PhD notes

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1 Summary of variations of different terms possible in a Lagrangian

Making sure to recall the Hodge operator rule: $\lambda \wedge \omega = \omega \wedge *\lambda$. The first result is

$$-\frac{1}{2}\delta\left(\mathrm{d}a_e \wedge *\mathrm{d}a_e\right) = \delta a_e \wedge \mathrm{d} * \mathrm{d}a_e - \mathrm{d}\left(\delta a_e \wedge *\mathrm{d}a_e\right) \tag{1}$$

as derived in appendix A. Next is the Abelian Chern-Simons term, which is derived in appendix B, gives the result

$$\delta\left(\frac{k}{4\pi}a_e \wedge \mathrm{d}a_e\right) = \frac{k}{2\pi}\delta a_e \wedge \mathrm{d}a_e + \frac{k}{4\pi}\mathrm{d}\left(a_e \wedge \delta a_e\right). \tag{2}$$

2 The Abelian Maxwell Chern-Simons Lagrangian

The Lagrangian of this theory is

$$L_{\mathrm{MCS}} = -\frac{1}{2g^2}\mathrm{d}a_e \wedge *\mathrm{d}a_e + \frac{k}{4\pi}a_e \wedge \mathrm{d}a_e, \quad a_e \in \mathfrak{u}_1 \tag{3}$$

This has the equation of motion (derived in appendix C) of

$$\mathrm{d} * \mathrm{d}a_e + \frac{g^2 k}{2\pi}\mathrm{d}a_e = 0, \tag{4}$$

meaning it has a mass $M = \frac{g^2 k}{2\pi}$. This is Yang Mills where the source is from the Chern-Simons term.

3 Some results on the path integrals of fields

Much like the integral

$$\int e^{-\frac{1}{2}ax^2 + bx}\mathrm{d}x = \sqrt{\frac{2\pi}{a}}e^{\frac{b^2}{2a}}, \tag{5}$$

by dropping the unnecessary factors out front, the path integral

$$\boxed{\int_{\mathfrak{g}\times M}\mathcal{D}F \exp i \int_M \left(-\frac{g^2}{2}F \wedge *F + F \wedge \mathrm{d}A\right) \sim \exp i \int_M \left(-\frac{1}{2g^2}\mathrm{d}A \wedge *\mathrm{d}A\right).} \tag{6}$$

This is fleshed out fully in the appendix D. Similarly, the integral

$$\boxed{\int \mathcal{D}F\mathcal{D}A \exp i \int_M \left(-\frac{g^2}{2}F \wedge *F + F \wedge \mathrm{d}A\right) \sim \int \mathcal{D}F\delta\left(\mathrm{d}F\right) \exp i \int_M \left(-\frac{g^2}{2}F \wedge *F\right).} \tag{7}$$

4 Abelian master partition function

Consider the "master" partition function

$$\mathcal{Z} = \int \mathcal{D}a_m\mathcal{D}a_e \; \exp i \int \left(-\frac{g^2}{2}a_m \wedge *a_m + a_m \wedge \mathrm{d}a_e + \frac{k}{4\pi}a_e \wedge \mathrm{d}a_e\right), \tag{8}$$

with a_m a gauge invariant one-form. The Lagrangian for this is

$$L = -\frac{g^2}{2}a_m \wedge *a_m + a_m \wedge \mathrm{d}a_e + \frac{k}{4\pi}a_e \wedge \mathrm{d}a_e. \tag{9}$$

This varies as

$$\delta L = \delta a_m \wedge \left(g^2 \mathrm{d} * \mathrm{d}a_m + \mathrm{d}a_e\right) + \delta a_e \wedge \left(\mathrm{d}a_m + \frac{k}{2\pi}\mathrm{d}a_e\right) + \mathrm{d}\left(g^2 * \mathrm{d}a_m \wedge \delta a_m + \frac{k}{4\pi} a_e \wedge \delta a_e\right) \tag{10}$$

upon the variation of both a_m and a_e independently. This is the result of varying the action *before* any integration over the fields.

5 Non-Abelian master partition function for MCS and its dual

The theory is

$$\begin{aligned}\mathcal{Z}_{\text{Master}} = \int \mathcal{D}a_m \mathcal{D}a_e \exp i\mathrm{tr}\left\{\int \left(-\frac{g^2}{2} a_m \wedge *a_m \right.\right. &+ a_m \wedge (\mathrm{d}a_e + a_e \wedge a_e) \\ &\left.\left. + \frac{k}{4\pi}\left(a_e \wedge \mathrm{d}a_e + \frac{2}{3} a_e \wedge a_e \wedge a_e\right)\right)\right\}.\end{aligned} \tag{11}$$

The Lagrangian for this is

$$L = -\frac{g^2}{2} a_m \wedge *a_m + a_m \wedge (\mathrm{d}a_e + a_e \wedge a_e) + \frac{k}{4\pi}\left(a_e \wedge \mathrm{d}a_e + \frac{2}{3} a_e \wedge a_e \wedge a_e\right). \tag{12}$$

This varies to give

$$Put the stuff here. \tag{13}$$

To get the electric theory (MCS), integrate over a_m (see Appendix E) to get

$$Z_{\text{Electric}} = \int \mathcal{D}a_m \exp i\mathrm{tr}\left\{\int \text{stuff}\right\}. \tag{14}$$

By substituting out the usual expression

$$a_e = b - \left(\frac{2\pi}{k}\right) a_m, \tag{15}$$

get the (non-integrated) magnetic theory

$$\begin{aligned}Z_{\text{Magnetic}} = \int \mathcal{D}a_m \mathcal{D}b \exp i\mathrm{tr}\bigg\{\int &- \frac{g^2}{2} a_m \wedge *a_m - \frac{\pi}{k} a_m \wedge \mathrm{d}a_m + \frac{k}{4\pi} b \wedge \mathrm{d}b \\ &- \left(\frac{2\pi}{k}\right) a_m \wedge b \wedge a_m \\ &+ \frac{2}{3}\left(\frac{2\pi}{k}\right)^2 a_m \wedge a_m \wedge a_m + \frac{k}{6\pi} b \wedge b \wedge b\bigg\},\end{aligned} \tag{16}$$

which is derived in F. Can organise this in various ways, for example

$$\boxed{Z_{\text{M}} = \int \mathcal{D}a_m \mathcal{D}b \exp i\mathrm{tr}\bigg\{\int -\frac{g^2}{2} a_m \wedge *a_m - \frac{\pi}{k} a_m \wedge \left[\mathrm{d}a_m + 2a_m \wedge \left(b - \frac{4\pi}{k} a_m\right)\right] + \frac{k}{4\pi} b \wedge \left(\mathrm{d}b + \frac{2}{3} b \wedge b\right)} \tag{17}$$

Might be useful to compare this to the Abelian case, where

$$Z_M^{(A)} = \int \mathcal{D}a_m \mathcal{D}b \exp i \int \left\{-\frac{g^2}{2} a_m \wedge *a_m - \frac{\pi}{k} a_m \wedge \mathrm{d}a_m + \frac{k}{4\pi} b \wedge \mathrm{d}b\right\}. \tag{18}$$

Interested in seeing the relationship as to when a_m and b couple with each other. Perhaps also non-zero b will change the apparent value of k and so on. Can draw Feynman diagrams for the theory we get. Aim for now is to get things as tidied up as possible. Things can get really confusing as to understand what we're looking at we must understand strongly coupled gauge theories.

6 Later bits

A Source free maxwell variation

The variation can be calculated as

$$
\begin{aligned}
\delta\left(\mathrm{d}a_e \wedge *\mathrm{d}a_e\right) &= \mathrm{d}\left(\delta a_e\right) \wedge *\mathrm{d}a_e + \mathrm{d}a_e \wedge *\mathrm{d}\left(\delta a_e\right) && \text{Product rule (over Lie algebra)} \\
&= \mathrm{d}\left(\delta a_e\right) \wedge *\mathrm{d}a_e + \mathrm{d}\left(\delta a_e\right) \wedge *\mathrm{d}a_e && \lambda \wedge *\omega = \omega \wedge *\lambda \\
&= 2\mathrm{d}\left(\delta a_e\right) \wedge *\mathrm{d}a_e \\
&= 2\mathrm{d}\left(\delta a_e \wedge *\mathrm{d}a_e\right) - 2\delta a_e \wedge \mathrm{d} * \mathrm{d}a_e && \text{Product rule (over manifold)},
\end{aligned}
$$

so

$$\boxed{-\frac{1}{2}\delta\left(\mathrm{d}a_e \wedge *\mathrm{d}a_e\right) = \delta a_e \wedge \mathrm{d} * \mathrm{d}a_e - \mathrm{d}\left(\delta a_e \wedge *\mathrm{d}a_e\right)} \tag{19}$$

B Abelian Chern-Simons variation

Calculating

$$
\begin{aligned}
\delta\left(\frac{k}{4\pi} A \wedge \mathrm{d}A\right) &= \frac{k}{4\pi}\delta\left(A \wedge \mathrm{d}A\right) \\
&= \frac{k}{4\pi}\left[\delta A \wedge \mathrm{d}A + A \wedge \delta\left(\mathrm{d}A\right)\right] \quad \text{Product rule (Lie algebra)} \\
&= \frac{k}{4\pi}\left[\delta A \wedge \mathrm{d}A + \mathrm{d}\left(A \wedge \delta A\right) - \mathrm{d}A \wedge \delta A\right] \quad \text{Product rule (Manifold)} \\
&= \frac{k}{4\pi}\left[\mathrm{d}\left(A \wedge \delta A\right) + 2\delta A \wedge \mathrm{d}A\right] \quad \text{Wedge product antisymmetry}.
\end{aligned}
$$

C Abelian Maxwell Chern-Simons equation of motion

Starting with the Action

$$S_{\mathrm{MCS}} = \int_{\mathcal{M}} -\frac{1}{2g^2}\mathrm{d}a_e \wedge *\mathrm{d}a_e + \frac{k}{4\pi}a_e \wedge \mathrm{d}a_e, \quad a_e \in \mathfrak{u}_1, \tag{20}$$

variation of the fields leaves

$$
\begin{aligned}
\delta S_{\mathrm{MCS}} &= \delta\left(\int_{\mathcal{M}} -\frac{1}{2g^2}\mathrm{d}a_e \wedge *\mathrm{d}a_e + \frac{k}{4\pi}a_e \wedge \mathrm{d}a_e\right) \\
&= \int_M -\frac{1}{2g^2}\delta\left(\mathrm{d}a_e \wedge *\mathrm{d}a_e\right) + \frac{k}{4\pi}\delta\left(a_e \wedge \mathrm{d}a_e\right)
\end{aligned}
$$

Then by equations (1) and (2),

$$
\begin{aligned}
\delta S_{\mathrm{MCS}} &= \int_M \frac{1}{g^2}\delta a_e \wedge \mathrm{d} * \mathrm{d}a_e - \frac{1}{g^2}\mathrm{d}\left(\delta a_e \wedge *\mathrm{d}a_e\right) + \frac{k}{2\pi}\delta a_e \wedge \mathrm{d}a_e + \frac{k}{4\pi}\mathrm{d}\left(a_e \wedge \delta a_e\right) \\
&= \int_M \frac{1}{g^2}\delta a_e \wedge \mathrm{d} * \mathrm{d}a_e + \frac{k}{2\pi}\delta a_e \wedge \mathrm{d}a_e + \int_{\partial M} -\frac{1}{g^2}\delta a_e \wedge *\mathrm{d}a_e + \frac{k}{4\pi}a_e \wedge \delta a_e \\
&= \int_M \delta a_e \wedge \left(\frac{1}{g^2}\mathrm{d} * \mathrm{d}a_e + \frac{k}{2\pi}\mathrm{d}a_e\right) - \int_{\partial M} \delta a_e \wedge \left(\frac{1}{g^2} * \mathrm{d}a_e + \frac{k}{4\pi}a_e\right).
\end{aligned}
$$

Imposing $\delta S_{\mathrm{MCS}} = 0$ for all δa_e with $\delta a_e = 0$ on ∂M yields the equation of motion.

$$\frac{1}{g^2}\mathrm{d} * \mathrm{d}a_e + \frac{k}{2\pi}\mathrm{d}a_e = 0. \tag{21}$$

D First path integral

This bit needs some real attention. Want to show that

$$\boxed{\int_{\mathfrak{g}\times M} \mathcal{D}F \exp i\int_M \left(-\frac{g^2}{2}F \wedge *F + F \wedge \mathrm{d}A\right) \sim \exp i\int_M \left(-\frac{1}{2g^2}\mathrm{d}A \wedge *\mathrm{d}A\right)} \tag{22}$$

convincingly

E Integrating the master theory to get the electric theory

Here is the integration of equation (11).

F Substitution in the master theory to get the magnetic theory

Substituting in b.

G Notes on differential geometry

For $p-$forms, write

$$\omega = \frac{1}{p!}\omega_{\mu_1 \ldots \mu_p}\omega_{\mu_1 \ldots \mu_p}\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}, \quad \omega \in \Lambda^p\left(\mathcal{M}\right). \tag{23}$$

H Notes on Lie-Algebra valued forms

Writing a Lie-algebra valued one-form as

$$A = A_\mu \mathrm{d}x^\mu, \tag{24}$$

it is the case that A_μ can decomposed as

$$A_\mu = A^a_\mu T^a, \tag{25}$$

because the value of A_μ is contained within the Lie-algebra. Note also $A^a_\mu = A^a_\mu(x)$, and $x \in \mathbb{R}^n$. So really,

$$A = A^a_\mu(x)\, T^a \otimes \mathrm{d}x^\mu, \tag{26}$$

with T^a elements of the Lie algebra defined at each point that x is evaluated for.

Wedge products need to be taken care of. Can use the trick when traces are taken where the generators of the lie algebra get factorised away from the coefficients as

$$\begin{aligned}
\mathrm{tr}\left(a \wedge b \wedge b\right) &= \mathrm{tr}\left(a^a t^a \wedge b^b t^b \wedge b^c t^c\right) \\
&= \mathrm{tr}\left(t^a t^b t^c\right)\left(a^a \wedge b^b \wedge b^c\right) \quad \text{(Factorising trick)} \\
&= \mathrm{tr}\left(t^c t^a t^b\right) a^a \wedge b^b \wedge b^c \quad \text{(Trace cyclicity)} \\
&= -\mathrm{tr}\left(t^c t^a t^b\right) a^a \wedge b^c \wedge b^b \quad \text{(Wedge product antisymmetry)} \\
&= \mathrm{tr}\left(t^c t^a t^b\right) b^c \wedge a^a \wedge b^b \quad \text{(Wedge product antisymmetry)} \\
&= \mathrm{tr}\left(b^c t^c \wedge a^a t^a \wedge b^b t^b\right) \\
&= \mathrm{tr}\left(b \wedge a \wedge b\right).
\end{aligned} \tag{27}$$

Similarly, find

$$\begin{aligned}
\mathrm{tr}\left(a \wedge b \wedge b\right) &= \mathrm{tr}\left(t^c t^a t^b\right) b^c \wedge a^a \wedge b^b \quad \text{(From above)} \\
&= \mathrm{tr}\left(t^b t^c t^a\right) b^c \wedge a^a \wedge b^b \quad \text{(Trace cyclicity)} \\
&= -\mathrm{tr}\left(t^b t^c t^a\right) b^c \wedge b^b \wedge a^a \quad \text{(Wedge product antisymmetry)} \\
&= \mathrm{tr}\left(t^b t^c t^a\right) b^b \wedge b^c \wedge a^a \quad \text{(Wedge product antisymmetry)} \\
&= \mathrm{tr}\left(b^b t^b \wedge b^c t^c \wedge a^a t^a\right) \qquad (28) \\
&= \mathrm{tr}\left(b \wedge b \wedge a\right). \qquad (29)
\end{aligned}$$

And so

$$\boxed{\mathrm{tr}\left(a \wedge b \wedge b\right) = \mathrm{tr}\left(b \wedge a \wedge b\right) = \mathrm{tr}\left(b \wedge b \wedge a\right).} \tag{30}$$

I Dimensional reduction

To reduce from 4D coordinates in $\mathbb{R}^4$ coordinates to 3D, first constrain the fields $\mathbb{A}_\mu(x)$ to be defined on $\mathbb{R}^3 \times I$, where I is some interval. Now things parameterised by $x \in \mathbb{R}^3$ and $\sigma \in I$. Eventually see that a 4D vector reduces to a 3D vector plus one scalar.

Take lowest mode as only it satisfies the boundary condition. This mode is either the vector or the scalar in our case. In 4D, $A_\mu \to A'_{\mu'} = \Lambda^\mu_{\mu'} A_\mu$. In 3D, we need a vector in a 3D subspace to remain in that 3D subspace under the transformations we are allowing.

To move from D dimensions to d dimensions, we make the $D-d$ dimensions we wish to remove compact, and then require that in the limit that their size goes to zero, that the energy remains finite.

For example, for a scalar field, consider $\phi(x)$ with a compact dimensions with period L. Can write any field of this sort as

$$\phi_n(x) = A_n \cos\left(\frac{2\pi n x}{L}\right). \tag{31}$$

According to quantum mechanics, this has a momentum $\pm nh/L$ along the x direction. Therefore, unless only the $n=0$ mode is non-zero as $L\to\infty$, then the momentum in this direction diverges. However, $\partial_x \phi_0 = 0$, and also $\partial_x \phi_x \neq 0$ for $n \neq 0$, and so removal of the ϕ dependence on x gives the desired behaviour. We still require the fields we do things like this to to transform correctly under new subgroups of the symmetry groups we had before, however this will be discussed later.

I.1 Dimensional reduction of electromagnetism

Following the work of the paper Dimensional reduction of electromagnetism, start with the Lagrangian of (3+1)-dimensional electromagnetism

$$\mathcal{L}_{3+1}(A^\mu) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{c}j^\mu A_\mu. \tag{32}$$

Here, the components of the field strength in Cartesian coordinates are

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{33}$$

and the corresponding components of the current are

$$J^\mu = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix}. \tag{34}$$

This gives the Lagrangian in terms of the electric and magnetic fields as

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B}\right) - \rho\phi + \frac{1}{c}\vec{J}\cdot\vec{A}. \tag{35}$$

Note here can also define the dual field strength tensor

$$*F^{\mu\nu} \equiv G^{\mu\nu} := \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{36}$$

which in turn gives back the field strength tensor via

$$F^{\mu\nu} = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}. \tag{37}$$

This gives the components

$$*F_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{38}$$

Note that these can be organised into blocks;

$$F_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & -E_x & -E_y & -E_z \\ \hline E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{array}\right) = \left(\begin{array}{c|c} & -\vec{E}^T \\ \hline \vec{E} & \tilde{B} \end{array}\right), \tag{39}$$

$$G_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} = \left(\begin{array}{c|c} & -\vec{B}^T \\ \hline \vec{B} & \tilde{E} \end{array}\right) \tag{40}$$

and

$$J^\mu = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} c\rho \\ \hline \\ \vec{J} \\ \\ \end{pmatrix}. \tag{41}$$

We are interested in proper Lorentz transformations $SO^+(1,3)$. This is the set of all Lorentz transformations that can be connected to the identity by a continuous curve lying in the group. This is composed of proper rotations and boosts. The set of all rotations forms a Lie subgroup isomorphic to the ordinary rotation group SO(3). These rotations leave $\vec{E}$ and $\vec{B}$ unmixed, and so their $3 \oplus 3$ structure preserved. Similarly, ρ and $\vec{J}$ are left unmixed by these transformations and so their $1 \oplus 3$ block structure is preserved too.

The equations constraining electromagnetism can be derived from the Lagrangian and are as following:

$$\text{Dynamics: } \partial_\mu F^{\mu\nu} = \frac{1}{c} j^\nu, \quad \partial_\mu G^{\mu\nu} = 0, \tag{42}$$

$$\text{Continuity: } \partial_t \rho + \partial_x J_x + \partial_y J_y + \partial_z J_z = 0. \tag{43}$$

Breaking the field strength tensor and its dual into $\vec{E}$ and $\vec{B}$ gives the Maxwell equations in their familiar form

$$\text{div}\vec{B} = 0 \quad \text{(Gauss's law for magnetism)}, \tag{44}$$

$$\frac{1}{c}\partial_t \vec{B} + \text{curl}\vec{E} = \vec{0} \quad \text{(Maxwell-Faraday law of induction)}, \tag{45}$$

$$\text{div}\vec{E} = \rho \quad \text{(Gauss's law)}, \tag{46}$$

$$-\frac{1}{c}\partial_t \vec{E} + \text{curl}\vec{B} = \frac{1}{c}\vec{J} \quad \text{(Ampère's circuital law)}. \tag{47}$$

Breaking these equations into Cartesian components gives the following set of equations. Firstly, (44) implies

$$\partial_x B_x + \partial_y B_y + \partial_z B_z = 0. \tag{48}$$

Equation (45) implies the three equations

$$\frac{1}{c}\partial_t B_x + \partial_y E_z - \partial_z E_y = 0, \tag{49}$$

$$\frac{1}{c}\partial_t B_y + \partial_z E_x - \partial_x E_z = 0, \tag{50}$$

$$\frac{1}{c}\partial_t B_z + \partial_x E_y - \partial_y E_x = 0. \tag{51}$$

Next, equation (46) implies

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = \rho. \tag{52}$$

Finally, equation (47) implies the three equations

$$-\frac{1}{c}\partial_t E_x + \partial_y B_z - \partial_z B_y = \frac{1}{c} J_x, \tag{53}$$

$$-\frac{1}{c}\partial_t E_y + \partial_z B_x - \partial_x B_z = \frac{1}{c} J_y, \tag{54}$$

$$-\frac{1}{c}\partial_t E_z + \partial_x B_y - \partial_y B_x = \frac{1}{c} J_z. \tag{55}$$

Now, a *descent* is performed along the z direction. This is done by requiring the z-independence of both sources and solutions. This is achieved by letting $\partial_z \to 0$. The result is that the equations of motion decouple into two independent sets of four equations. These sets are:

1: The $(\mathbf{E_x}, \mathbf{E_y}, \mathbf{B_z})$, $(\rho, \mathbf{J_x}, \mathbf{J_y})$ sector

This sector is made of 4 differential equations and a reduced continuity equation:

$$
\begin{aligned}
\frac{1}{c}\partial_t B_z + \partial_x E_y - \partial_y E_x &= 0, &(56)\\
\partial_x E_x + \partial_y E_y &= \rho, &(57)\\
-\frac{1}{c}\partial_t E_x + \partial_y B_z &= \frac{1}{c}J_x, &(58)\\
-\frac{1}{c}\partial_t E_y - \partial_x B_z &= \frac{1}{c}J_y, &(59)\\
& &(60)
\end{aligned}
$$

and

$$
\partial_t \rho + \partial_x J_x + \partial_y J_y = 0. \tag{61}
$$

2: The $(\mathbf{B_x}, \mathbf{B_y}, \mathbf{E_z})$, $(\mathbf{J_z})$ sector

This section is made of only 4 differential equations

$$
\begin{aligned}
\partial_x B_x + \partial_y B_y + \partial_z B_z &= 0, &(62)\\
\frac{1}{c}\partial_t B_x + \partial_y E_z &= 0, &(63)\\
\frac{1}{c}\partial_t B_y - \partial_x E_z &= 0, &(64)\\
-\frac{1}{c}\partial_t E_z + \partial_x B_y - \partial_y B_x &= \frac{1}{c}J_z. &(65)
\end{aligned}
$$

This partitioning can be highlighted in the field strength tensors

$$
F_{\mu\nu} = \left(\begin{array}{ccc|c} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ \hline E_z & -B_y & B_x & 0 \end{array}\right), \quad G_{\mu\nu} = \left(\begin{array}{ccc|c} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ \hline B_z & E_y & -E_x & 0 \end{array}\right), \quad J^{\mu} = \left(\begin{array}{c} c\rho \\ J_x \\ J_y \\ \hline J_z \end{array}\right). \tag{66}
$$

Note now that the elements of the first sector in G can be obtained by performing the correct alternating sum over the elements of the first sector in F. This done in terms of reduced spacetime indices a, $b \in \{0, 1, 2\}$ as

$$
G^{ab} = \frac{1}{2}\left(\varepsilon^{abc3}F_{c3} + \varepsilon^{ab3c}F_{3c}\right), \tag{67}
$$

which reduces to

$$
G^{ab} = \varepsilon^{abc}F_{c3}. \tag{68}
$$

Similarly, the second sector of G can be obtained from the second sector in F via

$$
G^{a3} = \frac{1}{2}\varepsilon^{a3bc}F_{bc}, \tag{69}
$$

which reduces to

$$
G^{a3} = \frac{1}{2}\varepsilon^{abc}F_{bc}. \tag{70}
$$

By recognising that the field strength tensors partition this way, the equations of motion for each sector can be written in tensor form. For the first (EEB) sector, these are

$$
\boxed{\varepsilon^{abc}\partial_a F_{bc} = 0, \quad \partial_a F^{ab} = \frac{1}{c}j^b} \tag{71}
$$

Now in this sector, F^{ab} is an antsymmetric (2+1) dimensional tensor, and its dual, G^{a3} is a (2+1) dimensional vector. Also, j^a is a (2+1) dimensional vector. This sector is considered the closer analog of 4D electromagnetism in 3D.

In the second (BBE) sector, the equations of motion in tensor form are

$$
\boxed{\varepsilon^{abc}\partial_b F_{c3} = 0, \quad \partial_a F^{a3} = \frac{1}{c}j^3.} \tag{72}
$$

In this sector, F^{a3} is a 3-component vector, and its dual G^{ab} is a (2+1) dimensional antisymmetric tensor. Finally note that j^3 is a scalar.

Now that a partitioning of the field components into sectors has been made clear, it is important to check that these are not mixed due to the application of the desired (Lorentz) transformations. Since z-dependence has been removed from the physics, the subgroup of Lorentz transformations that leaves the z components of vectors preserved is now the set of transformations of interest. This subgroup is represented in Cartesian coordinates by the set of matrices with the block diagonal form

$$\Lambda = \left(\begin{array}{c|c} L & \\ \hline & Q \end{array} \right), \tag{73}$$

where $L \in O\left(2,\, 1\right)$ and $Q \in O\left(1\right) = \{+1,\, -1\}$. When $Q = +1$, this leaves the z-axis preserved, whereas $Q = -1$ inverts the z-axis. It can be checked that applications of these matrices leave the sectors unmixed.

The above means that under dimensional reduction, $F^{\mu\nu}$ has become the (2+1)-dimensional two-form F^{ab} and the one form F^{a3}. Likewise, A^{μ} has become the 3D gauge field A^a and the scalar A^3. This is a key initial result used in Adi's paper.

Additionally, the original Lagrangian $\mathcal{L}_{3+1}\left(A\right)$ can be written

$$\mathcal{L}_{3+1}\left(A\right) = \left(-\frac{1}{4}F_{ab}F^{ab} - \frac{1}{c}j^a A_a\right) + \left(-\frac{1}{2}F_{a3}F^{a3} - \frac{1}{c}j^3 A_3\right). \tag{74}$$

Despite how things might look, this is not fully decoupled into A^a and A^3 in the way the derivatives of A are. First it is required that the gauge choice

$$\partial_3 A_a = 0 \tag{75}$$

(which can always be made) be made. This choice leaves the residual gauge freedom

$$A_a \rightarrow A_a + \partial_a f, \quad f = f\left(t,\, x,\, y\right). \tag{76}$$

Under this choice, the full Lagrangian can be written

$$\mathcal{L}_{3+1}\left(A\right) = \mathcal{L}_{\mathrm{EEB}}\left(A^0,\, A^1,\, A^2\right) + \mathcal{L}_{\mathrm{BBE}}\left(A^3\right), \tag{77}$$

where

$$\begin{aligned} \mathcal{L}_{\mathrm{EEB}}\left(A^0,\, A^1,\, A^2\right) &= -\frac{1}{4}\left(\partial_a A_b - \partial_b A_a\right)\left(\partial^a A^b - \partial^b A^a\right) - \frac{1}{c}j^a A_a & (78)\\ &= \frac{1}{2}\left(E_x^2 + E_y^2 - B_z^2\right) - \rho\phi + \frac{1}{c}\left(J_x A_x + J_y A_y\right), & (79) \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_{BBE}\left(A^3\right) &= -\frac{1}{2}\left(\partial_a A_3\right)\left(\partial^a A^3\right) - \frac{1}{c}j^3 A_3 & (80)\\ &= \frac{1}{2}\left(E_z^2 - B_x^2 - B_y^2\right) + \frac{1}{c}J_z A_z, & (81) \end{aligned}$$

which is truly decoupled into two pieces, with a full gauge freedom in the non-compact directions remaining.