Model Predictive Control Examples Sheet

Mark Cannon, Michaelmas Term 2017

Reading: Kouvaritakis & Cannon, Sections 2.1–2.6 and 3.1–3.3
or Maciejowski Chapters 2, 3, 6, 8

Prediction equations

1. A system with model

$$x(k+1) = Ax(k) + Bu(k), \qquad y(k) = Cx(k)$$
$$A = \begin{bmatrix} 1 & 0.1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

is to be controlled using an unconstrained predictive control law that minimizes the predicted performance cost

$$J(k) = \sum_{i=0}^{N-1} \Big(y^2(k+i|k) + \lambda u^2(k+i|k) \Big) + y^2(k+N|k), \qquad \lambda = 1.$$

(a). Show that the state predictions can be written in the form

$$\mathbf{x}(k) = \mathcal{M}x(k) + \mathcal{C}\mathbf{u}(k), \quad \mathbf{u}(k) = \begin{bmatrix} u(k|k) & \cdots & u(k+N-1|k) \end{bmatrix}^T$$

and evaluate $\mathcal{C}$ and $\mathcal{M}$ for a horizon of $N = 3$.

(b). For $N = 3$, determine the matrices H, F and G in

$$J(k) = \mathbf{u}^T(k)H\mathbf{u}(k) + 2x^T(k)F^T\mathbf{u}(k) + x^T(k)Gx(k).$$

(c). Give expressions for the derivatives $\partial J/\partial u(k+i|k)$ for $i = 0, 1, 2$. Hence verify that the gradient of J is $\nabla_{\mathbf{u}} J = 2H\mathbf{u} + 2Fx$.

2. (a). For the plant model and cost given in Question 1, show that the unconstrained predictive control law for $N = 3$ is linear feedback:

$$u(k) = K_3 x(k), \quad K_3 = -\begin{bmatrix} 0.1948 & 0.1168 \end{bmatrix}.$$

Hence show that the closed-loop system is unstable.

(b). Write some Matlab code to evaluate $\mathcal{C}_i$ for $i = 1, \ldots, N$, and to determine H and F for any given horizon length N. Show that the predictive control law does not stabilize the system if $N < 6$.

Infinite horizon cost and constraints

3. (a). Explain why the predictive control law of Question 1 necessarily stabilizes the system if the cost is minimized subject to $x(k+N|k) = 0$. (Hint: what is the infinite horizon cost when this constraint is used?)

(b). How would you modify the cost of Question 1 in order to achieve closed loop stability without including the constraint $x(k+N|k) = 0$? Why would this be preferable?

4. A predictive controller minimizes the predicted performance index:

$$J(k) = \sum_{i=0}^{\infty} \big[y^2(k+i|k) + u^2(k+i|k) \big]$$

at each time-step k subject to input constraints: $-1 \le u(k+i|k) \le 2$ for all $i \ge 0$. The system output y is related to the control input u via

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k)$$
$$A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

(a). Why is the MPC optimization performed repeatedly, at $k = 0, 1, 2 \ldots$, instead of just once, at $k = 0$?

(b). If the mode 2 feedback law is $u(k) = \begin{bmatrix} 2 & -1 \end{bmatrix} x(k)$, show that

$$J(k) = \sum_{i=0}^{N-1} \big[y^2(k+i|k) + u^2(k+i|k) \big] + x^T(k+N|k) \begin{bmatrix} 13 & -1 \\ -1 & 2 \end{bmatrix} x(k+N|k)$$

where N is the length of the mode 1 prediction horizon.

(c). Show that the constraints

$$-1 \le u(k+i|k) \le 2, \quad i = 0, 1, \ldots, N+1$$

ensure that the predictions satisfy $-1 \le u(k+i|k) \le 2$ for all $i \ge 0$.

(d). Derive a bound on $J^*(k+1) - J^*(k)$, where $J^*(k)$ is the optimal value of $J(k)$. Hence show that $\sum_{k=0}^{\infty}\big(y^2(k) + u^2(k)\big) \le J^*(0)$ along trajectories of the closed loop system.

(e). Is the closed loop system stable? Explain your answer.

5. (a). Explain the function of terminal constraints in a model predictive control strategy for a system with input or state constraints. Define two principal properties that must be satisfied by a terminal constraint set.

(b). A discrete time system has the state space model

$$x(k+1) = Ax(k) + Bu(k), \quad A = \begin{bmatrix} 0.3 & -0.9 \\ -0.4 & -2.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

and constraints

$$|x_1 + x_2| \le 1\,, \quad |x_1 - x_2| \le 1\,, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(i). If the terminal feedback law is $u(k) = Kx(k)$, $K = \begin{bmatrix} 0.4 & 1.8 \end{bmatrix}$, show that the following set is a valid terminal constraint set

$$\big\{x : |x_1 + x_2| \le 1\,,\ |x_1 - x_2| \le 1\big\}.$$

(ii). Describe a procedure for determining the largest terminal constraint set for the case of a general feedback gain K.

(c). What are the main considerations that govern the choice of the prediction horizon N?

Integral action and disturbances

6. The vertical position y of a machine tool positioning platform is controlled by a motor which applies a vertical force F to the platform (Figure 1). The platform has mass M and carries a variable load of mass m; the unloaded weight of the platform is balanced by a counter-weight. The force F is proportional to the voltage v applied to the motor, so that $F = K_v v$ where K_v is a fixed gain.

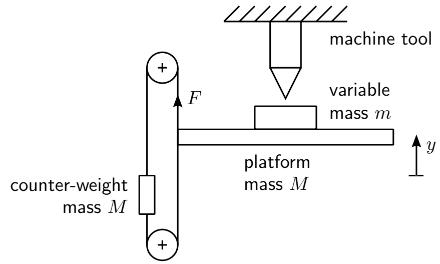

Figure 1. Machine tool and positioning platform

Assuming m is small enough that $M + m \approx M$, the unknown load constitutes a (constant) disturbance in the discrete-time model of the system for sampling interval T:

$$x(k+1) = Ax(k) + Bu(k) + Dd, \quad e(k) = Cx(k)$$
$$A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad B = \frac{K_v}{2M}\begin{bmatrix} T^2/2 \\ T \end{bmatrix}, \quad D = -\frac{g}{2M}\begin{bmatrix} T^2/2 \\ T \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

where e is the error in y relative to a desired steady-state height y^0, and

$$x(k) = \begin{bmatrix} y(kT) - y^0 \\ \dot{y}(kT) \end{bmatrix}, \quad u(k) = v(kT), \quad d = m.$$

(a). For the model parameters $M = 10\,\text{kg}$, $K_v = 7\,\text{N}\,\text{V}^{-1}$, $T = 0.1\,\text{s}$, the LQ-optimal feedback law with respect to the cost

$$J(k) = \sum_{i=0}^{\infty}\Big(e^2(k+i) + \lambda u^2(k+i)\Big), \quad \lambda = 10^{-4}$$

is $u(k) = Kx(k)$, $K = \begin{bmatrix} -66.0 & -19.4 \end{bmatrix}$. Determine the maximum steady state error $y - y^0$ with this controller if the mass of the load is limited to the range:

$$m \leq 0.5\,\text{kg}.$$

(b). Explain how to modify the cost and model dynamics in order to obtain a stabilizing LQ-optimal controller giving zero steady-state error.

(c). The motor input voltage is subject to the constraints

$$-1 \leq v \leq 1$$

A predictive controller is to be designed based on the predicted cost:

$$J(k) = \sum_{i=0}^{\infty} \Big(e^2(k+i|k) + e_I^2(k+i|k) + \lambda u^2(k+i|k) \Big), \quad \lambda = 10^{-4}$$

where $e_I(k+i+1|k) = e_I(k+i|k) + e(k+i|k)$.

(i). For a predicted input sequence with N degrees of freedom, show that $J(k)$ can be re-written as

$$J(k) = \sum_{i=0}^{N-1} \Big(e^2(k+i|k) + e_I^2(k+i|k) + \lambda u^2(k+i|k) \Big) + \|\xi^T(k+i|k)\|_P^2$$

and define ξ and P. What is the implied mode 2 feedback law?

(ii). Briefly explain how the constraints on v can be incorporated in a robust MPC strategy for this system (i.e. for all values of m in the range $m \leq 0.5\,\text{kg}$).

7. Assume that, for the given initial condition $x(0)$, the optimization of J subject to the robust constraints determined in Question 6 is initially feasible. Will the online optimization remain feasible at all future sampling times? What can be said about the steady-state value of y? Will the optimal value of the cost necessarily decrease monotonically, and what can be concluded about the convergence of the state $x(k)$ to zero in closed-loop operation?

8. A production planning problem involves optimizing the quantity u of stock manufactured in each week. The quantity x of stock that remains unsold at the start of week $k+1$ is given by

$$x_{k+1} = x_k + u_k - w_k, \quad k = 0, 1, \ldots$$

where the quantity w that is sold in each week is unknown in advance but is expected to be equal to a known constant $\hat{w}$. Limits on storage and manufacturing capacities imply that x and u can only take values in the intervals

$$0 \leq x_k \leq X, \quad 0 \leq u_k \leq U.$$

The desired level of stock in storage is x^*, and the planned values $u_{k|k}, u_{k+1|k}, \ldots$ are to be optimized at time k given a measurement of the value of x_k by minimizing a cost

$$J_k = \sum_{i=0}^{\infty} e_{k+i|k}^2\,, \quad e_{k+i|k} = x_{k+i|k} - x^*.$$

(a). What are the advantages of using a receding horizon control strategy in this application instead of an open-loop control sequence computed at $k = 0$?

(b). Assume that $w_k = \hat{w}$ for all $k = 0, 1, \ldots$

(i). Show that the unconstrained optimal control law is $u_k = \hat{w} - e_k$.

(ii). Show that, for a mode 1 horizon of N, the infinite horizon cost can be expressed

$$J_k = \sum_{i=0}^{N} e_{k+i|k}^2\,,$$

and state the corresponding mode 2 feedback law.

(iii). Show that constraints are satisfied over an infinite horizon if $0 \leq x_{k+i|k} \leq X$ and $0 \leq u_{k+i|k} \leq U$ for $0 \leq i \leq N-1$, and

$$\max\{0, \hat{w} + x^* - U\} \leq x_{k+N|k} \leq \min\{X, \hat{w} + x^*\}.$$

What assumptions on $\hat{w}$, x^*, U and X are needed?.

(c). Assume now that the future value of w is unknown and may take any value in an interval: $0 \le w_k \le W$. Suggest how to express the planned sequence $u_{k|k}, u_{k+1|k}, u_{k+2|k}$ in terms of the free variables in the receding horizon optimization problem, and justify your answer by determining the predictions $e_{k+1|k}, e_{k+2|k}, e_{k+3|k}$.

Some answers

1. (a). $\mathcal{C} = \begin{bmatrix} 0.15 & 0.05 & 0 \\ 2 & 1 & 0.5 \end{bmatrix}$

   (b). $H = \begin{bmatrix} 1.025 & 0.0075 & 0 \\ 0.0075 & 1.0025 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 0.2 & 0.12 \\ 0.05 & 0.035 \\ 0 & 0 \end{bmatrix}$, $G = \begin{bmatrix} 4 & 1.1 \\ 1.1 & 0.59 \end{bmatrix}$

2. (a). Closed loop poles for $N = 3$: eig$(A + BK_{N=3}) = 1.01, 1.93$

   (b).

N	4	5	6	7
eig$(A + BK_N)$	$1.03, 1.69$	1.11 ± 0.15i	0.86 ± 0.10i	0.95,0.58

3. (b). In $J(k)$, replace $y^2(k{+}N|k)$ with $\|x(k{+}N|k)\|_P^2$, $P = \begin{bmatrix} 22.46 & 4.098 \\ 4.098 & 12.79 \end{bmatrix}$

4. (d). $J^*(k+1) - J^*(k) \le -\big(y^2(k) + u^2(k)\big)$

   (e). $x = 0$ is locally asymptotically stable

6. (a). $|y - y^0| \le 0.0106$ m in steady state

   (c). ξ: augmented predicted state, $\xi = \begin{bmatrix} x & e_I \end{bmatrix}^T$, P: the solution of

   $$P - \left(\begin{bmatrix} A & 0 \\ C & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} K_\xi \right)^T P \left(\begin{bmatrix} A & 0 \\ C & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} K_\xi \right) = \begin{bmatrix} C^T C & 0 \\ 0 & 1 \end{bmatrix} + \lambda K_\xi^T K_\xi,$$

   Mode 2 feedback law: $u = K_\xi x$, e.g. LQ-optimal $K_\xi = -\begin{bmatrix} 201.4 & 29.6 & 48.2 \end{bmatrix}$

8. (c). $u_{k+i|k} = \hat{w} - e_{k+i|k} + c_{i|k}$, $c_{i|k} =$ decision variables for $i = 0, \ldots, N-1$, $c_{i|k} = 0$ for $i \ge N$