Adaptive Model Predictive Control: Robustness and Parameter Estimation

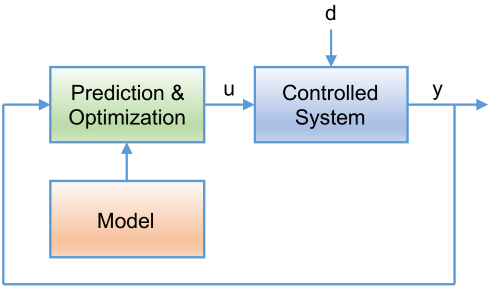
Mark Cannon

Joint work with:

Matthias Lorenzen	Stuttgart University
Xiaonan Lu	Oxford University
Sebastian East	Oxford/NNAISENSE

Motivation

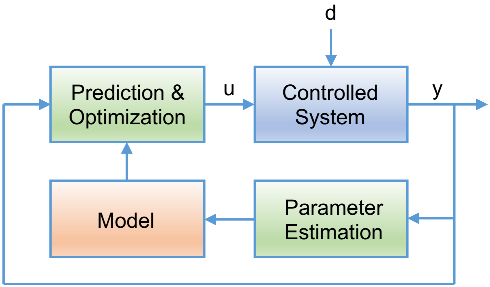
Robust MPC paradigm:



- Uncertain model & disturbances affect performance
- Large effort (time & money) spent on model identification offline

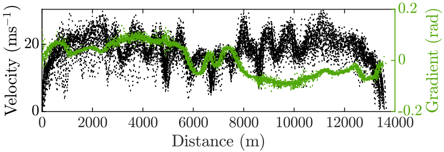
Motivation

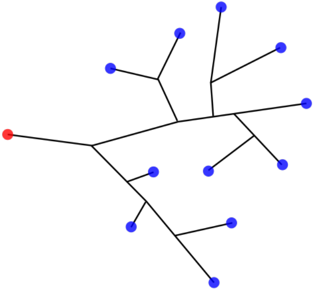
Adaptive MPC paradigm:



- Identify (or learn) model (or cost or constraints) online
- Require: robust constraint satisfaction
  closed loop stability & performance guarantees
  parameter convergence

Applications





- Uncertain parameters, uncertain demand
- Networks of interacting locally controlled systems

Overview

An idea with a long history: e.g. self-tuning control, DMC, GPC ...

[Clarke, Tuffs, Mohtadi, 1987]

Revisited with new tools:

- Set membership estimation

  [Bai, Cho, Tempo, 1998]

- Robust tube MPC

  [Langson, Chryssochoos, Rakovic, Mayne, 2004]

- Dual adaptive/predictive control

  [Lee & Lee, 2009]

Overview

Recent work on MPC with model adaptation

- Online learning & identification:
  - Persistency of Excitation constraints
    [Marafioti, Bitmead, Hovd, 2014]
  - RLS parameter estimation with covariance matrix in cost
    [Heirung, Ydstie, Foss, 2017]
  - Gaussian process regression, particle filtering
    [Klenske, Zeilinger, Scholkopf, Hennig, 2016]
    [Bayard & Schumitzky, 2010]

- Robust constraint satisfaction and performance:
  - Constraints based on prior uncertainty set, online update of cost only
    [Aswani, Gonzalez, Sastry, Tomlin, 2013]
  - Set-based identification, stable FIR plant model
    [Tanaskovic, Fagiano, Smith, Morari, 2014]

Overview

This talk:

1. Set membership parameter estimation
2. Polytopic tube robust MPC
3. Convex constraints for persistent excitation
4. Time varying model parameters
5. Differentiable MPC

Parameter set estimate

Plant model with unknown parameter vector $\theta^\star$ and disturbance w:

$$x_{k+1} = A(\theta^\star)x_k + B(\theta^\star)u_k + w_k$$

Assumption 1: model is affine in unknown parameters

$$x_{k+1} = D_k\theta^\star + d_k + w_k \qquad \begin{cases} D_k = D(x_k, u_k) \\ d_k = A_0 x_k + B_0 u_k \end{cases}$$

Assumption 2: stochastic disturbance $w_k \in \mathcal{W}$

$\mathcal{W} \ni 0$ is compact and convex

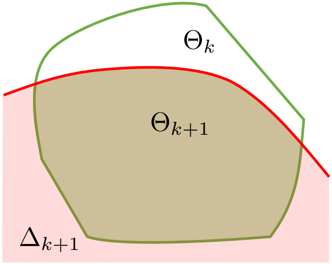
Unfalsified set: If x_k, x_{k-1}, u_{k-1} are known, then $\theta^\star \in \Delta_k$

$$\Delta_k = \{\theta : x_k = D_{k-1}\theta + d_{k-1} + w,\ w \in \mathcal{W}\}$$

Minimal parameter set estimate

Minimal parameter set update:

$$\Theta_{k+1} = \Theta_k \cap \Delta_{k+1}$$



Assumption 3: $\mathcal{W}$ is a 'tight' bound: for all $w^0 \in \partial\mathcal{W}$ and $\epsilon > 0$

$$\Pr\big\{\|w_k - w^0\| < \epsilon\big\} \geq p_w(\epsilon)$$

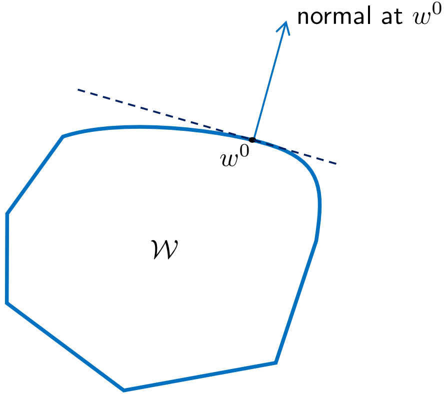
where $p_w(\epsilon) > 0 \ \forall \epsilon > 0$

Assumption 4: persistent excitation: $\exists\ \alpha$, $\beta > 0$, N such that

$$\|D_k\| \leq \alpha \quad \text{and} \quad \sum_{j=k}^{k+N-1} D_j^\top D_j \succeq \beta I \text{ for all } k$$

Minimal parameter set estimate

Unfalsified set:
$$\begin{aligned}\Delta_{k+1} &= \{\theta : x_{k+1} - D_k\theta - d_k \in \mathcal{W}\} \\ &= \{\theta : D_k(\theta^* - \theta) + w_k \in \mathcal{W}\}\end{aligned}$$



Minimal parameter set estimate

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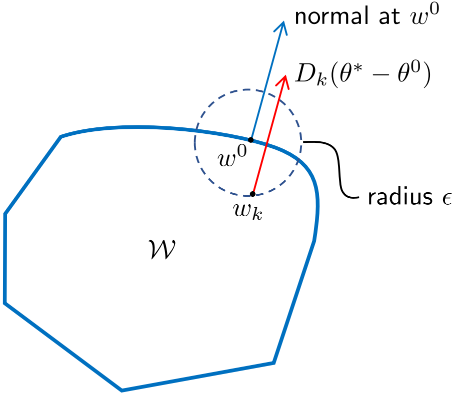
$$\begin{aligned}\Delta_{k+1} &= \{\theta : x_{k+1} - D_k\theta - d_k \in \mathcal{W}\}\\ &= \{\theta : D_k(\theta^* - \theta) + w_k \in \mathcal{W}\}\end{aligned}$$

For any given $\theta^0 \in \Theta_k$:

- pick $w^0 \in \partial\mathcal{W}$ so that $D_k(\theta^* - \theta^0)$ is normal to $\partial\mathcal{W}$ at w^0
- let $\epsilon = \|w_k - w^0\|$

then $\theta^0 \notin \Delta_{k+1}$ if $\epsilon < \|D_k(\theta^* - \theta^0)\|$



Minimal parameter set estimate

If Assumptions 1-4 hold, then $\Theta_k \to \{\theta^*\}$ as $k \to \infty$ w.p. 1

This follows from:

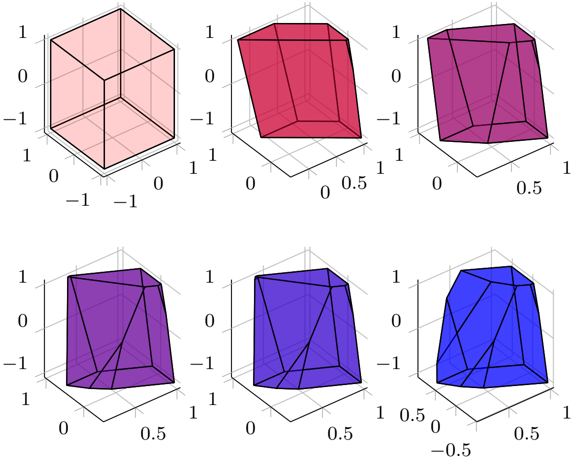
A. For any $\theta^0 \in \Theta_k$, if $\|\theta^* - \theta^0\| \geq \epsilon$, then
$$\Pr\{\theta^0 \notin \Delta_j\} \geq p_w\big(\epsilon\sqrt{\beta/N}\big)$$
for all k, all $\epsilon > 0$, and some $j \in \{k+1, \ldots, k+N\}$

B. For any $\theta^0 \in \Theta_0$ such that $\|\theta^0 - \theta^*\| \geq \epsilon$,
$$\Pr\{\theta^0 \in \Theta_k\} \leq \Big[1 - p_w\big(\epsilon\sqrt{\beta/N}\big)\Big]^{\lfloor k/N \rfloor}$$
for all k and all $\epsilon > 0$, so
$$\sum_{k=0}^{\infty} \Pr\{\theta^0 \in \Theta_k\} = 0 \ \overset{\text{Borel-Cantelli}}{\underset{\text{Lemma}}{\Longrightarrow}} \ \Pr\Big\{\theta^0 \in \bigcap_{k=0}^{\infty} \Theta_k\Big\} = 0$$

Minimal parameter set estimate

The complexity of Θ_k is unbounded in general

e.g. Minimal parameter set Θ_k for $k = 1, \ldots, 6$ with polytopic $\mathcal{W}$ and Θ_0



Fixed complexity polytopic parameter set estimate

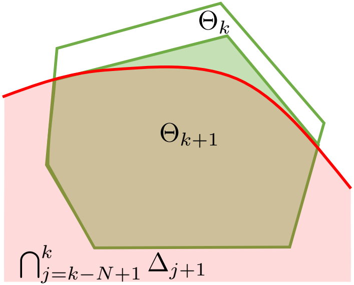
- Define $\Theta_k = \{\theta : H_\Theta \theta \leq h_k\}$ for a fixed matrix H_Θ

- Update Θ_{k+1} by solving, for each row i:

$$[h_{k+1}]_i = \max_{\substack{w_0 \in \mathcal{W}, \dots, w_{N-1} \in \mathcal{W} \\ \theta \in \Theta_k}} [H_\Theta]_i \theta$$

subject to

$$x_{k-N+2} = D_{k-N+1}\theta + d_{k-N+1} + w_0$$
$$\vdots$$
$$x_{k+1} = D_k \theta + d_k + w_{N-1}$$



- Then $\Theta_{k+1} \subseteq \Theta_k \subseteq \cdots \subseteq \Theta_0$,
and Θ_{k+1} is the minimum volume set such that

$$\Theta_{k+1} \supseteq \Theta_k \cap \bigcap_{j=k-N+1}^{k} \Delta_{j+1}$$

Fixed complexity polytopic parameter set estimate

If Assumptions 1-4 hold, then $\Theta_k \to \{\theta^*\}$ as $k \to \infty$ w.p. 1

This follows from:

- **A** If $[h_k]_i - [H_\Theta]_i\theta^* \geq \epsilon$, then

$$\Pr\left\{\{\theta : [H_\Theta]_i\theta = [h_k]_i\} \cap \bigcap_{j=k-N+1}^{k} \Delta_{j+1} = \emptyset\right\} \geq \left[p_w\left(\frac{\epsilon\beta}{\alpha N}\right)\right]^N$$

  for all i, k, and all $\epsilon > 0$

- **B** For any θ^0 such that $[H_\Theta]_i(\theta^0 - \theta^*) \geq \epsilon$ for some row i,

$$\Pr\{\theta^0 \in \Theta_k\} \leq \left\{1 - \left[p_w\left(\frac{\epsilon\beta}{N\alpha}\right)\right]^N\right\}^{\lfloor k/N \rfloor}$$

  for all k and all $\epsilon > 0$, so

$$\sum_{k=0}^{\infty} \Pr\{\theta^0 \in \Theta_k\} = 0 \quad \underset{\text{Lemma}}{\overset{\text{Borel-Cantelli}}{\Longrightarrow}} \quad \Pr\{\theta^0 \in \bigcap_{k=0}^{\infty} \Theta_k\} = 0$$

Example: fixed complexity parameter set estimate

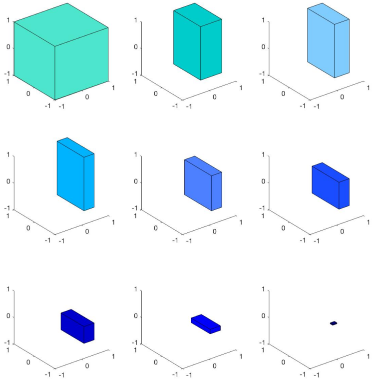


Figure: Parameter set Θ_k at time steps $k \in \{0, 1, 2; 10, 25, 50; 100, 500, 5000\}$

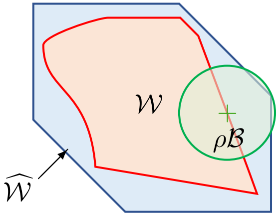
Θ set	Volume (%)	Cost*
Θ_0	100	62.22
Θ_1	26.1	61.13
Θ_2	18.3	61.03
Θ_{10}	12.7	60.96
Θ_{25}	8.3	60.93
Θ_{50}	6.3	60.77
Θ_{100}	3.4	59.45
Θ_{500}	0.7	57.94
Θ_{5000}	0.0089	53.95
$\theta^\star$	-	52.70

Table: Volume of Θ_k as $\Theta_k/\Theta_0 \times 100\%$; Cost* with same initial x_0 and constraints

Inexact disturbance bounds

What if $\mathcal{W}$ is not exactly known?

Suppose $w_k \in \widehat{\mathcal{W}}$ for all k, for known $\widehat{\mathcal{W}}$



Assumption 5: $\widehat{\mathcal{W}}$ is compact and convex, and $\mathcal{W} \subseteq \widehat{\mathcal{W}} \subseteq \mathcal{W} \oplus \rho\mathcal{B}$ for some $\rho \geq 0$, and $\mathcal{B} = \{x : \|x\| \leq 1\}$

Replace $\mathcal{W}$ with $\widehat{\mathcal{W}}$ in the fixed complexity polytopic parameter set update

then $\theta^\star \in \widehat{\Delta}_{k+1} = \{\theta : x_{k+1} = D_k\theta + d_k + w,\ w \in \widehat{\mathcal{W}}\}$, and

if Assumptions 1-5 hold, then $\Theta_k \to \{\theta^*\} \oplus \rho\sqrt{N/\beta}\,\mathcal{B}$ as $k \to \infty$ w.p. 1

Inexact disturbance bounds

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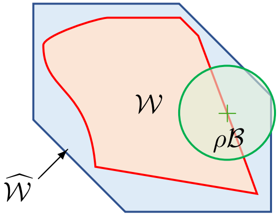

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Noisy measurements

Let $y_k = x_k + s_k$ be an estimate of x_k

Assumption 6: i.i.d. noise $s_k \in \mathcal{S}$ for all k
where $\mathcal{S} \ni 0$ is a compact, convex polytope

Assumption 7: the noise bound is tight, i.e. for all $s^0 \in \partial\mathcal{S}$ and $\epsilon > 0$

$$\Pr\big\{\|s_k - s^0\| < \epsilon\big\} \geq p_s(\epsilon)$$

where $p_s(\epsilon) > 0$ for all $\epsilon > 0$

Then $\mathcal{S} = \mathrm{co}\{s^{(j)}, \ldots, s^{(h)}\}$ implies $\theta^\star \in \mathrm{co}\{\widehat{\Delta}^{(1)}_{k+1}, \ldots, \widehat{\Delta}^{(h)}_{k+1}\}$, where

$$\widehat{\Delta}^{(j)}_{k+1} = \Big\{\theta : y_{k+1} - D(y_k - s^{(j)}, u_k)\theta - d(y_k - s^{(j)}_k, u_k) \in \widehat{\mathcal{W}} \oplus \mathcal{S}\Big\}$$

If Assumptions 1-7 hold, then $\Theta_k \to \{\theta^*\} \oplus \rho\sqrt{N/\beta}\,\mathcal{B}$ as $k \to \infty$ w.p. 1

Parameter point estimate

Define a point estimate $\hat{\theta}_k$ of $\theta^\star$

$\hat{\theta}_k$:	defines a nominal predicted performance index
Θ_k:	enforces constraints robustly

Given a parameter estimate $\hat{\theta}_k$:

- Least mean squares (LMS) filter estimate update is

$$\tilde{\theta}_{k+1} = \hat{\theta}_k + \mu D^\top(x_k, u_k)(x_{k+1} - \hat{x}_{1|k})$$
$$\hat{\theta}_{k+1} = \Pi_{\Theta_{k+1}}(\tilde{\theta}_{k+1})$$

  where

  - $\hat{x}_{1|k} = D(x_k, u_k)(\hat{\theta}_k)$
  - $\mu > 0$ satisfies $1/\mu > \sup_{(x,u)\in\mathcal{Z}} \|D(x,u)\|^2$
  - $\Pi_\Theta(\hat{\theta}) = \arg\min_{\theta\in\Theta} \|\theta - \hat{\theta}\|$ projects onto Θ

- For $\mu = 0$ the update is $\hat{\theta}_{k+1} = \Pi_{\Theta_{k+1}}(\hat{\theta}_k)$

Parameter point estimate

The LMS filter ($\mu > 0$) ensures the l^2 gain bound:

If $\sup_{k \in \mathbb{N}} \|x_k\| < \infty$ and $\sup_{k \in \mathbb{N}} \|u_k\| < \infty$, then $\hat{\theta}_k \in \Theta_k$ for all k and

$$\sup_{T \in \mathbb{N}, w_k \in \mathcal{W}, \hat{\theta}_0 \in \Theta_0} \frac{\sum_{k=0}^{T} \|\tilde{x}_{1|k}\|^2}{\frac{1}{\mu}\|\hat{\theta}_0 - \theta^\star\|^2 + \sum_{k=0}^{T} \|w_k\|^2} \leq 1$$

where $\tilde{x}_{1|k} = A(\theta^\star)x_k + B(\theta^\star)u_k - \hat{x}_{1|k}$ is the 1-step prediction error

Control Problem

Consider robust regulation of the system

$$x_{k+1} = A(\theta)x_k + B(\theta)u_k + w_k$$

with $\theta \in \Theta_k$, $w_k \in \mathcal{W}$, subject to the state and control constraints

$$Fx_k + Gu_k \leq \mathbf{1} = \begin{bmatrix}1 & \cdots & 1\end{bmatrix}^\top$$

Assumption (Robust stabilizability):
There exists a set $\mathcal{X} = \{x : Vx \leq \mathbf{1}\}$ and feedback gain K such that $\mathcal{X}$ is λ-contractive for some $\lambda \in [0, 1)$, i.e.

$$V\Phi(\theta)x \leq \lambda\mathbf{1}, \quad \text{for all } x \in \mathcal{X}, \theta \in \Theta_0.$$

where $\Phi(\theta) = A(\theta) + B(\theta)K$.

Control Problem

- Future state sequence predicted at time k: $x_{1|k}, x_{2|k}, \ldots$
- Control sequence predicted at time k: $u_{0|k}, u_{1|k}, \ldots$:

$$u_{i|k} = \begin{cases} Kx_{i|k} + v_{i|k} & i = 0, 1, \ldots, N-1 \\ Kx_{i|k} & i = N, N+1, \ldots \end{cases}$$

where $\mathbf{v} = (v_{0|k}, \ldots, v_{N|k})$ is a decision variable

Nominal predicted performance index

$$J_N(x_k, \hat{\theta}_k, \mathbf{v}_k) = \sum_{i=0}^{N-1} \Big(\|\hat{x}_{i|k}\|_Q^2 + \|\hat{u}_{i|k}\|_R^2 \Big) + \|\hat{x}_{N|k}\|_P^2$$

where $\hat{x}_{0|k} = x_k$

$$\hat{u}_{i|k} = K\hat{x}_{i|k} + v_{i|k}$$

$$\hat{x}_{i+1|k} = A(\hat{\theta}_k)\hat{x}_{i|k} + B(\hat{\theta}_k)\hat{u}_{i|k}, \qquad \hat{\theta}_k = \text{nominal parameter estimate}$$

and $P \succeq \Phi^\top(\theta) P \Phi(\theta) + Q + K^\top R K$ for all $\theta \in \Theta_k$

Tube MPC

A sequence of sets (a "tube") is constructed to bound the predicted state $x_{i|k}$, with ith cross section, $\mathcal{X}_{i|k}$:

$$\mathcal{X}_{i|k} = \{x : Vx \leq \alpha_{i|k}\}$$

where V is determined offline and $\alpha_{i|k}$ are online decision variables

(A) For robust satisfaction of $x_{i|k} \in \mathcal{X}_{i|k}$, we require

$$V\Phi(\theta)x + VB(\theta)v_{i|k} + \bar{w} \leq \alpha_{i+1|k} \quad \text{for all } x \in \mathcal{X}_{i|k},\ \theta \in \Theta_k$$

where $[\bar{w}]_i = \max_{w \in \mathcal{W}} [V]_i w$

(B) For robust satisfaction of $Fx_{i|k} + Gu_{i|k} \leq \mathbf{1}$, we require

$$(F + GK)x + Gv_{i|k} \leq \mathbf{1} \quad \text{for all } x \in \mathcal{X}_{i|k}$$

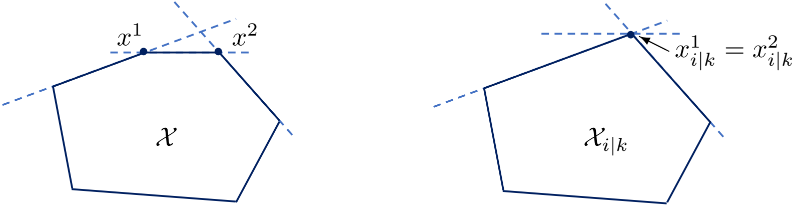
Condition (A) is bilinear in x and θ, but can be expressed in terms of linear inequalities using a vertex representation of either $\mathcal{X}_{i|k}$ or Θ_k

Tube MPC

We generate the vertex representation:

$$\mathcal{X}_{i|k} = \mathrm{co}\{x^1_{i|k}, \dots x^m_{i|k}\}$$

using the property that $\{x : [V]_r x \le [\alpha_{i|k}]_r\}$ is a supporting hyperplane of $\mathcal{X}_{i|k}$ for each r:



Hence each vertex $x^j_{i|k}$ is given by the intersection of hyperplanes corresponding to a fixed set of rows of V, and

$$x^j_{i|k} = U^j \alpha_{i|k}$$

for some U^j, determined offline from the vertices of $\mathcal{X} = \{x : Vx \le \mathbf{1}\}$

Tube MPC

Using the hyperplane and vertex descriptions of $\mathcal{X}_{i|k}$, the robust tube constraints become

- (A) $V\Phi(\theta)U^j\alpha_{i|k} + VB(\theta)v_{i|k} + \bar{w} \leq \alpha_{i+1|k}$ for all $\theta \in \Theta_k$, $j = 1, \ldots, m$
- (B) $(F + GK)U^j\alpha_{i|k} + Gv_{i|k} \leq \mathbf{1}$, $j = 1, \ldots, m$

Now condition (B) is linear and (A) can be equivalently written as linear constraints using

Polyhedral set inclusion lemma

Let $\mathcal{P}_i = \{x : F_i x \leq f_i\} \subset \mathbb{R}^n$ for $i = 1, 2$. Then $\mathcal{P}_1 \subseteq \mathcal{P}_2$ iff

$$\exists \Lambda \geq 0 \text{ such that } \Lambda F_1 = F_2 \text{ and } \Lambda f_1 \leq f_2$$

Robust MPC online optimization problem

Summary of constraints in the online MPC optimization at time k:

$$
\begin{aligned}
& V x_k \leq \alpha_{0|k} \\[1em]
& \Lambda^j_{i|k} H_\Theta = V D(U^j \alpha_{i|k}, K U^j \alpha_{i|k} + v_{i|k}) \\
& \Lambda^j_{i|k} h_k \leq \alpha_{i+1|k} - V d(u^j \alpha_{i|k}, K U^j \alpha_{i|k} + v_{i|k}) - \bar{w} \\
& \Lambda^j_{i|k} \geq 0 \\[1em]
& (F + GK) U^j \alpha_{i|k} + G v_{i|k} \leq \mathbf{1} \\[1em]
& \Lambda^j_{N|k} H_\Theta = V D(U^j \alpha_{N|k}, K U^j \alpha_{N|k}) \\
& \Lambda^j_{N|k} h_k \leq \alpha_{N|k} - V d(u^j \alpha_{N|k}, K U^j \alpha_{N|k}) - \bar{w} \\
& \Lambda^j_{N|k} \geq 0 \\[1em]
& (F + GK) U^j \alpha_{N|k} \leq \mathbf{1}
\end{aligned}
$$

for $i = 0, \ldots, N-1$, $j = 1, \ldots, m$

Let $\mathcal{F}(x_k, \Theta_k)$ be the feasible set for the decision variables $\mathbf{v}_k, \boldsymbol{\alpha}_k, \boldsymbol{\Lambda}_k$

Robust adaptive MPC algorithm

Offline: Choose Θ_0, $\mathcal{X}$, feedback gain K, and compute P

Online, at each time $k = 1, 2, \ldots$:

1. Given x_k, update the set (Θ_k) and point ($\hat{\theta}_k$) parameter estimates
2. Compute the solution $(\mathbf{v}_k^*, \boldsymbol{\alpha}_k^*, \boldsymbol{\Lambda}_k^*)$ of the QP:

$$\min_{\mathbf{v}_k, \boldsymbol{\alpha}_k, \boldsymbol{\Lambda}_k} J(x_k, \hat{\theta}_k, \mathbf{v}_k)$$
$$\text{subject to } (\mathbf{v}_k, \boldsymbol{\alpha}_k, \boldsymbol{\Lambda}_k) \in \mathcal{F}(x_k, \Theta_k)$$

3. Apply the control law $u_k^* = Kx_k + v_{0|k}^*$

Robust adaptive MPC algorithm

The MPC algorithm has the following closed loop properties:

If $\theta^\star \in \Theta_0$ and $\mathcal{F}(x_0, \Theta_0) \neq \emptyset$, then for all $k > 0$:

1. $\theta^\star \in \Theta_k$
2. $\mathcal{F}(x_k, \Theta_k) \neq \emptyset$
3. $F x_k + G u_k \leq \mathbf{1}$

If $\mu > 0$, then

5. the closed loop system is finite-gain l^2-stable, i.e.

$$\sum_{k=0}^{T} \|x_k\|^2 \leq c_0 \|x_0\|^2 + c_1 \|\hat{\theta}_0 - \theta^\star\|^2 + c_2 \sum_{k=0}^{T} \|w_k\|^2$$

for some constants $c_0, c_1, c_2 > 0$, for all T

Robust adaptive MPC algorithm

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If $\theta^\star \in \Theta_0$ and $\mathcal{F}(x_0, \Theta_0) \neq \emptyset$, then for all $k > 0$:

1. $\theta^\star \in \Theta_k$
2. $\mathcal{F}(x_k, \Theta_k) \neq \emptyset$
3. $Fx_k + Gu_k \leq \mathbf{1}$

If $\mu = 0$, then

5. the closed loop system is input-to-state stable, i.e.

$$\|x_T\| \leq \eta(\|x_k\|, T-k) + \psi(\max_{i \in \{k,\ldots,T-1\}} \|w_j\|) + \zeta(\|\hat{\theta}_k - \theta^*\|)$$

for some $\mathcal{KL}$-function η, some $\mathcal{K}$-functions ψ, ζ and all k, T.

Regulation example

2nd order linear system with

$$(A(\theta), B(\theta)) = (A_0, B_0) + \sum_{i=1}^{3} (A_i, B_i)\theta_i$$

$$A_0 = \begin{bmatrix} 0.5 & 0.2 \\ -0.1 & 0.6 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0.042 & 0 \\ 0.072 & 0.03 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.015 & 0.019 \\ 0.009 & 0.035 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 \\ -0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0.0397 \\ 0.059 \end{bmatrix}$$

- ▷ true parameter $\theta^\star = [0.8 \ \ 0.2 \ \ -0.5]^\top$, initial set $\Theta_0 = \{\theta : \|\theta\|_\infty \leq 1\}$.
- ▷ disturbance uniformly distributed on $\mathcal{W} = \{w \in \mathbb{R}^2 : \ \|w\|_\infty \leq 0.1\}$, w_k
- ▷ state and input constraints: $[x]_2 \geq -0.3$ and $u_k \leq 1$.

Regulation example: constraint satisfaction

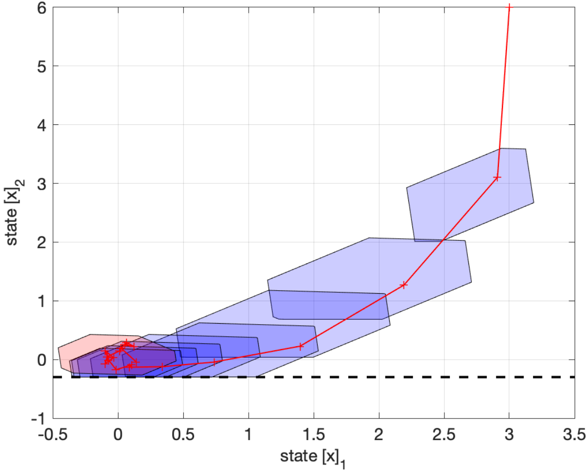

Figure: Realized closed-loop trajectory from initial condition $x_0 = [3\ 6]^\top$ (red line), predicted state tube at time $k = 0$ (tube cross-sections: blue, terminal set: pink)

Regulation example: constraint satisfaction

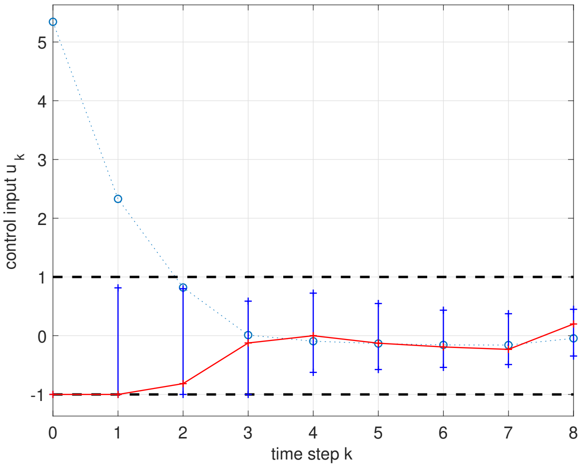

Figure: Realized closed-loop trajectory from initial condition $x_0 = [3\ 6]^\top$ (red line), predicted control tube at time $k = 0$ (tube cross-sections: blue)

Persistent excitation

PE condition evaluated over a future horizon is nonconvex in $u_{i|k}, x_{i|k}$:

$$\text{(PE):} \qquad \sum_{i=0}^{N-1} D^\top(x_{i|k}, u_{i|k})\, D(x_{i|k}, u_{i|k}) \succeq \beta I$$

Linearise:

- let $u_{i|k} = \bar{u}_{i|k} + \check{u}_{i|k}$ and $x_{i|k} = \bar{x}_{i|k} + \check{x}_{i|k}$, where $\bar{x}_{0|k} = x_k$ and

$$\bar{u}_{i|k} = K\bar{x}_{i|k} + v^*_{i+1|k-1}$$
$$\bar{x}_{i+1|k} = A(\hat{\theta}_k)\bar{x}_{i|k} + B(\hat{\theta}_k)\bar{u}_{i|k}$$

- then $D_{i|k} = \bar{D}_{i|k} + \check{D}_{i|k}$, where $\bar{D}_{i|k} = D(\bar{x}_{i|k}, \bar{u}_{i|k})$, $\check{D}_{i|k} = D(\check{x}_{i|k}, \check{u}_{i|k})$

$$\begin{aligned} D_{i|k}^\top D_{i|k} &= \check{D}_{i|k}^\top \bar{D}_{i|k} + \bar{D}_{i|k}^\top \check{D}_{i|k} + \bar{D}_{i|k}^\top \bar{D}_{i|k} + \check{D}_{i|k}^\top \check{D}_{i|k} \\ &\succeq \check{D}_{i|k}^\top \bar{D}_{i|k} + \bar{D}_{i|k}^\top \check{D}_{i|k} + \bar{D}_{i|k}^\top \bar{D}_{i|k} \end{aligned}$$

- so $\quad \check{D}_{i|k}^\top \bar{D}_{i|k} + \bar{D}_{i|k}^\top \check{D}_{i|k} + \bar{D}_{i|k}^\top \bar{D}_{i|k} \succeq \beta I \quad \Longrightarrow \quad D_{i|k}^\top D_{i|k} \succeq \beta I$

Persistent excitation

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$$\begin{aligned} D^\top_{i|k} D_{i|k} &= \check{D}^\top_{i|k}\bar{D}_{i|k} + \bar{D}^\top_{i|k}\check{D}_{i|k} + \bar{D}^\top_{i|k}\bar{D}_{i|k} + \check{D}^\top_{i|k}\check{D}_{i|k} \\ &\succeq \check{D}^\top_{i|k}\bar{D}_{i|k} + \bar{D}^\top_{i|k}\check{D}_{i|k} + \bar{D}^\top_{i|k}\bar{D}_{i|k} \end{aligned}$$

- so $\quad \check{D}^\top_{i|k}\bar{D}_{i|k} + \bar{D}^\top_{i|k}\check{D}_{i|k} + \bar{D}^\top_{i|k}\bar{D}_{i|k} \succeq \beta I \quad \Longrightarrow \quad D^\top_{i|k}D_{i|k} \succeq \beta I$

Persistent excitation

A sufficient condition for $\displaystyle\sum_{i=0}^{N-1} D_{i|k}^\top D_{i|k} \succeq \beta I$

is the LMI in $\check{x}_{i|k}, \check{u}_{i|k}, \beta$:

$$\text{(PE-LMI):} \qquad \sum_{i=0}^{N-1} \big(\check{D}_{i|k}^\top \bar{D}_{i|k} + \bar{D}_{i|k}^\top \check{D}_{i|k} + \bar{D}_{i|k}^\top \bar{D}_{i|k} \big) \succeq \beta I$$

This can be expressed in terms of

$$\begin{aligned} \check{x}_{i|k} &\in \mathcal{X}_{i|k} - \{\bar{x}_{i|k}\} \\ \check{u}_{i|k} &\in K(\mathcal{X}_{i|k} - \{\bar{x}_{i|k}\}) + \{v_{i|k}\} - \{v^*_{i+1|k-1}\} \end{aligned}$$

using

$$\check{D}_{i|k} \in \mathrm{co}\Big\{ D\big(U^j \alpha_{i|k} - \bar{x}_{i|k}, K(U^j \alpha_{i|k} - \bar{x}_{i|k}) + v_{i|k} - v^*_{i+1|k-1}\big) \Big\}$$

Hence (PE-LMI) is equivalent to an LMI in optimization variables $\mathbf{v}_k, \boldsymbol{\alpha}_k, \beta$

Robust adaptive MPC algorithm with PE condition

Offline: Choose Θ_0, $\mathcal{X}$, γ, feedback gain K, and compute P

Online, at each time $k = 1, 2, \ldots$:

1. Given x_k, update set (Θ_k) and point ($\hat{\theta}_k$) parameter estimates, and compute $\bar{x}_{i|k}, \bar{u}_{i|k}$, $i = 0, \ldots, N-1$
2. Compute the solution $(\mathbf{v}_k^*, \boldsymbol{\alpha}_k^*, \boldsymbol{\Lambda}_k^*)$ of the semidefinite program

$$\min_{\mathbf{v}_k, \boldsymbol{\alpha}_k, \boldsymbol{\Lambda}_k, \beta} J(x_k, \hat{\theta}_k, \mathbf{v}_k) - \gamma\beta$$

subject to $(\mathbf{v}_k, \boldsymbol{\alpha}_k, \boldsymbol{\Lambda}_k) \in \mathcal{F}(x_k, \Theta_k)$ and (PE-LMI)

3. Apply the control law $u_k^* = K x_k + v_{0|k}^*$

PE example

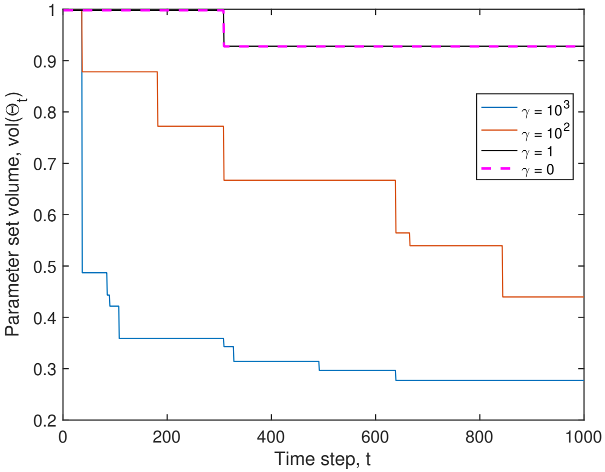

Figure: Parameter set volume $\mathrm{vol}(\Theta_t)$ vs cost weight γ

PE example

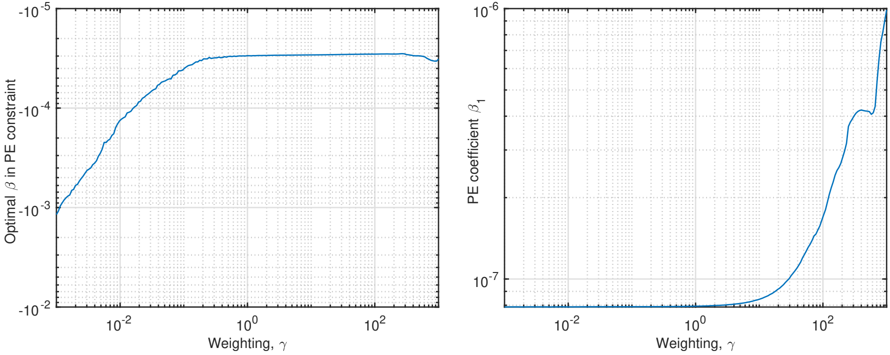

Figure: Minimum eigenvalue of information matrix vs cost weight γ

PE example

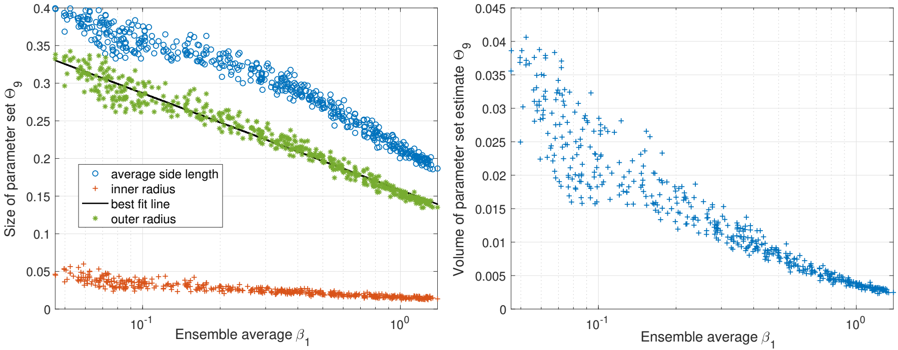

Figure: Size of parameter set after 10 time steps vs minimum eigenvalue of information matrix

Time-varying parameters

Assumption (time-varying parameters)

There exists a constant r_θ *such that the parameter vector* $\theta_k^\star$ *satisfies* $\theta_k^\star \in \Theta_0$ *for all* k *and* $\|\theta_{k+1}^\star - \theta_k^\star\| \leq r_\theta$

Define the dilation operator:

$$R_j(\Theta) = \{\theta : H_\Theta \theta \leq h + j r_\theta \mathbf{1}\}$$

Then the minimal parameter set at $k+1$ is

$$\Theta_{k+1} = R_1(\Theta_k \cap \Delta_{k+1}) \cap \Theta_0$$

and Θ_k is replaced in the tube MPC constraints by

$$\Theta_{i|k} = R_i(\Theta_k) \cap \Theta_0$$

Robust adaptive MPC algorithm with time-varying parameters

Parameter estimate bounds and recursive feasibility properties are unchanged:

Theorem (Closed loop properties)

If $\theta^\star \in \Theta_0$ *and* $\mathcal{F}(x_0, \Theta_0) \neq \emptyset$, *then for all* $k > 0$:

1. $\theta^\star \in \Theta_k$
2. $\mathcal{F}(x_k, \Theta_k) \neq \emptyset$
3. $Fx_k + Gu_k \leq \mathbf{1}$

But the LMS filter has an additional tracking error, which invalidates the l^2-stability properties, i.e. "certainty equivalence" no longer applies

Time-varying parameters example

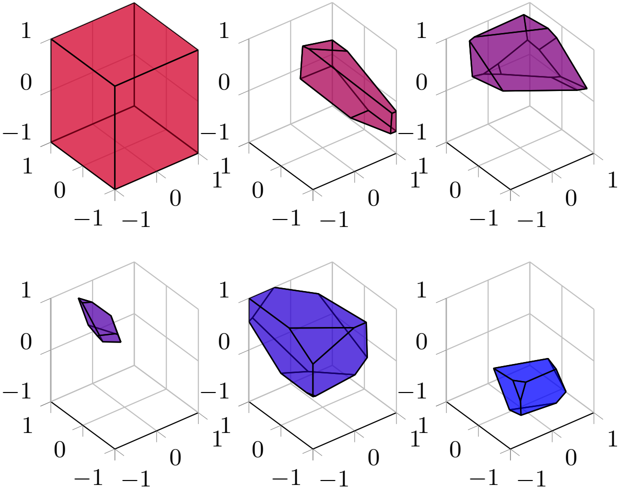

Figure: Parameter set Θ_k at times $k \in \{0, 100, 200, 300, 400, 500\}$ for the time-varying system with $r_\theta = 0.01$

Time-varying parameters example

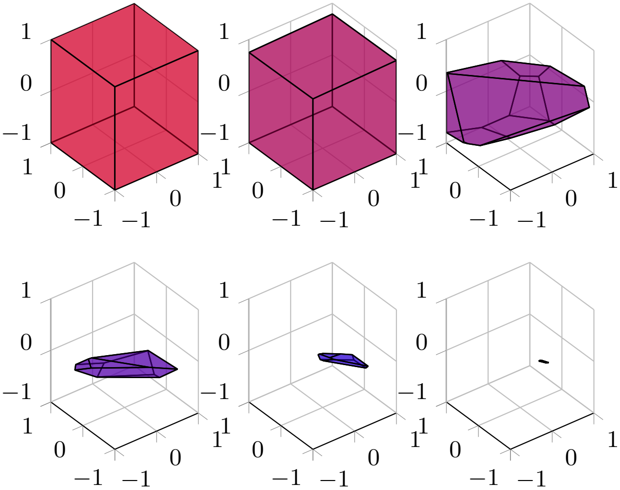

Figure: Parameter set Θ_k at times $k \in \{0, 5, 25, 70, 120, 500\}$ for the non-time-varying case for comparison

Differentiable MPC

- MPC law: $u_N(x_k, \hat{\theta}_k, \Theta_k)$ is the solution of a multiparametric programming problem
- Differentiable MPC uses the gradient $\nabla_{\hat{\theta}} u_N(\cdot)$ to train a neural network (NN) with weights $\hat{\theta}_k$ via back-propagation
- Update $\hat{\theta}_k$ as MPC optimization parameters embedded in a NN layer; retain parameter set estimate Θ_k for safe constraint handling

Differentiable MPC: learning model parameters

Linearly parameterised system model:

$$x_{k+1} = f(x_k, u_k, \theta^*) + w_k \qquad f(x_k, u_k, \theta) = D_k\theta + d_k \qquad \begin{cases} D_k = D(x_k, u_k) \\ d_k = d(x_k, u_k) \end{cases}$$

Parameter set estimate:

$$\Theta_{k+1} \supseteq \Theta_k \cap \Delta_{k+1}$$
$$\Delta_{k+1} = \{\theta : x_{k+1} - D_k\theta - d_k \in \mathcal{W}\}$$

Imitation learning problem: identify θ^* by observing an expert controller

Train $\hat{\theta}_k$ to minimize a loss function

$$\frac{1}{T}\sum_{t=k-T+1}^{k}\Big(\|\mathbf{u}_t - \mathbf{u}_N(x_t, \theta_k, \Theta_k)\|^2 + \sigma\|\hat{w}_t\|^2\Big)$$

where

$\mathbf{u}_t = \{u_t, \ldots, u_{t+N-1}\}$ = observed expert control sequence

$\mathbf{u}_N(x_t, \hat{\theta}_k, \Theta_k) = \{u_{0|t}, \ldots, u_{N-1|t}\}$ = MPC law for an initial state x_t

$\hat{w}_t = x_{t+1} - f(x_k, u_k, \hat{\theta}_k)$ = 1-step ahead error

Differentiable MPC: learning model parameters

Linearly parameterised system model:

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, \theta^*) + w_k \\ f(x_k, u_k, \theta) &= D_k\theta + d_k \end{aligned} \qquad \begin{cases} D_k = D(x_k, u_k) \\ d_k = d(x_k, u_k) \end{cases}$$

Parameter set estimate:

$$\begin{aligned} &\Theta_{k+1} \supseteq \Theta_k \cap \Delta_{k+1} \\ &\Delta_{k+1} = \{\theta : x_{k+1} - D_k\theta - d_k \in \mathcal{W}\} \end{aligned}$$

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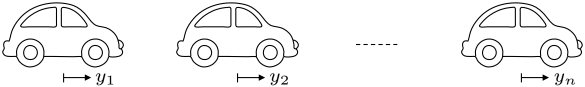
where

$$\begin{aligned} &\mathbf{u}_t = \{u_t, \ldots, u_{t+N-1}\} && = \text{observed expert control sequence} \\ &\mathbf{u}_N(x_t, \hat{\theta}_k, \Theta_k) = \{u_{0|t}, \ldots, u_{N-1|t}\} && = \text{MPC law for an initial state } x_t \\ &\hat{w}_t = x_{t+1} - f(x_k, u_k, \hat{\theta}_k) && = \text{1-step ahead error} \end{aligned}$$

Differentiable MPC: learning the MPC performance index

Platoon problem:



regulate $y_1, \ldots y_n$ a so that $\dot{y}_{i+1} - \dot{y}_i \to 0$ subject to $y_{i+1} - y_i \geq \underline{y}$

$$a \leq \ddot{y}_i \leq b$$

Prior assumptions:

- System model ($\ddot{y}_i = u_i$) is known
- $\underline{y}$, a, b are known

Unknown MPC cost to be learnt from observations of an expert controller

Conclusions

- Stable adaptive robust MPC is computationally tractable
- Set-membership parameter estimation and point estimates define MPC cost functions and robust constraints
- Nonconvex PE conditions can be relaxed to convex sufficient conditions

Future work

- How to ensure recursive feasibility while enforcing PE constraints?
- Can we relax the assumption of bounded disturbances?
- How to combine general adaptive parameter estimation with set-membership bounds?

References:

– M. Lorenzen, M. Cannon, F. Allgöwer, *Automatica*, 2019

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