# Lecture 2: Equilibria and stability

- An equilibrium is where the function in the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a zero solution, i.e. $\mathbf{x}^* \in \mathbb{R}^{\boldsymbol{n}}$ such that $\mathbf{f}(\mathbf{x}^*) = 0$.

- There may be many solutions to the equation $\mathbf{f}(\mathbf{x}^*) = 0$, but each is characterised by $\mathbf{f}(\mathbf{x}^*) = 0 \Rightarrow \dot{\mathbf{x}} = 0$, i.e. $\mathbf{x}$ does not change. Equilibrium points are sometimes be called ‘fixed points’.

- For a **linear** system with non-zero eigenvalues there is only one solution to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, i.e. $\mathbf{x} = 0$ . A nonlinear system can have many non-zero equilibria.

# Maps and equilibria

- In a similar way to differential equations, the equilibria of **maps** are the fixed points of the update equations, i.e.

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k) \text{ such that } \mathbf{x}_{k+1} = \mathbf{x}_k = \mathbf{x}^*$$

- In the case of differential equations, the equilibria are special points on a **flow** in phase space, whereas maps jump around without a flow – this property makes maps hard to understand and visualise.

# Flows and equilibria

- The solution of the non-linear differential equation in phase space can be thought of as a flow in an n-dimensional phase space, like a liquid. $\mathbf{f}(\mathbf{x})$ can be thought of as a vector field describing the flow or ‘fluid velocity’.

- Flows can end or start at equilibria, or can circulate around equilibria

- The stability of the flow near an equilibrium is an important property and something we will study today

# Stability of flow equilibria

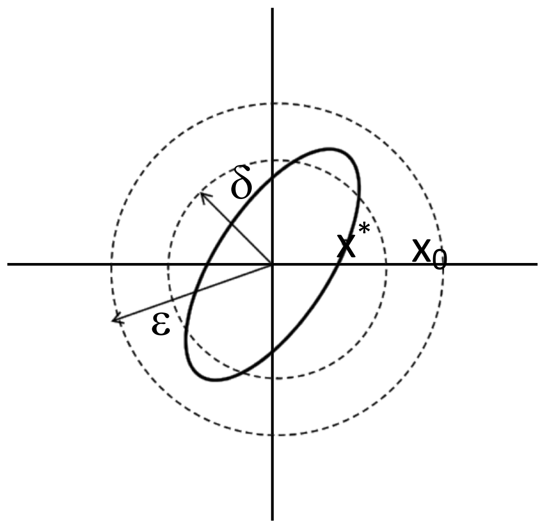
- **Definition**: An equilibrium point $\mathbf{x}^*$ is said to be **stable** if, given any real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that all solutions $\mathbf{y}(t)$ satisfy

$$|\mathbf{y}(t) - \mathbf{x}^*| < \epsilon \text{ for all } t \geq 0 \text{ whenever } |\mathbf{y}(0) - \mathbf{x}^*| < \delta$$

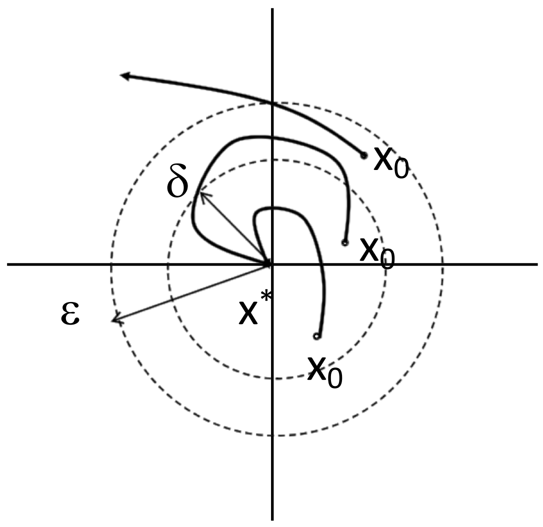
- Otherwise it is **unstable**

- **Definition**: An equilibrium point $\mathbf{x}^*$ is said to be **asymptotically stable** if it is stable and there is a real number $b > 0$ such that

$$\lim_{t\to\infty} |\mathbf{y}(t) - \mathbf{x}^*| = 0 \text{ whenever } |\mathbf{y}(0) - \mathbf{x}^*| < b$$




A stable equilibrium




An asymptotically stable equilibrium

- The solution cannot escape from a stable equilibrium
- The solution ends up ‘going down the hole’ if it starts close enough to an asymptotically stable equilibrium

# Epsilon-delta arguments

- The arguments in the definitions are in the form of a game
  - I give you a positive number $\varepsilon$ that I am free to choose
  - You respond with another number $\delta$ that satisfies some condition
  - If you can always find a number $\delta$ then you 'win'
- Many proofs in mathematics are based on this terminology

# Exponential stability

- **Definition**: An equilibrium point $\mathbf{x}^*$ is said to be exponentially stable if $\mathbf{x}^*$ is asymptotically stable and there exist finite $\alpha$, $\beta$, $\delta > 0$ such that if $|\mathbf{y}(0) - \mathbf{x}^*| < \delta$ then

$$|\mathbf{y}(t) - \mathbf{x}^*| \leq \alpha e^{-\beta t}\ |\mathbf{y}(0) - \mathbf{x}^*| \text{ for all } t \geq 0$$

- The above is a statement about the **rate** of convergence of the solution to the equilibrium point – how ‘fast’ it ‘goes down the hole’.

# Flows in 2x2 linear systems

- We will be studying the flows around equilibrium points using their local linearisations
- Each flow has a topology (shape) that falls into one of a number of distinct categories
- These flows can often be continuously distorted into the flows that solve the non-linear differential equations – so the topologies of a number of exemplar linear systems can provide families to which the non-linear solutions belong

# The uncoupled case

$$\dot{x}_1 = \alpha_1 x_1$$
$$\dot{x}_2 = \alpha_2 x_2$$

Solution:

$$x_1(t) = e^{\alpha_1 t} x_1(0)$$
$$x_2(t) = e^{\alpha_2 t} x_2(0)$$

These are parametric equations of curves in phase space

The equation of any such curve is of the form

$$x_1 = c {x_2}^{\alpha_1/\alpha_2}$$

# The uncoupled case

$$x_1(t) = e^{\alpha_1 t} x_1(0)$$
$$x_2(t) = e^{\alpha_2 t} x_2(0)$$

- The equilibrium point is the origin (it is a linear autonomous system)
- If $\alpha_1 < 0$ and $\alpha_2 < 0$, then the equilibrium is asymptotically (exponentially) stable
- If $\alpha_1 > 0$ or $\alpha_2 > 0$ the origin is unstable

# Coupled 2x2 linear systems

- The general form

$$\dot{x}_1 = ax_1 + bx_2$$
$$\dot{x}_2 = cx_1 + dx_2$$

Or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$
$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Solution

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0)$$

# The eigenvalues

The eigenvalues of **A** satisfy

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

or

$$\lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$$

The trace $\tau = \mathrm{tr}(\mathbf{A})$ of a matrix **A** is the sum of its eigenvalues:

$$\lambda_1 + \lambda_2 = \tau$$

the determinant $D = \det(\mathbf{A})$ of **A** is the product of its eigenvalues:

$$\lambda_1 \lambda_2 = D$$

# Solving using eigenvectors and values

- If the eigenvalues are real and distinct, then are two independent eigenvectors and we can re-write the initial condition as

$$\mathbf{x}(0) \; = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

Now work through the solution:

$$\begin{aligned}\mathbf{x}(t) &= \; e^{t\mathbf{A}}\mathbf{x}(0)\\ &= \left(\mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2\right)(c_1\mathbf{v}_1 + c_2\mathbf{v}_2)\\ &= c_1 e^{t\lambda_1}\mathbf{v}_1 + c_2 e^{t\lambda_2}\mathbf{v}_2\end{aligned}$$

(we have assumed neither eigenvalue is zero)

# Some characteristics

- If $\mathrm{Re}(\lambda) < 0$, then the component along the corresponding eigenvector decays to zero
- If $\mathrm{Re}(\lambda) > 0$, then the component along the corresponding eigenvector grows without bound
- If $\lambda = 0$, then the component along the corresponding eigenvector is constant
- If $\mathrm{Im}(\lambda) \neq 0$, then the solution spirals around the origin
- If $\mathrm{Im}(\lambda) = 0$, then the solution does not spiral around the origin
- If $\lambda$ is real, the solution will tend towards the eigenvector with the dominant eigenvalue

# Coordinate transformation

- We next discuss the characteristic shapes of trajectories in 2-D phase space using a coordinate transformation so that the matrix **A** is in a standard form:
  - Given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, define new coordinates $\mathbf{y} = \mathbf{V}^{-1}\mathbf{x}$
  - The new equations of motion are then $\dot{\mathbf{y}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}\mathbf{y}$
  - If the eigenvalues are real and distinct and **V** is the eigenvector matrix then $\dot{\mathbf{y}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\mathbf{y}$
  - Study the motion in the new coordinates. Motion in the original coordinates is obtained by multiplying $\mathbf{y}(t)$ by **V**
- The transformed matrix $\mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ is called the '**normal form**'

# The Saddle normal form

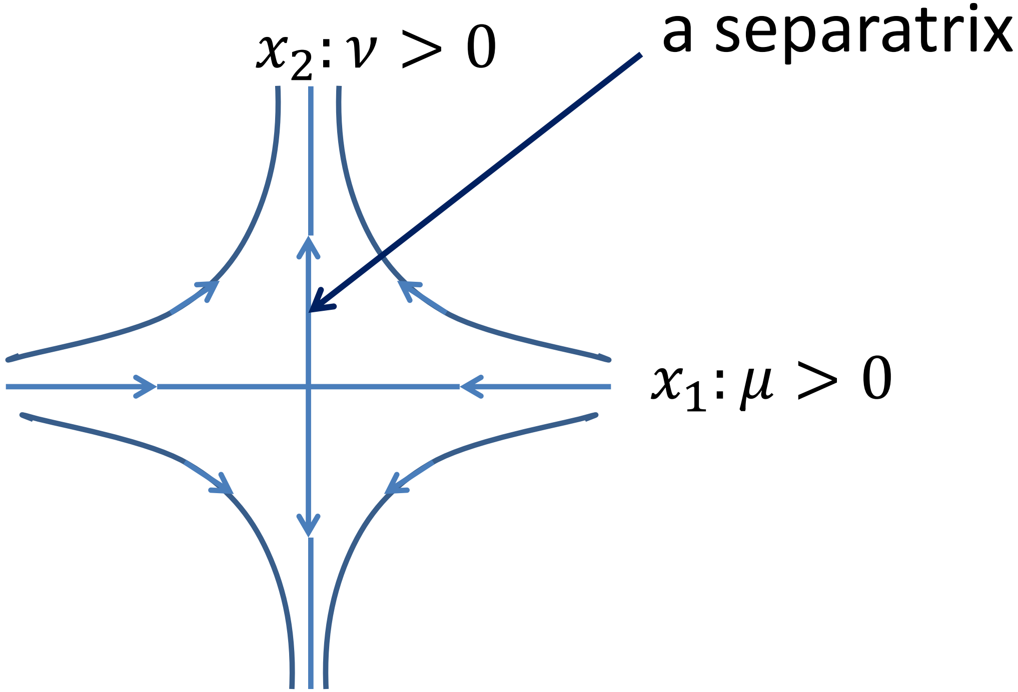
- If the eigenvalues are real and non-zero and $\lambda_1\lambda_2 < 0$, then the equilibrium is stable. In the transformed coordinates:

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \mu & 0 \\ 0 & \upsilon \end{bmatrix}$$

- There are four asymptotes that approach the origin, two are for $t \to \infty$ and two are for $t \to -\infty$

- These four non-zero trajectories are called **separatrices**

# The Saddle shape



# The Stable Node

- If the eigenvalues are real and both negative $\lambda_1 < 0, \lambda_2 < 0$, then
  - the eigenvalues are distinct:

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \mu & 0 \\ 0 & \nu \end{bmatrix}$$

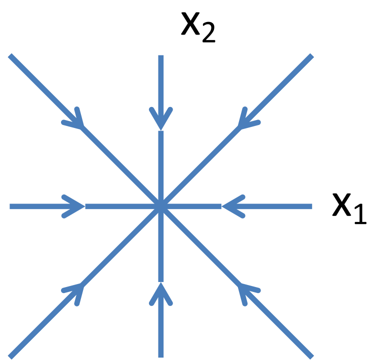
  - or $\mathbf{A}$ is diagonal: $\mathbf{A} = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}$
  - or the eigenvalues are equal, but there is only one eigenvector (the degenerate case):

$$\widetilde{\mathbf{V}}^{-1}\mathbf{A}\widetilde{\mathbf{V}} = \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix}$$

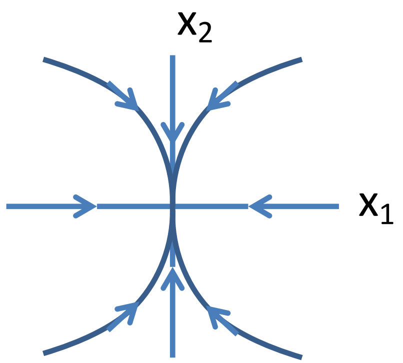
Here $\widetilde{\mathbf{V}}$ consists of 'generalized eigenvectors' (see Perko Section 1.3)

# Stable and unstable node shapes

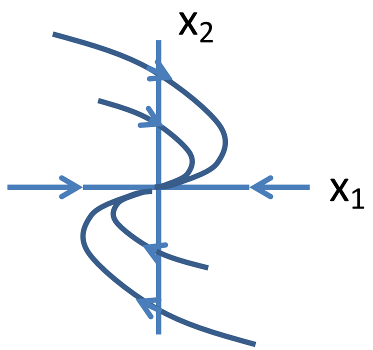
- Stable:




Diagonal matrix




Generic case $\mu < \nu$




Degenerate case

- Unstable: Both eigenvalues are positive. The same shapes as above, but reverse the directions of the arrows.

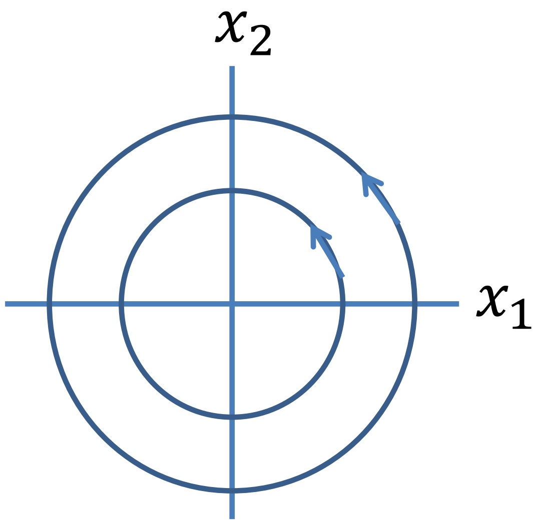
# The Centre Normal Form

- If the eigenvalues are purely imaginary (the real part is zero and the imaginary part is not zero). The equilibrium is neutrally stable
- There are no real eigenvectors and so use the construction from Lecture 1 to find the coordinate transformation matrix

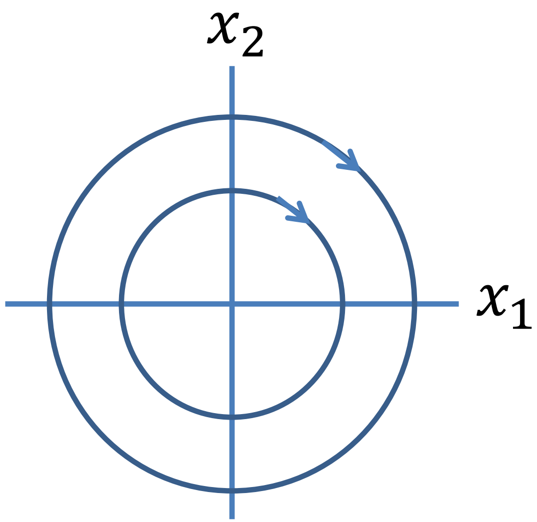
$$\widetilde{\mathbf{V}}^{-1}\mathbf{A}\widetilde{\mathbf{V}} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$$

- Matrix $\widetilde{\mathbf{V}}$ is constructed from the real and imaginary parts of the complex eigenvectors

# Centre Normal form Phase Space Portrait




$b > 0$




$b < 0$

# Stable and unstable spirals (foci)

The normal form of the matrix $\mathbf{A}$ is:

$$\mathbf{A} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\lambda_1 = a + jb, \qquad \lambda_2 = a - jb,$$

- $a < 0$: the spiral is stable
- $a > 0$: the spiral is unstable
- $b > 0$: the spiral is anti-clockwise
- $b < 0$: the spiral is clockwise

# Spirals or foci

$b > 0$ $\qquad\qquad$ $b < 0$

Stable focus

Unstable focus

# The Phase Space equations so far

- We have drawn the diagrams of the solutions of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$:
  - $\dot{\mathbf{x}} = \begin{bmatrix} \mu & 0 \\ 0 & \upsilon \end{bmatrix} \mathbf{x} \Rightarrow \mathbf{x}(t) = \begin{bmatrix} e^{\mu t} & 0 \\ 0 & e^{\upsilon t} \end{bmatrix} \mathbf{x}_0$
  - $\dot{\mathbf{x}} = \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix} \mathbf{x} \Rightarrow \mathbf{x}(t) = e^{\mu t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0$
  - $\dot{\mathbf{x}} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mathbf{x} \Rightarrow \mathbf{x}(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{x}_0$

# Changing coordinates

- Consider $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \mathbf{x}$

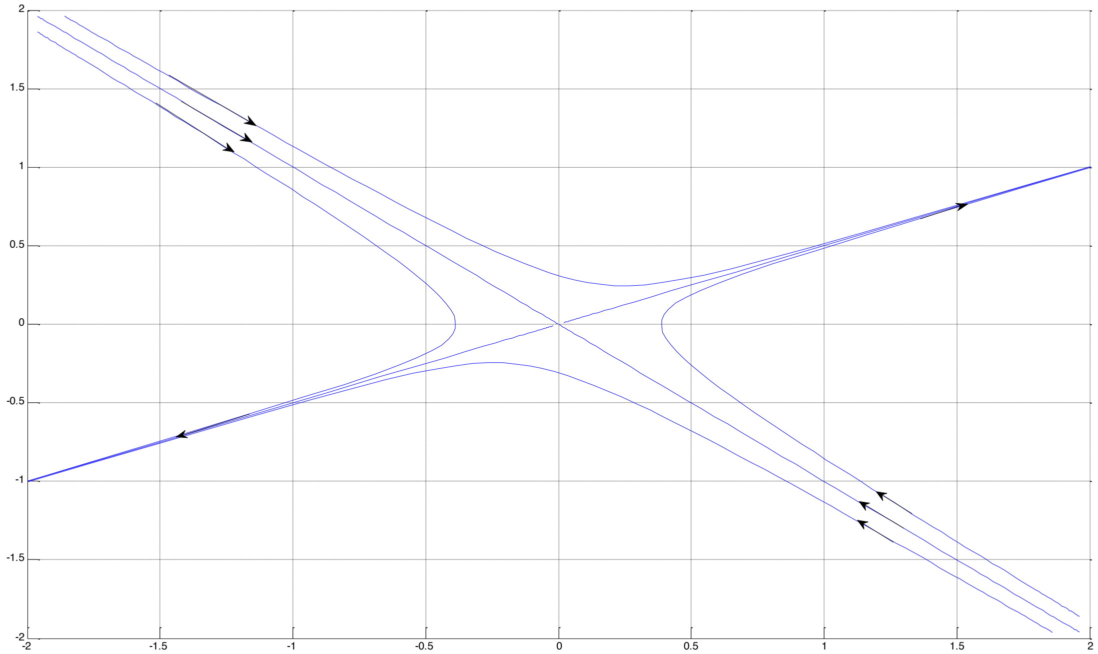
- Characteristic equation: $\lambda^2 + \lambda - 2 = 0$, eigenvalues: 1, -2 eigenvectors: $[2 \ \ 1]^T$ and $[-1 \ \ 1]^T$.

- Thus $\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$ i.e. $\mathbf{A} = \mathbf{V\Lambda V}^{-1}$

- Define new coordinates $\mathbf{y} = \mathbf{V}^{-1}\mathbf{x}$ defining new axes along $[2 \ \ 1]^T$ ($\lambda$=1 and thus unstable) and $[-1 \ \ 1]^T$ ($\lambda$=-2 and thus stable) .

$$\mathbf{x}(t) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-\mathbf{1}} \mathbf{x}_0$$

$$= \begin{bmatrix} 2/3\, e^{t} + 1/3\, e^{-2t} & 2/3\, e^{t} - 2/3\, e^{-2t} \\ 1/3\, e^{t} - 1/3\, e^{-2t} & 1/3\, e^{t} + 2/3\, e^{-2t} \end{bmatrix} \mathbf{x}_0$$



# Nonlinear systems

Consider the solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ near to where $\mathbf{f}(\mathbf{x}^*) = 0$.

Let $\mathbf{x} = \mathbf{x}^* + \mathbf{w}$, assume **f** is differentiable and note that the equilibrium point does not change with time. Taylor expansion of $f_j$ about $\mathbf{x}^*$:

$$f_j(\mathbf{x}^* + \mathbf{w}) = f_j(\mathbf{x}^*) + \sum_{k=1}^{n} \frac{\partial f_j}{\partial x_k} w_k + \mathcal{O}(|\mathbf{w}|^2)$$

For the entire vector of functions $\mathbf{f}$, also noting that $\mathbf{f}(\mathbf{x}^*) = 0$:

$$\dot{\mathbf{x}} = \dot{\mathbf{w}} = D\{\mathbf{f}(\mathbf{x}^*)\}\mathbf{w} + \mathcal{O}(|\mathbf{w}|^2)$$

$D\mathbf{f}$ is the Jacobian whose $(i, j)$th element is $\frac{\partial f_i}{\partial x_j}$

# Small perturbation model

Close to the equilibrium $\dot{\mathbf{w}} \cong D\{\mathbf{f}(\mathbf{x}^*)\}\mathbf{w}$ and is approximated by the linear system

$$\dot{\mathbf{w}} = D\{\mathbf{f}(\mathbf{x}^*)\}\mathbf{w} = \mathbf{A}\mathbf{w}$$

We consider the stability of this linear system near to the origin (**w**=0). Note we then have

$$\mathbf{w}(t) = e^{\mathbf{A}t}\mathbf{w}(\mathbf{0}) = e^{D\mathbf{f}(\mathbf{x}^*)t}\mathbf{w}(\mathbf{0})$$

The solution is asymptotically stable if all the eigenvalues of **A** have negative real parts.

# Hyperbolic equilibria

- **Definition** (Hyperbolic equilibrium): Let $\mathbf{x}^*$ be an equilibrium of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Then $\mathbf{x}^*$ is called a hyperbolic fixed point if none of the eigenvalues of $D\mathbf{f}(\mathbf{x}^*)$ real part equal to 0.

- **Theorem**: Suppose $\mathbf{x}^*$ is a hyperbolic fixed point and all the real parts of the eigenvalues are negative. Then the equilibrium solution $\mathbf{x} = \mathbf{x}^*$ is asymptotically stable.

# The properties of hyperbolic equilibria

- If the eigenvalues of the local linearisation all have negative real parts, then we have asymptotic stability of the non-linear system at that equilibrium. Note we say nothing about how big the region of stability is – this depends on your number $\delta$ that you responded with during the ‘game’.
- The properties of the solution of the non-linear system near to a hyperbolic equilibrium point are similar in shape to the linearised system – they are topologically identical (covered in Lecture 3).
- If any of the eigenvalues have zero real part, further investigation is needed.

# The Duffing Oscillator example

$$\dot{x} = y$$
$$\dot{y} = x - x^3 - \gamma y, \qquad \gamma \geq 0$$

Equilibria: $(x^*, y^*) = (0,0)$ and $(\pm 1,0)$

Jacobian: $D\mathbf{f} = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -\gamma \end{bmatrix}$

At $(0,0)$: $\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2+4}}{2}$ so the equilibrium point is unstable

At $(\pm 1,0)$: $\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2-8}}{2}$ so for $\gamma > 0$ we have asymptotic stability

**but** for $\gamma = 0$ we have a centre in this case the local behaviour of the linearization and the nonlinear system are different!