Lecture 3: Invariant Manifolds

- In this lecture we consider **differential manifolds**
  A manifold is a smooth surface that looks locally like Euclidean space

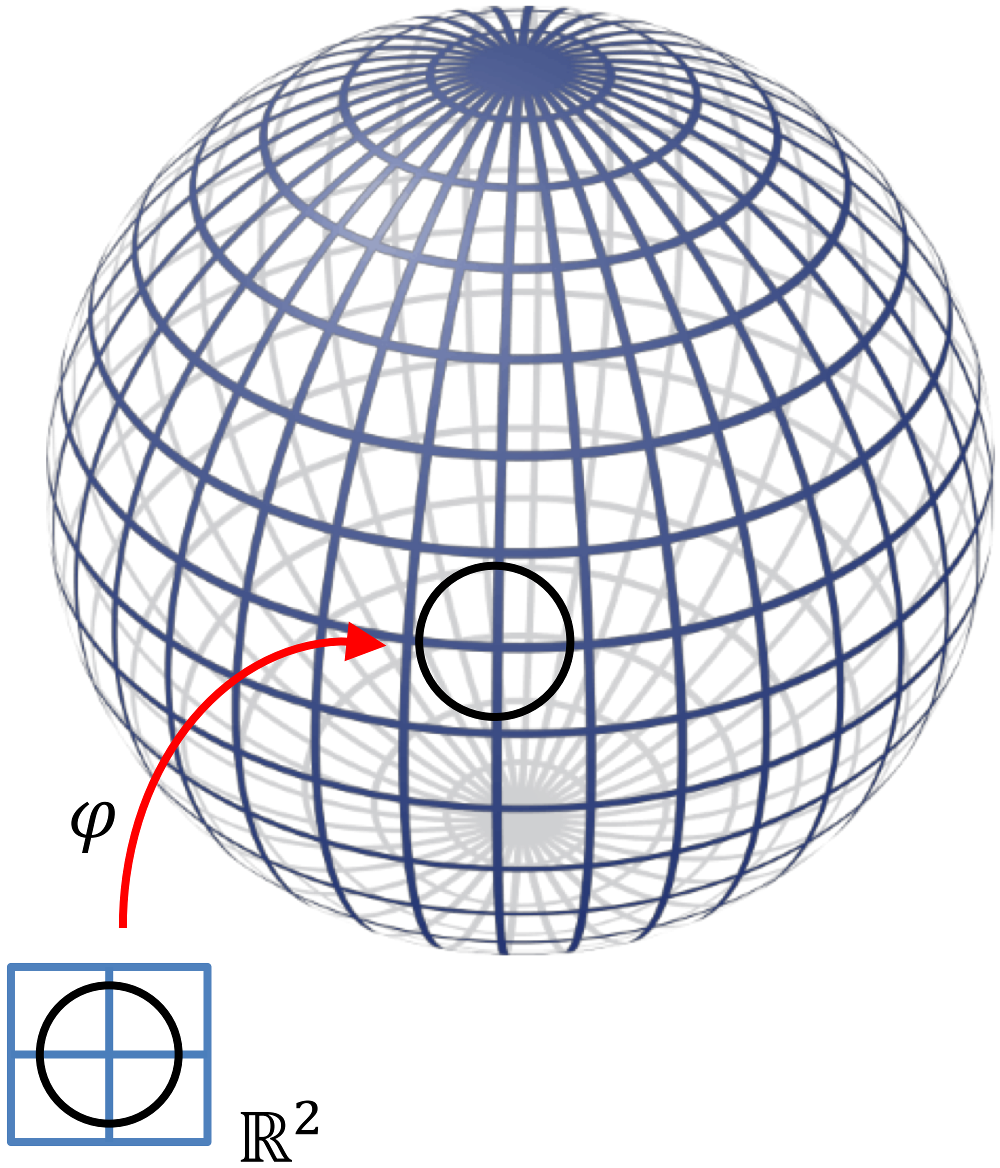
- We are interested in paths on these manifolds and how they characterise the solutions of differential equations;
  hence consider differential manifolds in phase space

- We will revisit hyperbolic equilibria and also consider non-hyperbolic equilibria and their associated manifolds

- We start with **vector subspaces** – generalisations of lines and planes.
  Manifolds are curved generalisations of vector subspaces.

Differential Manifolds

- Manifold M is locally made up of patches copied from $\mathbb{R}^n$
- e.g. M = surface of a sphere
- Definition of a differential manifold involves how local coordinate systems are placed on M and how neighbouring patches transform into each other



Linear systems: Stable, Unstable and Centre subspaces

- We studied the linear system

$$\dot{\mathbf{w}} = \mathbf{A}\mathbf{w}$$

- 3 distinct types of eigenvalue of interest:
  - the stable set, with $Re(\lambda) < 0$, denoted $\lambda_1{}^S, \lambda_2{}^S \cdots \lambda_s{}^S$
  - the unstable set, with $Re(\lambda) > 0$, denoted $\lambda_1{}^U, \lambda_2{}^U \cdots \lambda_u{}^U$
  - the centre set with, $Re(\lambda) = 0$, denoted $\lambda_1{}^C, \lambda_2{}^C \cdots \lambda_c{}^C$
- There are n eigenvalues, so $s + u + c = n$

Vector bases and the subspaces

- If $\mathbf{A}$ has real, distinct eigenvectors, the n eigenvectors span phase space and any vector can be made up from a weighted sum of eigenvectors (denoted $\mathbf{u}_1 \dots \mathbf{u}_n$)

  Then, given the initial condition $\mathbf{w}(0)$:

$$\mathbf{w}(0) = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n$$

  we can express the solution as

$$\mathbf{w}(t) = c_1\mathbf{u}_1 e^{\lambda_1 t} + c_2\mathbf{u}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{u}_n e^{\lambda_n t}$$

- If the eigenvalues are complex or repeated, other methods exist for defining $\mathbf{u}_1 \dots \mathbf{u}_n$

- Split phase space into three subspaces spanned by eigenvector sets:

  $\{\mathbf{u}_i{}^S\}$ eigenvectors associated with stable (negative real part) eigenvalues

  $\{\mathbf{u}_j{}^U\}$ eigenvectors associated with unstable (positive real part) eigenvalues

  $\{\mathbf{u}_k{}^C\}$ eigenvectors associated with eigenvalues having zero real part

- Each eigenvector, multiplied by $e^{\lambda t}$, $te^{\lambda t}$ or $t^2 e^{\lambda t}$ etc generates a component of the solution in $\{\mathbf{w}_i{}^S(t), \mathbf{w}_j{}^U(t), \mathbf{w}_k{}^C(t)\}$ of the form

$$\mathbf{w}(t) = \sum_{i=1}^{s} c_i \mathbf{w}_i{}^S(t) + \sum_{j=1}^{u} c_j \mathbf{w}_j{}^U(t) + \sum_{k=1}^{c} c_k \mathbf{w}_k{}^C(t)$$

  These components are written as

$$\mathbf{w}(t) = \mathbf{w}^S(t) + \mathbf{w}^U(t) + \mathbf{w}^C(t)$$

e.g. $\mathbf{w}_1{}^S(t) = e^{\lambda_1{}^S t}\mathbf{u}_1{}^S$.

Effect of changing coordinates on A

- Last lecture we considered a change of coordinates $\mathbf{y} = \mathbf{V}^{-1}\mathbf{x}$. Let the matrix $\mathbf{V}$ be made up of the three set of vectors $\{\mathbf{u}_i{}^S, \mathbf{u}_j{}^U, \mathbf{u}_k{}^C\}$ and call this matrix $\mathbf{T}$.

- Let $\mathbf{z} = \mathbf{T}^{-1}\mathbf{w}$ be the new coordinates:

$$\mathbf{z} = \mathbf{T}^{-1}\mathbf{w} = \begin{bmatrix} \mathbf{z}_S \\ \mathbf{z}_U \\ \mathbf{z}_C \end{bmatrix}$$

  This defines three decoupled subspaces, each with their individual coordinates

New equations of motion

- Change of coordinates:

$$\dot{\mathbf{z}} = \mathbf{T}^{-1}\dot{\mathbf{w}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{w} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{z}$$

- We chose the new coordinates so that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \mathbf{A}_S & 0 & 0 \\ 0 & \mathbf{A}_U & 0 \\ 0 & 0 & \mathbf{A}_C \end{bmatrix}$$

Where $Re[\lambda(\mathbf{A}_S)] < 0, Re[\lambda(\mathbf{A}_U)] > 0, Re[\lambda(\mathbf{A}_S)] = 0$

New equations of motion:

$$\dot{\mathbf{z}}_s = \mathbf{A}_S \boldsymbol{z}_S$$
$$\dot{\mathbf{z}}_U = \mathbf{A}_S \boldsymbol{z}_U$$
$$\dot{\mathbf{z}}_{\mathbf{C}} = \mathbf{A}_C \boldsymbol{z}_C$$

Consequences of the change of coordinates

- By changing coordinates we have split the solution of the linear differential equation into three independent subspaces:

$$E^{\mathrm{S}},\ E^{\mathrm{U}},\ \text{and}\ E^{\mathrm{C}}$$

- If the solution starts in one of these subspaces it will stay in it and cannot cross into one of the other subspaces

- These subspaces are **invariant with respect to the flow** $e^{\mathbf{A}t}$

Example system

Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}$$

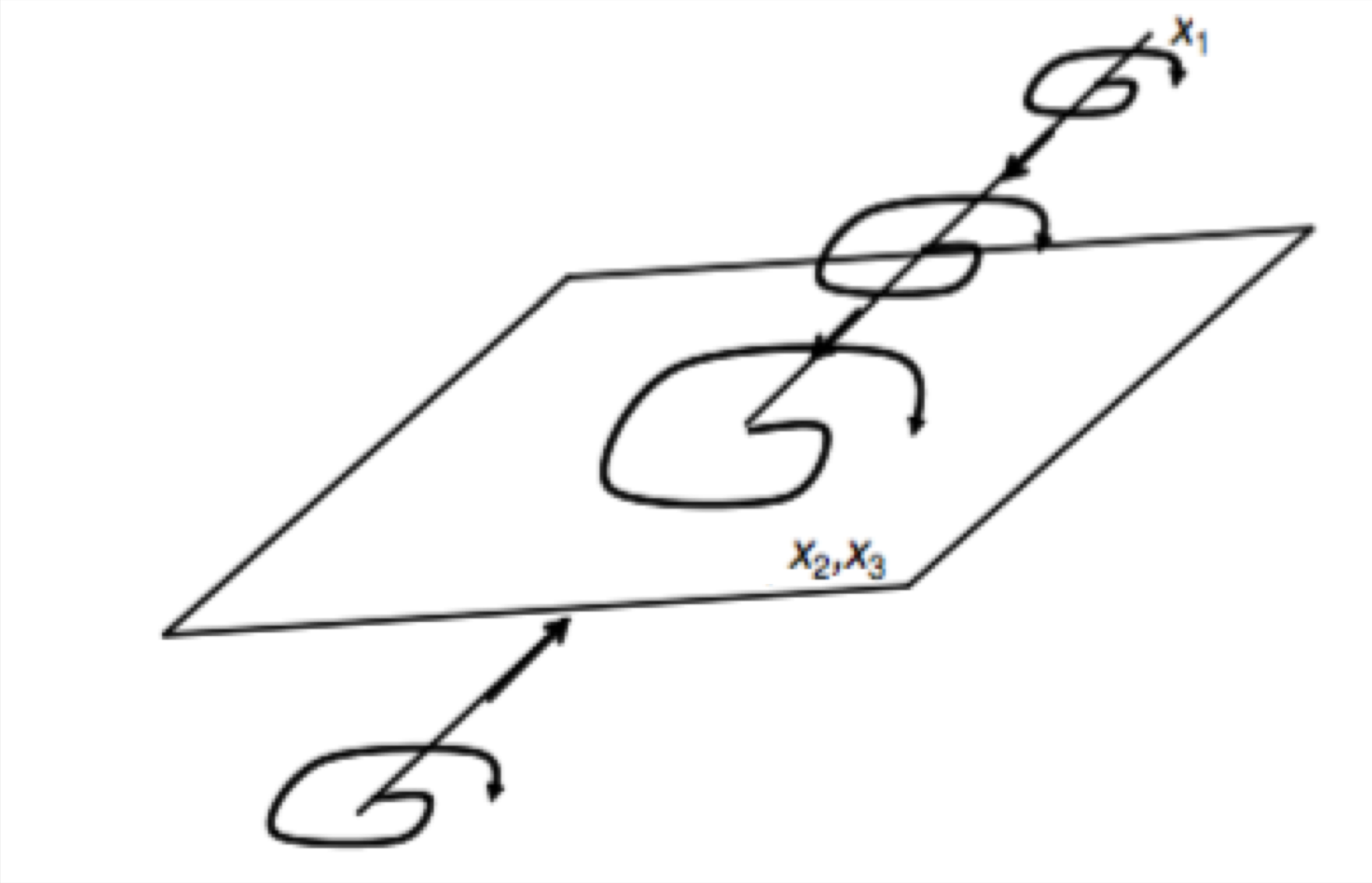
The eigenvalues are -3 and $2 \pm j$

with eigenvectors $[1 \quad 0 \quad 0]^T$ and $[0 \quad 1 \pm j \quad 1]^T$

The solution is

$$\mathbf{x}(t) = \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t}(\mathrm{cos}t + \mathrm{sin}t) & -2e^{2t}\mathrm{sin}t \\ 0 & e^{2t}\mathrm{sin}t & e^{2t}(\mathrm{cos}t - \mathrm{sin}t) \end{bmatrix} \mathbf{x}(0)$$

Stable subspace and unstable spiral



Degenerate example

Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \gamma \end{bmatrix} \mathbf{x}$$

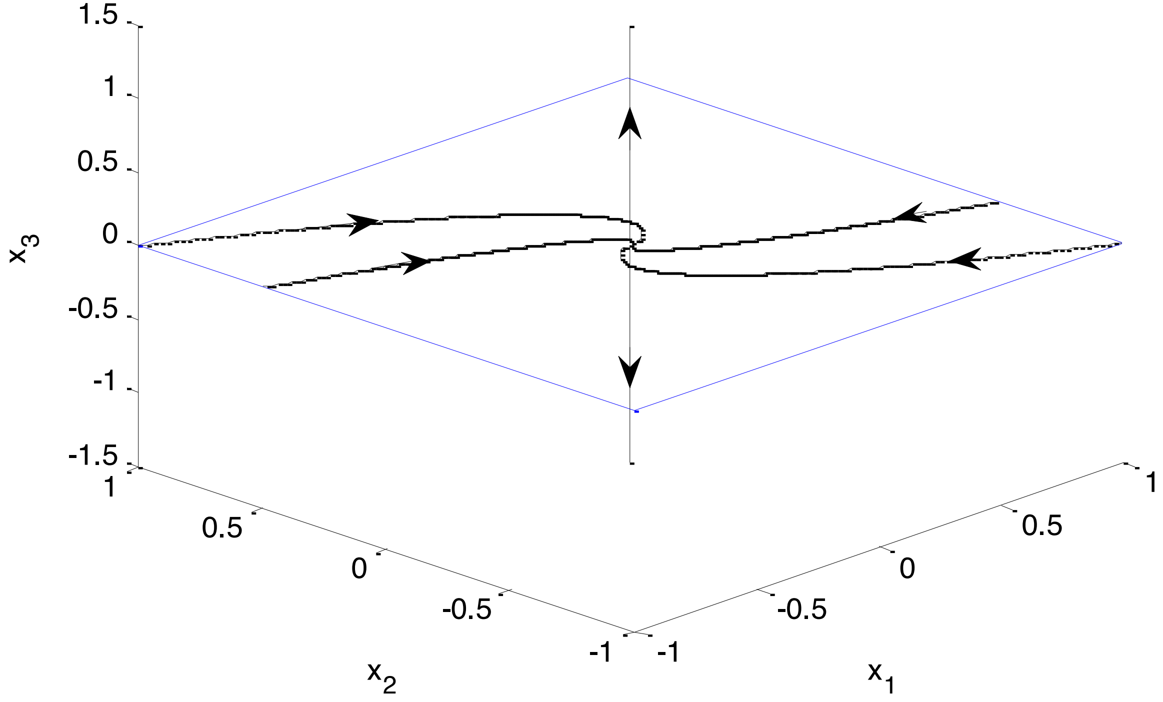
Assume $\lambda < 0$ and $\gamma > 0$.

We have a repeated eigenvalue λ and an eigenvalue γ.

The upper left block is degenerate and thus there is only one eigenvector for the eigenvalue λ.

$\mathbf{x}_3$ is an unstable subspace and $(\mathbf{x}_1, \mathbf{x}_2)$ spans a stable subspace.

Degenerate stable subspace and unstable subspace



Centre example

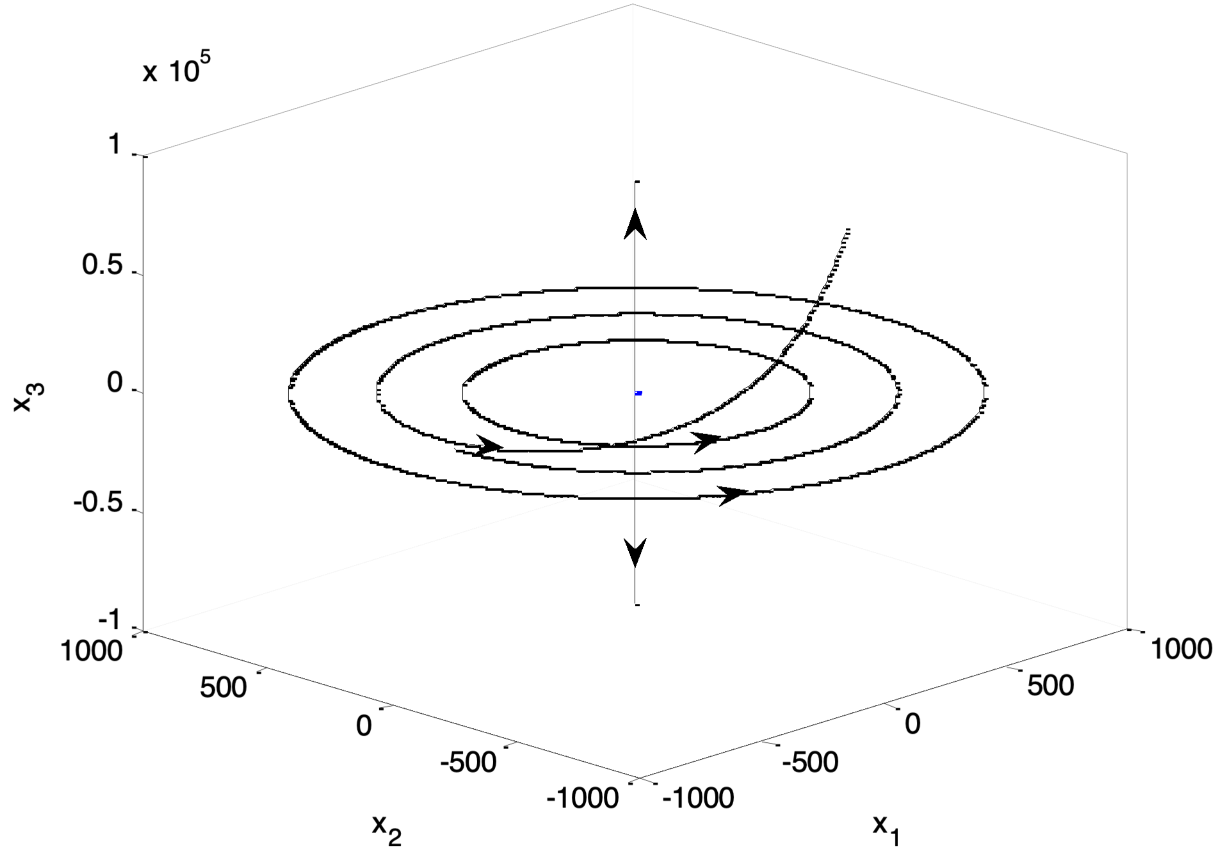
Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x}$$

The eigenvalues $\pm j$ have eigenvectors $[1 \quad \pm j \quad 0]^T$
and eigenvalue at 2 has eigenvector $[0 \;\; 0 \;\; 1]^T$

The $\mathbf{x}_3$-axis is thus an unstable subspace
and the $(\mathbf{x}_1 \; \mathbf{x}_2)$ plane is a centre subspace

Centre subspace and unstable subspace



Observations

- For the stable subspace
$$\lim_{t\to\infty} \mathbf{w}^S(t) = \mathbf{0}$$
All points in the stable subspace end up at the origin.
- For the unstable subspace
$$\lim_{t\to-\infty} \mathbf{w}^U(t) = \mathbf{0}$$
Tracing back points on the unstable subspace ends up at the origin.

Nothing can be said about the centre subspace without careful thought!
Look at the example $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ in the notes.

The Hartman-Grobman Theorem Introduction

Hyperbolic equilibrium points are in some way special (nothing can be said yet about the centre subspace made up from points with eigenvalues with zero real part).

We start by considering the local linearization of a nonlinear system about an equilibrium.

Nonlinear system local theory

- Consider the autonomous nonlinear differential equation
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$
Recall that $\mathbf{f}$ is a vector of functions and $\mathbf{x}$ is a vector of unknowns.
- Linearize about an equilibrium $\mathbf{x}^*$ by writing $\mathbf{x} = \mathbf{x}^* + \mathbf{w}$:
$$\dot{\mathbf{w}} = D\mathbf{f}(\mathbf{x}^*)\mathbf{w} + \mathcal{O}|\mathbf{w}|^2$$
- The phase space of the linearized system can be split into three subspaces by a coordinate transformation based on the eigenvectors of the Jacobian $D\mathbf{f}(\mathbf{x}^*)$:
$$\dot{\mathbf{z}}_s = \mathbf{A}_s\mathbf{z}_s + \mathbf{R}_s(\mathbf{z})$$
$$\dot{\mathbf{z}}_u = \mathbf{A}_u\mathbf{z}_u + \mathbf{R}_u(\mathbf{z})$$
$$\dot{\mathbf{z}}_c = \mathbf{A}_c\mathbf{z}_c + \mathbf{R}_c(\mathbf{z})$$
Each $\mathbf{A}$ is a square real matrix (stable, unstable or centre).
The $\mathbf{R}$'s are vector functions of the transformed errors that make up $\mathcal{O}|\mathbf{w}|^2$.

Hartman-Grobman Theorem

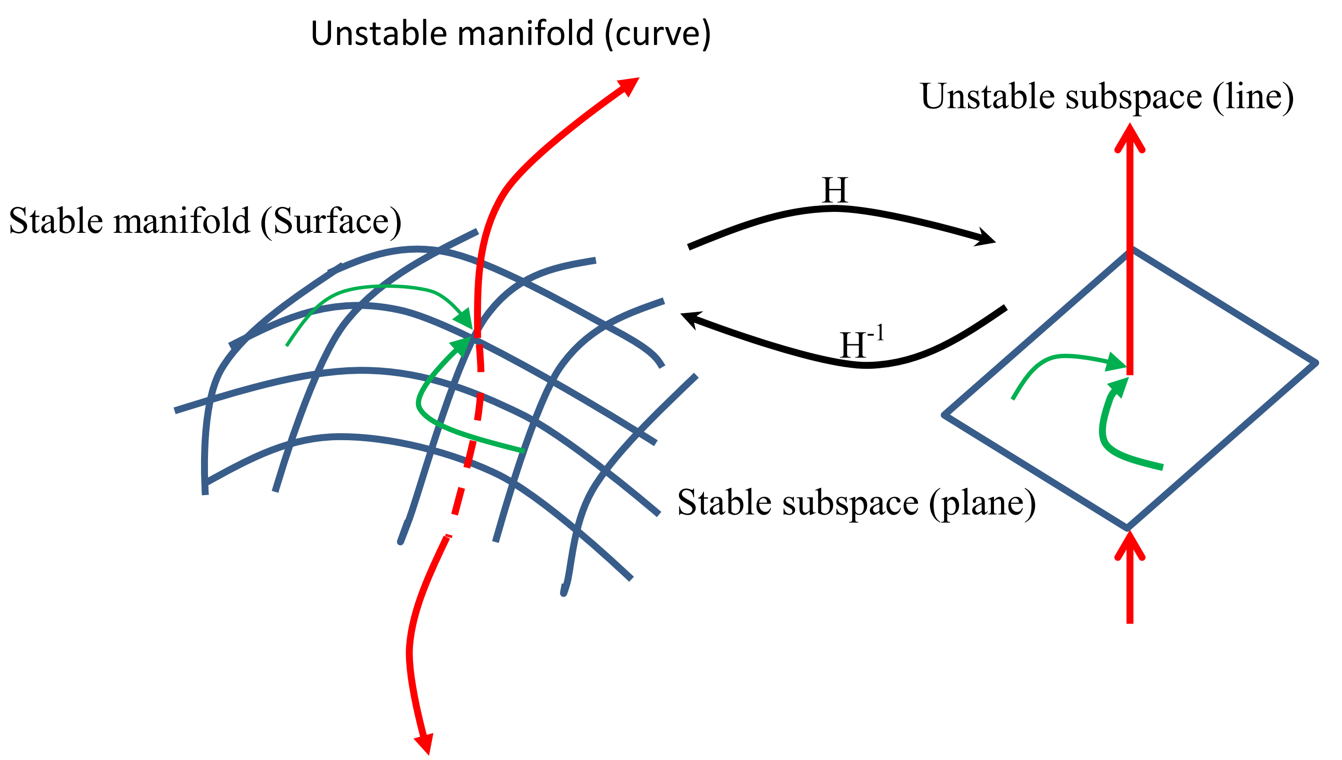
Theorem:

For each hyperbolic equilibrium point, there exists a bi-continuous function H (a mapping that is continuous and whose inverse is also continuous) between an open set containing the equilibrium point and an open set containing the origin of the linearized model so that trajectories are mapped exactly and the parameterisation of time is preserved.

Hartman-Grobman Theorem

- Near the origin, the stable linear subspace is mapped to a stable manifold in a region surrounding the equilibrium point.
- Near the origin, the unstable linear subspace is mapped to an unstable manifold in a region surrounding the equilibrium point.
- Nothing is said about non-hyperbolic equilibria!

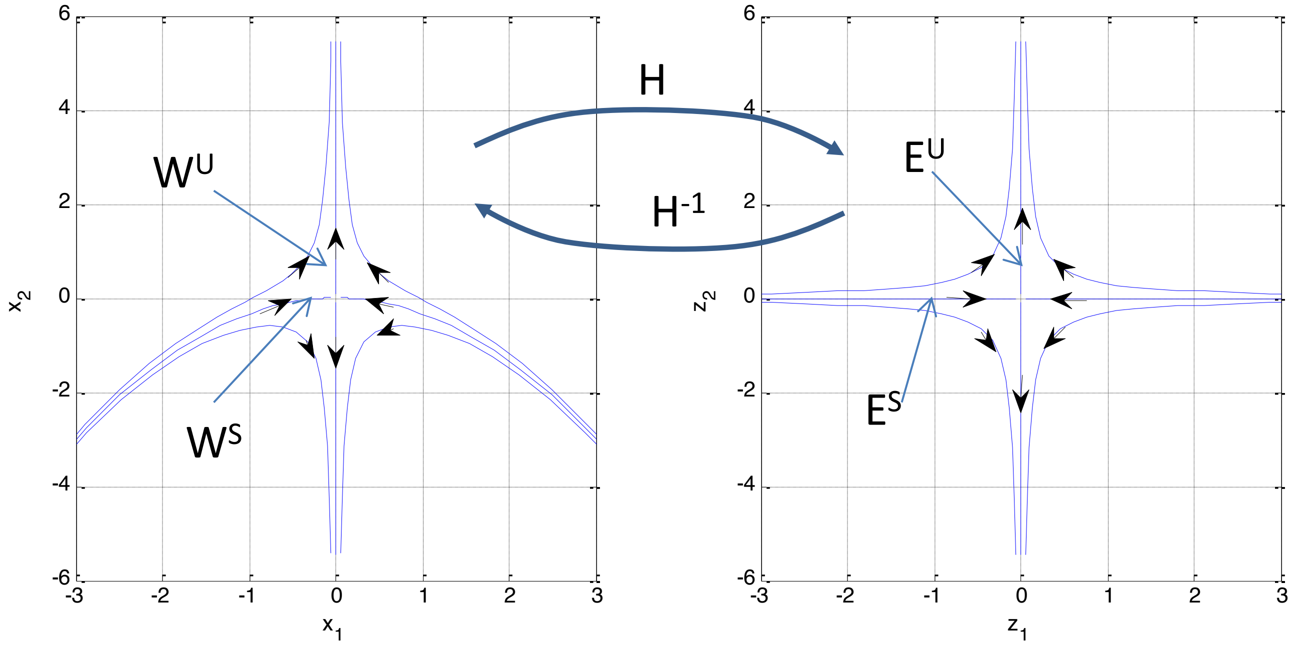
Illustration



Example of Hartman-Grobman Theorem

Consider the non-linear autonomous system

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = x_2 + {x_1}^2$$



Non-Hyperbolic Equilibria

- We cannot say much about non-linear equilibria whose linearized model is a centre.
- Use some tricks…

  e.g. transform a 2D system into polar coordinates.

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

  Let $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ then

$$\dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}, \dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}$$

  Does r grow, shrink or stay constant?

Example of polar transformation

Consider

$$\dot{x} = -y - xy$$
$$\dot{y} = x + x^2$$

Then

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{-x(y + xy) + y(x + x^2)}{r} = 0$$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{x(x + x^2) + y(y + xy)}{r^2} = 1 + x$$

Near the origin (the equilibrium point) we have a nonlinear centre.

Symmetry

- A 2D nonlinear system (x, y) is symmetric with respect to e.g. the x-axis if it is invariant under the transformation
$$(t, y) \rightarrow (-t, -y)$$

- **If** the system is symmetric with respect to either x or y, and **if** the origin is an equilibrium point, **then** centres will map to centres between the non-linear and linear approximation

Example of symmetry

Consider the system

$$\dot{x} = y - y^3 \stackrel{\text{def}}{=} f(x, y)$$
$$\dot{y} = -x - y^2 \stackrel{\text{def}}{=} g(x, y)$$

The equilibrium point at the origin has Jacobian $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and is thus a linear centre.

Now note:

$$f(x, -y) = -f(x, y)$$
$$g(x, -y) = g(x, y)$$

Therefore

$$\frac{dx}{d(-t)} = f(x, -y)$$
$$\frac{d(-y)}{d(-t)} \left(= \frac{dy}{dt} \right) = g(x, -y)$$

… so the nonlinear system has a centre

Conservative Systems

- If there exists a non-constant function $V(\mathbf{x})$ such that $dV/dt = 0$ along solutions of the nonlinear differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, the equations are called **conservative**. $V(\mathbf{x})$ does not change along the solution trajectories.

- If $\mathbf{x} = \mathbf{x}^*$ is an isolated equilibrium point and there is a $V(\mathbf{x})$ that has a local min or max at $\mathbf{x}^*$, then there is a region around that point that contains a **closed orbit**.

Conservative example

Consider the system

$$\dot{x} = v$$
$$\dot{v} = f(x)$$

Integrate

$$-f(x)\dot{x} + v\dot{v} = 0$$

to get

$$-\int_{x_0}^{x} f(s)ds + \frac{v^2}{2} = \text{constant}$$

Potential energy + kinetic energy = constant (see the examples sheet)

This is called a Newtonian system

Further example

Consider a system of the form

$$\dot{x} = f(x)g_1(y)$$
$$\dot{y} = f(y)g_2(x)$$

Then

$$\frac{g_2(x)}{f(x)}\dot{x} - \frac{g_1(y)}{f(y)}\dot{y} = 0$$

Which can be integrated to yield $V(x, y)$

Further example

Consider the system

$$\dot{x} = x - xy = x(1-y)$$
$$\dot{y} = -y + xy = y(x-1)$$

The equilibrium points are $(0,0)$ and $(1,1)$.

Jacobian at $(0,0)$ is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \implies$ nonlinear saddle point (by Hartman-Grobman Theorem).

Jacobian at $(1,1)$ is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; is $(1,1)$ a non-linear centre?

Here

$$\frac{x-1}{x}\dot{x} - \frac{1-y}{y}\dot{y} = 0$$
$$\implies x - \ln x + y - \ln y = V(x,y) = \text{constant}$$

At $(1,1)$, $\partial V/\partial x = 0$ and $\partial V/\partial y = 0$ and $\det(\partial^2 V/\partial x \partial y) > 0$ so $V(1,1)$ is a minimum point

Therefore (1,1) is a nonlinear centre.