C21 Model Predictive Control

Mark Cannon

4 lectures

Michaelmas Term 2017

Lecture 1

Introduction

Organisation

- 4 lectures:

week 3	Wednesday	12-1 pm	LR2
	Friday	12-1 pm	LR2
week 4	Wednesday	12-1 pm	LR2
	Friday	12-1 pm	LR2

- 1 class:

week 5	Friday	5-6 pm	LR5
week 6	Thursday	4-5 pm	LR5
	Friday	4-5 pm	LR6

Course outline

1. Introduction to MPC and constrained control
2. Prediction and optimization
3. Closed loop properties
4. Disturbances and integral action
5. Robust tube MPC

Books

1. B. Kouvaritakis and M. Cannon, *Model Predictive Control: Classical, Robust and Stochastic*, Springer 2015

   Recommended reading: Chapters 1, 2 & 3

2. J.B. Rawlings and D.Q. Mayne, *Model Predictive Control: Theory and Design.* Nob Hill Publishing, 2009

3. J.M. Maciejowski, *Predictive control with constraints.* Prentice Hall, 2002

   Recommended reading: Chapters 1–3, 6 & 8

Introduction

Classical controller design:

1. Determine plant model
2. Design controller (e.g. PID)
3. Apply controller

discard model

$x_{k+1} = f(x_k, u_k)$

$\downarrow$

Model predictive control (MPC):

1. Use model to predict system behaviour
2. Choose optimal trajectory
3. Repeat procedure (feedback)

Introduction

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Model predictive control (MPC):

1. Use model to predict system behaviour
2. Choose optimal trajectory
3. Repeat procedure (feedback)

user-defined optimality criterion

Overview of MPC

Model predictive control strategy:

1. Prediction
2. Online optimization
3. Receding horizon implementation

1. Prediction

⋆ Plant model: $x_{k+1} = f(x_k, u_k)$
⋆ Simulate forward in time (over a prediction horizon of N steps)

$$\text{input sequence:} \quad \mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ u_{1|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix} \quad \xrightarrow{\text{defines}} \quad \text{state sequence:} \quad \mathbf{x}_k = \begin{bmatrix} x_{1|k} \\ x_{2|k} \\ \vdots \\ x_{N|k} \end{bmatrix}$$

Notation: $(u_{i|k}, x_{i|k})$ = predicted i steps ahead | evaluated at time k
$x_{0|k} = x_k$

Overview of MPC

2. Optimization

   ⋆ Predicted quality criterion/cost: $J_k = \sum_{i=0}^{N} l_i(x_{i|k}, u_{i|k})$

   $l_i(x, u)$: stage cost

   ⋆ Solve numerically to determine optimal input sequence:

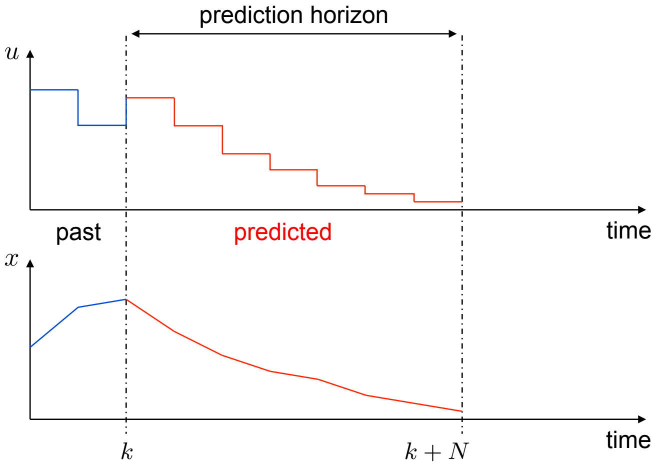
$$\begin{aligned} \mathbf{u}_k^* &= \arg\min_{\mathbf{u}_k} J_k \\ &= (u_{0|k}^*, \ldots, u_{N-1|k}^*) \end{aligned}$$

3. Implementation

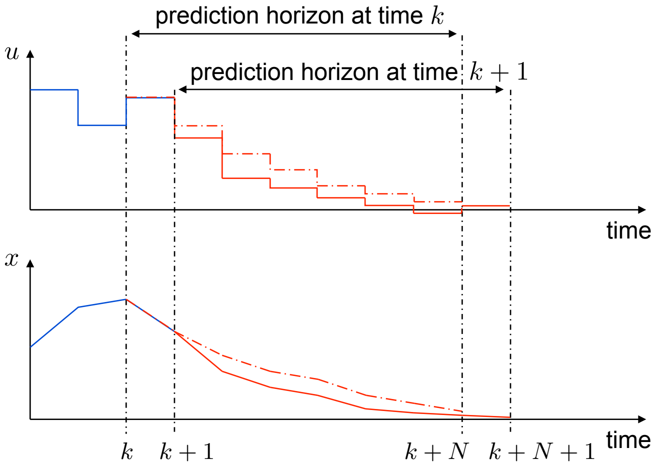
   ⋆ Use first element of $\mathbf{u}_k^*$ $\implies$ actual plant input $u_k = u_{0|k}^*$

   ⋆ Repeat optimization at each sampling instant $k = 0, 1, \ldots$

Overview of MPC



Overview of MPC



Overview of MPC

Optimization is repeated online at each sampling instant $k = 0, 1, \ldots$

$$\Downarrow$$

receding horizon:

$$\begin{aligned} \mathbf{u}_k &= (u_{0|k}, \ldots, u_{N-1|k}) & \mathbf{x}_k &= (x_{1|k}, \ldots, x_{N|k}) \\ \mathbf{u}_{k+1} &= (u_{0|k+1}, \ldots, u_{N-1|k+1}) & \mathbf{x}_{k+1} &= (x_{1|k+1}, \ldots, x_{N|k+1}) \\ &\vdots & &\vdots \end{aligned}$$

- $\star$ provides feedback since $\mathbf{u}_k^*, \mathbf{x}_k^*$ are functions of x_k
  - $\rightarrow$ so reduces effects of model error and measurement noise
- $\star$ and compensates for finite number of free variables in predictions
  - $\rightarrow$ so improves closed-loop performance

Example

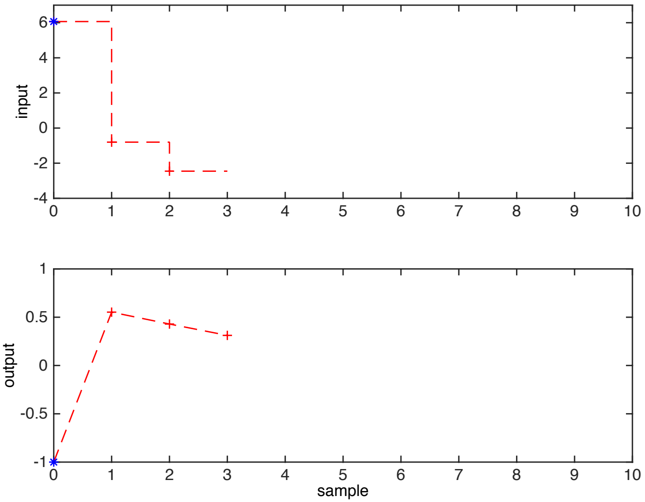
Plant model:
$$x_{k+1} = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u_k$$
$$y_k = \begin{bmatrix} -1 & 1 \end{bmatrix} x_k$$

Cost: $\displaystyle\sum_{i=0}^{N-1} (y_{i|k}^2 + u_{i|k}^2) + y_{N|k}^2$

Prediction horizon: $N = 3$

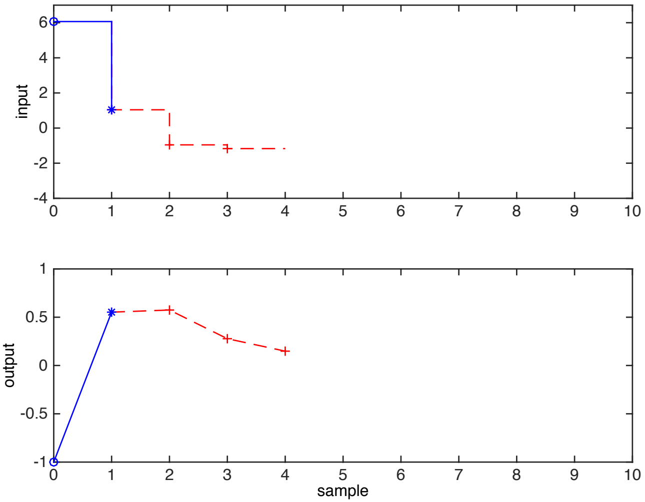
Free variables in predictions: $\mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ u_{1|k} \\ u_{2|k} \end{bmatrix}$

Example



input
6
4
2
0
-2
-4
0 1 2 3 4 5 6 7 8 9 10
output
1
0.5
0
-0.5
-1
0 1 2 3 4 5 6 7 8 9 10
sample

Example



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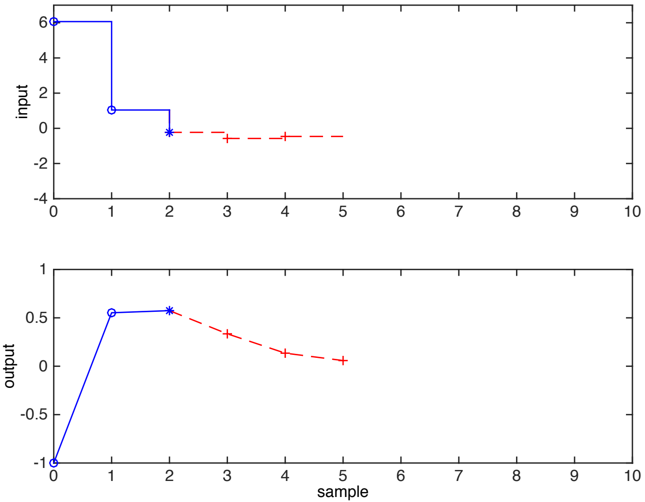

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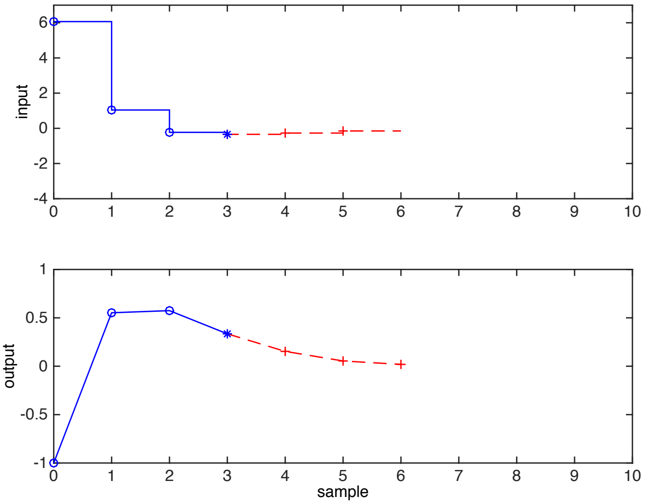
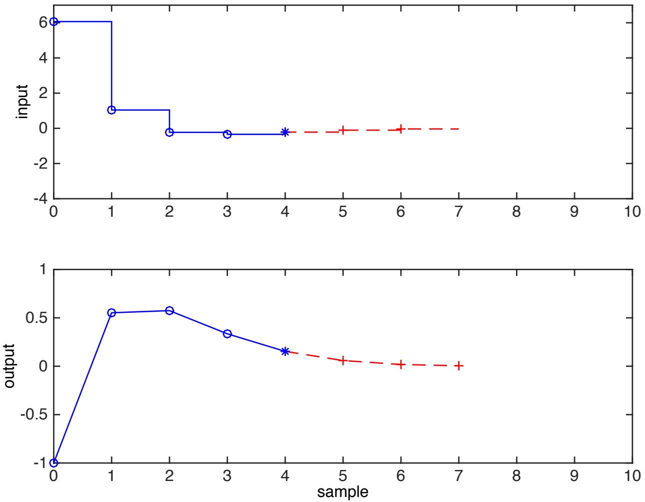
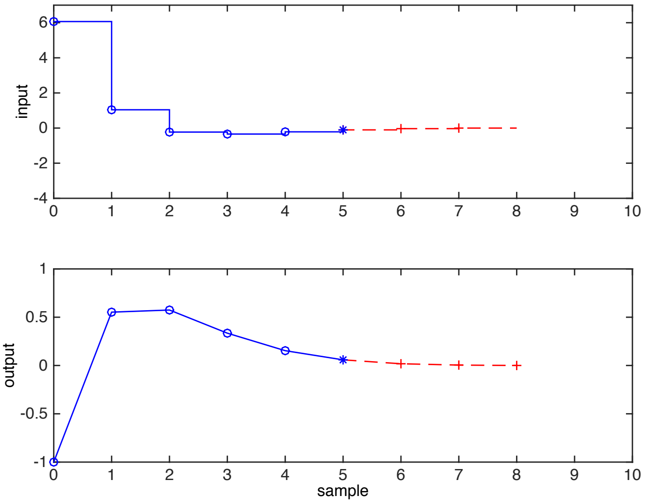
Example



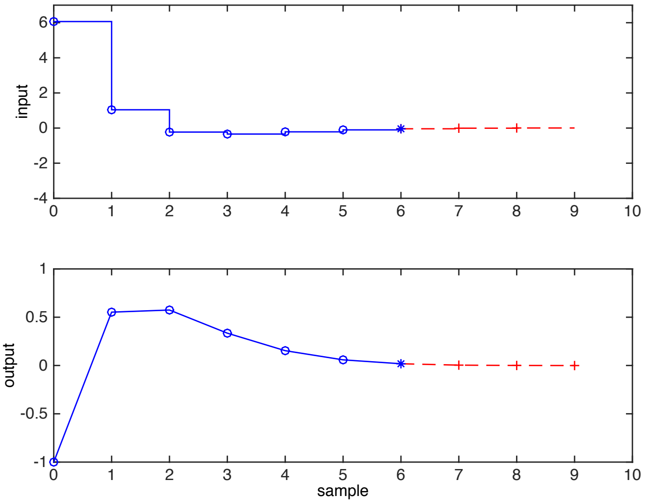
Example



Example



Example



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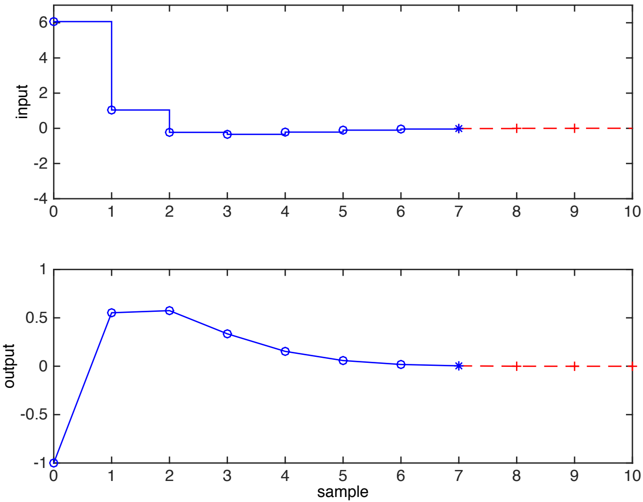
Example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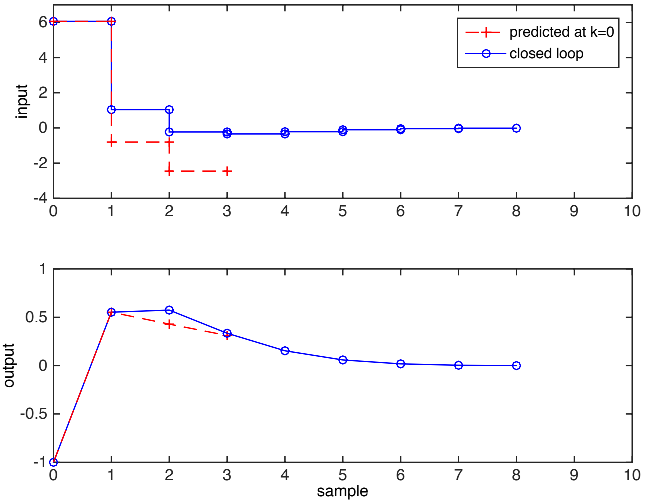
Example



predicted at k=0
closed loop
input
output
sample
6
4
2
0
-2
-4
1
0.5
0
-0.5
-1
0 1 2 3 4 5 6 7 8 9 10

Motivation for MPC

Advantages

- Flexible plant model

  e.g. multivariable
  linear or nonlinear
  deterministic, stochastic or fuzzy

- Handles constraints on control inputs and states

  e.g. actuator limits
  safety, environmental and economic constraints

- Approximately optimal control

Disadvantages

- Requires online optimization

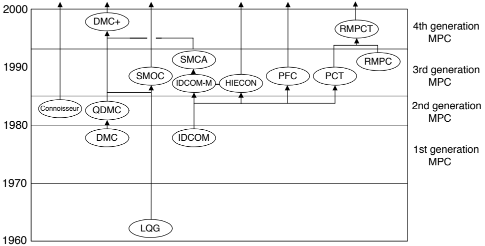
  e.g. large computation for nonlinear and uncertain systems

Historical development

Control strategy reinvented several times

LQ(G) optimal control	1950's–1980's
industrial process control	1980's
constrained nonlinear control	1990's–today

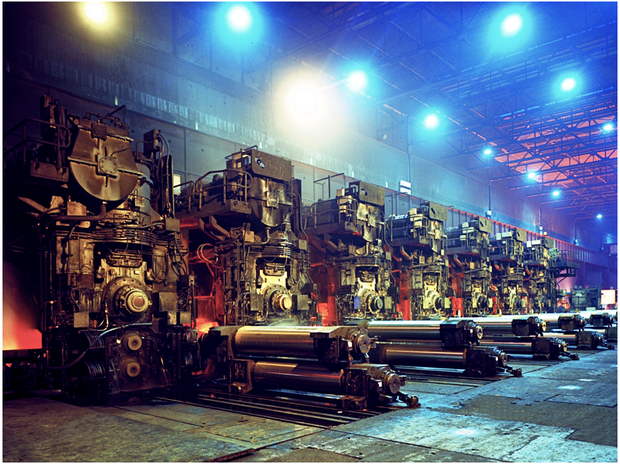
Development of commercial MPC algorithms:



[from Qin & Badgwell (2003)]

Applications: Process Control

Steel hot rolling mill



Applications: Process Control

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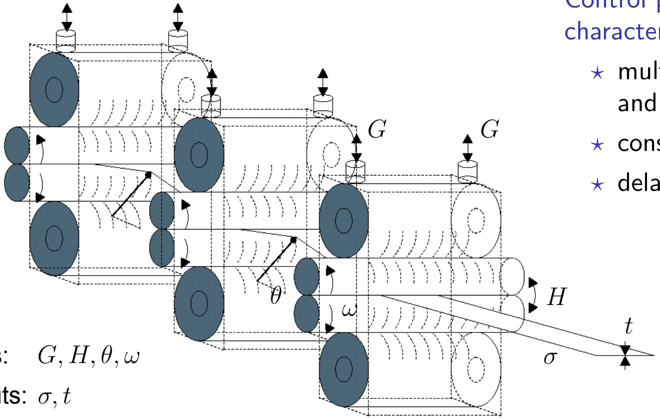


Inputs: G, H, θ, ω

Outputs: σ, t

Control problem characteristics:

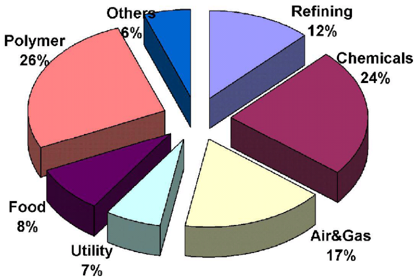
- ⋆ multiple inputs and outputs
- ⋆ constraints
- ⋆ delays

Objectives: control residual stress σ and thickness t

Applications: Chemical Process Control

- Applications of predictive control to more than 4,500 different chemical processes (based on a 2006 survey)

- MPC applications in the chemical industry



[from Nagy (2006)]

Typical control problems:

- ⋆ nonlinear dynamics
- ⋆ slow sampling rates
- ⋆ non-quadratic utility functions

Applications: Electromechanical systems

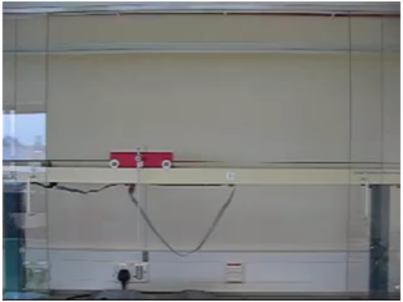
Variable-pitch wind turbines

Control problem characteristics:

- ⋆ stochastic uncertainty
- ⋆ fatigue constraints

Applications: Electromechanical systems

Predictive swing-up and balancing controllers



Autonomous racing for remote controlled cars

Control problem characteristics:

- ⋆ reference tracking
- ⋆ short sampling intervals
- ⋆ nonlinear dynamics

Prediction model

Linear plant model: $x_{k+1} = Ax_k + Bu_k$

- ▷ Predicted $\mathbf{x}_k$ depends linearly on $\mathbf{u}_k$ [details in Lecture 2]

- ▷ Therefore the cost is quadratic in $\mathbf{u}_k$ $\quad \mathbf{u}_k^T H \mathbf{u}_k + 2f^T \mathbf{u}_k + g(x_k)$

  and constraints are linear $\quad A_c \mathbf{u}_k \leq b(x_k)$

- ▷ Online optimization:

$$\min_{\mathbf{u}} \quad \mathbf{u}^T H \mathbf{u} + 2f^T \mathbf{u} \quad \text{s.t.} \quad A_c \mathbf{u} \leq b_c$$

is a convex Quadratic Program (QP),
which is reliably and efficiently solvable

Prediction model

Nonlinear plant model: $x_{k+1} = f(x_k, u_k)$

- ▷ Predicted $\mathbf{x}_k$ depends nonlinearly on $\mathbf{u}_k$

- ▷ In general the cost is nonconvex in $\mathbf{u}_k$ $\quad J_k(x_k, \mathbf{u}_k)$

  and the constraints are nonconvex $\quad g_c(x_k, \mathbf{u}_k) \leq 0$

- ▷ Online optimization:

$$\min_{\mathbf{u}} \quad J_k(x_k, \mathbf{u}) \quad \text{s.t.} \quad g_c(x_k, \mathbf{u}) \leq 0$$

is nonconvex

may have local minima

may not be solvable efficiently or reliably

Prediction model

Discrete time prediction model

- Predictions optimized periodically at $t = 0, T, 2T, \dots$
- Usually $T = T_s$ = sampling interval of model
- But $T = nT_s$ for $n > 1$ is also possible, e.g. if $T_s <$ time required to perform online optimization

n = integer so a time-shifted version of the optimal input sequence at time k can be implemented at time $k+1$

(allows a guarantee of stability – [Lecture 3])

e.g. if $n = 1$, then $\mathbf{u}_{k+1} = (u_{1|k}, \dots, u_{N-1|k}, u_{N|k})$ is possible,
where $(u_{0|k}, u_{1|k}, \dots, u_{N-1|k}) = \mathbf{u}_k^*$

Prediction model

Continuous time prediction model

- Predicted $u(t)$ need not be piecewise constant,

  e.g. 1st order hold gives continuous, piecewise linear $u(t)$
  or $u(t) =$ polynomial in t (piecewise quadratic, cubic etc)

- Continuous time prediction model can be integrated online,

  which is useful for nonlinear continuous time systems

- This course: discrete-time model and $T = T_s$ assumed

Constraints

Constraints are present in almost all control problems

- ▷ Input constraints, e.g. box constraints:

$$\underline{u} \le u_k \le \overline{u} \qquad \text{(absolute)}$$

$$\underline{\Delta u} \le u_k - u_{k-1} \le \overline{\Delta u} \qquad \text{(rate)}$$

  - ⋆ typically active during transients, e.g. valve saturation
    or d.c. motor saturation

- ▷ State constraints, e.g. box constraints

$$\underline{x} \le x_k \le \overline{x} \qquad \text{(linear)}$$

  - ⋆ can be active during transients, e.g. aircraft stall speed
  - ⋆ and in steady state, e.g. economic constraints

Constraints

Classify constraints as either hard or soft

- Hard constraints must be satisfied at all times,
  if this is not possible, then the problem is infeasible

- Soft constraints can be violated to avoid infeasibility

- Strategies for handling soft constraints:

  - impose (hard) constraints on the probability of violating each soft constraint
  - or remove active constraints until the problem becomes feasible

- This course: only hard constraints are considered

Constraint handling

Suboptimal methods for handling input constraints:

(a). Saturate the unconstrained control law

constraints are then usually ignored in controller design

(b). "De-tune" the unconstrained control law

increase the penalty on u in the performance objective

(c). Use an anti-windup strategy

to put limits on the state of a dynamic controller
(typically the integral term of a PI or PID controller)

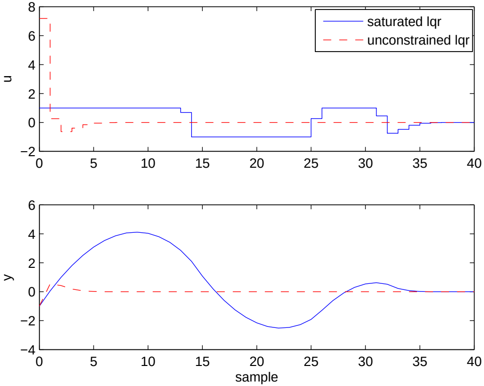
Constraint handling

Effects of input saturation, $\underline{u} \leq u_k \leq \overline{u}$

$$\text{unconstrained control law:} \quad u = u^0$$
$$\text{saturated control law:} \quad u = \max\{\min\{u^0, \overline{u}\}, \underline{u}\}$$

Example:

(A, B, C) as before
$\underline{u} = -1,\ \overline{u} = 1$
$u^0 = K_{LQ}x$



Input saturation causes

- ⋆ poor performance
- ⋆ possible instability (since the open-loop system is unstable)

Constraint handling

De-tuning of optimal control law:

$$K_{LQ} = \text{optimal gain for LQ cost } J^\infty = \sum_{k=0}^{\infty} \big(\|x_k\|_Q^2 + \|u_k\|_R^2\big)$$

Increase R until $u = K_{LQ}x$ satisfies constraints for all initial conditions

Example:

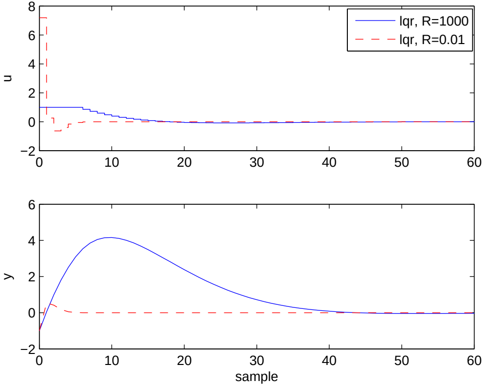
(A, B, C) as before

$10^{-2} \le R \le 10^3$

$\Downarrow$

settling time increased from 6 to 40

- ⋆ $y(t) \to 0$ slowly
- ⋆ stability can be ensured



Constraint handling

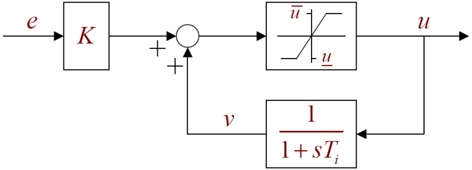
Anti-windup prevents instability of the controller while the input is saturated

Many possible approaches, e.g. anti-windup PI controller:

$$u = \max\{\min\{(Ke + v), \overline{u}\}, \underline{u}\}$$

$$T_i \dot{v} + v = u$$

$$\Downarrow$$



$$\underline{u} \leq u \leq \overline{u} \quad \Longrightarrow \quad u = K\Big(e + \frac{1}{T_i}\int^t e\,dt\Big)$$

$u = \underline{u}$ or $\overline{u}$ $\quad \Longrightarrow \quad$ $v(t) \to \underline{u}$ or $\overline{u}$ exponentially

Strategy is suboptimal and may not prevent instability

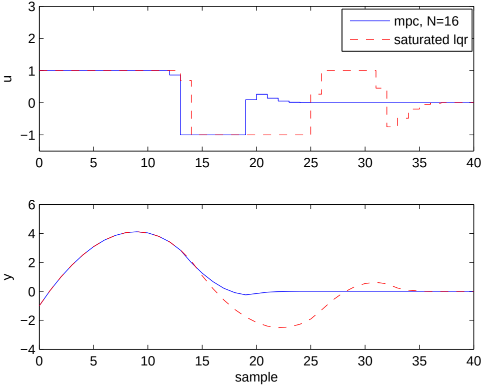
Constraint handling

Anti-windup is based in the past behaviour of the system, whereas MPC optimizes future performance

Example:
(A, B, C) as before

MPC vs saturated LQ
(both based on cost J^{∞}):

- ⋆ settling time reduced to 20 by MPC
- ⋆ stability is guaranteed with MPC



Summary

- Predict performance using plant model

  e.g. linear or nonlinear, discrete or continuous time

- Optimize future (open loop) control sequence

  computationally much easier than optimizing over feedback laws

- Implement first sample, then repeat optimization

  provides feedback to reduce effect of uncertainty

- Comparison of common methods of handling constraints:

  saturation, de-tuning, anti-windup, MPC

Lecture 2

Prediction and optimization

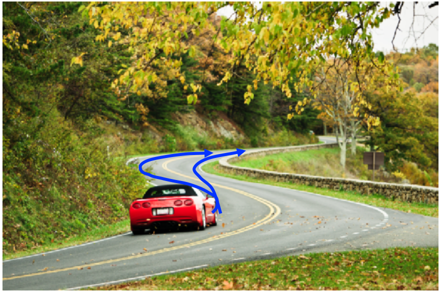
Prediction and optimization

- Input and state predictions
- Unconstrained finite horizon optimal control
- Infinite prediction horizons and connection with LQ optimal control
- Incorporating constraints
- Quadratic programming

Review of MPC strategy

At each sampling instant:

1. Use a model to predict system behaviour over a finite future horizon
2. Compute a control sequence by solving an online optimization problem
3. Apply the first element of optimal control sequence as control input



Advantages

- ⋆ flexible plant model
- ⋆ constraints taken into account
- ⋆ optimal performance

Disadvantage

- ⋆ online otimization required

Prediction equations

Linear time-invariant model: $x_{k+1} = Ax_k + Bu_k$

assume x_k is measured at $k = 0, 1, \ldots$

Predictions: $\mathbf{x}_k = \begin{bmatrix} x_{1|k} \\ \vdots \\ x_{N|k} \end{bmatrix} \qquad \mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix}$

Quadratic cost:
$$\begin{aligned} J_k &= J(x_k, \mathbf{u}_k) \\ &= \sum_{i=0}^{N-1} \bigl(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \bigr) + \|x_{N|k}\|_P^2 \end{aligned}$$

($\|x\|_Q^2 = x^T Q x, \quad \|u\|_R^2 = u^T R u$

$P =$ terminal weighting matrix)

Prediction equations

Linear time-invariant model: $\quad x_{i+1|k} = Ax_{i|k} + Bu_{i|k}$

assume x_k is measured at $k = 0, 1, \ldots$

$$
\begin{aligned}
x_{0|k} &= x_k \\
x_{1|k} &= Ax_k + Bu_{0|k} \\
&\vdots \\
x_{N|k} &= A^N x_k + A^{N-1}Bu_{0|k} + A^{N-2}Bu_{1|k} + \cdots + Bu_{N-1|k}
\end{aligned}
$$

$$\Downarrow$$

$$\boxed{\begin{gathered}\mathbf{x}_k = \mathcal{M}x_k + \mathcal{C}\mathbf{u}_k \\ \mathcal{C} = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \qquad \mathcal{M} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}\end{gathered}}$$

or

$x_{i|k} = A^i x_k + \mathcal{C}_i \mathbf{u}_k, \quad \mathcal{C}_i = i$th row of $\mathcal{C}$

Prediction equations

Linear time-invariant model: $x_{i+1|k} = Ax_{i|k} + Bu_{i|k}$

assume x_k is measured at $k = 0, 1, \ldots$

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\begin{aligned}
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&\vdots \\
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\end{aligned}
$$

$$\Downarrow$$

$$\mathbf{x}_k = \mathcal{M}x_k + \mathcal{C}\mathbf{u}_k$$

$$\mathcal{C} = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \qquad \mathcal{M} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}$$

or

$$x_{i|k} = A^i x_k + \mathcal{C}_i \mathbf{u}_k, \qquad \mathcal{C}_i = i\text{th row of } \mathcal{C}$$

Prediction equations

Predicted cost:

$$J_k = \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) + \|x_{N|k}\|_P^2$$

$$= x_k^T Q x_k + \mathbf{x}_k^T \mathbf{Q} \mathbf{x}_k + \mathbf{u}_k^T \mathbf{R} \mathbf{u}_k \qquad \begin{cases} \mathbf{Q} = \operatorname{diag}\{Q, \dots, Q, P\} \\ \mathbf{R} = \operatorname{diag}\{R, \dots, R, R\} \end{cases}$$

$$\Downarrow$$

$$\boxed{J_k = \mathbf{u}_k^T H \mathbf{u}_k + 2x_k^T F^T \mathbf{u}_k + x_k^T G x_k}$$

where

$$\begin{aligned} H &= \mathcal{C}^T \mathbf{Q} \mathcal{C} + \mathbf{R} && \leftarrow \ \mathbf{u} \times \mathbf{u} \text{ terms} \\ F &= \mathcal{C}^T \mathbf{Q} \mathcal{M} && \leftarrow \ \mathbf{u} \times x \text{ terms} \\ G &= \mathcal{M}^T \mathbf{Q} \mathcal{M} + Q && \leftarrow \ x \times x \text{ terms} \end{aligned}$$

time-invariant model $\Longrightarrow$ H, F, G can be computed offline

Prediction equations

Predicted cost:

$$J_k = \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) + \|x_{N|k}\|_P^2$$

$$= x_k^T Q x_k + \mathbf{x}_k^T \mathbf{Q} \mathbf{x}_k + \mathbf{u}_k^T \mathbf{R} \mathbf{u}_k \qquad \begin{cases} \mathbf{Q} = \mathrm{diag}\{Q, \ldots, Q, P\} \\ \mathbf{R} = \mathrm{diag}\{R, \ldots, R, R\} \end{cases}$$

$$\Downarrow$$

$$\boxed{J_k = \mathbf{u}_k^T H \mathbf{u}_k + 2 x_k^T F^T \mathbf{u}_k + x_k^T G x_k}$$

where

$$\begin{aligned} H &= \mathcal{C}^T \mathbf{Q} \mathcal{C} + \mathbf{R} && \leftarrow \ \mathbf{u} \times \mathbf{u} \text{ terms} \\ F &= \mathcal{C}^T \mathbf{Q} \mathcal{M} && \leftarrow \ \mathbf{u} \times x \text{ terms} \\ G &= \mathcal{M}^T \mathbf{Q} \mathcal{M} + Q && \leftarrow \ x \times x \text{ terms} \end{aligned}$$

time-invariant model $\Longrightarrow$ H, F, G can be computed offline

Prediction equations – example

Plant model: $x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k$

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.079 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

Prediction horizon $N = 4$: $$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.079 & 0 & 0 & 0 \\ 0.157 & 0 & 0 & 0 \\ 0.075 & 0.079 & 0 & 0 \\ 0.323 & 0.157 & 0 & 0 \\ 0.071 & 0.075 & 0.079 & 0 \\ 0.497 & 0.323 & 0.157 & 0 \\ 0.068 & 0.071 & 0.075 & 0.079 \end{bmatrix}$$

Cost matrices $Q = C^T C$, $R = 0.01$, and $P = Q$:

$$H = \begin{bmatrix} 0.271 & 0.122 & 0.016 & -0.034 \\ 0.122 & 0.086 & 0.014 & -0.020 \\ 0.016 & 0.014 & 0.023 & -0.007 \\ -0.034 & -0.020 & -0.007 & 0.016 \end{bmatrix} \qquad F = \begin{bmatrix} 0.977 & 4.925 \\ 0.383 & 2.174 \\ 0.016 & 0.219 \\ -0.115 & -0.618 \end{bmatrix}$$

$$G = \begin{bmatrix} 7.589 & 22.78 \\ 22.78 & 103.7 \end{bmatrix}$$

Prediction equations: LTV model

Aside: Linear time-varying model: $x_{k+1} = A_k x_k + B_k u_k$

assume x_k is measured at $k = 0, 1 \ldots$

Predictions:

$$
\begin{aligned}
x_{0|k} &= x_k \\
x_{1|k} &= A_k x_k + B_k u_{0|k} \\
x_{2|k} &= A_{k+1} A_k x_k + A_{k+1} B_k u_{0|k} + B_{k+1} u_{1|k} \\
&\vdots
\end{aligned}
$$

$$\Downarrow$$

$$x_{i|k} = \prod_{j=i-1}^{0} A_{k+j} x_k + \mathcal{C}_i(k) \mathbf{u}_k, \qquad i = 0, \ldots, N$$

$$\mathcal{C}_i(k) = \begin{bmatrix} \prod_{j=i-1}^{1} A_{k+j} B_k & \prod_{j=i-1}^{2} A_{k+j} B_{k+1} & \cdots & B_{k+i-1} & 0 & \cdots & 0 \end{bmatrix}$$

$$\Downarrow$$

$H(k)$, $F(k)$, $G(k)$ depend on k and must be computed online

Unconstrained optimization

Minimize cost: $\mathbf{u}^* = \arg\min_{\mathbf{u}} J, \quad J = \mathbf{u}^T H \mathbf{u} + 2x^T F^T \mathbf{u} + x^T G x$

differentiate w.r.t. $\mathbf{u}$: $\nabla_{\mathbf{u}} J = 2H\mathbf{u} + 2Fx = 0$

$$\Downarrow$$

$$\mathbf{u} = -H^{-1} F x$$

$$= \mathbf{u}^* \quad \text{if } H \text{ is positive definite} \;\; \text{i.e. if } H \succ 0$$

Here $H = \mathcal{C}^T \mathbf{Q} \mathcal{C} + \mathbf{R} \succ 0$ if: $\begin{cases} R \succ 0 \;\&\; Q, P \succeq 0 & \text{or} \\ R \succeq 0 \;\&\; Q, P \succ 0 \;\&\; \mathcal{C} \text{ is full-rank} \end{cases}$

$$\Updownarrow$$

(A, B) controllable

Receding horizon controller is linear state feedback:

$$u_k = -\begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} H^{-1} F x_k$$

is the closed loop response optimal? is it even stable?

Unconstrained optimization

Minimize cost: $\mathbf{u}^* = \arg\min_{\mathbf{u}} J, \quad J = \mathbf{u}^T H \mathbf{u} + 2x^T F^T \mathbf{u} + x^T G x$

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is the closed loop response optimal? is it even stable?

Example

Model: A, B, C as before, cost: $J_k = \sum_{i=0}^{N-1} (y_{i|k}^2 + 0.01 u_{i|k}^2) + y_{N|k}^2$

- For $N = 4$: $\mathbf{u}_k^* = -H^{-1} F x_k = \begin{bmatrix} -4.36 & -18.7 \\ 1.64 & 1.24 \\ 1.41 & 3.00 \\ 0.59 & 1.83 \end{bmatrix} x_k$

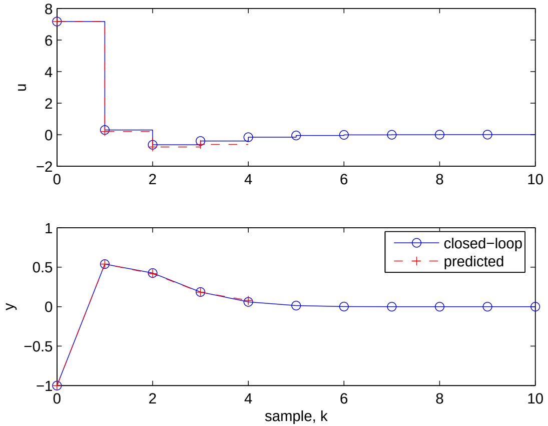
$$u_k = \begin{bmatrix} -4.36 & -18.7 \end{bmatrix} x_k$$

- For general N: $u_k = K_N x_k$

	$N = 4$	$N = 3$	$N = 2$	$N = 1$
K_N	$\begin{bmatrix} -4.36 & -18.69 \end{bmatrix}$	$\begin{bmatrix} -3.80 & -16.98 \end{bmatrix}$	$\begin{bmatrix} 1.22 & -3.95 \end{bmatrix}$	$\begin{bmatrix} 5.35 & 5.10 \end{bmatrix}$
$\lambda(A + BK_N)$	$0.29 \pm 0.17j$	$0.36 \pm 0.22j$	$1.36, 0.38$	$2.15, 0.30$
	stable	stable	unstable	unstable

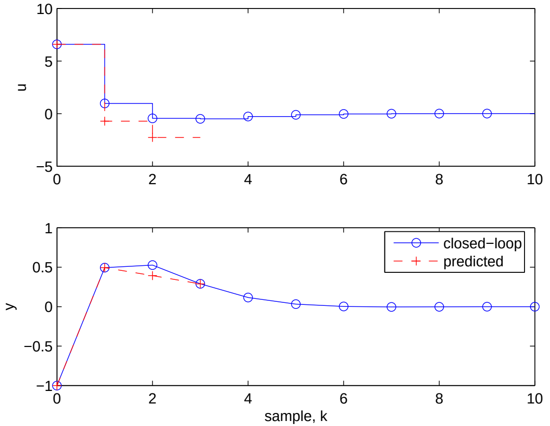
Example

Horizon: $N = 4$, $x_0 = (0.5, -0.5)$



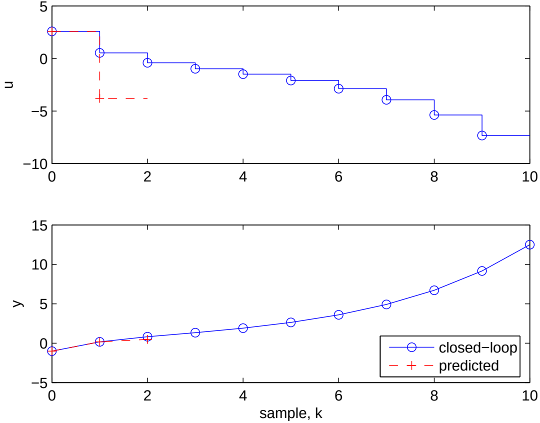
Example

Horizon: $N = 3$, $x_0 = (0.5, -0.5)$



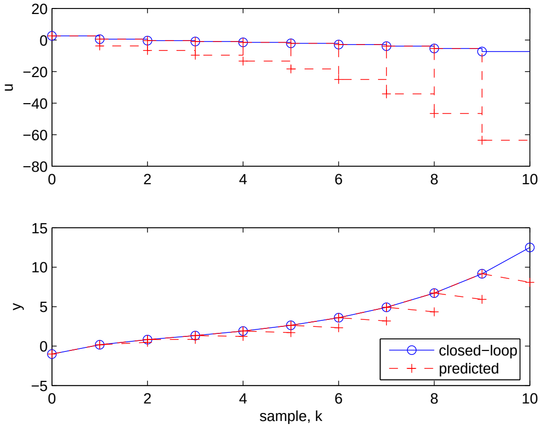
Example

Horizon: $N = 2,\ x_0 = (0.5, -0.5)$



Example

Horizon: $N = 2$, $x_0 = (0.5, -0.5)$



Observation: predicted and closed loop responses are different for small N

Receding horizon control

Why is this example unstable for $N \leq 2$?

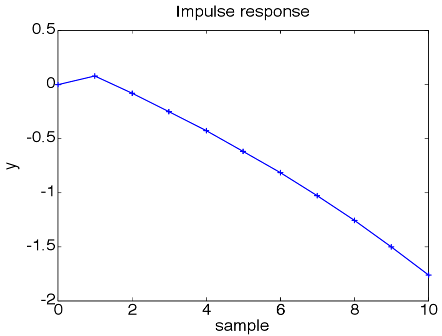
System is non-minimum phase

$\Downarrow$

impulse response changes sign

$\Downarrow$

hence short horizon causes instability



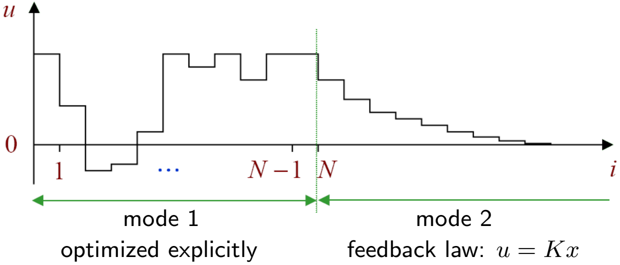
Solution:

- ⋆ use an infinite horizon cost
- ⋆ but keep a finite number of optimization variables in predictions

Dual mode predictions

An infinite prediction horizon is possible with dual mode predictions:

$$u_{i|k} = \begin{cases} \text{optimization variables} & i = 0, \ldots, N-1, \quad \text{mode 1} \\ K x_{i|k} & i = N, N+1, \ldots \quad \text{mode 2} \end{cases}$$



Feedback gain K: stabilizing and determined offline

unconstrained LQ optimal for $\sum_{i=0}^{\infty}(\|x_i\|_Q^2 + \|u_i\|_R^2)$

(usually)

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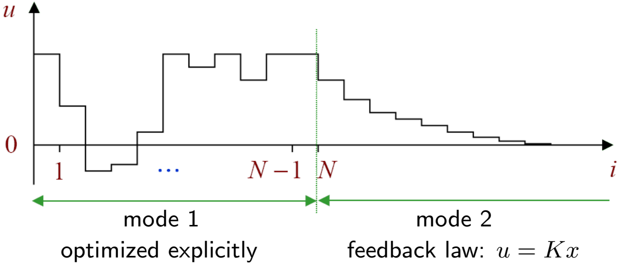

Feedback gain K: stabilizing and determined offline

unconstrained LQ optimal for $\sum_{i=0}^{\infty}(\|x_i\|_Q^2 + \|u_i\|_R^2)$

(usually)

Infinite horizon cost

If the predicted input sequence is

$$\{u_{0|k}, \ldots, u_{N-1|k}, Kx_{N|k}, K\Phi x_{N|k}, \ldots\}$$

then

$$\sum_{i=0}^{\infty}(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) = \sum_{i=0}^{N-1}(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2) + \|x_{N|k}\|_P^2$$

where

$$\boxed{P - (A+BK)^T P(A+BK) = Q + K^T RK}$$

Lyapunov matrix equation (discrete time)

Note:

- if $Q + K^T RK \succ 0$, then the solution P is unique and $P \succ 0$
- matlab: `P = dlyap(Phi',RHS);`
  `Phi = A+B*K; RHS = Q+K'*R*K;`
- P is the steady state Riccati equation solution if K is LQ-optimal

Infinite horizon cost

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P = dlyap(Phi',RHS);
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```
- P is the steady state Riccati equation solution if K is LQ-optimal

Infinite horizon cost

Proof that the predicted cost over the mode 2 horizon is $\|x_{N|k}\|_P^2$:

Let $J^\infty(x_0) = \sum_{i=0}^{\infty}\big(\|x_i\|_Q^2 + \|u_i\|_R^2\big)$, with $u_i = Kx_i$, $x_{i+1} = \Phi x_i \ \forall i$,

– then $J^\infty(x_0) = \sum_{i=0}^{\infty}\big(\|\Phi^i x_0\|_Q^2 + \|K\Phi^i x_0\|_R^2\big)$

$$= x_0^T \underbrace{\left[\sum_{i=0}^{\infty}(\Phi^i)^T(Q + K^T RK)\Phi^i\right]}_{=S} x_0$$

– but $\Phi^T S\Phi = \sum_{i=1}^{\infty}(\Phi^i)^T(Q + K^T RK)\Phi^i$

$$= S - (Q + K^T RK)$$

so if $\Phi = A + BK$, then $S = P$ and $J^\infty(x_{N|k}) = \|x_{N|k}\|_P^2$

Connection with LQ optimal control

Predicted cost:

$$J\big(x_k, (u_{0|k}, \dots, u_{N-1|k})\big) = \sum_{i=0}^{N-1} \big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\big) + \|x_{N|k}\|_P^2$$

where $\qquad P - (A+BK)^T P (A+BK) = Q + K^T R K$

and $K =$ LQ-optimal

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$u_{0|k}^* = K x_k \text{ where } \mathbf{u}_k^* = \arg\min_{\mathbf{u}} J(x_k, \mathbf{u}) = (u_{0|k}^*, \dots, u_{N-1|k}^*)$$

$$\Uparrow \quad \Uparrow \quad \Uparrow \quad \Uparrow \quad \Uparrow \quad \Uparrow$$

The Bellman principle of optimality implies:

$$\{u_{0|k}, u_{1,k}, \dots\} \text{ optimal} \iff \begin{cases} \{u_{0|k}, \dots, u_{N-1|k}\} \text{ optimal} \\ \text{and } K \text{ LQ-optimal} \end{cases}$$

Connection with LQ optimal control – example

- Model parameters (A, B, C) as before

  LQ optimal gain for $Q = C^T C$, $R = 0.01$: $K = \begin{bmatrix} -4.36 & -18.74 \end{bmatrix}$

  Lyapunov equation solution: $P = \begin{bmatrix} 3.92 & 4.83 \\ & 13.86 \end{bmatrix}$

- Cost matrices for $N = 4$:

$$H = \begin{bmatrix} 1.44 & 0.98 & 0.59 & 0.26 \\ & 0.72 & 0.44 & 0.20 \\ & & 0.30 & 0.14 \\ & & & 0.096 \end{bmatrix} \quad F = \begin{bmatrix} 3.67 & 23.9 \\ 2.37 & 16.2 \\ 1.36 & 9.50 \\ 0.556 & 4.18 \end{bmatrix} \quad G = \begin{bmatrix} 13.8 & 66.7 \\ & 413 \end{bmatrix}$$

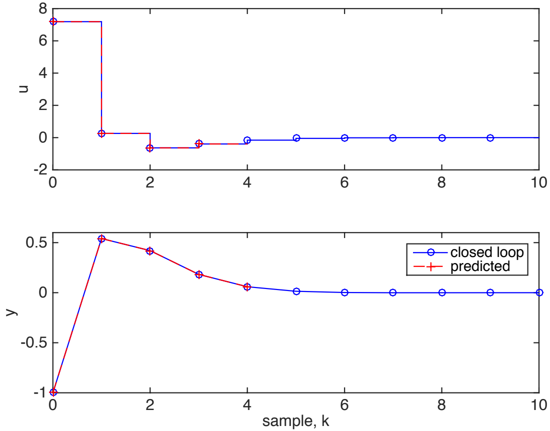
- Predictive control law: $u_k = -\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} H^{-1} F x_k$

$$= \begin{bmatrix} -4.35 & -18.74 \end{bmatrix} x_k$$

(identical to the LQ optimal controller)

Connection with LQ optimal control – example

- Response for $N = 4$, $x_0 = (0.5, -0.5)$



$$\left.\begin{array}{l}\text{Infinite horizon cost}\\ \text{no constraints}\end{array}\right\} \implies \text{identical predicted and closed loop responses}$$

Dual mode predictions

Aside: Pre-stabilized dual mode predictions with better numerical stability

- Inputs

$$
\begin{array}{lll}
\text{mode 1} & u_{i|k} = Kx_{i|k} + c_{i|k}, & i = 0, 1, \ldots, N-1 \\
\text{mode 2} & u_{i|k} = Kx_{i|k}, & i = N, N+1, \ldots
\end{array}
$$

- Dynamics

$$
\begin{array}{lll}
\text{mode 1} & x_{i+1|k} = \Phi x_{i|k} + Bc_{i|k}, & i = 0, 1, \ldots, N-1 \\
\text{mode 2} & x_{i+1|k} = \Phi x_{i|k}, & i = N, N+1, \ldots
\end{array}
$$

where $(c_{0|k}, \ldots, c_{N-1|k})$ are optimization variables

Dual mode predictions

Aside: Pre-stabilized dual mode predictions with better numerical stability

- Vectorized form: $\mathbf{x}_k = \mathcal{M} x_k + \mathcal{C}\mathbf{c}_k$

$$\mathbf{x}_k := \begin{bmatrix} x_{1|k} \\ \vdots \\ x_{N|k} \end{bmatrix}, \quad \mathbf{c}_k := \begin{bmatrix} c_{0|k} \\ \vdots \\ c_{N-1|k} \end{bmatrix}$$

$$\mathcal{M} = \begin{bmatrix} \Phi \\ \Phi^2 \\ \vdots \\ \Phi^N \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} B & 0 & \cdots & 0 \\ \Phi B & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi^{N-1} B & \Phi^{N-2} B & \cdots & B \end{bmatrix}$$

- Cost: $J\big(x_k, (u_{0|k}, \ldots, u_{N-1|k})\big) = J(x_k, \mathbf{c}_k)$

Input and state constraints

Infinite horizon unconstrained MPC = LQ optimal control

but MPC can also handle constraints

Consider constraints applied to mode 1 predictions:

- input constraints: $\underline{u} \leq u_{i|k} \leq \overline{u}, \quad i = 0, \ldots, N-1$

$$\iff \quad \begin{bmatrix} I \\ -I \end{bmatrix} \mathbf{u}_k \leq \begin{bmatrix} \overline{\mathbf{u}} \\ -\underline{\mathbf{u}} \end{bmatrix} \qquad \text{where} \qquad \begin{aligned} \overline{\mathbf{u}} &= \begin{bmatrix} \overline{u}^T & \cdots & \overline{u}^T \end{bmatrix}^T \\ \underline{\mathbf{u}} &= \begin{bmatrix} \underline{u}^T & \cdots & \underline{u}^T \end{bmatrix}^T \end{aligned}$$

- state constraints: $\underline{x} \leq x_{i|k} \leq \overline{x}, \quad i = 1, \ldots, N$

$$\iff \quad \begin{bmatrix} \mathcal{C}_i \\ -\mathcal{C}_i \end{bmatrix} \mathbf{u}_k \leq \begin{bmatrix} \overline{x} \\ -\underline{x} \end{bmatrix} + \begin{bmatrix} -A^i \\ A^i \end{bmatrix} x_k, \quad i = 1, \ldots, N$$

Input and state constraints

Constraints on mode 1 predictions can be expressed

$$A_c \mathbf{u}_k \le b_c + B_c x_k$$

where A_c, B_c, b_c can be computed offline since model is time-invariant

The online optimization is a quadratic program (QP):

$$\boxed{\begin{aligned} \underset{\mathbf{u}}{\text{minimize}} \quad & \mathbf{u}^T H \mathbf{u} + 2x_k^T F^T \mathbf{u} \\ \text{subject to} \quad & A_c \mathbf{u} \le b_c + B_c x_k \end{aligned}}$$

which is a convex optimization problem with a unique solution if

$$H = \mathcal{C}^T \mathbf{Q} \mathcal{C} + \mathbf{R} \;\; \text{is positive definite}$$

QP solvers: (a) Active set

Consider the QP:

$$\mathbf{u}^* = \arg\min_{\mathbf{u}} \quad \mathbf{u}^T H \mathbf{u} + 2 f^T \mathbf{u}$$
$$\text{subject to} \quad A\mathbf{u} \le b$$

and let $(A_i, b_i) = i$th row/element of (A, b)

- Individual constraints are active or inactive

active	inactive
$A_i \mathbf{u}^* = b_i, \quad \forall i \in \mathcal{I}$	$A_i \mathbf{u}^* \le b_i, \quad \forall i \notin \mathcal{I}$
b_i affects solution	b_i does not affect solution

- Equality constraint problem: $\mathbf{u}^* = \arg\min_{\mathbf{u}} \quad \mathbf{u}^T H \mathbf{u} + 2 f^T \mathbf{u}$

  $\text{subject to} \quad A_i \mathbf{u} = b_i, \quad \forall i \in \mathcal{I}$

- Solve QP by searching for $\mathcal{I}$
  - one equality constraint problem solved at each iteration
  - optimality conditions (KKT conditions) identify solution

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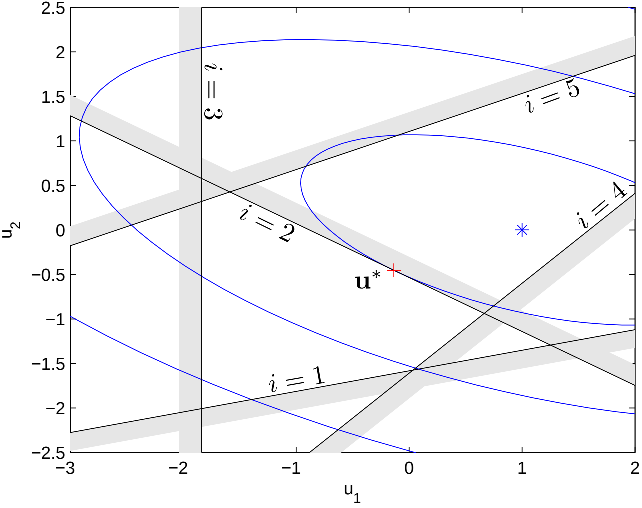
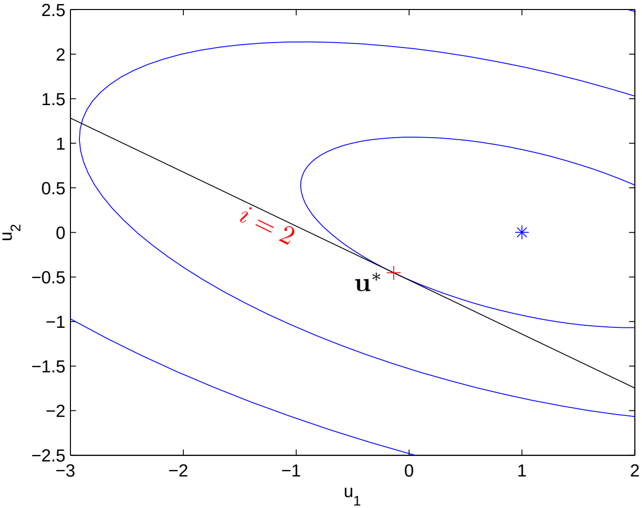
Active constraints – example



A QP problem with 5 inequality constraints
active set at solution: $\mathcal{I} = \{2\}$

Active constraints – example



An equivalent equality constraint problem

QP solvers: (a) Active set

- Computation:

  $O(N^3 n_u^3)$ additions/multiplications per iteration (conservative estimate)

  upper bound on number of iterations is exponential in problem size

- At each iteration choose trial active set using: cost gradient
  constraint sensitivities

  $$\Downarrow$$

  number of iterations needed is often small in practice

- In MPC $\mathbf{u}_k^* = \mathbf{u}^*(x_k)$ and $\mathcal{I}_k = \mathcal{I}(x_k)$

  hence initialize solver at time k using the solution computed at $k-1$

QP solvers: (b) Interior point

- Solve an unconstrained problem at each iteration:

$$\mathbf{u}(\mu) = \min_{\mathbf{u}} \mu\big(\mathbf{u}^T H \mathbf{u} + 2 f^T \mathbf{u}\big) + \phi(\mathbf{u})$$

  where

$$\phi(\mathbf{u}) = \text{barrier function} \quad (\phi \to \infty \text{ at constraints})$$
$$\mathbf{u} \to \mathbf{u}^* \text{ as } \mu \to \infty$$

  Increase μ until $\phi(\mathbf{u}^*) > 1/\epsilon$ ($\epsilon =$ user-defined tolerance)

- # operations per iterations is constant, e.g. $O(N^3 n_u^3)$
  # iterations for given ϵ is polynomial in problem size

$$\Downarrow$$

  Computational advantages for large-scale problems
  e.g. # variables $> 10^2$, # constraints $> 10^3$

- No general method for initializing at solution estimate

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$$\Downarrow$$

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  e.g. \# variables $> 10^2$, \# constraints $> 10^3$

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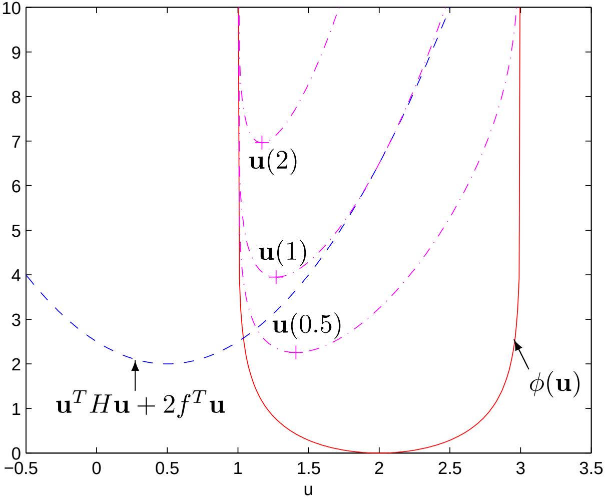
Interior point method – example



$\mathbf{u}(\mu) \to \mathbf{u}^* = 1$ as $\mu \to \infty$

but $\min_{\mathbf{u}} \mu\big(\mathbf{u}^T H\mathbf{u} + 2f^T\mathbf{u}\big) + \phi(\mathbf{u})$ becomes ill-conditioned as $\mu \to \infty$

QP solvers: (c) Multiparametric

Let $$\mathbf{u}^*(x) = \arg\min_{\mathbf{u}} \quad \mathbf{u}^T H \mathbf{u} + 2x^T F^T \mathbf{u}$$
$$\text{subject to} \quad A\mathbf{u} \le b + Bx$$

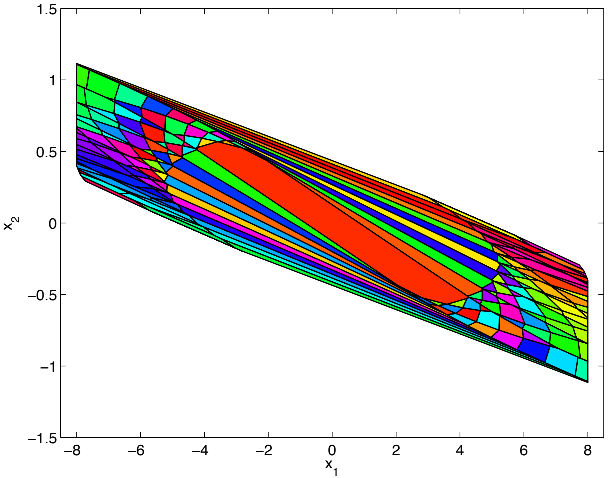
then:

- ⋆ $\mathbf{u}^*$ is a continous function of x
- ⋆ $\mathbf{u}^*(x) = K_j x + k_j$ for all x in a polytopic set $\mathcal{X}_j$

- ▷ In principle each K_j, k_j and $\mathcal{X}_j$ can be determined offline
- ▷ But number of sets $\mathcal{X}_j$ is usually large (depends exponentially on problem size)

  so online determination of j such that $x_k \in \mathcal{X}_j$ is difficult

Multiparametric QP – example



Model: (A, B, C) as before,
cost: $Q = C^T C$, $R = 1$, horizon: $N = 10$
constraints: $-1 \leq u \leq 1$, $-\mathbf{1} \leq x/8 \leq \mathbf{1}$

Summary

▷ Predicted control inputs: $\mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix}$

and states: $\mathbf{x}_k = \begin{bmatrix} x_{1|k} \\ \vdots \\ x_{N|k} \end{bmatrix} = \mathcal{M}x_k + \mathcal{C}\mathbf{u}_k$

▷ Predicted cost: $J(x_k, \mathbf{u}_k) = \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\right) + \|x_{N|k}\|_P^2$

$$= \mathbf{u}_k^T H \mathbf{u}_k + 2x_k^T F^T \mathbf{u}_k + x_k^T G x_k$$

▷ Online optimization of cost subject to linear state and input constraints is a QP problem:

$$\begin{aligned} \underset{\mathbf{u}}{\text{minimize}} \quad & \mathbf{u}^T H \mathbf{u} + 2x_k^T F^T \mathbf{u} \\ \text{subject to} \quad & A_c \mathbf{u} \le b_c + B_c x_k \end{aligned}$$

Lecture 3

Closed loop properties of MPC

Closed loop properties of MPC

- Review: infinite horizon cost
- Infinite horizon predictive control with constraints
- Closed loop stability
- Constraint-checking horizon
- Connection with constrained optimal control

Review: infinite horizon cost

Short prediction horizons cause poor performance and instability, so

- ⋆ use an infinite horizon cost: $J(x_k, \mathbf{u}_k) = \sum_{i=0}^{\infty} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\right)$

- ⋆ keep optimization finite-dimensional by using dual mode predictions:

$$u_{i|k} = \begin{cases} \text{optimization variables} & i = 0, \ldots, N-1, \quad \text{mode 1} \\ K x_{i|k} & i = N, N+1, \ldots \quad \text{mode 2} \end{cases}$$

mode 1: $\mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix}$ $\mathbf{u}_k$ optimized online

mode 2: $u_{i|k} = K x_{i|k}$ K chosen offline

Review: infinite horizon cost

- Cost for mode 2: $\displaystyle\sum_{i=N}^{\infty}\big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\big) = \|x_{N|k}\|_P^2$

  P is the solution of the Lyapunov equation

$$\boxed{P - (A+BK)^T P (A+BK) = Q + K^T R K}$$

- Infinite horizon cost:

$$\begin{aligned} J(x_k, \mathbf{u}_k) &= \sum_{i=0}^{N-1}\big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\big) + \|x_{N|k}\|_P^2 \\ &= \mathbf{u}_k^T H \mathbf{u}_k + 2x_k^T F^T \mathbf{u}_k + x_k^T G x_k \end{aligned}$$

Review: MPC online optimization

- Unconstrained optimization: $\nabla_{\mathbf{u}} J(x, \mathbf{u}^*) = 2H\mathbf{u}^* + 2Fx = 0$, so

$$\mathbf{u}^*(x) = -H^{-1}Fx$$

$\Longrightarrow$ linear controller: $u_k = K_{\mathsf{MPC}} x_k$

$$K_{\mathsf{MPC}} = \text{LQ-optimal if } K = \text{LQ-optimal (in mode 2)}$$

- Constrained optimization:

$$\begin{aligned} \mathbf{u}^*(x) = \arg\min_{\mathbf{u}} \quad & \mathbf{u}^T H \mathbf{u} + 2x^T F^T \mathbf{u} \\ \text{subject to} \quad & A_c \mathbf{u} \leq b_c + B_c x \end{aligned}$$

$\Longrightarrow$ nonlinear controller: $u_k = K_{\mathsf{MPC}}(x_k)$

Constrained MPC – example

- Plant model: $x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k$

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

  Constraints: $-1 \le u_k \le 1$

- MPC optimization (constraints applied only to mode 1 predictions):

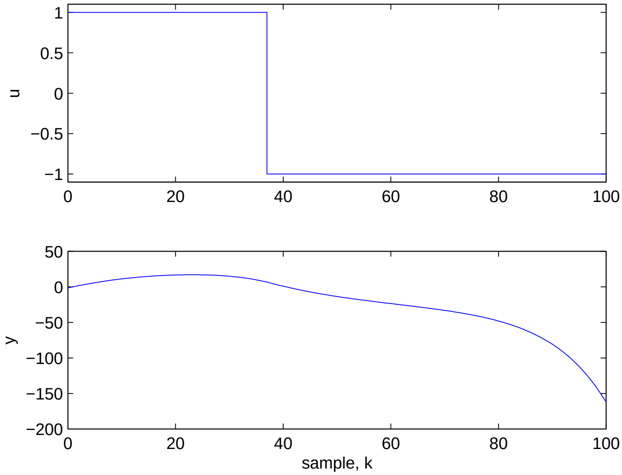
$$\begin{aligned} \underset{\mathbf{u}}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) + \|x_{N|k}\|_P^2 \\ \text{subject to} \quad & -1 \le u_{i|k} \le 1, \quad i = 0, \ldots, N-1 \end{aligned}$$

$$Q = C^T C, \; R = 0.01, \; N = 2$$

. . . performance? stability?

Constrained MPC – example

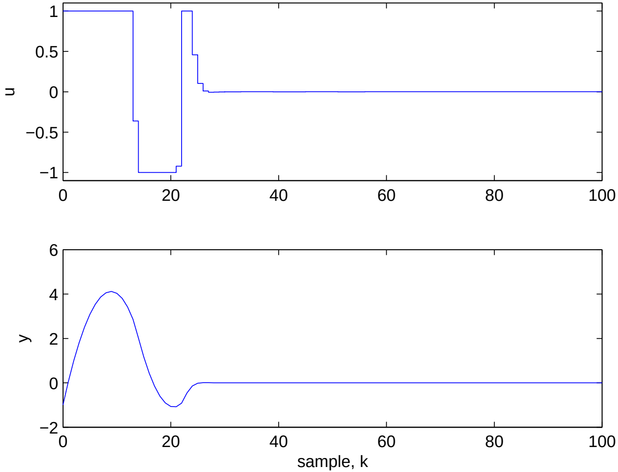
Closed loop response for $x_0 = (0.8, -0.8)$



unstable

Constrained MPC – example

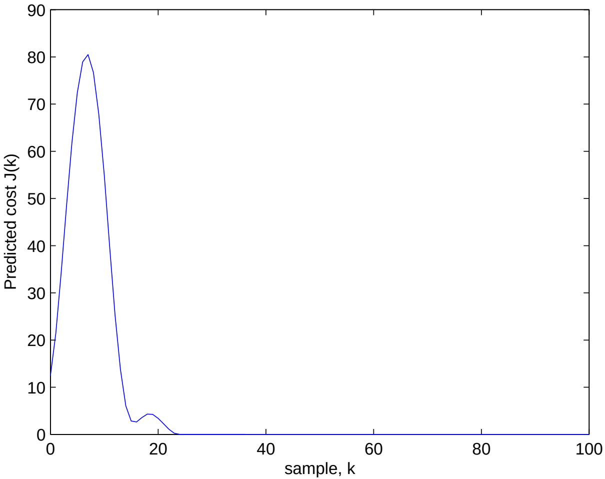
Closed loop response for $x_0 = (0.5, -0.5)$



stable, but ...

Constrained MPC – example

Optimal predicted cost $x_0 = (0.5, -0.5)$



...increasing $J_k \implies$ closed loop response does not follow predicted trajectory

Stability analysis

How can we guarantee the closed loop stability of MPC?

> (a). Show that a Lyapunov function exists demonstrating stability
> (b). Ensure that optimization feasible is at each time $k = 0, 1, \ldots$

- For Lyapunov stability analysis:
  - consider first the unconstrained problem
  - use predicted cost as a trial Lyapunov function

- Guarantee feasibility of the MPC optimization recursively

  by ensuring that feasibility at time $k \implies$ feasibility at $k + 1$

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Discrete time Lyapunov stability

Consider the system $x_{k+1} = f(x_k)$, with $f(0) = 0$

- Definition: $x = 0$ is a stable equilibrium point if

  $\max_k \|x_k\|$ can be made arbitrarily small
  by making x_0 sufficiently small

- If continuously differentiable $V(x)$ exists with

  (i). $V(x)$ is positive definite and
  (ii). $V(x_{k+1}) - V(x_k) \leq 0$

  then $x = 0$ is a stable equilibrium point

Discrete time Lyapunov stability

Consider the system $x_{k+1} = f(x_k)$, with $f(0) = 0$

- **Definition:** $x = 0$ is a stable equilibrium point if

$$\text{for all } R > 0 \text{ there exists } r \text{ such that}$$
$$\|x_0\| < r \implies \|x_k\| < R \text{ for all } k$$

- If continuously differentiable $V(x)$ exists with

  (i). $V(x)$ is positive definite and

  (ii). $V(x_{k+1}) - V(x_k) \leq 0$

  then $x = 0$ is a stable equilibrium point

Discrete time Lyapunov stability

Consider the system $x_{k+1} = f(x_k)$, with $f(0) = 0$

- Definition: $x = 0$ is an asymptotically stable equilibrium point if

  (i). $x = 0$ is stable and

  (ii). r exists such that $\|x_0\| < r \implies \lim_{k\to\infty} x_k = 0$

- If continuously differentiable $V(x)$ exists with

  (i). $V(x)$ is positive definite and

  (ii). $V(x_{k+1}) - V(x_k) < 0$ whenever $x_k \neq 0$

  then $x = 0$ is an asymptotically stable equilibrium point

Trial Lyapunov function:

$$\boxed{J^*(x_k) = J(x_k, \mathbf{u}_k^*)}$$

where $J(x_k, \mathbf{u}_k) = \sum_{i=0}^{\infty} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\right)$

- ⋆ $J^*(x)$ is positive definite if:

  (a). $R \succeq 0$ and $Q \succ 0$, or

  (b). $R \succ 0$ and $Q \succeq 0$ and (A, Q) is observable

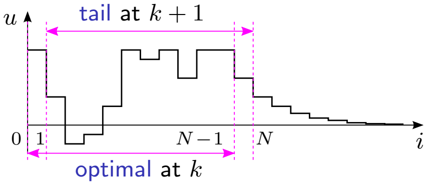
  since then $J^*(x_k) \geq 0$ and $J^*(x_k) = 0$ if and only if $x_k = 0$

- ⋆ $J^*(x)$ is continuously differentiable

  …from analysis of MPC optimization as a multiparametric QP

Lyapunov stability

Construct a bound on $J^*(x_{k+1}) - J^*(x_k)$ using the "tail" of the optimal prediction at time k



Optimal predicted sequences at time k:

$$\mathbf{u}_k^* = \begin{bmatrix} u_{0|k}^* \\ u_{1|k}^* \\ \vdots \\ u_{N-1|k}^* \\ Kx_{N|k}^* \\ \vdots \end{bmatrix} \qquad \mathbf{x}_k^* = \begin{bmatrix} x_{1|k}^* \\ x_{2|k}^* \\ \vdots \\ x_{N|k}^* \\ \Phi x_{N|k}^* \\ \vdots \end{bmatrix}$$

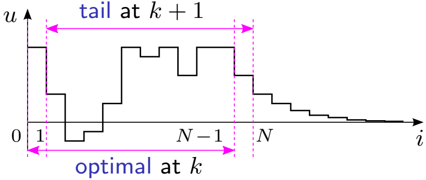
$$(\Phi = A + BK)$$

optimal at k : $\quad J^*(x_k) = J(x_k, \mathbf{u}_k^*) = \sum_{i=0}^{\infty} \left(\|x_{i|k}^*\|_Q^2 + \|u_{i|k}^*\|_R^2\right)$

tail at $k+1$: $\quad \tilde{J}(x_{k+1}) = J(x_{k+1}, \tilde{\mathbf{u}}_{k+1}) = \sum_{i=1}^{\infty} \left(\|x_{i|k}^*\|_Q^2 + \|u_{i|k}^*\|_R^2\right)$

Lyapunov stability

Construct a bound on $J^*(x_{k+1}) - J^*(x_k)$ using the "tail" of the optimal prediction at time k



Tail sequences at time $k+1$:

$$\tilde{\mathbf{u}}_{k+1} = \begin{bmatrix} u^*_{1|k} \\ \vdots \\ u^*_{N-1|k} \\ Kx^*_{N|k} \\ K\Phi x^*_{N|k} \\ \vdots \end{bmatrix} \quad \tilde{\mathbf{x}}_{k+1} = \begin{bmatrix} x^*_{2|k} \\ \vdots \\ x^*_{N|k} \\ \Phi x^*_{N|k} \\ \Phi^2 x^*_{N|k} \\ \vdots \end{bmatrix}$$

$(\Phi = A + BK)$

optimal at k : $\quad J^*(x_k) = J(x_k, \mathbf{u}^*_k) \quad = \sum_{i=0}^{\infty} \big(\|x^*_{i|k}\|^2_Q + \|u^*_{i|k}\|^2_R\big)$

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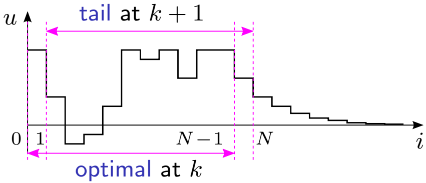

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Lyapunov stability

Construct a bound on $J^*(x_{k+1}) - J^*(x_k)$ using the "tail" of the optimal prediction at time k

Predicted cost for the tail:

$$\tilde{J}(x_{k+1}) = J^*(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2$$

but $\tilde{\mathbf{u}}_{k+1}$ is suboptimal at time $k+1$, so

$$J^*(x_{k+1}) \le \tilde{J}(x_{k+1})$$

Therefore

$$\boxed{J^*(x_{k+1}) \le J^*(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2}$$

Lyapunov stability

The bound $J^*(x_{k+1}) - J^*(x_k) \le -\|x_k\|_Q^2 - \|u_k\|_R^2$ implies:

1. the closed loop cost cannot exceed the initial predicted cost, since summing both sides over all $k \ge 0$ gives

$$\boxed{\sum_{k=0}^{\infty} \left(\|x_k\|_Q^2 + \|u_k\|_R^2\right) \le J^*(x_0)}$$

2. $x = 0$ is asymptotically stable

   - if $R \succeq 0$ and $Q \succ 0$, this follows from Lyapunov's direct method
   - if $R \succ 0$, $Q \succeq 0$ and (A, Q) observable, this follows from:

     (a). stability of $x = 0$ $\Leftarrow$ Lyapunov's direct method

     (b). $\lim_{k\to\infty} (\|x_k\|_Q^2 + \|u_k\|_R^2) = 0$ $\Leftarrow$ $\sum_{k=0}^{\infty} \left(\|x_k\|_Q^2 + \|u_k\|_R^2\right) < \infty$

Stability analysis

How can we guarantee the closed loop stability of MPC?

(a). Show that a Lyapunov function exists demonstrating stability
(b). Ensure that optimization feasible is at each time $k = 0, 1, \dots$

- For Lyapunov stability analysis:
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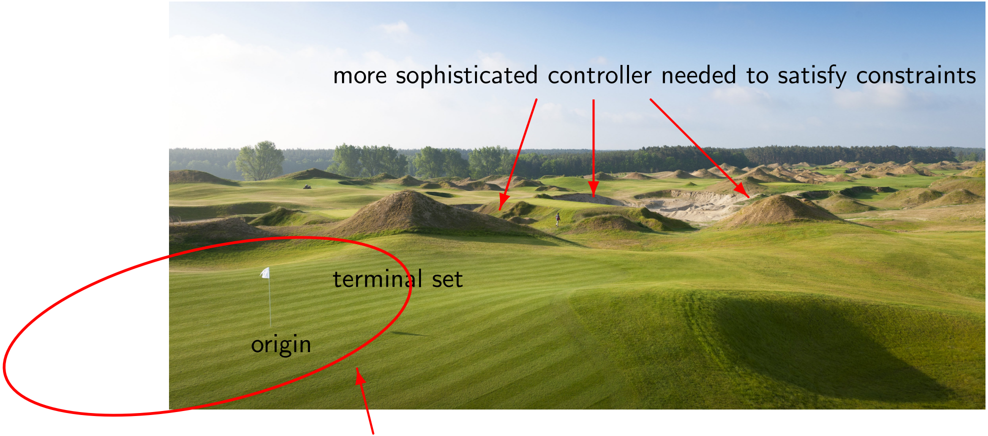
Terminal constraint

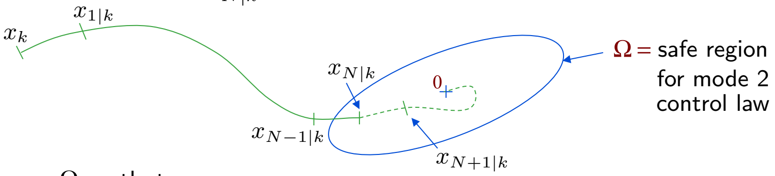
The basic idea



stabilizing linear controller satisfies constraints

Terminal constraint

Terminal constraint: $x_{N|k} \in \Omega$, where $\Omega =$ terminal set



Choose Ω so that:

(a). $x \in \Omega \quad \Longrightarrow \quad \begin{cases} \underline{u} \leq Kx \leq \overline{u} \\ \underline{x} \leq x \leq \overline{x} \end{cases}$

(b). $x \in \Omega \quad \Longrightarrow \quad (A + BK)x \in \Omega$

then Ω is invariant for the mode 2 dynamics and constraints, so

$$x_{N|k} \in \Omega \implies \begin{cases} \underline{u} \leq u_{i|k} \leq \overline{u} \\ \underline{x} \leq x_{i|k} \leq \overline{x} \end{cases} \text{for } i = N, N+1, \ldots$$

i.e. constraints are satisfied over
the infinite mode 2 prediction horizon

Stability of constrained MPC

Prototype MPC algorithm

At each time $k = 0, 1, \ldots$

(i). solve
$$\mathbf{u}_k^* = \arg\min_{\mathbf{u}_k} \; J(x_k, \mathbf{u}_k)$$
$$\text{s.t.} \;\; \underline{u} \le u_{i|k} \le \overline{u}, \;\; i = 0, \ldots, N-1$$
$$\underline{x} \le x_{i|k} \le \overline{x}, \;\; i = 1, \ldots, N$$
$$x_{N|k} \in \Omega$$

(ii). apply $u_k = u^*_{0|k}$ to the system

Asymptotically stabilizes $x = 0$ with region of attraction $\mathcal{F}_N$,

$$\mathcal{F}_N = \left\{ x_0 : \exists \, \{u_0, \ldots, u_{N-1}\} \text{ such that } \begin{array}{l} \underline{u} \le u_i \le \overline{u}, \;\; i = 0, \ldots, N-1 \\ \underline{x} \le x_i \le \overline{x}, \;\; i = 1, \ldots, N \\ x_N \in \Omega \end{array} \right\}$$

$=$ the set of all feasible initial conditions for N-step horizon

and terminal set Ω

Terminal constraints

Make Ω as large as possible so that the feasible set $\mathcal{F}_N$ is maximized, i.e.

$$\boxed{\Omega = \mathcal{X}_\infty = \lim_{j\to\infty} \mathcal{X}_j}$$

where

- ⋆ $\mathcal{X}_j$ = initial conditions for which constraints are satisfied for j steps with $u = Kx$

$$= \left\{ x : \quad \begin{array}{c} \underline{u} \le K(A+BK)^i x \le \overline{u} \\ \underline{x} \le (A+BK)^i x \le \overline{x} \end{array} \quad i = 0, \ldots, j \right\}$$

- ⋆ $\mathcal{X}_\infty = \mathcal{X}_\nu$ for some finite ν if $|\mathrm{eig}(A+BK)| < 1$

$$\Downarrow$$

$x \in \mathcal{X}_\infty$ if constraints are satisfied on a finite constraint checking horizon

Terminal constraints – Example

Plant model: $x_{k+1} = Ax_k + Bu_k, \qquad y_k = Cx_k$

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

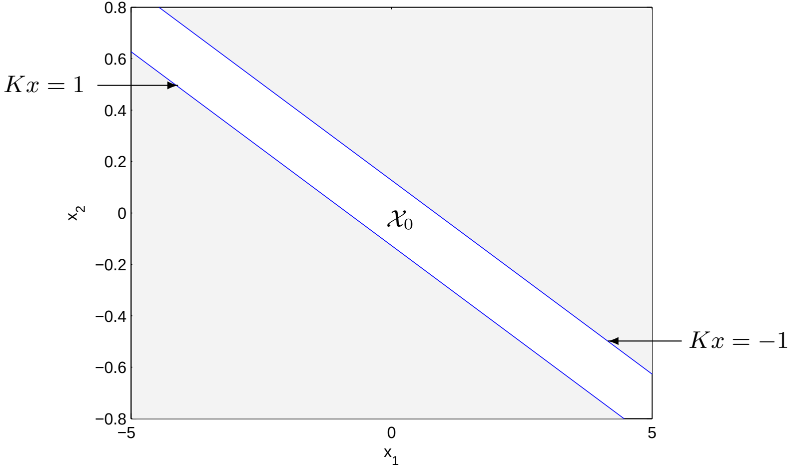
input constraints: $-1 \le u_k \le 1$

mode 2 feedback law: $K = \begin{bmatrix} -1.19 & -7.88 \end{bmatrix}$

$= K_{\mathsf{LQ}}$ for $Q = C^T C, \; R = 1$

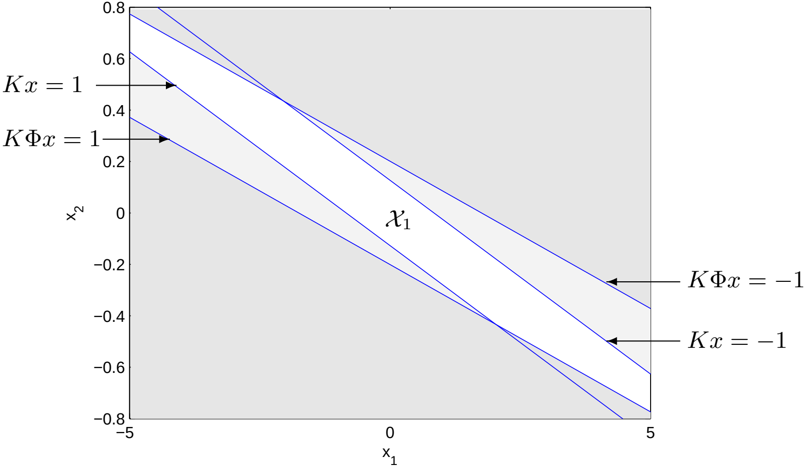
Terminal constraints – example

Constraints: $-1 \leq u \leq 1$



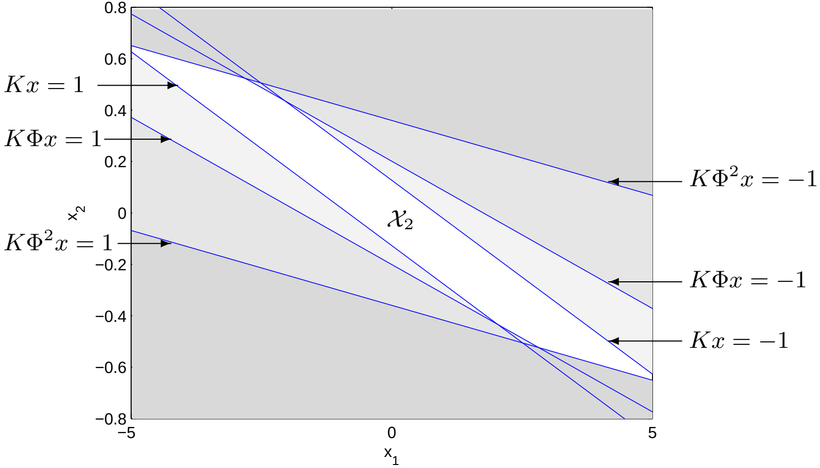
Terminal constraints – example

Constraints: $-1 \le u \le 1$



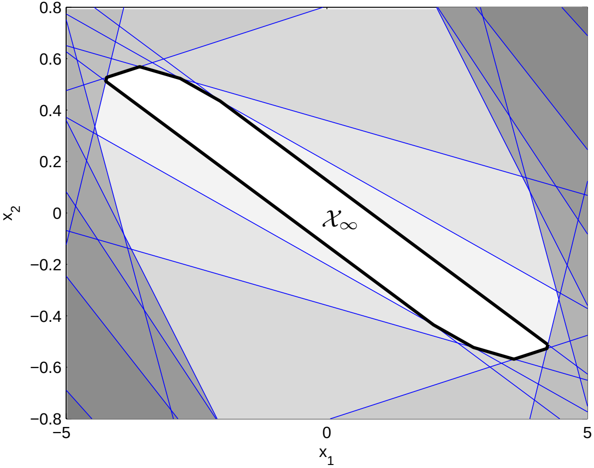
Terminal constraints – example

Constraints: $-1 \leq u \leq 1$



Terminal constraints – example

Constraints: $-1 \leq u \leq 1$



$\mathcal{X}_4 = \mathcal{X}_5 = \cdots = \mathcal{X}_j$ for all $j > 4$ so $\mathcal{X}_4 = \mathcal{X}_\infty$

Terminal constraints – example

In this example $\mathcal{X}_\infty$ is determined in a finite number of steps because

1. $(A + BK)$ is strictly stable, and
2. $\big((A + BK), K\big)$ is observable

1. $\Rightarrow$ $\left\{\begin{array}{l}\text{shortest distance of hyperplane} \\ K(A+BK)^i x \le 1 \text{ from origin}\end{array}\right\} = \dfrac{1}{\|K(A+BK)^i\|_2}$

$\rightarrow \infty \quad \text{as } i \rightarrow \infty$

2. $\Rightarrow$ $\mathcal{X}_\infty$ is bounded because $x_0 \notin \mathcal{X}_\infty$ if x_0 is sufficiently large

Here $\{x : -1 \le K(A+BK)^i x \le 1\}$ contains $\mathcal{X}_4$ for $i > 4$

$$\Downarrow$$

$$\mathcal{X}_\infty = \mathcal{X}_4$$

constraint checking horizon: $\nu = 4$

Terminal constraints – example

In this example $\mathcal{X}_\infty$ is determined in a finite number of steps because

1. $(A + BK)$ is strictly stable, and
2. $\big((A + BK), K\big)$ is observable

1 $\Rightarrow$ $\left\{\begin{array}{l}\text{shortest distance of hyperplane} \\ K(A+BK)^i x \le 1 \text{ from origin}\end{array}\right\} = \dfrac{1}{\|K(A+BK)^i\|_2}$

$\to \infty \quad$ as $i \to \infty$

2 $\Rightarrow$ $\mathcal{X}_\infty$ is bounded because $x_0 \notin \mathcal{X}_\infty$ if x_0 is sufficiently large

Here $\{x : -1 \le K(A+BK)^i x \le 1\}$ contains $\mathcal{X}_4$ for $i > 4$

$$\Downarrow$$

$$\mathcal{X}_\infty = \mathcal{X}_4$$

constraint checking horizon: $\nu = 4$

Terminal constraints

General case

Let $\mathcal{X}_j = \{x : F\Phi^i x \leq \mathbf{1},\ i = 0, \ldots j\}$ with $\begin{cases} \Phi \text{ strictly stable} \\ (\Phi, F) \text{ observable} \end{cases}$

then:

(i). $\mathcal{X}_\infty = \mathcal{X}_\nu$ for finite ν

(ii). $\mathcal{X}_\nu = \mathcal{X}_\infty$ iff $x \in \mathcal{X}_{\nu+1}$ whenever $x \in \mathcal{X}_\nu$

Proof of (ii)

(a). for any j, $\mathcal{X}_{j+1} = \mathcal{X}_j \cap \big\{x : F\Phi^{j+1}x \leq \mathbf{1}\big\}$

so $\mathcal{X}_j \supseteq \mathcal{X}_{j+1} \supseteq \lim_{j\to\infty} \mathcal{X}_j = \mathcal{X}_\infty$

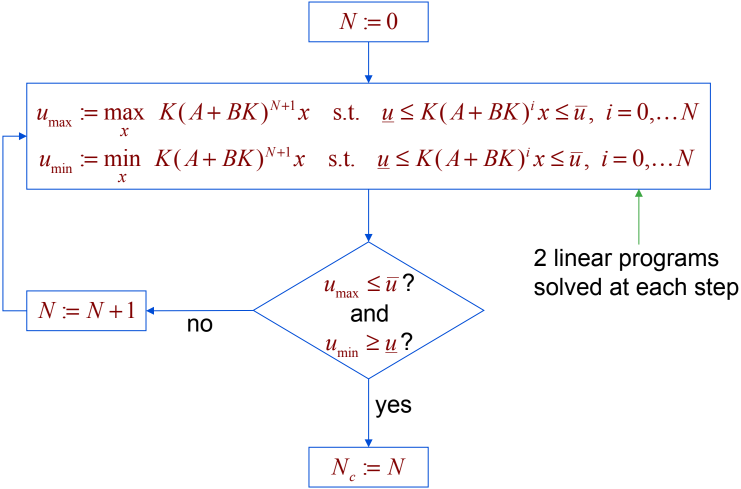
(b). if $x \in \mathcal{X}_{\nu+1}$ whenever $x \in \mathcal{X}_\nu$, then $\Phi x \in \mathcal{X}_\nu$ whenever $x \in \mathcal{X}_\nu$

but $\mathcal{X}_\nu \subseteq \big\{x : Fx \leq \mathbf{1}\big\}$ and it follows that $\mathcal{X}_\nu \subseteq \mathcal{X}_\infty$

(a) & (b) $\Rightarrow$ $\mathcal{X}_\nu = \mathcal{X}_\infty$

Terminal constraints – constraint checking horizon

Algorithm for computing constraint checking horizon N_c
for input constraints $\underline{u} \le u \le \overline{u}$:



Constrained MPC

Define the terminal set Ω as $\mathcal{X}_{N_c}$

MPC algorithm

At each time $k = 0, 1, \ldots$

(i). solve

$$
\begin{aligned}
\mathbf{u}_k^* = \arg\min_{\mathbf{u}_k} \quad & J(x_k, \mathbf{u}_k) \\
\text{s.t.} \quad & \underline{u} \le u_{i|k} \le \overline{u}, \ \ i = 0, \ldots, N + N_c \\
& \underline{x} \le x_{i|k} \le \overline{x}, \ \ i = 1, \ldots, N + N_c
\end{aligned}
$$

(ii). apply $u_k = u^*_{0|k}$ to the system

Note

- ⋆ predictions for $i = N, \ldots N + N_c$: $\begin{cases} x_{i|k} = (A + BK)^{i-N} x_{N|k} \\ u_{i|k} = K(A + BK)^{i-N} x_{N|k} \end{cases}$
- ⋆ $x_{N|k} \in \mathcal{X}_{N_c}$ implies linear constraints so online optimization is a QP

Closed loop performance

Longer horizon N ensures improved predicted cost $J^*(x_0)$

and is likely (but not certain) to give better closed-loop performance

Example: Cost vs N for $x_0 = (-7.5, 0.5)$

N	6	7	8	11	> 11
$J^*(x_0)$	364.2	357.0	356.3	356.0	356.0
$J_{\text{cl}}(x_0)$	356.0	356.0	356.0	356.0	356.0

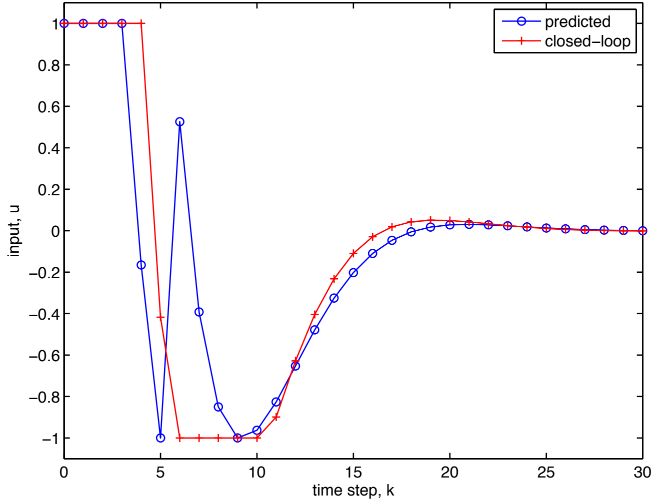
Closed loop cost: $J_{\text{cl}}(x_0) := \sum_{k=0}^{\infty} \bigl(\|x_k\|_Q^2 + \|u_k\|_R^2\bigr)$

For this initial condition:

MPC with $N = 11$ is identical to constrained LQ optimal control ($N = \infty$)!

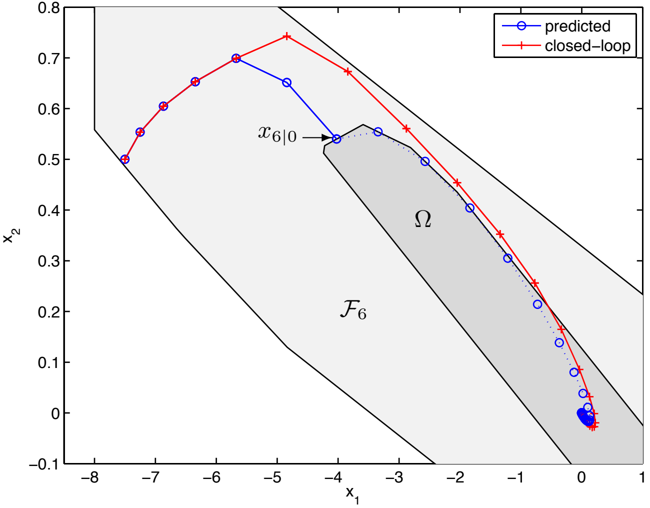
Closed loop performance – example

Predicted and closed loop inputs for $N = 6$



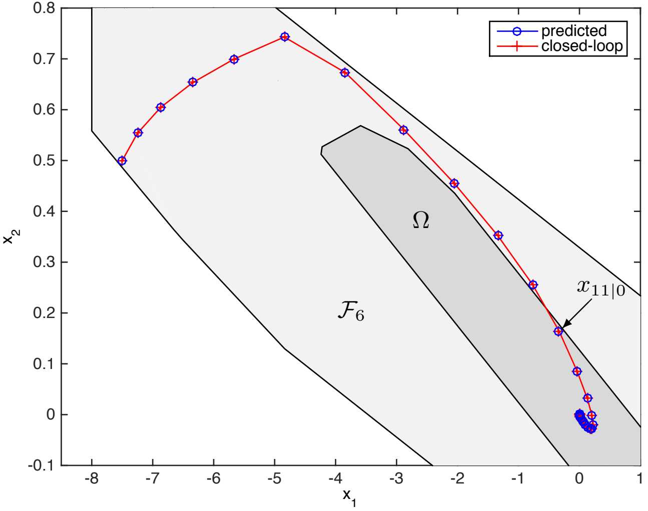
Closed loop performance – example

Predicted and closed loop states for $N = 6$



Closed loop performance – example

Predicted and closed loop states for $N = 11$



Choice of mode 1 horizon – performance

- For this x_0: $N = 11 \Rightarrow x_{N|0}$ lies in the interior of Ω

$$\Updownarrow$$

terminal constraint is inactive

$$\Downarrow$$

no reduction in cost for $N > 11$

- Constrained LQ optimal performance is always obtained with $N \geq N_\infty$ for some finite N_∞ dependent on x_0

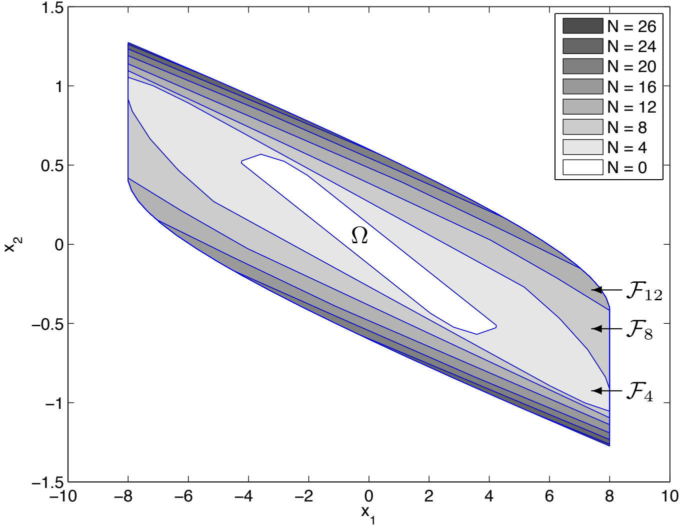
- N_∞ may be large, implying high computational load but closed loop performance is often close to optimal for $N < N_\infty$

  (due to receding horizon)

  in this example $J_{\text{cl}}(x_0) \approx$ optimal for $N \geq 6$

Choice of mode 1 horizon – region of attraction

Increasing N increases the feasible set $\mathcal{F}_N$



Summary

- Linear MPC ingredients:
  - Infinite cost horizon (via terminal cost)
  - Terminal constraints (via constraint-checking horizon)
- Constraints are satisfied over an infinite prediction horizon
- Closed-loop system is asymptotically stable
  with region of attraction equal to the set of feasible initial conditions
- Ideal optimal performance if mode 1 horizon N is large enough
  (but finite)

Lecture 4

Robustness to disturbances

Robustness to disturbances

- Review of nominal model predictive control
- Setpoint tracking and integral action
- Robustness to unknown disturbances
- Handling time-varying disturbances

Review

MPC with guaranteed stability – the basic idea

stabilizing linear controller: cost and constraints can be accounted for over an infinite prediction horizon

Review

MPC optimization for linear model $x_{k+1} = Ax_k + Bu_k$

$$
\begin{aligned}
\underset{\mathbf{u}_k}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \left(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 \right) + \|x_{N|k}\|_P^2 \\
\text{subject to} \quad & \underline{u} \le u_{i|k} \le \overline{u}, \ i = 0, \ldots, N + N_c \\
& \underline{x} \le x_{i|k} \le \overline{x}, \ i = 1, \ldots, N + N_c
\end{aligned}
$$

where

- $\star$ $u_{i|k} = Kx_{i|k}$ for $i \ge N$, with $K =$ unconstrained LQ optimal
- $\star$ terminal cost: $\|x_{N|k}\|_P^2 = \sum_{i=N}^{\infty} (\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2)$, with

$$P - \Phi^T P \Phi = Q + K^T R K, \quad \Phi = A + BK$$

- $\star$ terminal constraints are defined by the constraint checking horizon N_c:

$$\left.\begin{matrix} \underline{u} \le K\Phi^i x \le \overline{u} \\ \underline{x} \le \Phi^i x \le \overline{x} \end{matrix}\right\} \ i = 0, \ldots, N_c \implies \left\{\begin{matrix} \underline{u} \le K\Phi^{N_c+1} x \le \overline{u} \\ \underline{x} \le \Phi^{N_c+1} x \le \overline{x} \end{matrix}\right.$$

Review

MPC optimization for linear model $x_{k+1} = Ax_k + Bu_k$

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Review

MPC optimization for nonlinear model $x_{k+1} = f(x_k, u_k)$

$$\boxed{\begin{aligned} \underset{\mathbf{u}_k}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\big) + \|x_{N|k}\|_P^2 \\ \text{subject to} \quad & \underline{u} \le u_{i|k} \le \overline{u}, \ \ i = 0, \ldots, N-1 \\ & \underline{x} \le x_{i|k} \le \overline{x}, \ \ i = 1, \ldots, N-1 \\ & x_{N|k} \in \Omega \end{aligned}}$$

with

- ⋆ mode 2 feedback: $u_{i|k} = \kappa(x_{i|k})$ asymptotically stabilizing $x = 0$ (locally)
- ⋆ terminal cost: $\|x_{N|k}\|_P^2 \ge \displaystyle\sum_{i=N}^{\infty} \big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2\big)$

  for mode 2 dynamics: $x_{i+1|k} = f\big((x_{i|k}, \kappa(x_{i|k})\big)$
- ⋆ terminal constraint set Ω: invariant for mode 2 dynamics and constraints

$$\left.\begin{aligned} f\big(x, \kappa(x)\big) \in \Omega \\ \underline{u} \le \kappa(x) \le \overline{u}, \ \ \underline{x} \le x \le \overline{x} \end{aligned}\right\} \quad \text{for all } x \in \Omega$$

Review

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\end{array}\right\} \ \text{for all } x \in \Omega
$$

Comparison

- Linear MPC

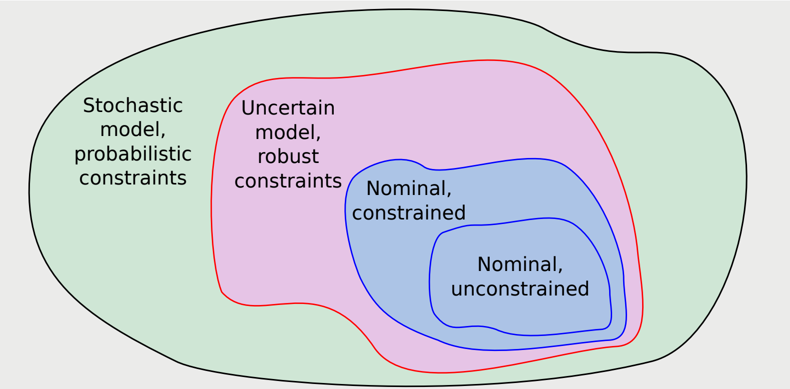
terminal cost	$\longleftarrow$	exact cost over the mode 2 horizon
terminal constraint set	$\longleftarrow$	contains all feasible initial conditions for mode 2

- Nonlinear MPC

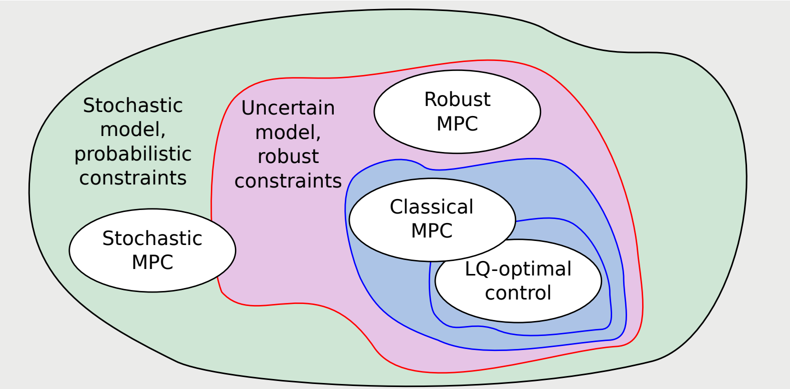
terminal cost	$\longleftarrow$	upper bound on cost over mode 2 horizon
terminal constraint set	$\longleftarrow$	invariant set (usually not the largest) for mode 2 dynamics and constraints

Model uncertainty

Model uncertainty



Model uncertainty



Model uncertainty

Common causes of model error and uncertainty

- Unknown or time-varying model parameters
  - unknown loads & inertias, static friction
  - unknown d.c. gain
- Random (stochastic) model parameters
  - random process noise or sensor noise
- Incomplete measurement of states
  - state estimation error

Setpoint tracking

Output setpoint: y^0

$$y \to y^0 \quad \Rightarrow \quad \begin{cases} x \to x^0 \\ u \to u^0 \end{cases} \quad \text{where} \quad \begin{aligned} x^0 &= Ax^0 + Bu^0 \\ y^0 &= Cx^0 \end{aligned}$$

$$\Downarrow$$

$$y^0 = C(I - A)^{-1}Bu^0$$

Setpoint for (u^0, x^0) is unique iff $C(I - A)^{-1}B$ is invertible

e.g. if $\dim(u) = \dim(y)$, then $\begin{cases} u^0 = \big(C(I - A)^{-1}B\big)^{-1} y^0 \\ x^0 = (I - A)^{-1}Bu^0 \end{cases}$

Tracking problem: $y_k \to y^0$ subject to $\begin{cases} \underline{u} \le u_k \le \overline{u} \\ \underline{x} \le x_k \le \overline{x} \end{cases}$

is only feasible if $\underline{u} \le u^0 \le \overline{u}$ and $\underline{x} \le x^0 \le \overline{x}$

Setpoint tracking

Unconstrained tracking problem:

$$\underset{\mathbf{u}_k^\delta}{\text{minimize}} \quad \sum_{i=0}^{\infty} \left(\|x_{i|k}^\delta\|_Q^2 + \|u_{i|k}^\delta\|_R^2 \right)$$

where $\begin{aligned} x^\delta &= x - x^0 \\ u^\delta &= u - u^0 \end{aligned}$

solution: $u_k = K x_k^\delta + u^0, \quad K = K_{\mathsf{LQ}}$

Constrained tracking problem:

$$\begin{aligned} \underset{\mathbf{u}_k^\delta}{\text{minimize}} \quad & \sum_{i=0}^{\infty} \left(\|x_{i|k}^\delta\|_Q^2 + \|u_{i|k}^\delta\|_R^2 \right) \\ \text{subject to} \quad & \underline{u} \leq u_{i|k}^\delta + u^0 \leq \overline{u}, \qquad i = 0, 1, \ldots \\ & \underline{x} \leq x_{i|k}^\delta + x^0 \leq \overline{x}, \qquad i = 1, 2, \ldots \end{aligned}$$

solution: $u_k = u_{0|k}^{\delta *} + u^0$

Setpoint tracking

If $\hat{u}^0$ is used instead of u^0 (e.g. d.c. gain $C(I-A)^{-1}B$ unknown)

then $u_k = u^{\delta *}_{0|k} + \hat{u}^0$ implies

$$u^\delta_k = u^{\delta *}_{0|k} + (\hat{u}^0 - u^0)$$
$$x^\delta_{k+1} = Ax^\delta_k + Bu^{\delta *}_{0|k} + B\underbrace{(\hat{u}^0 - u^0)}_{\text{constant disturbance}}$$

and if $u^{\delta *}_{0|k} \to Kx^\delta_k$ as $k \to \infty$, then

$$\lim_{k\to\infty} x^\delta_k = (I - A - BK)^{-1}B(\hat{u}^0 - u^0) \qquad \neq 0$$
$$\lim_{k\to\infty} y_k - y^0 = \underbrace{C(I - A - BK)^{-1}B(\hat{u}^0 - u^0)}_{\text{steady state tracking error}} \qquad \neq 0$$

Additive disturbances

Convert (constant) setpoint tracking problem into a regulation problem:

$$x \leftarrow x^\delta,\ y \leftarrow y^\delta,\ u \leftarrow u^\delta$$

Consider the effect of additive disturbance w:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Dw_k, \\ y_k &= Cx_k \end{aligned}$$

Assume that w_k is unknown at time k, but is known to be

- ⋆ constant: $w_k = w$ for all k
- ⋆ or time-varying within a known polytopic set: $w_k \in \mathcal{W}$ for all k

where $\quad \mathcal{W} = \operatorname{conv}\{w^{(1)}, \ldots, w^{(r)}\}$

or $\quad \mathcal{W} = \{w : Hw \leq \mathbf{1}\}$

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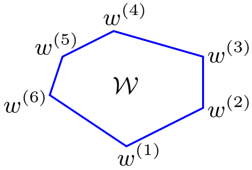

Integral action

Introduce integral action to remove steady state error in y
by considering the augmented system:

$$\boxed{z_k = \begin{bmatrix} x_k \\ v_k \end{bmatrix}, \qquad z_{k+1} = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} z_k + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k + \begin{bmatrix} D \\ 0 \end{bmatrix} w_k}$$

$$v_k = \text{ integrator state}$$
$$v_{k+1} = v_k + y_k$$

⋆ Linear feedback $\quad u_k = Kx_k + K_I v_k$

is stabilizing if $\quad \left| \mathrm{eig}\left(\begin{bmatrix} A + BK & BK_I \\ C & I \end{bmatrix} \right) \right| < 1$

⋆ If the closed-loop system is (strictly) stable and $w_k \to w =$ constant

then $\quad u_k \to u^{ss} \quad \Longrightarrow \quad v_k \to v^{ss} \quad \Longrightarrow \quad y_k \to 0$

even if $w \neq 0$!!

...but arbitrary K_I may destabilize the closed loop system

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Integral action

Ensure stability by using a modified cost:

$$\boxed{\underset{\mathbf{u}_k}{\text{minimize}} \quad \sum_{i=0}^{\infty} \left(\|z_{i|k}\|^2_{Q_z} + \|u_{i|k}\|^2_R \right) \qquad Q_z = \begin{bmatrix} Q & 0 \\ 0 & Q_I \end{bmatrix}}$$

with predictions generated by an augmented model

$$z_{i+1|k} = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} z_{i|k} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_{i|k}, \qquad z_{0|k} = \begin{bmatrix} x_k \\ v_k \end{bmatrix}$$

- ⋆ this is a nominal prediction model since $w_k = 0$ is assumed
- ⋆ unconstrained solution: $u_k = K_z z_k = K x_k + K_I v_k$
- ⋆ if $\left(\begin{bmatrix} A & 0 \\ C & I \end{bmatrix}, \begin{bmatrix} Q & 0 \\ 0 & Q_I \end{bmatrix} \right)$ is observable and $w_k \to w =$ constant

$$\text{then} \quad u_k \to u^{ss} \quad \Longrightarrow \quad v_k \to v^{ss} \quad \Longrightarrow \quad y_k \to 0$$

even if $w \neq 0$!!

Integral action

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even if $w \neq 0$!!

Integral action – example

Plant model:

$$x_{k+1} = Ax_k + Bu_k + Dw \qquad\qquad y_k = Cx_k$$

$$A = \begin{bmatrix} 1.1 & 2 \\ 0 & 0.95 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} \quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

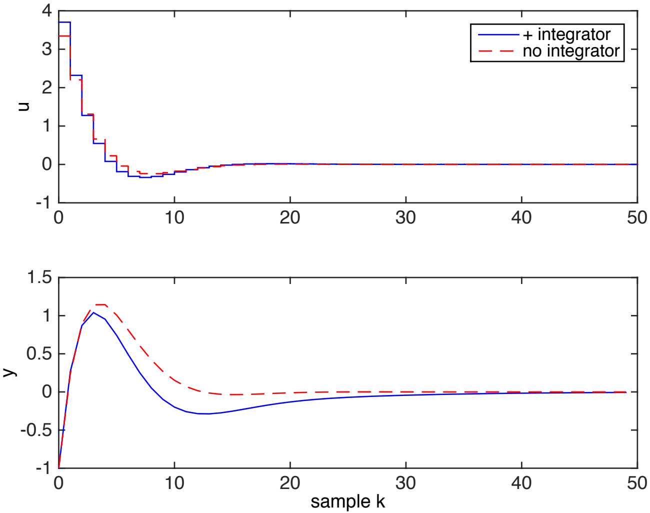
Constraints: none

Cost weighting matrices: $Q_z = \begin{bmatrix} C^T C & 0 \\ 0 & 0.01 \end{bmatrix}$, $R = 1$

Unconstrained LQ optimal feedback gain:

$$K_z = \begin{bmatrix} -1.625 & -9.033 & 0.069 \end{bmatrix}$$

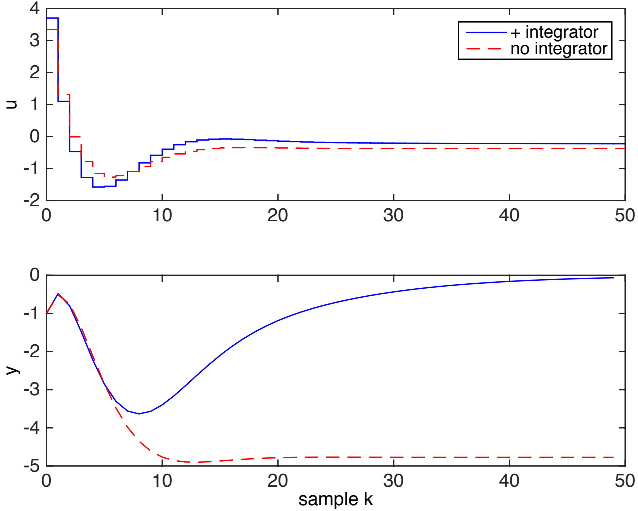
Integral action – example



Closed loop response for initial condition: $x_0 = (0.5, -0.5)$

no disturbance: $w = 0$

Integral action – example



Closed loop response for initial condition: $x_0 = (0.5, -0.5)$
constant disturbance: $w = 0.75$

Constrained MPC

Naive constrained MPC strategy: $w = 0$ assumed in predictions

$$
\begin{aligned}
\underset{\mathbf{u}_k}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \left(\|z_{i|k}\|^2_{Q_z} + \|u_{i|k}\|^2_R \right) + \|z_{N|k}\|^2_P \\
\text{subject to} \quad & \underline{u} \le u_{i|k} \le \overline{u}, \ \ i = 0, \ldots, N + N_c \\
& \underline{x} \le x_{i|k} \le \overline{x}, \ \ i = 1, \ldots, N + N_c
\end{aligned}
$$

with: P and N_c determined for mode 2 control law $u_{i|k} = K_z z_{i|k}$

initial prediction state: $z_{0|k} = \begin{bmatrix} x_k \\ v_k \end{bmatrix}$ where $v_k = v_{k-1} + y_k$

⋆ If closed loop system is stable

$$\text{then} \quad u_k \to u^{ss} \quad \Longrightarrow \quad v_k \to v^{ss} \quad \Longrightarrow \quad y_k \to 0$$

⋆ but disturbance w_k is ignored in predictions, so

$$\begin{cases} J^*(z_{k+1}) - J^*(z_k) \not\le 0 \\ \text{feasibility at time } k \not\Rightarrow \text{ feasibility at } k+1 \end{cases}$$

therefore no guarantee of stability

Constrained MPC

Naive constrained MPC strategy: $w = 0$ assumed in predictions

$$\begin{aligned} \underset{\mathbf{u}_k}{\text{minimize}} \quad & \sum_{i=0}^{N-1} \left(\|z_{i|k}\|_{Q_z}^2 + \|u_{i|k}\|_R^2 \right) + \|z_{N|k}\|_P^2 \\ \text{subject to} \quad & \underline{u} \le u_{i|k} \le \overline{u}, \ i = 0, \ldots, N + N_c \\ & \underline{x} \le x_{i|k} \le \overline{x}, \ i = 1, \ldots, N + N_c \end{aligned}$$

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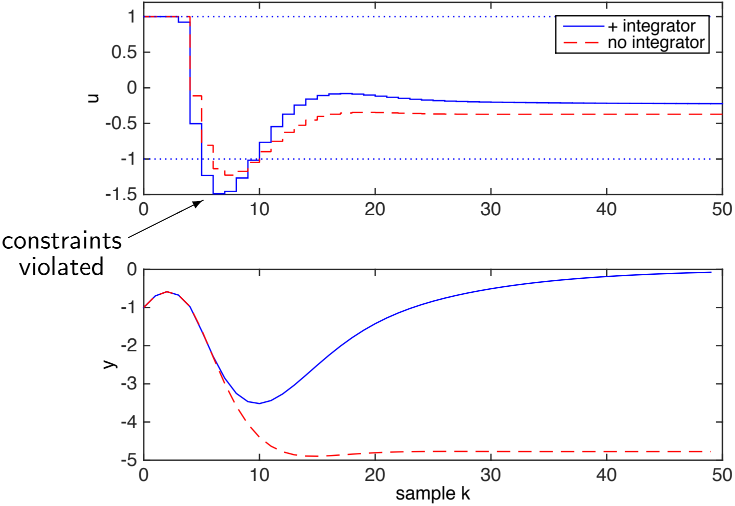
Constrained MPC – example



Closed loop response with initial condition: $x_0 = (0.5, -0.5)$

constraints: $-1 \le u \le 1$ disturbance: $w = 0.75$

Robust constraints

If predictions satisfy constraints $\begin{cases} \text{for all prediction times } i = 0, 1, \ldots \\ \text{for all disturbances } w_i \in \mathcal{W} \end{cases}$

then feasibility of constraints at time k ensures feasibility at time $k+1$

- Linear dynamics plus additive disturbance enables decomposition

  nominal predicted state $s_{i|k}$
  uncertain predicted state $e_{i|k}$

  where

$$x_{i|k} = s_{i|k} + e_{i|k} \qquad \begin{cases} s_{i+1|k} = \Phi s_{i|k} + B c_{i|k} & s_{0|k} = x_k \\ e_{i+1|k} = \Phi e_{i|k} + D w_{i|k} & e_{0|k} = 0 \end{cases}$$

- Pre-stabilized predictions:

$$u_{i|k} = K x_{i|k} + c_{i|k} \text{ and } \Phi = A + BK$$

  where $K = K_{\mathsf{LQ}}$ is the unconstrained LQ optimal gain

Pre-stabilized predictions – example

Scalar system: $x_{k+1} = 2x_k + u_k + w_k,$ constraint: $|x_k| \le 2$

uncertainty: $e_{i|k} = \sum_{j=0}^{i-1} 2^j w = (2^i - 1)w,$ disturbance: $w_k = w$, $|w| \le 1$

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Robust constraints:

$|s_{i|k} + e_{i|k}| \le 2$ for all $|w| \le 1$

$$\Updownarrow$$

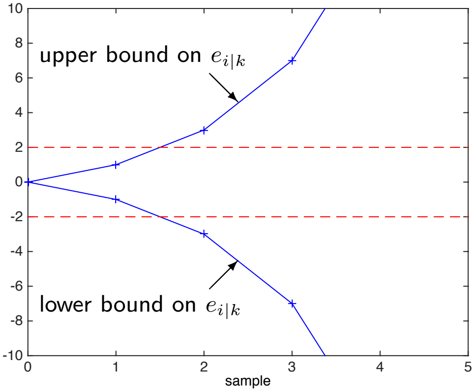
$$|s_{i|k}| \le 2 - \max_{|w|\le 1} |e_{i|k}|$$

$$\Downarrow$$

$$|s_{i|k}| \le 2 - (2^i - 1)$$

$$\Downarrow$$

infeasible for all $i > 1$



Pre-stabilized predictions – example

Avoid infeasibility by using pre-stabilized predictions:

$$u_{i|k} = Kx_{i|k} + c_{i|k}, \qquad K = -1.9, \qquad c_{i|k} = \begin{cases} \text{free} & i = 0, \ldots, N-1 \\ 0 & i \geq N \end{cases}$$

stable predictions: $e_{i|k} = \displaystyle\sum_{j=0}^{i-1} 0.1^j w = (1 - 0.1^i)w/0.9, \quad |w| \leq 1$

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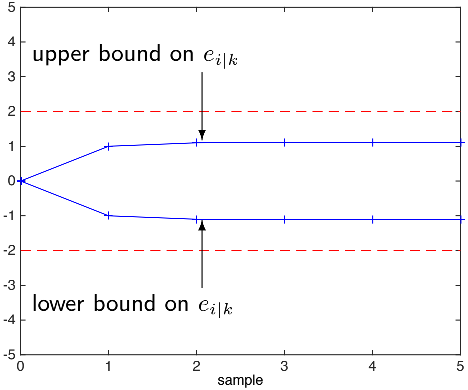
$$\Updownarrow$$

$$|s_{i|k}| \leq 2 - \max_{|w|\leq 1} |e_{i|k}|$$

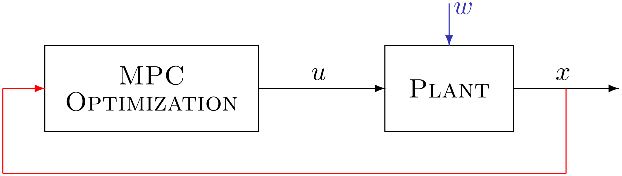
$$\Downarrow$$

$$|s_{i|k}| \leq \underbrace{2 - (1 - 0.1^i)/0.9}_{>0 \text{ for all } i}$$

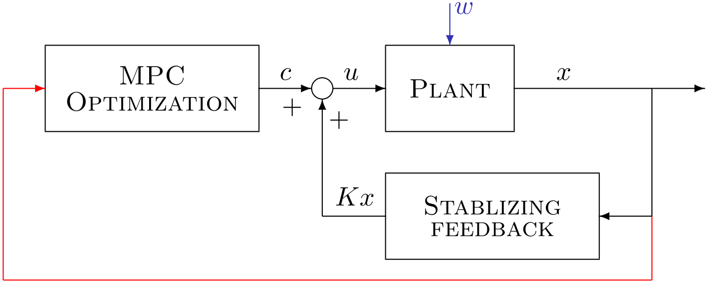


Pre-stabilized predictions

- Feedback structure of MPC with open loop predictions:



- Feedback structure of MPC with pre-stabilized predictions:



General form of robust constraints

How can we impose (general linear) constraints robustly?

⋆ Pre-stabilized predictions:

$$x_{i|k} = s_{i|k} + e_{i|k} \qquad \begin{cases} s_{i+1|k} = \Phi s_{i|k} + Bc_{i|k} & s_{0|k} = x_k \\ e_{i+1|k} = \Phi e_{i|k} + Dw_{i|k} & e_{0|k} = 0 \end{cases}$$

$$\Longrightarrow \quad e_{i|k} = Dw_{i-1} + \Phi Dw_{i-2} + \cdots + \Phi^{i-1} Dw_0$$

⋆ General linear constraints: $Fx_{i|k} + Gu_{i|k} \leq \mathbf{1}$

are equivalent to tightened constraints on nominal predictions:

$$\boxed{(F + GK)s_{i|k} + Gc_{i|k} \leq \mathbf{1} - h_i}$$

where
$$h_0 = 0$$
$$h_i = \max_{w_0,\ldots,w_{i-1}\in\mathcal{W}} (F + GK)e_{i|k}, \quad i = 1, 2, \ldots$$

(i.e. $h_i = h_{i-1} + \max_{w\in\mathcal{W}}(F + GK)w$
requiring one LP for each row of h_i)

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Tube interpretation

The uncertainty in predictions: $e_{i+1|k} = \Phi e_{i|k} + Dw_i$, $w_i \in \mathcal{W}$

evolves inside a tube (a sequence of sets): $e_{i|k} \in E_{i|k}$, where

$$E_{i|k} = D\mathcal{W} \oplus \Phi D\mathcal{W} \oplus \cdots \oplus \Phi^{i-1} D\mathcal{W}, \quad i = 1, 2, \ldots$$

Hence we can define:

- ⋆ a state tube $x_{i|k} = s_{i|k} + e_{i|k} \in \mathcal{X}_{i|k}$

$$\mathcal{X}_{i|k} = \{s_{i|k}\} \oplus E_{i|k}, \; i = 0, 1, \ldots$$

- ⋆ a control input tube $u_{i|k} = Kx_{i|k} + c_{i|k} = Ks_{i|k} + c_{i|k} + Ke_{i|k} \in \mathcal{U}_{i|k}$

$$\mathcal{U}_{i|k} = \{Ks_{i|k} + c_{i|k}\} \oplus KE_{i|k}, \; i = 0, 1, \ldots$$

and impose constraints robustly for the state and input tubes

(where $\oplus$ is Minkowski set addition)

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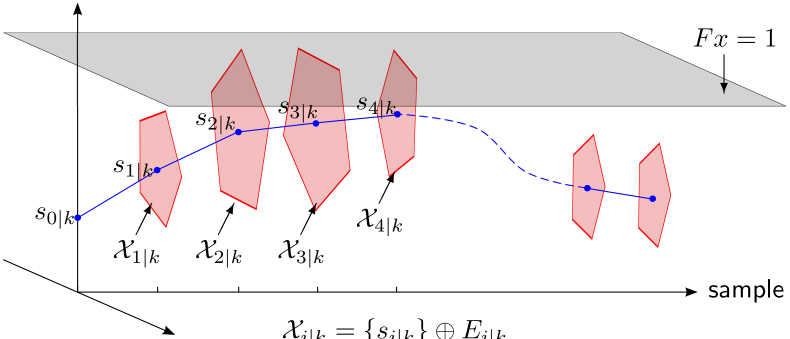
$$E_{i|k} = D\mathcal{W} \oplus \Phi D\mathcal{W} \oplus \cdots \oplus \Phi^{i-1} D\mathcal{W}, \quad i = 1, 2, \ldots$$

e.g. for constraints $Fx \leq \mathbf{1}$ $(G = 0)$



$$\mathcal{X}_{i|k} = \{s_{i|k}\} \oplus E_{i|k}$$

Prototype robust MPC algorithm

Offline: compute N_c and $h_1, \ldots, h_{N_c}$. Online at $k = 0, 1, \ldots$:

(i). solve
$$\mathbf{c}_k^* = \arg\min_{\mathbf{c}_k} \; J(x_k, \mathbf{c}_k)$$
$$\text{s.t.} \;\; (F + GK)s_{i|k} + Gc_{i|k} \le \mathbf{1} - h_i, \;\; i = 0, \ldots, N + N_c$$

(ii). apply $u_k = Kx_k + c^*_{0|k}$ to the system

- ⋆ tightened linear constraints are applied to nominal predictions
- ⋆ N_c is the constraint-checking horizon:

$$(F + GK)\Phi^i s \le \mathbf{1} - h_i, \quad i = 0, \ldots, N_c$$
$$\implies \quad (F + GK)\Phi^{N_c+1} s \le \mathbf{1} - h_{N_c+1}$$

- ⋆ the online optimization is robustly recursively feasible

Robust MPC

Prototype robust MPC algorithm

Offline: compute N_c and $h_1, \ldots, h_{N_c}$. Online at $k = 0, 1, \ldots$:

(i). solve
$$\mathbf{c}_k^* = \arg\min_{\mathbf{c}_k} \; J(x_k, \mathbf{c}_k)$$
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(ii). apply $u_k = Kx_k + c^*_{0|k}$ to the system

two alternative cost functions:

- ⋆ nominal cost (i.e. cost evaluated assuming $w_i = 0$ for all i)

$$J(x_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} \big(\|s_{i|k}\|_Q^2 + \|Ks_{i|k} + c_{i|k}\|_R^2 \big) = \|x_k\|_P^2 + \|\mathbf{c}_k\|_{W_c}^2$$

- ⋆ worst case cost, defined in terms of a desired disturbance gain γ:

$$J(x_k, \mathbf{c}_k) = \max_{w_i \in \mathcal{W}, \; i=0,1,\ldots} \sum_{i=0}^{\infty} \big(\|x_{i|k}\|_Q^2 + \|u_{i|k}\|_R^2 - \gamma^2 \|w_i\|^2 \big)$$

Convergence of robust MPC with nominal cost

If $u_{i|k} = Kx_{i|k} + c_{i|k}$ for $K = K_{\text{LQ}}$, then:

- the unconstrained optimum is $\mathbf{c}_k = 0$, so the nominal cost is
$$J(x_k, \mathbf{c}_k) = \|x_k\|_P^2 + \|\mathbf{c}_k\|_{W_c}^2$$
and W_c is block-diagonal: $W_c = \operatorname{diag}\{P_c, \ldots, P_c\}$

- recursive feasibility $\Rightarrow$ $\tilde{\mathbf{c}}_{k+1} = (c^*_{1|k}, \ldots, c^*_{N-1|k}, 0)$ feasible at $k+1$

- hence $\quad \|\mathbf{c}^*_{k+1}\|^2_{W_c} \leq \|\mathbf{c}^*_k\|^2_{W_c} - \|c^*_{0|k}\|^2_{P_c}$
$$\Rightarrow \quad \sum_{k=0}^{\infty} \|c_{0|k}\|^2_{P_c} \leq \|\mathbf{c}^*_0\|^2_{W_c} < \infty$$
$$\Rightarrow \quad \lim_{k\to\infty} c_{0|k} = 0$$

- therefore $\quad u_k \to Kx_k$ as $k \to \infty$

  $x_k \to$ the (minimal) robustly invariant set under unconstrained LQ optimal feedback

Robust MPC with constant disturbance

Assume $w_k = w =$ constant for all k

combine: pre-stabilized predictions
augmented state space model

- Predicted state and input sequences:

$$x_{i|k} = \begin{bmatrix} I & 0 \end{bmatrix} (s_{i|k} + e_{i|k})$$
$$u_{i|k} = K_z(s_{i|k} + e_{i|k}) + c_{i|k}$$

- Prediction model:

nominal $\quad s_{i+1|k} = \Phi s_{i|k} + \begin{bmatrix} B \\ 0 \end{bmatrix} c_{i|k} \qquad \Phi = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} K_z$

uncertain $\quad e_{i|k} = \sum_{j=0}^{i-1} \Phi^j \begin{bmatrix} D \\ 0 \end{bmatrix} w \qquad s_{0|k} = \begin{bmatrix} x_k \\ v_k \end{bmatrix}, \; e_{0|k} = 0$

- Nominal cost:

$$J(x_k, v_k, \mathbf{c}_k) = \sum_{i=0}^{\infty} \left(\|s_{i|k}\|^2_{Q_z} + \|K_z s_{i|k} + c_{i|k}\|^2_R \right)$$

Robust MPC with constant disturbance

Assume $w_k = w =$ constant for all k

combine: pre-stabilized predictions
augmented state space model

- $\star$ robust state constraints:

$$\underline{x} \le x_{i|k} \le \overline{x} \quad \Longleftrightarrow \quad \underline{x} + h_i \le s_{i|k} \le \overline{x} - h_i$$

$$h_i = \max_{w \in \mathcal{W}} \begin{bmatrix} I & 0 \end{bmatrix} \sum_{j=0}^{i-1} \Phi^j \begin{bmatrix} D \\ 0 \end{bmatrix} w$$

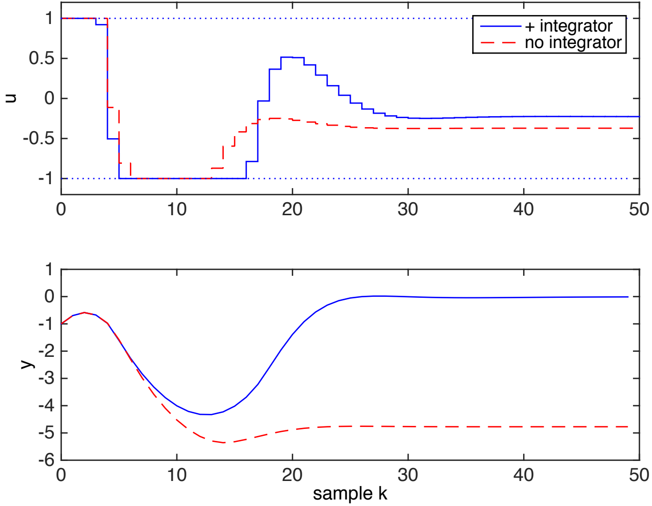
- $\star$ robust input constraints:

$$\underline{u} \le u_{i|k} \le \overline{u} \quad \Longleftrightarrow \quad \underline{u} + h_i' \le K_z s_{i|k} + c_{i|k} \le \overline{u} - h_i'$$

$$h_i' = \max_{w \in \mathcal{W}} K_z \sum_{j=0}^{i-1} \Phi^j \begin{bmatrix} D \\ 0 \end{bmatrix} w$$

- $\star$ N_c and h_i, h_i' for $i = 1, \dots, N_c$ can be computed offline

Robust MPC with constant disturbance – example



Closed loop response with initial condition: $x_0 = (0.5, -0.5)$

constraints: $-1 \leq u \leq 1$ disturbance: $w = 0.75$

Summary

▷ Integral action: augment model with integrated output error
include integrated output error in cost

then

(i). closed loop system is stable if $w = 0$

(ii). steady state error must be zero if response is stable for $w \neq 0$

▷ Robust MPC: use pre-stabilized predictions
apply constraints for all possible future uncertainty

then

(i). constraint feasibility is guaranteed at all times if initially feasible

(ii). closed loop system inherits the stability and convergence properties of unconstrained LQ optimal control (assuming nominal cost)

Overview of the course

1. ## Introduction and Motivation

   Basic MPC strategy; prediction models; input and state constraints; constraint handling: saturation, anti-windup, predictive control

2. ## Prediction and optimization

   Input/state prediction equations; unconstrained optimization. Infinite horizon cost; dual mode predictions. Incorporating constraints; quadratic programming.

3. ## Closed loop properties

   Lyapunov analysis based on predicted cost. Recursive feasibility; terminal constraints; the constraint checking horizon. Constrained LQ-optimal control.

4. ## Robustness to disturbances

   Setpoint tracking; MPC with integral action. Robustness to constant disturbances: prestabilized predictions and robust feasibility. Handling time-varying disturbances.

The End