Model Predictive Control Examples Sheet: Solutions

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Prediction equations

1. (a). The state predictions are:

$$\begin{aligned} x(k|k) &= x(k) \\ x(k+1|k) &= Ax(k) + Bu(k|k) \\ &\vdots \\ x(k+N|k) &= A^N x(k) + A^{N-1}Bu(k|k) + \cdots + Bu(k+N-1|k) \end{aligned}$$

so $x(k+i|k) = A^i x(k) + \mathcal{C}_i \mathbf{u}(k)$, $\mathbf{u}(k) = \begin{bmatrix} u(k|k) & \cdots & u(k+N-1|k) \end{bmatrix}^T$, where $\mathcal{C}_0 = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$, and $\mathcal{C}_i$ is the ith block-row of $\mathcal{C}$:

$$\mathcal{C} = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}$$

For the given A and B we get

$$\{B, AB, A^2B\} = \left\{ \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.05 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.15 \\ 2 \end{bmatrix} \right\}, \text{ so } \mathcal{C}_3 = \begin{bmatrix} 0.15 & 0.05 & 0 \\ 2 & 1 & 0.5 \end{bmatrix}.$$

(b). For $y(k) = Cx(k)$ with $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\lambda = 1$, the cost for $N = 3$ is:

$$\begin{aligned} J(k) &= \sum_{i=0}^{2} \big(y^2(k+i|k) + u^2(k+i|k)\big) + y^2(k+3|k) \\ &= x^T(k) C^T C x(k) + u^2(k|k) \\ &\quad + \big(Ax(k) + \mathcal{C}_1 \mathbf{u}(k)\big)^T C^T C \big(Ax(k) + \mathcal{C}_1 \mathbf{u}(k)\big) + u^2(k+1|k) \\ &\quad + \big(A^2x(k) + \mathcal{C}_2 \mathbf{u}(k)\big)^T C^T C \big(A^2x(k) + \mathcal{C}_2 \mathbf{u}(k)\big) + u^2(k+2|k) \end{aligned}$$

$$\begin{aligned} &\quad + \big(A^3x(k) + \mathcal{C}_3 \mathbf{u}(k)\big)^T C^T C \big(A^3x(k) + \mathcal{C}_3 \mathbf{u}(k)\big) \\ &= \mathbf{u}^T(k) \Big(I + \sum_{i=0}^{3} \mathcal{C}_i^T C^T C \mathcal{C}_i \Big) \mathbf{u}(k) + 2x^T(k) \sum_{i=0}^{3} A^{i^T} C^T C \mathcal{C}_i \mathbf{u}(k) \\ &\quad + x^T(k) \sum_{i=0}^{3} A^{i^T} C^T C A^i x(k) \\ &= \mathbf{u}^T(k) H \mathbf{u}(k) + 2x^T(k) F^T \mathbf{u}(k) + x^T(k) G x(k) \end{aligned}$$

with $H = \begin{bmatrix} 1.025 & 0.0075 & 0 \\ 0.0075 & 1.0025 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 0.2 & 0.12 \\ 0.05 & 0.035 \\ 0 & 0 \end{bmatrix}$, $G = \begin{bmatrix} 4 & 1.1 \\ 1.1 & 0.59 \end{bmatrix}$.

(c). Terms in $J(k)$ that depend on $u(k|k)$:

$$\begin{aligned} J(k) = 1.025u^2(k|k) &+ 2 \times 0.0075 u(k|k) u(k+1|k) + \cdots \\ &+ 2 \times 0.2 x_1 u(k|k) + 2 \times 0.12 x_2 u(k|k) + \cdots \end{aligned}$$

hence $\dfrac{\partial J}{\partial u(k|k)} = 2 \times 1.025 u(k|k) + 2 \times 0.0075 u(k+1|k) + 2 \times 0.2 x_1 + 2 \times 0.12 x_2$, which is the first element of $2H\mathbf{u}(k) + 2Fx(k)$. Repeating this for $u(k+1|k)$ and $u(k+2|k)$ gives

$$\nabla_{\mathbf{u}} J = \begin{bmatrix} \dfrac{\partial J}{\partial u(k|k)} & \dfrac{\partial J}{\partial u(k+1|k)} & \dfrac{\partial J}{\partial u(k+2|k)} \end{bmatrix}^T = 2H\mathbf{u}(k) + 2Fx(k).$$

2. (a). The optimal predicted input sequence is given by

$$\mathbf{u}^*(k) = -H^{-1} F x(k) = -\begin{bmatrix} 0.1948 & 0.1168 \\ 0.0484 & 0.0340 \\ 0 & 0 \end{bmatrix}$$

and the predictive control law is therefore

$$u(k) = K_{N=3} x(k), \quad K_{N=3} = -\begin{bmatrix} 0.1948 & 0.1168 \end{bmatrix}.$$

This is a linear feedback law so we can determine stability by checking the closed loop poles: $\text{eig}(A + BK_{N=3}) = \{1.01,\ 1.93\}$. Since these poles lie outside the unit circle, the closed loop system is unstable.

(b). Code to construct AA $= \begin{bmatrix} I & A^T & \cdots & A^{N^T} \end{bmatrix}^T$ and CC $= \begin{bmatrix} \mathcal{C}_0^T & \mathcal{C}^T \end{bmatrix}^T$:

```
AA = [eye(nx);zeros(N*nx,nx)]; CC = zeros((N+1)*nx,N*nu);
tmp = eye(nx);
for i = 1:N,
  rows = i*nx+(1:nx);
  CC(rows,:) = [tmp*B,CC(rows-nx,1:end-nu)];
  tmp = A*tmp;
  AA(rows,:) = tmp;
end
```

Code to construct H $= H$ and F $= F$:

```
H = eye(N); F = 0;
for i=1:N+1,
 rows = (i-1)*nx+(1:nx);
 H = H + CC(rows,:)'*C'*C*CC(rows,:);
 F = F + CC(rows,:)'*C'*C*AA(rows,:);
end
```

Prediction equations

3. (a). The predictive control law is now the first element of the predicted input sequence that is obtained by minimizing an infinite horizon predicted cost subject to the terminal equality constraint $x(k+N|k) = 0$:

$$\mathbf{u}^*(k) = \quad \arg\min_{\mathbf{u}} J = \sum_{i=0}^{\infty} y^2(k+i|k) + \lambda u^2(k+i|k)$$

subject to $x(k+N|k) = 0$

In this case the optimal cost $J^*\big(x(k)\big)$ is a Lyapunov function since: (i) $J(x)$ is positive definite in x if $N \geq 2$ (since (A, C) is observable); and (ii) $\{u^*(k+1|k), \ldots, u^*(k+N-1|k), 0, 0, \ldots\}$ is a feasible predicted input sequence at $k+1$ so $J^*\big(x(k+1)\big) \leq J^*\big(x(k)\big) - \big(y^2(k) + \lambda u^2(k)\big)$. This implies $x = 0$ is a stable equilibrium, while (ii) also implies that $y^2(k) + \lambda u^2(k) \to 0$ as $k \to \infty$, and hence the origin is asymptotically stable (since (A, C) is observable).

(b). In practice the constraint $x(k+N|k) = 0$ should be avoided because:

(i) it results in poor robustness to model errors and disturbances, and

(ii) it leads to very active predicted control sequences.

Closed-loop stability can be ensured instead by using the infinite horizon cost as described in Section 3.1 of the lecture notes (Lecture 3).

4. (a). Repeating the MPC optimization introduces feedback into the control law (which provides some robustness to model and measurement uncertainty), since the optimal predicted input sequence at time k depends on $x(k)$. It also removes some of the suboptimality that results from optimizing performance over a finite number of free variables.

(b). The cost over mode 2 is

$$\sum_{i=N}^{\infty} [y^2(k+i|k) + u^2(k+i|k)] = x^T(k+N|k)\bar{Q}x(k+N|k)$$

where matrix $\bar{Q}$ satisfies $\bar{Q} - (A+BK)^T\bar{Q}(A+BK) = C^TC + K^TK$.

For the given (A, B, C), K, and $\bar{Q}$ we get $A + BK = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ and

$$C^TC + K^TK = \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}, \quad (A+BK)^T\bar{Q}(A+BK) = \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix}$$

and hence $\bar{Q} - (A+BK)^T\bar{Q}(A+BK) = C^TC + K^TK$, as required.

(c). Over the mode 2 prediction horizon, the inputs can be expressed

$$u(k+N+i|k) = K(A+BK)^i x(k+N|k), \quad i = 0, 1, \ldots$$

For this system and mode 2 feedback gain K we get $(A+BK)^i = 0$ for all $i \geq 2$, and hence $K(A+BK)^i x = 0$ for all $i \geq 2$. Therefore the constraints need only be invoked over mode 1 and at prediction times N and $N+1$ in order to ensure that they are satisfied at every instant on an infinite prediction horizon.

(d). If the optimization is feasible at time k, then, since predictions at k satisfy constraints at all future times, the input sequence defined by $u(k+1+i|k+1) = u^*(k+1+i|k)$ for $i = 0, 1, \ldots$ must satisfy the constraints in the optimization at time $k+1$. This sequence gives a (suboptimal) cost of $J^*(k) - \big(y^2(k) + u^2(k)\big)$, and after optimization at time $k+1$ we therefore get $J^*(k+1) \leq J^*(k) - \big(y^2(k) + u^2(k)\big)$.

Summing this inequality for $k = 0, 1, \ldots$ gives

$$\sum_{k=0}^{\infty} \big(y^2(k) + u^2(k)\big) \leq J^*(0) - \lim_{k\to\infty} J^*(k) \leq J^*(0).$$

(e). The bound on $J^*(k+1) - J^*(k)$ in part (d) implies that the closed loop system is stable (i.e. $x = 0$ is locally asymptotically stable) if the pair (A, C) is observable, which is the case here since $CA = \begin{bmatrix} -2 & 2 \end{bmatrix}$ is not parallel with $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

5. (a). In a dual mode prediction strategy, terminal constraints ensure that predictions satisfy the system input and state constraints over the infinite horizon of mode 2 (i.e. at prediction times $i = N, N+1, \ldots$). This provides a recursive guarantee of feasibility by ensuring that, at each sampling instant k, the tail of the predicted input and state sequences that solve the constrained MPC optimization at time k will satisfy the constraints of the MPC optimization at time $k+1$. This makes it possible to derive a guarantee that the optimal predicted cost decreases with k, and hence a guarantee of closed loop stability.

A terminal constraint set $\mathcal{S}$ must be:

- invariant under the mode 2 feedback law, i.e. $x_{k+i} \in S$ implies $x_{k+i+1} \in S$, for all $i \geq N$.
- feasible, i.e. state constraints are instantaneously satisfied and input constraints are satisfied by the mode 2 feedback law, for all $x \in \mathcal{S}$.

(b). (i). It is easy to verify invariance here because $A + BK$ is diagonal. Specifically, $\mathcal{S} = \{x : |x_1 + x_2| \leq 1\,,\ |x_1 - x_2| \leq 1\}$ is invariant under $x(k+1) = (A+BK)x(k)$ because $A + BK = \text{diag}\{0.5, -0.3\}$ so the vertices of $\mathcal{S}$: $v = [\pm 1 \ \ 0]^T, [0 \ \ \pm 1]^T$ satisfy $(A+BK)v = [\pm 0.5 \ \ 0]^T, [0 \ \ \mp 0.3]^T \in \mathcal{S}$. Furthermore it is obvious that $\mathcal{S}$ is feasible with respect to the state constraints.

(ii). The maximal terminal set is obtained by checking the system constraints over a sufficiently long constraint checking horizon horizon N_c, i.e. for some finite N_c:

$$\mathcal{S}_{\max} = \{x : (A+BK)^i x \in \mathcal{X}, i = 0, 1, \ldots N_c - 1\}$$

where $\mathcal{X} = \{x : |x_1 + x_2| \leq 1\,,\ |x_1 - x_2| \leq 1\}$ is the constraint set for the system state. The minimum allowable value for N_c can be found by checking whether $(A+BK)^{M+1}x \in \mathcal{X}$ for all x such that $(A+BK)^i x \in \mathcal{X}$ for $i = 0, 1, \ldots M$. If this is true, then $N_c = M$, otherwise $N_c > M$. For given M this condition can be checked by solving a pair of linear programs:

$$z = \max\{\max_x\, [1 \ \ 1](A+BK)^{M+1}x,\ \ \max_x\, [1 \ \ -1](A+BK)^{M+1}x\}$$

where each of the two maximizations over x is performed subject to $(A+BK)^i x \in \mathcal{X}$, $i = 0, \ldots M$, and then checking whether $z < 1$.

(c). Increasing the mode 1 horizon N results in:

- improved performance, since the optimal value of predicted cost is non-increasing with increasing N (but note that, for each initial condition $x(0)$, there exists a finite value, N_∞, such that no improvement in closed loop performance can be obtained with $N > N_\infty$);
- larger operating region, since the set of feasible initial conditions is non-decreasing with increasing N;
- higher computational load, since the number of optimization variables increases.

Hence choosing N involves a trade-off between performance and computation.

Integral action and disturbances

6. (a). Using either $x_{ss} = (I - A - BK)^{-1} B_d d$ or

$$\dot{y}_{ss} = 0 \quad \Longrightarrow \quad u_{ss} = \frac{mg}{K_v} \quad \Longrightarrow \quad x_{ss} = \begin{bmatrix} mg/(K_v K_1) \\ 0 \end{bmatrix}$$

(where $K_1 = -66.0$), we get the maximum steady state error as:

$$d = m = 0.5\,\text{kg} \quad \Longrightarrow \quad e_{ss} = -0.0106\,\text{m}$$

(b). Augment the model to include integrated error e_I:

$$\xi(k+1) = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} \xi(k) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k), \quad \xi(k) = \begin{bmatrix} x(k) \\ e_I(k) \end{bmatrix}$$

and modify the performance index to include a term penalising e_I:

$$J(k) = \sum_{i=0}^{\infty} \big(e^2(k+i) + \lambda u^2(k+i) + \lambda_I e_I^2(k+i) \big).$$

Taking $\lambda = 10^{-4}$, and choosing e.g. $\lambda_I = 1$, this results in the LQ-optimal feedback law $u(k) = K_\xi \xi(k)$, with $K_\xi = -\begin{bmatrix} 201.4 & 29.6 & 48.2 \end{bmatrix}$. Figure 1 shows the closed loop responses for $m = 0.5\,\text{kg}$, when the platform is released from rest with $e(0) = 0$.

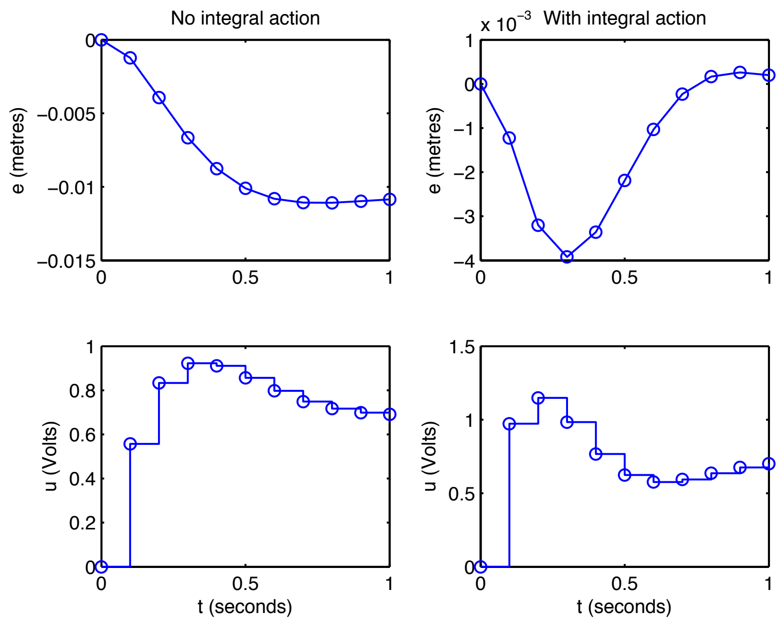

Figure 1. Closed loop platform responses for unconstrained LQ-optimal feedback without integral action (left) and with integral action (right)

(c). (i). The mode 2 feedback law is $u(k) = K_\xi \xi(k)$, where K_ξ is the LQ-optimal gain for the unconstrained minimization of J. The predicted cost can be written

$$J(k) = \sum_{i=0}^{\infty} \Big(\|\xi(k+i|k)\|^2_{Q_\xi} + \lambda u^2(k+i|k) \Big), \quad Q_\xi = \begin{bmatrix} C^T C & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, if P is the solution of the Lyapunov equation:

$$P - \left(\begin{bmatrix} A & 0 \\ C & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} K_\xi \right)^T P \left(\begin{bmatrix} A & 0 \\ C & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} K_\xi \right) = \begin{bmatrix} C^T C & 0 \\ 0 & 1 \end{bmatrix} + \lambda K_\xi^T K_\xi$$

then $\xi^T(k+N|k)P\xi(k+N|k)$ is is equal to the predicted cost over mode 2 for the nominal case of $d=0$, and hence

$$J(k) = \sum_{i=0}^{N-1} \Big(\|\xi(k+i|k)\|^2_{Q_\xi} + \lambda u^2(k+i|k) \Big) + \|\xi(k+N|k)\|^2_P.$$

(ii). In order to ensure robust constraint satisfaction (and hence guarantee that the MPC optimization remains feasible if it is initially feasible), the constraints must be imposed on the predictions that correspond to the worst-case disturbance values (i.e. $d=0$ and $d=0.5$). This can be achieved simply by generating two sets of predictions (i.e. the predictions that are obtained with $d=0$ and $d=0.5$), and ensuring that $u(k+i|k)$ lies in the allowable range for all $i=0,1,\ldots$ in both cases.

Since the open loop system is not strictly stable (it has repeated poles on the unit circle), the two sets of predictions will diverge over the mode 1 prediction horizon if the prediction dynamics are open loop in mode 1. This makes it difficult to satisfy constraints, and in fact it can be shown that the implied optimization is infeasible for all initial conditions whenever $N \geq 2$.

To overcome this problem it is necessary to pre-stabilize the mode 1 prediction dynamics by introducing feedback into mode 1 predictions:

$$u(k+i|k) = \begin{cases} K_\xi \xi(k+i|k) + c(i|k) & i = 0,\ldots,N-1 \\ K_\xi \xi(k+i|k) & i = N, N+1, \ldots \end{cases}$$

where $\mathbf{c}(k) = \{c(0|k),\ldots,c(N-1|k)\}$ are the degrees of freedom in predictions.

With this modification the input predictions corresponding to $d=0$ must lie between the solid lines in Figure 2 in order to satisfy the input constraints robustly. Since these upper and lower bounds do not overlap, the problem is now feasible for any N for some initial conditions $\xi(0)$.

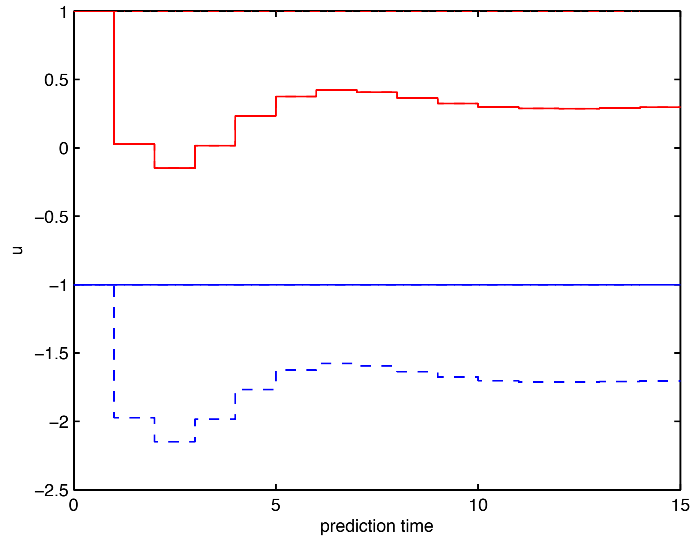

Figure 2. Bounds on nominal input predictions ensuring $|u(k+i|k)| \leq 1$ for all values of disturbance in the range $0 \leq d \leq 0.5$

7. Robust constraint satisfaction for all possible disturbance values implies recursive feasibility, so the robust constraints will be feasible at all times if they are initially feasible.

   The optimal predicted cost is not necessarily always decreasing over time since $d=0$ is assumed in evaluating the cost, and therefore (asymptotic) stability cannot be inferred from the optimal cost.

   However $c(0|k)$ necessarily converges to zero asymptotically, and it follows that the closed-loop response will be stable whenever the MPC problem is initially feasible.

   Furthermore, if stable, the response must have zero steady-state error since $e_I(k)$ must be finite in the steady state. This is illustrated in Figure 3 (compare the unconstrained responses of Figure 1 for same initial conditions).

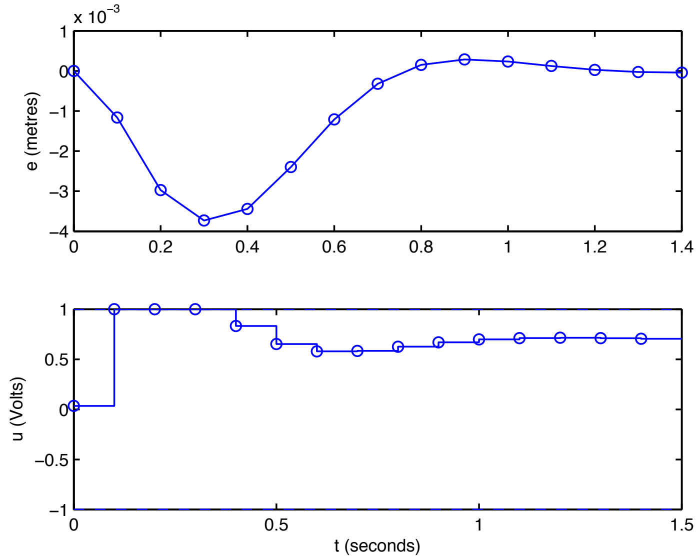

Figure 3. Closed-loop platform response under the MPC law that incorporates robust constraints and integral action.

8. (a). Two main advantages:

   (i). The receding horizon optimization is repeated at each time step, and this provides feedback (since the optimal predicted input sequence at k depends on the state x_k), reducing the effect of the uncertainty in w_k.

   (ii) Due to the presence of constraints, the optimization has to be performed over a finite number of free variables. Using a receding horizon optimization reduces the degree of suboptimality with respect to the infinite horizon optimal control problem.

(b). (i). From $x_{k+1} = x_k + u_k - w_k$ and $e_k = x_k - x^*$ we get the error dynamics $e_{k+1} = e_k + u_k - w_k$. Introducing the feedback law $u_k = \hat{w} - e_k$ and assuming $w_k = \hat{w}$ gives $e_k = 0$ for all $k \geq 1$, and since e_0 is independent of u_0, this necessarily provides the minimum cost: $J_0 = e_0^2$.

   (ii). The mode 2 feedback law $u_{k+i|k} = \hat{w} - e_{k+i|k}$ for $i \geq N$ gives $e_{k+i|k} = 0$ for all $i \geq N$. The associated cost is $J_k = \sum_{i=0}^{\infty} e_{k+i|k}^2 = \sum_{i=0}^{N-1} e_{k+i|k}^2 + e_{k+N|k}^2$.

   (iii). The constraints for the infinite prediction horizon are the union of the constraints for the horizons of mode 1 and mode 2.
   Mode 1 constraints: $0 \leq u_{k+i|k} \leq U$ and $0 \leq x_{k+i|k} \leq X$ for $i = 0, 1, \ldots, N-1$.
   Mode 2 constraints: $u_{k+i|k} = \hat{w} - e_{k+i|k}$ for $i \geq N$ implies $e_{k+i|k} = 0$ for $i \geq N+1$ and hence $x_{k+i|k} = x^*$ and $u_{k+i|k} = \hat{w}$ for $i \geq N+1$.
   Assuming the steady state to be feasible (i.e. $0 \leq \hat{w} \leq U$ and $0 \leq x^* \leq X$), the constraint checking horizon is therefore $N_c = 1$ and the mode 2 constraints are $0 \leq x_{k+N|k} \leq X$ and $0 \leq u_{k+N|k} = \hat{w} - e_{k+N|k} \leq U$, or equivalently $\max\{0, \hat{w} + x^* - U\} \leq x_{k+N|k} \leq \min\{X, \hat{w} + x^*\}$.

(c). Predictions: $u_{k+i|k} = \hat{w} - e_{k+i|k} + c_{i|k}$, where $c_{i|k}$ for $i = 0, \ldots, N-1$ are free variables (to be optimized online) and $c_{i|k} = 0$ for $i \geq N$. This gives pre-stabilized predictions, thus reducing the effect of the unknown disturbance $w_{k+i} - \hat{w}$ on the predicted sequence: $e_{k+i+1|k} = \hat{w} - w_{k+i|k} + c_{i|k}$, $i = 0, 1, \ldots$.