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# INFORMATION EXTRACTION USING PRONY'S METHOD

E. K. Miller and D. L. Lager

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# INFORMATION EXTRACTION USING PRONY'S METHOD

## Abstract

Prony's method is a technique for determining the coefficients (poles and residues in the frequency domain) of an exponential series. Recently, applications of this method have been considered in electromagnetics because it offers the possibility of obtaining the SEM poles of conducting objects from their transient response. In addition, Prony's method offers insight into the general problem of extracting useful information from one-dimensional data.

In particular, Prony's method can be viewed as a transformation of data from one form to another. The amount of information extractable from transformed data can be as much as or less than that in the original data. Clearly, retention of the greatest possible amount of information in the transformed data is desirable. Thus, in order to evaluate the information-transformation process, one must determine the amount of information in the original data and in the transformed data.

This report considers these questions of information content and data transformation. To define information content, we use concepts of data rank (number of poles in the exponential data) and data precision. Then, to study data transformation, we define a waveform in terms of specified poles and residues, sample the waveform to obtain the associated data, and use Prony's method to transform the data and compare it with the original information content. Other numerical processes, such as matrix inversion, can be viewed in the same way. We show that problems of ill-conditioning are related more to the data than to the process. Finally, we briefly consider applications of Prony's method to other physical problems with exponential solutions.

## Introduction

Systems identification is defined in general as "the determination from experimental data of a set of unknown parameters in a mathematical model of a physical process."[1] One specific tool we have found useful in systems identification for electromagnetics data is Prony's method,[2] a technique for obtaining the parameters (exponents and coefficients) of a complex exponential series. Its use in electromagnetics was instigated by development of the singularity expansion method (SEM),[3] which depends on knowledge of the exponents or poles associated with a given object. Prony's method can be used to find the SEM poles from transient data,[4] and a frequency-domain counterpart similarly can be used for spectral data.[5] Actually, these techniques are useful not only for experimental data, as implied above, but also for analytical or numerical results. Thus they provide alternatives to procedures first used in SEM for finding the poles. Because so much of physics can be described via complex exponentials, potential applications of Prony's method are very widespread.

Although we have found Prony's method beneficial in electromagnetics and related areas, it has not been as well received in general.[6] Evidently this is because results obtained from its use can be quite variable, depending on the assumed rank of the data, the distribution of poles in the complex plane, the presence of noise in the data, etc. Mathematically, Prony's method is regarded as ill-conditioned because, in some circumstances, small changes in the input data produce large changes in the output. It is this aspect of Prony processing that we particularly want to discuss here. Our experience in its use leads us to view it more favorably than do others processing complex exponential data.

The purpose of this discussion is to present some initial observations and results based on various computer experiments we have performed. In the next section, we describe our approach and the kinds of computations we have done. Then we present typical results of these computations. A concluding section summarizes our findings to date and includes recommendations for future study.

## Approach

### INFORMATION-CONTENT ANALYSIS OF THE DATA

Our evaluation of Prony is from an information-theoretic viewpoint. That is, we attempt to characterize it as an information transformer that accepts information of one kind (a transient waveform, for example) as an input and provides other information (the extracted poles and residues) as an output. The specific transient of concern is

$$f(t) = \sum_{\alpha=1}^{N} R_\alpha e^{s_\alpha t} \quad ,$$

where $R_\alpha$ and $s_\alpha$ are the residues and poles, respectively. Because f(t) is real, $s_\alpha$ and $R_\alpha$ occur in conjugate pairs, so only 2N real numbers are required to characterize the data, N of these associated with the poles and N with the residues.

We emphasize that we define the term *information* here in the general sense of knowledge acquired from data rather than as defined by Shannon,[7] because of the difficulty in quantitatively assessing the information content of data on a probablistic basis. Specifically, we intend to determine how much information might be preserved or lost when Prony's method is used to process the data, and to judge its performance on this basis.

We could work in terms of the total information content of the input data, defined as

$$I_I \equiv \frac{N}{M} \sum_{i=1}^{M} D(f_i) \quad , \tag{1a}$$

where $D(f_i)$ is the number of correct digits in input sample $f_i$ and M is the number of data samples. We find it more convenient, however, to use instead the average number of correct digits per data sample as given by

$$\overline{D}(f) = \frac{1}{M} \sum_{i=1}^{M} D(f_i) \tag{1b}$$

to obtain a rank-independent measure of input information.

Either of these measures of input information might be suitable when the residues are all of comparable amplitude. When the residues are significantly different, however, so that they do not contribute equally to $f_i$, then Eq. (1) may not be appropriate. In that case, we might instead attempt to define $I_I$ or $\overline{D}(f)$ from the parameters that determine $f_i$ rather than from $f_i$ itself.

Similarly, we could work in terms of the total output information content of the extracted parameters, defined by

$$I_O \equiv \sum_{i=1}^{N} D(\tilde{P}_i), \tag{2a}$$

where $D(\tilde{P}_i)$ is the number of correct digits in the extracted parameter $\tilde{P}_i$, the set of which includes both poles and residues. But this also involves the rank of the system, as does Eq. (1a), so we use an alternate definition,

$$\overline{D}(\tilde{P}) = \frac{1}{N} \sum_{i=1}^{N} D(\tilde{P}_i) \quad , \tag{2b}$$

which is the average number of correct digits per parameter extracted. Since it is sometimes more useful to evaluate separately the pole and residue results, we also define

$$\overline{D}(\tilde{s}) \equiv \frac{2}{N} \sum_{i=1}^{N/2} D(\tilde{s}_i) \quad , \qquad (2c)$$

$$\overline{D}(\tilde{R}) \equiv \frac{2}{N} \sum_{i=1}^{N/2} D(\tilde{R}_i) \quad , \qquad (2d)$$

which are the average number of correct digits per pole and residue value extracted.

Finally, we might define the performance of Prony as an information processor as

$$P \equiv \frac{I_I - I_0}{N} = \overline{D}(f) - \overline{D}(\tilde{P}) \quad , \qquad (3)$$

which represents the average number of digits of information lost per parameter.

## EIGENVALUE ANALYSIS OF THE DATA

It is also instructive to examine the eigenvalue characteristics of the data as they appear in the process.[8] A matrix of data values $f_i$ is first obtained that, for M = 2N, has the form,

$$\overline{\overline{F}} = \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_N \\ f_2 & f_3 & f_4 & \cdots & f_{N+1} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ f_N & f_{N+1} & f_{N+2} & \cdots & f_{2N-1} \end{bmatrix} , \quad (4)$$

the solution of which yields the coefficients of a polynomial whose roots give the poles. One might expect, then, that the vector of data values $\overline{f}$ strongly influences the success of the overall process. In essence, the samples $f_i$ constitute a basis set that is to represent all the waveform f(t), including the part not sampled; i.e., the sample space must span the waveform space. Therefore, it should be possible somehow to test the data matrix $\overline{\overline{F}}$ for its suitability in this regard. This is where the eigenvalue procedure is applicable.

The eigenvalues $\lambda$ and eigenvectors $\overline{m}$ of a matrix $\overline{\overline{M}}$ are defined by the relation[9]

$$\overline{\overline{M}} \cdot \overline{m} = \lambda_i \overline{m}; \; i = 1, \ldots, N \quad , \qquad (5a)$$

where the $\lambda_i$ are solutions of the determinental equation

$$\overline{\overline{M}} - \lambda \overline{\overline{I}} = 0 \quad . \qquad (5b)$$

For the given matrix $\overline{\overline{M}}$, the eigenvectors $\overline{m}_i$ represent a basis set for expanding the rows of $\overline{\overline{M}}$, and the eigenvalues represent their corresponding contribution to that expansion. For example, dominance of the rows of $\overline{\overline{M}}$ by one function, an exponential in our case, would result in only one eigenvalue of significant magnitude. Thus, an eigenvalue analysis of the matrix $\overline{\overline{F}}$ can provide the test mentioned above.

## OUTLINE OF COMPUTER EXPERIMENTS

The above quantities have been evaluated for data obtained from a specified pole set as a function of the various parameters involved. A 10-pole set having the nominal parameters ($s = \pm i\omega + \sigma$)

| $\alpha$ | $\omega_\alpha$ | $\sigma_\alpha$ | $R_\alpha$ |
|---|---|---|---|
| 1 | 1.0 | -0.1 | 1.0 |
| 2 | 2.5 | -0.25 | 1.0 |
| 3 | 5.0 | -0.50 | 1.0 |
| 4 | 7.5 | -0.75 | 1.0 |
| 5 | 10.0 | -1.0 | 1.0 |

was employed. Among the parameters that can be varied are two distinct sets. The one associated with the Prony process itself is comprised of:

1) $\Delta t$ - the intersample spacing;
2) T - the total data window, which equals $(M - 1)\Delta t$;
3) $N_p$ - the number of poles used in the processor, which for a square system results in $N_p = M/2$.

The other, much larger set of parameters is associated with the data being processed and includes:

1) N - the number of poles in the data
2) $R_\alpha$ - the set of residues
3) $s_\alpha$ - the set of poles
4) $D(f_i)$ - the specified accuracy of the input data.

In the study being reported here, we have conducted a series of computer experiments involving changes in both sets of parameters. These are:

1) For the nominal pole set, use $\Delta t_N/3 \lesssim \Delta t \lesssim 2\Delta t_N$ for a fixed value of $D(f_i)$, where $\Delta t_N$ is the Nyquist-rate sample spacing.

2) For the $\Delta t$ that produces the best $\overline{D}(\tilde{P})$ from (1), systematically vary $D(f_i)$ over the range of word sizes permitted by the computer (~12-14 digits) by truncating the exact values of $f_i$ to obtain a fixed $D(f)$.

3) Repeat (2) for the set of residues varying as $R_\alpha = 10^{1-\alpha}$ and $R_\alpha = 10^{\alpha-5}$.

4) Using the same $\Delta t$ as (2), systematically vary $D(f_i)$ by truncating the exact values of $f_i$ at a fixed level relative to the maximum to obtain an effective signal-to-noise ratio of $D(f)_{max}$.

5) Repeat (4) as in (3).

6) Using the same $\Delta t$ as (2), systematically shift the $\omega_\alpha$ according to $\omega_\alpha = \omega_5 - \Delta\omega(5 - \alpha)$, and increase the data window T by increasing M while keeping $N_p = M/2$.

7) Repeat (4) and (5), using the sliding-window procedure.

These experiments were performed to obtain a systematic set of results that might provide some insight into the Prony process as an information transformer. Experiment 1 simply establishes a baseline behavior for the process, while experiment 2 evaluates its input-output characteristics over a wider range of the input data. Experiment 3 examines the effects of widely varying residues on the output, because previous experience has indicated this to be an important aspect of the effect of noise on Prony processing. Experiments 4 and 5 approximate the effects of noise on Prony's performance. Experiment 6 determines how the pole locations, relative to the origin and to each other, might affect the data sampling that produces the best results. Finally, in experiment 7 we quantitatively examine some aspects of iterative data processing using a succession of overlapping data windows.

Clearly, many other experiments could be considered as well. In particular, the more general effect of noise on Prony's performance is of great significance. Another area of concern is the effective data rank, and how the data might best be sampled. These issues require further detailed study. The motivation and justification for doing so, we feel, may be provided by the initial results presented below.

## Numerical Results

Some results obtained from experiment 1 are given in Figs. 1-3. The quantities $\overline{D}(\tilde{s})$ and $\overline{D}(\tilde{R})$ are shown as functions of $\Delta t$ in Fig. 1. They both increase monotonically to $D(f_i)$ as $\Delta t$ approaches $\Delta t_N$.* An explanation for this result may be deduced from Fig. 2, where the eigenvalues of the data matrix are plotted for three values of $\Delta t$. The ratio of largest to smallest eigenvalue is a minimum for $\Delta t = 0.29$, but very much larger for the smaller $\Delta t$ values, showing that the effective rank of the data is very dependent on the intersample spacing.

This dependence is demonstrated in a different way in Fig. 3, where $\overline{D}(\tilde{s})$ and $\overline{D}(\tilde{R})$ are plotted as functions of the eigenvalue ratio, $E_R$. In this case $E_R$, which is sometimes used as a condition number, provides a clear indication of the data's information content. The increase of $\overline{D}(\tilde{s})$ and $\overline{D}(\tilde{R})$ and the decrease of $E_R$ with increasing $\Delta t$ might be surmised to come from the increasing linear independency of the $f_i$ that results. The reason for the downturn of both curves that occurs with decreasing $E_R$ past the peak is unclear. Since the numerical procedure employed here is the same for all

*Actually, sampling with $\Delta t > \Delta t_N$ can continue to produce good results if the poles whose frequencies exceed one-half the sampling frequency can be unfolded.

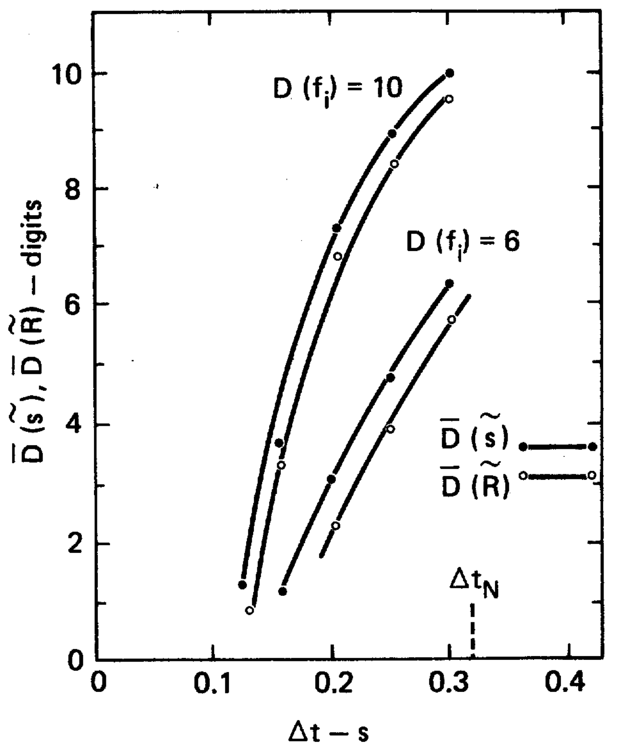


Fig. 1. The accuracy of Prony's method is very sensitive to the time interval $\Delta t$ between data samples, as illustrated here for noise-free data. We show the pole and residue accuracy as a function of $\Delta t$ for two values of $D(f_i)$. In this and all following cases, the solution is for a square system, where M = 2N.

$\Delta t$ used, and only the $f_i$ are changed, we conclude that in this case the data and not the Prony process is the limiting factor.

Upon selecting from experiment 1 $\Delta t = 0.95\ \Delta t_N$ as the optimum value for subsequent calculations, we find from experiment 2, where $D(f_i)$ is systematically varied, the results shown in Fig. 4. The relationship between input and output information content is essentially one-to-one. This outcome, which was somewhat surprising,

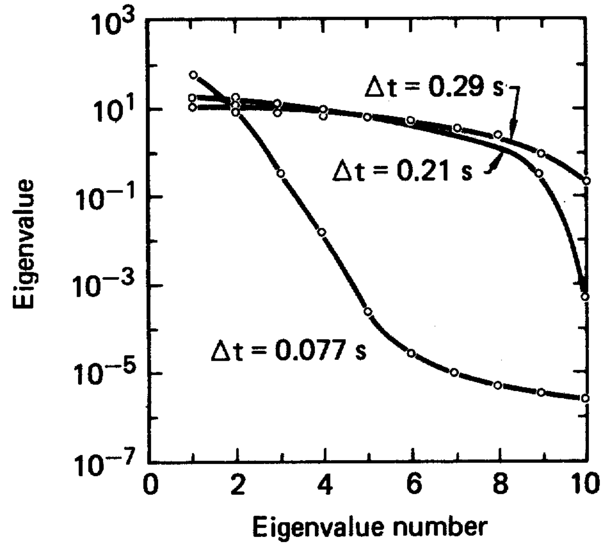


Fig. 2. Eigenvalues of the data matrix for three values of $\Delta t$ demonstrate the ill-conditioning of the data that can occur when the samples are too closely spaced.

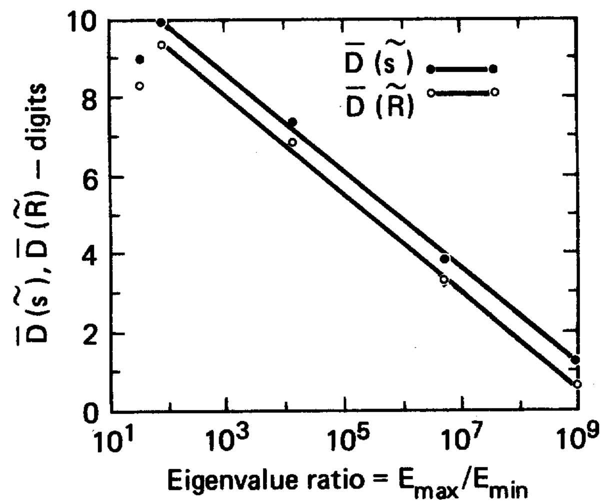


Fig. 3. The accuracy of Prony's method depends directly on the eigenvalue ratio.

indicates that Prony preserves the original information content of the sampled data in the transformed pole-residue output. Conversely, this result also implies that no further information could be extracted by, for

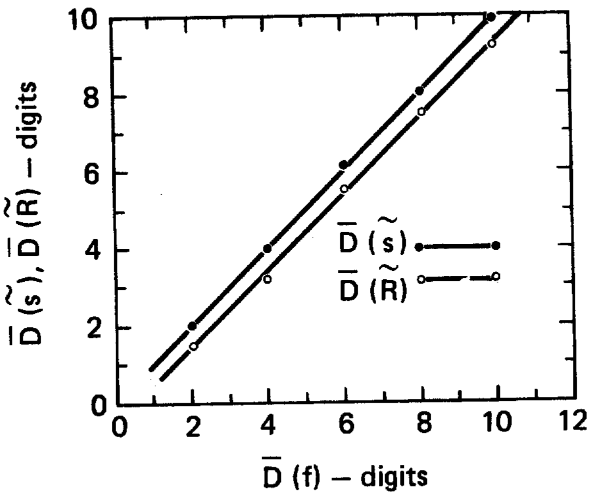


Fig. 4. In terms of pole and residue values, Prony's method can provide output information commensurate with input information. The pole values are typically obtained with greater accuracy than are the residue values, as these results indicate.

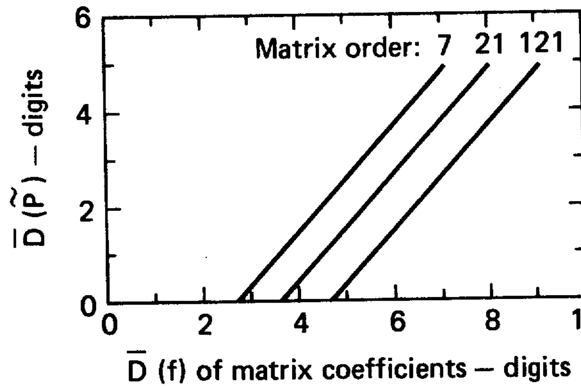


Fig. 5. By contrast to Prony's method, matrix inversion demonstrates a loss of information when evaluated in terms of pole and residue values.

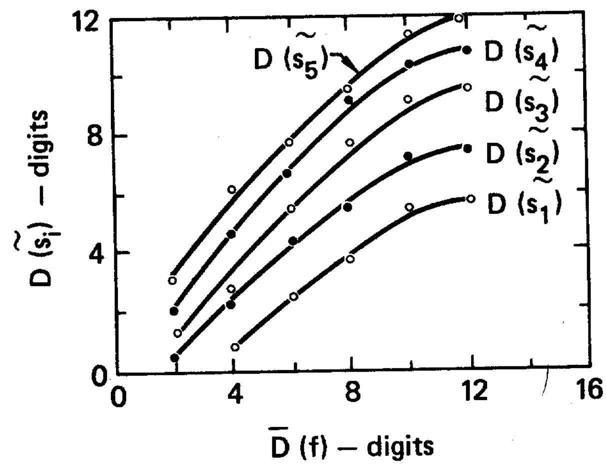


Fig. 6. The accuracy with which the poles are obtained is directly proportional to the amplitudes of their associated residues, as these results demonstrate for noise-free data.

example, using more data samples. By contrast, matrix inversion does not imply that, as shown in Fig. 5.[10]

Experiment 3, in which the $R_i$ differ by factors of 10, yields results like those in Fig. 6, where the individual $D(\tilde{s}_i)$ are shown as a function of $D(f_i)$. As expected from previous computations with noisy data, the poles of terms having the largest residues are computed most accurately. As a matter of fact, the decrease in relative accuracy is almost linearly dependent on the decrease in relative residue value.

The above results were obtained from, in effect, noise-free data. We now examine a more interesting and more realistic situation in which the data is contaminated in a noise-like way. To do this, we truncate the input $D(f_i)$ at a fixed point relative to the maximum. The result is a set of input samples whose accuracy is directly proportional to the amount by which they exceed the truncation point or "noise" level. The outcome of experiment 4, which depicts the influence of varying the truncation, or signal-to-noise, level for equal $R_i$ values, is shown in Fig. 7. We also include the result of experiment 2 there for comparison.

$\bar{D}(\tilde{P})$ is as linearly dependent on $\bar{D}(f)$ for the present case of a truncated $D(f_i)$ as it was for a constant value of $D(f_i)$. However, the present results are offset downward by an amount approximately equal to 3/4 digits in $\bar{D}(\tilde{P}_i)$. This apparent loss of information may be due to an inappropriate definition of $\bar{D}(f)$ used in this case. Recall that $D(f_i)$ is the number of correct digits in sample $f_i$. For this experiment, we have counted

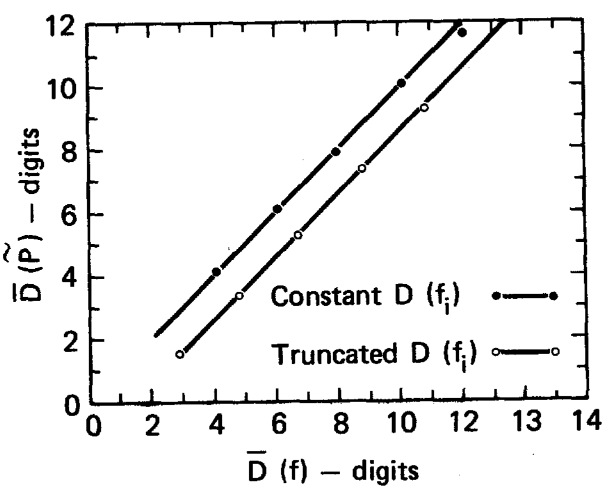


Fig. 7. The output information remains proportional to the input information when the data is noisy, as shown here. In this case, the noise was simulated by truncation of the input at a given level relative to the peak signal amplitude.

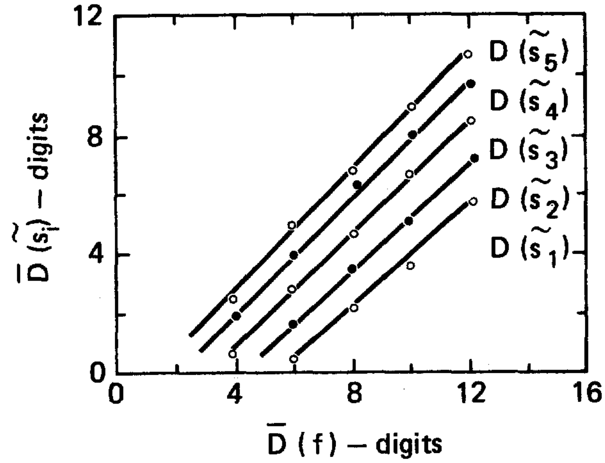


Fig. 8. The proportionality of the pole accuracy to residue amplitude is retained when noisy data like that of Fig. 7 is used.

all the digits that exceed the truncation level in computing $\bar{D}(f)$. But the information conveyed by a one-digit sample may be less than, for example, 1/Nth that of an N-digit sample. In other words, the utility of the data may decrease as the $D(f_i)$ are made unequal. Certainly, the information content of the $f_i$ cannot exceed that as defined here, but it could be significantly less.

Repetition of experiment 3 using truncated or noisy data as discussed above leads to the results shown in Fig. 8. These resemble the noise-free results of experiment 3 (Fig. 6), but are shifted downward in accuracy by about one digit. Again, we observe that the poles are lost in inverse order of their associated residue size. The minimum loss is approximately one digit for the largest residue pole and increases monotonically with decreasing residue value, which agrees approximately with the finding of experiment 4. Noise effects like these might be alleviated in several ways. A sliding-window technique (experiment 7) has been found useful, but requires further study to quantitatively assess it.

We now examine the effects on the Prony process of varying the imaginary components of the pole locations. This was done to assess the possible influence of the frequency separation of adjacent poles upon the data sampling required for achieving maximum accuracy in the extracted poles. One might anticipate, for example, that in addition to the Nyquist sampling criterion, which is dictated by the maximum frequency component in the waveform, there is also a sampling requirement related to some aspect of its minimum frequency behavior.

This is indeed the case, as demonstrated in Fig. 9. These results were obtained by fixing the frequency, $\omega_5$, of the highest-frequency pole, while varying the uniform interval $\Delta\omega$ among all the poles. The $\Delta t$ from experiment 2 was used throughout, and the data window, T, was systematically widened by increasing $M = 2N_p$. As the plots of Figs. 9-11 show, in order to achieve maximum accuracy, we must have $T \gtrsim 2\pi/\Delta\omega$; i.e., accuracy requires an observation time on the order of the minimum beat frequency. This is not surprising because, the closer together two poles are, the longer the

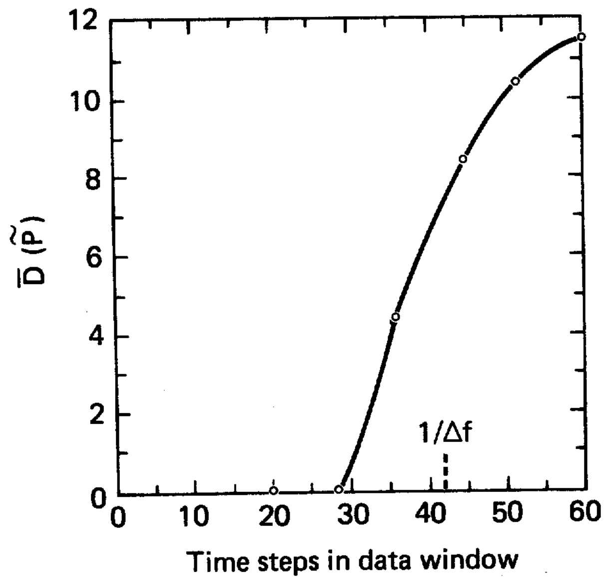


Fig. 9. When applied to exponential data, the accuracy of Prony's method is very sensitive not only to the interval between data samples, but also to the total data window available. As shown here, the data window must be comparable to the inverse beat frequency to obtain maximum accuracy.

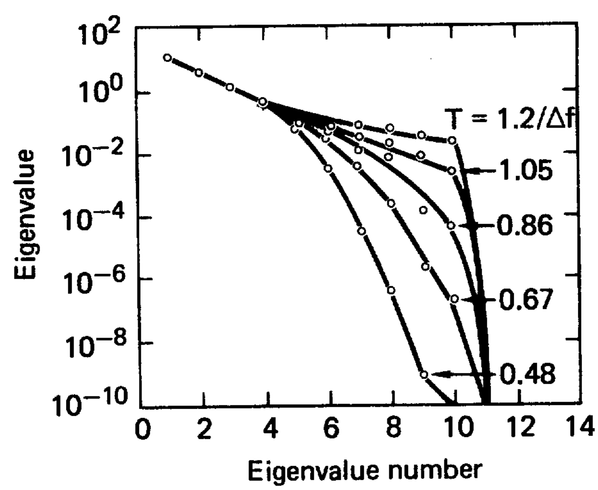


Fig. 10. The data window influences the range of eigenvalues of the data matrix.

waveform must be observed to resolve their frequency difference, a well-known result in other applications. We also found, however, that the minimum frequency, even when purely imaginary, has no such corresponding influence on T. The eigenvalue results are also useful in showing when the data window is wide enough, as shown in Figs. 10 and 11.

Because the sliding-window procedure has proven quite useful in reducing noise effects,[11] we conclude this section with application of the sliding window to data with white noise and a signal-to-noise ratio of 200. An example from this exercise is included in Fig. 12. A significant improvement in the pole values obtained from the noisy data occurs as a result of the sliding-window technique. By plotting the average pole values as a function of the number of iterations, we find a convergence towards the actual noise-free value, as demonstrated in Fig. 12.

The effect of the sliding window is essentially to average the data within a single waveform. Because the Prony process provides a model for the data, each separate application of Prony's method yields a smoothed representation of the waveform. A number, I, of such applications provides the equivalent of I individual representations for the same waveform, whose average leads to an improved model for the data. We have averaged here with respect to the poles, although other averages, such as the individual reconstructed waveforms, could also be used.

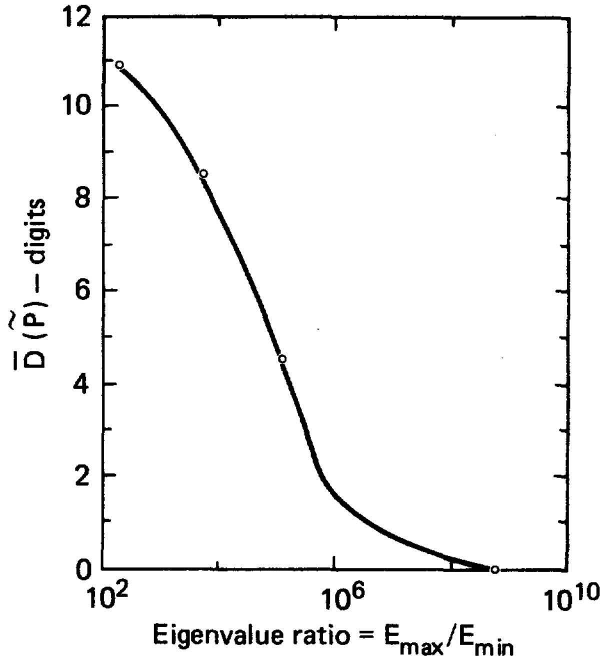


Fig. 11. The eigenvalue ratio indicates when the data window is adequate.

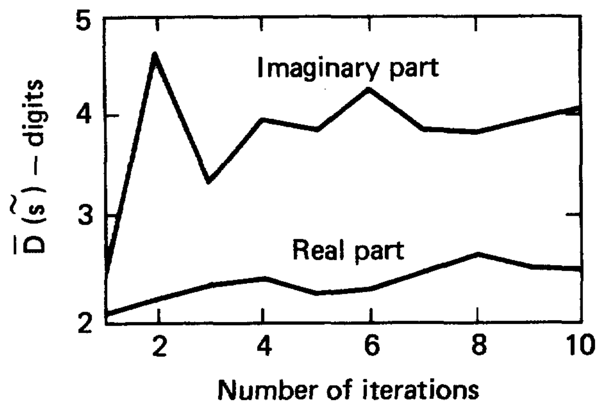


Fig. 12. Multiple processing can increase the accuracy of results obtained from noisy data. In the case shown here, the data window was moved one data-sample interval for each calculation.

It is important to mention that success of this technique depends very significantly on the data and how it is handled. First, it is clear that the method is best suited to data having uncorrelated noise. Second, it is very desirable that the data be oversampled, the more the better. This is because, when $\Delta t_N > \delta$, where $\delta$ is the sample interval of the data provided, only every second, or third, or ... data value is needed to implement Prony's method. Thus, when the data window is moved forward by $\delta$, a completely new set of samples is encountered. This can have the effect of decomposing the single measured waveform into two or more individual waveforms, which is like having two or more measurements. The benefit of this could not easily be attained without Prony's method, which, as mentioned above, permits averaging these separate waveforms extracted from one measurement. It is thus obviously to our benefit to oversample the minimum possible in using Prony's method but to oversample the maximum possible when taking the data.

Of course, even if every successive data point must be used because $\Delta t_N \sim \delta$, sliding the window is still worthwhile because of the new data value each iteration introduces. Clearly, one new data value cannot convey as much information as could 2N new values, but a sufficient number of iterations may ultimately be as effective. In this case, however, even when the noise is uncorrelated, the small change in the data vector from iteration to iteration can result in curve fitting and noise poles also forming clusters, although their variances will generally exceed those of the valid poles.[12]

## Observations and Conclusions

This study has systematically examined the performance of Prony's method as an information transformer. The input information $D(f_i)$ is the set of data samples of a complex exponential waveform, while the output information $D(\widetilde{P}_i)$ is the set of poles and residues extracted. Because Prony's method specifically, and the problem of fitting exponential data generally, have been considered mathematically ill-conditioned,[13] we have concentrated on evaluating Prony's method in terms of $D(f_i)$ measured relative to $D(\widetilde{P}_i)$. We were interested in finding the circumstances for which Prony worked best in this sense, and in understanding better what might cause it to fail.

Based upon results obtained to date as summarized above, we conclude first that Prony's method can indeed be made to appear ill-conditioned. But this outcome we further conclude is caused by the data rather than the processing technique. If, for example, the data is in any way underdetermined, Prony's performance will deteriorate. Underdetermined data can result from oversampling (too small a sampling interval), undersampling (too small a data window), or noise. In the first two cases, the problem is due to data of insufficient linear independency and inadequate rank. In the third case, reducing the information content per data sample can cause the problem. In all three cases, small changes in the data (or coefficients of the polynomial that gives the poles) can produce large changes in the poles.

Second, we conclude that Prony essentially can preserve the information content of data that is appropriately sampled and processed. Appropriate data sampling methods have already been mentioned. By appropriate processing, we mean the sliding-window technique and others that attempt to increase the total amount of information extractable from the data. Unambiguously defining $D(f_i)$ is difficult, so the above conclusion is somewhat qualitative at this stage.

Clearly, additional work is needed to further clarify these issues. Two main areas needing attention remain sampling strategies and noise effects. These are areas of general concern in systems identification, and their general resolution may not be realizable. But our sole interest is the handling of exponential phenomena, a specialization which can greatly reduce the breadth of the problem. Furthermore, we have the advantage of working in the context of a physical application, which provides a significant amount of insight and guidance into what could otherwise be a less productive mathematical exercise.

Finally, we want to emphasize that Prony's method is only one of many ways of treating exponential data, and that other techniques may be found to improve its performance. For now, however, it remains a leading candidate for treating the kinds of data which concern us.

## Acknowledgments

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