PhD notes

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January 2024

1 Summary of variations of different terms possible in a Lagrangian

Making sure to recall the Hodge operator rule: $\lambda \wedge \omega = \omega \wedge *\lambda$. The first result is

$$-\frac{1}{2}\delta\left(\mathrm{d}a_e \wedge *\mathrm{d}a_e\right) = \delta a_e \wedge \mathrm{d} * \mathrm{d}a_e - \mathrm{d}\left(\delta a_e \wedge *\mathrm{d}a_e\right) \tag{1}$$

as derived in appendix A. Next is the Abelian Chern-Simons term, which is derived in appendix B, gives the result

$$\delta\left(\frac{k}{4\pi}a_e \wedge \mathrm{d}a_e\right) = \frac{k}{2\pi}\delta a_e \wedge \mathrm{d}a_e + \frac{k}{4\pi}\mathrm{d}\left(a_e \wedge \delta a_e\right). \tag{2}$$

2 The Abelian Maxwell Chern-Simons Lagrangian

The Lagrangian of this theory is

$$L_{\mathrm{MCS}} = -\frac{1}{2g^2}\mathrm{d}a_e \wedge *\mathrm{d}a_e + \frac{k}{4\pi}a_e \wedge \mathrm{d}a_e, \quad a_e \in \mathfrak{u}_1 \tag{3}$$

This has the equation of motion (derived in appendix C) of

$$\mathrm{d} * \mathrm{d}a_e + \frac{g^2 k}{2\pi}\mathrm{d}a_e = 0, \tag{4}$$

meaning it has a mass $M = \frac{g^2 k}{2\pi}$. This is Yang Mills where the source is from the Chern-Simons term.

3 Some results on the path integrals of fields

Much like the integral

$$\int e^{-\frac{1}{2}ax^2+bx}\mathrm{d}x = \sqrt{\frac{2\pi}{a}}e^{\frac{b^2}{2a}}, \tag{5}$$

by dropping the unnecessary factors out front, the path integral

$$\boxed{\int_{\mathfrak{g}\times M}\mathcal{D}F \exp i\int_M\left(-\frac{g^2}{2}F \wedge *F + F \wedge \mathrm{d}A\right) \sim \exp i\int_M\left(-\frac{1}{2g^2}\mathrm{d}A \wedge *\mathrm{d}A\right).} \tag{6}$$

This is fleshed out fully in the appendix D. Similarly, the integral

$$\boxed{\int \mathcal{D}F\mathcal{D}A \exp i\int_M\left(-\frac{g^2}{2}F \wedge *F + F \wedge \mathrm{d}A\right) \sim \int \mathcal{D}F\delta\left(\mathrm{d}F\right)\exp i\int_M\left(-\frac{g^2}{2}F \wedge *F\right).} \tag{7}$$

4 Abelian master partition function

Consider the "master" partition function

$$\mathcal{Z} = \int \mathcal{D}a_m\mathcal{D}a_e \, \exp i\int\left(-\frac{g^2}{2}a_m \wedge *a_m + a_m \wedge \mathrm{d}a_e + \frac{k}{4\pi}a_e \wedge \mathrm{d}a_e\right), \tag{8}$$

with a_m a gauge invariant one-form.

5 Non-Abelian master partition function for MCS and its dual

The theory is

$$\mathcal{Z}_{\text{Master}} = \int \mathcal{D}a_m \mathcal{D}a_e \exp i\mathrm{tr} \left\{ \int \left(-\frac{g^2}{2} a_m \wedge *a_m \; + a_m \wedge (\mathrm{d}a_e + a_e \wedge a_e) \right.\right.$$
$$\left.\left. + \; \frac{k}{4\pi} \left(a_e \wedge \mathrm{d}a_e + \frac{2}{3} a_e \wedge a_e \wedge a_e \right) \right) \right\}. \tag{9}$$

To get the electric theory (MCS), integrate over a_m (see Appendix E) to get

$$Z_{\text{Electric}} = \int \mathcal{D}a_m \exp i\mathrm{tr} \left\{ \int \text{stuff} \right\}. \tag{10}$$

By substituting out the usual expression

$$a_e = b - \left(\frac{2\pi}{k} \right) a_m, \tag{11}$$

get the (non-integrated) magnetic theory

$$Z_{\text{Magnetic}} = \int \mathcal{D}b\mathcal{D}a_m \exp i\mathrm{tr} \left\{ \int - \frac{g^2}{2} a_m \wedge *a_m - \frac{\pi}{k} a_m \wedge \mathrm{d}a_m + \frac{k}{4\pi} b \wedge \mathrm{d}b \right.$$
$$- \left(\frac{2\pi}{k} \right) a_m \wedge b \wedge a_m$$
$$\left. + \frac{2}{3} \left(\frac{2\pi}{k} \right)^2 a_m \wedge a_m \wedge a_m + \frac{k}{6\pi} b \wedge b \wedge b \right\}, \tag{12}$$

which is derived in F. Can organise this in various ways, for example

$$\boxed{Z_{\text{M}} = \int \mathcal{D}b\mathcal{D}a_m \exp i\mathrm{tr} \left\{ \int -\frac{g^2}{2} a_m \wedge *a_m - \frac{\pi}{k} a_m \wedge \left[\mathrm{d}a_m + 2b \wedge a_m - 8 \left(\frac{\pi}{k} a_m \wedge a_m \right) \right] + \frac{k}{4\pi} b \wedge \left(\mathrm{d}b + \frac{2}{3} b \wedge b \right)\right.} \tag{13}$$

Interested in seeing the relationship as to when a_m and b couple with each other. Perhaps also non-zero b will change the apparent value of k and so on. Can draw Feynman diagrams for the theory we get. Aim for now is to get things as tidied up as possible. Things can get really confusing as to understand what we're looking at we must understand strongly coupled gauge theories.

6 Later bits

A Source free maxwell variation

The variation can be calculated as

$$\begin{aligned}\delta\left(\mathrm{d}a_e \wedge *\mathrm{d}a_e\right) &= \mathrm{d}\left(\delta a_e\right) \wedge *\mathrm{d}a_e + \mathrm{d}a_e \wedge *\mathrm{d}\left(\delta a_e\right) \quad \text{Product rule (over Lie algebra)}\\ &= \mathrm{d}\left(\delta a_e\right) \wedge *\mathrm{d}a_e + \mathrm{d}\left(\delta a_e\right) \wedge *\mathrm{d}a_e \quad \lambda \wedge *\omega = \omega \wedge *\lambda\\ &= 2\mathrm{d}\left(\delta a_e\right) \wedge *\mathrm{d}a_e\\ &= 2\mathrm{d}\left(\delta a_e \wedge *\mathrm{d}a_e\right) - 2\delta a_e \wedge \mathrm{d} * \mathrm{d}a_e \quad \text{Product rule (over manifold)},\end{aligned}$$

so

$$\boxed{-\frac{1}{2}\delta\left(\mathrm{d}a_e \wedge *\mathrm{d}a_e\right) = \delta a_e \wedge \mathrm{d} * \mathrm{d}a_e - \mathrm{d}\left(\delta a_e \wedge *\mathrm{d}a_e\right)} \tag{14}$$

B Abelian Chern-Simons variation

Calculating

$$\begin{aligned}\delta\left(\frac{k}{4\pi}A \wedge \mathrm{d}A\right) &= \frac{k}{4\pi}\delta\left(A \wedge \mathrm{d}A\right)\\ &= \frac{k}{4\pi}\left[\delta A \wedge \mathrm{d}A + A \wedge \delta\left(\mathrm{d}A\right)\right] \quad \text{Product rule (Lie algebra)}\\ &= \frac{k}{4\pi}\left[\delta A \wedge \mathrm{d}A + \mathrm{d}\left(A \wedge \delta A\right) - \mathrm{d}A \wedge \delta A\right] \quad \text{Product rule (Manifold)}\\ &= \frac{k}{4\pi}\left[\mathrm{d}\left(A \wedge \delta A\right) + 2\delta A \wedge \mathrm{d}A\right] \quad \text{Wedge product antisymmetry}.\end{aligned}$$

C Abelian Maxwell Chern-Simons equation of motion

Starting with the Action

$$S_{\mathrm{MCS}} = \int_{\mathcal{M}} -\frac{1}{2g^2}\mathrm{d}a_e \wedge *\mathrm{d}a_e + \frac{k}{4\pi}a_e \wedge \mathrm{d}a_e, \quad a_e \in \mathfrak{u}_1, \tag{15}$$

variation of the fields leaves

$$\begin{aligned}\delta S_{\mathrm{MCS}} &= \delta\left(\int_{\mathcal{M}} -\frac{1}{2g^2}\mathrm{d}a_e \wedge *\mathrm{d}a_e + \frac{k}{4\pi}a_e \wedge \mathrm{d}a_e\right)\\ &= \int_M -\frac{1}{2g^2}\delta\left(\mathrm{d}a_e \wedge *\mathrm{d}a_e\right) + \frac{k}{4\pi}\delta\left(a_e \wedge \mathrm{d}a_e\right)\end{aligned}$$

Then by equations (1) and (2),

$$\begin{aligned}\delta S_{\mathrm{MCS}} &= \int_M \frac{1}{g^2}\delta a_e \wedge \mathrm{d} * \mathrm{d}a_e - \frac{1}{g^2}\mathrm{d}\left(\delta a_e \wedge *\mathrm{d}a_e\right) + \frac{k}{2\pi}\delta a_e \wedge \mathrm{d}a_e + \frac{k}{4\pi}\mathrm{d}\left(a_e \wedge \delta a_e\right)\\ &= \int_M \frac{1}{g^2}\delta a_e \wedge \mathrm{d} * \mathrm{d}a_e + \frac{k}{2\pi}\delta a_e \wedge \mathrm{d}a_e + \int_{\partial M} -\frac{1}{g^2}\delta a_e \wedge *\mathrm{d}a_e + \frac{k}{4\pi}a_e \wedge \delta a_e\\ &= \int_M \delta a_e \wedge \left(\frac{1}{g^2}\mathrm{d} * \mathrm{d}a_e + \frac{k}{2\pi}\mathrm{d}a_e\right) - \int_{\partial M} \delta a_e \wedge \left(\frac{1}{g^2} * \mathrm{d}a_e + \frac{k}{4\pi}a_e\right).\end{aligned}$$

Imposing $\delta S_{\mathrm{MCS}} = 0$ for all δa_e with $\delta a_e = 0$ on ∂M yields the equation of motion.

$$\frac{1}{g^2}\mathrm{d} * \mathrm{d}a_e + \frac{k}{2\pi}\mathrm{d}a_e = 0. \tag{16}$$

D First path integral

E Integrating the master theory to get the electric theory

Here is the integration of equation (9).

F Substitution in the master theory to get the magnetic theory

Substituting in b.

G Notes on differential geometry

For $p-$forms, write

$$\omega = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p}\omega_{\mu_1\ldots\mu_p}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_p}, \quad \omega\in\Lambda^p\left(\mathcal{M}\right). \tag{17}$$

H Notes on Lie-Algebra valued forms

Writing a Lie-algebra valued one-form as

$$A = A_\mu \mathrm{d}x^\mu, \tag{18}$$

it is the case that A_μ can decomposed as

$$A_\mu = A_\mu^a T^a, \tag{19}$$

because the value of A_μ is contained within the Lie-algebra. Note also $A_\mu^a = A_\mu^a(x)$, and $x\in\mathbb{R}^n$. So really,

$$A = A_\mu^a(x)\, T^a \otimes \mathrm{d}x^\mu, \tag{20}$$

with T^a elements of the Lie algebra defined at each point that x is evaluated for.

Wedge products need to be taken care of. Can use the trick when traces are taken where the generators of the lie algebra get factorised away from the coefficients as

$$\begin{aligned}
\mathrm{tr}\left(a\wedge b\wedge b\right) &= \mathrm{tr}\left(a^a t^a \wedge b^b t^b \wedge b^c t^c\right) \\
&= \mathrm{tr}\left(t^a t^b t^c\right)\left(a^a\wedge b^b\wedge b^c\right) \quad \text{(Factorising trick)} \\
&= \mathrm{tr}\left(t^c t^a t^b\right) a^a\wedge b^b\wedge b^c \quad \text{(Trace cyclicity)} \\
&= -\mathrm{tr}\left(t^c t^a t^b\right) a^a\wedge b^c\wedge b^b \quad \text{(Wedge product antisymmetry)} \\
&= \mathrm{tr}\left(t^c t^a t^b\right) b^c\wedge a^a\wedge b^b \quad \text{(Wedge product antisymmetry)} \\
&= \mathrm{tr}\left(b^c t^c\wedge a^a t^a\wedge b^b t^b\right) \\
&= \mathrm{tr}\left(b\wedge a\wedge b\right).
\end{aligned} \tag{21}$$

Similarly, find

$$\begin{aligned}
\mathrm{tr}\left(a\wedge b\wedge b\right) &= \mathrm{tr}\left(t^c t^a t^b\right) b^c\wedge a^a\wedge b^b \quad \text{(From above)} \\
&= \mathrm{tr}\left(t^b t^c t^a\right) b^c\wedge a^a\wedge b^b \quad \text{(Trace cyclicity)} \\
&= -\mathrm{tr}\left(t^b t^c t^a\right) b^c\wedge b^b\wedge a^a \quad \text{(Wedge product antisymmetry)} \\
&= \mathrm{tr}\left(t^b t^c t^a\right) b^b\wedge b^c\wedge a^a \quad \text{(Wedge product antisymmetry)} \\
&= \mathrm{tr}\left(b^b t^b\wedge b^c t^c\wedge a^a t^a\right) \qquad (22) \\
&= \mathrm{tr}\left(b\wedge b\wedge a\right). \qquad (23)
\end{aligned}$$

And so

$$\boxed{\mathrm{tr}\left(a\wedge b\wedge b\right) = \mathrm{tr}\left(b\wedge a\wedge b\right) = \mathrm{tr}\left(b\wedge b\wedge a\right).} \tag{24}$$

I Dimensional reduction

To reduce from 4D coordinates in $\mathbb{R}^4$ coordinates to 3D, first constrain the fields $\mathbb{A}_\mu(x)$ to be defined on $\mathbb{R}^3\times I$, where I is some interval. Now things parameterised by $x\in\mathbb{R}^3$ and $\sigma\in I$. Eventually see that a 4D vector reduces to a 3D vector plus one scalar.

Take lowest mode as only it satisfies the boundary condition. This mode is either the vector or the scalar in our case. In 4D, $A_\mu \to A'_{\mu'} = \Lambda^\mu_{\ \mu'}A_\mu$. In 3D, we need a vector in a 3D subspace to remain in that 3D subspace under the transformations we are allowing.