraint checking horizon: $\nu = 4$

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3 - 24

Terminal constraints

General case

Let $\mathcal{X}_j = \{x : F\Phi^i x \leq \mathbf{1}, i = 0, \dots, j\}$ with $\begin{cases} \Phi \text{ strictly stable} \\ (\Phi, F) \text{ observable} \end{cases}$
 then: (i). $\mathcal{X}_\infty = \mathcal{X}_\nu$ for finite ν
 (ii). $\mathcal{X}_\nu = \mathcal{X}_\infty$ iff $x \in \mathcal{X}_{\nu+1}$ whenever $x \in \mathcal{X}_\nu$

Proof of (ii)

(a). for any j , $\mathcal{X}_{j+1} = \mathcal{X}_j \cap \{x : F\Phi^{j+1} x \leq \mathbf{1}\}$

so $\mathcal{X}_j \supseteq \mathcal{X}_{j+1} \supseteq \lim_{j \rightarrow \infty} \mathcal{X}_j = \mathcal{X}_\infty$

(b). if $x \in \mathcal{X}_{\nu+1}$ whenever $x \in \mathcal{X}_\nu$, then $\Phi x \in \mathcal{X}_\nu$ whenever $x \in \mathcal{X}_\nu$

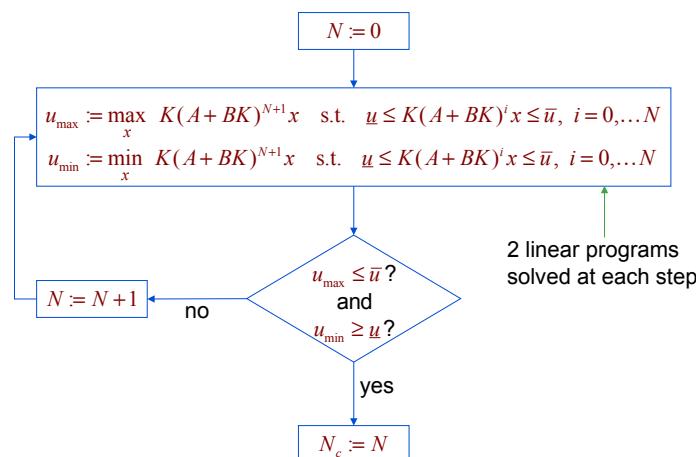
but $\mathcal{X}_\nu \subseteq \{x : Fx \leq \mathbf{1}\}$ and it follows that $\mathcal{X}_\nu \subseteq \mathcal{X}_\infty$

(a) & (b) $\Rightarrow \mathcal{X}_\nu = \mathcal{X}_\infty$

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Terminal constraints – constraint checking horizon

Algorithm for computing constraint checking horizon N_c
 for input constraints $\underline{u} \leq u \leq \bar{u}$:



Constrained MPC

Define the terminal set Ω as \mathcal{X}_{N_c}

MPC algorithm

At each time $k = 0, 1, \dots$

(i). solve $\mathbf{u}_k^* = \arg \min_{\mathbf{u}_k} J(x_k, \mathbf{u}_k)$

$$\begin{aligned} \text{s.t. } \underline{u} \leq u_{i|k} \leq \bar{u}, & i = 0, \dots, N + N_c \\ x \leq x_{i|k} \leq \bar{x}, & i = 1, \dots, N + N_c \end{aligned}$$

(ii). apply $u_k = u_{0|k}^*$ to the system

Note

* predictions for $i = N, \dots, N + N_c$: $\begin{cases} x_{i|k} = (A + BK)^{i-N} x_{N|k} \\ u_{i|k} = K(A + BK)^{i-N} x_{N|k} \end{cases}$

* $x_{N|k} \in \mathcal{X}_{N_c}$ implies linear constraints so online optimization is a QP

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Closed loop performance

Longer horizon N ensures improved predicted cost $J^*(x_0)$

and is likely (but not certain) to give better closed-loop performance

Example: Cost vs N for $x_0 = (-7.5, 0.5)$

N	6	7	8	11	> 11
$J^*(x_0)$	364.2	357.0	356.3	356.0	356.0
$J_{cl}(x_0)$	356.0	356.0	356.0	356.0	356.0

Closed loop cost: $J_{cl}(x_0) := \sum_{k=0}^{\infty} (\|x_k\|_Q^2 + \|u_k\|_R^2)$

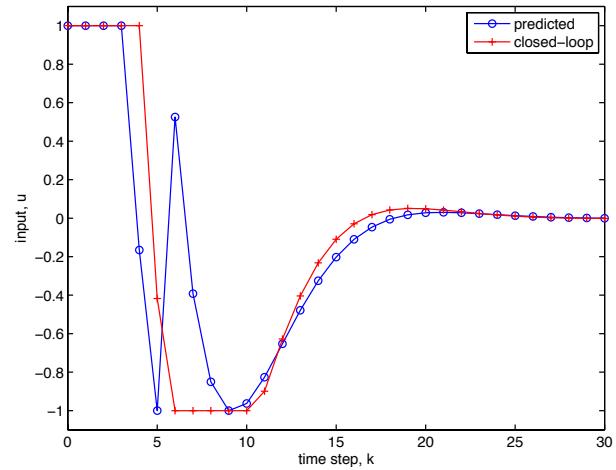
For this initial condition:

MPC with $N = 11$ is identical to constrained LQ optimal control ($N = \infty$)!

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Closed loop performance – example

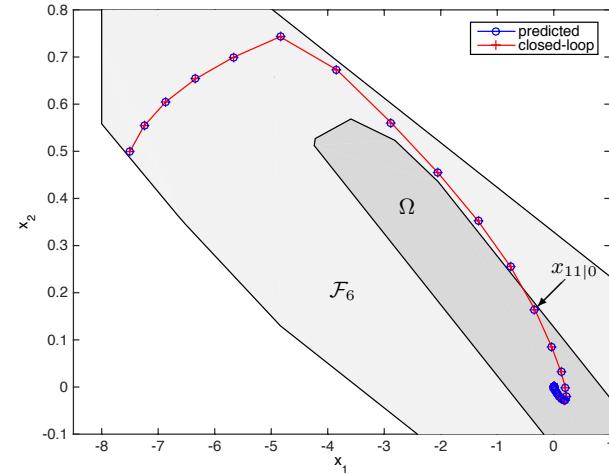
Predicted and closed loop inputs for $N = 6$



3 - 29

Closed loop performance – example

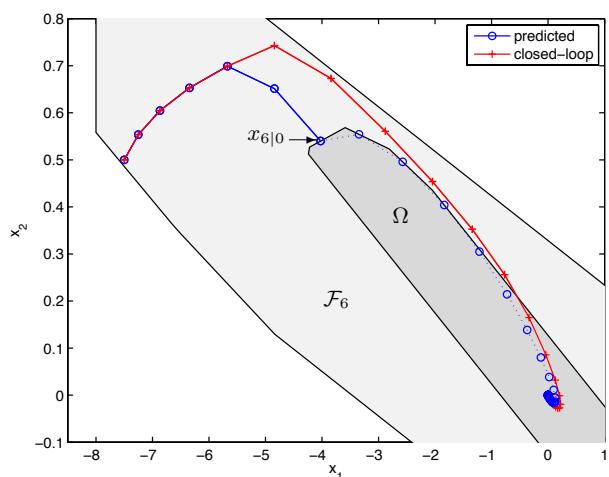
Predicted and closed loop states for $N = 11$



3 - 29

Closed loop performance – example

Predicted and closed loop states for $N = 6$



3 - 29

Choice of mode 1 horizon – performance

▷ For this $x_0: N = 11 \Rightarrow x_{N|0}$ lies in the interior of Ω



terminal constraint is inactive



no reduction in cost for $N > 11$

▷ Constrained LQ optimal performance is always obtained with $N \geq N_\infty$ for some finite N_∞ dependent on x_0

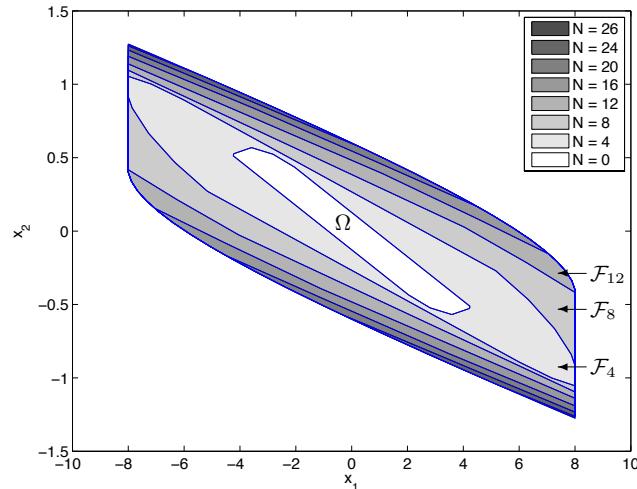
▷ N_∞ may be large, implying high computational load
but closed loop performance is often close to optimal for $N < N_\infty$
(due to receding horizon)

in this example $J_{\text{cl}}(x_0) \approx$ optimal for $N \geq 6$

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Choice of mode 1 horizon – region of attraction

Increasing N increases the feasible set \mathcal{F}_N



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Lecture 4

Robustness to disturbances

4 - 1

Summary

- ▷ Linear MPC ingredients:
 - * Infinite cost horizon (via terminal cost)
 - * Terminal constraints (via constraint-checking horizon)
- ▷ Constraints are satisfied over an infinite prediction horizon
- ▷ Closed-loop system is asymptotically stable with region of attraction equal to the set of feasible initial conditions
- ▷ Ideal optimal performance if mode 1 horizon N is large enough (but finite)

Robustness to disturbances

- Review of nominal model predictive control
- Setpoint tracking and integral action
- Robustness to unknown disturbances
- Handling time-varying disturbances

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4 - 2

Review

MPC with guaranteed stability – the basic idea



4 - 3

Review

MPC optimization for **nonlinear** model $x_{k+1} = f(x_k, u_k)$

$$\begin{aligned} & \underset{\mathbf{u}_k}{\text{minimize}} \quad \sum_{i=0}^{N-1} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) + \|x_{N|k}\|_P^2 \\ & \text{subject to} \quad \underline{u} \leq u_{i|k} \leq \bar{u}, \quad i = 0, \dots, N-1 \\ & \quad \underline{x} \leq x_{i|k} \leq \bar{x}, \quad i = 1, \dots, N-1 \\ & \quad x_{N|k} \in \Omega \end{aligned}$$

with

- * mode 2 feedback: $u_{i|k} = \kappa(x_{i|k})$ asymptotically stabilizes $x = 0$ (locally)
 - * terminal cost: $\|x_{N|k}\|_P^2 \geq \sum_{i=N}^{\infty} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2)$
for mode 2 dynamics: $x_{i+1|k} = f(x_{i|k}, \kappa(x_{i|k}))$
 - * terminal constraint set Ω : invariant for mode 2 dynamics and constraints
- $$\left. \begin{array}{l} f(x, \kappa(x)) \in \Omega \\ \underline{u} \leq \kappa(x) \leq \bar{u}, \quad \underline{x} \leq x \leq \bar{x} \end{array} \right\} \text{ for all } x \in \Omega$$

4 - 5

Review

MPC optimization for linear model $x_{k+1} = Ax_k + Bu_k$

$$\begin{aligned} & \underset{\mathbf{u}_k}{\text{minimize}} \quad \sum_{i=0}^{N-1} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) + \|x_{N|k}\|_P^2 \\ & \text{subject to} \quad \underline{u} \leq u_{i|k} \leq \bar{u}, \quad i = 0, \dots, N+N_c \\ & \quad \underline{x} \leq x_{i|k} \leq \bar{x}, \quad i = 1, \dots, N+N_c \end{aligned}$$

where

- * $u_{i|k} = Kx_{i|k}$ for $i \geq N$, with K = unconstrained LQ optimal
 - * terminal cost: $\|x_{N|k}\|_P^2 = \sum_{i=N}^{\infty} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2)$, with
 $P - \Phi^T P \Phi = Q + K^T R K, \quad \Phi = A + BK$
 - * terminal constraints are defined by the constraint checking horizon N_c :
- $$\left. \begin{array}{l} \underline{u} \leq K \Phi^i x \leq \bar{u} \\ \underline{x} \leq \Phi^i x \leq \bar{x} \end{array} \right\} \quad i = 0, \dots, N_c \quad \Rightarrow \quad \left. \begin{array}{l} \underline{u} \leq K \Phi^{N_c+1} x \leq \bar{u} \\ \underline{x} \leq \Phi^{N_c+1} x \leq \bar{x} \end{array} \right.$$

4 - 4

Comparison

▷ Linear MPC

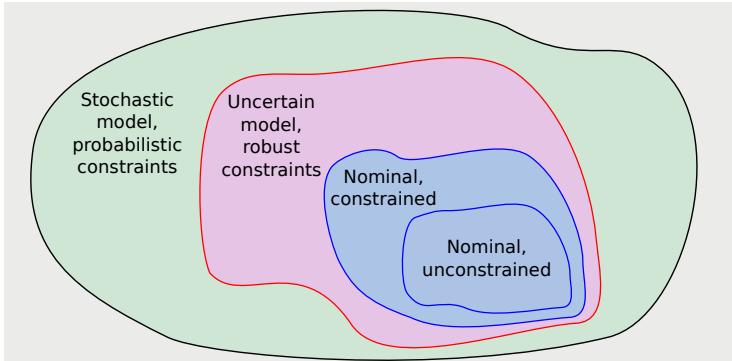
- | | | |
|-------------------------|---|---|
| terminal cost | ← | exact cost over the mode 2 horizon |
| terminal constraint set | ← | contains all feasible initial conditions for mode 2 |

▷ Nonlinear MPC

- | | | |
|-------------------------|---|---|
| terminal cost | ← | upper bound on cost over mode 2 horizon |
| terminal constraint set | ← | invariant set (usually not the largest) for mode 2 dynamics and constraints |

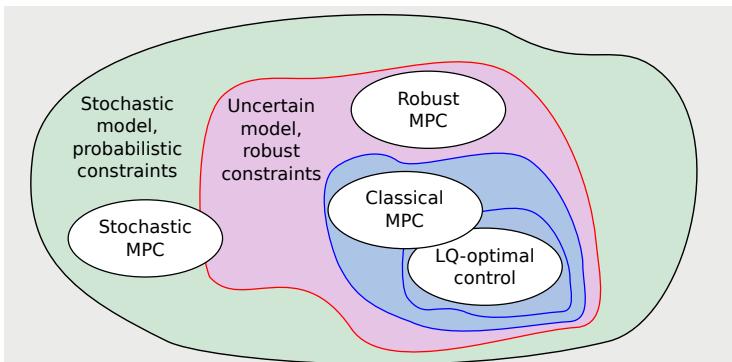
4 - 6

Model uncertainty



4 - 7

Model uncertainty



4 - 7

Model uncertainty

Common causes of model error and uncertainty:

- ▶ Unknown or time-varying model parameters
 - ▷ unknown loads & inertias, static friction
 - ▷ unknown d.c. gain
- ▶ Random (stochastic) model parameters
 - ▷ random process noise or sensor noise
- ▶ Incomplete measurement of states
 - ▷ state estimation error

4 - 7

Setpoint tracking

Output setpoint: y^0

$$y \rightarrow y^0 \Rightarrow \begin{cases} x \rightarrow x^0 \\ u \rightarrow u^0 \end{cases} \text{ where } \begin{aligned} x^0 &= Ax^0 + Bu^0 \\ y^0 &= Cx^0 \\ &\downarrow \\ y^0 &= C(I - A)^{-1}Bu^0 \end{aligned}$$

Setpoint for (u^0, x^0) is unique iff $C(I - A)^{-1}B$ is invertible

e.g. if $\dim(u) = \dim(y)$, then $\begin{cases} u^0 = (C(I - A)^{-1}B)^{-1}y^0 \\ x^0 = (I - A)^{-1}Bu^0 \end{cases}$

Tracking problem: $y_k \rightarrow y^0$ subject to $\begin{cases} \underline{u} \leq u_k \leq \bar{u} \\ \underline{x} \leq x_k \leq \bar{x} \end{cases}$
is only feasible if $\underline{u} \leq u^0 \leq \bar{u}$ and $\underline{x} \leq x^0 \leq \bar{x}$

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Setpoint tracking

Unconstrained tracking problem:

$$\underset{\mathbf{u}_k^\delta}{\text{minimize}} \quad \sum_{i=0}^{\infty} (\|x_{i|k}^\delta\|_Q^2 + \|u_{i|k}^\delta\|_R^2)$$

where $x^\delta = x - x^0$
 $u^\delta = u - u^0$

solution: $u_k = Kx_k^\delta + u^0, \quad K = K_{LQ}$

Constrained tracking problem:

$$\underset{\mathbf{u}_k^\delta}{\text{minimize}} \quad \sum_{i=0}^{\infty} (\|x_{i|k}^\delta\|_Q^2 + \|u_{i|k}^\delta\|_R^2)$$

subject to $\underline{u} \leq u_{i|k}^\delta + u^0 \leq \bar{u}, \quad i = 0, 1, \dots$
 $\underline{x} \leq x_{i|k}^\delta + x^0 \leq \bar{x}, \quad i = 1, 2, \dots$

solution: $u_k = u_{0|k}^{\delta*} + u^0$

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Setpoint tracking

If \hat{u}^0 is used instead of u^0

(e.g. d.c. gain $C(I - A)^{-1}B$ unknown)

then $u_k = u_{0|k}^{\delta*} + \hat{u}^0$ implies

$$u_k^\delta = u_{0|k}^{\delta*} + (\hat{u}^0 - u^0)$$

$$x_{k+1}^\delta = Ax_k^\delta + Bu_{0|k}^{\delta*} + B\underbrace{(\hat{u}^0 - u^0)}_{\text{constant disturbance}}$$

and if $u_{0|k}^{\delta*} \rightarrow Kx_k^\delta$ as $k \rightarrow \infty$, then

$$\lim_{k \rightarrow \infty} x_k^\delta = (I - A - BK)^{-1}B(\hat{u}^0 - u^0) \neq 0$$

$$\lim_{k \rightarrow \infty} y_k - y^0 = \underbrace{C(I - A - BK)^{-1}B(\hat{u}^0 - u^0)}_{\text{steady state tracking error}} \neq 0$$

Additive disturbances

Convert (constant) setpoint tracking problem into a regulation problem:

$$x \leftarrow x^\delta, \quad y \leftarrow y^\delta, \quad u \leftarrow u^\delta$$

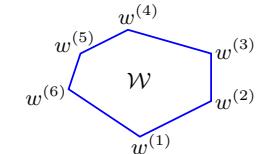
Consider the effect of additive disturbance w :

$$x_{k+1} = Ax_k + Bu_k + Dw_k,$$

$$y_k = Cx_k$$

Assume that w_k is unknown at time k , but is known to be

- ★ constant: $w_k = w$ for all k
 - ★ or time-varying within a known polytopic set: $w_k \in \mathcal{W}$ for all k
- where $\mathcal{W} = \text{conv}\{w^{(1)}, \dots, w^{(r)}\}$
 or $\mathcal{W} = \{w : Hw \leq \mathbf{1}\}$



4 - 11

Integral action (no constraints)

Introduce integral action to remove steady state error in y by considering the augmented system:

$$z_k = \begin{bmatrix} x_k \\ v_k \end{bmatrix}, \quad z_{k+1} = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} z_k + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} D \\ 0 \end{bmatrix} w_k$$

v_k = integrator state
 $v_{k+1} = v_k + y_k$

★ Linear feedback $u_k = Kx_k + K_I v_k$

is stabilizing if $\left| \text{eig} \left(\begin{bmatrix} A + BK & BK_I \\ C & I \end{bmatrix} \right) \right| < 1$

★ If the closed-loop system is (strictly) stable and $w_k \rightarrow w = \text{constant}$ then $u_k \rightarrow u^{ss} \implies v_k \rightarrow v^{ss} \implies y_k \rightarrow 0$ even if $w \neq 0$

... but arbitrary K_I may destabilize the closed loop system

4 - 10

4 - 12

Integral action (no constraints)

Ensure stability by using a modified cost:

$$\underset{\mathbf{u}_k}{\text{minimize}} \quad \sum_{i=0}^{\infty} (\|z_{i|k}\|_{Q_z}^2 + \|u_{i|k}\|_R^2) \quad Q_z = \begin{bmatrix} Q & 0 \\ 0 & Q_I \end{bmatrix}$$

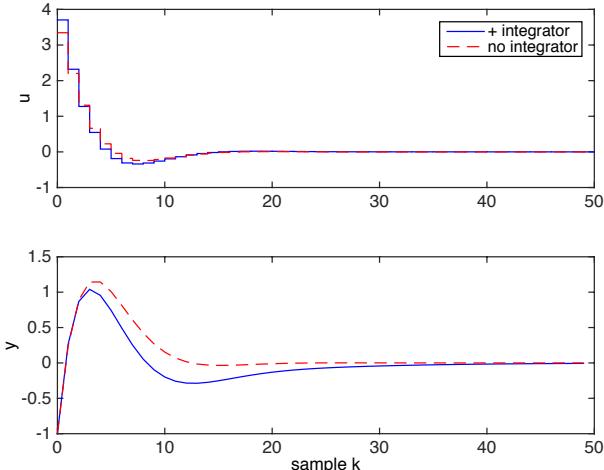
with predictions generated by an augmented model

$$z_{i+1|k} = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} z_{i|k} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_{i|k}, \quad z_{0|k} = \begin{bmatrix} x_k \\ v_k \end{bmatrix}$$

- * this is a nominal prediction model since $w_k = 0$ is assumed
- * unconstrained solution: $u_k = K_z z_k = K_x x_k + K_I v_k$
- * if $\left(\begin{bmatrix} A & 0 \\ C & I \end{bmatrix}, \begin{bmatrix} Q & 0 \\ 0 & Q_I \end{bmatrix} \right)$ is observable and $w_k \rightarrow w = \text{constant}$
then $u_k \rightarrow u^{ss} \implies v_k \rightarrow v^{ss} \implies y_k \rightarrow 0$
even if $w \neq 0!!$

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Integral action – example



Closed loop response for initial condition: $x_0 = (0.5, -0.5)$
no disturbance: $w = 0$

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Integral action – example

Plant model:

$$x_{k+1} = Ax_k + Bu_k + Dw \quad y_k = Cx_k$$

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} \quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = [-1 \quad 1]$$

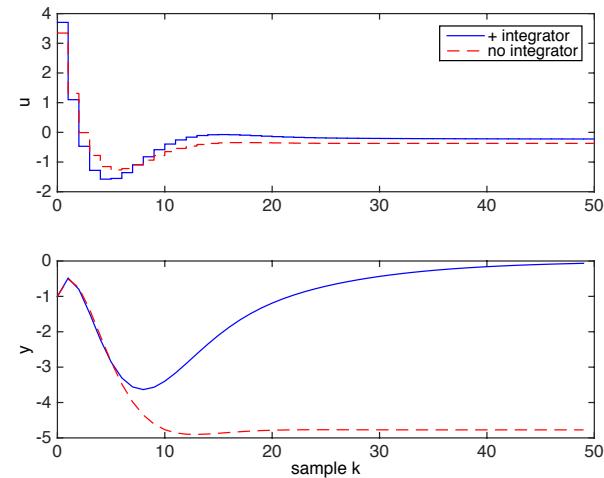
Constraints: none

Cost weighting matrices: $Q_z = \begin{bmatrix} C^T C & 0 \\ 0 & 0.01 \end{bmatrix}$, $R = 1$

Unconstrained LQ optimal feedback gain:

$$K_z = [-1.625 \quad -9.033 \quad 0.069]$$

Integral action – example



Closed loop response for initial condition: $x_0 = (0.5, -0.5)$
constant disturbance: $w = 0.75$

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Constrained MPC

Naive constrained MPC strategy: $w = 0$ assumed in predictions

$$\underset{\mathbf{u}_k}{\text{minimize}} \quad \sum_{i=0}^{N-1} (\|z_{i|k}\|_{Q_z}^2 + \|u_{i|k}\|_R^2) + \|z_{N|k}\|_P^2$$

$$\begin{aligned} \text{subject to} \quad & \underline{u} \leq u_{i|k} \leq \bar{u}, \quad i = 0, \dots, N + N_c \\ & \underline{x} \leq x_{i|k} \leq \bar{x}, \quad i = 1, \dots, N + N_c \end{aligned}$$

with: P and N_c determined for mode 2 control law $u_{i|k} = K_z z_{i|k}$

$$\text{initial prediction state: } z_{0|k} = \begin{bmatrix} x_k \\ v_k \end{bmatrix} \text{ where } v_{k+1} = v_k + y_k$$

* If closed loop system is stable

$$\text{then } u_k \rightarrow u^{ss} \implies v_k \rightarrow v^{ss} \implies y_k \rightarrow 0$$

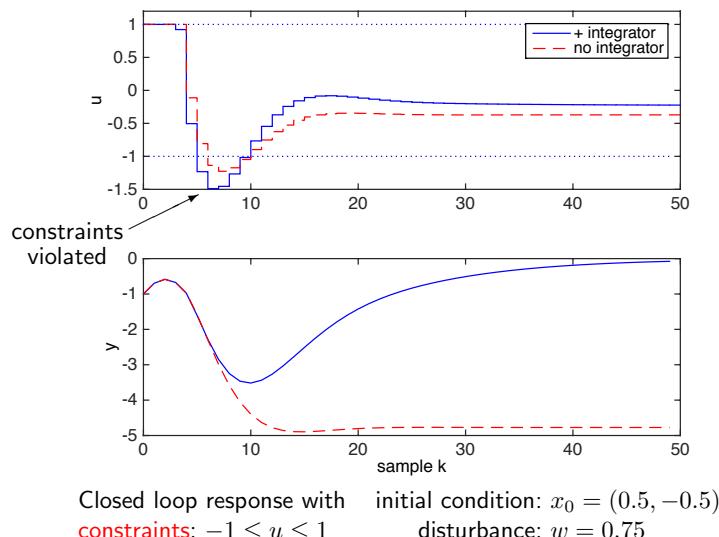
* but disturbance w_k is ignored in predictions, so

$$\begin{cases} J^*(z_{k+1}) - J^*(z_k) \not\leq 0 \\ \text{feasibility at time } k \not\Rightarrow \text{feasibility at } k+1 \end{cases}$$

therefore no guarantee of stability

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Constrained MPC – example



Robust constraints

If predictions satisfy constraints $\begin{cases} \text{for all prediction times } i = 0, 1, \dots \\ \text{for all disturbances } w_i \in \mathcal{W} \end{cases}$

then feasibility of constraints at time k ensures feasibility at time $k+1$

▷ Linear dynamics plus additive disturbance enables decomposition

nominal predicted state	$s_{i k}$
uncertain predicted state	$e_{i k}$

where

$$\begin{aligned} x_{i|k} &= s_{i|k} + e_{i|k} & \begin{cases} s_{i+1|k} = \Phi s_{i|k} + B c_{i|k} \\ e_{i+1|k} = \Phi e_{i|k} + D w_{i|k} \end{cases} & s_{0|k} = x_k \\ & & & e_{0|k} = 0 \end{aligned}$$

▷ Pre-stabilized predictions:

$$u_{i|k} = K x_{i|k} + c_{i|k} \text{ and } \Phi = A + BK$$

where $K = K_{LQ}$ is the unconstrained LQ optimal gain

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Pre-stabilized predictions – example

Scalar system: $x_{k+1} = \sum_{i=1}^{i-1} 2^i w + x_k + u_k + w_k$, constraint: $|x_k| \leq 2$
 uncertainty: $e_{i|k} = \sum_{j=0}^{j=i} 2^j w = (2^i - 1)w$, disturbance: $w_k = w$
 $|w| \leq 1$

Robust constraints:

$$|s_{i|k} + e_{i|k}| \leq 2 \text{ for all } |w| \leq 1$$

⇓

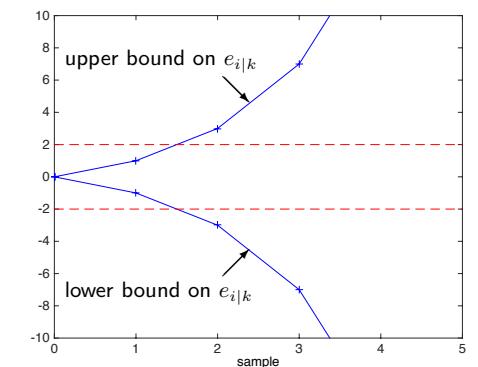
$$|s_{i|k}| \leq 2 - \max_{|w| \leq 1} |e_{i|k}|$$

⇓

$$|s_{i|k}| \leq 2 - (2^i - 1)$$

⇓

infeasible for all $i > 1$



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Pre-stabilized predictions – example

Avoid infeasibility by using pre-stabilized predictions:

$$u_{i|k} = Kx_{i|k} + c_{i|k}, \quad K = -1.9, \quad c_{i|k} = \begin{cases} \text{free} & i = 0, \dots, N-1 \\ 0 & i \geq N \end{cases}$$

$$\text{stable predictions: } e_{i|k} = \sum_{j=0}^{i-1} 0.1^j w = (1 - 0.1^i)w/0.9, \quad |w| \leq 1$$

Robust constraints:

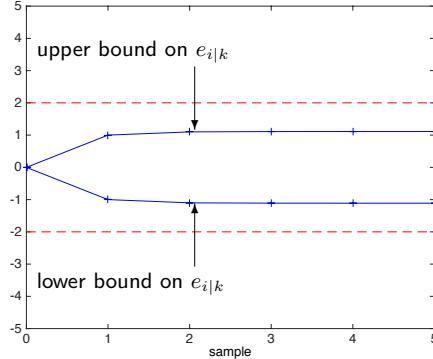
$$|s_{i|k} + e_{i|k}| \leq 2 \text{ for all } |w| \leq 1$$

\Downarrow

$$|s_{i|k}| \leq 2 - \max_{|w| \leq 1} |e_{i|k}|$$

\Downarrow

$$|s_{i|k}| \leq 2 - \underbrace{(1 - 0.1^i)/0.9}_{>0 \text{ for all } i}$$



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General form of robust constraints

How can we impose (general linear) constraints robustly?

* Pre-stabilized predictions:

$$\begin{aligned} x_{i|k} &= s_{i|k} + e_{i|k} & s_{0|k} &= x_k \\ e_{i+1|k} &= \Phi e_{i|k} + D w_{i|k} & e_{0|k} &= 0 \\ \implies e_{i|k} &= D w_{i-1} + \Phi D w_{i-2} + \dots + \Phi^{i-1} D w_0 \end{aligned}$$

* General linear constraints: $F x_{i|k} + G u_{i|k} \leq \mathbf{1}$

are equivalent to [tightened constraints](#) on nominal predictions:

$$(F + GK)s_{i|k} + Gc_{i|k} \leq \mathbf{1} - h_i$$

where $h_0 = 0$

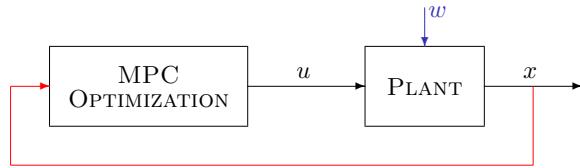
$$h_i = \max_{w_0, \dots, w_{i-1} \in \mathcal{W}} (F + GK)e_{i|k}, \quad i = 1, 2, \dots$$

(i.e. $h_i = h_{i-1} + \max_{w \in \mathcal{W}} (F + GK)w$
requiring one LP for each row of h_i)

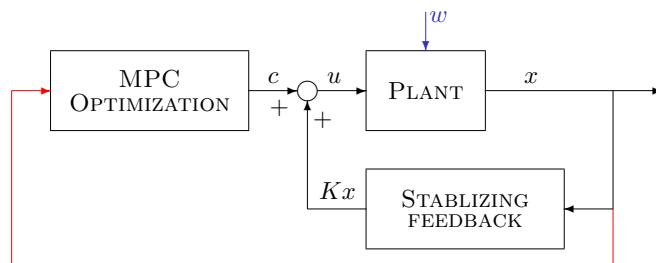
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Pre-stabilized predictions

▷ Feedback structure of MPC with open loop predictions:



▷ Feedback structure of MPC with pre-stabilized predictions:



Tube interpretation

The uncertainty in predictions: $e_{i+1|k} = \Phi e_{i|k} + D w_i, w_i \in \mathcal{W}$
evolves inside a [tube](#) (a sequence of sets): $e_{i|k} \in E_{i|k}$, where

$$E_{i|k} = DW \oplus \Phi DW \oplus \dots \oplus \Phi^{i-1} DW, \quad i = 1, 2, \dots$$

Hence we can define:

* a state tube $x_{i|k} = s_{i|k} + e_{i|k} \in \mathcal{X}_{i|k}$

$$\mathcal{X}_{i|k} = \{s_{i|k}\} \oplus E_{i|k}, \quad i = 0, 1, \dots$$

* a control input tube

$$u_{i|k} = Kx_{i|k} + c_{i|k} = Ks_{i|k} + c_{i|k} + Ke_{i|k} \in \mathcal{U}_{i|k}$$

$$\mathcal{U}_{i|k} = \{Ks_{i|k} + c_{i|k}\} \oplus KE_{i|k}, \quad i = 0, 1, \dots$$

and impose constraints robustly for the state and input tubes

(where \oplus is Minkowski set addition)

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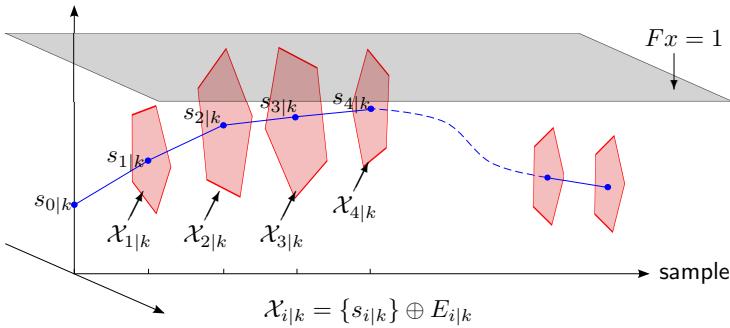
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Tube interpretation

The uncertainty in predictions: $e_{i+1|k} = \Phi e_{i|k} + D w_i$, $w_i \in \mathcal{W}$ evolves inside a **tube** (a sequence of sets): $e_{i|k} \in E_{i|k}$, where

$$E_{i|k} = D\mathcal{W} \oplus \Phi D\mathcal{W} \oplus \cdots \oplus \Phi^{i-1} D\mathcal{W}, \quad i = 1, 2, \dots$$

e.g. for constraints $Fx \leq \mathbf{1}$ ($G = 0$)



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Robust MPC

Prototype robust MPC algorithm

Offline: compute N_c and h_1, \dots, h_{N_c} . Online at $k = 0, 1, \dots$:

(i). solve

$$\mathbf{c}_k^* = \arg \min_{\mathbf{c}_k} J(x_k, \mathbf{c}_k)$$

$$\text{s.t. } (F + GK)s_{i|k} + Gc_{i|k} \leq \mathbf{1} - h_i, \quad i = 0, \dots, N + N_c$$

(ii). apply $u_k = Kx_k + c_{0|k}^*$ to the system

two alternative cost functions:

* nominal cost (i.e. cost evaluated assuming $w_i = 0$ for all i)

$$J(x_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} (\|s_{i|k}\|_Q^2 + \|Ks_{i|k} + c_{i|k}\|_R^2) = \|x_k\|_P^2 + \|\mathbf{c}_k\|_{W_c}^2$$

* worst case cost, defined in terms of a desired disturbance gain γ :

$$J(x_k, \mathbf{c}_k) = \max_{w_i \in \mathcal{W}, i=0,1,\dots} \sum_{i=0}^{\infty} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - \gamma^2 \|w_i\|^2)$$

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Robust MPC

Prototype robust MPC algorithm

Offline: compute N_c and h_1, \dots, h_{N_c} . Online at $k = 0, 1, \dots$:

(i). solve

$$\mathbf{c}_k^* = \arg \min_{\mathbf{c}_k} J(x_k, \mathbf{c}_k)$$

$$\text{s.t. } (F + GK)s_{i|k} + Gc_{i|k} \leq \mathbf{1} - h_i, \quad i = 0, \dots, N + N_c$$

(ii). apply $u_k = Kx_k + c_{0|k}^*$ to the system

* tightened linear constraints are applied to nominal predictions

* N_c is the constraint-checking horizon:

$$(F + GK)\Phi^i s \leq \mathbf{1} - h_i, \quad i = 0, \dots, N_c \implies (F + GK)\Phi^{N_c+1}s \leq \mathbf{1} - h_{N_c+1}$$

* the online optimization is robustly recursively feasible

Convergence of robust MPC with nominal cost

If $u_{i|k} = Kx_{i|k} + c_{i|k}$ for $K = K_{LQ}$, then:

* the unconstrained optimum is $\mathbf{c}_k = 0$, so the nominal cost is

$$J(x_k, \mathbf{c}_k) = \|x_k\|_P^2 + \|\mathbf{c}_k\|_{W_c}^2$$

and W_c is block-diagonal: $W_c = \text{diag}\{P_c, \dots, P_c\}$

* recursive feasibility $\Rightarrow \tilde{\mathbf{c}}_{k+1} = (c_{1|k}^*, \dots, c_{N-1|k}^*, 0)$ feasible at $k+1$

* hence $\|\mathbf{c}_{k+1}^*\|_{W_c}^2 \leq \|\mathbf{c}_k^*\|_{W_c}^2 - \|c_{0|k}^*\|_{P_c}^2$

$$\Rightarrow \sum_{k=0}^{\infty} \|c_{0|k}\|_{P_c}^2 \leq \|\mathbf{c}_0^*\|_{W_c}^2 < \infty$$

$$\Rightarrow \lim_{k \rightarrow \infty} c_{0|k} = 0$$

* therefore $u_k \rightarrow Kx_k$ as $k \rightarrow \infty$

$x_k \rightarrow$ the (minimal) robustly invariant set under unconstrained LQ optimal feedback

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Robust MPC with constant disturbance

Assume $w_k = w = \text{constant for all } k$

combine: pre-stabilized predictions
augmented state space model

- * Predicted state and input sequences:

$$x_{i|k} = [I \ 0] (s_{i|k} + e_{i|k})$$

$$u_{i|k} = K_z(s_{i|k} + e_{i|k}) + c_{i|k}$$

- * Prediction model:

$$\text{nominal} \quad s_{i+1|k} = \Phi s_{i|k} + \begin{bmatrix} B \\ 0 \end{bmatrix} c_{i|k} \quad \Phi = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} K_z$$

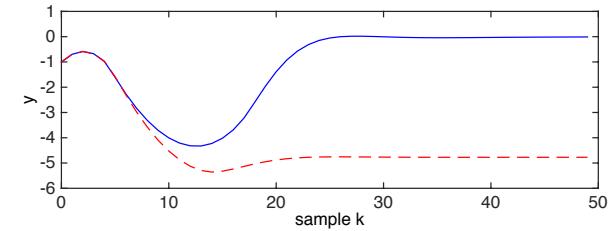
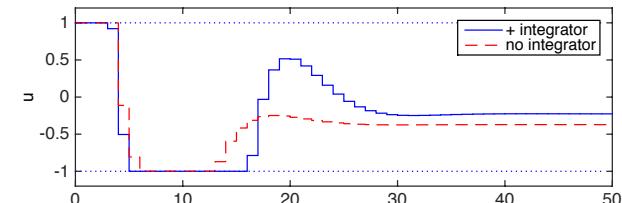
$$\text{uncertain} \quad e_{i|k} = \sum_{j=0}^{i-1} \Phi^j \begin{bmatrix} D \\ 0 \end{bmatrix} w \quad s_{0|k} = \begin{bmatrix} x_k \\ v_k \end{bmatrix}, \quad e_{0|k} = 0$$

- * Nominal cost:

$$J(x_k, v_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} (\|s_{i|k}\|_{Q_z}^2 + \|K_z s_{i|k} + c_{i|k}\|_R^2)$$

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Robust MPC with constant disturbance – example



Closed loop response with initial condition: $x_0 = (0.5, -0.5)$
constraints: $-1 \leq u \leq 1$ disturbance: $w = 0.75$

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Robust MPC with constant disturbance

Assume $w_k = w = \text{constant for all } k$

combine: pre-stabilized predictions
augmented state space model

- * robust state constraints:

$$\underline{x} \leq x_{i|k} \leq \bar{x} \iff \underline{x} + h_i \leq s_{i|k} \leq \bar{x} - h_i$$

$$h_i = \max_{w \in \mathcal{W}} [I \ 0] \sum_{j=0}^{i-1} \Phi^j \begin{bmatrix} D \\ 0 \end{bmatrix} w$$

- * robust input constraints:

$$\underline{u} \leq u_{i|k} \leq \bar{u} \iff \underline{u} + h'_i \leq K_z s_{i|k} + c_{i|k} \leq \bar{u} - h'_i$$

$$h'_i = \max_{w \in \mathcal{W}} K_z \sum_{j=0}^{i-1} \Phi^j \begin{bmatrix} D \\ 0 \end{bmatrix} w$$

- * N_c and h_i, h'_i for $i = 1, \dots, N_c$ can be computed offline

Summary

- ▷ Integral action: augment model with integrated output error
include integrated output error in cost

then

- (i). closed loop system is stable if $w = 0$
- (ii). steady state error must be zero if response is stable for $w \neq 0$

- ▷ Robust MPC: use pre-stabilized predictions
apply constraints for all possible future uncertainty

then

- (i). constraint feasibility is guaranteed at all times if initially feasible
- (ii). closed loop system inherits the stability and convergence properties of unconstrained LQ optimal control (assuming nominal cost)

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Overview of the course

① Introduction and Motivation

Basic MPC strategy; prediction models; input and state constraints;
constraint handling: saturation, anti-windup, predictive control

② Prediction and optimization

Input/state prediction equations; unconstrained optimization.
Infinite horizon cost; dual mode predictions. Incorporating
constraints; quadratic programming.

③ Closed loop properties

Lyapunov analysis based on predicted cost. Recursive feasibility;
terminal constraints; the constraint checking horizon. Constrained
LQ-optimal control.

④ Robustness to disturbances

Setpoint tracking; MPC with integral action. Robustness to
constant disturbances: prestabilized predictions and robust
feasibility. Handling time-varying disturbances.