Lecture 6: Limit Cycles and Index Theory

- In this lecture we consider limit cycles in detail.

- We exclude separatrix cycles (in particular homoclinic and heteroclinic connections connecting α to ω points – going from an α to an ω point takes an infinite time – not exactly periodic!)

- **Definition**: A solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ through $\mathbf{x}_0$ is said to be periodic if there exists a $T > 0$ such that $\phi(t, \mathbf{x}_0) = \phi(t + T, \mathbf{x}_0)$ for all $t \in \mathbb{R}$.

  The minimum such T is called the period of the periodic orbit.

Proving a periodic orbit does not exist

We may wish to prove that a particular second order dynamical system does not posses periodic orbits.

Bendixson's criterion: Let $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$.

If $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is **not identically zero** and **does not change sign** within a simply connected region D of the phase plane, then the system has **no closed orbits** in D.

Outline of proof

- Assume a closed orbit $\Gamma \subset D$ exists. Then the orbit is a parametric curve in t such that $\frac{dy}{dx} = \frac{g}{f}$.
- So on a closed orbit Γ we get $f dy = g dx \;\Rightarrow\; \oint_{\Gamma} (g dx - f dy) = 0$
- Using Stoke's theorem we then have

$$\int_{S} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dx\, dy = 0$$

  where $S \subset D$ is the region enclosed by Γ
- But if $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is never zero, then it must have the same sign all over D and so the integral cannot be zero.

Dulac's criterion

We consider the same differential equations but now allow both f and g to be multiplied by another function B.

Dulac's criterion: Let $B(x, y)$ be a continuously differentiable function, defined on a region $D \subset \mathbb{R}^2$.

If $\frac{\partial(Bf)}{\partial x} + \frac{\partial(Bg)}{\partial y}$ is not identically zero and does not change sign in D, then the system $\dot{x} = f(x, y), \dot{y} = g(x, y)$ has no closed orbits in D.

Example using Bendixson's criterion

Let

$$\dot{x} = y \stackrel{\text{def}}{=} f(x, y)$$
$$\dot{y} = x - x^3 - \gamma y \stackrel{\text{def}}{=} g(x, y), \qquad \gamma \geq 0$$

Then

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -\gamma$$

Thus for $\gamma \neq 0$ there are no closed orbits.

For $\gamma = 0$, there is an energy function $\frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$ and the system is Hamiltonian, so we can study the trajectories using level sets (see Lec. 4).

Second example

Let

$$\dot{x} = y \stackrel{\text{def}}{=} f(x,y)$$
$$\dot{y} = x - x^3 - \gamma y + x^2 y \stackrel{\text{def}}{=} g(x,y), \qquad \gamma \geq 0$$

then

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -\gamma + x^2.$$

Linearisation shows that $(0,0)$ is a saddle, and $(\pm 1,0)$ are stable nodes for $\gamma > 1$ and unstable nodes for $0 \leq \gamma < 1$.

There can be no closed orbits **within** regions where x is very large or very small compared to γ, but orbits may pass through these regions – we are thus undecided using Bendixson.

Gradient Systems

- Equations are such that $\dot{\mathbf{x}} = -\nabla V$
- Such systems **cannot** have closed orbits since

$$\dot{V} = \nabla V . \dot{\mathbf{x}} = -\nabla V . \nabla V = -|\dot{\mathbf{x}}|^2$$

implies that, if $\mathbf{x}(t)$ is on a closed orbit of period T, then

$$V\big(\mathbf{x}(T)\big) - V\big(\mathbf{x}(0)\big) = -\int_0^T |\dot{\mathbf{x}}|^2 dt \neq 0$$

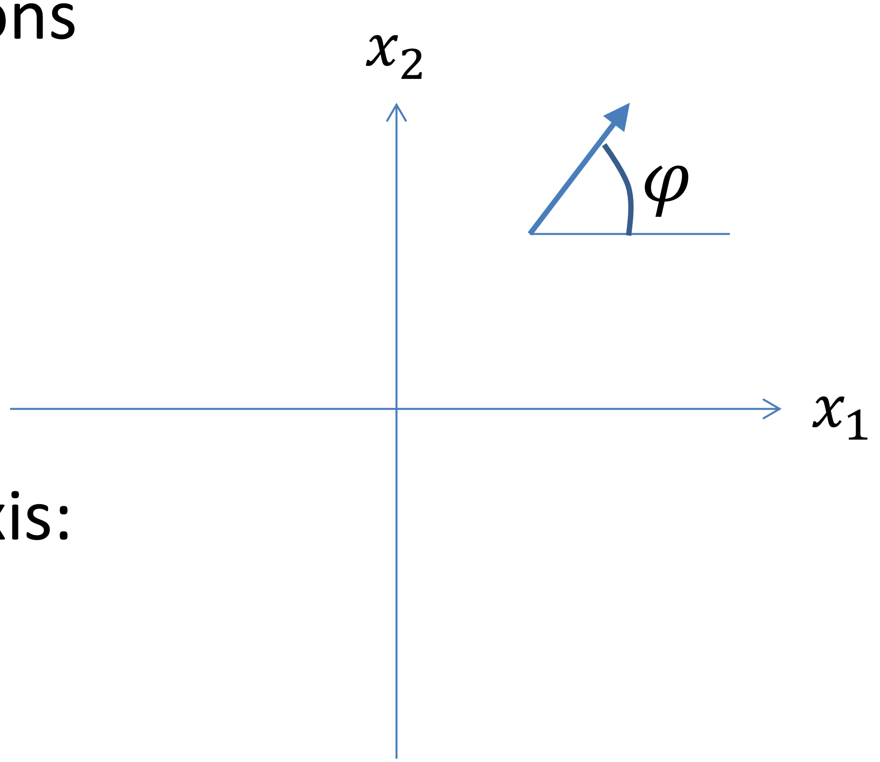
But on a closed orbit we must have $V\big(\mathbf{x}(T)\big) = V\big(\mathbf{x}(0)\big)$ implying $|\dot{\mathbf{x}}| = 0$ so $\mathbf{x}(0)$ must be a fixed point, not an orbit

Index Theory

For two-dimensional systems the equations

$$\dot{x}_1 = f(x_1, x_2)$$
$$\dot{x}_2 = g(x_1, x_2)$$

define a vector field or flow.



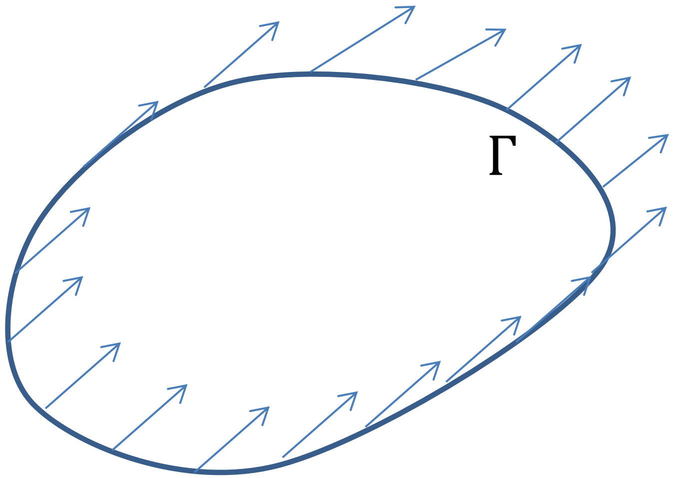
These vectors make an angle to the x_1-axis:

$$\varphi(x_1, x_2) = \tan^{-1}\left(\frac{g(x_1, x_2)}{f(x_1, x_2)}\right)$$

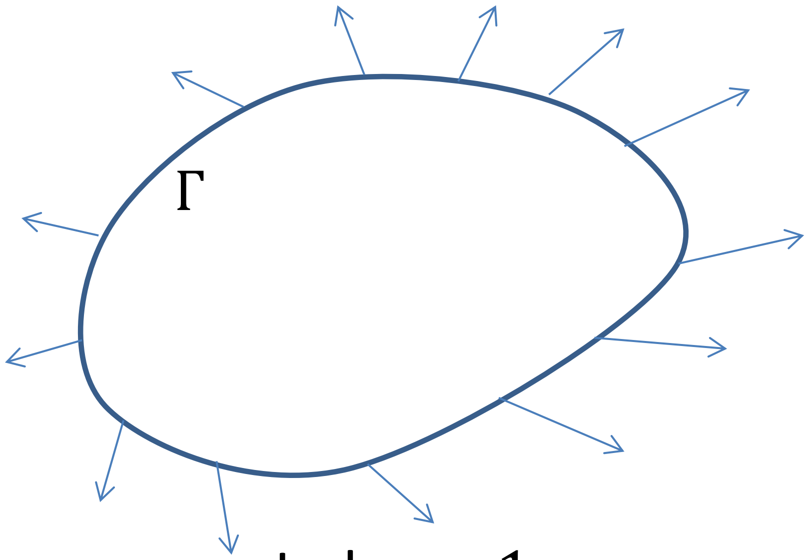
Index Theory

The index of a (non-intersecting, simple) curve Γ, $I(\Gamma)$ is defined by

$$I(\Gamma) = \frac{1}{2\pi} \oint_{\Gamma} d\varphi(x_1, x_2)$$



Index $= 0$



Index $= 1$

Follow the curve Γ around anti-clockwise and measure how many times the vectors on the curve rotate anti-clockwise during one rotation.

Properties of Indices

- The index is an integer (you must rotate by a multiple of 2π to get back to the start).
- If there are no equilibria inside Γ, then the index is zero ($I(\Gamma) = 0$).
- If Γ coincides with a closed orbit, then the index is 1.
- If Γ encloses an isolated saddle equilibrium point, then the index is -1. If Γ encloses any other isolated equilibrium point then the index is 1.
- The index of a curve Γ enclosing multiple isolated equilibrium points is the sum of the indices of the individual equilibrium points enclosed.

Some observations

- Given a closed curve that does not enclose any equilibrium points – can this be a trajectory?

  No: if this curve is a closed orbit then its index is 1, but as there are no equilibrium points the curve has index 0.

- Can there be a closed trajectory surrounding a single saddle node?

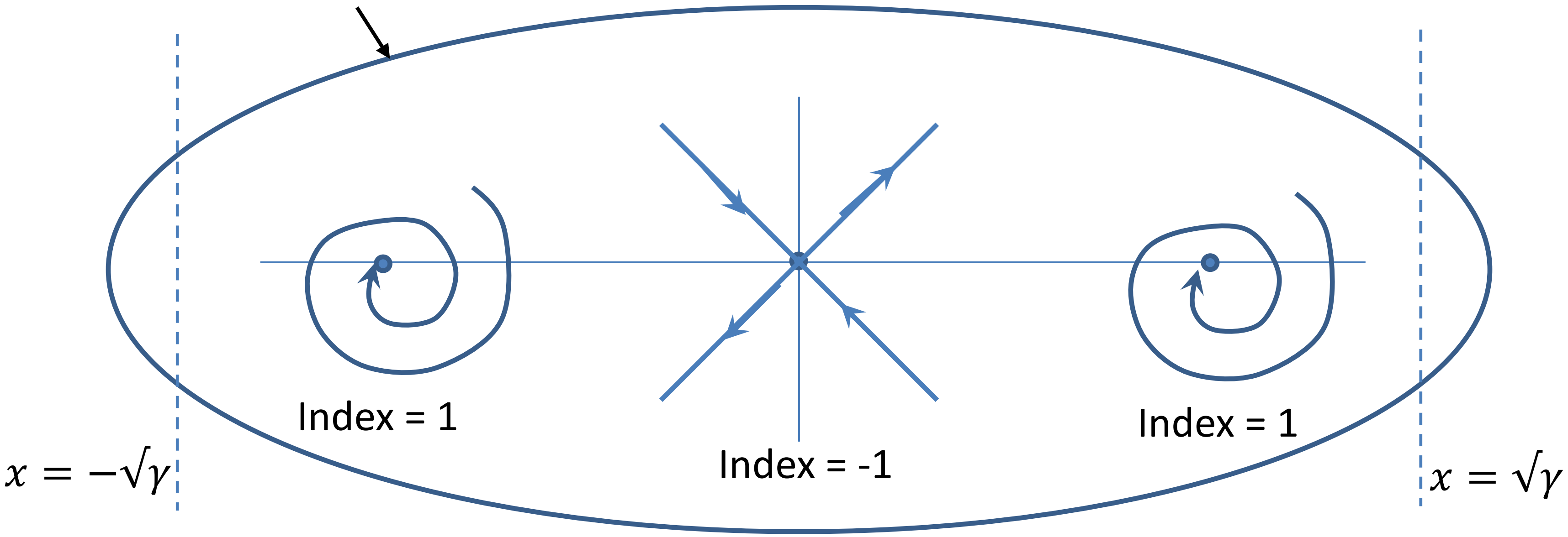
  No: the index of a saddle is -1 and that of a closed orbit is 1.

Return to Bendixson example

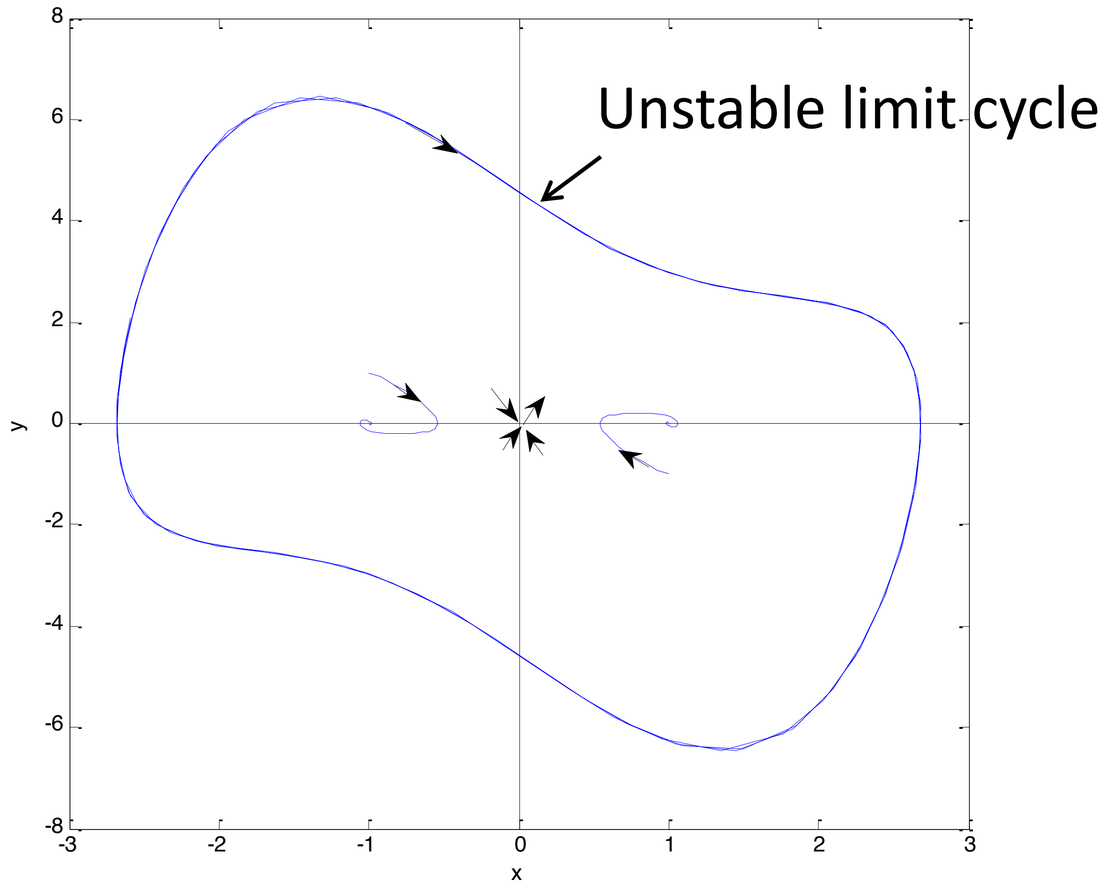
Earlier problem:

$$\dot{x} = y \stackrel{\text{def}}{=} f(x,y)$$
$$\dot{y} = x - x^3 - \gamma y + x^2 y \stackrel{\text{def}}{=} g(x,y), \qquad \gamma \geq 0$$
$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -\gamma + x^2$$

Index of curve = 1-1+1=1
this is the only possible closed orbit



Matlab simulation $\gamma = 2$



Another example of reasoning

Consider the system

$$\dot{x}_1 = x_1(3 - x_1 - 2x_2)$$
$$\dot{x}_2 = x_2(2 - x_1 - x_2)$$

with Jacobian: $\begin{bmatrix} 3 - 2x_1 - 2x_2 & -2x_2 \\ -x_1 & 2 - x_1 - 2x_2 \end{bmatrix}$

Equilibrium points:

$\mathbf{y}_1^* = (0,0)$, Jacobian: $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, positive eigenvalues (unstable node) so $I(\mathbf{y}_1^*) = 1$

$\mathbf{y}_2^* = (0,2)$, Jacobian: $\begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$, negative eigenvalues (stable node) so $I(\mathbf{y}_2^*) = 1$

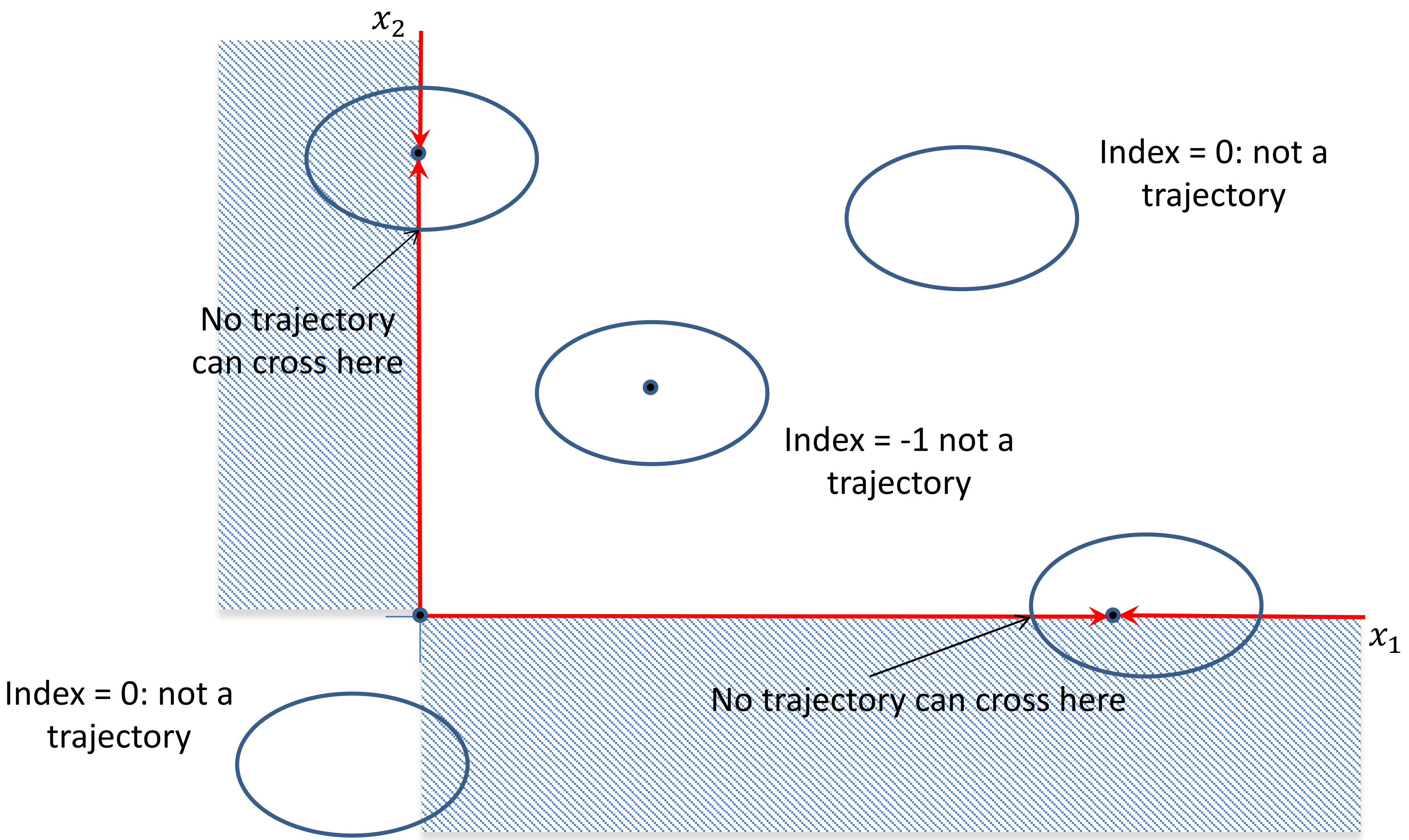
$\mathbf{y}_3^* = (3,0)$, Jacobian: $\begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}$, negative eigenvalues (stable node) so $I(\mathbf{y}_3^*) = 1$

$\mathbf{y}_4^* = (1,1)$, Jacobian: $\begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$, eigenvalues 0.414, -2.414 (saddle) so $I(\mathbf{y}_4^*) = -1$

Reasoning (continued)

- For closed trajectory we need an index of 1.
- Trajectories cannot cross (curves only meet at equilibrium points) and there are no equilibrium points in the 2nd, 3rd, or 4th quadrants.
- There are trajectories on the x_1- and x_2-axes, so no trajectory can cross into the 2nd, 3rd, or 4th quadrants. Hence these quadrants are free of any part of a closed trajectory. Thus $(0,0)$, $(0,2)$ and $(3,0)$ cannot lie inside a closed trajectory.
- The point $(1,1)$ is a saddle with index -1, so it cannot lie inside a closed trajectory.
- **Conclusion**: there are no closed trajectories.

Graphical illustration of reasoning



Limit Cycles

- Limit cycles are **isolated** periodic orbits that can be stable or unstable (a cycle around a linear centre is **not** isolated and thus not a limit cycle).
- In the plane a limit cycle is the α or ω limit set of some trajectory other than itself.
- **Definition**: A periodic orbit Γ is said to be **stable** if for every $\epsilon > 0$ there is a neighbourhood U of Γ such that for $\mathbf{x} \in U$ the distance between $\phi(t, \mathbf{x})$ and Γ is less than ϵ. Γ is called **asymptotically stable** if it is stable and, for all points $\mathbf{x} \in U$, this distance tends to zero as t tends to infinity.

Condition for stability

Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ have a periodic solution $\mathbf{x} = \gamma(t), 0 \leq t \leq T$, then the periodic orbit Γ lies on $\gamma(t)$.

- The periodic orbit is asymptotically stable only if

$$\int_0^T \nabla . \mathbf{f}\big(\gamma(t)\big) dt \leq 0$$

- For planar systems, if Γ is the ω limit set of all trajectories in the neighbourhood of Γ then it is a **stable** limit cycle.
- If it is the α limit set, then it is an **unstable** limit cycle.
- If it is the ω limit set for one trajectory and the α limit set for some other trajectory it is called a **semi-stable** limit cycle.

Limit cycle example

Consider

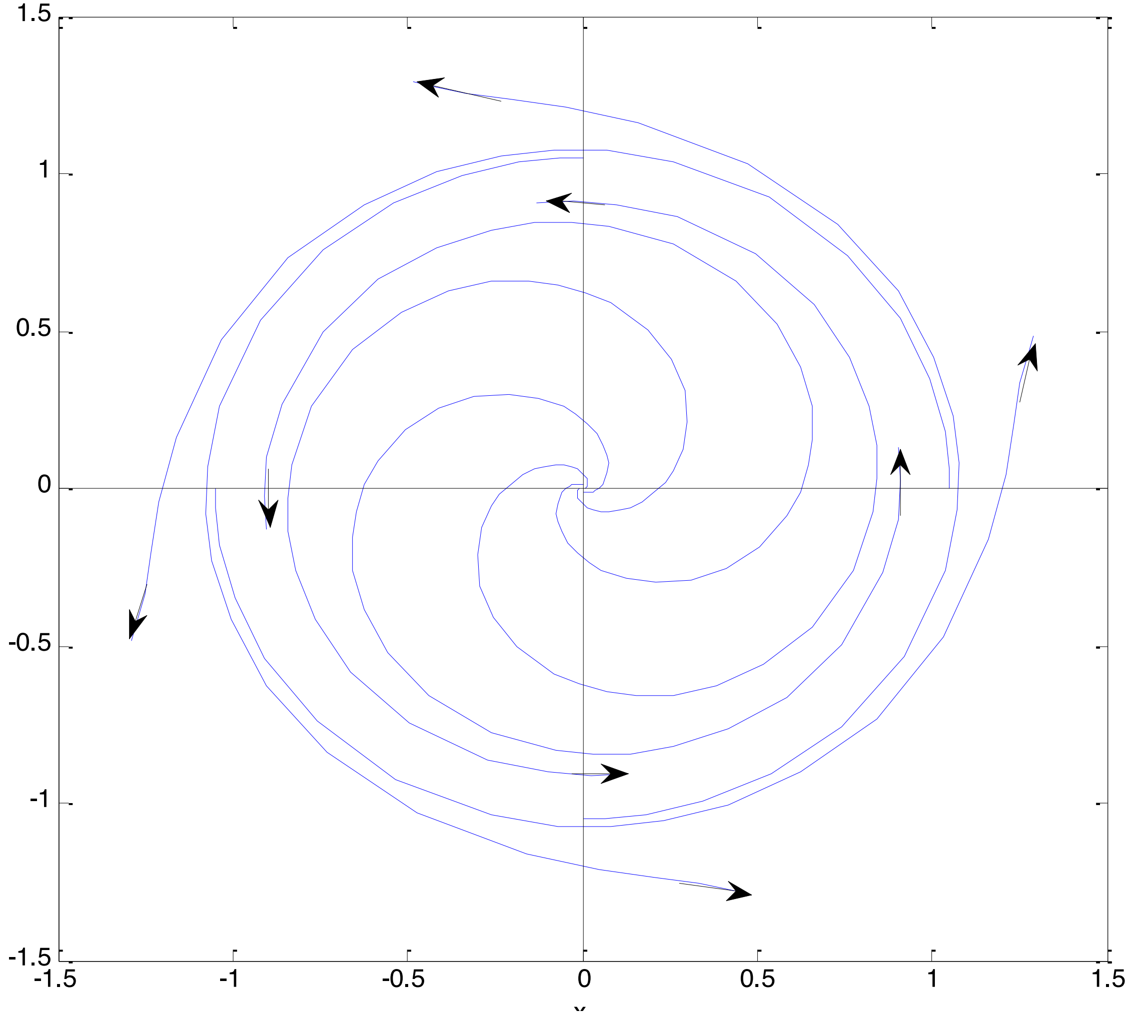
$$\dot{x} = -y + x(1 - x^2 - y^2)^2$$
$$\dot{y} = x + y(1 - x^2 - y^2)^2$$

In polar co-ords:

$$\dot{r} = r(1 - r^2)^2$$
$$\dot{\theta} = 1$$

For $r \neq 1, \dot{r} > 0$ and we spiral out.

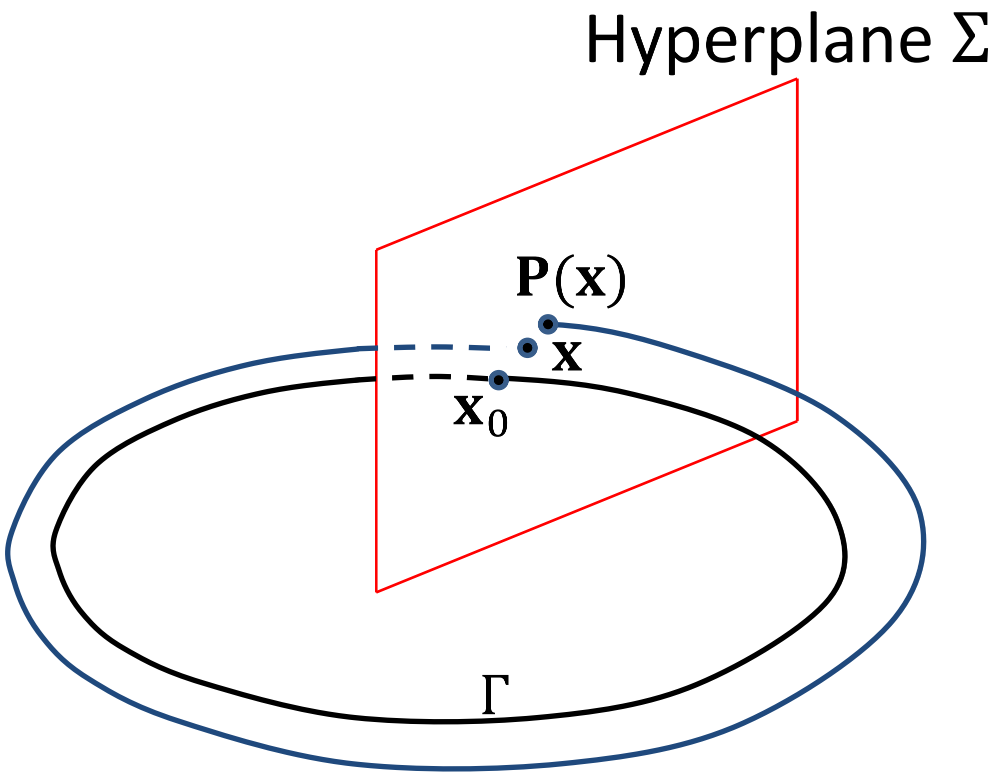
For $r = 1, \dot{r} = 0 \Rightarrow$ a limit cycle, which must be **semi-stable**.



The Poincaré Map

- An extremely important tool for the analysis of dynamical systems, sometimes called the 'return map'.
- Based on a hyperplane perpendicular to a periodic orbit.
- Consider points close to $\mathbf{x}_0$ on the orbit and where those points arrive back in the hyperplane after traversing the orbit. This defines a map
$$\mathbf{x} \mapsto \mathbf{P}(\mathbf{x})$$
- As the map is iterated, the intersection point $\mathbf{x}$ moves in the hyperplane.
- If we are on a periodic orbit then the return point is the original point.

The Poincaré Map



Example

$$\dot{x} = -y + x(1 - x^2 - y^2)$$
$$\dot{y} = x + y(1 - x^2 - y^2)$$

In polar co-ords:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$

This has stable limit cycle, which is an attractor for $\mathbb{R}^2 - \{0\}$ (see Lec. 5)

Solving the equations:

$$\int \frac{dr}{r(1 - r^2)} = \int dt \Rightarrow r = \frac{1}{\sqrt{1 - \left(\frac{1}{{r_0}^2} - 1\right) e^{-2t}}}$$

$$\theta = t + \theta_0$$

Poincaré Map

The hyperplane Σ is the ray $\theta = \theta_0$ through the origin that is crossed every 2π seconds.

Thus

$$P(r_0) = \frac{1}{\sqrt{1 - \left(\frac{1}{{r_0}^2} - 1\right) e^{-4\pi}}}$$

and $P(1) = 1$ (a fixed point).

Also

$$\left.\frac{dP}{dr}\right|_{r=1} = e^{-4\pi} < 1$$

so the limit cycle is stable.