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$$L = -\frac{1}{2g^2}\, da_e \wedge * da_e + \frac{k}{4\pi}\, a_e \wedge da_e$$

$$G \equiv \exp\left(i \oint \left(a_e - \frac{1}{m} * da_e\right)\right)$$

Consider $Z = \int DF_m\, DA_e \exp i \int \left(-\frac{g^2}{2} F_m \wedge * F_m + F_m \wedge dA_e\right)$

$$T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$$

$$T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$$

$$T^{(\mu\nu)\rho}{}_{\sigma} = \frac{1}{2}(T^{\mu\nu\rho}{}_{\sigma} + T^{\nu\mu\rho}{}_{\sigma})$$

If antisymmetrising over p objects, divide by $p!$

e.g.

$$T^{\mu}{}_{(\nu\rho\sigma)} = \frac{1}{3!}\left(T^{\mu}{}_{\nu\rho\sigma} + T^{\mu}{}_{\rho\nu\sigma} + T^{\mu}{}_{\rho\sigma\nu} + T^{\mu}{}_{\sigma\rho\nu} + T^{\mu}{}_{\sigma\nu\rho} + T^{\mu}{}_{\nu\sigma\rho}\right)$$

and

$$T^{\mu}{}_{[\nu\rho\sigma]} = \frac{1}{3!}\left(T^{\mu}{}_{\nu\rho\sigma} - T^{\mu}{}_{\rho\nu\sigma} + T^{\mu}{}_{\rho\sigma\nu} - T^{\mu}{}_{\sigma\rho\nu} + T^{\mu}{}_{\sigma\nu\rho} - T^{\mu}{}_{\nu\sigma\rho}\right)$$

$$w = \frac{1}{p!} w_{\mu_1 \cdots \mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$$dw = \frac{1}{p!} \frac{\partial w_{\mu_1 \cdots \mu_p}}{\partial x^{\nu}}\, dx^{\nu} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \iff (dw)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} w_{\mu_2 \cdots \mu_{p+1}]}$$

If $w = w_\mu dx$

$$(dw)_{\mu_1\mu_2} = (1+1)\, \partial_{[\mu_1} w_{\mu_2]}$$

$$= 2\, \partial_{[\mu_1} w_{\mu_2]}$$

$$= 2 \times \frac{1}{2}\left(\partial_{\mu_1} w_{\mu_2} - \partial_{\mu_2} w_{\mu_1}\right)$$

$$(dw)_{\mu_1\mu_2} = \partial_{\mu_1} w_{\mu_2} - \partial_{\mu_2} w_{\mu_1}$$

So $dw = \frac{1}{2!}\left(\partial_{\mu_1} w_{\mu_2} - \partial_{\mu_2} w_{\mu_1}\right) dx^{\mu_1} \wedge dx^{\mu_2}$.

$$\implies dw = \frac{1}{2}\left(\partial_{\mu_1} w_{\mu_2} - \partial_{\mu_2} w_{\mu_1}\right) dx^{\mu_1} \wedge dx^{\mu_2}.$$

$$A = A_\mu(x)\, dx^\mu$$

$$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

$$F = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu \wedge dx^\nu$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$F = dA$ so $dF = d^2 A = 0.$

Gauge, if $A \to A + d\alpha$

$$A_\mu dx^\mu \to (A_\mu + d\alpha)\, dx^\mu$$
$$= A_\mu dx^\mu + (\partial_\mu \alpha) dx^\mu$$
$$= (A_\mu + \partial_\mu \alpha) dx^\mu$$

And $F \to F + d(d\alpha) = F.$

$$S_{Maxwell} = \int d^3x \, -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu}, \quad S_{CS} = \int d^3x \, \frac{k}{4\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

Let $S_{MCS} = S_{Maxwell} + S_{CS}$

$$S_{MCS} = \int d^3x \quad -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

$$\delta S_{MCS} = \delta\left(\int d^3x \quad -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right)$$

$$= \int d^3x \; \delta\left(-\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right)$$

$$= \int d^3x \left[\frac{\partial}{\partial A_\alpha}\left(-\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right) \delta A_\alpha + \frac{\partial}{\partial_\beta A_\alpha}\left(-\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right) \delta(\partial_\beta A_\alpha) \right]$$

$$= \int d^3x \left[\frac{\partial}{\partial A_\alpha}\left(\frac{k}{4\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right) \delta A_\alpha + \frac{\partial}{\partial_\beta A_\alpha}\left(-\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right) \delta(\partial_\beta A_\alpha) \right]$$

$$\partial_\mu \frac{\partial(F^{\kappa\lambda}F_{\kappa\lambda})}{\partial(\partial_\mu A_\nu)} = \partial_\mu\left(\frac{\partial}{\partial(\partial_\mu A_\nu)} (\partial^\kappa A^\lambda - \partial^\lambda A^\kappa)(\partial_\kappa A_\lambda - \partial_\lambda A_\kappa) \right)$$

$$= \partial_\mu\left(\frac{\partial}{\partial(\partial_\mu A_\nu)} \left(\underline{\partial^\kappa A^\lambda \partial_\kappa A_\lambda} - \underline{\partial^\kappa A^\lambda \partial_\lambda A_\kappa} - \underline{\partial^\lambda A^\kappa \partial_\kappa A_\lambda} + \underline{\partial^\lambda A^\kappa \partial_\lambda A_\kappa} \right)\right).$$

$$= 2\, \partial_\mu\left(\frac{\partial}{\partial(\partial_\mu A_\nu)} \left(\underline{\partial^\kappa A^\lambda \partial_\kappa A_\lambda} - \partial^\kappa A^\lambda \partial_\lambda A_\kappa \right)\right).$$

$$= 2\, \partial_\mu \left(\partial^\kappa A^\lambda \frac{\partial}{\ldots} (\partial_\kappa A_\lambda) + \partial_\kappa A_\lambda \frac{\partial}{\ldots} (\partial^\kappa A^\lambda) - \frac{\partial}{\ldots} (\partial^\kappa A^\lambda \partial_\lambda A_\kappa \right) \right)$$

$$= 2\partial_\mu\left(\partial^\kappa A^\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial_\kappa A_\lambda) + \partial_\kappa A_\lambda \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial^\kappa A^\lambda) - \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial^\kappa A^\lambda \partial_\lambda A_\kappa)\right).$$

$$= 2\partial_\mu\left(\partial^\kappa A^\lambda \delta^\mu_\kappa \delta^\nu_\lambda + \partial_\kappa A_\lambda g^{\kappa\alpha} g^{\lambda\beta} \delta^\mu_\alpha \delta^\nu_\beta - \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial^\kappa A^\lambda \partial_\lambda A_\kappa)\right).$$

$$\frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial^\kappa A^\lambda \partial_\lambda A_\kappa) = 2\partial^\nu A^\mu$$

$$= 2\partial_\mu\left(\partial^\mu A^\nu + \partial_\kappa A_\lambda g^{\kappa\mu} g^{\lambda\nu} - \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial^\kappa A^\lambda \partial_\lambda A_\kappa)\right).$$

$$= 2\partial_\mu\left(2\partial^\mu A^\nu - \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial^\kappa A^\lambda \partial_\lambda A_\kappa)\right).$$

$$= 2\partial_\mu\left(2\partial^\mu A^\nu - 2\partial^\nu A^\mu\right)$$

$$= 4\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= 4\partial_\mu F^{\mu\nu}$$

So

$$\delta S_{MCS} = \int d^3x \left[\frac{\partial}{\partial A_\alpha}\left(\frac{k}{4\pi}\varepsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho\right)\delta A_\alpha + \frac{\partial}{\partial_\rho A_\alpha}\left(-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho\right)\delta(\partial_\rho A_\alpha)\right]$$

$$= \int d^3x \left[\frac{\partial}{\partial A_\alpha}\left(\frac{k}{4\pi}\varepsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho\right)\delta A_\alpha - \frac{1}{4e^2}\times 4F^{\rho\alpha}\delta(\partial_\rho A_\alpha) + \frac{\partial}{\partial_\rho A_\alpha}\left(\frac{k}{4\pi}\varepsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho\right)\delta(\partial_\rho A_\alpha)\right]$$

$$= \int d^3x \left[\frac{\partial}{\partial A_\alpha}\left(\frac{k}{4\pi}\varepsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho\right)\delta A_\alpha - \frac{1}{e^2}F^{\alpha\beta}\delta(\partial_\alpha A_\beta) + \frac{\partial}{\partial_\rho A_\alpha}\left(\frac{k}{4\pi}\varepsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho\right)\delta(\partial_\rho A_\alpha)\right]$$

$$= \int d^3x \left[\frac{k}{4\pi}\varepsilon^{\mu\nu\rho}\delta^\alpha_\mu \partial_\nu A_\rho (\delta A_\alpha) + \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}A_\mu \delta^\beta_\nu \delta^\alpha_\rho \,\delta(\partial_\beta A_\alpha) - \frac{1}{e^2}F^{\alpha\beta}\delta(\partial_\alpha A_\beta)\right]$$

$$= \int_M d^3x \left[\frac{k}{4\pi}\varepsilon^{\alpha\nu\rho}\partial_\nu A_\rho (\delta A_\alpha) + \frac{k}{4\pi}\varepsilon^{\mu\rho\alpha}A_\mu \delta(\partial_\rho A_\alpha) - \frac{1}{e^2}F^{\alpha\beta}\delta(\partial_\alpha A_\beta)\right]$$

$$= \int_M d^3x \left[\frac{k}{4\pi}\varepsilon^{\alpha\nu\rho}\partial_\nu A_\rho (\delta A_\alpha) + \frac{k}{4\pi}\varepsilon^{\mu\rho\alpha}A_\mu \delta(\partial_\rho A_\alpha) - \frac{1}{e^2}\left(\cancelto{0}{\partial_\alpha(F^{\alpha\beta}\delta A_\beta)} - \partial_\alpha F^{\alpha\beta})\delta A_\beta\right)\right]$$

$$= \int_M d^3x \left[\frac{k}{4\pi}\varepsilon^{\alpha\nu\rho}\partial_\nu A_\rho (\delta A_\alpha) + \frac{k}{4\pi}\varepsilon^{\mu\rho\alpha} A_\mu\, \delta(\partial_\rho A_\alpha) + \frac{1}{e^2}\partial_\alpha F^{\alpha\beta})\delta A_\beta\right]$$

$$= \int_M d^3x \left[\frac{k}{4\pi}\varepsilon^{\alpha\nu\rho}\partial_\nu A_\rho (\delta A_\alpha) + \frac{k}{4\pi}\varepsilon^{\mu\rho\alpha}\left(\cancel{\partial_\rho(A_\mu \delta A_\alpha)} - \partial_\rho A_\mu \delta A_\alpha\right) + \frac{1}{e^2}\partial_\alpha F^{\alpha\beta})\delta A_\beta\right]$$

0

$$= \int_M d^3x \left[\frac{k}{4\pi}\varepsilon^{\alpha\nu\rho}\partial_\nu A_\rho (\delta A_\alpha) - \frac{k}{4\pi}\varepsilon^{\mu\rho\alpha}\partial_\rho A_\mu (\delta A_\alpha) + \frac{1}{e^2}\partial_\alpha F^{\alpha\beta})\delta A_\beta\right]$$

$$= \int_M d^3x \left[\frac{k}{4\pi}\varepsilon^{\alpha\nu\rho}\partial_\nu A_\rho (\delta A_\alpha) - \frac{k}{4\pi}\varepsilon^{\mu\rho\alpha}\partial_\rho A_\mu (\delta A_\alpha) + \frac{1}{e^2}\partial_\beta F^{\beta\alpha})\delta A_\alpha\right]$$

$$= \int_M d^3x\, \delta A_\alpha \left[\frac{k}{4\pi}\left(\varepsilon^{\alpha\nu\rho}\partial_\nu A_\rho - \varepsilon^{\mu\rho\alpha}\partial_\rho A_\mu\right) + \frac{1}{e^2}\partial_\beta F^{\beta\alpha}\right]$$

$$= \int_M d^3x\, \delta A_\alpha \left[\frac{k}{4\pi}\left(\varepsilon^{\alpha\nu\rho}\partial_\nu A_\rho + \varepsilon^{\alpha\rho\mu}\partial_\rho A_\mu\right) + \frac{1}{e^2}\partial_\beta F^{\beta\alpha}\right]$$

$$= \int_M d^3x\, \delta A_\alpha \left[2\times\frac{k}{4\pi}\varepsilon^{\alpha\nu\rho}\partial_\nu A_\rho + \frac{1}{e^2}\partial_\beta F^{\beta\alpha}\right]$$

$$0 = \int_M d^3x\, \delta A_\alpha \left[2\times\frac{k}{4\pi}\varepsilon^{\alpha\nu\rho}\partial_\nu A_\rho + \frac{1}{e^2}\partial_\beta F^{\beta\alpha}\right] \quad \forall\ \delta A_\alpha \text{ so}$$

$$2\times\frac{k}{4\pi}\varepsilon^{\alpha\nu\rho}\partial_\nu A_\rho + \frac{1}{e^2}\partial_\beta F^{\beta\alpha} = 0$$

$$\partial_\beta F^{\beta\alpha} + 2\times\frac{ke^2}{4\pi}\varepsilon^{\alpha\nu\rho}\partial_\nu A_\rho = 0$$

$$\partial_\mu F^{\mu\nu} + 2\times\frac{ke^2}{4\pi}\varepsilon^{\nu\rho\sigma}\partial_\rho A_\sigma = 0$$

$$\partial_\mu F^{\mu\nu} + \frac{ke^2}{4\pi}\varepsilon^{\nu\rho\sigma}(2\partial_\rho A_\sigma) = 0$$

$$\varepsilon^{\nu\rho\sigma}F_{\rho\sigma} = \varepsilon^{\nu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)$$
$$= \varepsilon^{\nu\rho\sigma}\partial_\rho A_\sigma - \varepsilon^{\nu\rho\sigma}\partial_\sigma A_\rho$$

$$- \varepsilon^{\nu\rho\sigma}\partial_\rho A_\sigma + \varepsilon^{\nu\sigma\rho}\partial_\sigma A_\rho$$

$$- \varepsilon^{\nu\rho\sigma}\partial_\rho A_\sigma + \varepsilon^{\nu\rho\sigma}\partial_\rho A_\sigma$$

$$= 2\,\varepsilon^{\nu\rho\sigma}\partial_\rho A_\sigma$$

$$\boxed{\Rightarrow \quad \partial_\mu F^{\mu\nu} + \frac{k e^2}{4\pi}\varepsilon^{\nu\rho\sigma}F_{\rho\sigma} = 0. \qquad S_{MCS} = \int d^3x \; -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho}$$

Inner product
$$\langle \eta, \omega \rangle = \int_M \eta \wedge * \omega$$

$$a = a_\mu\, dx^\mu$$

$$F = da$$

$$F = d(a_\mu dx^\mu)$$

$$*F = *d(a_\mu dx^\mu)$$

$$F \wedge *F = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}$$

for a p-form $\omega \in \Lambda^p(M)$

$(*\omega) \in \Lambda^{n-p}(M)$

is defined by $(*\omega)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\sqrt{|g|}\,\varepsilon_{\mu_1\cdots\mu_{n-p}\nu_1\cdots\nu_p}\,\omega^{\nu_1\cdots\nu_p}$

$$\omega = \frac{1}{p!}\,\omega_{\mu_1\cdots\mu_p}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}.$$

So for a 2-form F in $n=4$.

$$(*F)_{\mu_1\mu_2} = \frac{1}{2!}\sqrt{|g|}\,\varepsilon_{\mu_1\mu_2\nu_1\nu_2}F^{\nu_1\nu_2}$$

$$\text{So } *F = \frac{1}{2!}(*F)_{\mu_1\mu_2}\,dx^{\mu_1}\wedge dx^{\mu_2}$$

$$= \frac{1}{4}\sqrt{|g|}\,\varepsilon_{\mu_1\mu_2\nu_1\nu_2}F^{\nu_1\nu_2}\,dx^{\mu_1}\wedge dx^{\mu_2}$$

$$F = \frac{1}{2}F_{\mu\nu}\,dx^\mu \wedge dx^\nu$$

$$\text{So } F \wedge *F = \left(\frac{1}{2}F_{\mu\nu}\,dx^\mu\wedge dx^\nu\right) \wedge \frac{1}{4}\sqrt{|g|}\,\varepsilon_{\mu_1\mu_2\nu_1\nu_2}F^{\nu_1\nu_2}\,dx^{\mu_1}\wedge dx^{\mu_2}$$

$$= \frac{1}{\ }\sqrt{|g|}\,\varepsilon_{\mu_1\mu_2\nu_1\nu_2}F_{\mu\nu}F^{\nu_1\nu_2}\,dx^\mu\wedge dx^\nu\wedge dx^{\mu_1}\wedge dx^{\mu_2}$$

$$= \frac{1}{8}\sqrt{|g|}\, \varepsilon_{\mu_1\mu_2\nu_1\nu_2} \ldots F_{\mu \ldots}$$

$$\overset{(?)}{=} \frac{1}{8}\sqrt{|g|}\, \varepsilon_{\mu_1\mu_2\nu_1\nu_2}\, g^{\nu_1\alpha} g^{\nu_2\beta} F_{\mu_1\mu_2} F_{\alpha\beta}\, dx^{\mu_1}\wedge dx^{\mu_2}\wedge dx^{\nu_1}\wedge dx^{\nu_2}$$

Maxwell equations
derivation
differential forms ☆.

$$S = -\frac{1}{2}\int F\wedge * F$$

$$F = dA$$

$$\delta S = -\frac{1}{2}\,\delta\int_M dA\wedge * dA$$

$$= -\frac{1}{2}\int_M d\delta A\wedge * dA + dA\wedge * d\delta A$$

$d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^p\,\omega\wedge d\eta$

If $\omega = d\lambda$

$d(\omega\wedge\eta) = 0 + (-1)^p\,\omega\wedge d\eta$

$$\langle\eta,\omega\rangle = \int_M \eta\wedge *\omega$$

Here for $\lambda,\omega\in\Lambda^k_U$ for $\lambda = d\delta A$ and $\omega = dA$

$\lambda\wedge *\omega = (-1)^q\langle\lambda,\omega\rangle e = (-1)^q\langle\omega,\lambda\rangle = \omega\wedge *\lambda.$

With $e = e_1\wedge e_2\wedge\cdots\wedge e_n \in \Lambda^n$

So $dA\wedge * d\delta A = d\delta A\wedge * dA$

$$\delta S = -\frac{1}{2}\int_M 2\, d\delta A\wedge * dA$$

Now, $d(\delta A\wedge * dA) = d\delta A\wedge * dA - \delta A\wedge d * dA$

Meaning $d\delta A\wedge * dA = d(\delta A\wedge * dA) + \delta A\wedge d * dA$

So,

$$\delta S = -\int_M d(\delta A\wedge * dA) - \int_M \delta A\wedge d * dA.$$

$$\delta S = -\int_{\partial M} \delta A\wedge * dA - \int_M \delta A\wedge d * dA.$$

$\delta A = 0$ on ∂M, so setting $\delta S = 0$;

$$0 = \int_M \delta A\wedge d * dA. \quad \forall\, \delta A$$

$$\Longrightarrow \quad d * dA = 0.$$

i.e. $d * F = 0.$

$$\boxed{\begin{array}{l} S = -\frac{1}{2}\int F \wedge * F \\ \text{gives} \quad d * F = 0. \end{array}}$$

$$S[A] = \frac{K}{4\pi}\int_M tr\left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A\right)$$

$$S[A+a] = \frac{K}{4\pi}\int_M tr\left((A+a)\wedge d(A+a) + \tfrac{2}{3}(A+a)\wedge(A+a)\wedge(A+a)\right).$$

So

$$S[A+a] - S[A] = \frac{K}{4\pi}\int_M tr\left[(A+a)\wedge d(A+a) + \tfrac{2}{3}(A+a)\wedge(A+a)\wedge(A+a) - \left(A \wedge dA + \tfrac{2}{3} A\wedge A\wedge A\right)\right].$$

$$= \frac{K}{4\pi}\int_M tr\left[A\wedge dA + a\wedge dA + A\wedge da + a\wedge da + \tfrac{2}{3}\left((A\wedge A + A\wedge a + a\wedge A + a\wedge a)\wedge(A+a) - \left(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A\right)\right]\right.$$

$$= \frac{K}{4\pi}\int_M tr\left[A\wedge dA + a\wedge dA + A\wedge da + a\wedge da + \tfrac{2}{3}\left(A\wedge A\wedge A + A\wedge a\wedge A + a\wedge A\wedge A + a\wedge a\wedge A + A\wedge A\wedge a + A\wedge a\wedge a + a\wedge A\wedge a + a\wedge a\wedge a\right)\right.$$

$$\left. - \left(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A\right)\right]$$

Alternatively,

Why is Adi's paper not including this?

$$CS(A) = \int_N tr(A\wedge dA) + \frac{2}{3}\int_N tr(A\wedge A\wedge A).$$

$$\delta CS(A) = \int_N tr(\delta A\wedge dA) + \int_N tr(A\wedge d\delta A) + 2\int_N tr(\delta A\wedge A\wedge A).$$

$$\int_N tr(A\wedge d\delta A)$$

$$\int_N d\,tr(A\wedge\delta A)$$

$$= \int_N d\,tr(A\wedge\delta A) + 2\int_N tr(\delta A\wedge(dA + A\wedge A)).$$

$$= \int_{\partial N} \underbrace{\cancel{tr(A\wedge\delta A)}}_{0} + 2\int_N tr(\delta A\wedge(dA + A\wedge A))$$

$$= \int_N tr(dA\wedge\delta A) + tr(A\wedge\delta dA)$$

$$= \int_N tr(\delta A\wedge dA) + tr(A\wedge\delta dA)$$

So

$$\int_N tr(A\wedge\delta dA) = \int_N d\,tr(A\wedge\delta A) + \int_N tr(\delta A\wedge dA)$$

$$\delta CS(A) = +2\int_N tr(\delta A\wedge(dA + A\wedge A))$$

Letting $\delta S = 0$,

$$\int_N tr(\delta A\wedge(dA + A\wedge A)) = 0 \quad \forall\,\delta A.$$

$$\Longrightarrow \quad dA + A\wedge A = 0$$

$$CS(A) = \int_N tr(A\wedge dA) + \frac{2}{3}\int_N tr(A\wedge A\wedge A).$$

gives $\quad dA + A\wedge A = 0$

Try $\displaystyle S = \int_M \frac{k}{4\pi} A\wedge dA$

$$\delta S = \int_M \frac{k}{4\pi}\,\delta(A\wedge dA)$$

$$= \int_M \frac{k}{4\pi}(\delta A\wedge dA + A\wedge\delta dA)$$

$$= \int_M \frac{k}{4\pi}(\delta A\wedge dA + d(A\wedge\delta A) - dA\wedge\delta A)$$

$$= \int_M \frac{k}{4\pi}\left[d(A\wedge\delta A) + 2\,\delta A\wedge dA\right]$$

$$= \int_{\partial M} \underbrace{\cancel{\frac{k}{4\pi} A\wedge\delta A}}_{0} + \int_M \frac{k}{2\pi}\,\delta A\wedge dA.$$

$$\delta\left(\frac{k}{4\pi} A\wedge dA\right) = \frac{k}{4\pi}\,\delta(A\wedge dA)$$

$$= \frac{k}{4\pi}\left[\delta A\wedge dA + A\wedge\delta(dA)\right]$$

$$= \frac{k}{4\pi}\left[\delta A\wedge dA + d(A\wedge\delta A) - dA\wedge\delta A\right]$$

$$= \frac{k}{4\pi}\left[d(A\wedge\delta A) + 2\,\delta A\wedge dA\right]$$

$dA = 0$ on ∂M

$$= \int_M \frac{k}{2\pi} \delta A \wedge dA. \quad \forall \delta A \qquad \Longrightarrow \frac{k}{2\pi} dA = 0 \text{ here.}$$

i.e., If $S_{cs} = \int_M \frac{k}{4\pi} A \wedge dA$

$$\delta S_{cs} = \int_M \frac{k}{2\pi} \delta A \wedge dA.$$

Now, for $S_M = -\frac{1}{2g^2} \int_M dA \wedge * dA$,

$$\delta S_M = -\frac{1}{2g^2} \delta\left(\int dA \wedge * dA\right)$$

$$= -\frac{1}{2g^2} \int_M d\delta A \wedge * dA + dA \wedge * d\delta A$$

But $dA \wedge * d\delta A = d\delta A \wedge * dA$

$$= -\frac{1}{g^2} \int_M d\delta A \wedge * dA$$

$$= -\frac{1}{g^2} \int_M \left(d(\delta A \wedge * dA) - \delta A \wedge d * dA \right)$$

$$= -\frac{1}{g^2} \int_{\partial M} \underbrace{\delta A \wedge * dA}_{0} + \frac{1}{g^2} \int_M \delta A \wedge d * dA$$

$$= \int_M \frac{1}{g^2} \delta A \wedge d * dA$$

So for

$$S_M = -\frac{1}{2g^2} \int_M dA \wedge * dA,$$

$$\delta S_M = \int_M \frac{1}{g^2} \delta A \wedge d * dA$$

Meaning if $S_{MCS} = S_M + S_{cs}$

$$S_{MCS} = \int_M -\frac{1}{2g^2}\, dA \wedge * dA + \int_M \frac{K}{4\pi}\, A \wedge dA$$

$$S_{MCS} = \int_M -\frac{1}{2g^2}\, dA \wedge * dA + \frac{K}{4\pi}\, A \wedge dA$$

then $\delta S_{MCS} = \int_M \frac{1}{g^2}\,\delta A \wedge d * dA + \int_M \frac{k}{2\pi}\,\delta A \wedge dA.$

$$\delta S_{MCS} = \int_M \delta A \wedge \left(\frac{1}{g^2}\, d * dA + \frac{k}{2\pi}\, dA \right)$$

That is, if

$$S_{MCS} = \int_M -\frac{1}{2g^2}\, dA \wedge * dA + \frac{K}{4\pi}\, A \wedge dA$$

then

$$\delta S_{MCS} = \int_M \delta A \wedge \left(\frac{1}{g^2}\, d * dA + \frac{k}{2\pi}\, dA \right)$$

Setting $\delta S_{MCS} = 0 \ \forall\, \delta A$,

See eom $$\frac{1}{g^2}\, d * dA + \frac{k}{2\pi}\, dA = 0$$

ie $$\boxed{d * dA + \frac{g^2 k}{2\pi}\, dA = 0} \ * \implies d * dA = -\frac{g^2 k}{2\pi}\, dA$$

Taking $*$ on both sides

$$*d*dA + \frac{g^2 k}{2\pi} * dA = 0$$

Then using $d^\dagger = *d*$

$$d^\dagger dA + \frac{g^2 k}{2\pi} * dA = 0$$

Now, $d^\dagger dA = -\frac{g^2 k}{2\pi} * dA$

So $d[d^\dagger dA] = -\frac{g^2 k}{2\pi} d * dA$

$$= -\frac{g^2 k}{2\pi} \times -\frac{g^2 k}{2\pi} dA$$

So $d[d^\dagger dA] = \left(\frac{g^2 k}{2\pi}\right)^2 dA$

$$\iff dd^\dagger F = \left(\frac{g^2 k}{2\pi}\right)^2 F$$

Mass $\frac{g^2 k}{2\pi}$

$$\boxed{Z = \int DF_m \, DA_e \exp\left(i\int -\frac{g^2}{2} F_m \wedge * F_m + F_m \wedge dA_e\right)}$$

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Define $G W_n = \exp\left(\frac{i 2\pi n}{k}\right) W_n$

with $W_n = \exp\left(i n \oint a_e\right)$

MCS eom is $d\left(\frac{1}{g^2} * da_e - \frac{k}{2\pi} a_e\right) = j_e \equiv dJ_e$.

with j_e the current on a disc D s.t. $C = \partial D$.

$$\int_c d\left(\frac{1}{g^2} * da_e - \frac{k}{2\pi} a_e\right) = \int_c dJ_e.$$

$$\frac{1}{g^2} * da_e - \frac{k}{2\pi} a_e = J_e.$$

$$\Rightarrow G = \exp\left(i \frac{2\pi}{k} \oint_c \frac{k}{2\pi} a_e - \frac{1}{g^2} * a_e\right)$$

is the generator of a $\mathbb{Z}_k$ symmetry.

$$G = \exp\left(i \oint_c b\right)$$

So $$G W_n = \exp\left(\frac{i 2\pi n}{k}\right) W_n$$

$$= \exp\left(\frac{i 2\pi n}{k}\right) \exp\left(i \oint_c b\right)$$

$$G W_n = \exp\left(i\left(\frac{2\pi n}{k} + \oint_c b\right)\right).$$

$$G W_{n+k} = \exp\left(\frac{i 2\pi k}{k} + \left(i\left(\frac{2\pi n}{k} + \oint_c b\right)\right)\right)$$

$$G W_{n+k} = G W_n$$

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$$\delta S_{CS} = \frac{k}{4\pi} \int_M d^3x \; \varepsilon^{\mu\nu\rho} \partial_\rho (\omega \partial_\nu A_\rho)$$

Choose $\omega = \frac{2\pi\tau}{\beta}$

$$= \frac{k}{4\pi} \int_M d^3x \; \varepsilon^{\mu\nu\rho} \partial_\rho (\omega \partial_\nu A_\rho)$$

$$= \frac{k}{4\pi} \frac{2\pi}{\beta} \int_M d^3x \; \varepsilon^{\mu\nu\rho} \partial_\rho (\tau \partial_\nu A_\rho)$$

$$= \frac{k}{4\pi} \frac{2\pi}{\beta} \int_{S^2} d^2x \int_{S^1} d\tau \; \varepsilon^{\mu\nu\rho} \partial_\rho (\tau \partial_\nu A_\rho)$$

$$= \frac{k}{4\pi} \frac{2\pi}{\beta} \int_{S^2} d^2x \int_{S^1} d\tau \; \varepsilon^{\mu\nu\rho} \left[(\partial_\rho \tau) \partial_\nu A_\rho + \tau \partial_\nu A_\rho\right]$$

$$= \frac{k}{2\beta} \int_{S^2} d^2x \int_{S^1} d\tau \left[\varepsilon^{0\nu\rho} \partial_\nu A_\rho + \varepsilon^{\mu\nu\rho} \partial_\mu \partial_\nu A_\rho\right]$$

$$= \frac{k}{2\beta} \times \beta \int_{S^2} d^2x \left[\varepsilon^{\nu\rho} \partial_\nu A_\rho + \varepsilon^{\mu\nu\rho} \cancel{\partial_\mu} \partial_\nu A_\rho \right]$$

0 as no boundary

$$= \frac{k}{2} \int_{S^2} \partial_1 A_2 - \partial_2 A_1 + 0$$

$$= \frac{k}{2} \int_{S^2} F_{12}$$

$$= \frac{k}{2} \times 2\pi n$$

$$= k n \pi$$

choose $n=1$, got $k\pi$

$$S_{CS}[A] = \frac{k}{4\pi} \int_M d^3x \; \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

$$A_\mu \to A_\mu' = A_\mu + \partial_\mu w$$

$$S_{CS}[A] \to S_{CS}[A'] = \frac{k}{4\pi} \int_M d^3x \; \varepsilon^{\mu\nu\rho} (A_\mu + \partial_\mu w) \partial_\nu (A_\rho + \partial_\rho w)$$

$$= \frac{k}{4\pi} \int_M d^3x \; \varepsilon^{\mu\nu\rho} \Big(A_\mu (\partial_\nu A_\rho + \partial \partial_\rho w) + (\partial_\mu w) \partial_\nu (A_\rho + \partial_\rho w) \Big)$$

$$= \frac{k}{4\pi} \int_M d^3x \; \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu \partial_\rho w + \varepsilon^{\mu\nu\rho} (\partial_\mu w) \partial_\nu A_\rho + \varepsilon^{\mu\nu\rho} (\partial_\mu w) \partial_\nu \partial_\rho w$$

0 by antisymmetry

$$= S_{CS}[A] + \frac{k}{4\pi} \int_M d^3x \; \varepsilon^{\mu\nu\rho} \left[A_\mu \partial_\nu \partial_\rho w + (\partial_\mu w) \partial_\nu A_\rho + \cancel{(\partial_\mu w) \partial_\nu \partial_\rho w} \right]$$

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$$= S_{cs}[A] + \frac{k}{4\pi}\int_M d^3x\ \varepsilon^{\mu\nu\rho}\left[\cancel{A_\mu \partial_\nu \partial_\rho w} + (\partial_\mu w)\partial_\nu A_\rho\right]$$

$$= S_{cs}[A] + \frac{k}{4\pi}\int_M d^3x\ \varepsilon^{\mu\nu\rho}\ (\partial_\mu w)\partial_\nu A_\rho$$

$$= S_{cs}[A] + \frac{k}{4\pi}\int_M d^3x\ \varepsilon^{\mu\nu\rho}\left[\partial_\mu(w\partial_\nu A_\rho) - \cancel{w\partial_\mu\partial_\nu A_\rho}\right]$$

$$= S_{cs}[A] + \frac{k}{4\pi}\int_M d^3x\ \varepsilon^{\mu\nu\rho}\ \partial_\mu(w\partial_\nu A_\rho)$$

$D_\mu w = \partial_\mu w + [A_\mu, w]$

$F_{\mu\nu} = [D_\mu, D_\nu]$

$F_{\mu\nu}\phi = [D_\mu, D_\nu]\phi$

Meeting with Adi 11/1/24

So far have done:

- Recap on forms.
- Derived eom for MCS
  got $dd^{+}F = \left(\frac{g^2 k}{2\pi}\right)^2 F$ $\quad \partial^2\phi = m^2\phi$

is this why the mass is $\frac{g^2 k}{2\pi}$?

- Intro to Wilson loops, Checked $G W_{n+k} = G W_n$ as from paper.
- Showed that $\delta S_{cs} = \frac{k}{4\pi} \int_M d^3x \, \varepsilon^{\mu\nu\rho} \partial_\nu (\omega \partial_\mu A_\rho)$

and that if $\omega = \frac{2\pi\tau}{\beta}$, $\delta S_{cs} = kn\pi$

provided $\int_{S^2} F_{12} = 2\pi n$

Is there a more general condition for this, say on S^3 that I should know about?

Charge quantisation.

- Question, how to approach the integration from (5) to (6) or from (5) to (7) in the paper?

Any pointers for getting practice on these kinds of integrals?

$$\int dx \, e^{-ax^2+bx} = e^{\frac{b^2}{2a}}$$

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$$L = -\frac{1}{2g^2} da_e \wedge * da_e + \frac{k}{4\pi} a_e \wedge da_e$$

has mass $M = \frac{g^2 k}{2\pi}$

Take $Z = \int DF_m \, DA_e \exp i \int -\frac{g^2}{2} F_m \wedge * F_m + F_m \wedge dA_e.$

Happy that $\int d^N\vec{x}\; e^{-\frac{1}{2}\vec{x}^T \underline{A}\vec{x} + \vec{b}^T\vec{x}} = \left(\frac{(2\pi)^N}{\det(\underline{A})}\right)^{\frac{1}{2}} e^{\frac{1}{2}\vec{b}^T\underline{A}^{-1}\vec{b}}$

Now want $Q = \int \mathcal{D}[f(x)]\; e^{-\frac{1}{2}\int dx\,dy\, f(x)A(x,y)f(y) + \int dx\, b(x)f(x)}$

$A(x,y) \to \vec{A}$, $f(x) \to \vec{x}$ as an infinite dimensional vector.

$$\int \mathcal{D}[f(x)]\; e^{-\frac{1}{2}\int dx\,dy\, f(x)A(x,y)f(y) + \int dx\, b(x)f(x)} \longrightarrow \int d^N x\; e^{-\frac{1}{2}\vec{x}^T A\vec{x} + \vec{b}^T\vec{x}}.$$

$\Rightarrow$

$$\int \mathcal{D}[f(x)]\; e^{-\frac{1}{2}\int dx\,dy\, f(x)A(x,y)f(y) + \int dx\, b(x)f(x)} = (2\pi)^N\left[\det(A(x,y))\right]^{-1/2} e^{\frac{1}{2}\int dx\,dy\; b(x)\,A^{-1}(x,y)\,b(y)}$$

where $\int dz\, A(x,z)\, A^{-1}(z,y) = \delta(x-y)$

i.e. $\int \mathcal{D}[f]\, e^{-\frac{1}{2}\int fAf + bf} = N e^{\frac{1}{2}\int bA^{-1}b}$

For $\mathcal{L} = -\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right) + \frac{1}{2}m^2 A_\mu A^\mu$

Need $\mathcal{L}$ in the form $\frac{1}{2}A^\mu K_{\mu\nu} A^\nu$

$$-\int \partial_\mu A_\nu \partial^\mu A^\nu = -\int \partial_\mu(A_\nu \partial^\mu A^\nu) + \int \underbrace{A_\nu \partial^2 A^\nu}_{\frac{1}{2}A^\mu \partial^2 g_{\mu\nu} A^\nu}$$

$$\text{So } \mathcal{L} = \frac{1}{2} A^\mu K_{\mu\nu} A^\nu + \text{Bounds}$$

$$\text{with } K_{\mu\nu} = (\partial^2 + m^2) g_{\mu\nu} - \partial_\mu \partial_\nu$$

$$\frac{\int DA\, e^{\frac{i}{2}\int d^4x\, A^\mu \hat{K}_{\mu\nu} A^\nu + i\int d^4x\, J_\mu A^\mu}}{\int DA\, e^{\frac{i}{2}\int d^4x\, A^\mu \hat{K}_{\mu\nu} A^\nu}} = e^{-\frac{i}{2}\int d^4x\, d^4y\, J^\mu(x)\, \hat{K}^{-1}_{\mu\nu}\, J^\nu(y)}$$

$$Z = \int DF_m\, DA_e\, \exp i \int -\frac{g^2}{2} F_m \wedge * F_m + F_m \wedge dA_e .$$

$$F_m = \frac{1}{2!} F_{(m)\mu\nu}\, dx^\mu \wedge dx^\nu$$

$$(* w)_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \sqrt{|g|}\, \varepsilon_{\mu_1 \cdots \mu_{n-p} \nu_1 \cdots \nu_p} w^{\nu_1 \cdots \nu_p}$$

In 4D

$$(* F)_{\alpha\beta} = \frac{1}{(4-2)!} \sqrt{|g|}\, \varepsilon_{\alpha\beta\rho\sigma} F^{\rho\sigma}$$

$$\text{So } * F = \frac{1}{2!} \cdot \frac{1}{2!} \sqrt{|g|}\, \varepsilon_{\alpha\beta\rho\sigma} F^{\rho\sigma} dx^\alpha \wedge dx^\beta$$

$$\text{So } F \wedge * F = \frac{1}{2!} F_{(m)\mu\nu}\, dx^\mu \wedge dx^\nu \wedge \frac{1}{2!} \cdot \frac{1}{2!} \sqrt{|g|}\, \varepsilon_{\alpha\beta\rho\sigma} F^{\rho\sigma} dx^\alpha \wedge dx^\beta$$

$$= \frac{1}{2!} \frac{1}{2!} \frac{1}{2!} \sqrt{|g|}\, F_{\mu\nu}\, \varepsilon_{\alpha\beta\rho\sigma} F^{\rho\sigma}\, dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta$$

$$= \frac{1}{4!} 3 \sqrt{|g|}\, \varepsilon_{\alpha\beta\rho\sigma} F_{\mu\nu} F^{\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta$$

$$W = \frac{1}{p!} w_{\mu_1 \cdots \mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$$\int_O f v = \int_{\mathbb{R}^n} dx_1 \cdots dx_n\, f(x) V(x)$$

$$\text{Naturally } V = \sqrt{-g}\, dx^0 \wedge \cdots \wedge dx^{n-1}$$

$$\text{giving } \int_M f v = \int_M d^n x \sqrt{-g}\, f$$

$$= \frac{1}{4!} 3\sqrt{|g|}\, \varepsilon_{\alpha\beta\rho\sigma} F_{\mu\nu} g^{\gamma\delta} F_{\gamma\delta}\, dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta$$

$$-\frac{1}{2}\int F \wedge * F = -\frac{1}{4}\int d^4x \sqrt{-g}\, g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}$$

$$\int_M f V = \int_M d^n x \sqrt{\pm g}\, f$$

$$V = \sqrt{-g}\, dx^0 \wedge \cdots \wedge dx^{n-1}$$

$$d(*\omega) = \frac{1}{p!(n-p)!} \partial_\alpha (\sqrt{|g|}\, \varepsilon_{\mu_1 \cdots \mu_{n-p} \nu_1 \cdots \nu_p} \omega^{\nu_1 \cdots \nu_p})\, dx^\alpha \wedge dx^{\mu_1} \wedge \cdots dx^{\mu_{n-p}}$$

$$\varepsilon_{\mu_1 \cdots \mu_{n-p} \nu_1 \cdots \nu_p} \partial_\alpha (\sqrt{|g|}\, F^{\nu_1 \cdots \nu_p}) = 0$$

$$\boxed{\varepsilon_{\mu\nu\rho\sigma} \partial_\alpha (\sqrt{|g|}\, F^{\rho\sigma}) = 0}$$

$$F \wedge (*F) = \underbrace{\left(\frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu\right)}_{F} \wedge \underbrace{\left(\frac{1}{2}\sqrt{g}\, \varepsilon_{\alpha\beta\rho\sigma} F^{\alpha\beta}\, dx^\rho \wedge dx^\sigma\right)}_{*F}$$

$$= \frac{1}{4} F_{\mu\nu} (*F)_{\rho\sigma}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$$

$$= \frac{1}{4} F_{\mu\nu} \sqrt{-g}\, \varepsilon_{\rho\sigma\alpha\beta} F^{\alpha\beta}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$$

$$= \frac{1}{4}\sqrt{-g}\, F_{\mu\nu} \varepsilon_{\rho\sigma\alpha\beta}\, g^{\gamma\alpha} g^{\delta\beta} F_{\gamma\delta}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$$

$$= \frac{1}{4}\sqrt{-g}\, \underline{\varepsilon_{\rho\sigma\alpha\beta} F_{\mu\nu}}\, F_{\gamma\delta} g^{\gamma\alpha} g^{\delta\beta}\, \underline{dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma}$$

$$= \frac{2}{4} F_{\alpha\beta} F_{\gamma\delta}\, g^{\gamma\alpha} g^{\delta\beta} \quad \checkmark$$

$$= \frac{1}{2} F_{\alpha\beta} F_{\gamma\sigma}\, g^{\gamma\alpha} g^{\sigma\beta} \sqrt{-g}\, dx^0 dx^1 dx^2 dx^3$$

$$= \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} \sqrt{-g}\, dx^0 dx^1 dx^2 dx^3$$

$$A_{\mu_1 \cdots \mu_n}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$$

$$= n!\, A_{123\cdots n}\, dx^1 dx^2 dx^3 dx^n$$

$$F_{\mu\nu} \varepsilon_{\rho\sigma\alpha\beta} = (F\varepsilon)_{\mu\nu\rho\sigma\alpha\beta}$$

$$F_{\mu\nu} \varepsilon_{\rho\sigma\alpha\beta}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$$

$$= (F\varepsilon)_{\mu\nu\rho\sigma\alpha\beta}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$$

$$= \frac{n!\, F_{\alpha\beta}\, V}{\sqrt{-g}}$$

$$\int d\alpha \wedge *J = \int \alpha \wedge * d * J = 0$$

$$*(d*J) = 0$$

$$\Leftrightarrow d*J = 0$$

$$V = \sqrt{|g|}\, dx^1 \wedge \cdots \wedge dx^n$$

$$= \frac{1}{n!}\sqrt{|g|}\, \varepsilon_{\mu_1 \cdots \mu_n}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$$

$$\Longrightarrow \frac{1}{n!}\sqrt{|g|}\,\varepsilon_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} = \sqrt{|g|}\quad dx^1\wedge\cdots\wedge dx^n$$

$$\varepsilon_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} = n!\, dx^1\wedge\cdots\wedge dx^n$$

So $F\wedge * F$

$$= \frac{1}{2}F_{\alpha\beta}\,dx^\alpha\wedge dx^\beta \wedge \frac{1}{2}\left(\frac{1}{2}\sqrt{|g|}\,\varepsilon_{\mu_1\mu_2\nu_1\nu_2}F^{\nu_1\nu_2}\right)dx^{\mu_1}\wedge dx^{\mu_2}$$

$$= \frac{1}{2}F_{\alpha\beta}\,\frac{1}{2}\,\frac{1}{2}\sqrt{|g|}\,\varepsilon_{\mu_1\mu_2\nu_1\nu_2}F^{\nu_1\nu_2}\,dx^\alpha\wedge dx^\beta\wedge dx^{\mu_1}\wedge dx^{\mu_2}$$

$$= \frac{1}{2}F_{\alpha\beta}\,\frac{1}{2}\,\frac{1}{2}\sqrt{|g|}\quad\varepsilon_{\nu_1\nu_2}F^{\nu_1\nu_2}\,dx^\alpha\wedge dx^\beta\wedge(2)\,dx^1\wedge dx^2$$

$$= \frac{1}{2}\,\frac{1}{2}F_{\alpha\beta}\sqrt{|g|}\,\varepsilon_{\nu_1\nu_2}F^{\nu_1\nu_2}\quad dx^\alpha\wedge dx^\beta\wedge dx^1\wedge dx^2$$

$$(*\omega)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\sqrt{|g|}\,\varepsilon_{\mu_1\cdots\mu_{n-p}\nu_1\cdots\nu_p}\,\omega^{\nu_1\cdots\nu_p}$$

$$(*\omega) = \frac{1}{(n-p)!}\,\frac{1}{p!}\sqrt{|g|}\,\varepsilon_{\mu_1\cdots\mu_{n-p}\nu_1\cdots\nu_p}\,\omega^{\nu_1\cdots\nu_p}\quad dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{n-p}}$$

$$F = \frac{1}{2}F_{\alpha\beta}\,dx^\alpha\wedge dx^\beta$$

$$(*F)_{\mu_1\mu_2} = \frac{1}{2!}\sqrt{|g|}\,\varepsilon_{\mu_1\mu_2\nu_1\nu_2}F^{\nu_1\nu_2}$$

$$\text{So } *F = \frac{1}{2!}\,\frac{1}{2!}\sqrt{|g|}\,\varepsilon_{\mu_1\mu_2\nu_1\nu_2}F^{\nu_1\nu_2}\,dx^{\mu_1}\wedge dx^{\mu_2}$$

$$(\omega\wedge\eta)_{\mu_1\cdots\mu_p\nu_1\cdots\nu_q} = \frac{(p+q)!}{p!\,q!}\,\omega_{[\mu_1\cdots\mu_p}\eta_{\nu_1\cdots\nu_q]}$$

$$(F\wedge *F)_{\mu_1\mu_2\nu_1\nu_2} = \frac{(2+2)!}{2!\;2!}\,F_{[\mu_1\mu_2}{*F}_{\nu_1\nu_2]}$$

$$= 6\,F_{[\mu_1\mu_2}{*F}_{\nu_1\nu_2]}$$

$$(*F)_{\alpha\beta} = -\sqrt{-g}\,\varepsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$$

$$(F\wedge *F)_{\alpha\beta\gamma\delta} = F_{\alpha\beta}(*F)_{\gamma\delta} - F_{\gamma\delta}(*F)_{\alpha\beta}$$

$$= F_{\alpha\beta}\times -\sqrt{-g}\,\varepsilon_{\gamma\delta\rho\sigma}F^{\rho\sigma} - F_{\gamma\delta}\times -\sqrt{-g}\cdot\varepsilon_{\alpha\beta\rho\sigma}F^{\rho\sigma}$$

$$(F_{\alpha\beta}-F_{\beta\alpha})({}^*F_{\gamma\delta}-{}^*F_{\delta\gamma})$$

$$= 2^2 F_{\alpha\beta}{}^*F_{\gamma\delta}$$

$$= 4\,F_{\alpha\beta}{}^*F_{\gamma\delta}$$

$$X = -\frac{g^2}{2}a_m\wedge *a_m + a_m\wedge da_e + \frac{k}{4\pi}a_e\wedge da_e$$

Let $a_e = b - \frac{2\pi}{k}a_m$

$$X = -\frac{g^2}{2}a_m\wedge *a_m + a_m\wedge d\left(b-\frac{2\pi}{k}a_m\right) + \frac{k}{4\pi}\left(b-\frac{2\pi}{k}a_m\right)\wedge d\left(b-\frac{2\pi}{k}a_m\right)$$

$$= -\frac{g}{2}a_m\wedge *a_m + a_m\wedge db - \frac{2\pi}{k}a_m\wedge da_m + \frac{k}{4\pi}\left(b\wedge db - \frac{2\pi}{k}b\wedge da_m - \frac{2\pi}{k}a_m\wedge db + \left(\frac{2\pi}{k}\right)^2 a_m\wedge da_m\right)$$

$$= -\frac{g}{2}a_m\wedge *a_m + a_m\wedge db - \frac{2\pi}{k}a_m\wedge da_m + \frac{k}{4\pi}b\wedge db - \frac{1}{2}b\wedge da_m - \frac{1}{2}a_m\wedge db + \frac{\pi}{k}a_m\wedge da_m$$

$$= -\frac{g}{2}a_m\wedge *a_m - \frac{\pi}{k}a_m\wedge da_m + \frac{k}{4\pi}b\wedge db \quad + a_m\wedge db - \frac{1}{2}b\wedge da_m - \frac{1}{2}a_m\wedge db$$

$$= -\frac{g}{2}a_m\wedge *a_m - \frac{\pi}{k}a_m\wedge da_m + \frac{k}{4\pi}b\wedge db + \frac{1}{2}\left(a_m\wedge db - b\wedge da_m\right)$$

$$= -\frac{g}{2}a_m\wedge *a_m - \frac{\pi}{k}a_m\wedge da_m + \frac{k}{4\pi}b\wedge db + \frac{1}{2}\left(d(a_m\wedge b) - da_m\wedge b - b\wedge da_m\right)$$

$$= -\frac{g}{2} a_m \wedge * a_m - \frac{\pi}{k} a_m \wedge da_m + \frac{k}{4\pi} b \wedge db + \frac{1}{2} d(a_m \wedge b)$$

0

$$\int F \wedge * F$$
$$= \int \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu \wedge \frac{1}{2} (*F)_{\alpha\beta} dx^\alpha \wedge dx^\beta$$

$$= \int \frac{1}{2} \frac{1}{2} f_{\mu\nu} (*F)_{\alpha\beta} dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta$$

$$F = B_3 dx \wedge dy + B_1 dy \wedge dz + B_2 dz \wedge dx + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt$$

$$F \wedge F = (B_3 dx \wedge dy + B_1 dy \wedge dz + B_2 dz \wedge dx + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt) \wedge (B_3 dx \wedge dy + B_1 dy \wedge dz + B_2 dz \wedge dx + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt)$$

$$= B_3 E_3 dx \wedge dy \wedge dz \wedge dt + B_1 E_1 dy \wedge dz \wedge dx \wedge dt + B_2 E_2 dz \wedge dx \wedge dy \wedge dt$$
$$+ E_1 B_1 dx \wedge + \cdots$$

$$= -2(E_1 B_1 + E_2 B_2 + E_3 B_3) dt \wedge dx \wedge dy \wedge dz$$

$$F = B_3 dx \wedge dy + B_1 dy \wedge dz + B_2 dz \wedge dx + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt$$

$$*(dt \wedge dx) = -dy \wedge dz$$
$$*(dt \wedge dy) = -dz \wedge dx$$
$$*(dt \wedge dz) = -dx \wedge dy$$
$$*(dx \wedge dy) = dt \wedge dz$$
$$*(dx \wedge dz) = -dt \wedge dy$$
$$*(dy \wedge dz) = dt \wedge dx$$

$$*F = E_3 dx \wedge dy + E_1 dy \wedge dz + E_2 dz \wedge dx - B_1 dx \wedge dt - B_2 dy \wedge dt - B_3 dz \wedge dt$$

So $F \wedge * F = (B_3 dx \wedge dy + B_1 dy \wedge dz + B_2 dz \wedge dx + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt)$

$$\wedge \left(E_3\, dx\wedge dy + E_1\, dy\wedge dz + E_2\, dz\wedge dx - B_1 dx\wedge dt - B_2 dy\wedge dt - B_3 dz\wedge dt\right)$$

$$= -B_3^2\, dx\wedge dy\wedge dz\wedge dt - B_1^2\, dy\wedge dz\wedge dx\wedge dt - B_2^2\, dz\wedge dx\wedge dy\wedge dt$$
$$+ E_1^2\, dx\wedge dt\wedge dy\wedge dz + E_2^2\, dy\wedge dt\wedge dz\wedge dx + E_3^2\, dz\wedge dt\wedge dx\wedge dy$$

$$= (E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2)\, dx\wedge dy\wedge dz\wedge dt$$

$$= (\vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B})\, dx\wedge dy\wedge dz\wedge dt$$

$$= F_{\mu\nu}F^{\mu\nu}\ dx\wedge dy\wedge dz\wedge dt$$

So $\int F\wedge F = \int F_{\mu\nu}F^{\mu\nu}\ dx\wedge dy\wedge dz\wedge dt$

$$= \int d^4x\, F_{\mu\nu}F^{\mu\nu}$$

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$$F\wedge * F \Longrightarrow *F\wedge F = -F\wedge * F$$
$$F \Longleftrightarrow * F$$

Type IIB : R-R Fields present

Have C, C_μ and $C_{\mu\nu\rho\sigma}$. And $F_5 = *F_5$ 'almost'.

'magnetically'

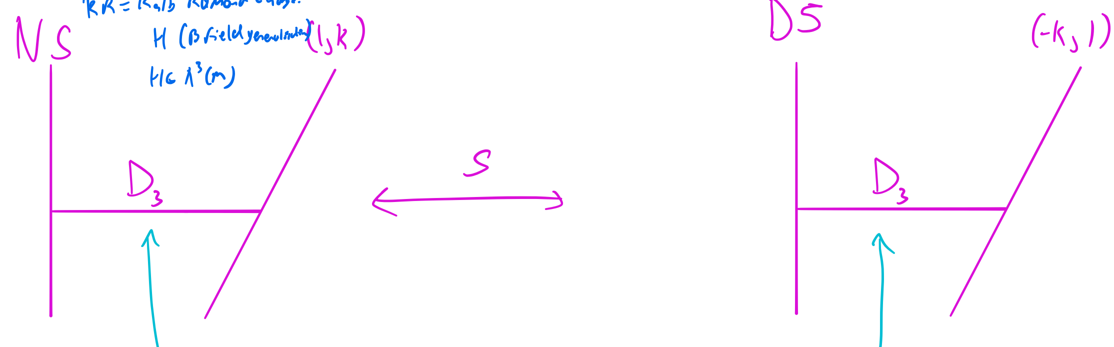
(Neveu-Schwarz) is charged under

RR = Kalb Ramond charge.

H (B field generalisation)

$H \in \Lambda^3(M)$

Charged under RR-field of the superstring C_p.



MCS here is S-Dual to magnetic MCS here.

NS5 $\longleftrightarrow$ D5 under S duality + Spatial rotation $(x^4, x^5, x^6) \longleftrightarrow (x^7, x^8, x^9)$.

$$\int DF_m \exp i\int -\frac{g^2}{2} F_{(m)}^{\mu\nu} F^{(m)}_{\mu\nu} + F_{(m)} \wedge dA_e$$

$$= N \exp i\int -\frac{1}{2g^2} dA_e \wedge * dA_e$$

$$\int e^{\frac{1}{2}ax^2+Jx} dx = \sqrt{\frac{\pi}{a}}\, e^{\frac{J^2}{2a}}$$

$$Z_{4D} = \int DF_m\, DA_e \exp i\int -\frac{g^2}{2} F_m \wedge * F_m + F_m \wedge dA_e$$

Upon 4D $\to$ 3D, get $F_m \to$ $f_m \in \Lambda^2(M)$, $a_m \in \Lambda^1(M)$.

and $A_e \to a_e$, gauge field. φ_e scalar

Stuck here.

$$Z \to Z_{3D} = \int Df_m\, D\varphi_e \exp i\int -\frac{g^2}{2} f_m \wedge * f_m + f_m \wedge d\varphi_e.$$

is it that $F_m \wedge * F_m \to f_m \wedge * f_m + a_m \wedge_{(3)} * a_m$

$F_m \wedge dA_e \to f_m \cancel{\wedge} dA_e + a_m \wedge dA_e{}_{(3)}$

$= f_m \wedge da_e + f_m \wedge d\varphi_e + a_m \wedge da_e + a_m \wedge \varphi_e$

$$\text{Taking } Z = \int Df_m\, D\varphi_e \exp i\int -\frac{g^2}{2} f_m \wedge * f_m + f_m \wedge d\varphi_e$$

$$= \int Df_m\, \delta(f_m) \exp i\int -\frac{g^2}{2} f_m \wedge * f_m$$

write $f_m = da_m$

$$= \int Da_m \exp i\int -\frac{g^2}{2} da_m \wedge * da_m$$

$$\int D\varphi_e \exp i\int -\frac{``d\varphi_e"^2}{2g^2}$$

$$= \int D\varphi_e \exp i \int -\frac{1}{2g^2} d\varphi_e \wedge * d\varphi_e$$

$$da_m \longleftrightarrow *d\varphi_e \quad \times$$

Should have $*$ in the duality

$$Z = \int Da_m Da_e \exp i \int -\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge da_e + \frac{k}{4\pi} a_e \wedge da_e)$$

So $S = \int -\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge da_e + \frac{k}{4\pi} a_e \wedge da_e)$

$$S[a_m + \delta a_m, a_e + \delta a_e] = \int -\frac{g^2}{2}(a_m + \delta a_m) \wedge *(a_m + \delta a_m) + (a_m + \delta a_m) \wedge d(a_e + \delta a_e) + \frac{k}{4\pi}(a_e + \delta a_e) \wedge * d(a_e + \delta a_e)$$

$$= \int -\frac{g^2}{2}\left(a_m \wedge * a_m + a_m \wedge * \delta a_m + \delta a_m \wedge * a_m + \delta a_m \wedge * \delta a_m\right) + a_m \wedge da_e + a_m \wedge d(\delta a_e) + \delta a_m \wedge da_e + \delta a_m \wedge d(\delta a_e)$$
$$+ \frac{k}{4\pi}\left(a_e \wedge * da_e + a_e \wedge * d(\delta a_e) + \delta a_e \wedge * da_e + \delta a_e \wedge * \delta a_e\right)$$

$$= S[a_m, a_e] + \int -\frac{g^2}{2}\left(a_m \wedge * \delta a_m + \delta a_m \wedge * a_m\right) + a_m \wedge d(\delta a_e) + \delta a_m \wedge da_e + \frac{k}{4\pi}\left(+ a_e \wedge * d(\delta a_e) + \delta a_e \wedge * da_e\right)$$
$$+ O(\delta^2)$$

So linear in δ,

$$\delta S[a_m, a_e] = \int -\frac{g^2}{2}\left(a_m \wedge * \delta a_m + \delta a_m \wedge * a_m\right) + a_m \wedge d(\delta a_e) + \delta a_m \wedge da_e + \frac{k}{4\pi}\left(+ a_e \wedge * d(\delta a_e) + \delta a_e \wedge * da_e\right)$$

Compare $Z = \int Da_m Da_e \exp i \int \left(-\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge da_e + \frac{k}{4\pi} a_e \wedge da_e\right)$

$$Z_e = \int Da_e \exp i \int -\frac{1}{2g^2} da_e \wedge * da_e + \frac{k}{4\pi} a_e \wedge da_e \qquad a_e = b - \left(\frac{2\pi}{k}\right) a_m$$

$$Z_n = \int Da_m Db \exp i \int -\frac{g^2}{2} a_m \wedge * a_m - \frac{\pi}{k} a_m \wedge da_m + \frac{k}{4\pi} b \wedge db$$

$$** \omega = (-1)^{k(n-k)} \omega$$
for $\omega \in \Lambda^k(M^n)$

$-\frac{g^2}{2} a_m \wedge * a_m$, Let $a_m \to -\frac{1}{g^2} * da_e$

get $-\frac{g^2}{2}\left(-\frac{1}{g^2} * da_e\right) \wedge *\left(-\frac{1}{g^2} * da_e\right)$

$= -\frac{1}{2g^2} * da_e \wedge ** da_e$

$= -\frac{1}{2g^2} * da_e \wedge da_e$

$= \frac{1}{2g^2} da_e \wedge * da_e$

Think $a_e \in \Lambda^1$ so $k=1, n=3$

$(-1)^{1(3-1)} = (-1)^2 = 1.$

for $da_e \in \Lambda^2(M^3)$
$k=2, n=3$
$(-1)^{2(3-2)} = 1$

$b = a_e - \frac{1}{M} * da_e$; $M = \frac{g^2 k}{2\pi}$

$a_e = b + \frac{1}{M} * da_e$

$= b + \frac{1}{\left(\frac{g^2 k}{2\pi}\right)} * - g^2 a_m$

$= b - \frac{k}{2\pi} a_m$

$-\frac{1}{2g^2} da_e \wedge * da_e + \frac{k}{4\pi} a_e \wedge da_e$

$= -\frac{1}{2g^2} d\left(b - \frac{2\pi}{k} a_m\right) \wedge * d\left(b - \frac{2\pi}{k}\right) a_m + \frac{k}{4\pi}\left(b - \frac{2\pi}{k} a_m\right) \wedge d\left(b - \frac{2\pi}{k} a_m\right)$

$= -\frac{1}{2g^2}\left(db \wedge * db - \frac{2\pi}{k} da_m \wedge * db - \left(\frac{2\pi}{k}\right) db \wedge * da_m + \left(\frac{2\pi}{k}\right)^2 da_m \wedge * da_m\right)$

$+ \frac{k}{4\pi}\left(b \wedge db - \frac{2\pi}{k} b \wedge da_m - \frac{2\pi}{k} a_m \wedge db + \left(\frac{2\pi}{k}\right)^2 a_m \wedge da_m\right).$

$= -\frac{1}{2g^2} db \wedge * db + \frac{\pi}{g^2 k}\left(da_m \wedge * db + db \wedge * da_m\right) - \frac{2\pi^2}{g^2 k^2} da_m \wedge * da_m$

$+ \frac{k}{4\pi} b \wedge db - \frac{1}{2}\left(b \wedge da_m + a_m \wedge db\right) + \frac{\pi}{k} a_m \wedge da_m.$

$-\frac{g^2}{2} a_m \wedge * a_m - \frac{\pi}{k} a_m \wedge da_m + \frac{k}{4\pi} b \wedge db$

$$Z = \int Da_m Da_e \exp i \int \left(-\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge da_e + \frac{k}{4\pi} a_e \wedge da_e \right).$$

$$Z_e = \int Da_e \exp i \int -\frac{1}{2g^2} da_e \wedge * da_e + \frac{k}{4\pi} a_e \wedge da_e$$

$$a_e = b - \left(\frac{2\pi}{k}\right) a_m$$

$$Z_n = \int Da_m Db \exp i \int -\frac{g^2}{2} a_m \wedge * a_m - \frac{\pi}{k} a_m \wedge da_m + \frac{k}{4\pi} b \wedge db$$

$$= \int Da_m Db \exp i \int -\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge d\left(b - \frac{2\pi}{k} a_m\right) + \frac{k}{4\pi} \left(b - \left(\frac{2\pi}{k}\right) a_m\right) \wedge d\left(b - \left(\frac{2\pi}{k}\right) a_m\right)$$

$$= \int Da_m Db \exp i \int -\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge db - \frac{2\pi}{k} a_m \wedge da_m + \frac{k}{4\pi} \left(b \wedge db - \frac{2\pi}{k} b \wedge da_m - \frac{2\pi}{k} a_m \wedge db + \frac{4\pi^2}{k^2} a_m \wedge da_m \right).$$

$$= \int Da_m D \quad \int -\frac{g^2}{2} a_m \wedge * a_m + \left(-\frac{2\pi}{k} + \frac{\pi}{k}\right) a_m \wedge da_m + \frac{k}{4\pi} b \wedge db + \left(1 - \frac{1}{2}\right) a_m \wedge db - \frac{1}{2} b \wedge da_m$$

$$= \int Da_m D \quad \int -\frac{g^2}{2} a_m \wedge * a_m + \left(-\frac{2\pi}{k} + \frac{\pi}{k}\right) a_m \wedge da_m + \frac{k}{4\pi} b \wedge db + \cancel{\frac{1}{2} a_m \wedge db} - \frac{1}{2} d(b \wedge a_m) + \cancel{\frac{1}{2} db \wedge a_m}$$

$$= \int Da_m D \quad \int -\frac{g^2}{2} a_m \wedge * a_m + \left(-\frac{2\pi}{k} + \frac{\pi}{k}\right) a_m \wedge da_m + \frac{k}{4\pi} b \wedge db - \cancel{-\frac{1}{2} d(b \wedge a_m)} \to 0$$

$$= \int D a_m D \quad \int -\frac{g^2}{2} a_m \wedge * a_m + -\frac{?}{i} a_m \wedge d a_m + \frac{k}{4\pi} b \wedge db$$

$$T^a \wedge T^b = \frac{1}{2}[T^a, T^b]$$

Write $A = A^a T^a$

$$[T^a, T^b] = i f^{abc} T^c$$

$$\text{So } A \wedge A = A^a T^a \wedge A^b T^b$$

$$= A^a A^b \, T^a \wedge T^b$$

$$= \frac{1}{2} A^a A^b \, [T^a, T^b]$$

$$A \wedge A = \frac{i}{2} A^a A^b f^{abd} T^d$$

$$(A \wedge A) \wedge A = \left(\frac{i}{2} A^a A^b f^{abd} T^d\right) \wedge A^c T^c$$

$$= \frac{i}{2} A^a A^b A^c \, f^{abd} \, T^d \wedge T^c$$

$$= \frac{i}{4} A^a A^b A^c \, f^{abd} \, [T^d, T^c]$$

$$= \frac{i}{4} A^a A^b A^c \, f^{abd} \, i f^{dce} T^e$$

$$= \frac{-1}{4} A^a A^b A^c \, f^{abd} \, f^{dce} T^e$$

— Dimensional reduction?

— Notice we never integrate over b in (13).

~~MCS~~ dual is $Z = \int Da\, Dc \, \exp i \int a \wedge * a + \frac{1}{2\pi} a \wedge dc + \frac{k}{4\pi} c \wedge dc$

MCS is dual $Z = \int Da_m \, Db \, \exp i \int -\frac{g^2}{2} a_m \wedge a_m - \frac{\pi}{k} a_m \wedge da_m + \frac{k}{4\pi} b \wedge db$

MCS is $Z = \int Da_m \, Da_e \, \exp i \int -\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge da_e + \frac{k}{4\pi} a_e \wedge da_e$

i.e. eqn 13 is the dual of eqn 11. ?

$$\boxed{L_{MCS}^{NA} = -\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge (da_e + a_e \wedge a_e) + \frac{k}{4\pi}\left(a_e \wedge da_e + \frac{2}{3} a_e \wedge a_e \wedge a_e\right). \quad \text{Try } \underline{a_e = b - \left(\frac{2\pi}{k}\right) a_m}}$$

$$\to L_{MCS}^{NA} = -\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge \left[\left(db - \left(\frac{2\pi}{k}\right) da_m\right) + \left(b - \left(\frac{2\pi}{k}\right) a_m\right) \wedge \left(b - \left(\frac{2\pi}{k}\right) a_m\right)\right]$$

$$+ \frac{k}{4\pi}\left[\left(b - \left(\frac{2\pi}{k}\right) a_m\right) \wedge \left(db - \left(\frac{2\pi}{k}\right) da_m\right) + \frac{2}{3}\left(b - \left(\frac{2\pi}{k}\right) a_m\right) \wedge \left(b - \left(\frac{2\pi}{k}\right) a_m\right) \wedge \left(b - \left(\frac{2\pi}{k}\right) a_m\right)\right]$$

Need to fix.

$$L_{MCS}^{NA} = -\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge db - \left(\frac{2\pi}{k}\right) a_m \wedge da_m + a_m \wedge b \wedge b - \left(\frac{2\pi}{k}\right) a_m \wedge b \wedge a_m - \left(\frac{2\pi}{k}\right) a_m \wedge a_m \wedge b + \left(\frac{2\pi}{k}\right)^2 a_m \wedge a_m \wedge a_m$$

$$+ \frac{k}{4\pi}\left[b \wedge db - \left(\frac{2\pi}{k}\right) b \wedge da_m - \left(\frac{2\pi}{k}\right) a_m \wedge db + \left(\frac{2\pi}{k}\right)^2 a_m \wedge da_m + \left(b \wedge b - \left(\frac{2\pi}{k}\right) b \wedge a_m - \left(\frac{2\pi}{k}\right) a_m \wedge b + \left(\frac{2\pi}{k}\right)^2 a_m \wedge a_m\right) \wedge \left(b - \left(\frac{2\pi}{k}\right) a_m\right)\right]$$

$$L_{MCS}^{NA} = -\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge db - \left(\frac{2\pi}{k}\right) a_m \wedge da_m + a_m \wedge b \wedge b - \left(\frac{2\pi}{k}\right)\left[a_m \wedge b \wedge a_m + a_m \wedge a_m \wedge b\right] + \left(\frac{2\pi}{k}\right)^2 a_m \wedge a_m \wedge a_m$$

$$+ \frac{k}{4\pi}\left[b \wedge db - \left(\frac{2\pi}{k}\right) b \wedge da_m - \left(\frac{2\pi}{k}\right) a_m \wedge db + \left(\frac{2\pi}{k}\right)^2 a_m \wedge da_m + b \wedge b \wedge b - \left(\frac{2\pi}{k}\right) b \wedge a_m \wedge b - \left(\frac{2\pi}{k}\right) a_m \wedge b \wedge b + \left(\frac{2\pi}{k}\right)^2 a_m \wedge a_m \wedge b\right.$$

$$\left. - \left(\frac{2\pi}{k}\right) b \wedge b \wedge a_m + \left(\frac{2\pi}{k}\right)^2 b \wedge a_m \wedge a_m + \left(\frac{2\pi}{k}\right)^2 a_m \wedge b \wedge a_m - \left(\frac{2\pi}{k}\right)^3 a_m \wedge a_m \wedge a_m\right]$$

$$L_{MCS}^{NA} = -\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge db - \left(\frac{2\pi}{k}\right) a_m \wedge da_m + a_m \wedge b \wedge b - \left(\frac{2\pi}{k}\right)\left[a_m \wedge b \wedge a_m + a_m \wedge a_m \wedge b\right] + \left(\frac{2\pi}{k}\right)^2 a_m \wedge a_m \wedge a_m$$

$$+\frac{k}{4\pi} b\wedge db - \frac{1}{2} b\wedge da_m - \frac{1}{2} a_m\wedge db + \frac{\pi}{k} a_m\wedge da_m + \frac{k}{4\pi} b\wedge b\wedge b - \frac{1}{2} b\wedge a_m\wedge b - \frac{1}{2} a_m\wedge b\wedge b + \frac{\pi}{k} a_m\wedge a_m\wedge b$$

$$-\frac{1}{2} b\wedge b\wedge a_m + \frac{\pi}{k} b\wedge a_m\wedge a_m + \frac{\pi}{k} a_m\wedge b\wedge a_m - \frac{2\pi^2}{k^2} a_m\wedge a_m\wedge a_m.$$

$$L_{\substack{MCS-\\NA}} = \cancel{-\frac{g^2}{2} a_m\wedge * a_m} \cancel{+ a_m\wedge db} \cancel{-\left(\frac{2\pi}{k}\right) a_m\wedge da_m} \cancel{+ a_m\wedge b\wedge b} - \left(\frac{2\pi}{k}\right)\left[\cancel{a_m\wedge b\wedge a_m} + \cancel{a_m\wedge a_m\wedge b}\right] \cancel{+\left(\frac{2\pi}{k}\right)^2 a_m\wedge a_m\wedge a_m}$$

$$\cancel{+\frac{k}{4\pi} b\wedge db} \cancel{-\frac{1}{2} b\wedge da_m} \cancel{-\frac{1}{2} a_m\wedge db} \cancel{+\frac{\pi}{k} a_m\wedge da_m} \cancel{+\frac{k}{4\pi} b\wedge b\wedge b} \cancel{-\frac{1}{2} b\wedge a_m\wedge b} \cancel{-\frac{1}{2} a_m\wedge b\wedge b} \cancel{+\frac{\pi}{k} a_m\wedge a_m\wedge b}$$

$$\cancel{-\frac{1}{2} b\wedge b\wedge a_m} \cancel{+\frac{\pi}{k} b\wedge a_m\wedge a_m} \cancel{+\frac{\pi}{k} a_m\wedge b\wedge a_m} \cancel{-\frac{2\pi^2}{k^2} a_m\wedge a_m\wedge a_m}.$$

$$L_{\substack{MCS-\\NA}} = -\frac{g^2}{2} a_m\wedge * a_m + \frac{1}{2}\left(a_m\wedge db - b\wedge da_m\right) - \frac{\pi}{k} a_m\wedge da_m + \frac{k}{4\pi} b\wedge db$$

$$+\left(a_m\wedge b\wedge b - \frac{1}{2} b\wedge a_m\wedge b - \frac{1}{2} b\wedge b\wedge a_m - \frac{1}{2} a_m\wedge b\wedge b\right)$$

$$+\left(-\left(\frac{2\pi}{k}\right) a_m\wedge b\wedge a_m - \left(\frac{2\pi}{k}\right) a_m\wedge a_m\wedge b + \frac{\pi}{k} a_m\wedge a_m\wedge b + \frac{\pi}{k} b\wedge a_m\wedge a_m + \frac{\pi}{k} a_m\wedge b\wedge a_m\right)$$

$$+\frac{k}{4\pi} b\wedge b\wedge b + \left(\frac{2\pi}{k}\right)^2 a_m\wedge a_m\wedge a_m - \frac{2\pi^2}{k^2} a_m\wedge a_m\wedge a_m$$

$$L_{\substack{MCS-\\NA}} = -\frac{g^2}{2} a_m\wedge * a_m + \frac{1}{2}\left(a_m\wedge db - b\wedge da_m\right) - \frac{\pi}{k} a_m\wedge da_m + \frac{k}{4\pi} b\wedge db$$

$$+\frac{1}{2}\left(a_m\wedge b\wedge b - b\wedge a_m\wedge b - b\wedge b\wedge a_m\right)$$

$$+\left(-\left(\frac{\pi}{K}\right)a_m\wedge b\wedge a_m-\frac{\pi}{K}\,a_m\wedge a_m\wedge b+\frac{\pi}{K}\,b\wedge a_m\wedge a_m\right)$$

$$+\frac{K}{4\pi}\,b\wedge b\wedge b\;+2\left(\frac{\pi}{K}\right)^2\,a_m\wedge a_m\wedge a_m$$

$$\boxed{\begin{aligned}
L_{MCS-NA} = {} & -\frac{g^2}{2}\,a_m\wedge * a_m+\frac{1}{2}\left(a_m\wedge db-b\wedge da_m\right)-\frac{\pi}{K}\,a_m\wedge da_m+\frac{K}{4\pi}\,b\wedge db\\
& +\frac{1}{2}\left(a_m\wedge b\wedge b-b\wedge a_m\wedge b-b\wedge b\wedge a_m\right)\\
& +\frac{\pi}{K}\left(b\wedge a_m\wedge a_m-a_m\wedge b\wedge a_m-a_m\wedge a_m\wedge b\right)\\
& +\frac{K}{4\pi}\,b\wedge b\wedge b\;+2\left(\frac{\pi}{K}\right)^2\,a_m\wedge a_m\wedge a_m
\end{aligned}}$$

$$+\frac{1}{2}\left(a_m\wedge db-b\wedge da_m\right)$$

$$a_m\wedge db-b\wedge da_m=d(a_m\wedge b)-da_m\wedge b-b\wedge da_m$$

$$\sim-\left(da_m\wedge b+b\wedge da_m\right)\qquad\Longrightarrow\;+\frac{1}{2}\left(a_m\wedge db-b\wedge da_m\right)\sim-\frac{1}{2}\left(da_m\wedge b+b\wedge da_m\right)$$

OR

$$a_m\wedge db-b\wedge da_m=a_m\wedge db-d(b\wedge a_m)+db\wedge a_m$$

$$=a_m\wedge db+db\wedge a_m\qquad\Longrightarrow\;+\frac{1}{2}\left(a_m\wedge db-b\wedge da_m\right)\sim\frac{1}{2}\left(a_m\wedge db+db\wedge a_m\right)$$

$$L_{\substack{MCS\\NA}} \sim -\frac{g^2}{2} a_m \wedge * a_m + \frac{1}{2}\left(a_m \wedge db + db \wedge a_m\right) - \frac{\pi}{K} a_m \wedge da_m + \frac{k}{4\pi} b \wedge db$$

$$+ \frac{1}{2}\left(a_m \wedge b \wedge b - b \wedge a_m \wedge b - b \wedge b \wedge a_m\right)$$

$$+ \frac{\pi}{K}\left(b \wedge a_m \wedge a_m - a_m \wedge b \wedge a_m - a_m \wedge a_m \wedge b\right)$$

$$+ \frac{k}{4\pi} b \wedge b \wedge b + 2\left(\frac{\pi}{K}\right)^2 a_m \wedge a_m \wedge a_m$$

$$L_{\substack{MCS\\NA}} \sim \boxed{-\frac{g^2}{2} a_m \wedge * a_m - \frac{\pi}{K} a_m \wedge da_m + \frac{k}{4\pi} b \wedge db}$$

Abelian part.

$$+ \frac{1}{2}\left(a_m \wedge db + db \wedge a_m\right)$$

$$+ \frac{1}{2}\left(a_m \wedge b \wedge b - b \wedge a_m \wedge b - b \wedge b \wedge a_m\right)$$

$$+ \frac{\pi}{K}\left(b \wedge a_m \wedge a_m - a_m \wedge b \wedge a_m - a_m \wedge a_m \wedge b\right)$$

$$+ \frac{k}{4\pi} b \wedge b \wedge b + 2\left(\frac{\pi}{K}\right)^2 a_m \wedge a_m \wedge a_m$$

Meeting with Adi

22/11/24

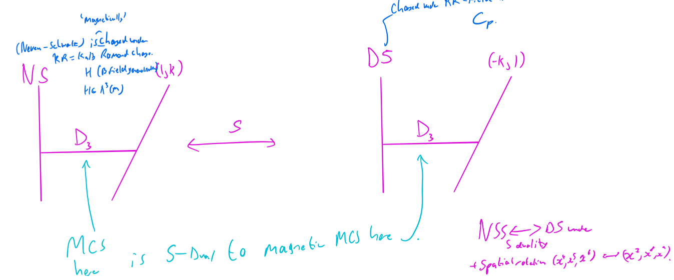
· Happy with the integration.

· Have spent some time familiarising myself with string theory (Went through Tong's notes).

→ Happy with all terminology used and implications of the Hanany-Witten paper, although I have not yet been through it to see how they get the desired result.

Type IIB : R-R Fields present

Have C, C_μ and $C_{\mu\nu\rho\sigma}$. And $F_5 = *F_5$ 'almost'.



· I don't understand why we don't integrate over b in equation 13.

We say $Z = \int Da_m Db \exp i \int -\frac{g^2}{2} a_m \wedge * a_m - \frac{\pi}{k} a_m \wedge da_m + \frac{k}{4\pi} b \wedge db$

Could integrate over b. Not nice to write. Consider b as having been integrated.

is the dual of MCS; does it matter that there are two fields?

· Should I solve for eom of eqn 13?

· Implications of a_m not being a gauge field? If $a_e = b - \left(\frac{2\pi}{k}\right) a_m$, surely b is a gauge field (because a_e is)?

· $\frac{\pi}{k} a_m \wedge da_m$ not ill defined because a_m is invariant under a gauge transformation? (For large gauge transformations)

Looks like a fractional level (not allowed), must be integer if a_m was a gauge field.

· I expanded the non-Abelian MCS partition function to get

All hats removed — trace is over the Lie algebra

$$L_{MCS_{NA}} \sim \boxed{-\frac{g^2}{2} a_m \wedge * a_m - \frac{\pi}{k} a_m \wedge da_m + \frac{k}{4\pi} b \wedge db}$$

Abelian part.

$$+\frac{1}{2}\left(a_m \wedge db + db \wedge a_m\right)$$

interacting:

$$+\frac{1}{2}\left(a_m \wedge b \wedge b - b \wedge a_m \wedge b - b \wedge b \wedge a_m\right)$$

Check if this cancels with first term or not.

$$+\frac{\pi}{k}\left(b \wedge a_m \wedge a_m - a_m \wedge b \wedge a_m - a_m \wedge a_m \wedge b\right)$$

$$+\frac{k}{4\pi} b \wedge b \wedge b + 2\left(\frac{\pi}{k}\right)^2 a_m \wedge a_m \wedge a_m$$

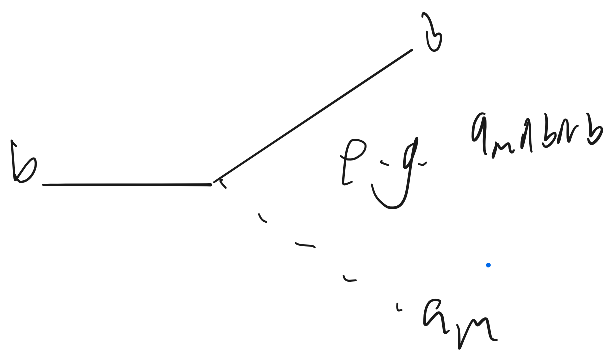
$a_m^a = a_m^a(x)$

$\mathrm{tr}\, t^a t^b t^c = \mathrm{tr}\, t^c t^a t^b$

$\mathrm{tr}\, a_m \wedge b \wedge b = \mathrm{tr}\, a_m^a t^a \wedge b^b t^b \wedge b^c t^c = (\mathrm{tr}\, t^a t^b t^c)(a_m^a \wedge b^b \wedge b^c)$

$= \mathrm{tr}\, b \wedge a_m \wedge b$

· Confused on the dimensional reduction part at the beginning.



$$a_m \wedge db + db \wedge a_m$$

$$\begin{aligned}
tr(db \wedge a_m) &= tr(db^a t^a \wedge a_m^b t^b) \\
&= tr(t^a t^b)\, db^a \wedge a_m^b \\
&= tr(t^b t^a)\, db^a \wedge a_m^b \\
&= -tr(t^b t^a)\, a_m^b \wedge db^a \\
&= -tr(t^a t^b)\, a_m^a \wedge db^b \\
&= -tr(a_m^a t^a \wedge db^b t^b) \\
&= -tr(a_m \wedge db)
\end{aligned}$$

So $tr(a_m \wedge db + db \wedge a_m) = tr(a_m \wedge db - a_m \wedge db) = 0.$

$\Longrightarrow$ No $a_m \wedge db$ term.

$$L = -\frac{1}{2g} da_e \wedge * da_e + \frac{k}{4\pi} a_e \wedge da_e$$

$$\delta L = \delta\left(-\frac{1}{2g} da_e \wedge * da_e\right) + \delta\left(\frac{k}{4\pi} a_e \wedge da_e\right)$$

$$= -\frac{1}{2g^2} d(\delta a_e) \wedge * da_e - \frac{1}{2g^2} da_e \wedge * d(\delta a_e) + \frac{k}{4\pi} \delta a_e \wedge da_e + \frac{k}{4\pi} a_e \wedge d(\delta a_e)$$

$$= -\frac{1}{2g^2} d(\delta a_e) \wedge * da_e - \frac{1}{2g^2} d\delta a_e \wedge * da_e + \frac{k}{4\pi} \delta a_e \wedge da_e + d\left(\frac{k}{4\pi} \cancel{a_e} \wedge \delta a_e\right) - \frac{k}{4\pi} da_e \wedge \delta a_e$$

$$\frac{-1}{g^2} d(\delta a_e)\wedge * da_e - \frac{k}{2\pi} da_e \wedge da_e$$

$$= -\frac{1}{g^2} d(\delta a_e \wedge * da_e) + \frac{1}{g^2}\delta a_e \wedge d * da_e - \frac{k}{2\pi} da_e \wedge \delta a_e$$

$$= \delta a_e \wedge \left(\frac{1}{g^2} d * da_e + \frac{k}{2\pi} da_e\right)$$

$$\Rightarrow d * da_e + \frac{g^2 k}{2\pi} da_e = 0$$

$$\langle \eta, w\rangle = \int_M \eta \wedge * w$$

Here for $\eta, w \in \Lambda^k_U$ for $\lambda = d\delta A$ and $w = dA$

$$\lambda \wedge * w = (-1)^q \langle \eta, w\rangle e = (-1)^q \langle w, \lambda\rangle = w \wedge * \lambda.$$

With $e = e_1 \wedge e_2 \wedge \cdots \wedge e_n \in \Lambda^n$ q is metric signature.

So $dA \wedge * d\delta A = d\delta A \wedge * dA$ ie, $\underbrace{\lambda \wedge * w = w \wedge * \lambda}_{\text{Hodge star rule}}$

$$\delta(da_e \wedge * da_e) = d(\delta a_e)\wedge * da_e + da_e \wedge * d(\delta a_e)$$

$$= d(\delta a_e)\wedge * da_e + d(\delta a_e)\wedge * da_e$$

$$= 2d(\delta a_e)\wedge * da_e$$

$$= 2\left(d(\delta a_e \wedge * da_e) - \delta a_e \wedge d * da_e\right)$$

$$= d(2\delta a_e \wedge * da_e) - 2\delta a_e \wedge d * da_e$$

So $-\frac{1}{2}\delta(da_e \wedge * da_e) = \delta a_e \wedge d * da_e - d(\delta a_e \wedge * da_e)$

Unification, Autumn Term 2020

Arttu Rajantie

October 4, 2020

Preface

These are the lecture notes for the fourth-year Unification course in the 2020-21 academic year at Imperial College London.

There are also some excellent textbooks, which I want to recommend as supplementary reading:

- W.N. Cottingham and D.A. Greenwood, "An Introduction to the Standard Model of Particle Physics, 2nd Edition" (Cambridge University Press, 2007)
- Matthew Robinson, "Symmetry and the Standard Model" (Springer, 2011)
- Dave Goldberg, "The Standard Model in a Nushell", (Princeton University Press, 2017)
- *More advanced:* Cliff Burgess and Guy Moore, "The Standard Model – A Primer" (Cambridge University Press, 2012)

The course assumes Advanced Classical Physics and Group Theory as background knowledge. The Advanced Classical Physics lecture notes are available on Blackboard and cover Lagrangian mechanics, relativistic electrodynamics and index notation.

Arttu Rajantie

Contents

Chapter 1

Introduction

1.1 Classical Field Theory

The Standard Model describes the elementary particles and their interactions using quantum fields. However, many properties of the theory can be understood in terms of classical fields.

Relativistic classical field theory is most conveniently described in the Lagrangian formulation. Let us therefore first briefly review the Lagrangian formulation in classical mechanics. The dynamics of a mechanical system is determined by the *action* S, which is a functional of the whole time evolution $x(t)$. The action can be written as an integral of the *Lagrangian function* $L(x, \dot{x})$, which depends on the coordinates and velocities at any given time,

$$S = \int dt\, L\left(x(t), \dot{x}(t)\right). \tag{1.1.1}$$

In simple mechanical systems, the Lagrangian is given by the difference of the kinetic energy $T = m\dot{x}^2/2$ and the potential energy $V(x)$,

$$L = \frac{1}{2}m\dot{x}^2 - V(x), \tag{1.1.2}$$

where m is the mass of the particle and the dot in $\dot{x}$ denotes the time derivative.

According to the *action principle*, the actual time evolution of the system follows the path of the lowest action. The equations of motion can therefore be obtained by requiring that the variation of the action δS vanishes when the time evolution is perturbed $x(t) \rightarrow x(t) + \delta x(t)$. The variation is given by

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt \delta L = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial x}\delta x(t) + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}(t) \right] \\ &= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial x}\delta x(t) - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) \delta x(t) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x(t) \right) \right] \\ &= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right] \delta x(t) + \left[\frac{\partial L}{\partial \dot{x}}\delta x(t) \right]_{t_1}^{t_2}. \end{aligned} \tag{1.1.3}$$

The substitution term vanishes because of fixed boundary conditions at initial and final times, $\delta x(t_1) = \delta x(t_2) = 0$. For the variation δS to vanish for any perturbation $\delta x(t)$, the quantity inside the brackets must vanish. This gives the *Euler-Lagrange equation*

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0. \tag{1.1.4}$$

To generalise this formalism from mechanics to classical field theory, consider a *field* $\phi(t,x)$ in 1+1 dimensions, in other words a function of spacetime position. Just like, say, they electric field, this field has some numerical value at every point in space at every time. In analogy with Eq. (1.1.1), we can write the action of the field evolution as as integral over space and time

$$S = \int dtdx\mathcal{L}(\phi,\dot{\phi},\phi'), \tag{1.1.5}$$

where the integrand $\mathcal{L}$ is known as the *Lagrangian density* as is assumed to be a local function of the field ϕ and its first derivatives $\dot{\phi} = \partial\phi/\partial t$ and $\phi' = \partial\phi/\partial x$.

Even without specifying what the functional form of $\mathcal{L}$ is, we can obtain the Euler-Lagrange equation in analogy with Eq. (1.1.3) by considering the variation δS of the action when the field evolution is perturbed $\phi \to \phi + \delta\phi$,

$$\delta S = \int dtdx \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\delta\dot{\phi} + \frac{\partial\mathcal{L}}{\partial\phi'}\delta\phi'\right] = \int dtdx \left[\frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\phi'}\right]\delta\phi = 0, \tag{1.1.6}$$

where we have again integrated by parts and dropped the boundary terms. This is justified if we consider fixed boundary conditions at the initial and final time, and also at spatial infinity. For the variation to vanish for any perturbation $\delta\phi$, the expression inside the brackets must vanish, which gives the equation

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\phi'} = 0. \tag{1.1.7}$$

(Note that here we use the notation d/dt and d/dx rather than partial derivatives to emphasize the fact that the t- and x-dependence of $\mathcal{L}$ arises through function $\phi(t,x)$.)

Because the Lagrangian $\mathcal{L}$ determines the equations of motion, it encodes all the information about the dynamics of the system. To make sense mathematically and physically, the Lagrangian has to satisfy some criteria. It has to be

(i) *Real*, because the action S has to be real. Otherwise it could not have a minimum, and we could not apply the action principle. (In fact, from quantum theory one would find that a complex action would correspond to loss of unitarity.)

(ii) It must have *dimensions* $[\mathcal{L}] = \mathrm{GeV}^d$ in d spacetime dimensions (expressed in natural units[1]). This is because the Lagrangian function L in mechanics has the units of energy, which means that the action is dimensionless in natural units, and because $[S] = [t] \times [x]^{d-1} \times [\mathcal{L}]$, we obtain $[\mathcal{L}] = \mathrm{GeV}^d$.

(iii) A *local* function of the field and its derivatives calculated at the same spacetime point. Otherwise the dynamics would involve action at a distance.

(iv) Invariant under any *symmetries*. As we will discuss later, a symmetry in physics means a transformation that does not affect the laws of physics. Because the laws of physics are determined by the Lagrangian, the transformation must therefore leave the Lagrangian unchanged.

[1]Natural units are defined by exploiting the observation that the speed of light c, Planck's constant $\hbar$, Boltzmann's constant k_B and vacuum permittivity ϵ_0 are all dimensionful constants. If one defines a unit system based on SI units but setting $c = \hbar = k_\mathrm{B} = \epsilon_0 = 1$, one can express all dimensionful quantities in terms of a single base unit, which one chooses to be a unit of energy, gigaelectronvolts, $\mathrm{GeV} \approx 1.6 \times 10^{-10}$ J. Then mass has units of $\mathrm{GeV} \approx 1.8 \times 10^{-27}$ kg, and time and length have units of $\mathrm{GeV}^{-1} \approx 6.6 \times 10^{-25}\ \mathrm{s} \approx 2.0 \times 10^{-16}$ m.

(v) In particular, as an important special case of criterion (iv), the Lagrangian $\mathcal{L}$ must be invariant under Lorentz transformations, in other words a Lorentz scalar.

1.2 Scalar Field Theory

Let us then consider relativistic field theory in 3+1-dimensional Minkowski space. I will follow the convention in which the Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

The simplest case is when the field ϕ is a Lorentz scalar. That means that the field value at a given spacetime point does not change in a Lorentz transformation, but the coordinates of the point change as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{1.2.1}$$

Therefore the requirement that the field is a scalar implies that $\phi'(x'^\mu) = \phi(x^\mu)$. Using the inverse Lorentz transformation (see PS1, Q2(d)), $x^\mu = \Lambda_\nu{}^\mu x'^\nu$, this can also be written as

$$\phi'(x^\mu) = \phi(\Lambda_\nu{}^\mu x^\nu). \tag{1.2.2}$$

To keep the expressions simple, I will usually not write the coordinates explicitly, so the transformation rule for a scalar field would be simply $\phi \to \phi$. From Eq. (1.2.2) it follows that the derivative $\partial_\mu \phi$ is a Lorentz vector, which means that it transforms as (see PS1, Q2(e))

$$\partial_\mu \phi \to \Lambda_\mu{}^\nu \partial_\nu \phi. \tag{1.2.3}$$

We assume that the Lagrangian is a function of the field and its derivative $\mathcal{L}(\phi, \partial_\mu \phi)$. Again, we use the action principle

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \partial_\mu \phi \right] = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \delta\phi + \text{boundary term} = 0, \tag{1.2.4}$$

to find the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0. \tag{1.2.5}$$

Considering terms that are lowest order in the field and its derivative, the simplest choice for the Lagrangian satisfying the criteria given in Section 1.1 is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \tag{1.2.6}$$

This is because a constant term would not contribute to the equation of motion, terms with a single derivative would not be Lorentz invariant, and a linear term ϕ can be removed by a constant shift in ϕ if a quadratic term is present. The overall normalisation of the Lagrangian $\mathcal{L}$ does not affect the equation of motion, and therefore we can choose the coefficient of the first term of be $1/2$ without loss of generality. The coefficient of the second term is an undetermined constant, which we choose to denote by $-m^2/2$ for reasons that will become clear. Substituting Eq. (1.2.6) into the Euler-Lagrange equation (1.2.5), we find

$$-m^2 \phi - \partial_\mu (\partial^\mu \phi) = 0, \tag{1.2.7}$$

which we can write as

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0. \tag{1.2.8}$$

This is a relativistic wave equation, and known as the *Klein-Gordon equation*.

The Klein-Gordon equation (1.2.8) was first introduced as a relativistic Schrödinger equation for a particle with mass m. If the particle has momentum $\mathbf{p}$, its energy is $E = (\mathbf{p}^2 + m^2)^{1/2}$, and therefore the time-dependent Schrödinder equation would be

$$i\partial_t \psi = \hat{H}\psi = \sqrt{\hat{\mathbf{p}}^2 + m^2}\psi. \tag{1.2.9}$$

To remove the square root, Klein and Gordon applied another time derivative,

$$-\partial_t^2 \psi = i\partial_t \left(\sqrt{\hat{\mathbf{p}}^2 + m^2}\psi \right) = \sqrt{\hat{\mathbf{p}}^2 + m^2} i\partial_t \psi = \left(\hat{\mathbf{p}}^2 + m^2 \right) \psi. \tag{1.2.10}$$

With the standard identification $\mathbf{p} = -i\boldsymbol{\nabla}$, this becomes

$$\partial_t^2 \psi - \boldsymbol{\nabla}^2 \psi + m^2 \psi = \partial^2 \psi + m^2 \psi = 0, \tag{1.2.11}$$

which is Eq. (1.2.8). Hence we conclude that in quantum theory, the Lagrangian (1.2.6) describes a particle with mass m. Further discussion of the quantum behaviour is a matter for the QFT course, and in this course we will simply assume this connection between classical and quantum field theory. Therefore, whenever we encounter a Klein-Gordon equation (1.2.8), we interpret it as a particle with mass m.

The Klein-Gordon equation (1.2.8) is linear, and it is easy to see that the independent solutions are plane waves

$$\phi(x^\mu) = \phi_0 e^{ik^\mu x_\mu}, \tag{1.2.12}$$

where ϕ_0 is a constant amplitude, and $k^\mu = (\omega, \mathbf{k})$ is the wave vector in four dimensions where ω is the frequency and $\mathbf{k}$ is the three-dimensional wave vector. Using the analogy with the Schrödinger equation, we can interpret them as the energy and the momentum of the particle, respectively,

$$\mathbf{p} = \mathbf{k}, \quad E = \omega = \sqrt{\mathbf{p}^2 + m^2}, \tag{1.2.13}$$

where the last relation follows by substituting the Ansatz (1.2.12) into the Klein-Gordon equation (1.2.8.

It is also useful to consider the Klein-Gordon equation (1.2.8) in momentum space by taking the Fourier transform,

$$\phi(t, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{x}} \phi(t, \mathbf{p}), \tag{1.2.14}$$

where the integration is over three-momenta $\mathbf{p}$. Substituting this into Eq. (1.2.8) gives (See PS1)

$$\partial_t^2 \phi(t, \mathbf{p}) + \left(\mathbf{p}^2 + m^2 \right) \phi(t, \mathbf{p}) = 0, \tag{1.2.15}$$

which shows that each Fourier mode oscillates independently of each other. In other words, two plane waves would travel through each other without interacting. Through the particle analogy, this implies that Eq. (1.2.6) describes non-interacting particles.

Let us now add more terms to Eq. (1.2.6). To be specific, we add a quartic term

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4}\lambda \phi^4. \tag{1.2.16}$$

The Euler-Lagrange equation (1.2.5) gives the equation of motion

$$\partial_\mu \partial^\mu \phi + m^2 \phi + \lambda \phi^3 = 0. \tag{1.2.17}$$

This equation is no longer linear. Indeed, taking again the Fourier transform (1.2.14), the equation becomes

$$\partial_t^2\phi(\mathbf{p})+\left(\mathbf{p}^2+m^2\right)\phi(\mathbf{p})+\lambda\int\frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}\frac{d^3q_3}{(2\pi)^3}\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\phi(\mathbf{q}_3)(2\pi)^3\delta(\mathbf{p}-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)=0. \tag{1.2.18}$$

This shows that the behaviour of the Fourier mode with momentum $\mathbf{p}$ depends on the modes $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$. Plane waves do not travel freely though each other but interact. Therefore the particles they correspond to in quantum theory, also interact, and the parameter λ characterises the strength of this interaction. The precise nature of this interaction is discussed in detail in the Quantum Field Theory course.

Similarly, one can consider adding other terms. For example, a cubic term $-g^3\phi^3$, which would lead to a three-point interaction. In principle, one could also consider higher orders in ϕ or its derivatives, but using *renormalisation theory* in quantum field theory one can show that in 3+1 dimensions, only terms with energy dimension of at most 4 are allowed. These are known as *renormalisable* terms. Terms with a higher dimension would lead to divergences which could not be removed by renormalisation. The proof of this is beyond the scope of this course, and therefore we simply adopt it as an additional criterion the Lagrangian has to satisfy,

(vi) The Lagrangian must be *renormalisable*, in other words have no terms of energy dimension higher than 4.

To see what this means for the scalar field ϕ, we have to find its dimensions. We already know that the Lagrangian has dimension $[\mathcal{L}]=\mathrm{GeV}^4$, and because it contains the term $m^2\phi^2$, that must have the same dimension. Furthermore, mass has dimensions of GeV, so $\mathrm{GeV}^4=[m]^2[\phi]^2=\mathrm{GeV}^2[\phi]^2$, from which we find $[\phi]=\mathrm{GeV}$. In other words, the field has units of energy. Therefore a term ϕ^n is only allowed by renormalisability if $n\leq 4$. The derivative has dimensions $[\partial_\mu\phi]=\mathrm{GeV}^2$, and therefore no further derivative terms are allowed either. The most general renormalisable Lagrangian for the real scalar theory is therefore

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi), \tag{1.2.19}$$

where the *potential* $V(\phi)$ is a polynomial of order ≤ 4.

We can also consider a complex scalar field $\phi\in\mathbb{C}$. Because the Lagrangian still has to be real, it becomes

$$\mathcal{L}=\partial_\mu\phi^*\partial^\mu\phi-m^2\phi^*\phi-\frac{1}{2}\lambda\left(\phi^*\phi\right)^2. \tag{1.2.20}$$

Note that the normalisation of the coefficients is chosen differently from the real scalar case (1.2.19). When using the action principle, it is convenient to treat ϕ and ϕ^* as independent variables, in line with the usual definition of complex differentiation in complex analysis. The Euler-Lagrange equation obtained by varying ϕ^* is

$$\frac{\partial\mathcal{L}}{\partial\phi^*}-\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}=0, \tag{1.2.21}$$

which leads to the equation of motion

$$\partial^2\phi+m^2\phi+\lambda\left(\phi^*\phi\right)\phi=0. \tag{1.2.22}$$

1.3 Vector Field

Let us now consider a real Lorentz vector field A_μ. Again we want to construct a Lorentz invariant Lagrangian involving the field and its derivatives. The derivative $\partial_\mu A_\nu$ is a rank 2 Lorentz tensor, and it can be decomposed into three pieces

$$\partial_\mu A_\nu = \frac{1}{4}\eta_{\mu\nu} S + \frac{1}{2}F_{\mu\nu} + \frac{1}{2}G_{\mu\nu}, \tag{1.3.1}$$

where

- S is a Lorentz scalar,
- $F_{\mu\nu}$ is an antisymmetric rank 2 tensor, which means that it satisfies $F_{\nu\mu} = -F_{\mu\nu}$, and
- $G_{\mu\nu}$ is a symmetric and traceless rank 2 tensor, which means that it satisfies $G_{\nu\mu} = G_{\mu\nu}$ and $\eta^{\mu\nu}G_{\mu\nu} = 0$.

These can be written explicitly as

$$S = \partial_\mu A^\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad G_{\mu\nu} = \partial_\mu A_\nu + \partial_\nu A_\mu - \frac{1}{2}\eta_{\mu\nu} S. \tag{1.3.2}$$

These three terms do not mix under Lorentz transformations, and scalar products of each pair of two different terms vanish,

$$\frac{1}{4}\eta^{\mu\nu} S F_{\mu\nu} = 0, \quad \frac{1}{4}\eta^{\mu\nu} S G_{\mu\nu} = 0, \quad F^{\mu\nu} G_{\mu\nu} = 0. \tag{1.3.3}$$

Therefore the only quadratic terms that are Lorentz invariant are squares of each term, and correspondingly the most general Lagrangian which is quadratic in field A_μ and its derivatives is

$$\mathcal{L} = aS^2 + bF_{\mu\nu}F^{\mu\nu} + cG_{\mu\nu}G^{\mu\nu} + dA_\mu A^\mu, \tag{1.3.4}$$

where a, b, c and d are arbitrary constants.

A concrete example of a vector field theory is electrodynamics written in terms of the four-vector potential $A^\mu = (\phi, \mathbf{A})$, ϕ is the scalar potential and $\mathbf{A}$ is the vector potential. Electric and magnetic fields are given by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi, \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{1.3.5}$$

Alternatively, the dynamics can be described in a Lorentz covariant way using the *field strength tensor* $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which coincides with the definition of the antisymmetric tensor $F_{\mu\nu}$ above.

As we know from electrodynamics, there are many different configurations of the potentials that correspond to the same electric and magnetic fields, in other words the same physical configuration. More precisely, $\mathbf{E}$ and $\mathbf{B}$ are invariant under *gauge transformations*

$$\phi \to \phi - \frac{\partial \lambda}{\partial t}, \quad \mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\lambda, \tag{1.3.6}$$

where λ is an arbitrary scalar function of spacetime. In terms of the four-vector potential, this *gauge invariance* can be written compactly as

$$A_\mu \to A_\mu - \partial_\mu \lambda. \tag{1.3.7}$$

The different terms in the general quadratic Lagrangian (1.3.4) transform under such a gauge transformation as

$$
\begin{aligned}
S = \partial_\mu A^\mu \quad &\to \quad \partial_\mu A^\mu + \partial^2\lambda = S + \partial^2\lambda, \\
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad &\to \quad \partial_\mu A_\nu + \partial_\mu\partial_\nu\lambda - \partial_\nu A_\mu - \partial_\nu\partial_\mu\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}, \\
G_{\mu\nu} = \partial_\mu A_\nu + \partial_\nu A_\mu - \frac{1}{2}\eta_{\mu\nu}S \quad &\to \quad \partial_\mu A_\nu + \partial_\mu\partial_\nu\lambda + \partial_\nu A_\mu + \partial_\nu\partial_\mu\lambda - \frac{1}{2}\eta_{\mu\nu}S - \frac{1}{2}\eta_{\mu\nu}\partial^2\lambda \\
&\quad\;\; = G_{\mu\nu} + 2\partial_\mu\partial_\nu\lambda - \frac{1}{2}\eta_{\mu\nu}\partial^2\lambda. \qquad (1.3.8)
\end{aligned}
$$

Hence the only term that is invariant under gauge transformations is $F_{\mu\nu}F^{\mu\nu}$. Therefore, for a vector field A_μ, the most general quadratic Lagrangian that is invariant under both Lorentz and gauge transformations is simply

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad (1.3.9)$$

where we have used the freedom to choose the normalisation of $\mathcal{L}$ to set $b = -1/4$.

This is an example of the power of symmetry in physics: The assumption of a symmetry can determine the form the Lagrangian and therefore the laws of physics. This is why symmetries play as central role in fundamental physics, where we are trying to find the Lagrangian that describes the fundamental laws of nature. If you can correctly identify the fundamental symmetries, then you know what the Lagrangian should be.

Of course, we can recognise Eq. (1.3.9) as the Maxwell Lagrangian for the free electromagnetic field. This means that the dynamics of the electromagnetic field is fully determined by the Lorentz and gauge symmetries. For completeness, the equation of motion is (see PS1 for details)

$$\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu A_\nu} = \partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu = 0. \qquad (1.3.10)$$

To understand the solutions of the equation it is convenient to choose the *radiation gauge* by imposing $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$. For any field configuration, it is always possible to find a gauge transformation that satisfies this gauge condition. Then the equation for $\nu = 0$ becomes

$$\partial^2 A^0 - \partial^0\partial_\mu A^\mu = \partial^2 A_0 - \partial^0\left(\partial^0 A^0 - \boldsymbol{\nabla}\cdot\mathbf{A}\right) = -\boldsymbol{\nabla}^2 A^0 = 0. \qquad (1.3.11)$$

Because the equationh has no time derivative, A_0 does not carry propagating waves. In fact, in the radiation gauge it is possible to impose an additional gauge condition $A^0 = 0$, and we assume that this has been done. For the spatial components $\nu = i$ we find

$$\partial^2 A^i - \partial^i\partial_\mu A^\mu = \partial^2 A_i - \partial^i\left(\partial^0 A^0 - \boldsymbol{\nabla}\cdot\mathbf{A}\right) = \partial^2 A_i = 0. \qquad (1.3.12)$$

This means that each spatial component satisfies a separate massless Klein-Gordon equation. Gauge invariance therefore implies masslessness of the photon.

Finally, by considering a plane wave solution $\mathbf{A} \propto \exp(ik^\mu x_\mu)$, we find that the gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ implies $\mathbf{k}\cdot\mathbf{A} = 0$, or equivalently $\mathbf{A} \perp \mathbf{k}$. Therefore the solution describes a transverse wave, with only two physical degrees of freedom, which correspond to the two spatial directions perpendicular to the direction of motion.

Chapter 2

Symmetry

2.1 Abelian Symmetry

Let us look again at the real scalar field Lagrangian (1.2.16). As we already discussed, it is invariant under Lorentz transformations. This Lorentz invariance is an example of a *spacetime symmetry*, by which we mean symmetry under transformations that change spacetime coordinates. Other examples of spacetime symmetries are rotation, translation and reflection symmetries. It is also a *continuous symmetry*, because one can consider an arbitrarily small Lorentz transformations. In other words, a Lorentz transformation can be continuously deformed to the trivial transformation.

The Lagrangian (1.2.16) is also invariant under the change of sign $\phi \to -\phi$. This is an *internal symmetry*, because it leaves spacetime coordinates unchanged and only changes the value of the field. It is also a *discrete symmetry*, because the effect of the transformation cannot be made arbitrarily small.

Consider, for constrast, the Lagrangian (1.2.20) of a complex scalar field. This Lagrangian is, again, invariant under Lorentz transformations, but it has a larger internal symmetry than the real scalar field. The Lagrangian is invariant under rotations of the complex phase

$$\phi(x) \to e^{i\theta}\phi(x), \tag{2.1.1}$$

where θ is an arbitrary real number. This is a continuous symmetry. It is also an *Abelian symmetry*, because any pair of symmetry transformations commute,

$$\mathrm{e}^{i\theta_1}e^{i\theta_2} = e^{i\theta_2}e^{i\theta_1}. \tag{2.1.2}$$

Other examples of Abelian symmetries are rotations in two dimensions and translations. Transformations that do not commute are called non-Abelian, and the corresponding symmetry is then called a *non-Abelian symmetry*. Lorentz transformations and three-dimensional rotations are non-Abelian symmetries. Non-Abelian symmetries play a central role in the Standard Model, but for simplicity, let us consider Abelian symmetries first.

The fact that the symmetry (2.1.1) is continuous has an important consequence. To see it, consider an infinitesimal transformation $\theta \ll 1$, so that we can ignore $O(\theta^2)$ terms. The field then transforms as

$$\phi \to \phi + \delta\phi = e^{i\theta}\phi = \phi + i\theta\phi + O(\theta^2), \quad \text{where } \delta\phi = i\theta\phi + O(\theta^2). \tag{2.1.3}$$

In general, when the field is perturbed as $\phi \to \phi + \delta\phi$, whether a symmetry or not, the Lagrangian changes by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\partial_\mu\phi + \frac{\partial\mathcal{L}}{\partial\phi^*}\delta\phi^* + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}\delta\partial_\mu\phi^*, \tag{2.1.4}$$

where ϕ and ϕ^* have again been treated as independent variables. If ϕ is a solution of the equations of motion, then it satisfies the Euler-Lagrange equation (1.2.5),[1] and we obtain

$$\begin{aligned}\delta\mathcal{L} &= \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right)\delta\phi+\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\mu\delta\phi+\left(\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}\right)\delta\phi^*+\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}\partial_\mu\delta\phi^* \\ &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi+\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}\delta\phi^*\right). \end{aligned} \tag{2.1.5}$$

In the current case, this becomes

$$\delta\mathcal{L}=\partial_\mu\left(\partial^\mu\phi^*\delta\phi+\partial^\mu\phi\delta\phi^*\right)=\partial_\mu\left(i\phi\partial^\mu\phi^*-i\phi^*\partial^\mu\phi\right)\theta. \tag{2.1.6}$$

Because the transformation is a symmetry, $\delta\mathcal{L}=0$, and therefore the quantity inside the brackets,[2]

$$j^\mu=2\mathrm{Im}\phi^*\partial^\mu\phi \tag{2.1.7}$$

must satisfy $\partial_\mu j^\mu = 0$, which means that it is a conserved current. More explicitly, if we write $j^\mu=(\rho,\mathbf{J})$, we obtain the continuity equation

$$\dot\rho+\boldsymbol{\nabla}\cdot\mathbf{J}=0. \tag{2.1.8}$$

We have therefore found that the continuous symmetry of the complex scalar Lagrangian (1.2.20) implies a conserved current. This is an example of *Noether's theorem*, according to which a continuous symmetry always gives rise to a conservation current. This a very powerful result, which is valid in both classical and quantum theories and gives us a vital guidance when trying to find more fundamental theories. Conservation laws are easy to discover experimentally, and Noether's theorem allows us to deduce from them the symmetries of the Lagrangian, which in turn restrict the form the Lagrangian can take.

2.2 Local Symmetry

The symmetry (2.1.1) discussed the previous section was a *global symmetry*: The symmetry transformation was the same at every spacetime point. It is easy to check that the Lagrangian (1.2.20) under position-dependent phase rotations

$$\phi(x)\rightarrow e^{i\theta(x)}\phi(x). \tag{2.2.1}$$

The potential terms are invariant, but the derivative transforms as

$$\partial_\mu\phi\rightarrow e^{i\theta}\partial_\mu\phi+i(\partial_\mu\theta)e^{i\theta}\phi, \tag{2.2.2}$$

and therefore the derivative term $\partial_\mu\phi^*\partial^\mu\phi$ is not invariant. If the Lagrangian was invariant under such a position-dependent transformation, it would be a *local symmetry*, and in fact we have already encountered one local symmetry: the gauge symmetry in Eq. (1.3.7).

However, considering the form of the gauge transformation in Eq. (1.3.7), we can see how to make the Lagrangian (1.2.20)invariant under local transformations (2.2.1). Let us introduce a vector field that transforms according to Eq. (1.3.7), that is,

$$A_\mu\rightarrow A_\mu-\partial_\mu\lambda. \tag{2.2.3}$$

[1] Note that the action principle does not imply $\delta\mathcal{L}=0$, only that $\delta S=0$

[2] Note that the transformation parameter θ does not appear in j^μ.

The combination

$$D_\mu\phi \equiv \partial_\mu\phi + ieA_\mu\phi, \tag{2.2.4}$$

where e is an arbitrary constant, would then transform as

$$D_\mu\phi \to e^{i\theta}\partial_\mu\phi + i(\partial_\mu\theta)e^{i\theta}\phi + ieA_\mu e^{i\theta}\phi - ie(\partial_\mu\lambda)e^{i\theta}\phi = e^{i\theta}D_\mu\phi + i(\partial_\mu\theta - e\partial_\mu\lambda)e^{i\theta}\phi. \tag{2.2.5}$$

If we choose $\lambda = \theta/e$, which means that we consider a simultaneous transformations of ϕ and A_μ,

$$\begin{aligned} \phi(x) &\to e^{i\theta(x)}\phi(x), \\ A_\mu(x) &\to A_\mu(x) - \frac{1}{e}\partial_\mu\theta(x), \end{aligned} \tag{2.2.6}$$

parameterised by the same scalar function $\theta(x)$, then

$$D_\mu\phi \to \mathrm{e}^{i\theta}D_\mu\phi. \tag{2.2.7}$$

The combination $D_\mu\phi$ defined in this way is known as the *covariant derivative*. From (2.2.7) we see that if we replace the derivatives ∂_μ with covariant derivatives $D_\mu\phi$, then the Lagrangian

$$\mathcal{L} = (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi - \frac{1}{2}\lambda\left(\phi^*\phi\right)^2 \tag{2.2.8}$$

is invariant under the local gauge transformations (2.2.6). As we will see later, one can use this same procedure more generally to turn global symmetries into local ones, and this is often referred to as "gauging" the symmetry. The vector field A_μ we introduced is called a *gauge field*, and local symmetries are often called *gauge symmetries*.

The new Lagrangian (2.2.8) is not the most general gauge invariant Lagrangian because, as we saw in Section 1.3, one can also add the term $F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Therefore, the most general renormalisable Lagrangian that is invariant under both Lorentz transformations and gauge transformations (2.2.6) is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi - \frac{1}{2}\lambda\left(\phi^*\phi\right)^2 \tag{2.2.9}$$

In a more explicit form, this Lagrangian can be written as

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \partial_\mu\phi^*\partial^\mu\phi + ieA^\mu\left(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right) + e^2 A_\mu A^\mu\phi^*\phi \\ &\quad -m^2\phi^*\phi - \frac{1}{2}\lambda\left(\phi^*\phi\right)^2. \end{aligned} \tag{2.2.10}$$

As discussed in Section 1.2, terms that contain more than two field factors correspond to interactions. This is the case with the last two terms on the first line, and therefore we can see that gauging the symmetry has given rise to an interaction between the scalar ϕ and the gauge field A_μ.

The equation of motion obtained from the Lagrangian (2.2.10) for the scalar field ϕ is

$$\frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*} = -ieA^\mu\partial_\mu\phi + e^2A^\mu A_\mu\phi - m^2\phi - \lambda\left(\phi^*\phi\right)\phi - \partial_\mu\left(\partial^\mu\phi + ieA^\mu\phi\right) = 0, \tag{2.2.11}$$

which can be written compactly as

$$D_\mu D^\mu\phi + m^2\phi + \lambda\left(\phi^*\phi\right)\phi = 0. \tag{2.2.12}$$

Comparison with Eq. (1.2.22) shows that this is the same equation as for the globally symmetric case, except that the derivatives have been replaced with covariant derivatives.

The equation of motion for the gauge field A_μ is more interesting,

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = ie\left(\phi \partial_\nu \phi^* - \phi^* \partial_\nu \phi\right) + 2e^2 A^\nu \phi^* \phi - \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = 0. \tag{2.2.13}$$

This can be written as

$$\partial_\mu F^{\mu\nu} = ie\left[\phi \left(D^\nu \phi\right)^* - \phi^* D^\nu \phi\right]. \tag{2.2.14}$$

Comparing this with the Lorentz covariant form of Maxwell's equations of electrodynamics,

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{2.2.15}$$

where j^ν is the electric current, we see that we can identify the gauge field with the electromagnetic field. If we do so, the left hand side Eq. (2.2.14) becomes the electric current

$$j^\nu = ie\left[\phi \left(D^\nu \phi\right)^* - \phi^* D^\nu \phi\right] = 2e\mathrm{Im}\phi^* D^\nu \phi. \tag{2.2.16}$$

This means that the scalar field ϕ is electrically charged. The constant parameter e we introduced in Eq. (2.2.4) is the charge of the field, although in order to actually show that the charge is quantised, one has to go beyond classical field theory. The theory (2.2.6) is known as *scalar electrodynamics*, and sometimes it is also called the *Abelian Higgs model*, because it is the theory that Peter Higgs studied when he discovered the Higgs mechanism.

The electric current (2.2.16) is he obvious generalisation of the Noether current (2.1.7), again obtained by replacing $\partial_\mu \to D_\mu$. It follows directly from Eq. (2.2.15) that it is conserved,

$$\partial_\nu j^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0, \tag{2.2.17}$$

and indeed, it is the Noether current associated with the local gauge symmetry (2.2.6).

To conclude, we have found that electrodynamics arises from gauging of the global symmetry of Eq. (1.2.20), and the scalar field acquires electric charge. This is a further demonstration of the power of symmetry in determining the nature of fundamental interactions.

2.3 Symmetry Groups

Consider now a real three-component scalar field

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{2.3.1}$$

with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^T \partial^\mu \phi - \frac{1}{2}m^2 \phi^T \phi - \frac{1}{4}\lambda \left(\phi^T \phi\right)^2. \tag{2.3.2}$$

This Lagrangian is invariant under internal transformations that correspond to multiplication by a 3×3 matrix M,

$$\phi \to M\phi, \tag{2.3.3}$$

provided that the transformation leaves the combination $\phi^T\phi$ invariant for any ϕ. Because

$$\phi^T\phi \to (M\phi)^T M\phi = \phi^T M^T M\phi, \tag{2.3.4}$$

this is true if $M^T M = \mathbb{1}$. In other words, the matrix M has to be an *orthogonal* 3×3 matrix. These matrices form a group called $O(3)$.

More generally a *group* is defined as a set G together with an operation $\cdot$, which satisfies the group axioms

(i) *closure*: If $a, b \in G$, then $a \cdot b \in G$.

(ii) *associativity*: If $a, b, c \in G$, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(iii) *identity*: There is an element $\mathbb{1} \in G$ (called the *identity element*) such that for any $a \in G$, $\mathbb{1} \cdot a = a \cdot \mathbb{1} = a$.

(iv) *invertibility*: If $a \in G$, then there is an element $a^{-1} \in G$ (the *inverse element*) such that $a \cdot a^{-1} = a^{-1} \cdot a = \mathbb{1}$.

The group consisting of set G and operation $\cdot$ can be expressed as the pair $(G, \cdot)$. Note that in general $a \cdot b \neq b \cdot a$, but if $a \cdot b = b \cdot a$, then the group is said to be *Abelian*.

We can check the $\mathrm{O}(3)$ group satisfies the group axioms:

(i) If $M, N \in \mathrm{O}(3)$, then $(MN)^T(MN) = N^T M^T MN = N^T N = \mathbb{1}$, so $MN \in \mathrm{O}(3)$.

(ii) Matrix multiplication is associative, so for $M, N, O \in \mathrm{O}(3)$, we have $(MN)O = MNO = M(NO)$.

(iii) The unit matrix $\mathbb{1}$ is orthogonal: $\mathbb{1}^T\mathbb{1} = \mathbb{1}$.

(iv) If $M \in \mathrm{O}(3)$, the M^T is its inverse because $M^T M = \mathbb{1}$. For square matrices, the inverse is unique, so this implies $MM^T = \mathbb{1}$.

Another, simpler example of a group is the group of integers $(\mathbb{Z}, +)$, where the group of operation is addition. Again, the group axioms are

(i) If $a, b \in \mathbb{Z}$, then $a + b \in \mathbb{Z}$.

(ii) If $a, b, c \in \mathbb{Z}$, then $(a + b) + c = a + b + c = a + (b + c)$.

(iii) The unit element is $0 \in \mathbb{Z}$: $0 + a = a + 0 = a$.

(iv) If $a \in \mathbb{Z}$, then the inverse element is $-a \in \mathbb{Z}$: $a + (-a) = -a + a = 0.$.

Groups can also be finite, for example the group $\mathbb{Z}_N$ of integers modulo N. The simplest case is $\mathbb{Z}_2 = (\{0, 1\}, +)$, which has the operations

$$0 + 0 = 0, \quad 0 + 1 = 1, \quad 1 + 0 = 1, \quad 1 + 1 = 0.$$

This is the same pattern as the group $(\{-1, 1\}, \times)$, where the group operation is multiplication,

$$1 \times 1 = 1, \quad 1 \times (-1) = -1, \quad (-1) \times 1 = -1, \quad (-1) \times (-1) = 1.$$

These two groups have a one-to-one correspondence between them, so we say they are *isomorphic*.

Many groups that we are interested in, such as the group $\mathrm{O}(3)$ mentioned above, are *Lie groups*. A Lie group is a group whose elements form a differentiable manifold, in such a way that the group operation and the inverse are differentiable.

For our purposes, the most important examples of Lie group are matrix groups, which are defined as subgroups of the *general linear group* $\mathrm{GL}(N, \mathbb{C})$. This is defined as the group of complex invertible $N \times N$ matrices. By restricting the elements to be real numbers, one obtains the real general linear group $\mathrm{GL}(N, \mathbb{R})$. Other subgroups one can form are

- The complex *special linear group* $\mathrm{SL}(N, \mathbf{C})$ defined as the group of complex $N \times N$ matrices with $\det M = 1$.
- The real *special linear group* $\mathrm{SL}(N, \mathbf{R})$ defined as the group of real $N \times N$ matrices with $\det M = 1$.
- The *unitary group* $\mathrm{U}(N)$, defined as the group of complex $N \times N$ matrices with $M^\dagger M = \mathbb{1}$.
- The *special unitary group* $\mathrm{SU}(N)$, defined as the group of complex $N \times N$ matrices with $M^\dagger M = \mathbb{1}$ and $\det M = 1$.
- The *orthogonal group* $\mathrm{O}(N)$, defined as the group of real $N \times N$ matrices with $M^T M = \mathbb{1}$.
- The *special orthogonal group* $\mathrm{SO}(N)$, defined as the group of real $N \times N$ matrices with $M^T M = \mathbb{1}$ and $\det M = 1$.

Phase rotations $\phi \to e^{i\theta}\phi$ form group $U(1)$. The same rotation can also be written is terms of the real and imaginary parts as

$$\begin{pmatrix} \phi_R \\ \phi_I \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_R \\ \phi_I \end{pmatrix}, \tag{2.3.5}$$

where the matrix

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.3.6}$$

is orthogonal, i.e., $M^T M = \mathbb{1}$ and special, i.e., $\det M = 1$. Therefore the phase rotations can also be seen as the group $\mathrm{SO}(2)$. This means that the groups $\mathrm{U}(1)$ and $\mathrm{SO}(2)$ are isomorphic.

For example, if we consider a complex N-component scalar field

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}, \tag{2.3.7}$$

we find that the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi^\dagger \phi) \tag{2.3.8}$$

is invariant under $\mathrm{U}(N)$ transformations. If the scalar field components are real, the symmetry group of the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^T \partial^\mu \phi - V(\phi^T \phi) \tag{2.3.9}$$

is $\mathrm{O}(N)$.

2.4 Group Generators

In many cases we are interested in *connected groups*, by which we mean a Lie group in which every element is connected to the identity $\mathbb{1}$ by a continuous path. For example, $\mathrm{SO}(N)$ is connected, but $\mathrm{O}(N)$ is not. This is because $M \in \mathrm{O}(N)$ have $\det M = \pm 1$, and those with $\det M = -1$ cannot be continuously connected to the identity matrix, which has $\det \mathbb{1} = 1$. $\mathrm{SO}(N)$ is a connected subgroup of $\mathrm{O}(N)$. More generally, any non-connected Lie group always has a connected subgroup.

Consider now a group element M that is in the infinitesimal neighbourhood of the identity $\mathbb{1}$. We can write $M = \mathbb{1} + i\delta\theta$, where we have included the imaginary factor i for future convenience[3]. For matrix groups, θ is a matrix that satisfies specific conditions depending on the group:

- $\mathrm{GL}(N, \mathbb{C})$: $\delta\theta$ can be any infinitesimal $N \times N$ matrix
- $\mathrm{GL}(N, \mathbb{R})$: $\delta\theta$ is an imaginary $N \times N$ matrix
- $\mathrm{SL}(N)$: The matrix M has to have determinant one. This means

$$\begin{aligned}\det(\mathbb{1} + i\delta\theta) &= \det e^{i\delta\theta} + O(\delta\theta^2) = e^{\mathrm{Tr}\, i\delta\theta} + O(\delta\theta^2) = 1 + i\mathrm{Tr}\,\delta\theta + O(\delta\theta^2) = 1, \\ &\Rightarrow \mathrm{Tr}\,\delta\theta = 0. \end{aligned} \tag{2.4.1}$$

  In other words, the matrix $\delta\theta$ has to be traceless.

- $\mathrm{U}(N)$: The matrix M has to satisfy $M^\dagger M = \mathbb{1}$. This means

$$M^\dagger M = \left(\mathbb{1} - i\delta\theta^\dagger\right)(\mathbb{1} + i\delta\theta) = \mathbb{1} + i\left(\delta\theta - \delta\theta^\dagger\right) + O(\delta\theta^2) = \mathbb{1}. \tag{2.4.2}$$

  Hence $\delta\theta$ needs to be Hermitean, $\delta\theta^\dagger = \delta\theta$.

- $\mathrm{O}(N)$: The matrix M has to satisfy $M^T M = \mathbb{1}$. This means

$$M^T M = \left(\mathbb{1} + i\delta\theta^T\right)(\mathbb{1} + i\delta\theta) = \mathbb{1} + i\left(\delta\theta + \delta\theta^T\right) + O(\delta\theta^2) = \mathbb{1}. \tag{2.4.3}$$

  Therefore $\delta\theta$ needs to be antisymmetric, $\delta\theta^T = -\delta\theta$. Note also that because M is real, $\delta\theta$ is imaginary.

Above, we had made the assumption that $\delta\theta$ needs to be infinitesimal, but consider now a non-infinitesimal matrix θ satisfying the conditions above. The set of such matrices is the *Lie algebra* corresponding to the Lie group.

Any element of the connected group can be reached by repeating infinitesimal steps, so let us define $\delta\theta = \theta/N$, which becomes infinitesimal in the limit of large N. Repeating N times the transformation given by $\delta\theta$ gives us

$$\lim_{N\to\infty} \left(\mathbb{1} + i\frac{\theta}{N}\right)^N = e^{i\theta}. \tag{2.4.4}$$

Therefore an arbitrary element M of the Lie group can therefore be expressed as $M = e^{i\theta}$ where θ is an element of the Lie algebra. Note, however, that several elements θ of the Lie algebra can give the same element M of the group.

[3] This is the usual convention in physics. In mathematics literature, the i is usually not included.

The sum $\theta_1 + \theta_2$ of any two elements of the Lie algebra θ_1 and θ_2 is also an element of the algebra, and so it any element multiplied by a scalar. Therefore the elements of the Lie algebra form a *vector space*. This means that we can choose a basis, i.e., a set of linearly independent elements of the algebra, which we denote by t^a, and express any element θ as a linear combination $\theta = \theta^a t^a$ with real coefficients $\theta^a \in \mathbb{R}$.[4] Here index $a \in \{1, \ldots, D\}$, where D is the number of lineary independent matrices in the algebra and is called the *dimensionality* of the group. The basis matrices t^a are called the *generators* of the group. For the different matrix groups we have

- $\mathrm{GL}(N, \mathbb{C})$: generators t^a are complex $N \times N$ matrices, $D = 2N^2$.
- $\mathrm{GL}(N, \mathbb{R})$: generators t^a are imaginary $N \times N$ matrices, $D = N^2$.
- $\mathrm{SL}(N, \mathbb{C})$: generators t^a are traceless complex $N \times N$ matrices, $D = 2(N^2 - 1)$.
- $\mathrm{SL}(N, \mathbb{R})$: generators t^a are traceless imaginary $N \times N$ matrices, $D = N^2 - 1$.
- $\mathrm{U}(N)$: generators t^a are Hermitean $N \times N$ matrices, $D = N^2$.
- $\mathrm{SU}(N)$: generators t^a are traceless Hermitean $N \times N$ matrices, $D = N^2 - 1$.
- $\mathrm{SO}(N)$: generators t^a are antisymmetric imaginary $N \times N$ matrices, $D = N(N-1)/2$.

Note that antisymmetric imaginary matrix is always traceless, so $\mathrm{O}(N)$ has the same generators as $\mathrm{SO}(N)$. This is because $\mathrm{SO}(N)$ is the connected subgroup of $\mathrm{O}(N)$. In practice, the generators are chosen to be orthogonal in the sense that $\mathrm{Tr}\, t^a t^b = C\delta^{ab}$ with some constant C, which depends on the group. For $\mathrm{SU}(N)$, it is conventionally chosen to be $C = 1/2$ and for $\mathrm{SO}(N)$, $C = 2$.

For example, the group $\mathrm{SO}(2)$ has dimensionality $D = 2 \times 1/2 = 1$, so it has only one generator, which is the appropriately normalised imaginary antisymmetric matrix,

$$t^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{2.4.5}$$

$\mathrm{SO}(3)$ has dimensionality $D = 3 \times 2/2 = 3$, so it has three generators which can be chosen to be

$$t^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad t^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.4.6}$$

$\mathrm{SU}(2)$ has dimensionality $D = 2 \times 2 - 1 = 3$. The generators can be chosen to be $t^a = \sigma^a/2$, where σ^a are the Pauli matrices,

$$t^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4.7}$$

2.5 Lie Algebras

In the last section, the Lie algebra was introduced as a set of matrices θ that give rise to elements M of the corresponding matrix group through $M = e^{i\theta}$. However, just like a group can be thought of as an abstract set instead of matrices, we can also give an abstract definition of the Lie algebra. For

[4] Repeated indices are summed unless otherwise stated. Except in the case of Lorentz indices, we do not make a distinction between superscript and subscript indices.

an abstract group, the defining property of the group is the group operation, i.e., the product M_1M_2 of two group elements M_1 and M_2. Assuming again infinitesimal transformations, we write them in terms of infinitesimal elements θ_1 and θ_2 of the Lie algebra as $M_1 = e^{i\theta_1}$ and $M_2 = e^{i\theta_2}$. We want to find the element of the Lie algebra that corresponds to the product M_1M_2.

We can Taylor expand the product as

$$\begin{aligned} M_1M_2 &= \left(\mathbb{1} + i\theta_1 - \frac{1}{2}\theta_1^2\right)\left(\mathbb{1} + i\theta_2 - \frac{1}{2}\theta_2^2\right) + O(\theta^3) \\ &= \mathbb{1} + i(\theta_1+\theta_2) - \frac{1}{2}(\theta_1^2+\theta_2^2+2\theta_1\theta_2) + O(\theta^3) \\ &= \mathbb{1} + i(\theta_1+\theta_2) - \frac{1}{2}(\theta_1+\theta_2)^2 - \frac{1}{2}[\theta_1,\theta_2] + O(\theta^3) \\ &= e^{i\left(\theta_1+\theta_2+i\frac{1}{2}[\theta_1,\theta_2]\right)} + O(\theta^3). \end{aligned} \quad (2.5.1)$$

This means that the element of the Lie algebra that corresponds to the product M_1M_2 is $\theta_1 + \theta_2 + i\frac{1}{2}[\theta_1, \theta_2]$, which shows that $i[\theta_1, \theta_2]$ has to be an element of the Lie algebra, too. Furthermore, this shows that group product rules are determined by the commutators $[\theta_1, \theta_2]$.

The general definition of the Lie algebra captures these properties. It is defined as a vector space with a non-associative operation $[X, Y]$, known as the Lie bracket, such that if X and Y are elements of the algebra, $i[X, Y]$ is also an element. The operation is anticommutative $[X, Y] = -[Y, X]$ and it satisfies the *Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \quad (2.5.2)$$

These properties are satisfied by the commutator of matrices.

The generators t^a correspond to the basis vector of the vector space, which means that we can write any elements θ_1 and θ_2 of the algebra in terms of the generators t^a, as $\theta_1 = \theta_1^a t^a$ and $\theta_2^a t^a$. Correspondingly, the commutator is given by

$$[\theta_1,\theta_2] = \theta_1^a\theta_2^b[t^a,t^b]. \quad (2.5.3)$$

Hence the algebra, and therefore also the structure of the Lie group, is determined by the commutators $[t^a, t^b]$ of the generators. Because $i[t^a, t^b]$ is an element of the algebra, we can write it as a linear combination of the generators t^a,

$$[t^a,t^b] = if^{abc}t^c, \quad (2.5.4)$$

where the coefficients f^{abc} are real and known as *structure constants*.

It follows directly from the definition (2.5.4) that the structure constants are antisymmetric with respect to the first two indices. Often (more precisely, when the Lie algebra is semi-simple), they are also cyclic,

$$f^{abc} = f^{bca} = f^{cab}, \quad (2.5.5)$$

from which it follows that they are actually antisymmetric with respect to any pair of indices. They also satisfy the Jacobi identity (see Problem Sheet 2)

$$f^{abe}f^{cde} + f^{ace}f^{dbe} + f^{ade}f^{bce} = 0. \quad (2.5.6)$$

Every Lie algebra corresponds to one or more Lie groups. The Lie groups that share the same Lie algebra are *locally isomorphic*. For example, $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$ have the same structure constants $f^{abc} = \epsilon^{abc}$, where ϵ^{abc} is the Levi-Civita symbol defined by $\epsilon^{123} = 1$. Therefore $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$ are locally isomorphic. They are not, however, globally isomorphic, because an $\mathrm{SO}(3)$ rotation by angle 2π around any axis corresponds to a trivial operation whereas the corresponding $\mathrm{SU}(2)$ operation corresponds to multiplication by -1 (see Problem Sheet 2).

2.6 Representations

So far we have considered matrix groups which are subgroups of $\mathrm{GL}(N)$ and whose elements are therefore $N\times N$ matrices. The generators t^a, $a\in\{1,\ldots,D\}$ of the group are therefore also $N\times N$ matrices. However, if one finds another set of D matrices T^a that satisfy the same algebra,

$$[T^a,T^b]=if^{abc}T^c, \tag{2.6.1}$$

with the same structure constants f^{abc}, then they form a different *representation* of the same Lie algebra. The original representation $\{t^a\}$ is called the *fundamental representation*. We will use the upper case T^a to denote group generators in general, and the lower case t^a to denote specifically the generators of the fundamental representation.

If one can simultaneously block diagonalise all D matrices T^a, then the representation is said to be *reducible*. Otherwise it is *irreducible*. A simple example of a reducible representation is

$$T^a=\begin{pmatrix}t^a & 0\\ 0 & t^a\end{pmatrix}. \tag{2.6.2}$$

In addition to the fundamental representation $\{t^a\}$, one can also always define the *adjoint representation* as $D\times D$ matrices T^a

$$(T^a)_{bc}=-if^{abc}. \tag{2.6.3}$$

It follows from Eq. (2.5.6) that that these matrices satisfy the relation (2.6.1), and therefore they are a representation of the algebra.

In analogy with the representation of the algebra, we define a *representation* of the Lie group as a set of matrices such that

1. to each group element g corresponds a matrix $M(g)$,
2. for any two group elements g_1 and g_2, $M(g_1)M(g_2)=M(g_1\cdot g_2)$.

If $M(g_1)=M(g_2)$ implies $g_1=g_2$, then the representation is *faithful*. The *trivial representation* corresponds to having $M(g)=1$ for every group element g. This is clearly not a faithful representation.

If $\{T^a\}$ are a representation of the algebra, then the matrices $M=e^{i\theta^aT^a}$ satisfy the same product rules as the original group elements $e^{i\theta^at^a}$, and therefore they form a representation of the group. If $\{T^a\}$ are $n\times n$ matrices, then these matrices M act on an n-component vector $\phi=(\phi_1,\ldots,\phi_n)^T$ as

$$\phi_i\to M_{ij}\phi_j,\quad i,j\in\{1,\ldots,n\}, \tag{2.6.4}$$

or, for an infinitesimal transformation $M=\mathbb{1}+i\theta^aT^a$, $|\theta^a|\ll 1$ as

$$\phi_i\to\phi_i+i\theta^aT^a_{ij}\phi_j+O(\theta^2),\quad i,j\in\{1,\ldots,n\}. \tag{2.6.5}$$

Then we say that the field ϕ is (or transforms) in the representation $\{T^a\}$.

It is worth looking at the adjoint representation (2.6.3) as a specific example. From Eq. (2.6.5) we see that a field in the adjoint representation transforms as

$$\phi_b\to\phi_b+\theta^af^{abc}\phi_c+O(\theta^2),\quad a,b,c\in\{1,\ldots,D\}. \tag{2.6.6}$$

The adjoint representation has the special property that a vector ϕ_b transforming under the adjoint representation can be expressed as an element of the Lie algebra, i.e., as an $N\times N$ matrix, as $\Phi=$

$\phi_a t^a$, where t^a are the generators of the fundamental representation. Then the transformation law is $\Phi \to M\Phi M^{-1}$, where the matrices M are in the fundamental representation. This can be seen by considering an infinitesimal transformation $M = \mathbb{1} + i\theta^a t^a$, $|\theta^a| \ll 1$,

$$\begin{aligned} M\Phi M^{-1} &= (\mathbb{1} + i\theta^a t^a)\phi_b t^b(\mathbb{1} - i\theta^c t^c) + O(\theta^2) \\ &= \phi_b t^b + i\theta^a \phi_b [t^a, t^b] + O(\theta^2) \\ &= \phi_b t^b - \theta^a \phi_b f^{abc} t^c + O(\theta^2) \\ &= \left(\phi_b + \theta^a f^{abc}\phi_c\right) t^b + O(\theta^2), \end{aligned} \tag{2.6.7}$$

where in the last step we have swapped the indices $b \leftrightarrow c$. A field in the adjoint representation can therefore be expressed equivalently either has a D-component vector or a $N \times N$ matrix, whichever is more convenient for the given calculation.

Chapter 3

Symmetry in Field Theory

3.1 Noether's Theorem

Let us now assume a scalar field ϕ with n complex components ϕ_i, with a Lagrangian

$$\mathcal{L}(\phi_i, \phi_i^*, \partial_\mu \phi_i, \partial_\mu \phi_i^*) \tag{3.1.1}$$

that is invariant under a symmetry group G, in other words under transformations

$$\phi_i \to M_{ij}\phi_j, \tag{3.1.2}$$

where the matrices M_{ij} form some representation of the group G.

Consider now an infinitesimal transformation

$$M_{ij} = \left(\delta_{ij} + i\theta_a T^a_{ij}\right). \tag{3.1.3}$$

Under such a transformation, the field changes as $\phi_i \to \phi_i + \delta\phi_i$, where

$$\delta\phi_i = i\theta^a T^a_{ij}\phi_j. \tag{3.1.4}$$

The complex conjugate transforms as

$$\delta\phi_i^* = -i\theta^a \phi_j^* \left(T^a_{ij}\right)^*. \tag{3.1.5}$$

In general, whether G is a symmetry or not, and whether $\phi(x)$ is a solution or not, the Lagrangian changes under this transformation by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\partial_\mu\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\phi_i^*}\delta\phi_i^* + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i^*}\partial_\mu\delta\phi_i^*. \tag{3.1.6}$$

However, if we assume that $\phi(x)$ is a solution of the equations of motion, then each component satisfies the Euler-Lagrange equation (1.2.5),

$$\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} = 0 \tag{3.1.7}$$

and its complex conjugate. Substituting this into Eq. (3.1.6), we find that for a solution of the equations of motion,

$$\delta\mathcal{L} = \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right)\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\partial_\mu\delta\phi_i + \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i^*)}\right)\delta\phi_i^* + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i^*}\partial_\mu\delta\phi_i^*. \tag{3.1.8}$$

Now, we notice that first pair of terms and the second pair of terms both have the form of a derivative of a product, and therefore we can write more compactly

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i^*)}\delta\phi_i^* \right). \tag{3.1.9}$$

Substituting Eqs. (3.1.4) and (3.1.5), this becomes

$$\delta\mathcal{L} = i\theta^a\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}T^a_{ij}\phi_j - \phi_j^* \left(T^a_{ij}\right)^* \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i^*)} \right). \tag{3.1.10}$$

This expression is valid whether the transformation is a symmetry or not. If it is a symmetry, then by definition $\delta\mathcal{L} = 0$, and Eq. (3.1.10) implies that

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}T^a_{ij}\phi_j - \phi_j^* \left(T^a_{ij}\right)^* \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i^*)} \right) = 0. \tag{3.1.11}$$

This shows that the *Noether currents*

$$j^{\mu a} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}T^a_{ij}\phi_j - \phi_j^* \left(T^a_{ij}\right)^* \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i^*)}, \tag{3.1.12}$$

are conserved, i.e., satisfy $\partial_\mu j^{\mu a} = 0$. These currents are labelled by the index $a \in \{1, \ldots, D\}$, which means that there are D conserved currents, one for each generator T^a.

For example, if the Lagrangian has the form

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - V(\phi^\dagger\phi), \tag{3.1.13}$$

so that it has an $\mathrm{U}(N)$ symmetry, we find $D = N^2$ conserved currents

$$j^a_\mu = (\partial_\mu\phi_i^*)\, T^a_{ij}\phi_j - \phi_i^* T^a_{ij}\partial_\mu\phi_j = (\partial_\mu\phi)^\dagger\, T^a\phi - \phi^\dagger T^a\partial_\mu\phi, \tag{3.1.14}$$

where we have used the Hermiticity of the $\mathrm{U}(N)$ generators, $\left(T^a_{ij}\right)^* = T^a_{ji}$.

3.2 Non-Abelian Gauge Fields

As in Section 2.2, consider now a local (i.e. position dependent) transformation $M(x)$,

$$\phi_i(x) \to M_{ij}(x)\phi_j(x). \tag{3.2.1}$$

The potential term $V(\phi^\dagger\phi)$ in the Lagrangian (3.1.13) is invariant, but because the derivative transforms as

$$\partial_\mu\phi_i \to (\partial_\mu M_{ij})\phi_j + M_{ij}\partial_\mu\phi_j, \tag{3.2.2}$$

the derivative terms $\partial_\mu\phi^\dagger\partial^\mu\phi$ is not.

As in the Abelian case, we define the *covariant derivative* as

$$D_\mu\phi \equiv \partial_\mu + igA_\mu, \tag{3.2.3}$$

where g is a constant and A_μ is a gauge field whose transformation properties will be chosen in such a way that

$$D_\mu\phi \to MD_\mu\phi. \tag{3.2.4}$$

If this is the case, then the $(D_\mu\phi)^\dagger D^\mu\phi$ is invariant.

Let us therefore assume a transformation

$$\begin{aligned}\phi &\to M\phi,\\ A_\mu &\to A'_\mu,\end{aligned} \tag{3.2.5}$$

where the transformed gauge field A'_μ has not yet been specified. We wish to determine what it needs to be so that we achive the transformation law (3.2.4). The covariant derivative of ϕ transforms as

$$D_\mu\phi = \partial_\mu\phi+igA_\mu\phi \to (\partial_\mu M)\phi+M\partial_\mu\phi+igA'_\mu M\phi = MD_\mu\phi+(\partial_\mu M)\phi+igA'_\mu M\phi-igMA_\mu\phi. \tag{3.2.6}$$

To achieve the desired transformation law (3.2.4), the last three terms must cancel for any ϕ, which means

$$igA'_\mu M\phi = igMA_\mu\phi - (\partial_\mu M)\phi. \tag{3.2.7}$$

For this to be true for any ϕ, we need

$$A'_\mu M = MA_\mu + \frac{i}{g}(\partial_\mu M), \tag{3.2.8}$$

and multiplying from the right with the inverse transformation M^{-1}, we obtain the transformation law for A_μ,

$$A'_\mu = MA_\mu M^{-1} + \frac{i}{g}(\partial_\mu M)M^{-1}. \tag{3.2.9}$$

For example, the Lagrangian (3.1.13) has $\mathrm{U}(N)$ symmetry, so $M^{-1} = M^\dagger$. With gradients replaced by covariant derivatives, the Lagrangian

$$\mathcal{L} = (D_\mu\phi)^\dagger D^\mu\phi - V(\phi^\dagger\phi), \tag{3.2.10}$$

is invariant under the non-Abelian local gauge transformations

$$\begin{aligned}\phi &\to M\phi,\\ A_\mu &\to MA_\mu M^\dagger + \frac{i}{g}(\partial_\mu M)M^\dagger,\end{aligned} \tag{3.2.11}$$

Next, we need to determine what kind of an object A_μ needs to be to have the transformation law (3.2.11). If we consider an infinitesimal transformation $M = \mathbb{1} + i\theta$, where θ is an infinitesimal element of the Lie algebra, we note that

$$\frac{i}{g}(\partial_\mu M)M^{-1} = \frac{i}{g}(i\partial_\mu\theta) = -\frac{1}{g}\partial_\mu\theta. \tag{3.2.12}$$

Because θ is an element of the Lie algebra, this means that $\frac{i}{g}(\partial_\mu M)M^{-1}$ is an element of the Lie algebra, too.

This suggests that A_μ could be chosen to be an element of the Lie algebra. To check this, we have to confirm that A'_μ in Eq. (3.2.9) would then be an element of the algebra, too. Considering again an infinitesimal transformation $M = \mathbb{1} + i\theta$,

$$\begin{aligned}A'_\mu &= MA_\mu M^{-1} + \frac{i}{g}(\partial_\mu M)M^{-1}\\ &= (\mathbb{1}+i\theta)A_\mu(\mathbb{1}-i\theta) - \frac{1}{g}\partial_\mu\theta = A_\mu + i[\theta, A_\mu] - \frac{1}{g}\partial_\mu\theta.\end{aligned} \tag{3.2.13}$$

The second term is the Lie bracket and is therefore an element of the algebra, and therefore the whole expression in an element of the algebra, too.

We have therefore shown that the transformation law (3.2.11) can be realised if A_μ is an element of the Lie algebra. It can therefore be written in term of the generators t^a as $A_\mu = A_\mu^a t^a$, $a \in \{1, \ldots, D\}$ with real coefficients A_μ^a. Therefore we can think of the gauge field as a D-component real vector field. When considering the covariant derivative $D_\mu\phi$ of a field ϕ in a particular representation, then the gauge field would be written as a linear combination of the generators of that representation, $A_\mu = A_\mu^a T^a$.

For future reference, let us also determine how the vector components A_μ^a transform under an infinitesimal gauge transformation. Writing $A_\mu = A_\mu^a t^a$ and $\theta = \theta^a t^a$, we obtain from Eq. (3.2.13)

$$\begin{aligned} A_\mu^a t^a &\rightarrow A_\mu^a t^a + i\theta^a A_\mu^b [t^a, t^b] - \frac{1}{g}\partial_\mu \theta^a t^a = A_\mu^a t^a - f^{abc}\theta^a A_\mu^b t^c - \frac{1}{g}\partial_\mu \theta^a t^a \\ &= A_\mu^a t^a - f^{abc}\theta^b A_\mu^c t^a - \frac{1}{g}\partial_\mu \theta^a t^a = \left(A_\mu^a - f^{abc}\theta^b A_\mu^c - \frac{1}{g}\partial_\mu\theta^a\right) t^a. \end{aligned} \tag{3.2.14}$$

Hence the component A_μ^a transforms as

$$A_\mu^a \rightarrow A_\mu^a - f^{abc}\theta^b A_\mu^c - \frac{1}{g}\partial_\mu \theta^a.$$

Comparing with the Abelian case Eq. (2.2.6) we see that this equation has an additional term that involves the structure constants and therefore reflects the non-Abelian nature of the gauge group.

In Section 2.2 we found that in the Abelian case the kinetic term for the gauge field is $-(1/4)F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In the non-Abelian case this combination is not gauge invariant, and therefore it is not a suitable kinetic term. However, consider the commutator of the Abelian covariant derivatives,

$$\begin{aligned} [D_\mu, D_\nu]\phi &= D_\mu D_\nu \phi - D_\nu D_\mu \phi \\ &= (\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu)\phi - (\partial_\nu + ieA_\nu)(\partial_\mu + ieA_\mu)\phi \\ &= \partial_\mu\partial_\nu\phi + ieA_\mu\partial_\nu\phi + ie(\partial_\mu A_\nu)\phi + ieA_\nu\partial_\mu\phi - e^2 A_\mu A_\nu \phi \\ &\quad -\partial_\nu\partial_\mu\phi - ieA_\nu\partial_\mu\phi - ie(\partial_\nu A_\mu)\phi - ieA_\mu\partial_\nu\phi + e^2 A_\nu A_\mu\phi \\ &= ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\phi = ieF_{\mu\nu}\phi. \end{aligned} \tag{3.2.15}$$

We can therefore write the Abelian field strength tensor as $F_{\mu\nu} = -(i/e)[D_\mu, D_\nu]$. Note that even though D_μ is a differential operator, $F_{\mu\nu}$ is not because the derivatives of ϕ cancel in Eq. (3.2.15).

Based on this, we can try if the same works in the non-Abelian case. Note first that because $D_\mu\phi \rightarrow MD_\mu\phi$ under a gauge transformation M, we have

$$[D_\mu, D_\nu]\phi \rightarrow M[D_\mu, D_\nu]\phi = M[D_\mu, D_\nu]M^{-1}M\phi. \tag{3.2.16}$$

Therefore if we define again

$$F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu], \tag{3.2.17}$$

then $F_{\mu\nu} \rightarrow MF^{\mu\nu}M^{-1}$, which means that it is in the adjoint representation. Hence $F_{\mu\nu}F^{\mu\nu} \rightarrow MF_{\mu\nu}M^{-1}MF^{\mu\nu}M^{-1} = MF_{\mu\nu}F^{\mu\nu}M^{-1}$, so this combination is still not gauge invariant. Furthermore, it is a matrix so it cannot be a term in the Lagrangian. However, its trace is a real number and also gauge invariant,

$$\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \rightarrow \mathrm{Tr}\, MF_{\mu\nu}F^{\mu\nu}M^{-1} = \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}. \tag{3.2.18}$$

Therefore it is a natural non-Abelian generalisation of the Abelian kinetic term.

To express $F_{\mu\nu}$ explicitly in terms of the gauge field A_μ, we write

$$\begin{aligned}
[D_\mu, D_\nu]\phi &= D_\mu D_\nu\phi - D_\nu D_\mu\phi \\
&= (\partial_\mu + igA_\mu)(\partial_\nu + igA_\nu)\phi - (\partial_\nu + igA_\nu)(\partial_\mu + igA_\mu)\phi \\
&= \partial_\mu\partial_\nu\phi + igA_\mu\partial_\nu\phi + ig(\partial_\mu A_\nu)\phi + igA_\nu\partial_\mu\phi - g^2 A_\mu A_\nu\phi \\
&\quad -\partial_\nu\partial_\mu\phi - igA_\nu\partial_\mu\phi - ig(\partial_\nu A_\mu)\phi - igA_\mu\partial_\nu\phi + g^2 A_\nu A_\mu\phi \\
&= ig(\partial_\mu A_\nu - \partial_\nu A_\mu)\phi - g^2[A_\mu, A_\nu]\phi.
\end{aligned} \tag{3.2.19}$$

Hence

$$F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]. \tag{3.2.20}$$

Because A_μ is an element of the Lie algebra, this means that $F_{\mu\nu}$ is an element of the Lie algebra as well. Note that the derivatives of ϕ cancel again in Eq. (3.2.19), so $F_{\mu\nu}$ is again not a differential operator.

As for any element of the Lie algebra, it is often useful to write $F_{\mu\nu}$ is terms of the generators, $F_{\mu\nu} = F^a_{\mu\nu}t^a$. We can find the components $F^a_{\mu\nu}$ by substituting $A_\mu = A^a_\mu t^a$ into Eq. (3.2.20),

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A^a_\nu t^a - \partial_\nu A^a_\mu t^a + igA^b_\mu A^c_\nu[t^b, t^c] \\
&= \partial_\mu A^a_\nu t^a - \partial_\nu A^a_\mu t^a - gf^{abc}A^b_\mu A^c_\nu t^a \equiv F^a_{\mu\nu}t^a.
\end{aligned} \tag{3.2.21}$$

Hence

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu. \tag{3.2.22}$$

We can see that for each value of a, $F^a_{\mu\nu}$ looks like the Abelian field strength tensor with an additional second-order term.

In terms of these components, we can write

$$\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} = F^a_{\mu\nu}F^{\mu\nu b}\mathrm{Tr}\, t^a t^b = \frac{1}{2}F^a_{\mu\nu}F^{\mu\nu a}, \tag{3.2.23}$$

where we used the normalisation $\mathrm{Tr}\, t^a t^b = (1/2)\delta^{ab}$ (see Section 2.4). It is convenient to normalise the kinetic term in such that each component $F^a_{\mu\nu}$ has the same coefficient $-(1/4)$ as in the Abelian case (1.3.9). Therefore the scalar field Lagrangian (3.1.13) with a local gauge symmetry is written as

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi - V(\phi^\dagger\phi). \tag{3.2.24}$$

The first term alone,

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}, \tag{3.2.25}$$

gives a theory of a non-Abelian gauge field only, with no matter fields. This theory is called the *Yang-Mills theory* or pure gauge theory. Writing the Lagrangian in terms of the gauge field components A^a_μ, we obtain

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} \\
&= -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}\right) \\
&\quad +\frac{g}{2}f^{abc}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)A^{\mu b}A^{\nu c} - \frac{g^2}{4}f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e}.
\end{aligned} \tag{3.2.26}$$

The first line is D copies of the Maxwell term, and therefore the theory consists of D photon-like massless vector fields. The second line, which has no analogue in the Abelian theory, describes self-interactions of the gauge field. Therefore a non-Abelian gauge field is never free.

The Standard Model has two separate non-Abelian gauge fields: Strong interactions are described by an $\mathrm{SU}(3)$ gauge field, and therefore it has $D = 3^2 - 1 = 8$ gauge bosons called gluons. Weak interactions are described by an $\mathrm{SU}(2)$ gauge field, giving rise to $D = 2^2 - 1 = 3$ gauge bosons. However, at this level the picture does not match with experiments, because both strong and weak interactions are short-range forces, and therefore quite different from electrodynamics.

In the case of the strong interactions, the solution lies in the self-interaction term, which is so strong that gluons (or quarks) can never exist as free particles. This effect, known as *confinement*, means that the actual particle states of the theory (such as protons and neutrons) are neutral under the strong force.

The short range of the weak interactions, on the other hand, is due a non-zero mass of the weak gauge bosons, which seems to be in contradiction with our finding that they should be massless. This puzzle is solved by the *Higgs mechanism*, which gives mass to them, as we will see later.

3.3 Equations of Motion

Let us now find the equation of motion for the gauge field in the Yang-Mills theory (3.2.26). The Euler-Lagrange equation is

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial A^e_\rho} - \partial_\sigma\frac{\partial\mathcal{L}}{\partial\partial_\sigma A^e_\rho} &= -\frac{1}{2}F^{\mu\nu a}\frac{\partial F^a_{\mu\nu}}{\partial A^e_\rho} - \partial_\sigma\left(-\frac{1}{2}F^{\mu\nu a}\frac{\partial F^a_{\mu\nu}}{\partial\partial_\sigma A^e_\rho}\right) \\
&= -\frac{1}{2}F^{\mu\nu a}\left(-2gf^{abe}A^b_\mu\delta^\rho_\nu\delta_{ec}\right) - \partial_\sigma\left(-\frac{1}{2}F^{\mu\nu a}2\delta^\sigma_\mu\delta^\rho_\nu\delta_{ae}\right) \\
&= gF^{\mu\rho a}f^{abe}A^b_\mu + \partial_\mu F^{\mu\rho e} = \left(\delta_{ae}\partial_\mu + gf^{bea}A^b_\mu\right)F^{\mu\rho a} = 0.
\end{aligned} \tag{3.3.1}$$

Comparing with Eq. (2.6.3), we can write this in terms of the generators T^b of the adjoint representation,

$$\left(\delta_{ae}\partial_\mu + igA^b_\mu T^b_{ea}\right)F^{\mu\rho a} = 0. \tag{3.3.2}$$

The expression inside the brackets is the covariant derivative in the adjoint representation D^{ae}_μ, so we can write the equation of motion in a compact form,

$$D^{ab}_\mu F^{\mu\nu b} = 0, \tag{3.3.3}$$

where $D^{ab}_\mu = \delta_{ab}\partial_\mu + igA^c_\mu T^c_{ab}$.

The same equation can also be written in terms of the covariant derivative in the fundamental representation, $D_\mu = \partial_\mu + igA^a_\mu t^a$, as

$$[D_\mu, F^{\mu\nu}] = 0, \tag{3.3.4}$$

as the explicit calculation shows,

$$\begin{aligned}
[D_\mu, F^{\mu\nu}] &= \partial_\mu F^{\mu\nu} + ig[A_\mu, F^{\mu\nu}] \\
&= \partial_\mu F^{\mu\nu e}t^e + igA^b_\mu F^{\mu\nu a}[t^b, t^a] \\
&= \partial_\mu F^{\mu\nu e}t^e + igA^b_\mu F^{\mu\nu a}if^{bae}t^e \\
&= \left(\partial_\mu\delta_{ae} - gf^{bae}A^b_\mu\right)F^{\mu\nu a}t^e = 0,
\end{aligned} \tag{3.3.5}$$

where the last equality follows from Eq. (3.3.1).

We also obtain another equation for the gauge field by considering the Jacobi identity of the covariant derivatives D_μ (in the fundamental representation),

$$[D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]] + [D_\lambda, [D_\mu, D_\nu]] = 0, \tag{3.3.6}$$

which implies

$$[D_\mu, F_{\nu\lambda}] + [D_\nu, F_{\lambda\mu}] + [D_\lambda, F_{\mu\nu}] = 0, \tag{3.3.7}$$

or more compactly as

$$\epsilon^{\mu\nu\lambda\rho}[D_\nu, F_{\lambda\rho}] = 0, \tag{3.3.8}$$

where $\epsilon^{\mu\nu\lambda\rho}$ is the fully antisymmetric Levi-Civita tensor. This is known as the *Bianchi identity*. By defining the *dual field strength tensor*

$$\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}, \tag{3.3.9}$$

the Bianchi identity can be written as $[D_\mu, \tilde{F}^{\mu\nu}] = 0$. Alternatively, in terms of the adjoint covariant derivative D_μ^{ab}, this can be written as

$$D_\mu^{ab}\tilde{F}^{\mu\nu b} = 0. \tag{3.3.10}$$

If we consider the theory with the scalar field (3.2.24), the gauge field Euler-Lagrange equation has an additional term,

$$D_\mu^{ab}F^{\mu\nu b} = -\frac{\partial}{\partial A_\nu^b}\left[(D_\mu\phi)^\dagger D^\mu\phi\right] = ig\left[\phi^\dagger T^b D^\nu\phi - (D^\nu\phi)^\dagger T^b\phi\right]. \tag{3.3.11}$$

Defining the non-Abelian current

$$j^{\nu b} = ig\left[\phi^\dagger T^b D^\nu\phi - (D^\nu\phi)^\dagger T^b\phi\right], \tag{3.3.12}$$

this becomes

$$D_\mu^{ab}F^{\mu\nu b} = j^{\nu b}. \tag{3.3.13}$$

Note that, up to a constant factor, the current $j^{\nu b}$ is the natural generalisation of the Noether current (3.1.14) obtained by replacing $\partial_\mu \to D_\mu$. However, it is not conserved, $\partial_\mu j^{\mu a} \neq 0$.

On the other hand, the current j_μ^a is covariantly conserved. To see this, write first

$$D_\mu^{ab}j^{\mu b} = D_\mu^{ab}D_\nu^{bc}F^{\nu\mu c} = \frac{1}{2}\left(D_\mu^{ab}D_\nu^{bc} - D_\nu^{ab}D_\mu^{bc}\right)F^{\nu\mu c}, \tag{3.3.14}$$

where we used the fact that $F^{\mu\nu a}$ is antisymmetric. Now, the relation $[D_\mu, D_\nu] = igF_{\mu\nu}$ is true in any representation, and therefore specifically in the adjoint representation we find

$$D_\mu^{ab}D_\nu^{bc} - D_\nu^{ab}D_\mu^{bc} = igF_{\mu\nu}^d T_{ac}^d = gf^{acd}F_{\mu\nu}^d. \tag{3.3.15}$$

Substituting this in Eq. (3.3.14), we obtain

$$D_\mu^{ab}j^{\mu b} = -\frac{1}{2}gf^{acd}F^{\mu\nu c}F_{\mu\nu}^d = 0, \tag{3.3.16}$$

where the last equality follows because f^{acd} is antisymmetric.

The scalar field equation of motion is obtained from the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi_i^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i^*} = 0. \tag{3.3.17}$$

It gives (see Problem Sheet 3)

$$D_\mu D^\mu \phi + \frac{\partial V}{\partial \phi^*} = 0, \tag{3.3.18}$$

where D_μ is the covariant derivative corresponding to the representation the scalar field ϕ is in.

3.4 Spontaneous Symmetry Breaking

Let us determine the particle spectrum in the theory (3.2.24) with an $\mathrm{SU}(N)$ gauge field and a fundamental scalar field. To do this, we assume that A_μ and ϕ are both small, and keep only quadratic terms. This gives the quadratic Lagrangian

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}\right) \\ &\quad + \partial_\mu \phi_i^* \partial^\mu \phi_i - m^2 \phi_i^* \phi_i. \end{aligned} \tag{3.4.1}$$

From this we see that the theory contains $D = N^2 - 1$ massless vector bosons A_μ^a, each carrying two transverse degrees of freedom, and N complex scalar bosons ϕ_i, each carrying two real degrees of freedom.

Note, however, that we assumed that the ground state of the theory corresponds to $\phi = 0$. This is the case if $m^2 > 0$, but what if $m^2 < 0$? For simplicity, consider first a single real scalar field ϕ with a $\mathbb{Z}_2$ symmetry $\phi \to -\phi$ and with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi), \quad V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4, \tag{3.4.2}$$

with $\mu^2 > 0$. In this case the potential has two degenerate minima at $\phi = \pm v$, where $v = \mu/\sqrt{\lambda}$. These correspond to two degenerate vacuum states. The field value $\pm v$ at the minimum is called the *vacuum expectation value* or simply *vev*.

Because the minima are related by a symmetry, they are physically identical, and therefore we can choose to focus on the positive minimum $\phi = +v$ without any loss of generality. Let us denote the deviation of field from its minimum by φ and write $\phi = v + \varphi$. In terms of φ, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - V(v + \varphi), \tag{3.4.3}$$

and

$$\begin{aligned} V(v+\varphi) &= -\frac{1}{2}\mu^2(v+\varphi)^2 + \frac{1}{4}\lambda(v+\varphi)^4 \\ &= -\frac{1}{2}\frac{\mu^4}{4\lambda} + \mu^2\varphi^2 + \sqrt{\lambda}\mu\varphi^3 + \frac{1}{4}\lambda\varphi^4. \end{aligned} \tag{3.4.4}$$

The first term is a constant and therefore it does not contribute to the Euler-Lagrange equation and can be ignored. The second term has the form of a mass term for the field φ, and corresponds to the mass $m = \sqrt{2}\mu$. The last two terms describe interactions.

Note that because of the cubic term φ^3, the potential is not symmetric under $\varphi \to -\varphi$. Therefore the $\mathbb{Z}_2$ symmetry is no longer apparent. The reason for this is, obviously, that the choice of the ground state $\phi = +v$ breaks the symmetry. However, the symmetry is still there and it is just hidden. Therefore we say that the symmetry is spontaneously broken. In general, *spontaneous symmetry breaking* (SSB) refers to the situation in which the state of the system is not invariant under the full symmetry of the Lagrangian, or in other words, that $\mathcal{L}$ symmetric but the state is not.

As a second example, consider a theory of a complex scalar field ϕ with a global U(1) symmetry $\phi \to e^{i\theta}\phi$ and Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi^*, \phi), \quad V(\phi^*, \phi) = -\mu^2 \phi^* \phi + \frac{1}{2}\lambda \left(\phi^* \phi\right)^2. \tag{3.4.5}$$

The extrema of the potential are found by differentiating

$$\frac{\partial V}{\partial \phi^*} = -\mu^2 \phi + \lambda \left(\phi^* \phi\right) \phi = 0, \tag{3.4.6}$$

which shows that the condition for the minimum is

$$\phi^* \phi = \frac{\mu^2}{\lambda} \equiv \frac{v^2}{2}. \tag{3.4.7}$$

This means that there is a circle of minima

$$\phi = \frac{v}{\sqrt{2}} e^{i\theta}, \quad \theta \in [0, 2\pi). \tag{3.4.8}$$

In general, a continuous set of minima is called the *vacuum manifold*.

As in the discrete case, the different vacua are all identical, and therefore we can choose $\phi = \phi_0 = v/\sqrt{2}$ with real v. We expand the field around this vacuum state,

$$\phi(x) = \frac{1}{\sqrt{2}} \left(v + \varphi(x) + i\chi(x)\right), \quad \varphi, \chi \in \mathbb{R}. \tag{3.4.9}$$

In terms of φ and χ, the potential is

$$\begin{aligned} V(\phi^*, \phi) &= -\frac{1}{2}\mu^2 \left[(v+\varphi)^2 + \chi^2\right] + \frac{1}{8}\lambda \left[(v+\varphi)^2 + \chi^2\right]^2 \\ &= -\frac{\mu^4}{2\lambda} + \mu^2 \varphi^2 + \frac{1}{2}\lambda v \varphi \left(\varphi^2 + \chi^2\right) + \frac{1}{8}\lambda \left(\varphi^2 + \chi^2\right)^2. \end{aligned} \tag{3.4.10}$$

Again, the first term is an irrelevant constant, and the last two terms are interactions. The second term is a mass term for the field φ, which again has mass $m_\varphi = \sqrt{2}\mu$. However, note that there is no mass term for the field χ, which is therefore massless. This means that, as a result of the spontaneous symmetry breaking, the theory contains a massless scalar particle, known as a *Goldstone boson* or a Goldstone mode.

More generally, we can consider an n-component real scalar field $\phi = (\phi_1, \ldots, \phi_n)^T$, with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^T \partial^\mu \phi - V(\phi). \tag{3.4.11}$$

This form of the kinetic term requires the symmetry transformarion matrices to be orthogonal, and we will assume it for simplicity. It would be possible to consider other kinetic terms.

The vacuum state $\phi = \phi_0$ can be found from the condition

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\phi=\phi_0} = 0 \quad \text{for all } i. \tag{3.4.12}$$

Taylor expanding around the vacuum state $\phi = \phi_0 + \varphi$, we obtain

$$\begin{aligned} V(\phi) &= V(\phi_0+\varphi) = V(\phi_0) + \varphi_i \left.\frac{\partial V}{\partial \phi_i}\right|_{\phi_0} + \frac{1}{2}\varphi_i\varphi_j \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right|_{\phi_0} + O(\varphi^3) \\ &= V(\phi_0) + \frac{1}{2}m_{ij}^2\varphi_i\varphi_j + O(\varphi^3), \end{aligned} \tag{3.4.13}$$

where the linear term cancels because of Eq. (3.4.12) and we have defined the *scalar mass matrix*

$$m_{ij}^2 \equiv \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right|_{\phi_0}. \tag{3.4.14}$$

The eigenvectors of this matrix determine the different particle species, and their masses are the square roots of the corresponding eigenvalues.

Assume now that the theory is invariant under some symmetry group G, under which the field transforms as $\phi \to M\phi$, where M is an element of the relevant representation of the group. Considering an infinitesimal transformation,

$$\phi_i \to \phi_i + i\theta^a T_{ij}^a \phi_j, \tag{3.4.15}$$

we find that the vacuum expectation value ϕ_0 transforms as $\phi_0 \to \phi_0 + \delta\phi$, where

$$\delta\phi = i\theta^a T_{ij}^a \phi_{0j}. \tag{3.4.16}$$

As a simple example, consider the SO(3) group and the vev $\phi_0 = (0, 0, v)^T$. In this case,

$$\begin{aligned} t^1\phi_0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -iv \\ 0 \end{pmatrix}, \\ t^2\phi_0 &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = \begin{pmatrix} iv \\ 0 \\ 0 \end{pmatrix}, \\ t^3\phi_0 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.4.17}$$

From this we see that the generators t^1 and t^2 change the vev ϕ_0, but the generator t^3 does not. We say that the generators t^1 and t^2 are *broken*, whereas t^3 is an *unbroken generator*. In general, T is an unbroken generator if $T\phi_0 = 0$.

In general, depending on the choice of the generators and the vev ϕ_0, it is possible that even though $T^a\phi_0 \neq 0$ for all T^a, there is some linear combination $\hat{T} = c^a T^a$ such that $\hat{T}\phi_0 = 0$. Therefore we define a $D \times D$ *symmetry breaking matrix*

$$S^{ab} = -\left(T^a\phi_0\right)^T T^b\phi_0 = \phi_0^T T^a T^b \phi_0 = \phi_{0i} T_{ij}^a T_{jk}^b \phi_{0k}, \tag{3.4.18}$$

where in the second equality we have used the fact that the generators of orthogonal transformation are antisymmetric, $(T^a)^T = -T^a$. For future reference it is useful to note that this matrix is real and symmetric. Therefore its eigenvalues are real, and in fact, non-negative.

If $\hat{T}\phi_0 = 0$ for some $\hat{T} = c^a T^a$, then

$$S^{ab}c^b = \phi_0^T T^a c^b T^b \phi_0 = \phi_0^T T^a \hat{T}\phi_0 = 0, \tag{3.4.19}$$

which means that c^b is an eigenvector of S^{ab} with eigenvalue zero. This means that the unbroken generators are given by eigenvectors with zero eigenvalue. Conversely, if c^a is an eigenvector with eigenvalue zero, then

$$c^a S^{ab} c^b = -\left(c^a T^a \phi_0\right)^T c^b T^b \phi_0 = 0, \tag{3.4.20}$$

which means that $c^a T^a \phi_0 = 0$, and therefore $\hat{T} \equiv c^a T^a$ is an unbroken generator. Therefore, there is a one-to-one correspondence between unbroken generators and eigenvectors with zero eigenvalues. That also means that any eigenvector with a non-zero eigenvalue corresponds to a broken generator.,

Overall, the matrix has D orthonormal eigenvectors c^A and corresponding eigenvalues $\lambda^A \geq 0$, where $A \in \{1, \ldots, D\}$. Let us order them in such a way that any zero eigenvalues come first, i.e., $\lambda^A = 0$ for $A \leq D'$, where D' is the number of zero eigenvalues. Correspondingly, $\lambda^A > 0$ for $D' < A \leq D$. Using the eigenvectors c^A, we can define a new set of generators

$$\hat{T}^A = (c^A)^a T^a, \tag{3.4.21}$$

and vectors $\hat{\phi}^A \equiv i\hat{T}^A \phi_0$. Then

$$\begin{aligned} \hat{\phi}^A &= 0 \quad \text{for } A \leq D', \\ \hat{\phi}^A &\neq 0 \quad \text{for } D' < A \leq D. \end{aligned} \tag{3.4.22}$$

This means that $\hat{T}^A$ for $A \leq D'$ are the unbroken generators. They generate a Lie group H, which is a subgroup of the original group G and under which the vacuum state is symmetric. This subgroup is called the *residual symmetry group*. We say that the *symmetry breaking pattern* is $G \to H$.

In the example of the $\mathrm{SO}(3)$ group considered above, there is a single unbroken generator t^3. It generates the $U(1)$ (or $\mathrm{SO}(2)$) group, and therefore the symmetry breaking pattern is $\mathrm{SO}(3) \to \mathrm{U}(1)$.

In terms of the new set of generators $\hat{T}^A$, the transformation law (3.4.15) can be written as

$$\phi_i \to \phi_i + \delta\phi_i, \quad \delta\phi_i = i\hat{\theta}^A \hat{T}^A_{ij}\phi_j, \quad A \in \{1, \ldots, D\}. \tag{3.4.23}$$

Because this is a symmetry, the potential does not change under the transformation,

$$\delta V = \delta\phi_i \frac{\partial V}{\partial \phi_i} = i\hat{\theta}^A \hat{T}^A_{ij}\phi_j \frac{\partial V}{\partial \phi_i} = 0. \tag{3.4.24}$$

Differentiating this equation with respect to ϕ_k gives

$$i\hat{\theta}^A \hat{T}^A_{ik} \frac{\partial V}{\partial \phi_i} + i\hat{\theta}^A \hat{T}^A_{ij}\phi_j \frac{\partial^2 V}{\partial \phi_k \partial \phi_i} = 0. \tag{3.4.25}$$

Now, evaluate this at the minimum $\phi = \phi_0$, using

$$\left.\frac{\partial V}{\partial \phi_i}\right|_{\phi_0} = 0, \quad \left.\frac{\partial^2 V}{\partial \phi_k \partial \phi_i}\right|_{\phi_0} = m^2_{ki}, \tag{3.4.26}$$

to obtain

$$i\hat{\theta}^A \hat{T}^A_{ij}\phi_{0j}m^2_{ki} = \hat{\theta}^A m^2_{ki}\hat{\phi}^A_i = 0 \quad \text{for any coefficients } \hat{\theta}^A. \tag{3.4.27}$$

This implies that

$$m^2_{ij}\hat{\phi}^A_j = 0 \quad \text{for all } A \in \{1,\ldots,D\}. \tag{3.4.28}$$

If $\hat{T}^A$ is an unbroken generator, then $\hat{\phi}^A = 0$, and this equation is satisfied trivially. On the other hand, if $\hat{T}^A$ is a broken generator, then $\hat{\phi}^A \neq 0$, and the result shows that $\hat{\phi}^A$ has to be an eigenvector of the mass matrix m^2_{ij} with zero eigenvalue. In other words, it corresponds to a massless Goldstone particle. This can be summarised as *Goldstone's theorem*: Every broken generator gives rise to a massless Goldstone particle.

Considering again the example of $\mathrm{SO}(3)$ discussed earlier, we find that because there are two broken generators, t^1 and t^2, there are two massless Goldstone particles. Indeed, an explicit calculation of the mass matrix gives

$$m^2_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\mu^2 \end{pmatrix}. \tag{3.4.29}$$

3.5 Higgs Mechanism

In the previous section we assumed that the broken symmetry is global, meaning that the symmetry transformation are position-independent, and we found that it leads to massless Goldstone particles. If we consider a local symmetry instead, we find a very different result.

For simplicity, let us first consider a theory with a complex scalar field and a local $\mathrm{U}(1)$ symmetry,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^* D^\mu\phi - V(\phi^*,\phi), \quad V(\phi^*,\phi) = -\mu^2\phi^*\phi + \frac{1}{2}\lambda\left(\phi^*\phi\right)^2. \tag{3.5.1}$$

As before, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu + ieA_\mu$.

The vacuum states of the theory are given by $A_\mu = 0$, $|\phi| = \mu/\sqrt{\lambda}$ (or any gauge transformation of this state). Therefore as in Eq. (3.4.9), we again expand the field as $\phi = (v+\varphi+i\chi)/\sqrt{2}$, and obtain the same potential (3.4.10). This time, we also have the covariant derivative to consider,

$$D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi = \frac{1}{\sqrt{2}}\left(\partial_\mu\varphi + i\partial_\mu\chi + ievA_\mu + ieA_\mu\varphi - eA_\mu\chi\right). \tag{3.5.2}$$

Therefore the derivative term in the Lagrangian is

$$\begin{aligned}(D_\mu\phi)^* D^\mu\phi &= \frac{1}{2}\left(\partial_\mu\varphi - eA_\mu\chi\right)^2 + \frac{1}{2}\left(\partial_\mu\chi^2 + evA_\mu + eA_\mu\varphi\right)^2 \\ &= \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - e(\partial_\mu\varphi)A^\mu\chi + \frac{1}{2}e^2A_\mu A^\mu\chi^2 \\ &\quad +\frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}e^2v^2A_\mu A^\mu + \frac{1}{2}e^2A_\mu A^\mu\varphi^2 \\ &\quad +ev(\partial_\mu\chi)A^\mu + e^2vA_\mu A^\mu\varphi + e(\partial_\mu\chi)A^\mu\varphi.\end{aligned} \tag{3.5.3}$$

Keeping only the quadratic terms, we therefore obtain

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \frac{1}{2}e^2v^2A_\mu A^\mu \\ &\quad +\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \mu^2\varphi^2 + ev(\partial_\mu\chi)A^\mu + \mathcal{L}_{\text{int}}.\end{aligned} \tag{3.5.4}$$

Because of the mixing term $ev(\partial_\mu\chi)A^\mu$, the equations of motion for A_μ and χ do not decouple and therefore we cannot immediately determine the particle spectrum. To deal with this problem, we note that we have the freedom to carry out a gauge transformation (2.2.6). In particular, we can choose $\theta(x) = -\arg\phi(x) \approx -\chi/v$. This makes the field ϕ real and corresponds to a particular gauge choice called the *unitary gauge*. In this gauge, $\chi = 0$, and therefore the Lagrangian is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \frac{1}{2}e^2v^2A_\mu A^\mu \\ &\quad +\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \mu^2\varphi^2 + \mathcal{L}_{\rm int}. \end{aligned} \tag{3.5.5}$$

This Lagrangian has only two fields, A_μ and φ, and is not gauge invariant, but it is physically equivalent to the original gauge-invariant Lagrangian (3.5.4). It has no mixing term between A_μ and φ, and therefore two fields describe two distinct particles.

The φ terms in the Lagrangian are familiar and describe a massive real scalar field with mass $m_\varphi = \sqrt{2}\mu$. This is known as the Higgs scalar. To analyse the gauge field part of the Lagrangian, find the equation of motion,

$$\frac{\partial\mathcal{L}}{\partial A_\rho} - \partial_\sigma\frac{\partial\mathcal{L}}{\partial\partial_\sigma A_\rho} = e^2v^2A^\rho - \partial_\sigma(-F^{\sigma\rho}) = 0, \tag{3.5.6}$$

which gives

$$\partial_\mu F^{\mu\nu} + e^2v^2A^\nu = 0. \tag{3.5.7}$$

To interpret this equation, it is useful to differentiate it with ∂_ν, which gives

$$\partial_\nu\partial_\mu F^{\mu\nu} + e^2v^2\partial_\nu A^\nu = 0. \tag{3.5.8}$$

Because $\partial_\nu\partial_\mu$ is symmetric but $F^{\mu\nu}$ is antisymmetric, the first term cancels, and therefore the equation implies

$$\partial_\mu A^\mu = 0. \tag{3.5.9}$$

This looks like the Lorenz gauge condition, but remember that we had already fixed the unitary gauge. Here it is therefore an actual physical equation which follows from the equations of motion in the unitary gauge. It has the effect of removing one of the four degrees of freedom that a vector field would otherwise carry, and therefore brings the number of degrees of freedom down to three.

Let us now write the equation of motion (3.5.7) explicitly in terms of A_μ,

$$\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + e^2v^2A^\nu = \partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu + e^2v^2A^\nu = 0. \tag{3.5.10}$$

Because of Eq. (3.5.9), the second term vanishes, and we are left with

$$\partial_\mu\partial^\mu A^\nu + e^2v^2A^\nu = 0, \tag{3.5.11}$$

which means that each component of the four-vector A_μ satisfies the massive Klein-Gordon equation with mass $m = ev$. Therefore A_μ has become a massive vector particle.

In summary, the particle spectrum of the theory consists of a massive vector field A_μ, which consists of three real components and has mass $m_\gamma = ev$, and a single real scalar field with mass $m_\varphi = \sqrt{2}\mu$, so the total number of real degrees of freedom is $3 + 1 = 4$. There are no massless particles, neither a massless vector nor a massless Goldstone scalar. This way of giving a mass to the vector field through spontaneous symmetry breaking is known as the *Higgs mechanism*.

In comparison, when the symmetry is not broken, the particle spectrum consists of a massless vector (photon), which consists of two real degrees of freedom as we found in Section 1.3, and a complex scalar, which also consists of two real degrees of freedom. The total number of degrees of freedom is therefore $2 + 2 = 4$. As we can see, the total number of degrees of freedom is therefore unchanged by the symmetry breaking, but one of the scalar degrees of freedom (more specifically, the Goldstone mode) is "eaten" by the photon, and this allows it to become massive.

3.6 Non-Abelian Higgs Mechanism

Let us now generalise the result of the previous section to the non-Abelian case. Consider now a theory with an n-component real scalar field $\phi = (\phi_1, \ldots, \phi_n)^T$ with some local gauge symmetry group G, and Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi)_i(D^\mu\phi)_i - V(\phi), \tag{3.6.1}$$

where $D_\mu = \partial_\mu + igA_\mu$. We assume that the scalar field has a non-zero vev $\phi = \phi_0 \neq 0$, and that we have used Eq. (3.4.21) to construct the new set of generators $\hat{T}^A$ and the corresponding vectors $\hat{\phi}^A = i\hat{T}^A\phi_0$.

We now write the gauge field as $A_\mu = A_\mu^A\hat{T}^A$, and the scalar field as $\phi(x) = \phi_0 + \varphi(x)$. The covariant derivative is then

$$(D_\mu\phi)_i = \partial_\mu\varphi_i + igA_\mu^A\hat{T}_{ij}^A\phi_{0j} + igA_\mu^A\hat{T}_{ij}^A\varphi_j = \partial_\mu\varphi_i + gA_\mu^A\hat{\phi}_i^A + igA_\mu^A\hat{T}_{ij}^A\varphi_j, \tag{3.6.2}$$

and the quadratic part of the derivative term in the Lagrangian is

$$\frac{1}{2}(D_\mu\phi)_i(D^\mu\phi)_i = \frac{1}{2}(\partial_\mu\varphi_i)\partial^\mu\varphi_i + g(\partial_\mu\varphi_i)A^{\mu A}\hat{\phi}_i^A + \frac{1}{2}g^2A_\mu^A A^{\mu B}\hat{\phi}_i^A\hat{\phi}_i^B + \text{(higher order)}. \tag{3.6.3}$$

Note that the vectors $\hat{\phi}^A$ are orthogonal,

$$\begin{aligned} \hat{\phi}_i^A\hat{\phi}_i^B &= (\hat{\phi}^A)^T\hat{\phi}^B = -\phi_0^T(\hat{T}^A)^T\hat{T}^B\phi_0 = -(c^A)^a\phi_0^T(T^a)^T T^b\phi_0(c^B)^b \\ &= (c^A)^a S^{ab}(c^B)^b = \lambda_B(c^A)^a(c^B)^a = \lambda^B\delta_{AB}, \end{aligned} \tag{3.6.4}$$

where we have used the fact that c^B is an eigenvector of the symmetry breaking matrix S^{ab} with eigenvalue λ^B, i.e., $S^{ab}(c^B)^b = \lambda^B(c^B)^a$ (Note that here one should not sum over the index B!) and that the eigenvectors c^A are orthonormal. Eq, (3.6.4) also confirms that $\lambda^B \geq 0$ as previously stated. Therefore

$$\frac{1}{2}(D_\mu\phi)_i(D^\mu\phi)_i = \frac{1}{2}(\partial_\mu\varphi_i)\partial^\mu\varphi_i + g(\partial_\mu\varphi_i)A^{\mu A}\hat{\phi}_i^A + \frac{1}{2}g^2\lambda_A A_\mu^A A^{\mu A} + \text{(higher order)}. \tag{3.6.5}$$

As in the Abelian case, the mixing term between φ and A_μ means that we cannot determine the particle spectrum directly. Again, we remove it with a gauge choice. An infinitesimal gauge transformation $M = \mathbb{1} + i\theta^A\hat{T}^A$ would transform the field as

$$\phi = \phi_0 + \varphi \to (\mathbb{1} + i\theta^A\hat{T}^A)(\phi_0 + \varphi) = \phi_0 + \varphi + \theta^A\hat{T}^A\phi_0 = \phi_0 + \varphi + i\theta^A\hat{\phi}^A, \tag{3.6.6}$$

where we assumed that $\varphi \ll \phi_0$. Therefore φ_i transforms as

$$\varphi_i \to \varphi_i' = \varphi_i + \theta^A\hat{\phi}_i^A. \tag{3.6.7}$$

Let us now choose

$$\theta^A = -\frac{1}{\lambda^A}\left(\varphi^T\hat{\phi}^A\right), \quad \text{(no sum over } A\text{)}, \tag{3.6.8}$$

Then the transformed field is

$$\varphi \to \varphi' = \varphi - \sum_{A=D'+1}^{D}\frac{1}{\lambda^A}\left(\varphi^T\hat{\phi}^A\right)\hat{\phi}^A. \tag{3.6.9}$$

Taking a scalar product with vector $\hat{\phi}^B$, we find

$$\varphi'_i\hat{\phi}^B_i = \varphi_i\hat{\phi}^B_i - \sum_A\frac{1}{\lambda^A}\left(\varphi_j\hat{\phi}^A_j\right)\hat{\phi}^A_i\hat{\phi}^B_i. \tag{3.6.10}$$

Now we use the orthogonality of the vectors $\hat{\phi}^A$ from Eq. (3.6.4) to obtain

$$\varphi'_i\hat{\phi}^B_i = \varphi_i\hat{\phi}^B_i - \sum_A\frac{1}{\lambda^A}\left(\varphi_j\hat{\phi}^A_j\right)\lambda^B\delta_{AB} = \varphi_i\hat{\phi}^B_i - \frac{\lambda^B}{\lambda^B}\left(\varphi_j\hat{\phi}^B_j\right) = 0. \tag{3.6.11}$$

This means that with the transformation (3.6.8) we have obtained a gauge in which

$$\varphi^T\hat{\phi}^A = 0 \quad \text{for any } A. \tag{3.6.12}$$

This is known the *unitary gauge* in the non-Abelian theory. It removes the mixing term in Eq. (3.6.5), and therefore it allows us to identify the particle states. Furthermore, if $\hat{\phi}^A \neq 0$, then Eq. (3.6.12) implies that φ has no component in that direction. In Section 3.4 we found that non-zero $\hat{\phi}^A$ correspond massless to Goldstone modes, and therefore the condition (3.6.12) eliminates all Goldstone modes from the theory.

Finally, we write the quadratic part of the Lagrangian in the unitary gauge,

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\left(\partial_\mu A^A_\nu - \partial_\nu A^A_\mu\right)\left(\partial^\mu A^{\nu A} - \partial^\nu A^{\mu A}\right) + \frac{1}{2}g^2\lambda^A A^A_\mu A^{\mu A} \\ &\quad +\frac{1}{2}\partial_\mu\varphi_i\partial^\mu\varphi_i - \frac{1}{2}m^2_{ij}\varphi_i\varphi_j,\end{aligned} \tag{3.6.13}$$

where the scalar field mass term is the same as in the global theory. Note that this Lagrangian is only valid in the unitary gauge (3.6.12), which removes $D - D'$ Goldstone modes from the scalar field.

From this Lagaragian we can determine the particle spectrum: In total, the theory has D vector fields A^A_μ, each with mass $M^A = g\sqrt{\lambda_A}$. Hence D' of them are massless and contain two degrees of freedom each, and $D - D'$ are massive and contain three degrees of freedom each. In addition the theory has $n - (D - D')$ massive real scalar fields. The total number of degrees of freedom is therefore $2D' + 3(D - D') + n - (D - D') = 2D + n$.

In comparison, without spontaneous symmetry breaking the theory would have D massless vector fields with two degrees of freedom each, and n real scalar fields. The total number of degrees of freedom is therefore $2D + n$, which is the same as in the broken phase. Again, we can see that spontaneous symmetry breaking does not change the number of degrees of freedom.

As an example, consider again $\mathrm{SO}(3)$ broken by a three-component field $\phi = (0, 0, v)^T$. This was actually proposed as a theory for electroweak unification by Georgi and Glashow in 1972. In this case the matrix S^{ab} is (see Problem Sheet 4)

$$S^{ab} = \begin{pmatrix} v^2 & 0 & 0 \\ 0 & v^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.6.14}$$

and therefore we have one massless gauge boson, which we can identify with the photon, and two massive ones, which would be the two W bosons. However, in 1973 experiments showed evidence for a neutral weak gauge boson (i.e. the Z boson), which was not compatible with this theory.

It is also useful to see explicitly how our results translate to the case of complex scalar fields $\phi_i \in \mathbb{C}$. For simplicity will assume the kinetic term $(D_\mu\phi)^\dagger D^\mu\phi$, which requires unitary transformation matrices and, therefore, Hermitean generators. The relevant term that gives the mass to the gauge field is again the derivative term, and keeping only the relevant term, we have

$$(D_\mu\phi)^\dagger D^\mu\phi \sim g^2\phi_0^\dagger A_\mu A^\mu\phi_0 = g^2 A_\mu^a A^{\mu b}\phi_0^\dagger (T^a)^\dagger T^b\phi_0, = \frac{1}{2}g^2 A_\mu^a A^{\mu b}\phi_0^\dagger \left\{T^a, T^b\right\}\phi_0, \quad (3.6.15)$$

where we have used the fact that $A_\mu^a A^{\mu b}$ is symmetric with respect to the indices a and b. Therefore we define the symmetric matrix

$$S^{ab} = \phi_0^\dagger \left\{T^a, T^b\right\}\phi_0, \quad (3.6.16)$$

which is then used in exactly the same way as in the real case, so that the vector boson masses are again given by the eigenvalues λ^A of S^{ab} as $M^A = g\sqrt{\lambda^A}$.

As an example of a complex field, consider the $\mathrm{SU}(2)$ gauge group broken by a scalar field in the fundamental representation. The vev is

$$\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}. \quad (3.6.17)$$

The generators are $t^a = \sigma^a/2$, where the Pauli matrices satisfy $\{\sigma^a, \sigma^b\} = 2\delta_{ab}\mathbb{1}$. Therefore the symmetry breaking matrix S^{ab} is

$$S^{ab} = \phi_0^\dagger\{t^a, t^b\}\phi_0 = \frac{1}{8}\begin{pmatrix}0 & v\end{pmatrix}\sigma^a\sigma^b\begin{pmatrix}0\\ v\end{pmatrix} = \frac{1}{4}\delta_{ab}\begin{pmatrix}0 & v\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix} = \frac{v^2}{4}\delta_{ab}. \quad (3.6.18)$$

All three eigenvalues are equal $\lambda^A = v^2/4$, and correspondingly the theory contains three massive gauge bosons, all with same mass $M^A = gv/2$. This is encouraging for the weak interactions, because experiments have shown that there are indeed three massive gauge bosons $\mathrm{W}_\pm$ and Z. However, the Z is heavier than the W's, $m_Z \approx 91\mathrm{GeV} > m_W \approx 80$ GeV, so the agreement is not perfect.

3.7 Electroweak Symmetry Breaking

The correct theory of electroweak unification was proposed by Weinberg and Salam in 1967-68, and it is based on the gauge group $\mathrm{SU}(2)\times\mathrm{U}(1)$. Just like in the $\mathrm{SU}(2)$ example, the Higgs field is a complex doublet, and the $\mathrm{U}(1)$ corresponds to overall gauge rotations. The transformation is therefore

$$\phi \to e^{i\theta^a t^a + i\eta}\phi, \quad (3.7.1)$$

where θ^a and η are real numbers. When the symmetry is gauged, the two gauge groups are assumed to have different couplings, g_2 for the $\mathrm{SU}(2)$ group, and (for future convenience) $g_1/2$ for the $\mathrm{U}(1)$ group. Therefore the covariant derivative is

$$D_\mu\phi = \partial_\mu\phi + ig_2 A_\mu\phi + i(g_1/2)B_\mu\phi, \quad (3.7.2)$$

where $A_\mu = A_\mu^a t^a$ and B_μ are the $\mathrm{SU}(2)$ and $\mathrm{U}(1)$ gauge fields. It is convenient to define a fourth generator

$$t^4 = \frac{g_1}{2g_2}\mathbb{1}, \quad (3.7.3)$$

and write $A_\mu^4 \equiv B_\mu$, because then the covariant derivative is simply

$$D_\mu \phi = \partial_\mu \phi + i g_2 A_\mu^{a'} t^{a'} \phi, \tag{3.7.4}$$

where $a' \in \{1, \ldots, 4\}$. This form allows us to use the formalism developed in Section 3.6.

Again, we assume the vev

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{3.7.5}$$

and we obtain the matrix

$$S^{ab} = \frac{1}{2} \begin{pmatrix} 0 & v \end{pmatrix} \{t^a, t^b\} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{v^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -g_1/g_2 \\ 0 & 0 & -g_1/g_2 & g_1^2/g_2^2 \end{pmatrix}. \tag{3.7.6}$$

The eigenvalues and eigenvectors are

$$\begin{aligned}
\lambda^1 &= 0, & c^1 &= \frac{(0,0,g_1,g_2)}{\sqrt{g_1^2+g_2^2}} = (0,0,\sin\theta_W, \cos\theta_W), \\
\lambda^2 &= \frac{v^2}{4}, & c^2 &= (1,0,0,0), \\
\lambda^3 &= \frac{v^2}{4}, & c^3 &= (0,1,0,0), \\
\lambda^4 = \frac{g_1^2+g_2^2}{g_2^2}\frac{v^2}{4} &= \frac{v^2}{4\cos^2\theta_W}, & c^4 &= \frac{(0,0,g_2,-g_1)}{\sqrt{g_1^2+g_2^2}} = (0,0,\cos\theta_W, -\sin\theta_W),
\end{aligned} \tag{3.7.7}$$

where we have introduced the *Weinberg angle* θ_W, also known as the weak mixing angle, defined by

$$\sin\theta_W = \frac{g_1}{\sqrt{g_1^2+g_2^2}}, \quad \cos\theta_W = \frac{g_2}{\sqrt{g_1^2+g_2^2}}. \tag{3.7.8}$$

We can see that there is one zero eigenvalue, which means one unbroken generator and correspondingly a residual $\mathrm{U}(1)$ symmetry. The symmetry breaking pattern is, therefore $\mathrm{SU}(2) \times \mathrm{U}(1) \to \mathrm{U}(1)$. Because of the non-zero mixing angle, the residual $\mathrm{U}(1)$ is, however, not the same as the original $\mathrm{U}(1)$ but rather a linear combination of it and an $\mathrm{U}(1)$ subgroup of the $\mathrm{SU}(2)$ group.

More explicitly, expressing the gauge fields in terms of the new generators $\hat{T}^A$ through $A_\mu^{a'} t^{a'} = \hat{A}_\mu^A \hat{T}^A$, we identify the different particle species,

$$\begin{aligned}
\text{photon}: \quad & \mathcal{A}_\mu = \hat{A}_\mu^1 = \cos\theta_W B_\mu + \sin\theta_W A_\mu^3, & m_\gamma &= 0, \\
\mathrm{W}^\pm: \quad & W_\mu^\pm = \tfrac{1}{\sqrt{2}}(\hat{A}_\mu^2 \mp i\hat{A}_\mu^3) = \tfrac{1}{\sqrt{2}}(A_\mu^1 \mp i A_\mu^2), & m_W &= \frac{g_2 v}{2}, \\
\mathrm{Z}: \quad & Z_\mu = \hat{A}_\mu^4 = \cos\theta_W A_\mu^3 - \sin\theta_W B_\mu, & m_Z &= \sqrt{g_1^2+g_2^2}\frac{v}{2}.
\end{aligned} \tag{3.7.9}$$

We can determine the Weinberg angle θ_W from the measured W and Z boson masses,

$$\cos\theta_W = \frac{m_W}{m_Z} \approx \frac{80.385\ \text{GeV}}{91.1876\ \text{GeV}}, \tag{3.7.10}$$

from which we obtain

$$\sin^2\theta_W = 0.22336 \pm 0.00010. \tag{3.7.11}$$

The value of the gauge coupling g_2 can be determined from the QED coupling e. To do this, we write the covariant derivative in terms of the physical gauge fields. To be more general, we can consider that the covariant derivative acts on a field in an arbitrary representation of $\mathrm{SU}(2)$ and has $\mathrm{U}(1)$ charge Y. Then the covariant derivative is

$$\begin{aligned} D_\mu &= \partial_\mu + ig_2 A^a_\mu T^a + ig_1 Y B_\mu \\ &= \partial_\mu + i\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2}}(T^3+Y)\mathcal{A}_\mu + i\frac{\left(g_2^2 T^3 - g_1^2 Y\right)}{\sqrt{g_1^2+g_2^2}} Z_\mu + \frac{ig_2}{\sqrt{2}}\left(W^+_\mu T^+ + W^-_\mu T^-\right), \end{aligned} \tag{3.7.12}$$

where $T^\pm = T^1 \pm iT^2$. The second term gives the coupling strength to the photon field $\mathcal{A}_\mu$, and therefore

$$e = \frac{g_1 g_2}{\sqrt{g_1^2+g_2^2}} = g_2 \sin\theta_W. \tag{3.7.13}$$

The generator of the residual $\mathrm{U}(1)$ gauge group, which corresponds to electromagnetism, is the unbroken generator

$$Q = T^3 + Y, \tag{3.7.14}$$

and the electric charges of different particles are given by the eigenvalues of the matrix eQ. We will see later that for the electron $Q = -1$, and therefore e corresponds to the unit electric charge. Because $\alpha = e^2/4\pi \approx 1/137$, we have $e \approx 0.30$, and correspondingly $g_2 = e/\sin\theta_W \approx 0.64$. From $m_W \approx 80.385$ GeV, we also obtain the value of the Higgs vev,

$$v = \frac{2m_W}{g_2} \approx 246\ \mathrm{GeV}. \tag{3.7.15}$$

Chapter 4

Fermions

4.1 Dirac Equation

So far we have discussed scalar and vector fields, which describe particles with spin zero and one, respectively. In the Standard Model, the Higgs field is a scalar and the gauge fields are vectors. However, the matter fields, leptons and quarks, have spin $1/2$ and are neither scalars not vectors but rather something we call *spinors*.

Paul Dirac first discovered spinors as a way to find a first-order equation to replace the Klein-Gordon equation (1.2.8). He considered an Ansatz that can we write in modern notation as

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0, \tag{4.1.1}$$

where m would be the mass of the particle, and γ^μ are (possibly non-commuting) coefficients whose properties we need to determine from the condition that we would like the field ψ to satisfy the Klein-Gordon equation (1.2.8). To achieve this, we operate again from the left with the same differential operator $(i\gamma^\mu\partial_\mu - m)$. This gives

$$(i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu - m)\psi = 0. \tag{4.1.2}$$

Expanding the brackets, we obtain

$$(-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - 2mi\gamma^\mu\partial_\mu + m^2)\psi = 0. \tag{4.1.3}$$

We can simplify the second term with the help of Eq. (4.1.1), which shows that $i\gamma^\mu\partial_\mu\psi = m\psi$. This yields

$$(-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi = 0. \tag{4.1.4}$$

Now, we write

$$\gamma^\mu\gamma^\nu = \frac{1}{2}[\gamma^\mu, \gamma^\nu] + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}, \tag{4.1.5}$$

where $\{,\}$ is the anticommutator, i.e., $\{A, B\} = AB + BA$. Because partial derivatives ∂_μ and ∂_ν commute, the commutator drops out and we are left with

$$\left(-\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu - m^2\right)\psi = 0. \tag{4.1.6}$$

In order for this to reduce to the Klein-Gordon equation (1.2.8), the coefficients γ^μ must satisfy

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}\mathbb{1}. \tag{4.1.7}$$

This is known as the *Clifford algebra*.

It is clear that real or complex numbers cannot satisfy the Clifford algebra, but it is easy to find a set of four matrices that do so. These matrices are called *gamma matrices*. The matrices have to be at least 4×4 (or $2^{d/2} \times 2^{d/2}$ in d spacetime dimensions). If γ^μ are 4×4 matrices, then the field ψ in Eq. (4.1.1) has to have four components. However, it is not a four-vector because, as we will see, it does not transform under Lorentz transformations the same way four-vectors do. Instead, it is a *spinor*.

In general, we label the components ψ_α of this four-component spinor ψ and the elements $\gamma^\mu_{\alpha\beta}$ of the 4×4 matrices γ^μ by *spinor indices* $\alpha, \beta \in \{1, 2, 3, 4\}$.

There are many different ways to choose a set of four 4×4 matrices that satisfy the Clifford algebra (4.1.7). In addition, they are usually chosen to satisfy

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i. \tag{4.1.8}$$

The three most common ones are the *chiral representation*

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \; i \in \{1, 2, 3\}, \tag{4.1.9}$$

the *Dirac representation*

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \; i \in \{1, 2, 3\}, \tag{4.1.10}$$

and the *Majorana representation*

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}. \tag{4.1.11}$$

In these expressions, σ^i are the Pauli matrices, and $\mathbb{1}$ is the 2×2 unit matrix.

4.2 Dirac Spinors

To understand how spinors such as ψ transform under Lorentz transformations, we first have to return to Lorentz transformations of four-vectors and discuss that in terms of the general theory discussed in Chapter 2. Remember that Lorentz transformations (or contravariant four-vectors) are defined as matrices $\Lambda^\mu{}_\nu$ that satisfy the condition

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}. \tag{4.2.1}$$

In matrix notation, this can be written as

$$\Lambda \eta \Lambda^T = \eta. \tag{4.2.2}$$

Note that if the metric η was a unit matrix, this would mean that the Lorentz transformation matrices Λ are orthogonal. However, as it happens, $\eta = \mathrm{diag}\,(1, -1, -1, -1)$, which means that we have to generalise our concept of orthogonal matrices. The Lie group of Lorentz transformations is called $O(3, 1)$, indicating a metric with three -1s and one 1.

Consider now an infinitesimal transformation

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu, \tag{4.2.3}$$

where $\omega^\mu{}_\nu$ are real and infinitesimal so that we can ignore any second or higher power of ω. Inserting Eq. (4.2.3) into Eq. (4.2.1), we find

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu} + \omega^{\mu\nu} + \omega^{\nu\mu} = \eta^{\mu\nu}. \tag{4.2.4}$$

This means that we must have $\omega^{\nu\mu} = -\omega^{\mu\nu}$, or in other words, $\omega^{\mu\nu}$ must be antisymmetric. Conversely, any antisymmetric $\omega^{\nu\mu}$ gives rise to a Lorentz transformation that satisfies Eq. (4.2.1). This means that the Lorentz transformations are generated by antisymmetric 4×4 matrices.

A general antisymmetric 4×4 matrix has six free real parameters, and therefore the Lorentz group has six generators. This agrees with the physical intuition, because a general Lorentz transformation is a combination of rotations and boosts, and a rotation requires three parameters (Euler angles) and a boost requires three parameters (boost velocity).

To find the generators, we would need to find six linearly independent antisymmetric 4×4 matrices. We could denote them by $(M^a)^\mu{}_\nu$, where $a \in \{1, \ldots, 6\}$, so that we would write

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \Omega^a (M^a)^\mu{}_\nu, \tag{4.2.5}$$

where Ω^a are six real numbers that parameterise the Lorentz transformation.

However, it is actually more convenient to label the generators with antisymmetric pairs of Lorentz indices ρ, σ,

$$(M^{\rho\sigma})^\mu{}_\nu = -(M^{\sigma\rho})^\mu{}_\nu. \tag{4.2.6}$$

Because there are six such pairs, this is equivalent to $a \in \{1, \ldots, 6\}$. Using this labelling we write

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu, \tag{4.2.7}$$

where the coefficients $\Omega_{\rho\sigma} \in \mathbb{R}$ are antisymmetric with respect to indices ρ, σ, and we have included a factor $1/2$ because each pair of indices appears twice in the sum.

A convenient explicit choice of the matrices $M^{\rho\sigma}$ is

$$(M^{\rho\sigma})^{\mu\nu} = -\eta^{\rho\mu}\eta^{\sigma\nu} + \eta^{\sigma\mu}\eta^{\rho\nu}. \tag{4.2.8}$$

Lowering the second index, we obtain equivalently

$$(M^{\rho\sigma})^\mu{}_\nu = -\eta^{\rho\mu}\delta^\sigma_\nu + \eta^{\sigma\mu}\delta^\rho_\nu. \tag{4.2.9}$$

It is straightforward (see Problem Sheet 5) to show that these generators satisfy the Lie algebra

$$[M^{\rho\sigma}, M^{\tau\chi}] = -\eta^{\sigma\tau}M^{\rho\chi} + \eta^{\rho\tau}M^{\sigma\chi} - \eta^{\rho\chi}M^{\sigma\tau} + \eta^{\sigma\chi}M^{\rho\tau}. \tag{4.2.10}$$

The matrices $(M^{\rho\sigma})^\mu{}_\nu$ generate the vector representation of the Lorentz group. It describes how four-vectors and tensors transform under Lorentz transformations. However, spinors such as ψ correspond to a different representation. To see this, we define another set of matrices labeled by ρ and σ,

$$S^{\rho\sigma} = -\frac{1}{4}[\gamma^\rho, \gamma^\sigma]. \tag{4.2.11}$$

We note that, just like $M^{\rho\sigma}$, these are antisymmetric with respect to $\rho \leftrightarrow \sigma$, and therefore there are again six independent matrices. Using the Clifford algebra (4.1.7), one can verify (see Problem Sheet 5) that these matrices satisfy the same Lie algebra as $M^{\rho\sigma}$,

$$[S^{\rho\sigma}, S^{\tau\chi}] = -\eta^{\sigma\tau}S^{\rho\chi} + \eta^{\rho\tau}S^{\sigma\chi} - \eta^{\rho\chi}S^{\sigma\tau} + \eta^{\sigma\chi}S^{\rho\tau}. \tag{4.2.12}$$

Therefore they generate a different representation of the Lorentz group called the *spinor representation*.

Like the gamma matrices, $S^{\rho\sigma}$ are 4×4 matrices labelled by spinor indices $\alpha, \beta \in \{1,2,3,4\}$, and therefore it is natural to expect that they would act on spinors and describe their Lorentz transformations. In other words, spinors would transform as

$$\psi \to S[\Lambda]\psi, \quad \text{where } S[\Lambda] = \mathbb{1} + \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}, \tag{4.2.13}$$

where we use the notation $S[\Lambda]$ to indicate that this is the spinor Lorentz transformation that correponds to the vector Lorentz transformation Λ. Note that the Lorentz transformation matrix $S[\Lambda]$ is a 4×4 matrix with elements $S[\Lambda]_{\alpha\beta}$ labelled by spinor indices.

To confirm that Eq. (4.2.13) is indeed the correct transformation law for the spinor ψ, we consider a Lorentz transformation of the left-hand-side of the Dirac equation (4.1.1). As shown in Problem Sheet 5,

$$(i\gamma^\mu\partial_\mu - m)\psi \to (i\gamma^\mu\Lambda_\mu{}^\nu\partial_\nu - m)S[\Lambda]\psi = S[\Lambda](i\gamma^\mu\partial_\mu - m)\psi. \tag{4.2.14}$$

Hence, if spinors transforms as Eq. (4.2.13), then the Dirac equation is Lorentz covariant: If it is valid in one frame, it is valid in any other frame. This shows that the transformation law (4.2.13) is correct. As always, a finite Lorentz transformation can be obtained by exponentiating the infinitesimal transformation, and therefore the complete transformation law for spinors is

$$\psi \to S[\Lambda]\psi, \quad \text{where } S[\Lambda] = e^{\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}}. \tag{4.2.15}$$

To understand the spinor representation better, let us consider a rotation around the z axis, which corresponds to $\Omega_{12} = -\Omega_{21} = \theta$, which other coefficients $\Omega_{\rho\sigma} = 0$. For vectors, the transformation matrix is

$$\Lambda^\mu{}_\nu = \left(e^{\theta M^{12}}\right)^\mu{}_\nu = \left(\exp\begin{pmatrix}0&0&0&0\\0&0&-\theta&0\\0&\theta&0&0\\0&0&0&0\end{pmatrix}\right)^\mu{}_\nu = \begin{pmatrix}1&0&0&0\\0&\cos\theta&-\sin\theta&0\\0&\sin\theta&\cos\theta&0\\0&0&0&1\end{pmatrix}^\mu{}_\nu, \tag{4.2.16}$$

which confirms that this is indeed a rotation by angle θ. In particular, a rotation $\theta = 2\pi$ leads to $\Lambda = \mathbb{1}$, as it should, because a 2π rotation leaves a vector unchanged.

Similarly, the transformation matrix for spinors is

$$S[\Lambda] = e^{\theta S^{12}} = e^{-\frac{1}{4}\theta[\gamma^1,\gamma^2]}. \tag{4.2.17}$$

Choosing the Dirac representation,

$$[\gamma^1, \gamma^2] = -2i\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}, \tag{4.2.18}$$

so

$$S[\Lambda] = \begin{pmatrix}e^{i\theta/2}&0&0&0\\0&e^{-i\theta/2}&0&0\\0&0&e^{i\theta/2}&0\\0&0&0&e^{-i\theta/2}\end{pmatrix}. \tag{4.2.19}$$

Interestingly, choosing $\theta = 2\pi$ gives $S[\Lambda] = -\mathbb{1}$. This means that a 2π rotation does *not* leave a spinor unchanged but instead flips its sign $\psi \to -\psi$. You need to rotate a spinor by 4π to return it to its original value. In quantum mechanics this can be understood as the corresponding particles having spin $1/2$.

Next, we would like to write a Lagrangian for the spinor field ψ that would give the Dirac equation (4.1.1) as its Euler-Lagrange equation. Because the Lagrangian $\mathcal{L}$ must be a Lorentz scalar, we need to find scalar combinations of spinors ψ. It would be natural to guess that $\psi^\dagger\psi$ would be a Lorentz scalar, but this turns out to be incorrect. It transforms as

$$\psi^\dagger\psi \to \psi^\dagger S[\Lambda]^\dagger S[\Lambda]\psi, \tag{4.2.20}$$

and would therefore be invariant if $S[\Lambda]$ was unitary. By considering an infinitesimal transformation (4.2.13), we see that this is equivalent to the generators $S^{\rho\sigma}$ being anti-Hermitean,

$$(S^{\rho\sigma})^\dagger = -S^{\rho\sigma}, \tag{4.2.21}$$

which requires

$$[(\gamma^\rho)^\dagger, (\gamma^\sigma)^\dagger] = [\gamma^\rho, \gamma^\sigma]. \tag{4.2.22}$$

It is, however, not possible to choose the gamma matrices in such a way that this would be satisfied for all ρ and σ. For example, if the gamma matrices are chosen according to Eq. (4.1.8), one finds that $(S^{00})^\dagger = -S^{00}$ and $(S^{ij})^\dagger = -S^{ij}$ but $(S^{0i})^\dagger = S^{0i}$.

On the other hand, the transformation $S[\Lambda]$ satisfies

$$S[\Lambda]^\dagger\gamma^0 S[\Lambda] = \gamma^0. \tag{4.2.23}$$

This means that if one defines the *Dirac adjoint* $\bar{\psi}$ as

$$\bar{\psi} \equiv \psi^\dagger\gamma^0, \tag{4.2.24}$$

then $\bar{\psi}\psi$ is a Lorentz scalar,

$$\bar{\psi}\psi = \psi^\dagger\gamma^0\psi \to \psi^\dagger S[\Lambda]^\dagger\gamma^0 S[\Lambda]\psi = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi. \tag{4.2.25}$$

Similarly, one can easily show that for any pair of spinors ψ and χ, $\bar{\psi}\gamma^\mu\chi$ is a Lorentz vector. For a Lorentz vector, say a_μ, contracted with the gamma matrices, it is conventional to use the notation

$$\not{a} = \gamma^\mu a_\mu, \tag{4.2.26}$$

so that $\bar{\psi}\not{a}\chi$ is a scalar. This applies also when the vector is a derivative, so that $\bar{\psi}\not{\partial}\psi = \bar{\psi}\gamma^\mu\partial_\mu\psi$ is a Lorentz scalar.

4.3 Dirac Lagrangian

Using the Lorentz transformation properties derived in the previous section, we can write down a Lorentz invariant Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi = \bar{\psi}\left(i\not{\partial} - m\right)\psi, \tag{4.3.1}$$

which is known as the *Dirac Lagrangian*. For this to make sense as a Lagrangian, the corresponding action $S = \int d^4x\mathcal{L}$ must be real. We can easily check that $\bar\psi\psi$ is real,

$$(\bar\psi\psi)^* = \left(\psi^\dagger\gamma^0\psi\right)^* = \psi^\dagger(\gamma^0)^\dagger\psi = \psi^\dagger\gamma^0\psi = \bar\psi\psi, \tag{4.3.2}$$

but the derivative term $\bar\psi i\partial\!\!\!/\psi$ is a little bit more subtle,

$$(\bar\psi i\partial\!\!\!/\psi)^* = \left(i\psi^\dagger\gamma^0\gamma^\mu\partial_\mu\psi\right)^\dagger = -i(\partial_\mu\psi^\dagger)(\gamma^\mu)^\dagger(\gamma^0)^\dagger\psi. \tag{4.3.3}$$

Using Eq. (4.1.8), we find that for $\mu = 0$,

$$(\gamma^\mu)^\dagger(\gamma^0)^\dagger = (\gamma^0)^\dagger(\gamma^0)^\dagger = \gamma^0\gamma^0, \tag{4.3.4}$$

and for $\mu = i$,

$$(\gamma^\mu)^\dagger(\gamma^0)^\dagger = (\gamma^i)^\dagger(\gamma^0)^\dagger = -\gamma^i\gamma^0 = \gamma^0\gamma^i, \tag{4.3.5}$$

where the last equality follows from the Clifford algebra (4.1.7). Together, these mean that for any μ,

$$(\gamma^\mu)^\dagger(\gamma^0)^\dagger = \gamma^0\gamma^\mu. \tag{4.3.6}$$

Using this, Eq. (4.3.3) simplifies to

$$\left(\bar\psi i\partial\!\!\!/\psi\right)^* = -i(\partial_\mu\psi^\dagger)\gamma^0\gamma^\mu\psi = -i(\partial_\mu\bar\psi)\gamma^\mu\psi. \tag{4.3.7}$$

This does not imply that the derivative term is real, but its imaginary part is

$$\mathrm{Im}\,\left(\bar\psi i\partial\!\!\!/\psi\right) = \frac{1}{2}\left[\bar\psi i\partial\!\!\!/\psi - \left(\bar\psi i\partial\!\!\!/\psi\right)^*\right] = \frac{i}{2}\left[\bar\psi\gamma^\mu\partial_\mu\psi + (\partial_\mu\bar\psi)\gamma^\mu\psi\right] = \frac{i}{2}\partial_\mu\left(\bar\psi\gamma^\mu\psi\right). \tag{4.3.8}$$

This is a total derivative, and therefore it vanishes when integrated over spacetime if ψ is chosen to behave suitably at infinity. Hence, even if the Lagrangian is not real in general, the action S is, and therefore the action principle and the Euler-Lagrange equation are valid.

For the purpose of the Euler-Lagrange equation, we can use the components ψ_α and $\bar\psi_\alpha$ as free variables. Because the Lagrangian has no derivatives of $\bar\psi_\alpha$, its variation gives simply

$$\frac{\partial\mathcal{L}}{\partial\bar\psi_\alpha} = \left(i\gamma^\mu_{\alpha\beta}\partial_\mu - m\delta_{\alpha\beta}\right)\psi_\beta = 0, \tag{4.3.9}$$

or more compactly

$$(i\partial\!\!\!/ - m)\psi = 0, \tag{4.3.10}$$

which is just the Dirac equation (4.1.1) and therefore demonstrates that this is indeed the correct Lagrangian.

It is worth noting the number of degrees of freedom in the spinor ψ. Because it has four complex components ψ_α, $\alpha \in \{1,2,3,4\}$, one might naively expect it to correspond to eight real degrees of freedom. Note, however, that the Klein-Gordon equation (1.2.8) is second order in time. Hence, to describe the state of a single real scalar degree of freedom one needs to specify two real numbers at each point: it value and its time derivative. In contrast, the Dirac equation (4.1.1) is first order in time, and therefore the value of the spinor determines its time derivative. Hence, the state of the spinor field is specified by four complex numbers, or equivalently eight real numbers, at each point. That is the same amount information as in four real scalar fields. Therefore the Dirac spinor contains four degrees of freedom. Physically these four degrees of freedom correspond to a particle with spin up, a particle with spin down, an antiparticle with spin up, and an antiparticle with spin down.

We can also note that the Lagrangian (4.3.1) has a global $\mathrm{U}(1)$ symmetry $\psi \to e^{i\theta}\psi$, which is known as the *vector symmetry*. Through Noether's theorem this symmetry implies a conserved current, which we can find in the same way as in Section 3.1: Under an infinitesimal transformation, the spinor ψ changes by

$$\delta\psi_\alpha = i\theta\psi_\alpha, \tag{4.3.11}$$

and hence the Lagrangian changes by

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\partial_\mu\psi_\alpha}\delta\psi_\alpha + \frac{\partial \mathcal{L}}{\partial\partial_\mu\bar{\psi}_\alpha}\delta\bar{\psi}_\alpha \right) = i\theta\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\partial_\mu\psi_\alpha}\psi_\alpha \right). \tag{4.3.12}$$

Because the transformation (4.3.11) is a symmetry, this must vanish, and therefore we have found a conserved current

$$j^\mu = -i\frac{\partial \mathcal{L}}{\partial\partial_\mu\psi_\alpha}\psi_\alpha = \bar{\psi}\gamma^\mu\psi, \tag{4.3.13}$$

in other words, $\partial_\mu j^\mu = 0$. Physically, this current corresponds to the particle number.

As in Section 2.2, the symmetry can be gauged and turned into a local symmetry, by replacing the derivatives by covariant derivatives

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu, \tag{4.3.14}$$

where A_μ is the gauge field. This leads to the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi, \tag{4.3.15}$$

where we have added a Maxwell term in the same way as in Eq. (2.2.9). This is the Lagrangian of *quantum electrodynamics*,[1] which describes electrons ψ interacting with the electromagnetic field A_μ. Note that represents a very important step of unification: In classical electrodynamics, particles are treated in completely different way from the electromagnetic field, and the same is true even when the theory is quantised in terms of a Schrodinger equation for a charged particle. In contrast, in Eq. (4.3.15) electrons and photons appear on the same footing as two different types of fields, spinors and vectors.

The Dirac Lagrangian (4.3.1) has also three other important symmetries: parity, charge conjugation and time reversal.

By *parity* we mean inverting the sign of the spatial coordinates, $(t, \mathbf{x}) \to (t, -\mathbf{x})$ or, in terms of the four-vector

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \to \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}. \tag{4.3.16}$$

If, at the same time, the spinor ψ is transformed as

$$\psi \to \gamma^0\psi, \tag{4.3.17}$$

[1] Of course, our treatment is not quantum but classical, but the Lagrangian is the same.

the Lagrangian (4.3.1) remains unchanged,

$$\begin{aligned}
\mathcal{L} &= \bar{\psi}(i\partial\!\!\!/ - m)\psi = \bar{\psi}\left(i\gamma^0\partial_0 + i\gamma^i\partial_i - m\right)\psi \\
&\to \bar{\psi}\gamma^0\left(i\gamma^0\partial_0 - i\gamma^i\partial_i - m\right)\gamma^0\psi = \bar{\psi}\left(i\gamma^0\partial_0 + i\gamma^i\partial_i - m\right)\gamma^0\gamma^0\psi \\
&= \bar{\psi}\left(i\gamma^0\partial_0 + i\gamma^i\partial_i - m\right)\psi = \bar{\psi}(i\partial\!\!\!/ - m)\psi,
\end{aligned} \tag{4.3.18}$$

where we used the Clifford algebra (4.1.7) in the second line. This shows that parity is a symmetry of the theory, and that spinors transform under parity according to Eq. (4.3.17).

Charge conjugation is discussed in Problem Sheet 5, and corresponds to the transformation

$$\psi \to C\psi^*, \tag{4.3.19}$$

where C is a 4×4 matrix that satisfies

$$C^\dagger C = \mathbb{1}, \quad C^\dagger\gamma^\mu C = -(\gamma^\mu)^*. \tag{4.3.20}$$

One can easily check this is a symmetry of the Dirac Lagrangian (4.3.1), and that it is also a symmetry of quantum electrodynamics (4.3.15), if at the same time either $e \to -e$ or $A_\mu \to -A_\mu$. This shows that the physical interpretation is indeed charge conjugation, i.e., swapping positive and negative charges.

Time reversal is a slightly more subtle symmetry. It corresponds to inverting the sign of the time coordinate $(t, \mathbf{x}) \to (-t, \mathbf{x})$. One can show generally that any Lorentz invariant quantum field theory is symmetric under the combined parity, charge conjugation and time reversal operations. Therefore time reversal is equivalent to a CP transformation which consists of a parity transformation and charge conjugation.

4.4 Weyl Spinors

As we saw in the previous section, the QED Lagrangian (4.3.15) is invariant under parity. However, experiments show that in reality, weak interactions violate it. This was first seen in 1957 in beta decay of $^{60}\mathrm{Co}$ by Chien-Shiung Wu and has later been confirmed in many experiments. Therefore Eq. (4.3.15) cannot be correct, and instead, one needs to find a theory that is not fully invariant under parity.

In fact, the treatment of spinors can be easily generalised to allow for parity violation. We first introduce a fifth gamma matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{4.4.1}$$

This matrix anticommutes with all other gamma matrices,

$$\{\gamma^5, \gamma^\mu\} = 0 \quad \text{for all } \mu, \tag{4.4.2}$$

and satisfies $(\gamma^5)^2 = \mathbb{1}$ and $(\gamma^5)^\dagger = \gamma^5$.

Using γ^5 we define two further matrices

$$P_L = \frac{1}{2}(\mathbb{1} - \gamma^5), \quad P_R = \frac{1}{2}(\mathbb{1} + \gamma^5). \tag{4.4.3}$$

They satisfy the relations

$$P_L + P_R = \mathbb{1}, \quad P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0, \tag{4.4.4}$$

which mean that the matrices as projection operators. We call P_L the left-handed projection and P_R the right-handed projection. These projection operators are also Hermitean,

$$P_L^\dagger = P_L, \quad P_R^\dagger = P_R, \tag{4.4.5}$$

and they satify the relation

$$P_{L/R}\gamma^\mu = \frac{1}{2}(\mathbb{1} \mp \gamma^5)\gamma^\mu = \gamma^\mu \frac{1}{2}(\mathbb{1} \pm \gamma^5) = \gamma^\mu P_{R/L}. \tag{4.4.6}$$

Using the projection operators, we can decompose the Dirac spinor ψ into two pieces ψ_L and ψ_R as

$$\psi = \psi_L + \psi_R, \quad \text{where } \psi_L = P_L\psi \text{ and } \psi_R = P_R\psi. \tag{4.4.7}$$

Here ψ_L satisfies $P_L\psi_L = \psi_L$ and $P_R\psi_L = 0$, and ψ_R satisfies $P_L\psi_R = 0$ and $P_R\psi_R = \psi_R$. Spinors that satisfy these conditions are called left-handed and right-handed *Weyl spinors*, respectively.

To understand Weyl spinors more concretely, it is convenient to use the chiral representation (4.1.9), where we have

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{4.4.8}$$

and

$$P_L = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad P_R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{4.4.9}$$

If

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \text{then} \quad \psi_L = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_R = \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{4.4.10}$$

Hence we see that in this basis, the left-handed Weyl spinor simply contains only the top two components, and the right-handed Weyl spinor only the bottom two components. Since the full Dirac spinor ψ contains four degrees of freedom, this means that ψ_L and ψ_R contain two degrees of freedom each.

Of course, the split into ψ_L and ψ_R is only meaningful if it is respected by the physics. In particular, it must be preserved by Lorentz transformations, so that split is not dependent on the choice of frame. To show this we first note that for any ρ and σ,

$$\gamma^\rho\gamma^\sigma\gamma^5 = -\gamma^\rho\gamma^5\gamma^\sigma = \gamma^5\gamma^\rho\gamma^\sigma, \tag{4.4.11}$$

and therefore

$$[S^{\rho\sigma}, \gamma^5] = -\frac{1}{4}\left[[\gamma^\rho, \gamma^\sigma], \gamma^5\right] = -\frac{1}{4}\left[\gamma^\rho\gamma^\sigma, \gamma^5\right] + \frac{1}{4}\left[\gamma^\sigma\gamma^\rho, \gamma^5\right] = 0. \tag{4.4.12}$$

Hence the left/right projections commute with Lorentz transformations,

$$\left[S[\Lambda], P_{L/R}\right] = \mp\frac{1}{2}\left[S[\Lambda], \gamma^5\right] = 0. \tag{4.4.13}$$

Under Lorentz transformation, the Weyl spinors therefore transform as

$$\psi_{L/R} = P_{L/R}\psi \to P_{L/R}S[\Lambda]\psi = S[\Lambda]P_{L/R}\psi = S[\Lambda]\psi_{L/R}, \tag{4.4.14}$$

which means that a left-handed Weyl spinor remains left-handed under Lorentz transformation and a right-handed Weyl spinor remain right-handed.

On the other, under parity (4.3.17), the behaviour is different,

$$\psi_{L/R} = P_{L/R}\psi \to P_{L/R}\gamma^0\psi = \gamma^0 P_{R/L}\psi = \gamma^0\psi_{R/L}. \tag{4.4.15}$$

Therefore a parity transformation turns a left-handed Weyl spinor into right-handed and a right-handed one into left-handed. This explains the terms left- and right-handed.

The fact that Weyl spinors are not invariant under parity means that if we want to have a theory that violates parity, we can achieve it without breaking Lorentz invariance, by writing a Lagrangian in which the left- and right-handed spinors appear differently.

Let us, however, first write the Dirac Lagrangian (4.3.1) in terms of the left- and right-handed spinors. We first note that the Dirac adjoint of a Weyl spinor is

$$\bar\psi_{L/R} \equiv \psi^\dagger_{L/R}\gamma^0 = \psi^\dagger P_{L/R}\gamma^0 = \psi^\dagger\gamma^0 P_{R/L} = \bar\psi P_{R/L}. \tag{4.4.16}$$

If we decompose a Dirac spinor ψ as

$$\psi = P_L\psi_L + P_R\psi_R, \tag{4.4.17}$$

then

$$\bar\psi = \bar\psi_L P_R + \bar\psi_R P_L. \tag{4.4.18}$$

For two Dirac spinors ψ and χ, we can therefore write

$$\bar\psi\chi = (\bar\psi_L P_R + \bar\psi_R P_L)(P_L\chi_L + P_R\chi_R) = \bar\psi_L\chi_R + \bar\psi_R\chi_L. \tag{4.4.19}$$

On the other hand,

$$\begin{aligned} \bar\psi\gamma^\mu\chi &= (\bar\psi_L P_R + \bar\psi_R P_L)\gamma^\mu(P_L\chi_L + P_R\chi_R) = (\bar\psi_L P_R + \bar\psi_R P_L)(P_R\gamma^\mu\chi_L + P_L\gamma^\mu\chi_R) \\ &= \bar\psi_L\gamma^\mu\chi_L + \bar\psi_R\gamma^\mu\chi_R, \end{aligned} \tag{4.4.20}$$

where we used Eq. (4.4.6). In particular, this means that the Dirac Lagrangian (4.3.1), written in terms of ψ_L and ψ_R, is

$$\mathcal{L} = \bar\psi_L i\partial\!\!\!/\psi_L + \bar\psi_R i\partial\!\!\!/\psi_R - m\left(\bar\psi_L\psi_R + \bar\psi_R\psi_L\right). \tag{4.4.21}$$

From Eq. (4.4.21) we see that if $m = 0$, the is no term that involves both ψ_L and ψ_R. Therefore the left- and right-handed spinors decouple and evolve independently of each other. The physical reason for this is that the right- and left-handed spinors corresponds to the spin of the particle being parallel or antiparallel to the momentum, respectively. This property is known as the *helicity* of the particle. For a massive particle, it is possible to carry out a Lorentz boost to take the momentum to zero, in which case the helicity of the particle is not well defined. However, if the mass is zero, such a boost is not possible, and therefore the helicity is a well-defined intrinsic property of the particle.

Furthermore, if $m = 0$, the Lagrangian has an additional global U(1) symmetry known as *axial symmetry*. It corresponds to rotating the phases of the left-handed and right-handed components in the opposite directions,

$$\begin{aligned} \psi_L &\to e^{i\theta}\psi_L, \\ \psi_R &\to e^{-i\theta}\psi_R. \end{aligned} \tag{4.4.22}$$

In terms of the full Dirac spinor $\psi = \psi_L + \psi_R$, this transformation can be written as $\psi \to \exp(-i\theta\gamma^5)\psi$. If the mass is non-zero, we see that the mass term breaks the axial symmetry explicitly. If the mass is small compared with the other relevant energy scales, the axial symmetry is approximate.

The fact that the left-handed and right-handed components decouple if $m = 0$ also means that we can consistently write down a Lagrangian that has only, say, a left-handed spinor ψ_L without a corresponding right-handed spinor ψ_R,

$$\mathcal{L} = \bar{\psi}_L i\partial\!\!\!/ \psi_L,$$

and which, therefore violates parity. However then the corresponding particle has to be massless. This is very important for the Standard Model of particle physics.

4.5 Leptons

As discussed earlier, the parity symmetry (4.3.17) is violated in nature. In the Standard Model of particle physics this arises because the left-handed and right-handed fields transform differently under gauge transformations. Therefore they should be seen as two completely separate set of fields, rather than two parts of a four-component Dirac spinor.

The Standard Model has three families of leptons, each consisting of a charged lepton and the corresponding neutrino. We start by considering the first family, which contains the electron e^- and the electron neutrino ν_e. Their interactions are determined by the way they transform under the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge symmetry, through the covariant derivative

$$D_\mu = \partial_\mu + ig_3 G^c_\mu T^c_{\mathrm{SU}(3)} + ig_2 A^a_\mu T^a_{\mathrm{SU}(2)} + ig_1 Y B_\mu \mathbb{1}, \quad c \in \{1, \ldots, 8\},\ a \in \{1, 2, 3\}, \tag{4.5.1}$$

where G^c_μ, A^a_μ and B_μ are the $\mathrm{SU}(3)$, $\mathrm{SU}(2)$ and $\mathrm{U}(1)$ gauge fields, $T^c_{\mathrm{SU}(3)}$ are the generators of the $\mathrm{SU}(3)$ representation of the field, $T^a_{\mathrm{SU}(2)}$ are the generators of the $\mathrm{SU}(2)$ representation of the field, and Y is its $\mathrm{U}(1)$ hypercharge. Different particle species can be in different representations of the three groups, and would have different sets of generators $T^c_{\mathrm{SU}(3)}$, $T^a_{\mathrm{SU}(2)}$ and Y. We will now determine for each species what they should be to match the observed physics.

Comparing with the Higgs covariant derivative (3.7.2), we can see that for the Higgs field $T^c_{\mathrm{SU}(3)} = 0$, $T^a_{\mathrm{SU}(2)} = t^a$, and $Y = 1/2$.

The left-handed lepton field, which we denote by ℓ_L, should be neutral under $\mathrm{SU}(3)$, because leptons do not interact with the strong force, and hence $T^c_{\mathrm{SU}(3)} = 0$. On the other hand, weak interactions can change electrons into neutrinos, so we assume that they are the two components of the $\mathrm{SU}(2)$ fundamental representation. Therefore we choose $T^a_{\mathrm{SU}(2)} = t^a$ and write

$$\ell_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}. \tag{4.5.2}$$

The field will also have a $\mathrm{U}(1)$ charge Y, which we wish to determine. The covariant derivative (4.5.1) therefore reduces to

$$D_\mu = \partial_\mu + ig_2 A^a_\mu t^a + ig_1 Y B_\mu. \tag{4.5.3}$$

In accordance with Eq. (4.4.21), the kinetic term of ℓ_L is

$$\mathcal{L} = \ldots \bar{\ell}_L i D\!\!\!\!/\, \ell_L. \tag{4.5.4}$$

According to Eq. (3.7.14), the electric charge is given by

$$eQ\ell_L = e\left(t^3 + Y\mathbb{1}\right)\ell_L = e\begin{pmatrix} Y+1/2 & 0 \\ 0 & Y-1/2 \end{pmatrix}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \begin{pmatrix} (Y+1/2)e\times\nu_L \\ (Y-1/2)e\times e_L \end{pmatrix}. \tag{4.5.5}$$

To identify ν_L with the neutrino, it needs to be electrically neutral, which implies $Y = -1/2$. It then follows that e_L has electric charge $(Y-1/2)e = -e$, in line with the assumption that it corresponds to the electron. This justifies the choice of notation in Eq. (4.5.2).

In contrast, the right-handed lepton field ℓ_R is chosen to be neutral under both $\mathrm{SU}(3)$ and $\mathrm{SU}(2)$, so $T^c_{\mathrm{SU}(3)} = T^a_{\mathrm{SU}(2)} = 0$. It is therefore a one-component field, whose electric charge is given by

$$eQ\ell_R = e(0+Y)\ell_R = Ye\times\ell_R. \tag{4.5.6}$$

To identify this with the electron, $\ell_R = e_R$, it needs to have electric charge $-e$, so we set $Y = -1$. Its kinetic term is therefore

$$\mathcal{L} = \ldots\bar{\ell}_R i\not{D}\ell_R, \tag{4.5.7}$$

where

$$D_\mu = \partial_\mu - ig_1 B_\mu. \tag{4.5.8}$$

Note that because the left-handed and right-handed leptons transform differently under $\mathrm{SU}(2)$ and $\mathrm{U}(1)$, the mass term (4.4.21)

$$\mathcal{L} = \ldots - m(\bar{e}_L e_R + \bar{e}_R e_L) \tag{4.5.9}$$

is not gauge invariant and is therefore not allowed. However, the electron is known to have a mass $m_e \approx 0.511\mathrm{MeV}$. How does this mass arise if there is no mass term?

Of course we have all heard the answer: The mass is generated by the Higgs field. Specifically, we can write a so-called *Yukawa term*,

$$\mathcal{L} = \ldots - y_e(\bar{\ell}_L\phi\ell_R + \bar{\ell}_R\phi^\dagger\ell_L), \tag{4.5.10}$$

which is invariant under both $\mathrm{SU}(2)$ and $\mathrm{U}(1)$ gauge transformations and is therefore allowed. In the broken phase, with the Higgs vev

$$\phi = \phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{4.5.11}$$

this becomes

$$\mathcal{L} = \ldots - \frac{y_e}{\sqrt{2}}\left(\begin{pmatrix} \bar{\nu}_L & \bar{e}_L \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} e_R + \bar{e}_R\begin{pmatrix} 0 & v \end{pmatrix}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}\right) = \ldots - \frac{1}{\sqrt{2}}y_e v\left(\bar{e}_L e_R + \bar{e}_R e_L\right). \tag{4.5.12}$$

This has the exact form the mass term in Eq. (4.4.21), with mass $m_e = y_e v/\sqrt{2}$. We see that under spontaneous symmetry breaking, the two Weyl spinors e_L and e_R which would otherwise be two separate massless fields, have combined to form something that looks like a massive Dirac spinor and which we can interpret as the physical electron field.

We also note that there is no mass term (and in fact no right-handed component) for the neutrino, so the theory predicts that it should be massless. In reality, this is not quite true, because experiments show that the neutrino has a small mass. It is possible to modify the theory to allow for such a mass, and that means that the neutrino masses are very interesting as they are a hint about physics beyond the minimal Standard Model.

Furthermore because the Standard Model has three families of leptons, we actually have three sets of fields $\ell^f_{L/R}$, labelled by $f \in \{1, 2, 3\}$,

$$\begin{aligned}
\ell^1_L &= \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \quad \ell^1_R = e_R, \\
\ell^2_L &= \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \quad \ell^2_R = \mu_R, \\
\ell^3_L &= \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}, \quad \ell^3_R = \tau_R.
\end{aligned} \tag{4.5.13}$$

Each family has the same structure but a different Yukawa coupling, which leads to different e, μ and τ masses.

The most general Yukawa term for the free families involves a complex 3×3 matrix Y_ℓ of Yukawa couplings, y^{fg}_ℓ, $f, g \in \{1, 2, 3\}$,

$$\mathcal{L} = \ldots - \sum_{f,g} \left[y^{fg}_\ell \bar{\ell}^f_L \phi \ell^g_R + (y^{fg}_\ell)^* \bar{\ell}^g_R \phi^\dagger \ell^f_L \right]. \tag{4.5.14}$$

The Yukawa matrix y^{fg}_ℓ would therefore appear to consist of 18 free real parameters. However, most of these parameters can be removed by redefinition of the fields. There is a general result in linear algebra that for any complex matrix Y, one can find a *singular value decomposition* in terms of two unitary matrices V_L and V_R, such that $V^\dagger_L Y V_R$ is real and diagonal. In particular, for the Yukawa matrix Y_ℓ, this means that there are unitary matrices $V_{\ell L}$ and $V_{\ell R}$ such that $V^\dagger_{\ell L} Y_\ell V_{\ell R}$ is real and diagonal,

$$V^\dagger_{\ell L} Y_\ell V_{\ell R} = V^\dagger_{\ell L} \begin{pmatrix} y^{11}_\ell & y^{12}_\ell & y^{13}_\ell \\ y^{21}_\ell & y^{22}_\ell & y^{23}_\ell \\ y^{31}_\ell & y^{32}_\ell & y^{33}_\ell \end{pmatrix} V_{\ell R} = \begin{pmatrix} y^1_\ell & 0 & 0 \\ 0 & y^2_\ell & 0 \\ 0 & 0 & y^3_\ell \end{pmatrix}. \tag{4.5.15}$$

Using these matrices, let us redefine the lepton fields,

$$\ell^f_L \to V^{fg}_{\ell L} \ell^g_L, \quad \ell^f_R \to V^{fg}_{\ell R} \ell^g_R. \tag{4.5.16}$$

Because $V_{\ell L}$ and $V_{\ell R}$ are unitary, the lepton kinetic terms do not change, but the Yukawa term (4.5.14) becomes diagonal,

$$\mathcal{L} = \ldots - \sum_f y^f_\ell \left(\bar{\ell}^f_L \phi \ell^f_R + \bar{\ell}^f_R \phi^\dagger \ell^f_L \right). \tag{4.5.17}$$

Therefore the lepton Yukawa term contains only three free real parameters y^f_ℓ, which determine the masses of the three charged leptons through $m^f_\ell = y^f_\ell v/\sqrt{2}$.

4.6 Quarks

The Standard Model also has six flavours of quarks, which are charged under the $\mathrm{SU}(3)$ of strong interactions, and which again fall into three families: up and down, charm and strange, top and bottom.

Let us first consider the left-handed quarks q^f_L, $f \in \{1, 2, 3\}$. Because they feel the strong force, we assume that they are in the fundamental representation of $\mathrm{SU}(3)$, which means that they have three "colours". In principle we should write a colour index to keep track of this, but we will omit it for simplicity because it does not affect any of our results. Like with left-handed leptons, because weak interactions can change the quark flavour within a family, we assume that they are in the fundamental representation of $\mathrm{SU}(2)$, and hence write them as two-component "doublets" consisting of an "up-like" quark u_L^f and a "down-like" quark d_L^f,

$$q_L^f = \begin{pmatrix} u_L^f \\ d_L^f \end{pmatrix}. \tag{4.6.1}$$

Because proton (two up quarks and one down quark) has charge $+e$ and neutron (one up quark and two down quarks) is neutral, the electric charges need to be

$$eQq_L^f = e\begin{pmatrix} Y+1/2 & 0 \\ 0 & Y+1/2 \end{pmatrix}\begin{pmatrix} u_L^f \\ d_L^f \end{pmatrix} = \begin{pmatrix} (Y+1/2)e \times u_L^f \\ (Y-1/2)e \times d_L^f \end{pmatrix} = \begin{pmatrix} 2e/3 \times u_L^f \\ -e/3 \times d_L^f \end{pmatrix}. \tag{4.6.2}$$

Hence the $\mathrm{U}(1)$ hypercharge has to be $Y = 1/6$.

Right-handed quarks are also in the fundamental representation of $\mathrm{SU}(3)$, but just like right-handed leptons, they are neutral under $\mathrm{SU}(2)$. However, it contrast to leptons, we want to both up and down-type quarks to be massive, so we need two right-handed quark fields u_R^f and d_R^f, with $\mathrm{U}(1)$ charges

$$Y(u_R^f) = 2/3, \quad Y(d_R^f) = -1/3. \tag{4.6.3}$$

Quark-masses are again generated by the symmetry breaking through the Yukawa couplings. For simplicity, we first focus on the first family $f = 1$. For the down quark, the Yukawa term is identical to the leptons,

$$\mathcal{L} = \ldots - y_d\left(\bar{q}_L\phi d_R + \bar{d}_R\phi^\dagger q_L\right). \tag{4.6.4}$$

This is invariant under $\mathrm{SU}(2)$ transformations $\phi \to M\phi$, $q_L \to Mq_L$, where $M \in \mathrm{SU}(2)$, because

$$\bar{q}_L\phi d_R \to \bar{q}_L M^\dagger M\phi d_R = \bar{q}_L\phi d_R. \tag{4.6.5}$$

It is also invariant under $\mathrm{U}(1)$ transformation $\phi \to e^{i\theta/2}\phi$, $q_L \to e^{i\theta/6}q_L$, $d_R \to e^{-i\theta/3}d_R$, because

$$\bar{q}_L\phi d_R \to e^{-i\theta/6+i\theta/2-i\theta/3}\bar{q}_L\phi d_R = \bar{q}_L\phi d_R. \tag{4.6.6}$$

Substituting

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{4.6.7}$$

this gives the mass term

$$\mathcal{L} = \ldots - \frac{y_d v}{\sqrt{2}}\left(\bar{d}_L d_R + \bar{d}_R d_L\right), \tag{4.6.8}$$

and hence the mass of the down quark is $m_d = y_d v/\sqrt{2}$.

To write a Yukawa term for up-type quarks, we define

$$\tilde{\phi} \equiv i\sigma_2\phi^*, \tag{4.6.9}$$

and note that if the Higgs field transforms us $\phi \to U\phi$ under an $\mathrm{SU}(2)$ transformation, then

$$\tilde{\phi} = i\sigma_2\phi^* \to= i\sigma_2 M^*\phi^* = Mi\sigma_2\phi^* = M\tilde{\phi}, \tag{4.6.10}$$

where we have used the property that $M^* = \sigma_2 M \sigma_2$ for any SU(2) matrix M. Therefore $\tilde{\phi}$ transforms in the fundamental representation of SU(2). Yet, under a U(1) transformation $\phi \to e^{i\theta/2}\phi$, we have $\tilde{\phi} \to e^{-i\theta/2}\tilde{\phi}$, and therefore $\tilde{\phi}$ has U(1) hypercharge $Y = -1/2$. For the vev, we find

$$\phi = \phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} \quad \Rightarrow \quad \tilde{\phi} = \tilde{\phi}_0 = i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}v\\ 0\end{pmatrix}. \tag{4.6.11}$$

This allows us to write a Yukawa term for the up-type quarks,

$$\mathcal{L} = \ldots - y_u\left(\bar{q}_L\tilde{\phi}u_R + \bar{u}_R\tilde{\phi}^\dagger q_L\right), \tag{4.6.12}$$

which is invariant under SU(2) in the same way as Eq. (4.6.4) and invariant under U(1) because

$$\bar{q}_L\tilde{\phi}u_R \to e^{-i\theta/6-i\theta/2+2i\theta/3}\bar{q}_L\tilde{\phi}u_R = \bar{q}_L\tilde{\phi}u_R. \tag{4.6.13}$$

Substituting the vev (4.6.11), Eq. (4.6.12) becomes a mass term for the up quarks,

$$\mathcal{L} = \ldots - \frac{y_u v}{\sqrt{2}}\left(\bar{u}_L u_R + \bar{u}_R u_L\right), \tag{4.6.14}$$

with mass $m_u = y_u v/\sqrt{2}$.

Generalising Eqs. (4.6.4) and (4.6.12) to three families, we obtain

$$\mathcal{L} = \ldots - \sum_{f,g=1}^{3}\left(y_d^{fg}\bar{q}_L^f\phi d_R^g + (y_d^{fg})^*\bar{d}_R^g\phi^\dagger q_L^f + y_u^{fg}\bar{q}_L^f\tilde{\phi}u_R^g + (y_u^{fg})^*\bar{u}_R^g\tilde{\phi}^\dagger q_L^f\right), \tag{4.6.15}$$

where, in the same way as in Eq. (4.5.14), y_d^{fg} and y_u^{fg} are the elements of the 3×3 complex Yukawa matrices Y_d and Y_u.

Again, as in Eq. (4.5.16), we want to use the singular value decomposition to turn the Yukawa matrices into diagonal form, so we introduce unitary matrices V_{dL}, V_{dR}, V_{uL}, V_{uR} such that $V_{dL}^\dagger Y_d V_{dR}$ and $V_{uL}^\dagger Y_u V_{uR}$ are diagonal. Then we redefine the quark fields

$$\begin{aligned} &d_L^f \to V_{dL}^{fg}d_L^g, \quad d_R^f \to V_{dR}^{fg}d_R^g,\\ &u_L^f \to V_{uL}^{fg}u_L^g, \quad u_R^f \to V_{uR}^{fg}u_R^g. \end{aligned} \tag{4.6.16}$$

This makes the Yukawa term diagonal,

$$\mathcal{L} = \ldots - \sum_{f=1}^{3} y_d^f\left(\bar{q}_L^f\phi d_R^f + \bar{d}_R^f\phi^\dagger q_L^f\right) - \sum_{f=1}^{3} y_u^f\left(\bar{q}_L^f\tilde{\phi}u_R^f + \bar{u}_R^f\tilde{\phi}^\dagger q_L^f\right), \tag{4.6.17}$$

but because the two components of the left-handed quark field q_L^f are rotated differently, it changes the kinetic term,

$$\begin{aligned} \bar{q}_L^f i\not{D} q_L^f &= \ldots - \frac{g_2}{\sqrt{2}}\left(\bar{u}_L^f\gamma^\mu W_\mu^+ d_L^f + \text{h.c.}\right)\\ &\to \ldots - \frac{g_2}{\sqrt{2}}\left(\bar{u}_L^f(V_{uL}^\dagger)^{fg}\gamma^\mu W_\mu^+ V_{dL}^{gh}d_L^h + \text{h.c.}\right)\\ &= \ldots - \frac{g_2}{\sqrt{2}}\left(\bar{u}_L^f(V_{uL}^\dagger V_{dL})^{fg}\gamma^\mu W_\mu^+ d_L^g + \text{h.c.}\right). \end{aligned} \tag{4.6.18}$$

This means that the interaction term with W bosons is not diagonal with respect to the family index f, which physically mean that weak interaction (more specifically, the *charged current* interaction mediated by W bosons) can change quark family. This is described by the CKM (Cabibbo-Kobayashi-Maskawa) matrix

$$V_{\rm CKM} \equiv V_{uL}^{\dagger} V_{dL}. \tag{4.6.19}$$

A general unitary matrix has 9 free parameters, but five of them can be removed by rotations of the relative phases of the six quarks, which means that $V_{\rm CKM}$ has four free physical parameters, which have to be determined by experiment. These correspond to three mixing angles between quark families, which describe transition probabilities, and one complex phase, which leads to CP violation.

Under symmetry breaking, by setting $\phi = \phi_0$ in Eq. (4.6.17) we find mass terms

$$\mathcal{L} = \ldots - \sum_{f=1}^{3} \left[\frac{y_d^f v}{\sqrt{2}} \left(\bar{d}_L^f d_R^f + \bar{d}_R^f d_L^f \right) + \frac{y_u^f v}{\sqrt{2}} \left(\bar{u}_L^f u_R^f + \bar{u}_R^f u_L^f \right) \right], \tag{4.6.20}$$

which gives the quark masses

$$m_d^f = \frac{y_d^f v}{\sqrt{2}}, \quad m_u^f = \frac{y_u^f v}{\sqrt{2}}. \tag{4.6.21}$$

Finally, combining Eq. (3.6.1) including the gauge kinetic terms for the three gauge groups, with the kinetic terms (4.5.4), (4.5.7) for leptons and quarks and the Yukawa terms (4.5.12), (4.6.15), we obtain the full *Standard Model Lagrangian*

$$\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{2}{\rm Tr}\, G_{\mu\nu}G^{\mu\nu} - \frac{1}{2}{\rm Tr}\, W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
& + (D_\mu\phi)^\dagger D^\mu\phi + \mu^2\phi^\dagger\phi - \frac{1}{2}\lambda\left(\phi^\dagger\phi\right)^2 \\
& + \sum_{f=1}^{3} \left(\bar{\ell}_L^f i\not{D}\ell_L^f + \bar{\ell}_R^f i\not{D}\ell_R^f + \bar{q}_L^f i\not{D} q_L^f + \bar{d}_R^f i\not{D} d_R^f + \bar{u}_R^f i\not{D} u_R^f \right) \\
& - \sum_{f=1}^{3} y_\ell^f \left(\bar{\ell}_L^f \phi \ell_R^f + \bar{\ell}_R^f \phi^\dagger \ell_L^f \right) \\
& - \sum_{f,g=1}^{3} \left(y_d^{fg}\bar{q}_L^f \phi d_R^g + (y_d^{fg})^* \bar{d}_R^g \phi^\dagger q_L^f + y_u^{fg}\bar{q}_L^f \tilde{\phi} u_R^g + (y_u^{fg})^* \bar{u}_R^g \tilde{\phi}^\dagger q_L^f \right),
\end{aligned} \tag{4.6.22}$$

where $G_{\mu\nu}$, $W_{\mu\nu}$ and $F_{\mu\nu}$ are the SU(3), SU(2) and U(1) field strength tensors, respectively. This equation is the most general $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge invariant renormalisable Lagrangian for the fields ℓ_L, ℓ_R, q_L, u_R, u_L and ϕ. Therefore it is not possible to add any further terms to it without introducing new fields.

Overall, ignoring the neutrino masses, the theory has 19 free parameters. These are 3 gauge couplings g_1, g_2, g_3; 1 Higgs self coupling λ; 1 Higgs mass term μ^2; 3 lepton Yukawa couplings y_e^f; 10 parameters (6 masses, 3 mixing angles, 1 complex phase) in the quark Yukawa matrices Y_d and Y_u; and one more parameter called the strong CP angle θ, which we have not discussed. All of these parameters have been measured, and remarkably, the theory agrees perfectly with all particle accelerator experiments, although there are hints from neutrino experiments and cosmology showing that it cannot be the complete theory of everything.

$$L = -\frac{1}{2g} da_e \wedge * da_e + \frac{k}{4\pi} a_e \wedge da_e$$

$$\delta L = \delta\left(-\frac{1}{2g} da_e \wedge * da_e\right) + \delta\left(\frac{k}{4\pi} a_e \wedge da_e\right)$$

$$= -\frac{1}{2g^2} d(\delta a_e) \wedge * da_e - \frac{1}{2g^2} da_e \wedge * d(\delta a_e) + \frac{k}{4\pi} \delta a_e \wedge da_e + \frac{k}{4\pi} a_e \wedge d(\delta a_e)$$

$$= -\frac{1}{2g^2} d(\delta a_e) \wedge * da_e - \frac{1}{2g^2} d\delta a_e \wedge * da_e + \frac{k}{4\pi} \delta a_e \wedge da_e + d\left(\frac{k}{4\pi} \cancel{a_e} \wedge \delta a_e\right) - \frac{k}{4\pi} d a_e \wedge \delta a_e$$

$$= -\frac{1}{g^2} d(\delta a_e) \wedge * da_e - \frac{k}{2\pi} da_e \wedge \delta a_e$$

$$= -\frac{1}{g^2} d(\delta a_e \wedge * da_e) + \frac{1}{g^2} \delta a_e \wedge d * da_e - \frac{k}{2\pi} da_e \wedge \delta a_e$$

$$= \delta a_e \wedge \left(\frac{1}{g^2} d * da_e + \frac{k}{2\pi} da_e\right)$$

$$\Rightarrow d * da_e + \frac{g^2 k}{2\pi} da_e = 0$$

S-Dual of Maxwell–Chern-Simons Theory

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We discuss the dynamics of three-dimensional Maxwell theory coupled to a level-k Chern-Simons term. Motivated by S-duality in string theory, we argue that the theory admits an S-dual description. The S-dual theory contains a nongauge one-form field, previously proposed by Deser and Jackiw [Phys. Lett. **139B**, 371 (1984).] and a level-k $U(1)$ Chern-Simons term, $\mathcal{Z}_{\mathrm{MCS}} = \mathcal{Z}_{\mathrm{DJ}} \mathcal{Z}_{\mathrm{CS}}$. The couplings to external electric and magnetic currents and their string theory realizations are also discussed.

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Introduction.—Chern-Simons theory is vastly used in mathematical physics, in condensed-matter physics, and in string theory [1]. It was studied intensively in the past three decades, yet the dynamics of Yang-Mills–Chern-Simons theory is not fully understood in the strong coupling regime.

Four-dimensional S-duality is an exact duality between two $\mathcal{N} = 4$ super-Yang-Mills theories, enabling us to calculate quantities in the strong coupling regime using a dual weakly coupled theory. In the Abelian case, it reduces to the old electric-magnetic duality which swaps electric and magnetic fields:

$$F \leftrightarrow * F. \tag{1}$$

In 3D, Abelian S-duality relates the electric field to a dual scalar:

$$f \leftrightarrow * d\phi. \tag{2}$$

The purpose of this note is to extend S-duality to 3D Maxwell Chern-Simons (MCS) theory, with either a compact or noncompact $U(1)$ gauge group. It should hold on any spin manifold. The Lagrangian of the theory is given by

$$L = -\frac{1}{2g^2} da_e \wedge * da_e + \frac{k}{4\pi} a_e \wedge da_e. \tag{3}$$

MCS theory contains a vector boson of mass $M = (g^2 k / 2\pi)$. At low energies, the kinetic term is irrelevant, and the theory flows to a pure level-k Chern-Simons theory. As explained in the section on the derivation of the duality, the theory admits a global $\mathbb{Z}_k$ one-form symmetry generated by

$$G \equiv \exp\left(i \oint \left(a_e - \frac{1}{M} * da_e\right)\right). \tag{4}$$

When the theory is compactified on a torus, the global $\mathbb{Z}_k$ one-form symmetry is spontaneously broken, resulting in k degenerate vacua.

Several attempts were made to find the S-dual of Eq. (3). In Ref. [2], Deser and Jackiw proposed a "self-dual model" (SDM) which describes a massive vector. While SDM describes a massive vector, it does not admit a $\mathbb{Z}_k$ one-form symmetry, and neither does it flow to a pure Chern-Simons theory at low energies; hence, it cannot be an exact dual of MCS theory.

A closely related problem concerns the open string dynamics on a certain Hanany-Witten brane configuration. It is well known [3] that MCS theory lives on the left brane configuration shown in Fig. 1. Type-IIB S-duality maps the left configuration into the right configuration. Thus, knowing the field theory that lives on the right configuration will solve the problem of finding the S-dual. In early attempts [3,4], the authors found gauge theories with a fractional-level Chern-Simons term. While the theories

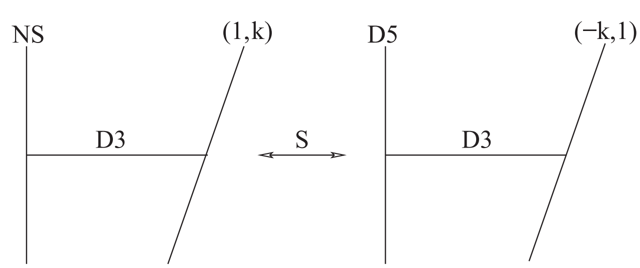

FIG. 1. The electric theory on the left brane configuration is Maxwell–Chern-Simons. The magnetic theory, obtained by type-IIB S-duality, lives on the right brane configuration.

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they found are classically equivalent to MCS, it cannot be the full answer, as it does not admit the symmetries or the same dynamics as the electric theory.

Derivation of the duality.—We may use 4D S-duality between Maxwell theories to derive the 3D duality. In 4D, a pure Maxwell theory with a coupling g is dual to a pure Maxwell theory with a coupling $1/g$.

Consider the following partition function: $[A, A]A$

$A\wedge A\wedge A = 0$ in Abelian case

$$\mathcal{Z} = \int \underset{\text{QFT. M}}{DF_m} DA_e \exp i \int \left(-\frac{g^2}{2}\underbrace{F_m \wedge *F_m}_{F^2} + \underbrace{F_m \wedge dA_e}_{F\,dA}\right), \tag{5}$$

A_e is the "electric" gauge field, while F_m is a "magnetic" gauge-invariant two-form. g is the "electric" gauge coupling.

Upon integrating over F_m, we obtain the electric theory

$$\mathcal{Z} = \int DA_e \exp i \int \left(-\frac{1}{2g^2} dA_e \wedge *dA_e\right). \tag{6}$$

If instead we integrate over A_e, we obtain

$$\mathcal{Z} = \int DF_m \delta(dF_m) \exp i \int \left(-\frac{g^2}{2} F_m \wedge *F_m\right); \tag{7}$$

hence, it can be written in terms of A_m such that $F_m = dA_m$:

$$\mathcal{Z} = \int DA_m \exp i \left(-\frac{g^2}{2} dA_m \wedge *dA_m\right). \tag{8}$$

This is the magnetic theory dual to the electric theory.

Let us use the dimensional reduction of Eq. (5) in order to derive the 3D duality. Upon reducing to 3D, the 4D two-form F_m becomes a 3D two-form f_m and a one-form a_m. The 4D gauge field A_e becomes a 3D gauge field a_e and a scalar ϕ_e. The two-form f_m and the scalar ϕ_e decouple from the rest of the action and admit

$$\mathcal{Z} = \int Df_m D\phi_e \exp i \int \left(-\frac{g^2}{2} f_m \wedge *f_m + f_m \wedge d\phi_e\right), \tag{9}$$

which leads to the well-known S-duality

$$d\hat{a}_m \leftrightarrow *d\phi_e, \tag{10}$$

where $f_m = d\hat{a}_m$.

Let us focus on the duality between a_e and a_m, which is the prime purpose of this Letter. We add to the action a Chern-Simons term [5]. Our proposal is the following "master" partition function:

$$\mathcal{Z} = \int Da_m Da_e \exp i \int \left(-\frac{g^2}{2} a_m \wedge *a_m + a_m \wedge da_e + \frac{k}{4\pi} a_e \wedge da_e\right). \tag{11}$$

Note that a_m is a *gauge-invariant* one-form. Upon integration over a_m, we obtain the electric theory

$$\mathcal{Z} = \int Da_e \exp i \int \left(-\frac{1}{2g^2} da_e \wedge *da_e + \frac{k}{4\pi} a_e \wedge da_e\right), \tag{12}$$

$$\Rightarrow \frac{1}{g^2} d*da + \frac{k}{2\pi} da = 0$$

namely Maxwell–Chern-Simons theory.

In order to derive the magnetic theory, we should use Eq. (11) and integrate over a_e. This is a subtle point. Instead, let us use a change of variables, $a_e = b - (2\pi/k)a_m$, to obtain the following partition function:

$$\mathcal{Z} = \int Da_m Db \exp i \int \left(-\frac{g^2}{2} a_m \wedge *a_m - \frac{\pi}{k} a_m \wedge da_m + \frac{k}{4\pi} b \wedge db\right). \tag{13}$$

Equation (13) is our proposal for the S-dual of Maxwell–Chern-Simons theory. The partition function of the magnetic theory is a product of the Deser-Jackiw theory and a level-k Chern-Simons term,

$$\mathcal{Z}_{\rm MCS} = \mathcal{Z}_{\rm DJ}\mathcal{Z}_{\rm CS}. \tag{14}$$

Note that a_m is not a gauge field, and therefore the term $(\pi/k)a_m \wedge da_m$ is not ill-defined.

Both the electric and the magnetic theories describe a massive vector of mass $M = (g^2 k/2\pi)$ and a decoupled level-k Chern-Simons theory. Both theories exhibit a one-form $\mathbb{Z}_k$ symmetry.

Let us now provide another argument in favor of our proposal in Eq. (13). We begin with the magnetic brane configuration of Fig. 1. It was argued by Gaiotto and Witten [6] that the theory which lives on the intersection of the 3-brane and the tilted 5-brane (without a D5 brane) is

$$\mathcal{Z} = \int Da Dc \exp i \int \left(\frac{1}{2\pi} a \wedge dc + \frac{k}{4\pi} c \wedge dc\right). \tag{15}$$

In order to understand what happens when we add a D5 brane, let us assume that the terms that we need to add to the action are k-independent. Indeed, the information about k is encoded in the tilted 5-brane, not in the 3-brane. Let us use $k = 0$, since in this case the duality is well understood: the electric theory is pure Maxwell, and the magnetic (mirror) theory is a massless scalar. The brane realization of the duality was provided in the seminal work of Hanany and Witten [7].

We may write the theory of a free massless scalar as follows:

$$\mathcal{Z} = \int Da\, Dc \exp i \int \left(a \wedge *a + \frac{1}{2\pi} a \wedge dc \right), \quad (16)$$

with a being a gauge-invariant one-form. The equation of motion for c is $*da = 0$—namely, that $a = d\chi$. Thus, for $k = 0$ we obtain a theory of a free scalar $(d\chi)^2$, as expected.

We find that adding a term $a \wedge *a$ to the action yields a theory that describes the correct dual of Maxwell theory. We propose that $a \wedge *a$ is the missing term in Eq. (15)—namely, that by adding it to Eq. (15) we obtain the dual of MCS for any k. Note that

$$\mathcal{Z} = \int Da\, Dc \exp i \int \left(a \wedge *a + \frac{1}{2\pi} a \wedge dc + \frac{k}{4\pi} c \wedge dc \right) \quad (17)$$

is almost identical to Eq. (11). An important difference is that Gaitto and Witten introduced a *gauge field* a, whereas in Eq. (17) we added a term that breaks gauge invariance. We may reintroduce gauge invariance in Eq. (17) by transforming the fixed gauge vector a into a gauge-invariant term by adding a scalar η as follows:

$$\mathcal{Z} = \int DaDcD\eta \exp i \int \Big((a - d\eta) \wedge *(a - d\eta) + \frac{1}{2\pi} a \wedge dc + \frac{k}{4\pi} c \wedge dc \Big), \quad (18)$$

such that under a gauge transformation $a \to a + d\lambda$, $\eta \to \eta + \lambda$, with a being a $U(1)$ gauge field. Equation (17) may be viewed as the fixed-gauge version of Eq. (18) with $d\eta = 0$.

Our proposal in Eq. (13) passes all the requirements from a dual theory: it admits a $\mathbb{Z}_k$ global symmetry, it flows to pure Chern-Simons theory in the IR, it contains a massive vector of mass M, and finally, when $k = 0$, it agrees with the results of Hanany and Witten [7]. As we shall see, the brane realizations of both electric and magnetic theories predict the existence of k-degenerate vacua.

We summarize this section by writing the precise map between the electric and magnetic variables using Eq. (11):

$$-g^2 a_m = *da_e, \quad (19)$$

$$b = a_e - \frac{1}{M} * da_e, \quad (20)$$

or

$$a_e = b - \frac{2\pi}{k} a_m. \quad (21)$$

Comments on $\mathbb{Z}_k$.—Let us introduce a Wilson loop in MCS theory. We wish to measure the $\mathbb{Z}_k$ charge of the loop —namely, the number of fundamental strings, n, that pass through a certain contour C. We will define an operator G such that

$$GW_n = \exp\left(i \frac{2\pi n}{k} \right) W_n, \quad (22)$$

with W_n being a Wilson loop of charge n, $W_n = \exp(in \oint a_e)$.

In order to define G, let us consider the equation of motion in MCS:

$$d\left(\frac{1}{g^2} * da_e - \frac{k}{2\pi} a_e \right) = j_e \equiv dJ_e, \quad (23)$$

where J_e is the integral of the electric current j_e over a disk D such that $C = \partial D$. The setup is depicted in Fig. 2. By integrating Eq. (23), we learn that

$$\frac{1}{g^2} * da_e - \frac{k}{2\pi} a_e = J_e. \quad (24)$$

We can therefore define a generator of a $\mathbb{Z}_k$ symmetry as follows:

$$G = \exp\left(i \frac{2\pi}{k} \oint_C \left(\frac{k}{2\pi} a_e - \frac{1}{g^2} * da_e \right) \right) = \exp\left(i \oint_C b \right). \quad (25)$$

Note that the implication of $\mathbb{Z}_k$ symmetry is a symmetry $n \to n + k$: namely, that a collection of k strings is topologically isomorphic to a singlet—namely, to no strings at all. This is supported by string theory: suppose that we attempt to place the end points of k coincident strings on the worldvolume of the D3 brane. The collection of k fundamental strings can transform itself into an anti-D-string and a $(k, 1)$ string. Instead of ending on the

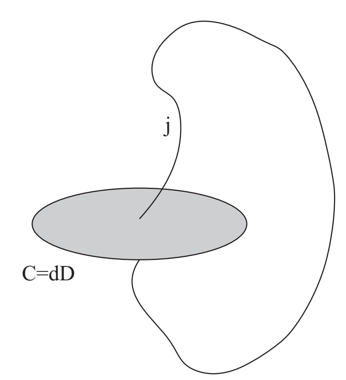

FIG. 2. A Wilson loop passing through a domain D (shaded region) whose boundary is $C = \partial D$.

worldvolume of the D3 brane, the D-string can end on an NS5 brane, and a $(k,1)$ string can end on a $(1,k)$ 5-brane. Thus, string theory predicts that a collection of k strings can be removed from the worldvolume of the 3D gauge theory. A similar phenomenon happens in the magnetic dual if we attempt to introduce k coincident D-strings in the worldvolume of the magnetic theory.

When the theory is defined on the torus, the $\mathbb{Z}_k$ symmetry is broken, resulting in k vacua [8]. An intuitive explanation is as follows: The level-k $U(1)$ Chern-Simons theory is equivalent (using level-rank duality) to a level-$1SU(k)$ theory that admits a $\mathbb{Z}_k$ center symmetry. When it is defined on the torus, the $SU(k)$ theory deconfines, resulting in k degenerate vacua, parametrized by the eigenvalues of the 't Hooft loop.

The k vacua manifest themselves in both the electric and magnetic brane configurations as follows: the D3 brane may end on any of the k constituents of the 5-branes. Each one of the k choices corresponds to a vacuum.

Coupling to external sources, Wilson and magnetic loops.—Consider the coupling of the electric gauge field to a source j_e, namely $a_e j_e$. It translates to the coupling $(b-(\pi/k)a_m)j_e$ in the magnetic side. We therefore suggest that the Wilson loop

$$W_e = \exp i \oint a_e \tag{26}$$

in the electric side is mapped into a magnetic loop of the form

$$M_m = \exp i \oint \left(b - \frac{2\pi}{k} a_m \right) \tag{27}$$

in the magnetic side.

We may use the above map between the Wilson loop in the electric side and its magnetic counterpart to study the dynamics of 3D QED-CS. Using the worldline formalism [9], we can write the partition function of MCS theory coupled to N_f massless fields as follows:

$$\mathcal{Z}_{\text{QED-CS}} = \int Da_e \exp(iS_{\text{MCS}}) \sum_n \frac{(N_f \Gamma_e)^n}{n!}, \tag{28}$$

with

$$\Gamma_e = \int \frac{dt}{t^{\frac{5}{2}}} \int Dx \exp\left(- \int_0^t d\tau (\dot{x})^2 \right) \exp i \oint a_e. \tag{29}$$

The duality yields the following partition function:

$$\mathcal{Z}_{\text{magnetic}} = \int Da_m Db \exp(iS_{\text{DJ-CS}}) \sum_n \frac{(N_f \Gamma_m)^n}{n!}, \tag{30}$$

with

$$\Gamma_m = \int \frac{dt}{t^{\frac{5}{2}}} \int Dx \exp\left(- \int d\tau (\dot{x})^2 \right) \exp i \oint \left(b - \frac{2\pi}{k} a_m \right). \tag{31}$$

This suggests that the dynamics of QED with N_f massless flavors is captured by a dual DJ-CS theory coupled to N_f massless "monopoles." The precise coupling of a_m and b to the monopoles is given by Eq. (31). We may write the dual magnetic theory in a more "standard" form:

$$\mathcal{Z} = \int Da_m Db\, D\bar{\psi}_m D\psi_m \exp i\, S_{\text{magnetic}}, \tag{32}$$

with S_{magnetic} given by

$$S_{\text{magnetic}} = \int \left(-\frac{g^2}{2} a_m \wedge *a_m - \frac{\pi}{k} a_m \wedge da_m + \frac{k}{4\pi} b \wedge db + \bar{\psi}_m \gamma \wedge \star \left(i\partial + b - \frac{2\pi}{k} a_m \right) \psi_m \right). \tag{33}$$

It is interesting to note that the QED-CS theory is mapped to a theory of interacting massless magnetic "monopoles," with a coupling $2\pi/gk$. Thus, when the electrons couple strongly to a_e, the "monopoles" couple weakly to a_m, and we may use perturbation theory in the magnetic side to study the strongly coupled electric theory.

Following Itzhaki [10], let us define a magnetic ("disorder") loop in the electric theory

$$M_e = \exp\left(i \oint_C \left(k a_e - \frac{2\pi}{g^2} * da_e \right) \right), \tag{34}$$

which is mapped into the electric loop in the magnetic side,

$$W_m = \exp\left(ik \oint_C b \right). \tag{35}$$

The magnetic loop in the electric side and the electric (Wilson) loop in the magnetic side are trivial [10].

We suggest that a rectangular Wilson loop (or magnetic loop) should be identified with the end points of an F-string and an anti-F-string (or a D-string and an anti-D-string) that end on the 3-branes of Fig. 3.

A D-string can end on an NS5 brane instead of a 3-brane; hence, the magnetic loop in the electric theory should be trivial. Similarly, a F-string can end on a D5 brane instead of a 3-brane; hence, a Wilson loop in the magnetic theory should be trivial. This is consistent with our definitions of the magnetic loop [Eq. (34)] and the Wilson loop [Eq. (35)].

Summary.—The purpose of this Letter is to find the S-dual of MCS theory. We found that the dual theory [Eq. (13)] contains a nongauge vector of mass M and a decoupled pure TQFT. The magnetic theory nicely captures

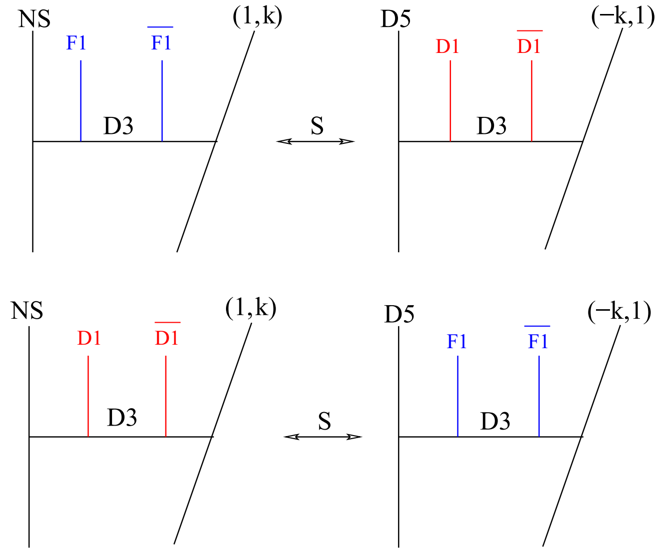

FIG. 3. Rectangular Wilson loops can be realized in string theory by ending a pair of an F-string and an anti-F-string on the 3-brane. Similarly, rectangular 't Hooft loops can be realized by ending a pair of a D-string and an anti-D-string on the 3-brane. The end of the F-string represents a heavy quark, whereas the end of the D-string represents a heavy monopole.

the dynamics of the electric theory: a theory with a mass gap that flows in the IR to a TQFT. The duality we uncovered in this Letter is a precise manifestation of the duality between a topological insulator and a topological superconductor outlined in Ref. [11].

It will be interesting to find the S-dual of the non-Abelian $U(N)$ theory that lives on a collection of N coincident D3 branes, suspended between tilted 5-branes. The master field of that theory may be obtained by replacing the Abelian one-forms of Eq. (11) with non-Abelian one-forms as follows [12]:

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$$\mathcal{Z} = \int Da_m Da_e \exp\, i\,\mathrm{tr} \int \left(-\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge (da_e + a_e \wedge a_e) + \frac{k}{4\pi}\left(a_e \wedge da_e + \frac{2}{3} a_e \wedge a_e \wedge a_e \right)\right), \tag{36}$$

together with $a_e = b - (2\pi/k)a_m$. Other dualities that involve SO/Sp (and an orientifold in string theory) could also be derived. The generalization to supersymmetric QED or QCD theories with a CS term [4] is also interesting and can be written down using the worldline formalism, as in the previous section on coupling. The duality found in this Letter is useful for studying the strong coupling regime of those theories.

Finally, it is well known that MCS theory admits Seiberg duality. The manifestation of the duality using an exchange of 5-branes in the magnetic theory might teach us about 5-branes' dynamics.

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[1] G. V. Dunne, Aspects of Chern-Simons theory, arXiv:hep-th/9902115.
[2] S. Deser and R. Jackiw, "Self-duality" of topologically massive gauge theories, Phys. Lett. **139B**, 371 (1984).
[3] T. Kitao and N. Ohta, Spectrum of Maxwell-Chern-Simons theory realized on type IIB Brane configurations, Nucl. Phys. **B578**, 215 (2000); O. Bergman, A. Hanany, A. Karch, and B. Kol, Branes and supersymmetry breaking in three-dimensional gauge theories, J. High Energy Phys. 10 (1999) 036.
[4] A. Kapustin and M. J. Strassler, On mirror symmetry in three-dimensional Abelian gauge theories, J. High Energy Phys. 04 (1999) 021.
[5] The Chern-Simons term can be obtained by a dimensional reduction of a space-dependent theta term $\int d^4x \theta(x) F_e \wedge F_e$, such that $\theta(x) = (k/4\pi)H(x^3)$, where $H(x^3)$ is the Heaviside step function.
[6] D. Gaiotto and E. Witten, S-duality of boundary conditions in $N = 4$ super Yang-Mills theory, Adv. Theor. Math. Phys. **13**, 721 (2009).
[7] A. Hanany and E. Witten, Type-IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics, Nucl. Phys. **B492**, 152 (1997).
[8] C. L. Ho and Y. Hosotani, Operator Algebra in Chern-Simons Theory on a Torus, Phys. Rev. Lett. **70**, 1360 (1993); D. Tong, Lecture notes, https://www.damtp.cam.ac.uk/user/tong/gaugetheory/83d.pdf.
[9] M. J. Strassler, Field theory without Feynman diagrams: One loop effective actions, Nucl. Phys. **B385**, 145 (1992).
[10] N. Itzhaki, Anyons, 't Hooft loops and a generalized connection in three-dimensions, Phys. Rev. D **67**, 065008 (2003).
[11] J. Murugan and H. Nastase, Particle-vortex duality in topological insulators and superconductors, J. High Energy Phys. 05 (2017) 159.
[12] I thank Shigeki Sugimoto for this suggestion.

$$\mathcal{Z} = \int Da_m Da_e \exp\left(i\mathrm{tr}\int\left(-\frac{g^2}{2} a_m \wedge * a_m + a_m \wedge (da_e + a_e \wedge a_e) + \frac{k}{4\pi}\left(a_e \wedge da_e + \frac{2}{3} a_e \wedge a_e \wedge a_e\right)\right)\right).$$

$$\wedge * (da_e + a_e \wedge a_e) + \frac{k}{4\pi}\left(a_e \wedge da_e + \frac{2}{3} a_e \wedge a_e \wedge a_e\right)\Big)\Big).$$

$$= \int Da_e \exp\Big(i\,\mathrm{tr} \int -\frac{1}{2g^2}\big(da_e + a_e \wedge a_e\big)\wedge \ldots \frac{\ \ }{4\pi}$$

$$= \int Da_e \exp\Big(i\,\mathrm{tr} \int -\frac{1}{2g^2}\big(da_e \wedge * da_e + da_e \wedge (a_e \wedge a_e) + (a_e \wedge a_e) \wedge * da_e + (a_e \wedge a_e) \wedge *(a_e \wedge a_e)\big) + \frac{K}{4\pi}\Big(a_e \wedge da_e + \frac{2}{3} a_e \wedge a_e \wedge a_e\Big)\Big).$$