C21 Nonlinear Systems

Mark Cannon

4 lectures

Michaelmas Term 2018

Lecture 1

Introduction and Concepts of Stability

Organisation

▷ 4 lectures:	week 1	Monday	3-4 pm	LR2
		Thursday	3-4 pm	LR2
	week 2	Monday	3-4 pm	LR2
		Thursday	3-4 pm	LR2
▷ 1 class:	week 3	Thursday	4-5 pm or 5-6 pm	LR5
	or week 4	Friday	2-3 pm	LR6

Course outline

1. Types of stability
2. Linearization
3. Lyapunov's direct method
4. Regions of attraction
5. Linear systems and passive systems

Books

- J.-J. Slotine & W. Li *Applied Nonlinear Control*, Prentice-Hall 1991.
  - ⋆ Stability
  - ⋆ Interconnected systems and passive systems
- H.K. Khalil *Nonlinear Systems*, Prentice-Hall 1996.
  - ⋆ Stability
  - ⋆ Passive systems
- M. Vidyasagar *Nonlinear Systems Analysis*, Prentice-Hall 1993.
  - ⋆ Stability & passivity (more technical detail)

Why use nonlinear control?

- Real systems are nonlinear
  - ⋆ friction, non-ideal components
  - ⋆ actuator saturation
  - ⋆ sensor nonlinearity
- Analysis via linearization
  - ⋆ accuracy of approximation?
  - ⋆ conservative?
- Account for nonlinearities in high performance applications
  - ⋆ Robotics, Aerospace, Petrochemical industries, Process control, Power generation . . .
- Account for nonlinearities if linear models inadequate
  - ⋆ large operating region
  - ⋆ model properties change at linearization point

Linear vs nonlinear system properties

Free response

Linear system	Nonlinear system
$\boxed{\dot{x} = Ax}$	$\boxed{\dot{x} = f(x)}$
• Unique equilibrium point: $Ax = 0 \iff x = 0$	• Multiple equilibrium points $f(x) = 0$
• Stability independent of initial conditions	• Stability dependent on initial conditions

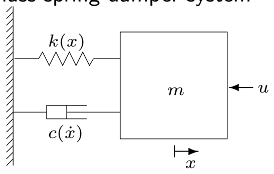
Linear vs nonlinear system properties

Forced response

Linear system	Nonlinear system
$\boxed{\dot{x} = Ax + Bu}$	$\boxed{\dot{x} = f(x, u)}$
• $\|u\|$ finite $\Rightarrow$ $\|x\|$ finite if open-loop stable	• $\|u\|$ finite $\not\Rightarrow$ $\|x\|$ finite
• Frequency response: $u = U \sin \omega t \;\Rightarrow\; x = X \sin(\omega t + \phi)$	• No frequency response $u = U \sin \omega t \;\not\Rightarrow\; x$ sinusoidal
• Superposition: $u = u_1 + u_2 \;\Rightarrow x = x_1 + x_2$	• No linear superposition $u = u_1 + u_2 \;\not\Rightarrow x = x_1 + x_2$

Example: step response

Mass-spring-damper system

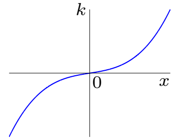


Equation of motion:

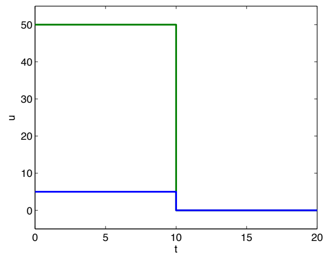
$$\ddot{x} + c(\dot{x}) + k(x) = u$$

$c(\dot{x}) = \dot{x}$

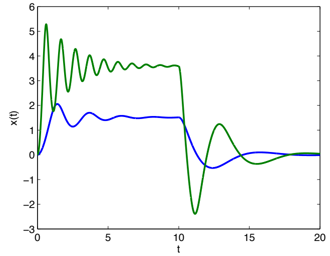
$k(x)$ nonlinear:



Input $u(t)$



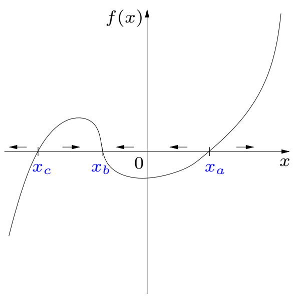
Response $x(t)$



apparent damping ratio depends on size of input step

Example: multiple equilibria

First order system: $\dot{x} = f(x)$



$x > x_a$	$\Longrightarrow$	$f(x) > 0$	$\Longrightarrow$	$x(t)$ increases
$x_b < x < x_a$	$\Longrightarrow$	$f(x) < 0$	$\Longrightarrow$	$x(t)$ decreases
$x_c < x < x_b$	$\Longrightarrow$	$f(x) > 0$	$\Longrightarrow$	$x(t)$ increases
$x < x_c$	$\Longrightarrow$	$f(x) < 0$	$\Longrightarrow$	$x(t)$ decreases

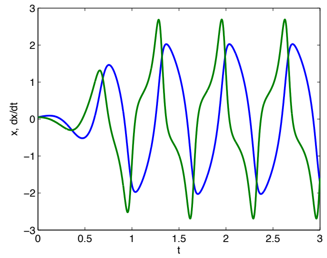
- x_a, x_c are unstable equilibrium points
- x_b is a stable equilibrium point

Example: limit cycle

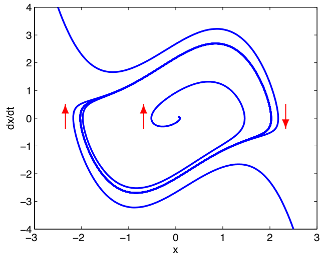
Van der Pol oscillator:

$$\ddot{x} + (x^2 - 1)\dot{x} + x = 0$$

- Response $x(t)$ tends to a limit cycle (= trajectory forming a closed curve)
- Amplitude independent of initial conditions



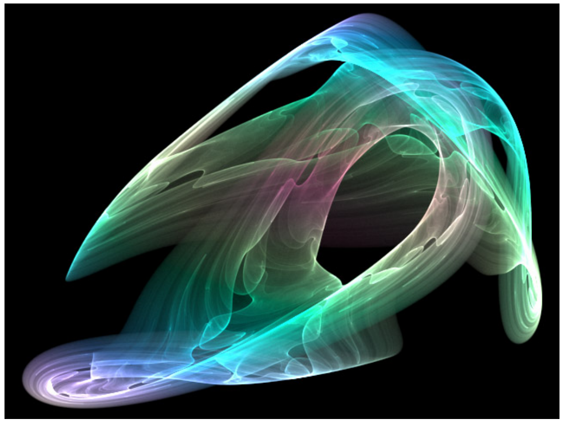
Response with $x(0) = 0.05$, $\dot{x}(0) = 0.05$



State trajectories $(x(t), \dot{x}(t))$

Example: chaotic behaviour

Strange attractor



Example: chaotic behaviour

Lorenz attractor

- Simplified model of atmospheric convection:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

- State variables

$x(t)$: fluid velocity

$y(t)$: difference in temperature of acsending and descending fluid

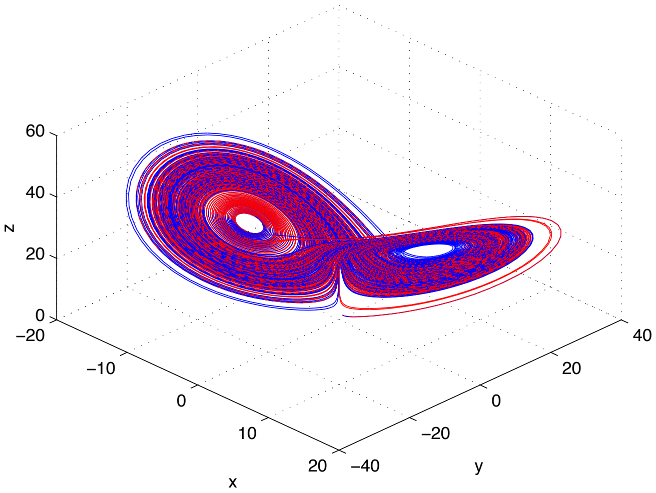
$z(t)$: characterizes distortion of vertical temperature profile

- Parameters $\sigma = 10,\ \beta = 8/3,\ \rho =$ variable

Example: chaotic behaviour

Lorenz attractor

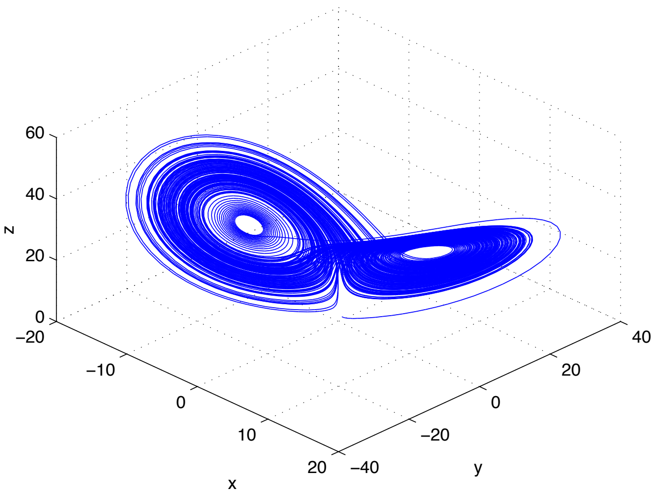
sensitivity to initial conditions



Example: chaotic behaviour

Lorenz attractor

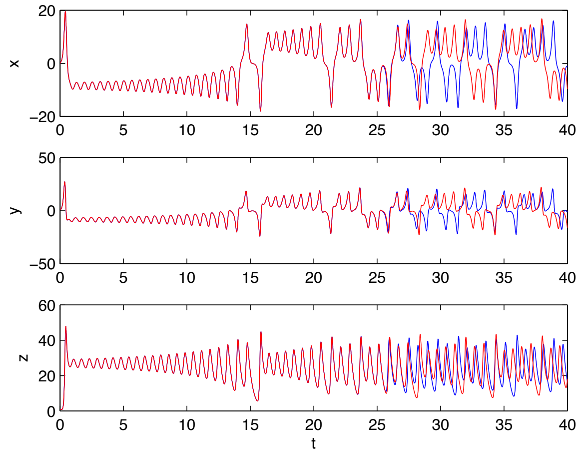
$\rho = 28 \implies$ "strange attractor":



Example: chaotic behaviour

Lorenz attractor

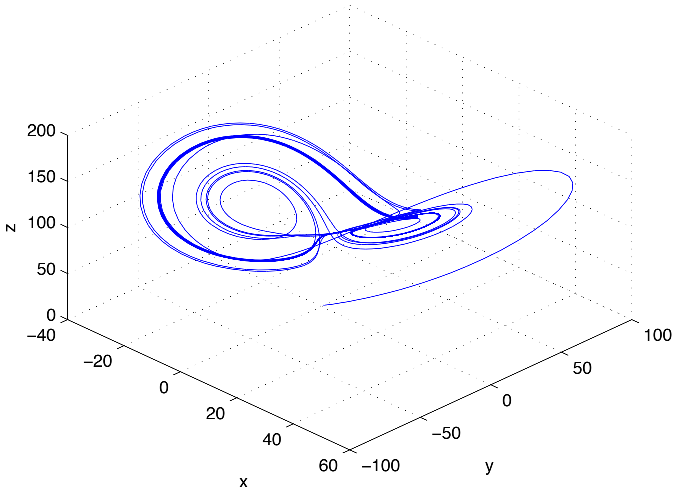
sensitivity to initial conditions blue: $(x, y, z) = (0, 1, 1.05)$
red: $(x, y, z) = (0, 1, 1.050001)$



Example: chaotic behaviour

Lorenz attractor

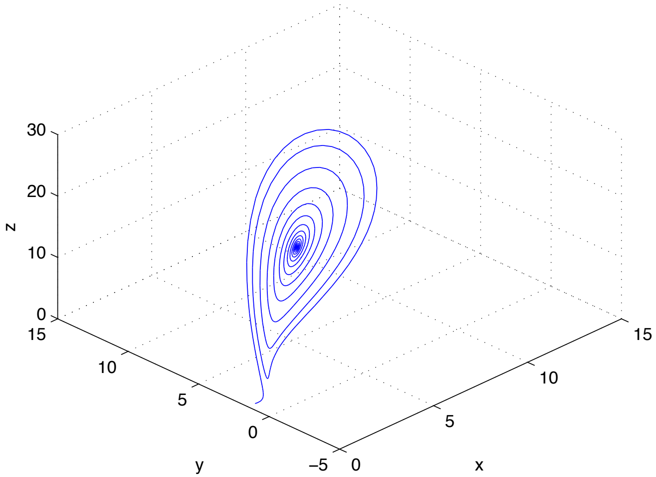
$\rho = 99.96 \implies$ limit cycle:



Example: chaotic behaviour

Lorenz attractor

$\rho = 14 \implies$ convergence to a stable equilibrium:



State space equations

$$\dot{x} = f(x, u, t) \qquad \begin{aligned} x &: \text{ state} \\ u &: \text{ input} \end{aligned}$$

e.g. nth order differential equation:

$$\frac{d^n y}{dt^n} = h\Big(y, \ldots, \frac{d^{n-1}y}{dt^{n-1}}, u, t\Big)$$

has state vector (one possible choice)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ d^{n-1}y/dt^{n-1} \end{bmatrix}$$

and state space dynamics:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ h\big(x_1, x_2, \ldots, x_n, u, t\big) \end{bmatrix} = f(x, u, t)$$

Equilibrium points

x^* is an equilibrium point of system $\dot{x} = f(x)$ iff:

$$x(0) = x^* \quad \text{implies} \quad x(t) = x^* \quad \forall t > 0$$

i.e. $f(x^*) = 0$

Examples:

(a) $\ddot{y} + \alpha\dot{y}^2 + \beta \sin y = 0$ (damped pendulum)

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad x^* = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}, \; n = 0, \pm 1$$

(b) $\ddot{y} + (y-1)^2\dot{y} + y - \sin(\pi y/2) = 0$

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \; \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- ⋆ Consider local stability of individual equilibrium points
- ⋆ Convention: define f so that $x = 0$ is equilibrium point of interest
- ⋆ Autonomous system: $\dot{x} = f(x) \implies x^* =$ constant

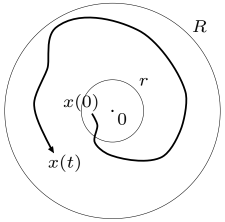
Stability definition

An equilibrium point $x = 0$ is stable iff:

$\max_t \|x(t)\|$ can be made arbitrarily small
by making $\|x(0)\|$ small enough

$$\Updownarrow$$

$$\boxed{\begin{array}{c}\text{for any } R > 0, \text{ there exists } r > 0 \text{ so that} \\ \|x(0)\| < r \implies \|x(t)\| < R \quad \forall t > 0\end{array}}$$



- Is $x = 0$ a stable equilibrium for the Van der Pol oscillator example?
- No guarantee of convergence to the equilibrium point

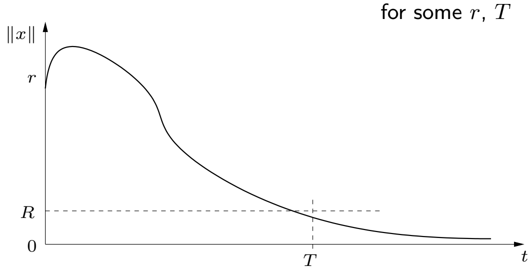
Asymptotic stability definition

An equilibrium point $x = 0$ is asymptotically stable iff:

$$\boxed{\begin{array}{rl}\text{(i).} & x = 0 \text{ is stable} \\ \text{(ii).} & \|x(0)\| < r \implies \|x(t)\| \to 0 \text{ as } t \to \infty\end{array}}$$

(ii) is equivalent to:

for any $R > 0$,
$\|x(0)\| < r \implies \|x(t)\| < R \quad \forall t > T$
for some r, T

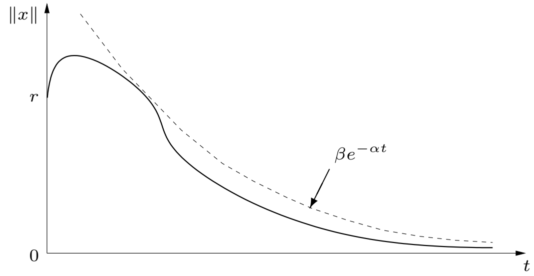


Exponential stability definition

An equilibrium point $x = 0$ is exponentially stable iff:

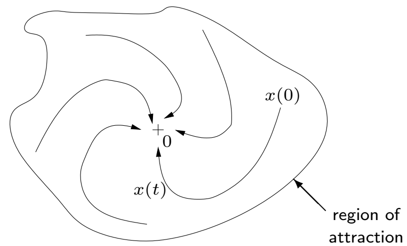
$$\boxed{\|x(0)\| < r \implies \|x(t)\| \le \beta e^{-\alpha t} \quad \forall t > 0}$$

exponential stability is a special case of asymptotic stability



Region of attraction

The region of attraction of $x = 0$ is the set of all initial conditions $x(0)$
for which $x(t) \to 0$ as $t \to \infty$



- Every asymptotically stable equilibrium point has a region of attraction
- $r = \infty \implies$ entire state space is a region of attraction
  $\implies x = 0$ is globally asymptotically stable
- Are stable linear systems asymptotically stable?

Summary

- Nonlinear state space equations: $\dot{x} = f(x, u)$
  x = state vector, u = control input
- Equilibrium points: x^* is an equilibrium point
  of $\dot{x} = f(x)$ if $f(x^*) = 0$
- Stable equilibrium point: x^* is stable if state trajectories starting
  close to x^* remain near x^* at all times
- Asymptotically stable equilibrium point: x^* must be stable and state trajectories starting near x^* must tend to x^* asympotically
- Region of attraction: the set of initial conditions from which state trajectories converge asymptotically to equilibrium x^*

Lecture 2

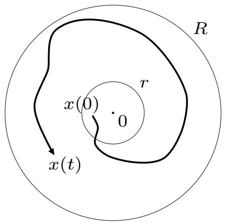
Linearization and Lyapunov's direct method

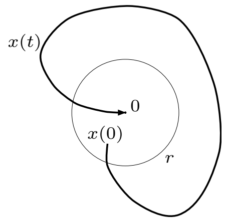
Linearization and Lyapunov's direct method

- Review of stability definitions
- Linearization method
- Direct method for stability
- Direct method for asymptotic stability
- Linearization method revisited

Review of stability definitions

System: $\dot{x} = f(x)$ ⋆ unforced system (i.e. closed-loop)
⋆ consider stability of individual equilibrium points





0 is a stable equilibrium if:

$$\|x(0)\| \leq r \implies \|x(t)\| \leq R$$
for any $R > 0$

0 is asymptotically stable if:

$$\|x(0)\| \leq r \implies \|x(t)\| \to 0$$
as $t \to \infty$

Stability	$\rightarrow$	local property
Asymptotic stability	$\rightarrow$	global if $r = \infty$ allowed

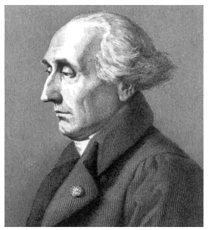
Historical development of Stability Theory

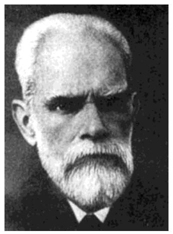
- Potential energy in conservative mechanics (Lagrange 1788):

  An equilibrium point of a conservative system is stable if it corresponds to a minimum of the potential energy stored in the system

- Energy storage analogy for general ODEs (Lyapunov 1892)

- Invariant sets (Lefschetz, La Salle 1960s)


J-L. Lagrange 1736-1813


A. M. Lyapunov 1857-1918


S. Lefschetz 1884-1972

Lyapunov's linearization method

- Determine stability of equilibrium at $x = 0$ by analyzing the stability of the linearized system at $x = 0$.

- Jacobian linearization:

$\dot{x} = f(x)$	original nonlinear dynamics
$= f(0) + \left.\frac{\partial f}{\partial x}\right\vert_{x=0} x + R_1$	Taylor's series expansion, $R_1 = O(\|x\|^2)$
$\approx Ax$	since $f(0) = 0$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \qquad \frac{\partial f}{\partial x} \text{ assumed continuous at } x = 0$$

Lyapunov's linearization method

Conditions on A for stability of original nonlinear system at $x = 0$:

stability of linearization	stability of nonlinear system at $x = 0$
$\text{Re}\big(\lambda(A)\big) < 0$	asymptotically stable (locally)
$\max \text{Re}\big(\lambda(A)\big) = 0$	stable or unstable
$\max \text{Re}\big(\lambda(A)\big) > 0$	unstable

Lyapunov's linearization method

- Some examples

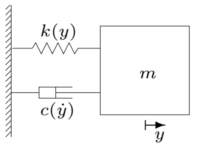
(stable)	$\dot{x} = -x^3$	$\xrightarrow{\text{linearize}}$	$\dot{x} = 0$	(indeterminate)
(unstable)	$\dot{x} = x^3$	$\xrightarrow{\text{linearize}}$	$\dot{x} = 0$	(indeterminate)

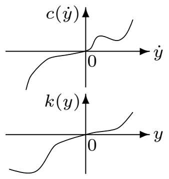
$\uparrow$
higher order terms determine stability

- Why does linear control work?
  1. Linearize the model:
  $$\begin{aligned}\dot{x} &= f(x,u)\\ &\approx Ax + Bu, \qquad A = \frac{\partial f}{\partial x}(0,0),\ B = \frac{\partial f}{\partial u}(0,0)\end{aligned}$$
  2. Design a linear feedback controller using the linearized model:
  $$u = -Kx, \quad \max \mathrm{Re}\big(\lambda(A - BK)\big) < 0$$
  closed-loop linear model strictly stable

nonlinear system $\dot{x} = f(x, -Kx)$ is locally asymptotically stable at $x = 0$

Lyapunov's direct method: mass-spring-damper example





Equation of motion: $\quad m\ddot{y} + c(\dot{y}) + k(y) = 0$

Stored energy: $\quad V = \text{K.E.} + \text{P.E.} \quad \begin{cases} \text{K.E.} = \frac{1}{2}m\dot{y}^2 \\ \text{P.E.} = \int_0^y k(y)\,dy \end{cases}$

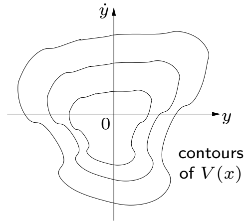
Rate of energy dissipation
$$\begin{aligned}\dot{V} &= \tfrac{1}{2}m\ddot{y}\frac{d}{d\dot{y}}\dot{y}^2 + \dot{y}\frac{d}{dy}\left[\int_0^y k(y)\,dy\right]\\ &= m\ddot{y}\dot{y} + \dot{y}k(y)\end{aligned}$$

but $m\ddot{y} + k(y) = -c(\dot{y})$, so
$$\begin{aligned}\dot{V} &= -c(\dot{y})\dot{y}\\ &\le 0 \qquad \leftarrow \text{since } \mathrm{sign}\big(c(\dot{y})\big) = \mathrm{sign}(\dot{y})\end{aligned}$$

Mass-spring-damper example contd.

- System state: e.g. $x = [y \ \ \dot{y}]^T$
- $\dot{V}(x) \le 0$ implies that $x = 0$ is stable

$\uparrow$
$V(x(t))$ must decrease over time
but
$V(x)$ increases with increasing $\|x\|$



- Formal argument:

for any given $R > 0$:

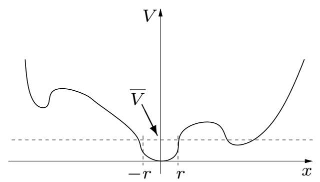
$\|x\| < R$	whenever	$V(x) < \overline{V}$	for some $\overline{V}$
and $V(x) < \overline{V}$	whenever	$\|x\| < r$	for some r

$$\begin{aligned}\therefore\ \|x(0)\| < r \quad &\Longrightarrow \quad V\big(x(0)\big) < \overline{V}\\ &\Longrightarrow \quad V\big(x(t)\big) < \overline{V} \quad \text{for all } t > 0\\ &\Longrightarrow \quad \|x(t)\| < R \quad \text{for all } t > 0\end{aligned}$$

Positive definite functions

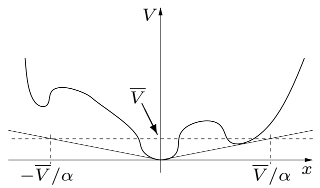
- What if $V(x)$ is not monotonically increasing in $\|x\|$?
- Same arguments apply if $V(x)$ is continuous and positive definite, i.e.

$$\boxed{\begin{array}{lll}\text{(i).} & V(0) = 0 & \\ \text{(ii).} & V(x) > 0 & \text{for all } x \neq 0\end{array}}$$



for any given $\overline{V} > 0$,
can always find r so that

$V(x) < \overline{V}$ whenever $\|x\| < r$



$V(x) \ge \alpha\|x\|^n$
for some constants α, n, so

$\|x\| < (\overline{V}/\alpha)^{1/n}$ whenever $V(x) < \overline{V}$

Lyapunov stability theorem

If there exists a continuous function $V(x)$ such that

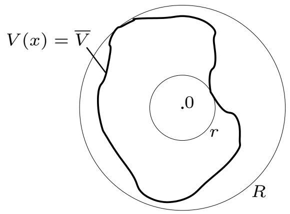
$$\boxed{\begin{array}{l} V(x) \text{ is positive definite} \\ \dot{V}(x) \le 0 \end{array}}$$

then $x = 0$ is stable.

To show that this implies $\|x(t)\| < R$ for all $t > 0$ whenever $\|x(0)\| < r$ for any R and some r:

1. choose $\overline{V}$ as the minimum of $V(x)$ for $\|x\| = R$
2. find r so that $V(x) < \overline{V}$ whenever $\|x\| < r$
3. then $\dot{V}(x) \le 0$ ensures that

$$V\big(x(t)\big) < \overline{V} \quad \forall t > 0 \quad \text{if } \|x(0)\| < r$$
$$\therefore \|x(t)\| < R \quad \forall t > 0$$



Lyapunov stability theorem

- Lyapunov's direct method also applies if $V(x)$ is locally positive definite, i.e. if

$$\begin{array}{ll} \text{(i).} & V(0) = 0 \\ \text{(ii).} & V(x) > 0 \quad \text{for } x \neq 0 \text{ and } \|x\| < R_0 \end{array}$$

  then $x = 0$ is stable if $\dot{V}(x) \le 0$ whenever $\|x\| < R_0$.

- Apply the theorem without determining R, r
  – only need to find p.d. $V(x)$ satisfying $\dot{V}(x) \le 0$.

- Examples

(i). $\dot{x} = -a(t)x, \quad a(t) > 0$

$$V = \tfrac{1}{2}x^2 \quad \Longrightarrow \quad \dot{V} = x\dot{x} = -a(t)x^2 \le 0$$

(ii). $\dot{x} = -a(x), \quad \mathrm{sign}\big(a(x)\big) = \mathrm{sign}(x)$

$$V = \tfrac{1}{2}x^2 \quad \Longrightarrow \quad \dot{V} = x\dot{x} = -a(x)x \le 0$$

Lyapunov stability theorem

- More examples

(iii). $\dot{x} = -a(x), \quad \int_0^x a(x)\,dx > 0$

$$V = \int_0^x a(x)\,dx \quad \Longrightarrow \quad \dot{V} = a(x)\dot{x} = -a^2(x) \le 0$$

(iv). $\ddot{\theta} + \sin\theta = 0$

$$V = \tfrac{1}{2}\dot{\theta}^2 + \int_0^\theta \sin\theta\,d\theta \quad \Longrightarrow \quad \dot{V} = \ddot{\theta}\dot{\theta} + \dot{\theta}\sin\theta = 0$$

Asymptotic stability theorem

If there exists a continuous function $V(x)$ such that

$$\boxed{\begin{array}{ll} V(x) & \text{is positive definite} \\ \dot{V}(x) & \text{is negative definite} \end{array}}$$

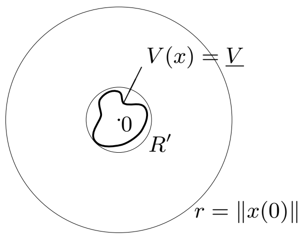
then $x = 0$ is locally asymptotically stable.
($\dot{V}$ negative definite $\iff$ $-\dot{V}$ positive definite)

Asymptotic convergence $x(t) \to 0$ as $t \to \infty$ can be shown by contradiction:

if $\|x(t)\| > R'$ for all $t \ge 0$, then

$$\left.\begin{array}{l} \dot{V}(x) < -W \\ \\ V(x) \ge \underline{V} \end{array}\right\} \quad \text{for all } t \ge 0$$

↑
contradiction



Linearization method and asymptotic stability

- Asymptotic stability result also applies if $\dot{V}(x)$ is only locally negative definite.

- Why does the linearization method work?

  - ⋆ consider 1st order system: $\dot{x} = f(x)$
    linearize about $x = 0$: $\quad = -ax + R \qquad R = O(x^2)$

  - ⋆ assume $a > 0$ and try Lyapunov function V:

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^2 \\ \dot{V}(x) &= x\dot{x} = -ax^2 + Rx = -x^2(a - R/x) \\ &\leq -x^2(a - |R/x|) \end{aligned}$$

  - ⋆ but $R = O(x^2)$ implies $|R| \leq \beta x^2$ for some constant β, so

$$\begin{aligned} \dot{V} &\leq -x^2(a - \beta|x|) \\ &\leq -\gamma x^2 \quad \text{if } |x| \leq (a-\gamma)/\beta \end{aligned}$$

$\implies$ $\dot{V}$ negative definite for $|x|$ small enough
$\implies$ $x = 0$ locally asymptotically stable

Generalization to nth order systems is straightforward

Global asymptotic stability theorem

If there exists a continuous function $V(x)$ such that

$$\boxed{\begin{array}{l} \left.\begin{array}{ll} V(x) & \text{is positive definite} \\ \dot{V}(x) & \text{is negative definite} \end{array}\right\} \text{for all } x \\ V(x) \to \infty \text{ as } \|x\| \to \infty \end{array}}$$

then $x = 0$ is globally asymptotically stable

- If $V(x) \to \infty$ as $\|x\| \to \infty$, then $V(x)$ is radially unbounded

- Test whether $V(x)$ is radially unbounded by checking if $V(x) \to \infty$ as each individual element of x tends to infinity (necessary).

Global asymptotic stability theorem

- Global asymptotic stability requires:

$$\|x(t)\| \text{ finite} \begin{cases} \text{for all } t > 0 \\ \text{for all } x(0) \end{cases}$$

$\uparrow$
not guaranteed by $\dot{V}$ negative definite

in addition to asymptotic stability of $x = 0$

- Hence add extra condition: $V(x) \to \infty$ as $\|x\| \to \infty$

$\updownarrow$ equiv. to

level sets $\{x \ : \ V(x) \leq \overline{V}\}$ are finite

$\updownarrow$ equiv. to

$\|x\|$ is finite whenever $V(x)$ is finite

$\uparrow$
prevents $x(t)$ drifting away from 0 despite $\dot{V} < 0$

Asymptotic stability example

System:
$$\begin{aligned} \dot{x}_1 &= (x_2 - 1)x_1^3 \\ \dot{x}_2 &= -\frac{x_1^4}{(1+x_1^2)^2} - \frac{x_2}{1+x_2^2} \end{aligned}$$

- Trial Lyapunov function $V(x) = x_1^2 + x_2^2$:

$$\begin{aligned} \dot{V}(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= -2x_1^4 + \underbrace{2x_2x_1^4 - 2\frac{x_2x_1^4}{(1+x_1^2)^2}} - 2\frac{x_2^2}{1+x_2^2} \quad \not\leq 0 \end{aligned}$$

$\uparrow$
change V to make
these terms cancel

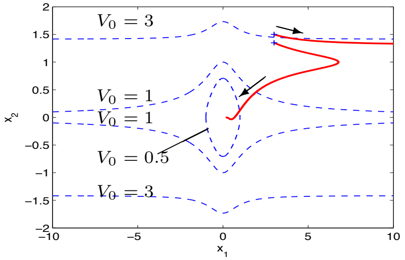
Asymptotic stability example

- New trial Lyapunov function $V(x) = \dfrac{x_1^2}{1+x_1^2} + x_2^2$:

$$\dot V(x) = 2\Big[\frac{x_1}{1+x_1^2} - \frac{x_1^3}{(1+x_1^2)^2}\Big]\dot x_1 + 2x_2\dot x_2$$
$$= -2\frac{x_1^4}{(1+x_1^2)^2} - 2\frac{x_2^2}{1+x_2^2} \le 0$$

$V(x)$ positive definite, $\dot V(x)$ negative definite $\implies$ $x=0$ a.s.
But $V(x)$ not radially unbounded, so cannot conclude global asymptotic stability

State trajectories:



Convergence and invariant sets

Summary

- Positive definite functions
- Derivative of $V(x)$ along trajectories of $\dot x = f(x)$
- Lyapunov's direct method for: stability
  asymptotic stability
  global stability
- Lyapunov's linearization method

Convergence and invariant sets

- Review of Lyapunov's direct method
- Convergence analysis using Barbalat's Lemma
- Invariant sets
- Global and local invariant set theorem
- Example

Review of Lyapunov's direct method

Positive definite functions

- If
$$V(0) = 0$$
$$V(x) > 0 \quad \text{for all } x \neq 0$$
then $V(x)$ is positive definite

- If $\mathcal{S}$ is a set containing $x = 0$ and
$$V(0) = 0$$
$$V(x) > 0 \quad \text{for all } x \neq 0,\ x \in \mathcal{S}$$
then $V(x)$ is locally positive definite (within S)

- e.g.
$$V(x) = x^T x \qquad \leftarrow \qquad \text{positive definite}$$
$$V(x) = x^T x(1 - x^T x) \qquad \leftarrow \qquad \text{locally positive definite within } \mathcal{S} = \{x \ : \ x^T x < 1\}$$

Review of Lyapunov's direct method

System: $\dot{x} = f(x), \quad f(0) = 0$

Storage function: $V(x)$

Time-derivative of V: $\dot{V}(x) = \dfrac{\partial V}{\partial x}\dfrac{dx}{dt} = \nabla V(x)^T \dot{x} = \nabla V(x)^T f(x)$

- If
$$\left.\begin{array}{l}\text{(i). } V(x) \text{ is positive definite} \\ \text{(ii). } \dot{V}(x) \leq 0\end{array}\right\} \text{ for all } x \in \mathcal{S}$$
then the equilibrium $x = 0$ is stable

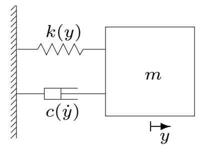
- If
$$\text{(iii). } \dot{V}(x) \text{ is negative definite} \qquad \text{for all } x \in \mathcal{S}$$
then the equilibrium $x = 0$ is asymptotically stable

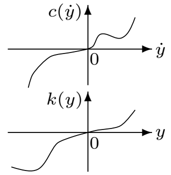
- If
$$\text{(iv). } S = \text{entire state space}$$
$$\text{(v). } V(x) \to \infty \text{ as } \|x\| \to \infty$$
then the equilibrium $x = 0$ is globally asymptotically stable

Convergence analysis

- What can be said about convergence of $x(t)$ to 0
  if $\dot{V}(x) \leq 0$ but $\dot{V}(x)$ is not negative definite?

- Revisit m-s-d example:





Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$

Storage function: $V = \text{K.E.} + \text{P.E.} = \frac{1}{2}m\dot{y}^2 + \int_0^y k(y)\, dy$
$$\dot{V} = -c(\dot{y})\dot{y}$$

Convergence analysis

- V is p.d. and $\dot{V} \leq 0$ so: $(y, \dot{y}) = (0, 0)$ is stable
  and $V(y, \dot{y})$ tends to a finite limit as $t \to \infty$

- but does $(y, \dot{y})$ converge to $(0, 0)$?

$\updownarrow$ equivalent to

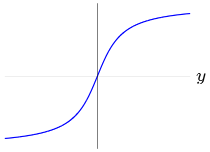
can $V(y, \dot{y})$ "get stuck" at $V = V_0 \neq 0$ as $t \to \infty$?

$\downarrow$

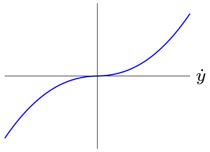
need to consider motion at points $(y, \dot{y})$ for which $\dot{V} = 0$

Example

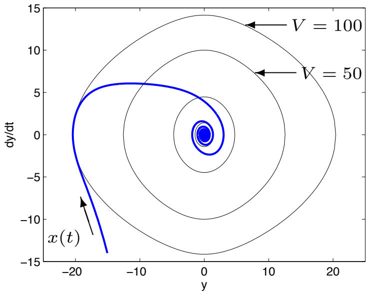
Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$



$k(y) = 5\tan^{-1}(y/5)$



$c(\dot{y}) = 0.1\dot{y}(0.5 + |\dot{y}|)(2 - e^{-0.1|\dot{y}|})$



Storage function:

$V = \frac{1}{2}\dot{y}^2 + \int_0^y 5\tan^{-1}(y/5)\,dy$

$\dot{V} = -c(\dot{y})\dot{y} \le 0$

$\downarrow$

$\dot{V} = 0$ when $\dot{y} = 0$

but $k(y) \neq 0 \implies \ddot{y} \neq 0 \implies \ddot{V} \neq 0$

$\Downarrow$

V continues to decrease until $y = \dot{y} = 0$

Convergence analysis

Summary of method:

1. show that $\dot{V}(x) \to 0$ as $t \to \infty$
2. determine the set $\mathcal{R}$ of points x for which $\dot{V}(x) = 0$
3. identify the subset $\mathcal{M}$ of $\mathcal{R}$ for which $\dot{V}(x) = 0$ at all future times

$\boxed{\text{then } x(t) \text{ has to converge to } \mathcal{M} \text{ as } t \to \infty}$

This approach is the basis of the invariant set theorems

Barbalat's Lemma

Barbalat's lemma: For any function $\phi(t)$, if

(i). $\int_0^t \phi(\tau)\,d\tau$ converges to a finite limit as $t \to \infty$

(ii). $\dot{\phi}(t)$ is finite for all t

then $\lim_{t\to\infty} \phi(t) = 0$

- Obvious for the case that $\phi(t) \ge 0$ for all t
- Condition (ii) is needed to ensure that $\phi(t)$ remains continuous for all t

$\uparrow$

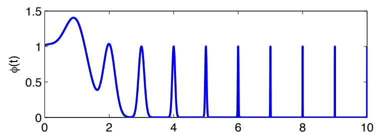
Can construct discontinuous $\phi(t)$ for which $\int_0^t \phi(\tau)\,d\tau$ converges

but $\phi(t) \not\to 0$ as $t \to \infty$

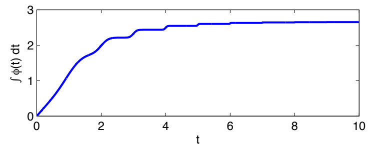
Barbalat's Lemma

Example: pulse train $\phi(t) = \sum_{k=0}^{\infty} e^{-4^k(t-k)^2}$:

$\phi(t)$:



$\int_0^t \phi(\tau)\,d\tau$:



From the plots it is clear that

$\int_0^t \phi(s)\,ds$ tends to a finite limit

but $\phi(t) \not\to 0$ as $t \to \infty$ because $\dot{\phi}(t) \to \infty$ as $t \to \infty$

Barbalat's Lemma contd.

Apply Barbalat's Lemma to $\dot V(x(t)) = \phi(t) \le 0$:

- Integrate:

$$\int_0^t \phi(s)\,ds = V(x(t)) - V(x(0)) \qquad \leftarrow \text{finite limit as } t \to \infty$$

- Differentiate:

$$\dot\phi(t) = \ddot V(x(t)) = f^T(x)\frac{\partial^2 V}{\partial x^2}(x)f(x) + \nabla V(x)\frac{\partial f}{\partial x}(x)f(x)$$

$$= \text{finite for all } t \text{ if } f(x) \text{ continuous and } V(x) \text{ continuously differentiable}$$

$$\Downarrow$$

$$\boxed{\dot V(x) \to 0 \text{ as } t \to \infty}$$

The above arguments rely on $\|x(t)\|$ remaining finite for all t, which is implied by:

$V(x)$ positive definite
$\dot V(x) \le 0$
$V(x) \to \infty$ as $\|x\| \to \infty$

Convergence analysis

Summary of method:

1. show that $\dot V(x) \to 0$ as $t \to \infty$
   $\to$ true whenever $\dot V \le 0$ & V, f are smooth & $\|x(t)\|$ is bounded
   [by Barbalat's Lemma]
2. determine the set $\mathcal{R}$ of points x for which $\dot V(x) = 0$
   $\to$ algebra!
3. identify the subset $\mathcal{M}$ of $\mathcal{R}$ for which $\dot V(x) = 0$ at all future times
   $\to$ $\mathcal{M}$ must be invariant

$$\boxed{\text{then } x(t) \text{ has to converge to } \mathcal{M} \text{ as } t \to \infty}$$

This approach is the basis of the invariant set theorems

Invariant sets

- A set of points $\mathcal{M}$ in state space is invariant if

$$\boxed{x(t_0) \in \mathcal{M} \quad \Longrightarrow \quad x(t) \in \mathcal{M} \quad \text{for all } t > t_0}$$

Examples:

- ⋆ Equilibrium points
- ⋆ Limit cycles
- ⋆ Level sets of $V(x)$ provided $\dot V(x) \le 0$ $\quad\leftarrow\quad$ i.e. $\{x \;:\; V(x) \le V_0\}$ for constant V_0

- If $\dot V(x) \to 0$ as $t \to \infty$, then

$$\boxed{\text{$x(t)$ must converge to an invariant set $\mathcal{M}$ contained within the set of points on which $\dot V(x) = 0$}}$$

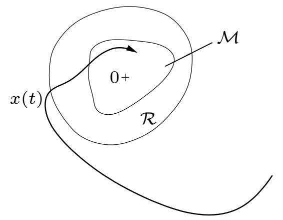
as $t \to \infty$

Global invariant set theorem

If there exists a continuously differentiable function $V(x)$ such that

$V(x)$ is positive definite
$\dot V(x) \le 0$
$V(x) \to \infty$ as $\|x\| \to \infty$

then: (i). $\dot V(x) \to 0$ as $t \to \infty$
(ii). $x(t) \to \mathcal{M} =$ the largest invariant set contained in $\mathcal{R}$
where $\mathcal{R} = \{x \;:\; \dot V(x) = 0\}$



- $\dot V(x)$ negative definite $\Longrightarrow \mathcal{M} = 0$ (c.f. Lyapunov's direct method)
- Determine $\mathcal{M}$ by considering system dynamics within $\mathcal{R}$

Global invariant set theorem

Revisit m-s-d example (for the last time)

- $V(x)$ is positive definite, $V(x) \to \infty$ as $\|x\| \to \infty$, and
$$\dot{V}(y, \dot{y}) = -c(\dot{y})\dot{y} \le 0$$
- therefore $\dot{V} \to 0$, implying $\dot{y} \to 0$ as $t \to \infty$
i.e. $\mathcal{R} = \{(y, \dot{y}) \ : \ \dot{y} = 0\}$
- but $\dot{y} = 0$ implies $\ddot{y} = -k(y)/m$
- therefore $\ddot{y} \ne 0$ unless $y = 0$, so $\dot{y}(t) = 0$ for all t only if $y(t) = 0$
i.e. $\mathcal{M} = \{(y, \dot{y}) \ : \ (y, \dot{y}) = (0, 0)\}$

$$\Downarrow$$

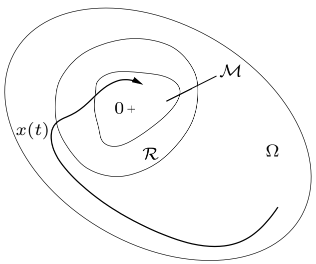
$(y, \dot{y}) = (0, 0)$ is a globally asymptotically stable equilibrium!

Local invariant set theorem

If there exists a continuously differentiable function $V(x)$ such that

> the level set $\Omega = \{x \ : \ V(x) \le V_0\}$ is bounded for some V_0
> and $\dot{V}(x) \le 0$ whenever $x \in \Omega$

then:
(i). Ω is an invariant set
(ii). $x(0) \in \Omega \quad \Longrightarrow \quad \dot{V}(x) \to 0$ as $t \to \infty$
(iii). $x(t) \to \mathcal{M}$ = largest invariant set contained in $\mathcal{R}$
where $\mathcal{R} = \{x \ : \ \dot{V}(x) = 0\}$



Local invariant set theorem

- $V(x)$ doesn't have to be positive definite or radially unbounded
- Result is based on Barbalat's Lemma applied to $\dot{V}$

$$\uparrow$$

applies here because finite Ω implies $\|x(t)\|$ finite for all t
since $x(0) \in \Omega$ and $\dot{V} \le 0$

- Ω is a region of attraction for $\mathcal{M}$

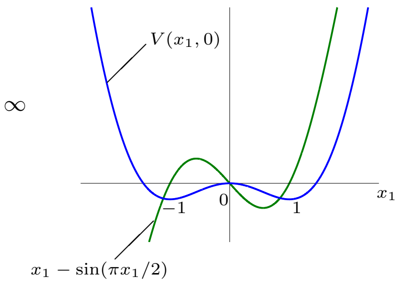
Example: local invariant set theorem

- Second order system:
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -(x_1 - 1)^2 x_2^3 - x_1 + \sin(\pi x_1/2)$$
- Equilibrium points: $(x_1, x_2) = (0, 0), (1, 0), (-1, 0)$
- Trial storage function:
$$V(x) = \tfrac{1}{2}x_2^2 + \int_0^{x_1} \big(y - \sin(\pi y/2)\big)\, dy$$

V is not positive definite
but $V(x) \to \infty$ if $x_1 \to \infty$ or $x_2 \to \infty$

$$\Downarrow$$

level sets of V are finite



Example: local invariant set theorem contd.

- Differentiate: $\dot{V}(x) = -(x_1 - 1)^2 x_2^4 \leq 0$

$$\dot{V}(x) = 0 \quad \Longleftrightarrow \quad x \in \mathcal{R} = \{x \ : \ x_1 = 1 \ \text{ or } \ x_2 = 0\}$$

- From the system model, $x \in \mathcal{R}$ implies:

$$x_1 = 1 \quad \Longrightarrow \quad (\dot{x}_1, \dot{x}_2) = (x_2, 0)$$
and
$$x_2 = 0 \quad \Longrightarrow \quad (\dot{x}_1, \dot{x}_2) = (0, \sin(\pi x_1/2) - x_1)$$

therefore $\begin{cases} x(t) \text{ remains on line } x_1 = 1 \text{ only if } x_2 = 0 \\ x(t) \text{ remains on line } x_2 = 0 \text{ only if } x_1 = 0, \ 1 \text{ or } -1 \end{cases}$

$$\Longrightarrow \ \mathcal{M} = \{(0,0), (1,0), (-1,0)\}$$

- Apply local invariant set theorem to any level set $\Omega = \{x \ : \ V(x) \leq V_0\}$:

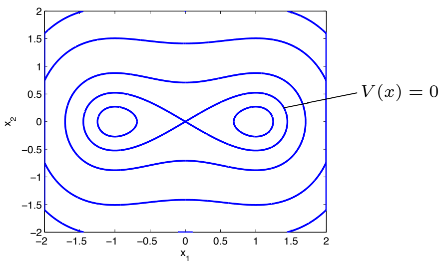
$$\left.\begin{array}{l} \Omega \text{ is finite} \\ \dot{V} \leq 0 \end{array}\right\} \Longrightarrow \quad x(t) \to \mathcal{M} = \{(0,0), (1,0), (-1,0)\} \text{ as } t \to \infty$$

Example: local invariant set theorem contd.

- From any initial condition, $x(t)$ converges asymptotically to $(0,0)$, $(1,0)$ or $(-1,0)$

  but $x = (0,0)$ is unstable
  (linearized system at $(0,0)$ has poles $\pm\sqrt{\frac{\pi}{2} - 1}$ so is unstable)

- Contours of $V(x)$:



Use local invariant set theorem on level sets $\Omega = \{x : V(x) \leq V_0\}$ for $V_0 < 0$

$$\downarrow$$

$x = (1,0)$, $x = (-1,0)$ are stable equilibrium points

Summary

- Convergence analysis using Barbalat's lemma
- Invariant sets
- Invariant set methods for convergence: local invariant set theorem
  global invariant set theorem

Lecture 4

Linear systems, passivity, and the circle criterion

Linear systems, passivity, and the circle criterion

- Summary of stability methods
- Lyapunov functions for linear systems
- Passive systems
- Passive linear systems
- The circle criterion
- Example

Summary of stability methods

- **Linearization** method

$$\dot{x} = Ax \text{ is strictly stable, } A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$
$$\Downarrow$$
$$x = 0 \text{ locally asymptotically stable}$$

- **Lyapunov's direct** method

$V(x)$ locally p.d. $\dot{V}(x) \leq 0$ locally	$V(x)$ locally p.d. $\dot{V}(x)$ locally n.d.	$V(x)$ p.d. $\dot{V}(x)$ n.d. $V(x) \to \infty$ as $\lVert x \rVert \to \infty$
$\Downarrow$	$\Downarrow$	$\Downarrow$
$x = 0$ stable	$x = 0$ locally asymptotically stable	$x = 0$ globally asymptotically stable

- **Invariant set** theorems

$V(x)$ p.d. $\dot{V}(x) \leq 0$ $V(x) \to \infty$ as $\lVert x \rVert \to \infty$	$\Omega = \{x \;:\; V(x) \leq V_0\}$ bounded $\dot{V}(x) \leq 0$ for all $x \in \Omega$

$$\Downarrow$$

$x(t)$ converges to the union of invariant sets contained in $\{x \;:\; \dot{V}(x) = 0\}$

Summary of stability methods

- **Instability theorems** analogous to Lyapunov's direct method, e.g.

$$\left.\begin{array}{l} V(x) \text{ p.d.} \\ \dot{V}(x) \text{ p.d.} \end{array}\right\} \implies x = 0 \text{ unstable}$$

- Lyapunov stability criteria are only **sufficient**, e.g.

$$\left.\begin{array}{l} V(x) \text{ p.d.} \\ \dot{V}(x) \not\leq 0 \end{array}\right\} \;\not\Longrightarrow\; x = 0 \text{ unstable}$$

(some other $V(x)$ demonstrating stability may exist)

- **Converse theorems**

$$x = 0 \text{ stable} \implies V(x) \text{ demonstrating stability exists}$$

(can swap premises and conclusions in Lyapunov's direct method)

$$\uparrow$$

But **no general method** for constructing $V(x)$

Linear systems

- Systematic method for constructing storage function $V(x) = x^T P x$

$\dot{x} = Ax$ strictly stable $\implies$ can always find constant matrix P so that $\dot{V}(x)$ is negative definite

- Only need consider symmetric P

$$x^T P x = \tfrac{1}{2} x^T P x + \tfrac{1}{2} x^T P^T x = \tfrac{1}{2} x^T \underbrace{(P + P^T)}_{\text{SYMMETRIC}} x$$

- Need $\lambda(P) > 0$ for positive definite $V(x) = x^T P x$

$P = U \Lambda U^T$ — eigenvector/value decomposition

$\Downarrow$

$x^T P x = z^T \Lambda z$ — $z = U^T x$

$\Downarrow$

$x^T P x$ positive definite iff Λ strictly positive — $\left\{ \begin{array}{l} \text{notation: } P > 0 \\ \text{or "}P\text{ is positive definite"} \end{array} \right.$

Linear systems

- How is P computed?

$$\left.\begin{array}{r} \dot{x} = Ax \\ V(x) = x^T P x \end{array}\right\} \implies \begin{array}{l} \dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x \\ \quad\;\;\; = x^T (PA + A^T P) x \end{array}$$

$\therefore x = 0$ is globally asymptotically stable if, for some Q:

$$\boxed{PA + A^T P = -Q \qquad Q = Q^T > 0}$$

Lyapunov matrix equation

- Pick $Q > 0$ and solve $PA + A^T P = -Q$ for P, then

$$\boxed{\mathrm{Re}[\lambda(A)] < 0 \quad \iff \quad \text{unique solution for } P \text{ and } P = P^T > 0}$$

Proof:

$\Longleftarrow$ due to $\dot{V}(x) = -x^T Q x$ negative definite

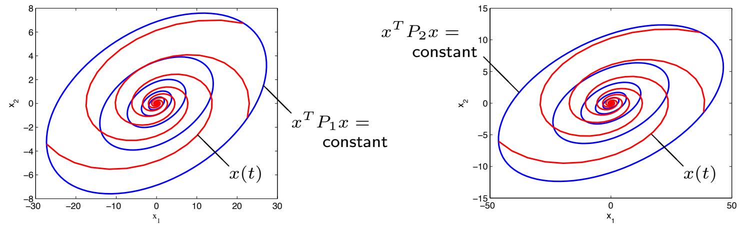
$\Longrightarrow$ follows from integrating $\dot{V}$ w.r.t. t: $P = \displaystyle\int_0^\infty e^{A^T t} Q e^{At}\, dt$

Example: Lyapunov matrix equation

Stable linear system $\dot{x} = Ax$: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -16 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\qquad \lambda(A) = -1 \pm i\sqrt{15}$

Solve $PA + A^T P = -Q$ for P:

$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow P_1 = \begin{bmatrix} 0.33 & -0.5 \\ -0.5 & 4.25 \end{bmatrix}$ $\qquad Q_2 = \begin{bmatrix} 0.41 & -0.19 \\ -0.19 & 0.11 \end{bmatrix} \Rightarrow P_2 = \begin{bmatrix} 0.12 & -0.21 \\ -0.21 & 1.67 \end{bmatrix}$

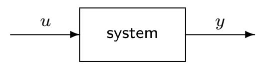


Here:

- ⋆ any choice of $Q > 0$ gives $P > 0$ (since A is strictly stable)
- ⋆ but not every $P > 0$ gives $Q > 0$

Passive systems

- Systematic method for constructing storage functions
- Input-output representation of system:



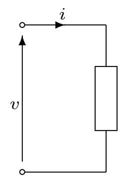
input : u
output : y

The system is passive if

$$\boxed{\dot{V} = yu - g \quad \text{for some } V(t) \geq 0, \;\; g(t) \geq 0}$$

also the system is dissipative if $\displaystyle\int_0^\infty yu\, dt \neq 0 \implies \int_0^\infty g\, dt > 0$

- Motivated by electrical networks with no internal power generation

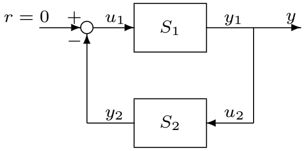


$\left.\begin{array}{l} \text{input: } i \\ \text{output: } v \end{array}\right\}$ stored energy: $V = \displaystyle\int_0^t vi\, dt$, $\quad \dot{V} = iv$

Passive systems

Passivity is useful for determining storage functions for feedback systems

- Closed-loop system with passive subsystems S_1, S_2:



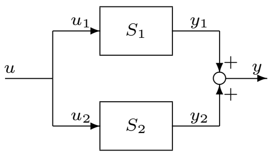
$$S_1: \quad V_1 \geq 0 \quad \dot V_1 = y_1 u_1 - g_1$$
$$S_2: \quad V_2 \geq 0 \quad \dot V_2 = y_2 u_2 - g_2$$

$$\begin{aligned} V_1 + V_2 &\geq 0 \\ \dot V_1 + \dot V_2 &= y_1 u_1 + y_2 u_2 - g_1 - g_2 \\ &= y_1(-y_2) + y_2 y_1 - g_1 - g_2 \\ &= -g_1 - g_2 \\ &\leq 0 \end{aligned}$$

$\Longrightarrow$ $V = V_1 + V_2$ is a Lyapunov function for the closed-loop system
if V is a p.d. function of the system state

Interconnected passive systems

- Parallel connection:

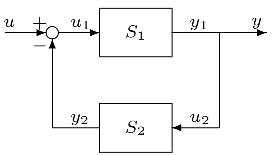


$$\begin{aligned} V_1 + V_2 &\geq 0 \\ \dot V_1 + \dot V_2 &= y_1 u_1 + y_2 u_2 - g_1 - g_2 \\ &= (y_1 + y_2)u - g_1 - g_2 \\ &= yu - g_1 - g_2 \end{aligned}$$

$\Downarrow$

Overall system from u to y is passive

- Feedback connection:



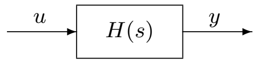
$$\begin{aligned} V_1 + V_2 &\geq 0 \\ \dot V_1 + \dot V_2 &= y_1 u_1 + y_2 u_2 - g_1 - g_2 \\ &= y(u - y_2) + y_2 y - g_1 - g_2 \\ &= yu - g_1 - g_2 \end{aligned}$$

$\Downarrow$

Overall system from u to y is passive

Passive linear systems

Transfer function : $\dfrac{Y(s)}{U(s)} = H(s)$



- H is passive if and only if

$$\boxed{\begin{array}{ll} \text{(i).} & \mathrm{Re}(p_i) \leq 0, \text{ where } \{p_i\} \text{ are the poles of } H(s) \\ \text{(ii).} & \mathrm{Re}\big[H(j\omega)\big] \geq 0 \text{ for all } 0 \leq \omega \leq \infty \end{array}}$$

  - ⋆ H must be stable, otherwise $V(t) = \displaystyle\int_0^t yu\,dt$ is not defined for all u
  - ⋆ From Parseval's theorem:

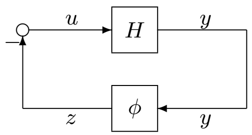
$$\mathrm{Re}\big[H(j\omega)\big] \geq 0 \quad \Longleftrightarrow \quad \int_0^t yu\,dt \geq 0 \text{ for all } u(t) \text{ and } t$$

$\nearrow$

frequency domain criterion for passivity

- H is dissipative if and only if $\mathrm{Re}(p_i) \leq 0$ and
$\mathrm{Re}\big[H(j\omega)\big] > 0$ for all $0 \leq \omega < \infty$

Linear system + static nonlinearity



H linear: $\dfrac{Y(s)}{U(s)} = H(s)$

ϕ static nonlinearity: $z = \phi(y)$

What are the conditions on H and ϕ for closed-loop stability?

- A common problem in practice, due to e.g.
  - ⋆ actuator saturation (valves, dc motors, etc.)
  - ⋆ sensor nonlinearity

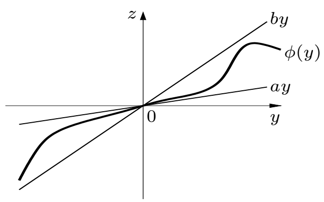
- Determine closed-loop stability given:

ϕ belongs to sector $[a, b]$

$\Updownarrow$

$$\boxed{a \leq \frac{\phi(y)}{y} \leq b}$$

"Absolute stability problem"



Linear system + static nonlinearity

- Aizerman's conjecture (1949):

  Closed-loop system is stable if stable for $\phi(y) = ky$, $a \le k \le b$

  false (necessary but not sufficient)

- Sufficient conditions for closed-loop stability:

  $\left.\begin{array}{l}\text{Popov criterion (1960)}\\ \text{Circle criterion}\end{array}\right\}$ based on passivity

- The passivity approach:

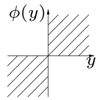
  (1). If H is dissipative (i.e. if $\mathrm{Re}[H(j\omega)] > 0$ and H is stable), then:

  $$\left.\begin{array}{l} V = x^T P x \\ \dot V = yu - x^T Q x \end{array}\right\} \text{for some } P > 0,\ Q > 0$$

  $$= -y\phi(y) - x^T Q x \qquad \leftarrow x = \text{state of } H$$

  (2). If ϕ belongs to sector $[0, \infty)$, then:

  $$y\phi(y) \ge 0 \qquad \leftarrow$$

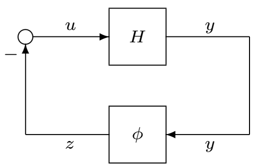


  (1) & (2) $\implies$ $\dot V \le -x^T Q x$

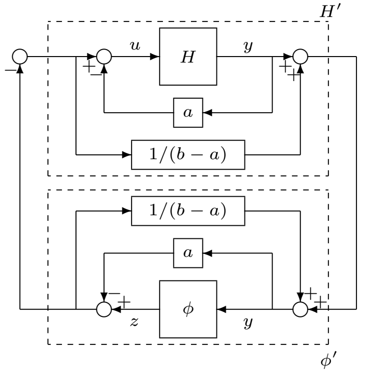
  $\implies$ $x = 0$ is globally asymptotically stable

Circle criterion

Use loop transformations to generalize the approach for $\left\{\begin{array}{l} H \text{ not passive} \\ \phi \notin [0, \infty) \end{array}\right.$



$\longleftrightarrow$ equiv. to



$\phi \in [a, b]$ $\quad a, b$ arbitrary

$\phi \in [a, b] \implies \phi' \in [0, \infty]$

$$H'(j\omega) = \frac{H(j\omega)}{1 + aH(j\omega)} + \frac{1}{b - a}$$

Circle criterion

To make $H'(j\omega) = \dfrac{H(j\omega)}{1 + aH(j\omega)} + \dfrac{1}{b-a}$ dissipative, need:

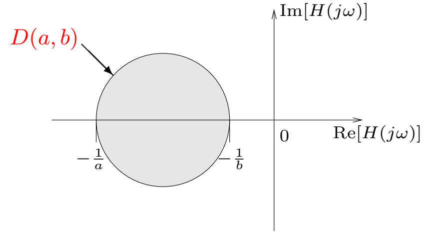
(i). H' stable $\iff$ $\dfrac{H(j\omega)}{1 + aH(j\omega)}$ stable

$\Updownarrow$

Nyquist plot of $H(j\omega)$ goes through ν anti-clockwise encirclements of $-1/a$ as ω goes from $-\infty$ to ∞

(ν = no. poles of $H(j\omega)$ in RHP)

(ii). $\mathrm{Re}[H'(j\omega)] > 0$ $\iff$ $\left\{\begin{array}{ll} H(j\omega) \text{ lies outside } D(a,b) & \text{if } ab > 0 \\ H(j\omega) \text{ lies inside } D(a,b) & \text{if } ab < 0 \end{array}\right.$

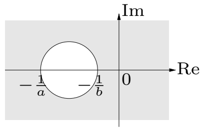


Graphical interpretation of circle criterion

$x = 0$ is globally asymptotically stable if:

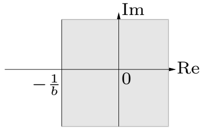
⋆ $0 < a < b$

$H(j\omega)$ lies in shaded region and does ν anti-clockwise encirclements of $D(a, b)$



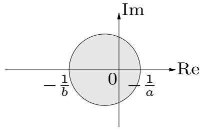
⋆ $b > a = 0$

$H(j\omega)$ lies in shaded region and $\nu = 0$ (can't encircle $-1/a$)



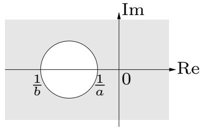
⋆ $a < 0 < b$

$H(j\omega)$ lies in shaded region and $\nu = 0$ (can't encircle $-1/a$)



⋆ $a < b < 0$

$-H(j\omega)$ lies in shaded region and does ν anti-clockwise encirclements of $D(-b, -a)$



Circle criterion

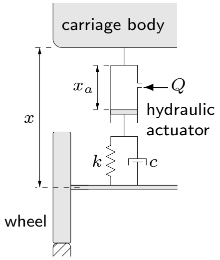
- Circle criterion is equivalent to Nyquist criterion for $a = b > 0$
  $\uparrow$
  then $D(a,b) = -\dfrac{1}{a}$
  (single point)

- Circle criterion is only sufficient for closed-loop stability for general a, b

- Results apply to time-varying static nonlinearity: $\phi(y,t)$

Example: Active suspension system

- Active suspension system for high-speed train:



$$\begin{aligned} Q &= \phi(u) \\ \dot{x}_a &= Q/A \end{aligned}$$

u : valve input signal
Q : flow rate
ϕ : valve characteristics, $\phi \in [0.005, 0.1]$
A : actuator working area

- Force exerted by suspension system on carriage body: F_{susp}

$$\begin{aligned} F_{\text{susp}} &= k(x_a - x) + c(\dot{x}_a - \dot{x}) \\ &= \big(k \int^t Q\,dt + cQ\big)/A - kx - c\dot{x}, \qquad Q = \phi(u) \end{aligned}$$

- Design controller to compensate for the effects of (constant) unknown load on displacement x despite uncertain valve characteristics $\phi(u)$.

Active suspension system contd.

- Dynamics:

$$\begin{aligned} & F_{\text{susp}} - F = m\ddot{x} \\ \implies \quad & m\ddot{x} + c\dot{x} + kx = \big(k \int^t Q\,dt + cQ\big)/A - F, \qquad Q = \phi(u) \end{aligned}$$

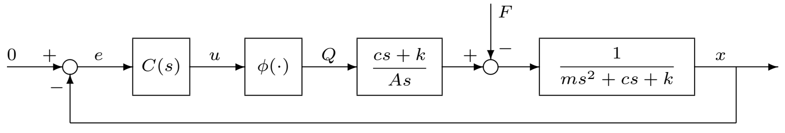
F : unknown load on suspension unit
m : effective carriage mass

- Transfer function model:

$$X(s) = \frac{cs+k}{ms^2+cs+k} \cdot \frac{Q(s)}{As} - \frac{F}{ms^2+cs+k} \qquad Q = \phi(u)$$

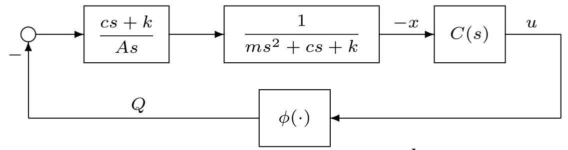
- Try linear compensator $C(s)$:

$$U(s) = C(s)E(s) \qquad e = -x, \quad \text{setpoint: } x = 0$$



Active suspension system contd.

- For constant F, we need to stabilize the closed-loop system:



linear system: $H(s) = \dfrac{cs+k}{As(ms^2+cs+k)} \cdot C(s)$

static nonlinearity: $\phi \in [0.005, 0.1]$

- P+D compensator (no integral term needed):

$$C(s) = K(1+\alpha s) \quad \implies \quad H(s) = \frac{K(1+\alpha s)(cs+k)}{As(ms^2+cs+k)}$$

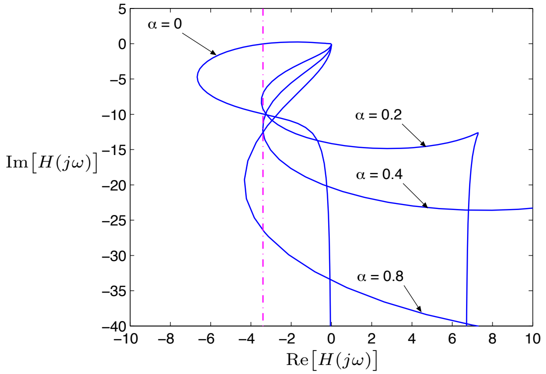
H open-loop stable ($\nu = 0$)

- From the circle criterion, closed-loop (global asymptotic) stability is ensured if:

$H(j\omega)$ lies outside $D(0.005, 0.1)$
$\uparrow$
sufficient condition: $\mathrm{Re}\big[H(j\omega)\big] > -10$

Active suspension system contd.

- Nyquist plot of $H(j\omega)$ for $K = 1$ and $\alpha = 0, 0.2, 0.4, 0.8$:



- To maximize gain margin:

  choose $\alpha = 0.2$ $\quad\leftarrow$ allows for largest K
  $K \le 10/3.4 = 2.94$

Summary

At the end of the course you should be able to do the following:

- Understand the basic Lyapunov stability definitions (lecture 1)
- Analyse stability using the linearization method (lecture 2)
- Analyse stability by Lyapunov's direct method (lecture 2)
- Determine convergence using Barbalat's Lemma (lecture 3)
- Understand how invariant sets can determine regions of attraction (lecture 3)
- Construct Lyapunov functions for linear systems and passive systems (lecture 4)
- Use the circle criterion to design controllers for systems with static nonlinearities (lecture 4)

Addendum – Limit cycle stability analysis

System has state $x = (x_1, x_2)$ and dynamics:

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_1 h(x)\\ \dot{x}_2 &= x_1 - x_2 h(x)\end{aligned}$$

where $h(x) = x_1^2 + x_2^2 - 1$.

Limit cycle

Differentiate $h(x)$ w.r.t. t using system dynamics:

$$\dot{h} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2(x_1^2 + x_2^2)h(x) = -2(h+1)h$$

hence $h = 0 \implies \dot{h} = 0$, so $\{x : x_1^2 + x_2^2 = 1\}$ must contain a limit cycle.

Addendum – Limit cycle stability analysis

System has state $x = (x_1, x_2)$ and dynamics:

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_1 h(x)\\ \dot{x}_2 &= x_1 - x_2 h(x)\end{aligned}$$

where $h(x) = x_1^2 + x_2^2 - 1$.

Limit cycle stability

Let $V(x) = h^2(x)$, then $\dot{V} = 2h\dot{h} = -4h^2(h+1)$
$\qquad = -4h^2(x)(x_1^2 + x_2^2) \le 0$

- $\{x : V(x) \le c\}$ is an invariant set for any constant c and $\{x : V(x) = 0\} = \{x : x_1^2 + x_2^2 = 1\}$ is stable
- $\dot{V} = 0 \implies h = 0$ (or $x_1 = x_2 = 0$)
  $h = 0 \implies \dot{h} = 0$
  $\implies$ the limit cycle $\{x : h = 0\}$ is the largest invariant set contained in $\{x : V(x) < 1$ and $\dot{V}(x) = 0\}$, so is asymptotically stable