C21 Nonlinear Systems

Mark Cannon

4 lectures

Michaelmas Term 2017

Lecture 1

Introduction and Concepts of Stability

Course outline

1. Types of stability
2. Linearization
3. Lyapunov's direct method
4. Regions of attraction
5. Linear systems and passive systems

Books

- J.-J. Slotine & W. Li *Applied Nonlinear Control*, Prentice-Hall 1991.
  - ⋆ Stability
  - ⋆ Interconnected systems and passive systems
- H.K. Khalil *Nonlinear Systems*, Prentice-Hall 1996.
  - ⋆ Stability
  - ⋆ Passive systems
- M. Vidyasagar *Nonlinear Systems Analysis*, Prentice-Hall 1993.
  - ⋆ Stability & passivity (more technical detail)

Why use nonlinear control?

- Real systems are nonlinear
  - ⋆ friction, non-ideal components
  - ⋆ actuator saturation
  - ⋆ sensor nonlinearity
- Analysis via linearization
  - ⋆ accuracy of approximation?
  - ⋆ conservative?
- Account for nonlinearities in high performance applications
  - ⋆ Robotics, Aerospace, Petrochemical industries, Process control, Power generation . . .
- Account for nonlinearities if linear models inadequate
  - ⋆ large operating region
  - ⋆ model properties change at linearization point

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Linear vs nonlinear system properties

Free response

Linear system	Nonlinear system
$\boxed{\dot{x} = Ax}$	$\boxed{\dot{x} = f(x)}$

- Unique equilibrium point:
  $$Ax = 0 \iff x = 0$$
- Stability independent of initial conditions

- Multiple equilibrium points
  $$f(x) = 0$$
- Stability dependent on initial conditions

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Linear vs nonlinear system properties

Forced response

Linear system

$$\boxed{\dot{x} = Ax + Bu}$$

- $\|u\|$ finite $\Rightarrow$ $\|x\|$ finite if open-loop stable
- Frequency response:
  $u = U \sin \omega t \;\; \Rightarrow \;\; x = X \sin(\omega t + \phi)$
- Superposition:
  $u = u_1 + u_2 \;\; \Rightarrow x = x_1 + x_2$

Nonlinear system

$$\boxed{\dot{x} = f(x, u)}$$

- $\|u\|$ finite $\not\Rightarrow$ $\|x\|$ finite
- No frequency response
  $u = U \sin \omega t \;\; \not\Rightarrow \;\; x$ sinusoidal
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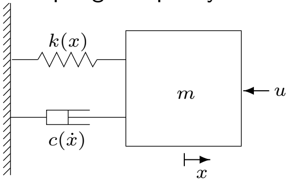
Example: step response

Mass-spring-damper system

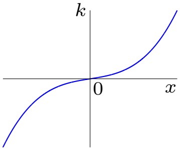


Equation of motion:

$$\ddot{x} + c(\dot{x}) + k(x) = u$$

$c(\dot{x}) = \dot{x}$

$k(x)$ nonlinear:



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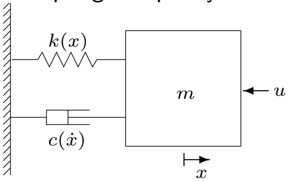

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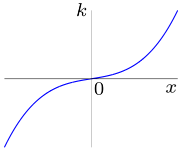
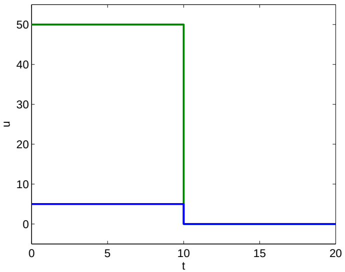
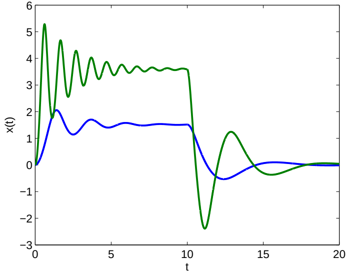


Input $u(t)$



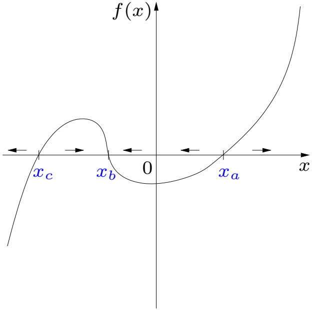
Response $x(t)$



apparent damping ratio depends on size of input step

Example: multiple equilibria

First order system: $\dot{x} = f(x)$



$$
\begin{array}{lcccl}
x > x_a & \Longrightarrow & f(x) > 0 & \Longrightarrow & x(t) \text{ increases} \\
x_b < x < x_a & \Longrightarrow & f(x) < 0 & \Longrightarrow & x(t) \text{ decreases} \\
x_c < x < x_b & \Longrightarrow & f(x) > 0 & \Longrightarrow & x(t) \text{ increases} \\
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- x_a, x_c are unstable equilibrium points
- x_b is a stable equilibrium point

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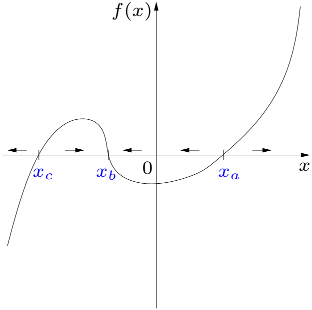

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Example: limit cycle

Van der Pol oscillator:

$$\ddot{x} + (x^2 - 1)\dot{x} + x = 0$$

- Response $x(t)$ tends to a limit cycle ($=$ trajectory forming a closed curve)
- Amplitude independent of initial conditions

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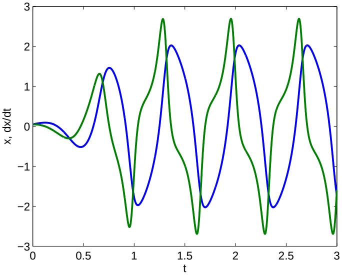
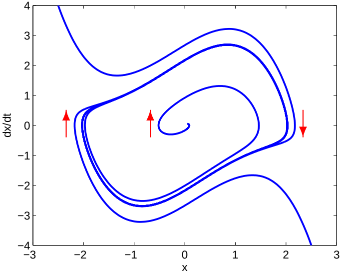


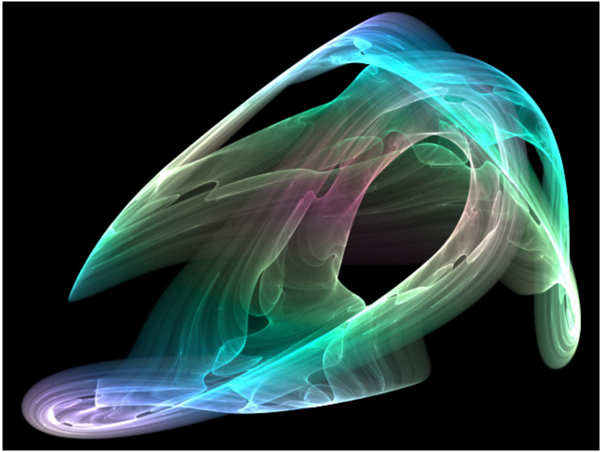
Response with $x(0) = 0.05$, $\dot{x}(0) = 0.05$



State trajectories $(x(t), \dot{x}(t))$

Example: chaotic behaviour

Strange attractor



Example: chaotic behaviour

Lorenz attractor

- Simplified model of atmospheric convection:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{aligned}$$

- State variables

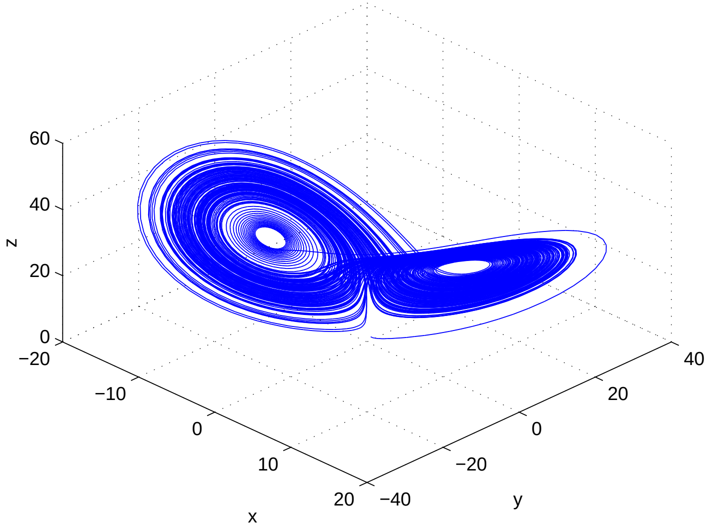
$x(t)$:	fluid velocity
$y(t)$:	difference in temperature of acsending and descending fluid
$z(t)$:	characterizes distortion of vertical temperature profile

- Parameters $\sigma = 10$, $\beta = 8/3$, $\rho =$ variable

Example: chaotic behaviour

Lorenz attractor

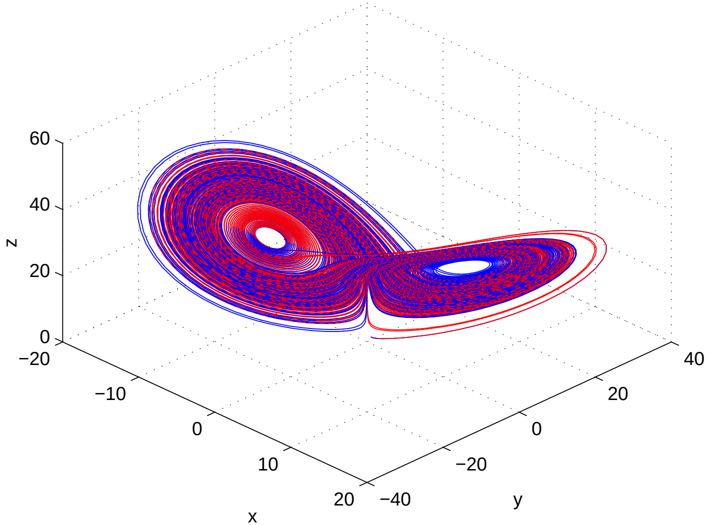
$\rho = 28 \implies$ "strange attractor":



Example: chaotic behaviour

Lorenz attractor

sensitivity to initial conditions



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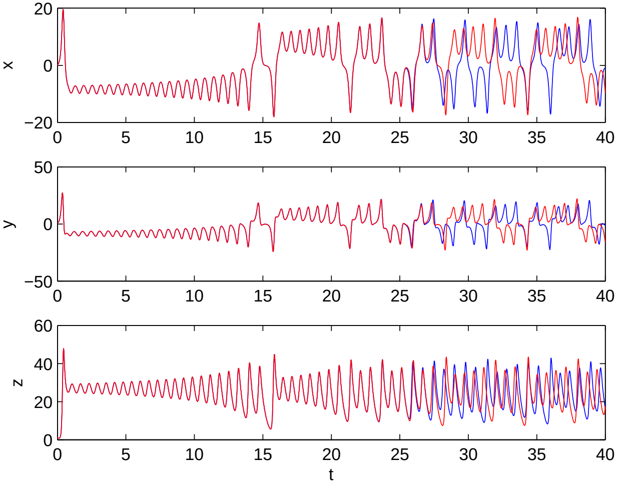
sensitivity to initial conditions

blue: $(x, y, z) = (0, 1, 1.05)$

red: $(x, y, z) = (0, 1, 1.050001)$



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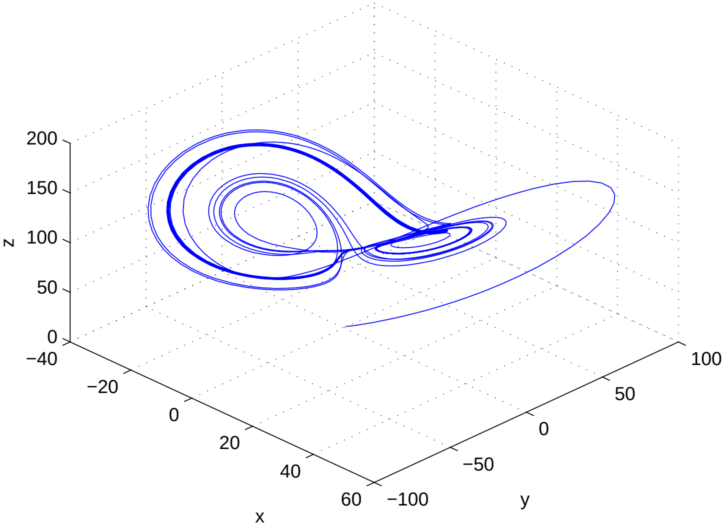
$\rho = 99.96 \implies$ limit cycle:



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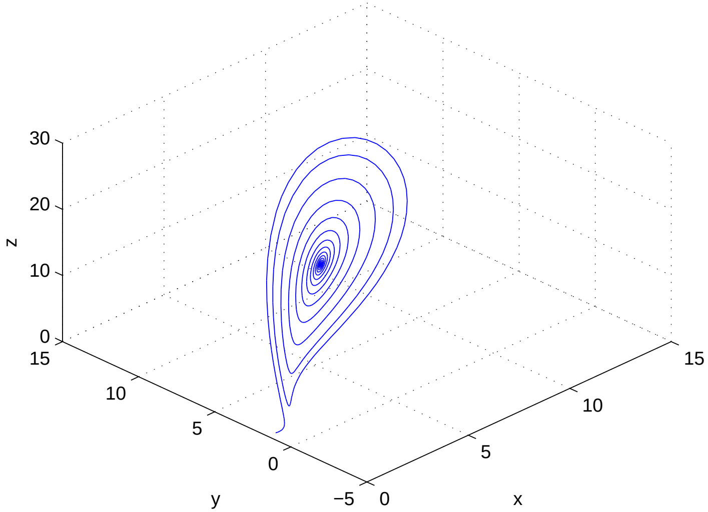
$\rho = 14 \implies$ convergence to a stable equilibrium:



State space equations

$$\dot{x} = f(x, u, t) \qquad \begin{aligned} x &: \text{ state} \\ u &: \text{ input} \end{aligned}$$

e.g. nth order differential equation:

$$\frac{d^n y}{dt^n} = h\left(y, \ldots, \frac{d^{n-1}y}{dt^{n-1}}, u, t\right)$$

has state vector (one possible choice)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ d^{n-1}y/dt^{n-1} \end{bmatrix}$$

and state space dynamics:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ h\big(x_1, x_2, \ldots, x_n, u, t\big) \end{bmatrix} = f(x, u, t)$$

Equilibrium points

x^* is an equilibrium point of system $\dot{x} = f(x)$ iff:

$$x(0) = x^* \quad \text{implies} \quad x(t) = x^* \quad \forall t > 0$$

i.e. $f(x^*) = 0$

Examples:

(a) $\ddot{y} + \alpha \dot{y}^2 + \beta \sin y = 0$ (damped pendulum)

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad x^* = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}, \; n = 0, \pm 1$$

(b) $\ddot{y} + (y-1)^2 \dot{y} + y - \sin(\pi y/2) = 0$

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- ⋆ Consider local stability of individual equilibrium points
- ⋆ Convention: define f so that $x = 0$ is equilibrium point of interest
- ⋆ Autonomous system: $\dot{x} = f(x) \quad \Longrightarrow \quad x^* =$ constant

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Stability definition

An equilibrium point $x = 0$ is stable iff:

$$\max_t \|x(t)\| \text{ can be made arbitrarily small}$$
$$\text{by making } \|x(0)\| \text{ small enough}$$

- Is $x = 0$ a stable equilibrium for the Van der Pol oscillator example?
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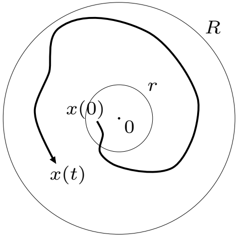

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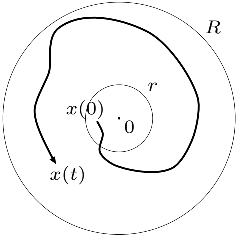

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Asymptotic stability definition

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(ii) is equivalent to:

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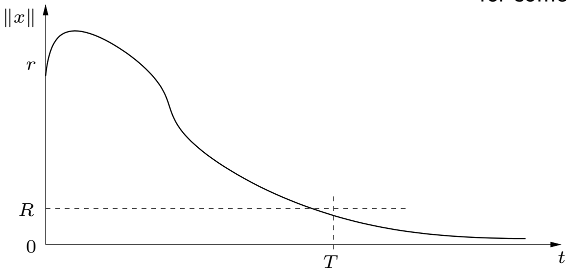
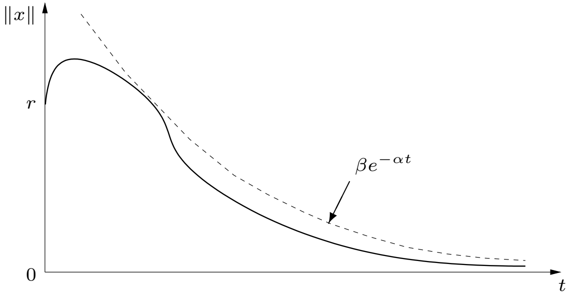


Exponential stability definition

An equilibrium point $x = 0$ is exponentially stable iff:

$$\boxed{\|x(0)\| < r \implies \|x(t)\| \le \beta e^{-\alpha t} \quad \forall t > 0}$$

exponential stability is a special case of asymptotic stability



Region of attraction

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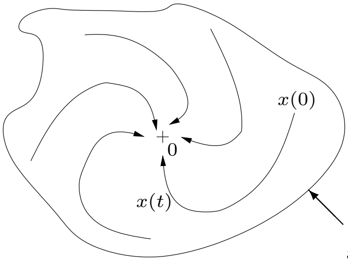

- Every asymptotically stable equilibrium point has a region of attraction
- $r = \infty \quad \implies \quad$ entire state space is a region of attraction
  $\implies \quad x = 0$ is globally asymptotically stable
- Are stable linear systems asymptotically stable?

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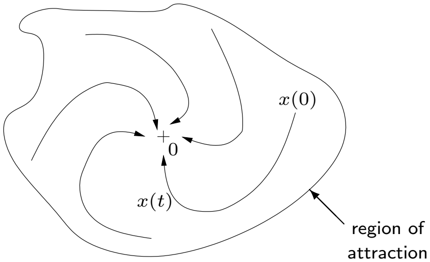

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- Are stable linear systems asymptotically stable?

Summary

- Nonlinear state space equations: $\dot{x} = f(x, u)$
  $x =$ state vector, $u =$ control input
- Equilibrium points: x^* is an equilibrium point
  of $\dot{x} = f(x)$ if $f(x^*) = 0$
- Stable equilibrium point: x^* is stable if state trajectories starting
  close to x^* remain near x^* at all times
- Asymptotically stable equilibrium point: x^* must be stable and state trajectories starting near x^* must tend to x^* asymptotically
- Region of attraction: the set of initial conditions from which state trajectories converge asymptotically to equilibrium x^*

Lecture 2

Linearization and Lyapunov's direct method

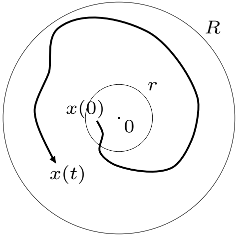
Linearization and Lyapunov's direct method

- Review of stability definitions
- Linearization method
- Direct method for stability
- Direct method for asymptotic stability
- Linearization method revisited

Review of stability definitions

System: $\dot{x} = f(x)$

- ⋆ unforced system (i.e. closed-loop)
- ⋆ consider stability of individual equilibrium points



0 is a stable equilibrium if:

$$\|x(0)\| \le r \implies \|x(t)\| \le R$$

for any $R > 0$

0 is asymptotically stable if:

$$\|x(0)\| \le r \implies \|x(t)\| \to 0$$

as $t \to \infty$

Stability → local property

Asymptotic stability → global if $r = \infty$ allowed

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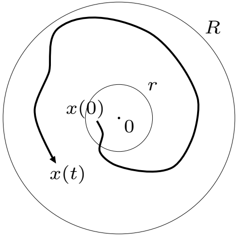

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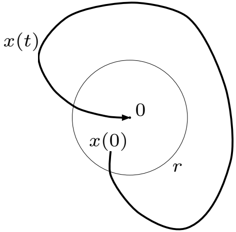

0 is asymptotically stable if:

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Stability $\rightarrow$ local property

Asymptotic stability $\rightarrow$ global if $r = \infty$ allowed

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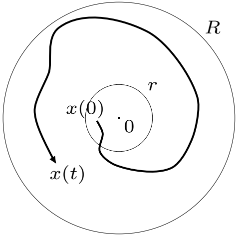

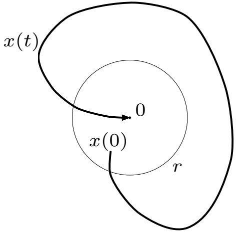

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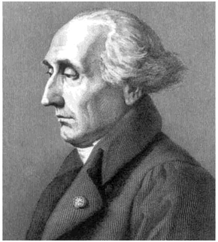
Historical development of Stability Theory

- Potential energy in conservative mechanics (Lagrange 1788):

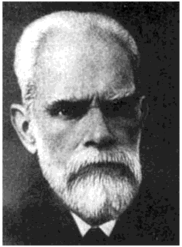
  An equilibrium point of a conservative system is stable if it corresponds to a minimum of the potential energy stored in the system

- Energy storage analogy for general ODEs (Lyapunov 1892)

- Invariant sets (Lefschetz, La Salle 1960s)



J-L. Lagrange 1736-1813



A. M. Lyapunov 1857-1918

S. Lefschetz 1884-1972

Lyapunov's linearization method

- Determine stability of equilibrium at $x = 0$ by analyzing the stability of the linearized system at $x = 0$.

- Jacobian linearization:

$$\begin{aligned} \dot{x} &= f(x) && \text{original nonlinear dynamics} \\ &= f(0) + \frac{\partial f}{\partial x}\Big|_{x=0} x + R_1 && \text{Taylor's series expansion, } R_1 = O(\|x\|^2) \\ &\approx Ax && \text{since } f(0) = 0 \end{aligned}$$

where

$$A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \qquad \frac{\partial f}{\partial x} \text{ assumed continuous at } x = 0$$

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Lyapunov's linearization method

Conditions on A for stability of original nonlinear system at $x = 0$:

stability of linearization	stability of nonlinear system at $x = 0$
$\text{Re}\big(\lambda(A)\big) < 0$	asymptotically stable (locally)
$\max \text{Re}\big(\lambda(A)\big) = 0$	stable or unstable
$\max \text{Re}\big(\lambda(A)\big) > 0$	unstable

Lyapunov's linearization method

- Some examples

$$
\begin{array}{rcccr}
\text{(stable)} & \dot{x} = -x^3 & \xrightarrow{\text{linearize}} & \dot{x} = 0 & \text{(indeterminate)} \\
\text{(unstable)} & \dot{x} = x^3 & \xrightarrow{\text{linearize}} & \dot{x} = 0 & \text{(indeterminate)} \\
 & & & \uparrow & \\
\end{array}
$$

higher order terms determine stability

- Why does linear control work?

  1. Linearize the model:

$$
\begin{aligned}
\dot{x} &= f(x, u) \\
&\approx Ax + Bu, \qquad A = \frac{\partial f}{\partial x}(0,0), \;\; B = \frac{\partial f}{\partial u}(0,0)
\end{aligned}
$$

  2. Design a linear feedback controller using the linearized model:

$$
u = -Kx, \quad \max \mathrm{Re}\big(\lambda(A - BK)\big) < 0
$$

closed-loop linear model strictly stable

nonlinear system $\dot{x} = f(x, -Kx)$ is locally asymptotically stable at $x = 0$

Lyapunov's linearization method

- Some examples

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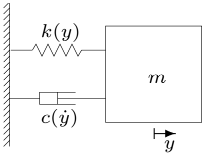
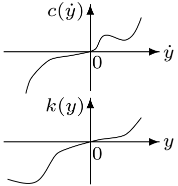
Lyapunov's direct method: mass-spring-damper example





Equation of motion: $m\ddot y + c(\dot y) + k(y) = 0$

Stored energy: $V = \text{K.E.} + \text{P.E.}$ $\begin{cases} \text{K.E.} = \frac{1}{2}m\dot y^2 \\ \text{P.E.} = \int_0^y k(y)\,dy \end{cases}$

Rate of energy dissipation
$$\dot V = \tfrac{1}{2}m\ddot y\frac{d}{d\dot y}\dot y^2 + \dot y\frac{d}{dy}\left[\int_0^y k(y)\,dy\right]$$
$$= m\ddot y\dot y + \dot y k(y)$$

but $m\ddot y + k(y) = -c(\dot y)$, so
$$\dot V = -c(\dot y)\dot y$$
$$\leq 0 \qquad \leftarrow \text{since } \operatorname{sign}\big(c(\dot y)\big) = \operatorname{sign}(\dot y)$$

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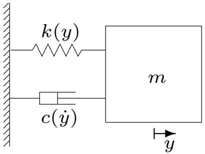

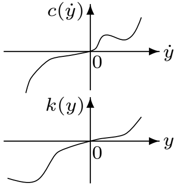

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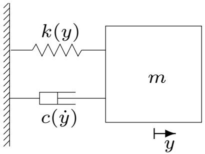

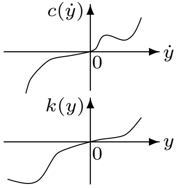

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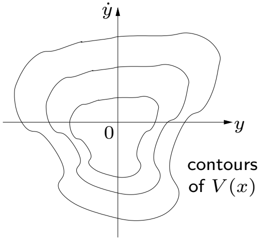
Mass-spring-damper example contd.

- System state: e.g. $x = [y \quad \dot{y}]^T$

- $\dot{V}(x) \leq 0$ implies that $x = 0$ is stable
$$\uparrow$$
$V(x(t))$ must decrease over time

but

$V(x)$ increases with increasing $\|x\|$



- Formal argument:

for any given $R > 0$:

$$\|x\| < R \quad \text{whenever} \quad V(x) < \overline{V} \quad \text{for some } \overline{V}$$

$$\text{and } V(x) < \overline{V} \quad \text{whenever} \quad \|x\| < r \quad \text{for some } r$$

$$\begin{aligned} \therefore\ \|x(0)\| < r \quad &\implies \quad V\big(x(0)\big) < \overline{V} \\ &\implies \quad V\big(x(t)\big) < \overline{V} \quad \text{for all } t > 0 \\ &\implies \quad \|x(t)\| < R \quad \text{for all } t > 0 \end{aligned}$$

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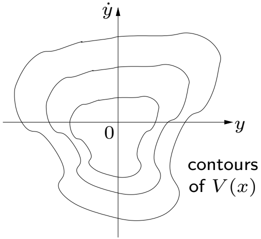

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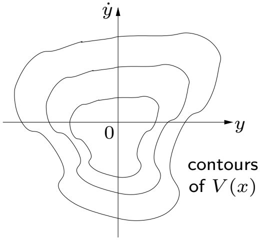

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Positive definite functions

- What if $V(x)$ is not monotonically increasing in $\|x\|$?
- Same arguments apply if $V(x)$ is continuous and positive definite, i.e.

$$\boxed{\begin{array}{rll} \text{(i).} & V(0) = 0 & \\ \text{(ii).} & V(x) > 0 & \text{for all } x \neq 0 \end{array}}$$

for any given $\overline{V} > 0$,
can always find r so that

$$V(x) < \overline{V} \quad \text{whenever} \quad \|x\| < r$$

$V(x) \geq \alpha \|x\|^n$
for some constants α, n, so

$$\|x\| < (\overline{V}/\alpha)^{1/n} \quad \text{whenever} \quad V(x) < \overline{V}$$

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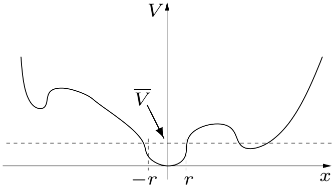

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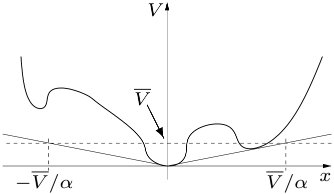

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Lyapunov stability theorem

If there exists a continuous function $V(x)$ such that

$$\boxed{\begin{array}{l} V(x) \text{ is positive definite} \\ \dot{V}(x) \le 0 \end{array}}$$

then $x = 0$ is stable.

To show that this implies $\|x(t)\| < R$ for all $t > 0$ whenever $\|x(0)\| < r$ for any R and some r:

1. choose $\overline{V}$ as the minimum of $V(x)$ for $\|x\| = R$
2. find r so that $V(x) < \overline{V}$ whenever $\|x\| < r$
3. then $\dot{V}(x) \le 0$ ensures that

$$V(x(t)) < \overline{V} \quad \forall t > 0 \quad \text{if } \|x(0)\| < r$$
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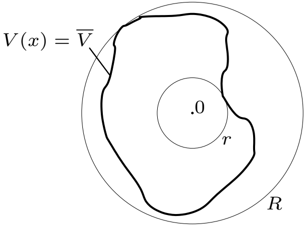

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- Lyapunov's direct method also applies if $V(x)$ is locally positive definite, i.e. if

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- Apply the theorem without determining R, r

  – only need to find p.d. $V(x)$ satisfying $\dot{V}(x) \leq 0$.

- Examples

  (i). $\dot{x} = -a(t)x, \quad a(t) > 0$

$$V = \tfrac{1}{2}x^2 \quad \Longrightarrow \quad \begin{aligned} \dot{V} &= x\dot{x} \\ &= -a(t)x^2 \leq 0 \end{aligned}$$

  (ii). $\dot{x} = -a(x), \quad \mathrm{sign}\big(a(x)\big) = \mathrm{sign}(x)$

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Lyapunov stability theorem

- More examples

(iii). $\dot{x} = -a(x), \quad \int_0^x a(x)\,dx > 0$

$$V = \int_0^x a(x)\,dx \quad \Longrightarrow \quad \begin{aligned}\dot{V} &= a(x)\dot{x} \\ &= -a^2(x) \le 0\end{aligned}$$

(iv). $\ddot{\theta} + \sin\theta = 0$

$$V = \tfrac{1}{2}\dot{\theta}^2 + \int_0^\theta \sin\theta\,d\theta \quad \Longrightarrow \quad \begin{aligned}\dot{V} &= \ddot{\theta}\dot{\theta} + \dot{\theta}\sin\theta \\ &= 0\end{aligned}$$

Asymptotic stability theorem

If there exists a continuous function $V(x)$ such that

$$\boxed{\begin{array}{ll} V(x) & \text{is positive definite} \\ \dot{V}(x) & \text{is negative definite} \end{array}}$$

then $x = 0$ is locally asymptotically stable.

($\dot{V}$ negative definite $\iff$ $-\dot{V}$ positive definite)

Asymptotic convergence $x(t) \to 0$ as $t \to \infty$ can be shown by contradiction:

if $\|x(t)\| > R'$ for all $t \geq 0$, then

$$\left.\begin{array}{l} \dot{V}(x) < -W \\ \\ V(x) \geq \underline{V} \end{array}\right\} \quad \text{for all } t \geq 0$$

$\uparrow$

contradiction

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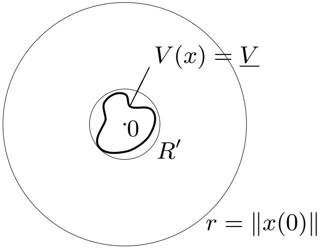

Linearization method and asymptotic stability

- Asymptotic stability result also applies if $\dot{V}(x)$ is only locally negative definite.

- Why does the linearization method work?

  ⋆ consider 1st order system: $\dot{x} = f(x)$
  linearize about $x = 0$: $\quad = -ax + R \qquad R = O(x^2)$

  ⋆ assume $a > 0$ and try Lyapunov function V:

$$
\begin{aligned}
V(x) &= \tfrac{1}{2}x^2 \\
\dot{V}(x) &= x\dot{x} = -ax^2 + Rx = -x^2(a - R/x) \\
&\le -x^2(a - |R/x|)
\end{aligned}
$$

  ⋆ but $R = O(x^2)$ implies $|R| \le \beta x^2$ for some constant β, so

$$
\begin{aligned}
\dot{V} &\le -x^2(a - \beta|x|) \\
&\le -\gamma x^2 \quad \text{if } |x| \le (a-\gamma)/\beta
\end{aligned}
$$

$\Longrightarrow$ $\dot{V}$ negative definite for $|x|$ small enough
$\Longrightarrow$ $x = 0$ locally asymptotically stable

Generalization to nth order systems is straightforward

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$\implies$ $\dot{V}$ negative definite for $|x|$ small enough
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Generalization to nth order systems is straightforward

Global asymptotic stability theorem

If there exists a continuous function $V(x)$ such that

$$\boxed{\begin{array}{l} \left.\begin{array}{ll} V(x) & \text{is positive definite} \\ \dot{V}(x) & \text{is negative definite} \end{array}\right\} \text{for all } x \\ V(x) \to \infty \text{ as } \|x\| \to \infty \end{array}}$$

then $x = 0$ is globally asymptotically stable

- If $V(x) \to \infty$ as $\|x\| \to \infty$, then $V(x)$ is radially unbounded
- Test whether $V(x)$ is radially unbounded by checking if $V(x) \to \infty$ as each individual element of x tends to infinity (necessary).

Global asymptotic stability theorem

- Global asymptotic stability requires:

$$\|x(t)\| \text{ finite} \begin{cases} \text{for all } t > 0 \\ \text{for all } x(0) \end{cases}$$

$$\uparrow$$

not guaranteed by $\dot{V}$ negative definite

in addition to asymptotic stability of $x = 0$

- Hence add extra condition: $V(x) \to \infty$ as $\|x\| \to \infty$

$$\updownarrow \text{ equiv. to}$$

$$\text{level sets } \{x \ : \ V(x) \leq \overline{V}\} \text{ are finite}$$

$$\updownarrow \text{ equiv. to}$$

$$\|x\| \text{ is finite whenever } V(x) \text{ is finite}$$

$$\uparrow$$

prevents $x(t)$ drifting away from 0 despite $\dot{V} < 0$

Asymptotic stability example

System:
$$\begin{aligned}\dot{x}_1 &= (x_2 - 1)x_1^3\\ \dot{x}_2 &= -\frac{x_1^4}{(1+x_1^2)^2} - \frac{x_2}{1+x_2^2}\end{aligned}$$

- Trial Lyapunov function $V(x) = x_1^2 + x_2^2$:

$$\begin{aligned}\dot{V}(x) &= 2x_1\dot{x_1} + 2x_2\dot{x_2}\\ &= -2x_1^4 + \underbrace{2x_2x_1^4 - 2\frac{x_2x_1^4}{(1+x_1^2)^2}}_{\text{change } V \text{ to make these terms cancel}} - 2\frac{x_2^2}{1+x_2^2} \qquad \not\leq 0\end{aligned}$$

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change V to make
these terms cancel

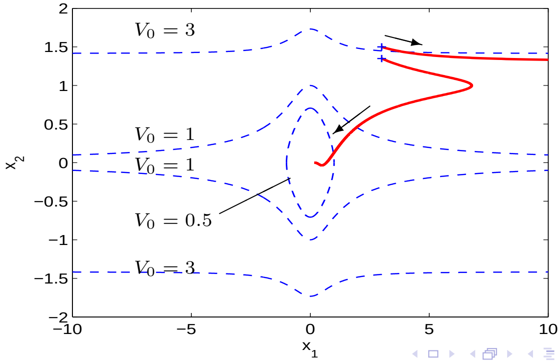
Asymptotic stability example

- New trial Lyapunov function $V(x) = \dfrac{x_1^2}{1+x_1^2} + x_2^2$:

$$\dot{V}(x) = 2\left[\frac{x_1}{1+x_1^2} - \frac{x_1^3}{(1+x_1^2)^2}\right]\dot{x}_1 + 2x_2\dot{x}_2$$
$$= -2\frac{x_1^4}{(1+x_1^2)^2} - 2\frac{x_2^2}{1+x_2^2} \le 0$$

$V(x)$ positive definite, $\dot{V}(x)$ negative definite $\implies$ $x = 0$ a.s.
But $V(x)$ not radially unbounded, so cannot conclude global asymptotic stability

State trajectories:



Summary

- Positive definite functions
- Derivative of $V(x)$ along trajectories of $\dot{x} = f(x)$
- Lyapunov's direct method for: stability
  asymptotic stability
  global stability
- Lyapunov's linearization method

Lecture 3

Convergence and invariant sets

Convergence and invariant sets

- Review of Lyapunov's direct method
- Convergence analysis using Barbalat's Lemma
- Invariant sets
- Global and local invariant set theorem
- Example

Review of Lyapunov's direct method

Positive definite functions

- If
$$\begin{aligned} &V(0) = 0 \\ &V(x) > 0 \quad \text{for all } x \neq 0 \end{aligned}$$
then $V(x)$ is positive definite

- If $\mathcal{S}$ is a set containing $x = 0$ and
$$\begin{aligned} &V(0) = 0 \\ &V(x) > 0 \quad \text{for all } x \neq 0,\ x \in \mathcal{S} \end{aligned}$$
then $V(x)$ is locally positive definite (within $\mathcal{S}$)

- e.g.

$$V(x) = x^T x \qquad \leftarrow \qquad \text{positive definite}$$

$$V(x) = x^T x(1 - x^T x) \qquad \leftarrow \qquad \text{locally positive definite within } \mathcal{S} = \{x \ : \ x^T x < 1\}$$

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Review of Lyapunov's direct method

System: $\dot{x} = f(x), \quad f(0) = 0$

Storage function: $V(x)$

Time-derivative of V: $\dot{V}(x) = \dfrac{\partial V}{\partial x}\dfrac{dx}{dt} = \nabla V(x)^T \dot{x} = \nabla V(x)^T f(x)$

- If

$$\left.\begin{array}{r l} \text{(i).} & V(x) \text{ is positive definite} \\ \text{(ii).} & \dot{V}(x) \le 0 \end{array}\right\} \text{ for all } x \in \mathcal{S}$$

  then the equilibrium $x = 0$ is stable

- If

$$\text{(iii).} \quad \dot{V}(x) \text{ is negative definite} \qquad \text{for all } x \in \mathcal{S}$$

  then the equilibrium $x = 0$ is asymptotically stable

- If

$$\begin{array}{r l} \text{(iv).} & S = \text{entire state space} \\ \text{(v).} & V(x) \to \infty \text{ as } \|x\| \to \infty \end{array}$$

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- What can be said about convergence of $x(t)$ to 0
  if $\dot{V}(x) \leq 0$ but $\dot{V}(x)$ is not negative definite?

- Revisit m-s-d example:

Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$

Storage function: $$V = \text{K.E.} + \text{P.E.} = \tfrac{1}{2}m\dot{y}^2 + \int_0^y k(y)\, dy$$
$$\dot{V} = -c(\dot{y})\dot{y}$$

Convergence analysis

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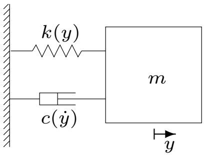

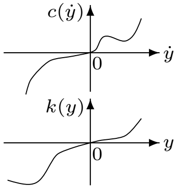

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$$\dot{V} = -c(\dot{y})\dot{y}$$

Convergence analysis

- V is p.d. and $\dot{V} \le 0$ so: $(y, \dot{y}) = (0, 0)$ is stable
  and $V(y, \dot{y})$ tends to a finite limit as $t \to \infty$
- but does $(y, \dot{y})$ converge to $(0, 0)$?

$\updownarrow$ equivalent to

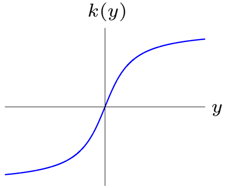
can $V(y, \dot{y})$ "get stuck" at $V = V_0 \neq 0$ as $t \to \infty$?

$\downarrow$

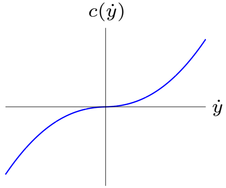
need to consider motion at points $(y, \dot{y})$ for which $\dot{V} = 0$

Example

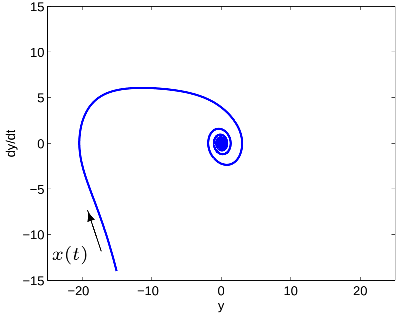
Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$



$k(y) = 5\tan^{-1}(y/5)$



$c(\dot{y}) = 0.1\dot{y}(0.5 + |\dot{y}|)(2 - e^{-0.1|\dot{y}|})$



Storage function:

$$V = \tfrac{1}{2}\dot{y}^2 + \int_0^y 5\tan^{-1}(y/5)\,dy$$

$$\dot{V} = -c(\dot{y})\dot{y} \le 0$$

$$\downarrow$$

$$\dot{V} = 0 \text{ when } \dot{y} = 0$$

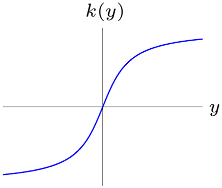
but $k(y) \neq 0 \implies \ddot{y} \neq 0 \implies \ddot{V} \neq 0$

$$\Downarrow$$

V continues to decrease until $y = \dot{y} = 0$

Example

Equation of motion: $m\ddot{y} + c(\dot{y}) + k(y) = 0$



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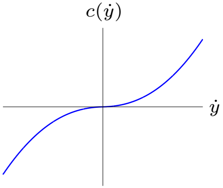

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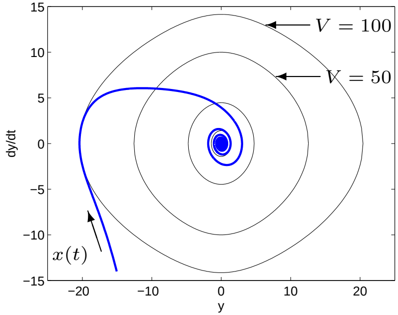

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$$\Downarrow$$

V continues to decrease until $y = \dot{y} = 0$

Convergence analysis

Summary of method:

1. show that $\dot{V}(x) \to 0$ as $t \to \infty$
2. determine the set $\mathcal{R}$ of points x for which $\dot{V}(x) = 0$
3. identify the subset $\mathcal{M}$ of $\mathcal{R}$ for which $\dot{V}(x) = 0$ at all future times

$$\boxed{\text{then } x(t) \text{ has to converge to } \mathcal{M} \text{ as } t \to \infty}$$

This approach is the basis of the invariant set theorems

Barbalat's Lemma

Barbalat's lemma: For any function $\phi(t)$, if

$$\boxed{\begin{array}{l} \text{(i). } \displaystyle\int_0^t \phi(\tau)\,d\tau \text{ converges to a finite limit as } t \to \infty \\ \text{(ii). } \dot{\phi}(t) \text{ is finite for all } t \end{array}}$$

then $\lim_{t\to\infty} \phi(t) = 0$

- Obvious for the case that $\phi(t) \geq 0$ for all t
- Condition (ii) is needed to ensure that $\phi(t)$ remains continuous for all t

$\uparrow$

Can construct discontinuous $\phi(t)$ for which $\displaystyle\int_0^t \phi(\tau)\,d\tau$ converges

but $\phi(t) \not\to 0$ as $t \to \infty$

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Can construct discontinuous $\phi(t)$ for which $\displaystyle\int_0^t \phi(\tau)\,d\tau$ converges

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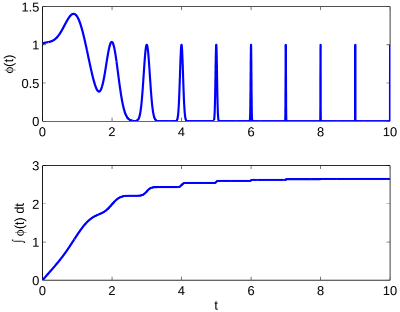
Barbalat's Lemma

Example: pulse train $\phi(t) = \sum_{k=0}^{\infty} e^{-4^k (t-k)^2}$:

$\phi(t)$:

$\int_0^t \phi(\tau)\, d\tau$:



From the plots it is clear that

$$\int_0^t \phi(s)\, ds \text{ tends to a finite limit}$$

but $\quad \phi(t) \not\to 0$ as $t \to \infty \quad$ because $\dot{\phi}(t) \to \infty$ as $t \to \infty$

Barbalat's Lemma contd.

Apply Barbalat's Lemma to $\dot{V}\big(x(t)\big) = \phi(t) \le 0$:

- Integrate:
  $$\int_0^t \phi(s)\,ds = V\big(x(t)\big) - V\big(x(0)\big) \qquad \leftarrow \text{finite limit as } t \to \infty$$
- Differentiate:
  $$\dot{\phi}(t) = \ddot{V}\big(x(t)\big) = f^T(x)\frac{\partial^2 V}{\partial x^2}(x) f(x) + \nabla V(x) \frac{\partial f}{\partial x}(x) f(x)$$
  $$= \text{finite for all } t \text{ if } f(x) \text{ continuous and } V(x) \text{ continuously differentiable}$$

$$\Downarrow$$

$$\boxed{\dot{V}(x) \to 0 \text{ as } t \to \infty}$$

The above arguments rely on $\|x(t)\|$ remaining finite for all t, which is implied by:

$V(x)$ positive definite
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The above arguments rely on $\|x(t)\|$ remaining finite for all t, which is implied by:

$$\begin{aligned} &V(x) \text{ positive definite} \\ &\dot{V}(x) \leq 0 \\ &V(x) \to \infty \text{ as } \|x\| \to \infty \end{aligned}$$

Convergence analysis

Summary of method:

1. show that $\dot{V}(x) \to 0$ as $t \to \infty$
   $\to$ true whenever $\dot{V} \leq 0$ & V, f are smooth & $\|x(t)\|$ is bounded

   [by Barbalat's Lemma]

2. determine the set $\mathcal{R}$ of points x for which $\dot{V}(x) = 0$
   $\to$ algebra!

3. identify the subset $\mathcal{M}$ of $\mathcal{R}$ for which $\dot{V}(x) = 0$ at all future times
   $\to$ $\mathcal{M}$ must be invariant

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This approach is the basis of the invariant set theorems

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Summary of method:

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Invariant sets

- A set of points $\mathcal{M}$ in state space is invariant if

$$\boxed{x(t_0) \in \mathcal{M} \quad \Longrightarrow \quad x(t) \in \mathcal{M} \quad \text{for all } t > t_0}$$

Examples:

- ⋆ Equilibrium points
- ⋆ Limit cycles
- ⋆ Level sets of $V(x)$ provided $\dot{V}(x) \leq 0$ $\leftarrow$ i.e. $\{x \ : \ V(x) \leq V_0\}$ for constant V_0

- If $\dot{V}(x) \to 0$ as $t \to \infty$, then

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Global invariant set theorem

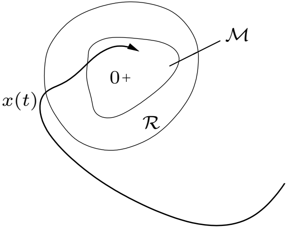
If there exists a continuously differentiable function $V(x)$ such that

$$\boxed{\begin{array}{l} V(x) \text{ is positive definite} \\ \dot{V}(x) \le 0 \\ V(x) \to \infty \text{ as } \|x\| \to \infty \end{array}}$$

then: (i). $\dot{V}(x) \to 0$ as $t \to \infty$

(ii). $x(t) \to \mathcal{M} =$ the largest invariant set contained in $\mathcal{R}$

where $\mathcal{R} = \{x \ : \ \dot{V}(x) = 0\}$



- $\dot{V}(x)$ negative definite $\implies \mathcal{M} = 0$ (c.f. Lyapunov's direct method)
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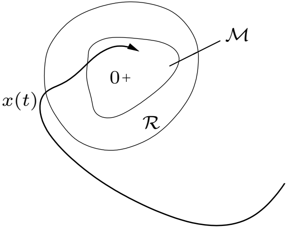

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Revisit m-s-d example (for the last time)

- $V(x)$ is positive definite, $V(x) \to \infty$ as $\|x\| \to \infty$, and
$$\dot{V}(y, \dot{y}) = -c(\dot{y})\dot{y} \leq 0$$
- therefore $\dot{V} \to 0$, implying $\dot{y} \to 0$ as $t \to \infty$
i.e. $\mathcal{R} = \{(y, \dot{y}) \ : \ \dot{y} = 0\}$
- but $\dot{y} = 0$ implies $\ddot{y} = -k(y)/m$
- therefore $\ddot{y} \neq 0$ unless $y = 0$, so $\dot{y}(t) = 0$ for all t only if $y(t) = 0$
i.e. $\mathcal{M} = \{(y, \dot{y}) \ : \ (y, \dot{y}) = (0, 0)\}$

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$(y, \dot{y}) = (0, 0)$ is a globally asymptotically stable equilibrium!

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Local invariant set theorem

If there exists a continuously differentiable function $V(x)$ such that

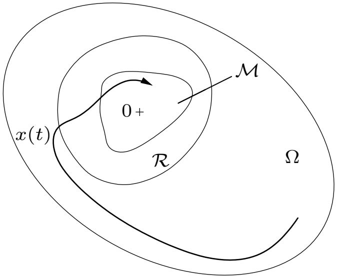
> the level set $\Omega = \{x \;:\; V(x) \le V_0\}$ is bounded for some V_0
> and $\dot{V}(x) \le 0$ whenever $x \in \Omega$

then: (i). Ω is an invariant set

(ii). $x(0) \in \Omega \quad \Longrightarrow \quad \dot{V}(x) \to 0$ as $t \to \infty$

(iii). $x(t) \to \mathcal{M}$ = largest invariant set contained in $\mathcal{R}$

where $\mathcal{R} = \{x \;:\; \dot{V}(x) = 0\}$



Local invariant set theorem

- $V(x)$ doesn't have to be positive definite or radially unbounded

- Result is based on Barbalat's Lemma applied to $\dot{V}$

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  applies here because finite Ω implies $\|x(t)\|$ finite for all t

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Example: local invariant set theorem

- Second order system: $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(x_1-1)^2 x_2^3 - x_1 + \sin(\pi x_1/2) \end{aligned}$$
- Equilibrium points: $(x_1, x_2) = (0,0), (1,0), (-1,0)$
- Trial storage function:

$$V(x) = \tfrac{1}{2}x_2^2 + \int_0^{x_1} \big(y - \sin(\pi y/2)\big)\, dy$$

V is not positive definite
but $V(x) \to \infty$ if $x_1 \to \infty$ or $x_2 \to \infty$

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level sets of V are finite

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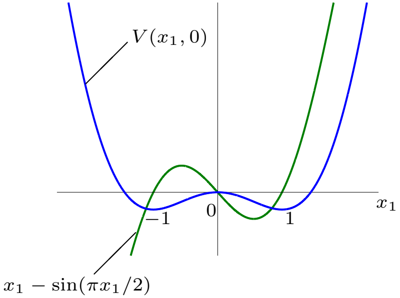

Example: local invariant set theorem contd.

- Differentiate: $\dot{V}(x) = -(x_1 - 1)^2 x_2^4 \leq 0$

$$\dot{V}(x) = 0 \quad \iff \quad x \in \mathcal{R} = \{x \ : \ x_1 = 1 \ \text{ or } \ x_2 = 0\}$$

- From the system model, $x \in \mathcal{R}$ implies:

$$\begin{aligned} x_1 = 1 \quad &\implies \quad (\dot{x}_1, \dot{x}_2) = (x_2, 0) \\ &\text{and} \\ x_2 = 0 \quad &\implies \quad (\dot{x}_1, \dot{x}_2) = (0, \sin(\pi x_1/2) - x_1) \end{aligned}$$

$$\text{therefore} \begin{cases} x(t) \text{ remains on line } x_1 = 1 \text{ only if } x_2 = 0 \\ x(t) \text{ remains on line } x_2 = 0 \text{ only if } x_1 = 0, \ 1 \text{ or } -1 \end{cases}$$

$$\implies \mathcal{M} = \{(0,0), (1,0), (-1,0)\}$$

- Apply local invariant set theorem to any level set $\Omega = \{x \ : \ V(x) \leq V_0\}$:

$$\left.\begin{array}{l} \Omega \text{ is finite} \\ \dot{V} \leq 0 \end{array}\right\} \implies \quad x(t) \to \mathcal{M} = \{(0,0), (1,0), (-1,0)\} \text{ as } t \to \infty$$

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  but $x = (0,0)$ is unstable

  (linearized system at $(0,0)$ has poles $\pm\sqrt{\frac{\pi}{2} - 1}$ so is unstable)

- Contours of $V(x)$:

Use local invariant set theorem on level sets $\Omega = \{x : V(x) \leq V_0\}$ for $V_0 < 0$

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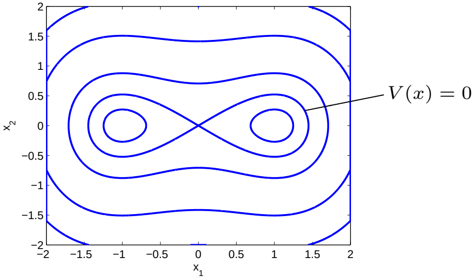

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Summary

- Convergence analysis using Barbalat's lemma
- Invariant sets
- Invariant set methods for convergence: local invariant set theorem
  global invariant set theorem

Lecture 4

Linear systems, passivity, and the circle criterion

Linear systems, passivity, and the circle criterion

- Summary of stability methods
- Lyapunov functions for linear systems
- Passive systems
- Passive linear systems
- The circle criterion
- Example

Summary of stability methods

- Linearization method

$$\dot{x} = Ax \text{ is strictly stable, } A = \frac{\partial f}{\partial x}\Big|_{x=0}$$
$$\Downarrow$$
$$x = 0 \text{ locally asymptotically stable}$$

- Lyapunov's direct method

$V(x)$ locally p.d. $\dot{V}(x) \le 0$ locally $\Downarrow$ $x = 0$ stable	$V(x)$ locally p.d. $\dot{V}(x)$ locally n.d. $\Downarrow$ $x = 0$ locally asymptotically stable	$V(x)$ p.d. $\dot{V}(x)$ n.d. $V(x) \to \infty$ as $\lVert x\rVert \to \infty$ $\Downarrow$ $x = 0$ globally asymptotically stable

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- Instability theorems analogous to Lyapunov's direct method, e.g.

$$\left.\begin{array}{l} V(x) \text{ p.d.} \\ \dot{V}(x) \text{ p.d.} \end{array}\right\} \quad \Longrightarrow \quad x = 0 \text{ unstable}$$

- Lyapunov stability criteria are only sufficient, e.g.

$$\left.\begin{array}{l} V(x) \text{ p.d.} \\ \dot{V}(x) \not\leq 0 \end{array}\right\} \quad \not\Longrightarrow \quad x = 0 \text{ unstable}$$

(some other $V(x)$ demonstrating stability may exist)

- Converse theorems

$$x = 0 \text{ stable} \quad \Longrightarrow \quad V(x) \text{ demonstrating stability exists}$$

(can swap premises and conclusions in Lyapunov's direct method)

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Linear systems

- Systematic method for constructing storage function $V(x) = x^T P x$

$\dot{x} = Ax$ strictly stable $\implies$ can always find constant matrix P so that $\dot{V}(x)$ is negative definite

- Only need consider symmetric P

$$x^T P x = \tfrac{1}{2} x^T P x + \tfrac{1}{2} x^T P^T x = \tfrac{1}{2} x^T \underbrace{(P + P^T)}_{\text{SYMMETRIC}} x$$

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$P = U \Lambda U^T$ — eigenvector/value decomposition

$\Downarrow$

$x^T P x = z^T \Lambda z$ — $z = U^T x$

$\Downarrow$

$x^T P x$ positive definite iff Λ strictly positive — $\left\{ \begin{array}{l} \text{notation: } P > 0 \\ \text{or "}P\text{ is positive definite"} \end{array} \right.$

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$P = U\Lambda U^T$ eigenvector/value decomposition

$\Downarrow$

$x^T P x = z^T \Lambda z$ $\qquad z = U^T x$

$\Downarrow$

$x^T P x$ positive definite iff Λ strictly positive $\qquad \begin{cases} \text{notation: } P > 0 \\ \text{or ``}P\text{ is positive definite''} \end{cases}$

Linear systems

- Systematic method for constructing storage function $V(x) = x^T P x$

$$\dot{x} = Ax \text{ strictly stable} \quad \Longrightarrow \quad \begin{array}{l} \text{can always find constant matrix } P \\ \text{so that } \dot{V}(x) \text{ is negative definite} \end{array}$$

- Only need consider symmetric P

$$x^T P x = \tfrac{1}{2} x^T P x + \tfrac{1}{2} x^T P^T x = \tfrac{1}{2} x^T \underbrace{\left(P + P^T\right)}_{\text{SYMMETRIC}} x$$

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$$\begin{array}{ll} P = U \Lambda U^T & \text{eigenvector/value decomposition} \\ \Downarrow & \\ x^T P x = z^T \Lambda z & z = U^T x \\ \Downarrow & \\ x^T P x \text{ positive definite} & \left\{ \begin{array}{l} \text{notation: } P > 0 \\ \text{or ``} P \text{ is positive definite''} \end{array} \right. \\ \text{iff } \Lambda \text{ strictly positive} & \end{array}$$

- How is P computed?

$$\left.\begin{aligned} \dot{x} &= Ax \\ V(x) &= x^T P x \end{aligned}\right\} \implies \begin{aligned} \dot{V}(x) &= x^T P \dot{x} + \dot{x}^T P x \\ &= x^T (PA + A^T P) x \end{aligned}$$

$\therefore x = 0$ is globally asymptotically stable if, for some Q:

$$\boxed{PA + A^T P = -Q \qquad Q = Q^T > 0}$$

Lyapunov matrix equation

- Pick $Q > 0$ and solve $PA + A^T P = -Q$ for P, then

$$\boxed{\mathrm{Re}[\lambda(A)] < 0 \iff \text{unique solution for } P \text{ and } P = P^T > 0}$$

Proof:

$\Longleftarrow$ due to $\dot{V}(x) = -x^T Q x$ negative definite

$\implies$ follows from integrating $\dot{V}$ w.r.t. t: $P = \int_0^\infty e^{A^T t} Q e^{At}\, dt$

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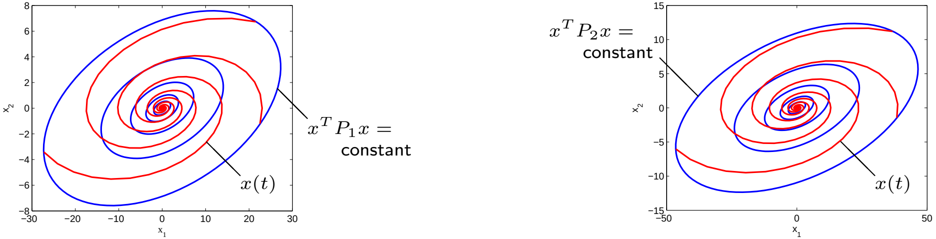
Example: Lyapunov matrix equation

Stable linear system $\dot{x} = Ax$: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -16 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\qquad \lambda(A) = -1 \pm i\sqrt{15}$

Solve $PA + A^T P = -Q$ for P:

$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow P_1 = \begin{bmatrix} 0.33 & -0.5 \\ -0.5 & 4.25 \end{bmatrix}$ $\qquad Q_2 = \begin{bmatrix} 0.41 & -0.19 \\ -0.19 & 0.11 \end{bmatrix} \Rightarrow P_2 = \begin{bmatrix} 0.12 & -0.21 \\ -0.21 & 1.67 \end{bmatrix}$

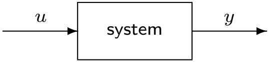


Here:

- ⋆ any choice of $Q > 0$ gives $P > 0$ (since A is strictly stable)
- ⋆ but not every $P > 0$ gives $Q > 0$

Passive systems

- Systematic method for constructing storage functions
- Input-output representation of system:



input : u
output : y

The system is passive if

$$\boxed{\dot{V} = yu - g \quad \text{for some } V(t) \geq 0, \quad g(t) \geq 0}$$

also the system is dissipative if $\displaystyle\int_0^\infty yu\,dt \neq 0 \implies \int_0^\infty g\,dt > 0$

- Motivated by electrical networks with no internal power generation

input: i
output: v } stored energy: $V = \int_0^t vi\,dt$

$\dot{V} = iv$

Passive systems

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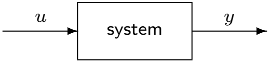

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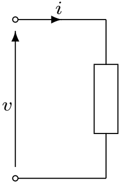
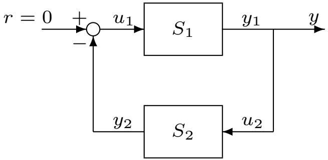


$$\left.\begin{array}{r} \text{input: } i \\ \text{output: } v \end{array}\right\} \quad \text{stored energy:} \quad \begin{array}{l} V = \displaystyle\int_0^t vi\,dt \\ \dot{V} = iv \end{array}$$

Passive systems

Passivity is useful for determining storage functions for feedback systems

- Closed-loop system with passive subsystems S_1, S_2:



$$\begin{aligned} S_1: \quad & V_1 \geq 0 \quad \dot{V}_1 = y_1 u_1 - g_1 \\ S_2: \quad & V_2 \geq 0 \quad \dot{V}_2 = y_2 u_2 - g_2 \end{aligned}$$

$$V_1 + V_2 \geq 0$$

$$\begin{aligned} \dot{V}_1 + \dot{V}_2 &= y_1 u_1 + y_2 u_2 - g_1 - g_2 \\ &= y_1(-y_2) + y_2 y_1 - g_1 - g_2 \\ &= -g_1 - g_2 \\ &\leq 0 \end{aligned}$$

$\implies$ $V = V_1 + V_2$ is a Lyapunov function for the closed-loop system

if V is a p.d. function of the system state

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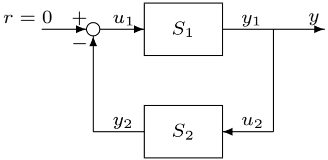

$$
\begin{aligned}
S_1 :& \quad V_1 \geq 0 \quad \dot{V}_1 = y_1 u_1 - g_1 \\
S_2 :& \quad V_2 \geq 0 \quad \dot{V}_2 = y_2 u_2 - g_2
\end{aligned}
$$

$$
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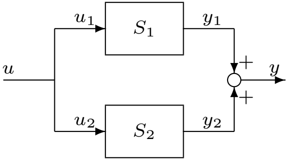
Interconnected passive systems

- Parallel connection:



$$
\begin{aligned}
V_1 + V_2 &\ge 0 \\
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&= (y_1 + y_2)u - g_1 - g_2 \\
&= yu - g_1 - g_2
\end{aligned}
$$

$\Downarrow$

Overall system from u to y is passive

- Feedback connection:

$$
\begin{aligned}
V_1 + V_2 &\ge 0 \\
\dot{V}_1 + \dot{V}_2 &= y_1 u_1 + y_2 u_2 - g_1 - g_2 \\
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$\Downarrow$

Overall system from u to y is passive

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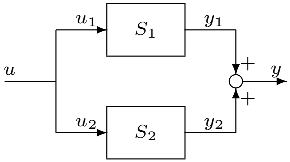

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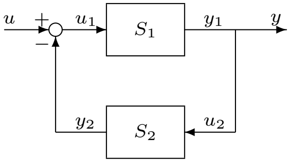

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Overall system from u to y is passive

Passive linear systems

Transfer function : $\dfrac{Y(s)}{U(s)} = H(s)$

$$u \longrightarrow \boxed{H(s)} \longrightarrow y$$

- H is passive if and only if

(i).	$\mathrm{Re}(p_i) \le 0$, where $\{p_i\}$ are the poles of $H(s)$
(ii).	$\mathrm{Re}\big[H(j\omega)\big] \ge 0$ for all $0 \le \omega \le \infty$

⋆ H must be stable, otherwise $V(t) = \displaystyle\int_0^t yu\,dt$ is not defined for all u

⋆ From Parseval's theorem:

$$\mathrm{Re}\big[H(j\omega)\big] \ge 0 \quad \Longleftrightarrow \quad \int_0^t yu\,dt \ge 0 \text{ for all } u(t) \text{ and } t$$

frequency domain criterion for passivity

- H is dissipative if and only if $\mathrm{Re}(p_i) \le 0$ and

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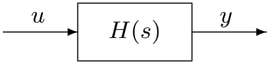

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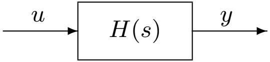

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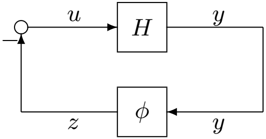
Linear system + static nonlinearity



H linear: $\dfrac{Y(s)}{U(s)} = H(s)$

ϕ static nonlinearity: $z = \phi(y)$

What are the conditions on H and ϕ for closed-loop stability?

- A common problem in practice, due to e.g.
  - ⋆ actuator saturation (valves, dc motors, etc.)
  - ⋆ sensor nonlinearity

- Determine closed-loop stability given:

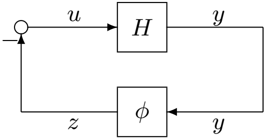
ϕ belongs to sector $[a, b]$

$$\Updownarrow$$

$$\boxed{a \le \frac{\phi(y)}{y} \le b}$$

"Absolute stability problem"

Linear system + static nonlinearity



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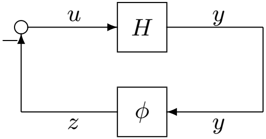

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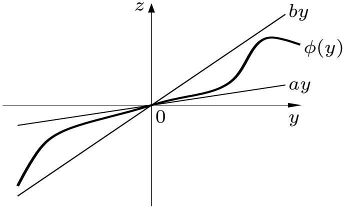

Linear system + static nonlinearity

- Aizerman's conjecture (1949):

  Closed-loop system is stable if stable for $\phi(y) = ky$, $a \le k \le b$

  false (necessary but not sufficient)

- Sufficient conditions for closed-loop stability:

  $$\left.\begin{array}{l}\text{Popov criterion (1960)}\\ \text{Circle criterion}\end{array}\right\}\ \text{based on passivity}$$

- The passivity approach:

  (1). If H is dissipative (i.e. if $\mathrm{Re}[H(j\omega)] > 0$ and H is stable), then:

  $$\left.\begin{aligned} V &= x^T P x \\ \dot{V} &= yu - x^T Q x \end{aligned}\right\}\ \text{for some } P > 0,\ \ Q > 0$$

  $$= -y\phi(y) - x^T Q x \qquad \leftarrow x = \text{state of } H$$

  (2). If ϕ belongs to sector $[0, \infty)$, then:

  $$y\phi(y) \ge 0 \qquad \leftarrow$$

$$\begin{aligned}(1)\ \&\ (2) \quad &\Longrightarrow \quad \dot{V} \le -x^T Q x \\ &\Longrightarrow \quad x = 0 \text{ is globally asymptotically stable}\end{aligned}$$

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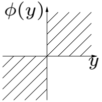

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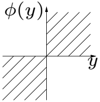
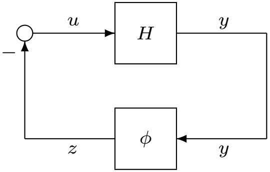
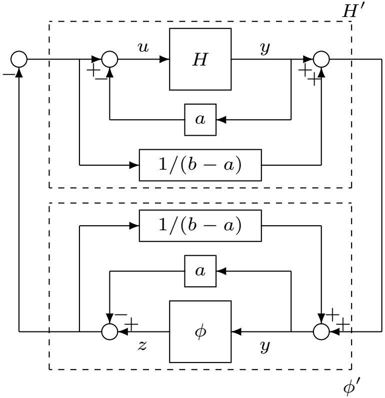


$$\begin{aligned} \text{(1) \& (2)} \quad &\Longrightarrow \quad \dot V \le -x^T Q x \\ &\Longrightarrow \quad x = 0 \text{ is globally asymptotically stable} \end{aligned}$$

Use loop transformations to generalize the approach for $\begin{cases} H \text{ not passive} \\ \phi \notin [0, \infty) \end{cases}$



$\longleftrightarrow$
equiv. to



$\phi \in [a, b] \quad a, b$ arbitrary

$$\phi \in [a, b] \implies \phi' \in [0, \infty]$$

$$H'(j\omega) = \frac{H(j\omega)}{1 + aH(j\omega)} + \frac{1}{b - a}$$

Circle criterion

To make $H'(j\omega) = \dfrac{H(j\omega)}{1 + aH(j\omega)} + \dfrac{1}{b-a}$ dissipative, need:

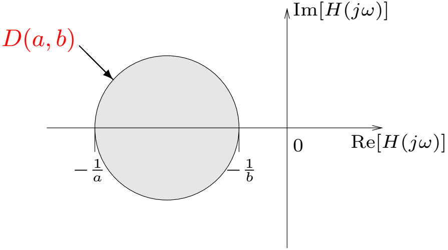
(i). H' stable $\iff$ $\dfrac{H(j\omega)}{1 + aH(j\omega)}$ stable

$\Updownarrow$

Nyquist plot of $H(j\omega)$ goes through ν anti-clockwise encirclements of $-1/a$ as ω goes from $-\infty$ to ∞

(ν = no. poles of $H(j\omega)$ in RHP)

(ii). $\mathrm{Re}\big[H'(j\omega)\big] > 0 \iff \begin{cases} H(j\omega) \text{ lies outside } D(a,b) & \text{if } ab > 0 \\ H(j\omega) \text{ lies inside } D(a,b) & \text{if } ab < 0 \end{cases}$

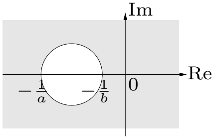


Graphical interpretation of circle criterion

$x = 0$ is globally asymptotically stable if:

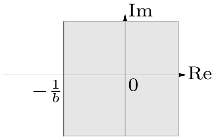
⋆ $0 < a < b$

$H(j\omega)$ lies in shaded region and does ν anti-clockwise encirclements of $D(a, b)$



⋆ $b > a = 0$

$H(j\omega)$ lies in shaded region and $\nu = 0$ (can't encircle $-1/a$)



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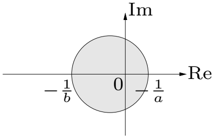

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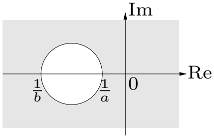
$-H(j\omega)$ lies in shaded region and does ν anti-clockwise encirclements of $D(-b, -a)$



Circle criterion

- Circle criterion is equivalent to Nyquist criterion for $a = b > 0$

$$\uparrow$$

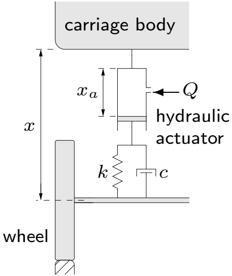
then $D(a,b) = -\dfrac{1}{a}$ (single point)

- Circle criterion is only sufficient for closed-loop stability for general a, b

- Results apply to time-varying static nonlinearity: $\phi(y,t)$

Example: Active suspension system

- Active suspension system for high-speed train:



$$
\begin{aligned}
Q &= \phi(u) \\
\dot{x}_a &= Q/A
\end{aligned}
$$

u : valve input signal
Q : flow rate
ϕ : valve characteristics, $\phi \in [0.005, 0.1]$
A : actuator working area

- Force exerted by suspension system on carriage body: F_{susp}

$$
\begin{aligned}
F_{\text{susp}} &= k(x_a - x) + c(\dot{x}_a - \dot{x}) \\
&= \Big(k \int^t Q\, dt + cQ\Big)/A - kx - c\dot{x}, \qquad Q = \phi(u)
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- Design controller to compensate for the effects of (constant) unknown load on displacement x despite uncertain valve characteristics $\phi(u)$.

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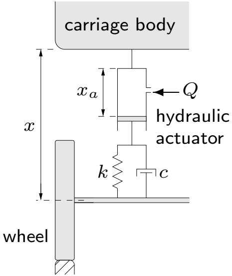

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Active suspension system contd.

- Dynamics:

$$
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& F_{\text{susp}} - F = m\ddot{x} \\
\implies \quad & m\ddot{x} + c\dot{x} + kx = \Big(k \int^t Q \, dt + cQ\Big)/A - F, \qquad Q = \phi(u)
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$$

F : unknown load on suspension unit
m : effective carriage mass

- Transfer function model:

$$
X(s) = \frac{cs + k}{ms^2 + cs + k} \cdot \frac{Q(s)}{As} - \frac{F}{ms^2 + cs + k} \qquad Q = \phi(u)
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- Try linear compensator $C(s)$:

$$
U(s) = C(s)E(s) \qquad e = -x, \quad \text{setpoint: } x = 0
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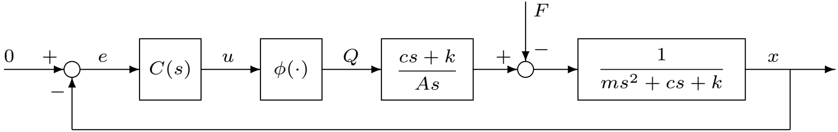
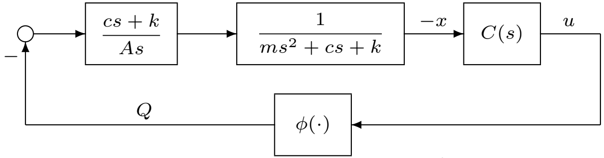


Active suspension system contd.

- For constant F, we need to stabilize the closed-loop system:



$$\text{linear system:} \quad H(s) = \frac{cs+k}{As(ms^2+cs+k)} \cdot C(s)$$

$$\text{static nonlinearity:} \quad \phi \in [0.005, 0.1]$$

- P+D compensator (no integral term needed):

$$C(s) = K(1+\alpha s) \quad \Longrightarrow \quad H(s) = \frac{K(1+\alpha s)(cs+k)}{As(ms^2+cs+k)}$$

H open-loop stable ($\nu = 0$)

- From the circle criterion, closed-loop (global asymptotic) stability is ensured if:

$H(j\omega)$ lies outside $D(0.005, 0.1)$

$\Uparrow$

sufficient condition: $\mathrm{Re}\big[H(j\omega)\big] > -10$

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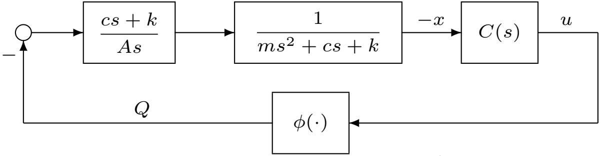

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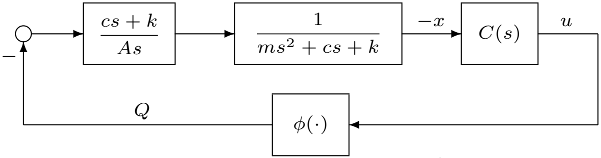

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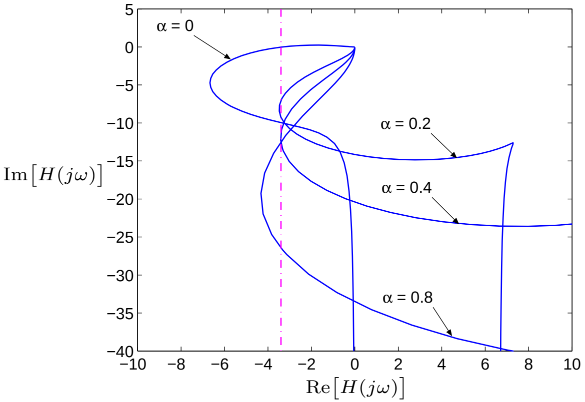
Active suspension system contd.

- Nyquist plot of $H(j\omega)$ for $K = 1$ and $\alpha = 0, 0.2, 0.4, 0.8$:



- To maximize gain margin:

choose $\alpha = 0.2$ $\leftarrow$ allows for largest K

$K \leq 10/3.4 = 2.94$

Summary

At the end of the course you should be able to do the following:

- Understand the basic Lyapunov stability definitions (lecture 1)
- Analyse stability using the linearization method (lecture 2)
- Analyse stability by Lyapunov's direct method (lecture 2)
- Determine convergence using Barbalat's Lemma (lecture 3)
- Understand how invariant sets can determine regions of attraction (lecture 3)
- Construct Lyapunov functions for linear systems and passive systems (lecture 4)
- Use the circle criterion to design controllers for systems with static nonlinearities (lecture 4)

Addendum – Limit cycle stability analysis

System has state $x = (x_1, x_2)$ and dynamics:

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_1 h(x)\\ \dot{x}_2 &= x_1 - x_2 h(x)\end{aligned}$$

where $h(x) = x_1^2 + x_2^2 - 1$.

Limit cycle

Differentiate $h(x)$ w.r.t. t using system dynamics:

$$\dot{h} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2(x_1^2 + x_2^2)h(x) = -2(h+1)h$$

hence $h = 0 \implies \dot{h} = 0$, so $\{x : x_1^2 + x_2^2 = 1\}$ must contain a limit cycle.

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where $h(x) = x_1^2 + x_2^2 - 1$.

Limit cycle stability

Let $V(x) = h^2(x)$, then
$$\begin{aligned}\dot{V} &= 2h\dot{h} = -4h^2(h+1)\\ &= -4h^2(x)(x_1^2 + x_2^2) \le 0\end{aligned}$$

- $\{x : V(x) \le c\}$ is an invariant set for any constant c and $\{x : V(x) = 0\} = \{x : x_1^2 + x_2^2 = 1\}$ is stable
- $\dot{V} = 0 \implies h = 0$ (or $x_1 = x_2 = 0$)
  $h = 0 \implies \dot{h} = 0$
  $\implies$ the limit cycle $\{x : h = 0\}$ is the largest invariant set contained in $\{x : V(x) < 1$ and $\dot{V}(x) = 0\}$, so is asymptotically stable