C24 DS2 Dynamical Systems 2

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You are expected to refer to textbooks as well as the lecture notes when attempting the questions on this sheet. Some questions suggest using numerical methods in MATLAB or Mathematica.

Questions

1. Asymptotic Behaviour

(a) Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + (1 - x_1^2 - x_2^2)x_2.\end{aligned}$$

(i) Is the origin stable?

(ii) Show that this system has a limit cycle and find it.

(iii) Show that all trajectories not starting from the origin converge to the limit cycle.

(b) Show that the $(0,0)$ equilibrium of

$$\begin{aligned}\dot{x} &= -x^3 + 2y^3\\ \dot{y} &= -2xy^2\end{aligned}$$

is asymptotically stable, using the candidate Lyapunov function $V(x,y) = \frac{1}{2}(x^2 + y^2)$.

2. Constructing a Trapping Region

Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + ax_2 + x_1^2 x_2\\ \dot{x}_2 &= b - ax_2 - x_1^2 x_2\end{aligned}$$

where $a, b > 0$.

(a) Find the stability properties of the equilibrium at $\big(b, b/(a+b^2)\big)$. When is this unstable?

(b) Show that the region defined by $x_1 \geq 0$, $x_2 \geq 0$, $x_2 \leq b/a$ and $x_2 \leq -x_1 + (b/a + b)$ is a trapping region.

(c) Use Poincaré Bendixson to show that this system can admit a limit cycle.

3. Nonexistence of Periodic Orbits

(a) Show that the system

$$\dot{x}_1 = x_1(2 - x_1 - x_2)$$
$$\dot{x}_2 = x_2(4x_1 - x_1^2 - 3)$$

has no closed orbits in the first quadrant using Dulac's theorem with $B = 1/(x_1^a x_2^b)$ for some a, b. Find the four equilibria, classify their stability and sketch the phase portrait.

(b) Use the Dulac function $B(x, y) = be^{-2\beta x}$ to show that the system

$$\dot{x} = y$$
$$\dot{y} = -ax - by + \alpha x^2 + \beta y^2$$

has no limit cycle on the plane.

(c) Consider the system

$$\dot{x}_1 = -\varepsilon \left(\frac{x_1^3}{3} - x_1 + x_2 \right), \quad \varepsilon > 0$$
$$\dot{x}_2 = -x_1$$

Show that there is no limit cycle within the strip $|x_1| < 1$, using Bendixson's theorem.

4. Index Theory

(a) Using Index theory where possible, and Dulac-Bendixson otherwise, show that the following system

$$\dot{x}_1 = x_1(4 - x_2 - x_1^2)$$
$$\dot{x}_2 = x_2(x_1 - 1)$$

does not have any closed orbits.

(b) Show that the system

$$\dot{x}_1 = x_1 e^{-x_1}$$
$$\dot{x}_2 = 1 + x_1 + x_2^2$$

does not have any closed trajectories.

(c) A system has three closed trajectories, C_1, C_2 and C_3, all anticlockwise, with C_2 and C_3 enclosed by C_1. C_2 does not enclose C_3 and vice-versa. Show that there is at least one fixed point enclosed by C_1 but not by C_2 and C_3.

(d) Determine the index of the zero equilibrium of

$$\dot{x} = x^2$$
$$\dot{y} = -y$$

[Hint: plot the vector field in MATLAB or Mathematica.]

(e) Determine the index of the zero equilibrium of

$$\dot{x} = x^2 - y^2$$
$$\dot{y} = 2xy.$$

[Hint: plot the vector field in MATLAB or Mathematica.]

5. One-dimensional bifurcations

For the following systems, classify the number and type of bifurcations:

(a) $\dot{x} = \mu - x - e^{-x}$.

(b) $\dot{x} = -x + \mu \tanh x$.

(c) $\dot{x} = \mu x + x^4$.

6. Hopf bifurcations

Show that the following system

$$\dot{x}_1 = \mu x_1 - x_2 + \sigma x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + \mu x_2 + \sigma x_2(x_1^2 + x_2^2)$$

undergoes a Hopf bifurcation at $(0, 0)$ when $\mu = 0$ and that this bifurcation is

(a) Supercritical if $\sigma = -1$.

(b) Subcritical if $\sigma = 1$.

(c) Degenerate if $\sigma = 0$.

7. Real Quadratic Map

Consider the system

$$x \mapsto x^2 + c$$

(a) Find all equilibria as a function of c.

(b) Considering c as a bifurcation parameter, find values of c where bifurcations occur.

(c) For which values of c does this system admit a stable 2-cycle?

(d) Draw an orbit diagram for this map, using e.g. MATLAB or Mathematica

8. The Rikitake system

Consider the Rikitake system

$$\begin{aligned}\dot{x} &= -\mu x + zy \\ \dot{y} &= -\mu y + (z-a)x \\ \dot{z} &= 1 - xy\end{aligned}$$

(a) Show that the system is dissipative (i.e. that $\nabla \cdot \mathbf{f}(\mathbf{x}) < 0$ for all $\mathbf{x}$, where $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} = \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$).

(b) Find the fixed points and show that they can be written in the form $x^* = \pm k$, $y^* = \pm 1/k$ and $z^* = \mu k^2$ where $\mu(k^2 - k^{-2}) = a$.

(c) Classify the stability of the fixed points.

(d) Simulate this system with parameters $a = 5, \mu = 2$, using e.g. MATLAB or Mathematica.

Some Answers and Hints

1(b). Hint: use a Lyapunov function of the form $V(x,y) = \frac{1}{2}(x^2 + y^2)$.

3(a). $a = b = 1$.

4(a). Hint: in one of the quadrants you may have to use Dulac's theorem (see also 3(a)).

4(d). Index = 0.

5(a). A saddle-node bifurcation at $\mu = 1$.

5(b). A pitchfork bifurcation at $\mu = 1$.

6. Hint: change to polar coordinates.

7(c). Hint: consider the two-hop map, $x_{k+2} = f(f(x_k))$.

8(a). $\nabla \cdot \mathbf{f} = -2\mu$, hence the system is dissipative.