Lecture 5: Asymptotic behaviour

Global properties of trajectories:

- Suppose $\phi(t, \mathbf{x}_0)$ is the flow of $\mathbf{f}(\mathbf{x})$ with $\phi(0, \mathbf{x}_0) = \mathbf{x}_0$
  (i.e. $\mathbf{x}(t) = \phi(t, \mathbf{x}_0)$ is a solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ such that $\phi(0, \mathbf{x}_0) = \mathbf{x}_0$)

- This solution defines a path or trajectory in some set D containing $\mathbf{x}_0$:
$$\Gamma_{\mathbf{x}_0} = \{\mathbf{x} \in D : \mathbf{x} = \phi(t, \mathbf{x}_0), t \in \mathbb{R}\}$$

- We want to determine the asymptotic behaviour of this solution
  Hence define the α and ω limit points of the trajectory

Limit points

Definition: A point $\mathbf{p} \in D$ is called an ω **limit point** of the trajectory $\phi(t, \mathbf{x})$ if there exists a sequence of times $\{t_i\}$, $t_i \to \infty$, such that

$$\lim_{i \to \infty} \phi(t_i, \mathbf{x}) = \mathbf{p}$$

this point is denoted $\omega(\mathbf{x})$.

- Note that we may need to choose the times $\{t_i\}$ carefully in order to get a limit point (e.g. consider $\phi(t, \mathbf{x}) = \mathrm{e}^{-2t} + \cos(t)$)
- An α **limit point** is defined in the same way, but with $t_i \to -\infty$. This point is denoted $\alpha(\mathbf{x})$

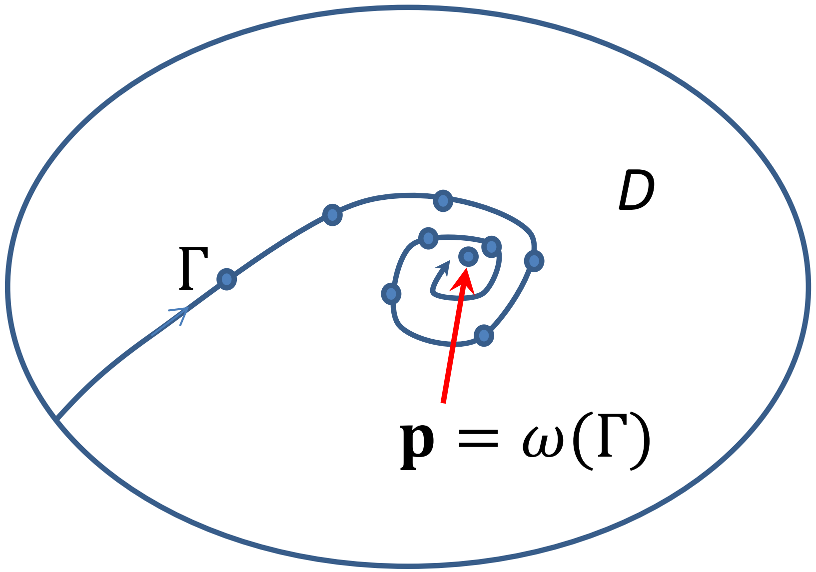
Limit points

Definition: $\alpha(\Gamma)$ and $\omega(\Gamma)$ are called the α-**limit set** and ω-**limit set** respectively. These are the sets of all α limit points and ω limit points for the trajectory Γ.

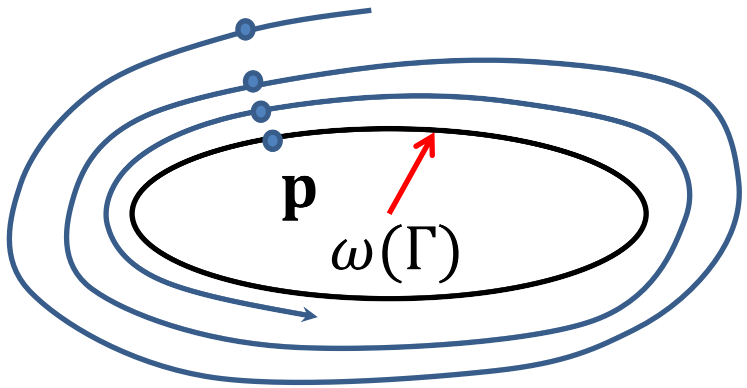
- Hence $\alpha(\Gamma)$ is the set of points from which the trajectory Γ originates (at $t = -\infty$) and $\omega(\Gamma)$ is the set of points to which it tends (at $t = \infty$)

- The set of all limit points is called the **limit set of** Γ

Example limit sequences



A sequence of points leading to an isolated point ω-limit set



A carefully chosen sequence of points to an ω-limit point when the ω-limit set is not an isolated point

Equilibrium points

- An equilibrium point $\mathbf{x}^*$ is its own α and ω limit point.
  Conversely, if a trajectory has a unique ω limit point $\mathbf{x}^*$, then $\mathbf{x}^*$ must be an equilibrium point.

- Not all ω limit points are equilibrium points (e.g. see the previous slide). If a point $\mathbf{p}$ is a limit point and $\dot{\mathbf{p}} \neq 0$ then this trajectory is a closed orbit. Note that we had to choose the sequence of points on the trajectory carefully in order to find a limit point and that there are infinitely many points on the ω limit set.

Invariance

Definition: Let $\phi(t, \mathbf{x})$ be the flow of $\mathbf{f}(\mathbf{x})$ on a set D, then a set $S \subset D$ is called **positively invariant** if $\phi(t, \mathbf{x}) \in S$ for all $\mathbf{x} \in S$ and all $t \geq 0$.

- All points in S stay in S under the action of the flow – the solution cannot 'escape' from S. We saw an example of this in the case of the stable and unstable invariant manifolds in earlier lectures.

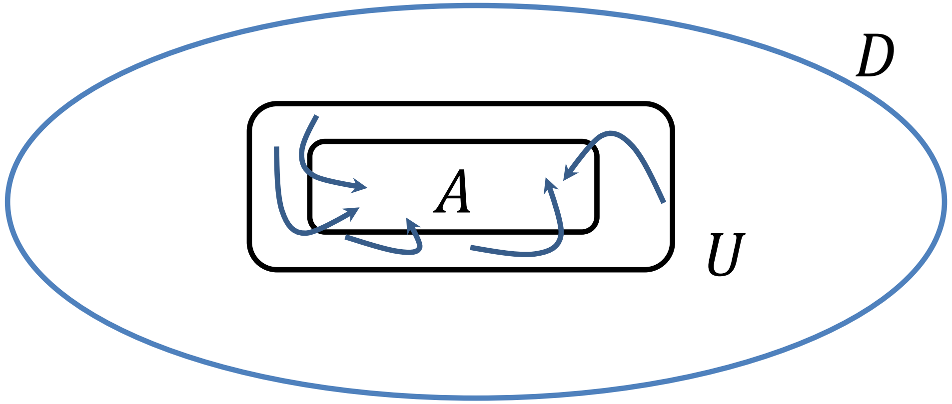
- If a region M is positively invariant, closed and bounded, then the ω limit set is not empty (i.e. you have to go somewhere!). The limit set itself is positively invariant (since once there, you stay in the set).

Attraction

Definition: An invariant set $A \subset D$ is **attracting** if

1. there is some neighbourhood U of A which is positively invariant, and
2. all trajectories starting in U tend to A as $t \to \infty$.

U is called a **trapping region** of A.



Neighbourhood: A set surrounding a point $\mathbf{x}$ so that the distance to all points in the neighbourhood from the point $\mathbf{x}$ is less than some positive number ϵ.

Attractors

Definition: An **attractor** is an invariant attracting set (e.g. a limit cycle or equilibrium point) such that no subset of the invariant set is itself an invariant attracting set.

Example: A stable node or focus is the ω limit set of all trajectories passing through points in a neighbourhood of the equilibrium point – the equilibrium point is an attractor. A saddle point on its own cannot be an attractor as trajectories leave the saddle point's neighbourhood.

Example

$$\dot{x} = -y + x(1 - x^2 - y^2)$$
$$\dot{y} = x + y(1 - x^2 - y^2)$$

In polar co-ords:

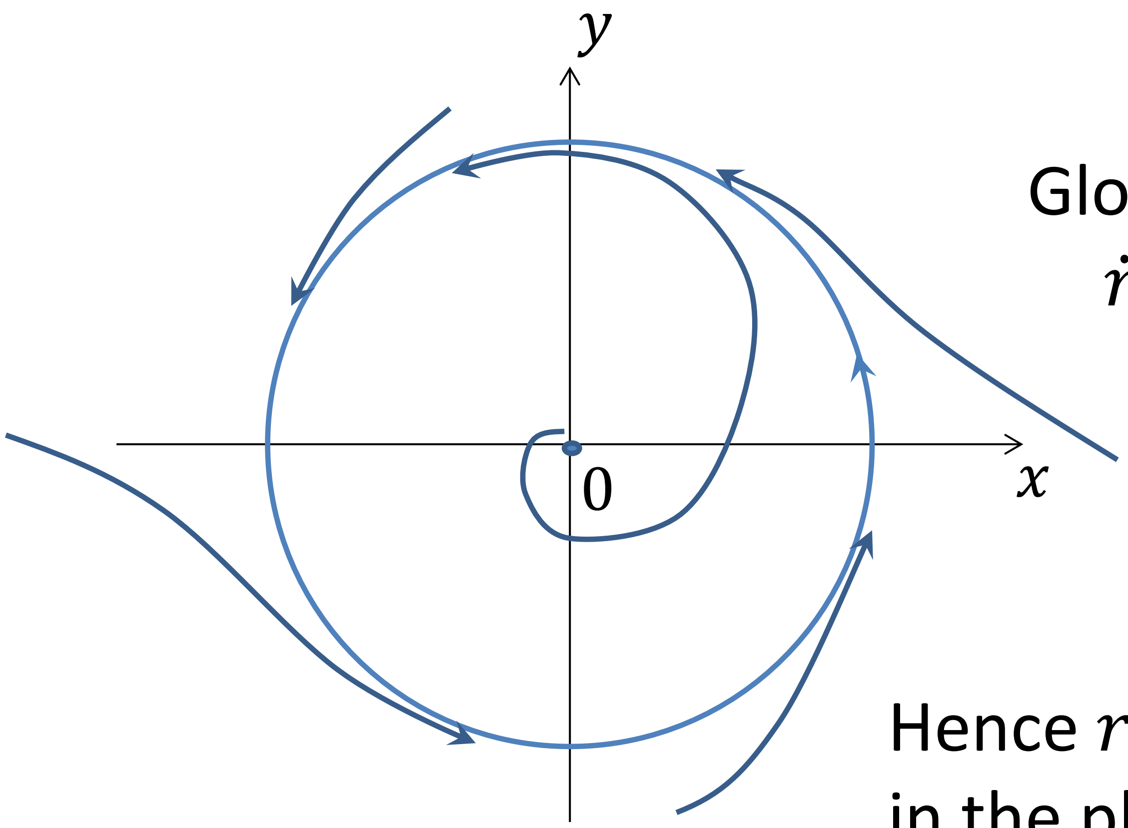
$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$

Here: $r = 0$ is an unstable hyperbolic equilibrium point

$r = 1$ is a limit cycle (since $\dot{r} = 0$)

Setting $r = 1 + \delta r$ gives $\dot{\delta r} \approx -2\delta r$, so $r = 1$ is a stable limit cycle.

Example



Global behaviour:

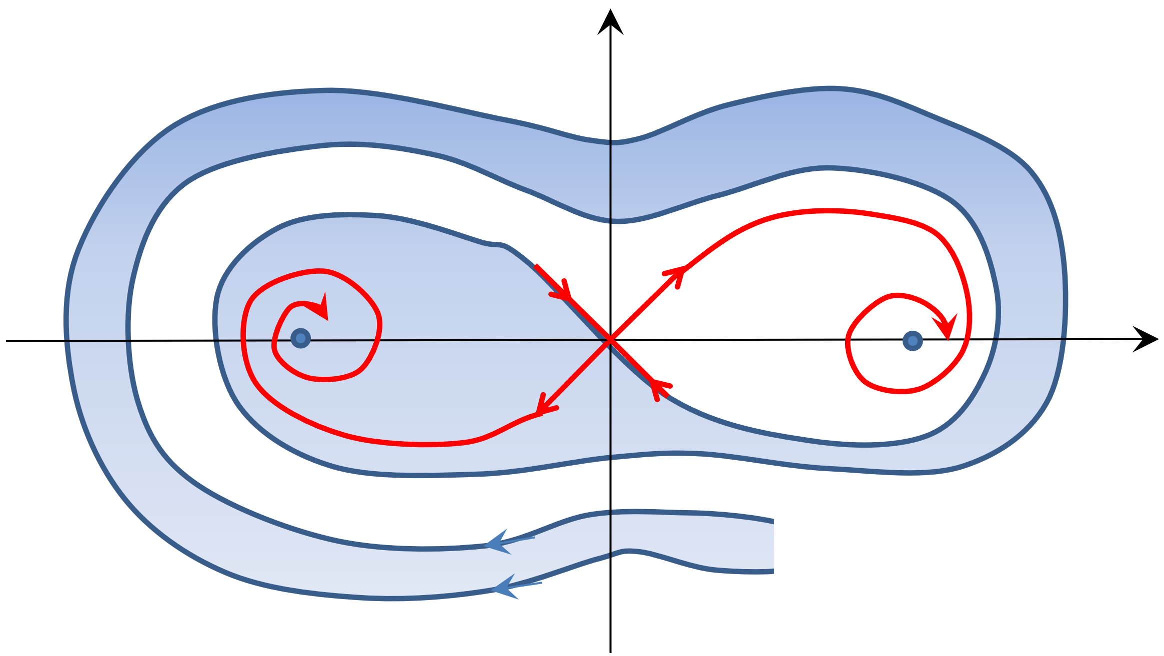
$$\dot{r} = r(1 - r^2) \begin{cases} > 0 & \text{if } r > 1 \\ < 0 & \text{if } 0 < r < 1 \end{cases}$$

Hence $r = 1$ is the ω limit set for all points in the plane except for the origin (which is its own ω limit set). The trapping region U is the whole of the plane minus the origin.

Basin of attraction

The domain or basin of attraction of an attracting set A is the union of all trajectories forming a trapping region of A.

The domain of attraction of the left hand equilibrium point of the Duffing oscillator



La Salle's Invariance Principle

- Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n$ and let $D \subset \mathbb{R}^n$ be a positively invariant set (all points starting in D remain in D). If the boundary of D is differentiable and D has a non-empty interior, then D is a trapping region.

- Suppose there exists a $V(\mathbf{x})$ that satisfies $\dot{V}(\mathbf{x}) \leq 0$ within D and consider the following two sets:

$$E = \left\{\mathbf{x} \in D : \ \dot{V}(\mathbf{x}) = 0\right\}$$
$$M = \{\text{union of all positively invariant sets in } E\}$$

The Principle: Every trajectory starting at $\mathbf{x} \in D$ tends to M as $t \to \infty$.

Example: Duffing Oscillator

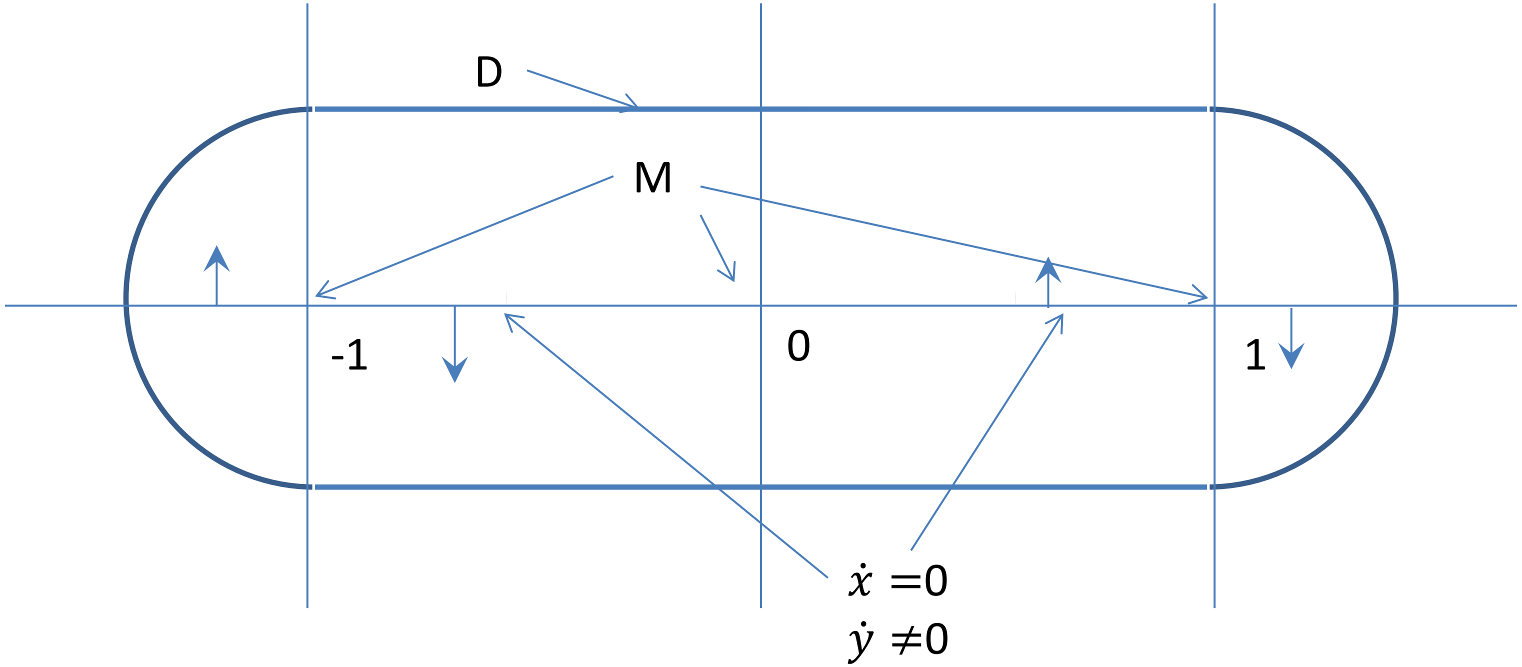
$$\dot{x} = y$$
$$\dot{y} = x - x^3 - \gamma y, \qquad \gamma > 0$$

Let $V(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$, then

$$\dot{V} = -\gamma y^2$$

- Let $D = \{(x, y) : V(x, y) \leq c\}$ for $c > 0$, then is positively invariant since $\dot{V} \leq 0$.
- Here $E = \{(x, y) : y = 0\}$ and $M = \{(-1,0), (0,0), (1,0)\}$.
- LaSalle's principle implies that all trajectories starting in D converge to M, and hence to one of the three equilibria.

The Duffing Oscillator and LaSalle



Types of orbits – Preparation for The Poincaré Bendixson Theorem

The Poincaré Bendixson concerns the range of attractors that can exist in the **Phase Plane**. We will consider possible attractors, but first define some terms:

- A **homoclinic orbit** is a trajectory that joins a saddle point equilibrium point to itself (it moves out on an unstable manifold and comes back on a stable manifold).
- A **heteroclinic orbit** (or **heteroclinic connection**) joins two different equilibrium points.
- A **separatrix cycle** partitions phase space into two regions with different characteristics and there are many ways to construct such cycles.

Homoclinic orbit example

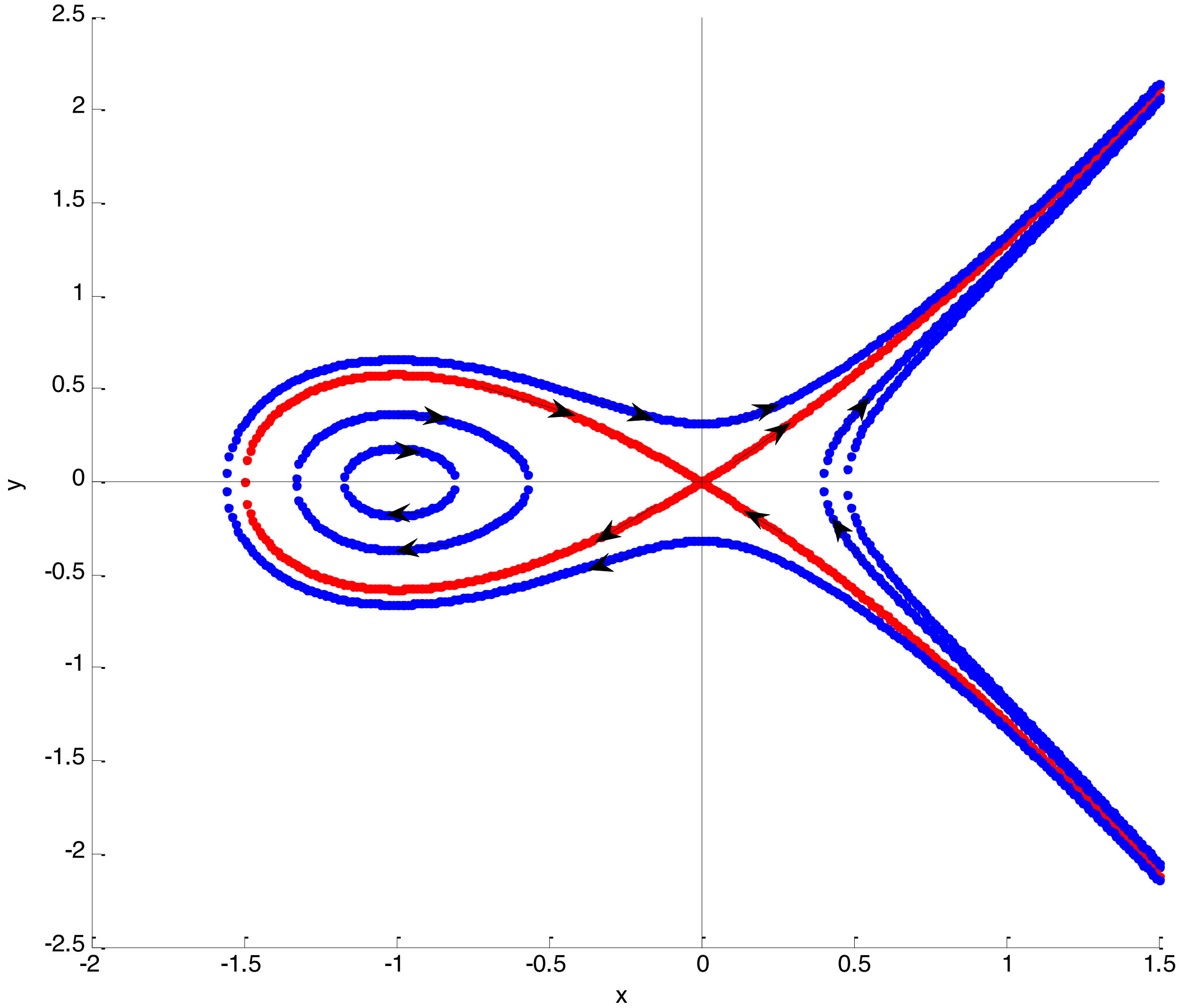
Consider the Hamiltonian system:

$$\dot{x} = y$$
$$\dot{y} = x + x^2$$
$$H(x, y) = \frac{y^2}{2} - \frac{x^2}{2} - \frac{x^3}{3}$$

- The solution curves are the level sets of the Hamiltonian (energy is constant) and are defined by $y^2 - x^2 - \frac{2}{3}x^3 = c$
- If $c = 0$, then $y^2 = x^2 + \frac{2}{3}x^3$, which goes through a saddle point at $(x, y) = (0,0)$ (the Jacobian is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with eigenvalues 1, -1).

Homoclinic orbit example

The trajectory shown in red leaves the origin on the local unstable manifold and returns on the local stable manifold.



Heteroclinic example

- The undamped simple pendulum

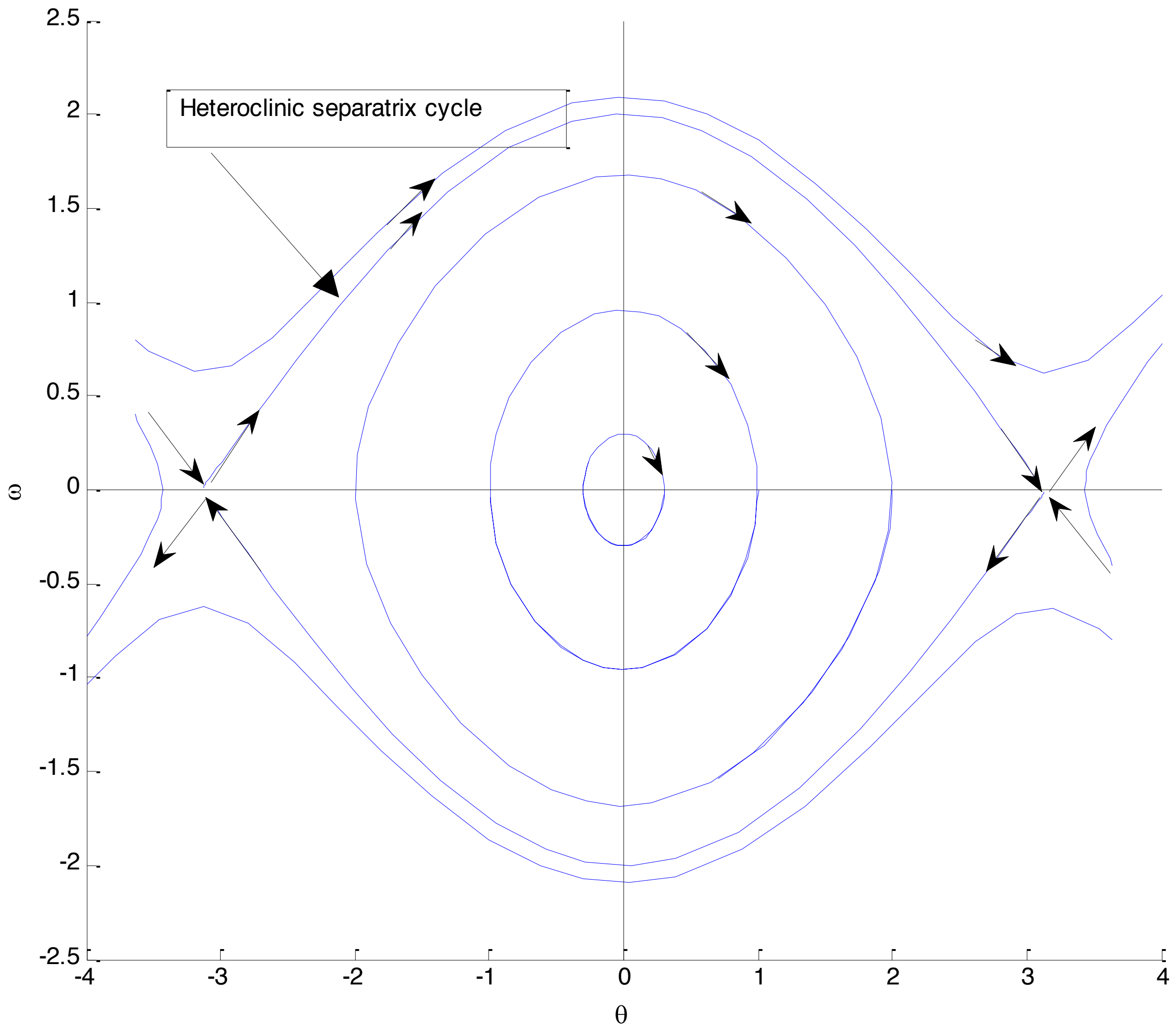
$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\sin\theta$$

- There are saddle points when the pendulum is pointing upwards – these saddle points are connected by a heteroclinic orbit.

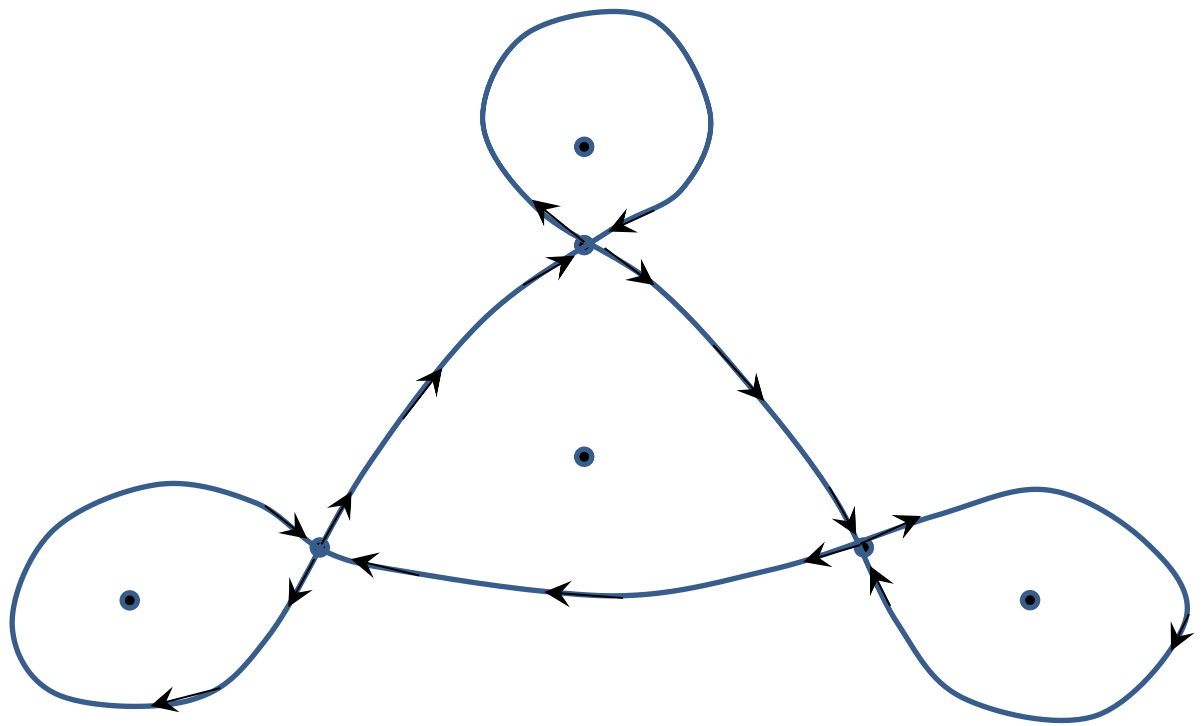
- The two heteroclinic orbits in the upper and lower half of the plane define a heteroclinic separatrix cycle.

- A number of ‘compound’ separatrix cycles are shown in the notes. Note: At any point where the trajectory appears to cross itself there must be an equilibrium point.

Simple pendulum phase portrait



Example compound separatrix cycle



Poincaré Bendixson Theorem in the plane

- Let M be a positively invariant region of a vector field in $\mathbb{R}^2$ containing only a finite number of equilibria. Let $\mathbf{x} \in M$ and consider $\omega(\mathbf{x})$. Then one of the following possibilities holds:
  i. $\omega(\mathbf{x})$ is an equilibrium.
  ii. $\omega(\mathbf{x})$ is a closed orbit.
  iii. $\omega(\mathbf{x})$ consists of a finite number of equilibria $\mathbf{x}_1^*, \dots, \mathbf{x}_n^*$ and orbits γ with $\alpha(\gamma) = \mathbf{x}_i^*$ and $\omega(\gamma) = \mathbf{x}_j^*$. (Note: this defines a set of heteroclinic connections – consider the pendulum phase portrait.)

- If there are only stable equilibria inside M, then there can only be one. If there are no equilibria, then there is a closed orbit inside it.

Example application of Poincaré Bendixson Thm.

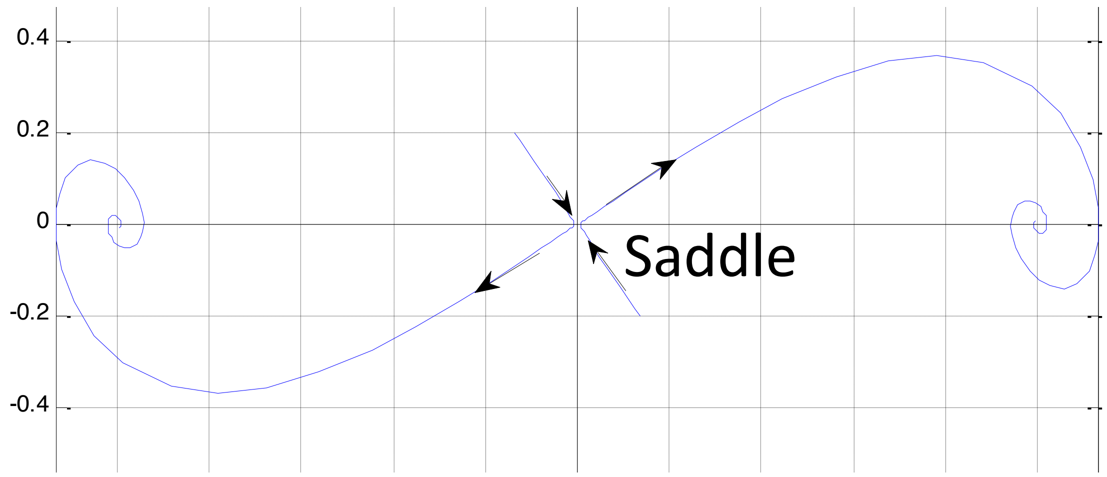
- Consider the Duffing oscillator:

$$\dot{x} = y$$
$$\dot{y} = x - x^3 - \gamma y, \qquad \gamma > 0$$

The level sets of $V(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} = c$ are positively invariant since $\dot{V}(x,y) = -\gamma y^2 \leq 0$.

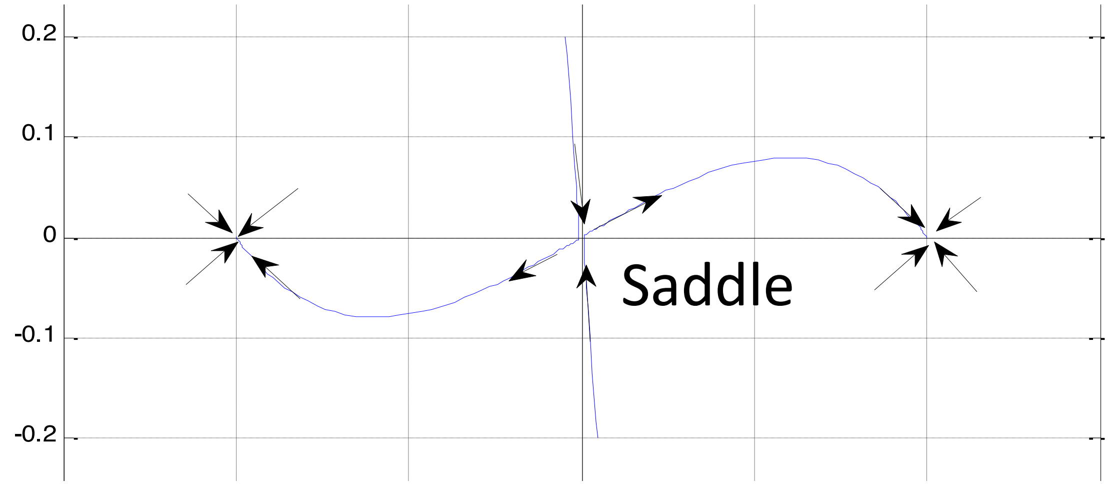
- For $c > 0$, three equilibria lie inside $\{(x,y) : V(x,y) \leq c\}$: an unstable equilibrium at $(0,0)$, and two stable equilibria at $(-1,0), (1,0)$.
- For $c = 0$, the level sets split into two with a common point at $(0,0)$. Hence trajectories leaving the unstable equilibrium point at $(0,0)$ must end up at the stable equilibria.

Duffing Oscillator example illustrating two types of stable equilibria



$0 < \gamma \leq \sqrt{8}$

Foci



$\gamma > \sqrt{8}$

Nodes

Further example

$$\dot{x}_1 = x_1 + x_2 - x_1({x_1}^2 + {x_2}^2)$$
$$\dot{x}_2 = -x_1 + x_2 - x_2({x_1}^2 + {x_2}^2)$$

The origin is an equilibrium point, linearisation gives Jacobian $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ with eigenvalues at $1 \pm j$ (an unstable spiral).

If we consider the region defined by

$$V = \frac{{x_1}^2}{2} + \frac{{x_2}^2}{2} = c$$

If $c > 1$, then $\dot{V} = c - c^2 < 0$ so $\{(x, y) : V \leq c\}$ is an invariant region. As there is only an unstable equilibrium point inside the region, by Poincaré Bendixson the region must contain a stable limit cycle.

Unstable spiral within a stable limit cycle