PDE: ASSIGNMENT 1

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1. Problem 1

If B is a positive-definite, Hermitian matrix, show that there is a unique matrix $\sqrt{B}$ that is Hermitian positive-definite and $\left(\sqrt{B}\right)^2 = B$.

Let M be the matrix of eigenvectors of B. Since B is Hermitian and positive definite, M and it's inverse is Hermitian and positive-definite. Then $B' = M^{-1}BM$, and B' is a diagonal matrix of the eigenvalues of B.

$M\sqrt{B'}M^{-1} = \sqrt{B}$, which is Hermitian and positive-definite.

Even M is not unique (any αM is a matrix of eigenvectors of B), B' is unique since $(\alpha M)^{-1}B\alpha M = \frac{1}{\alpha}\alpha M^{-1}BM$. Therefore, $\sqrt{B}$ is unique.

2. Problem 2

A and B are Hermitian and B is positive definite.

2.1. Part 1. Show that $B^{-1}A$ is similar to a Hermitian.

C is similar to $B^{-1}A \Rightarrow C = MB^{-1}AM^{-1}$ for any invertible M. From Problem 1, $\sqrt{B}$ is invertible so let $M = \sqrt{B}$.

$C = \sqrt{B}B^{-1}A\sqrt{B}^{-1} = B^{-1/2}AB^{-1/2}$, which is Hermitian since A and $\sqrt{B}$ are Hermitian. Therefore, $B^{-1}A$ is similar to a Hermitian.

2.2. Part 2. What does this say about the eigenvalues of $B^{-1}A$?

The eigenvalues are real.

2.3. Part 3. Are the eigenvectors orthogonal?

No. The eigenvectors of C are orthagonal, but $B^{-1}A$ is not Hermitian. It is *similar* to a Hermitian matrix. So it will not necessarily have orthagonal eigenvectors.

2.4. Part 4. Verify the answers above in Julia with 5×5 matrices.

see notebook.

2.5. **Part 5.** What is special about $C = M^T BM$? Show that the elements of C are a kind of dot product of the eigenvectors with a factor of B in the middle.

see notebook.

3. Problem 3

The solutions of ODE $y'' - 2y' - cy = 0$ take the form $y(t) = C_1 e^{(1+\sqrt{1+c})t} + C_2 e^{(1-\sqrt{1+c})t}$ for some constants C_1 and C_2 determined by the initial conditions. Suppose that A is a real-symmetric 4×4 matrix with eigenvalues 3, 8, 15, and 24, which correspond to eigenvectors $\mathbf{x_1}$, $\mathbf{x_2}$, $\mathbf{x_3}$, $\mathbf{x_4}$.

3.1. **Part 1.** If $\mathbf{x}(t)$ solves the system of ODEs $\frac{d^2}{dx^2}\mathbf{x} - 2\frac{d}{dx}\mathbf{x} = A\mathbf{x}$, with initial conditions $\mathbf{x}(0) = \mathbf{a}_0$ and $\mathbf{x}'(0) = \mathbf{b}_0$, find the closed-form expression of $\mathbf{x}(t)$ in terms of the eigenvectors and initial conditions.

Since A is Hermitian and positive-definite, the eigenvectors are orthogonal. Thus, the solution can be expressed as linear combination of the eigenvectors.

$$\mathbf{x}(t) = \sum_{n=1}^{4} c_n(t)\mathbf{x}_n$$

$$\frac{d^2}{dx^2}\mathbf{x}(t) - 2\frac{d}{dx}\mathbf{x}(t) - A\mathbf{x}(t) = \sum_{n=1}^{4} \left(\ddot{c_n} - 2\dot{c_n} - \lambda_n\right)\mathbf{x}_n = 0$$

Since $\mathbf{x}_n$ are linearly independent, $\ddot{c_n} - 2\dot{c_n} - \lambda_n = 0$ for all n.

$$c_n(t) = \alpha_n e^{(1+\sqrt{1+\lambda_n})t} + \beta_n e^{(1-\sqrt{1+\lambda_n})t}$$

$$\mathbf{x}(0) = \sum_{n=1}^{4} (\alpha_n + \beta_n)\mathbf{x}_n = \mathbf{a}_0$$

$$\alpha_n + \beta_n = \frac{\mathbf{x}_x^* \mathbf{a}_0}{||\mathbf{x}_n||^2}$$

$$\mathbf{x}'(0) = \sum_{n=1}^{4} \Big(\alpha_n(1 + \sqrt{1+\lambda_n}) + \beta_n(1 - \sqrt{1+\lambda_n})\Big)\mathbf{x}_n = \mathbf{b}_0$$

$$\mathbf{x}'(0) = \sum_{n=1}^{4} \Big(\alpha_n + \beta_n + \sqrt{1+\lambda_n}(\alpha_n - \beta_n)\Big)\mathbf{x}_n = \mathbf{b}_0$$

$$\frac{\mathbf{x}_n^* \mathbf{a}_0}{||\mathbf{x}_n||^2} + \sqrt{1+\lambda_n}(\alpha_n - \beta_n) = \frac{\mathbf{x}_n^* \mathbf{b}_0}{||\mathbf{x}_n||^2}$$

$$\alpha_n - \beta_n = \frac{\mathbf{x}_n^*(\mathbf{b}_0 - \mathbf{a}_0)}{||\mathbf{x}_n||^2\sqrt{1+\lambda_n}}$$

$$\mathbf{x}(t) = \sum_{n=1}^{4} \left(\left(\mathbf{x}_n^* \mathbf{a}_0 + \frac{\mathbf{x}_n^*(\mathbf{b}_0 - \mathbf{a}_0)}{\sqrt{1+\lambda_n}}\right) e^{(1+\sqrt{1+\lambda_n})t} + \left(\mathbf{x}_n^* \mathbf{a}_0 - \frac{\mathbf{x}_n^*(\mathbf{b}_0 - \mathbf{a}_0)}{\sqrt{1+\lambda_n}}\right) e^{(1-\sqrt{1+\lambda_n})t}\right) \frac{\mathbf{x}_n}{2||\mathbf{x}_n||^2}$$

3.2. **Part 2.** After a long time ($t >> 0$), what is the expected approximation of the solution?

$\mathbf{x}(t)$ will be dominated by the fastest growing term in the mode with the largest eigenvalue ($\lambda_4 = 24$).

$$\mathbf{x}(t >> 0) \cong \left(\mathbf{x}_4^* \mathbf{a}_0 + \frac{\mathbf{x}_4^*(\mathbf{b}_0 - \mathbf{a}_0)}{\sqrt{1+\lambda_4}} \right) e^{(1+\sqrt{1+\lambda_4})t} \frac{\mathbf{x}_4}{2||\mathbf{x}_4||^2}$$

$$\mathbf{x}(t >> 0) \cong \left(\mathbf{x}_4^* \mathbf{a}_0 + \frac{\mathbf{x}_4^*(\mathbf{b}_0 - \mathbf{a}_0)}{5} \right) e^{6t} \frac{\mathbf{x}_4}{2||\mathbf{x}_4||^2}$$

4. Problem 4

Consider the 1d Poisson equation $\frac{d^2}{dx^2}u(x) = f(x)$ for the vector space of functions $u(x)$ on $x \in [0, L]$ with the Dirichlet boundary conditions $u(0) = u(L) = 0$.

4.1. **Part 1.** Suppose the boundary conditions are changed to *periodic* boundary condition $u(0) = u(L)$. What are the eigenfunctions of $\frac{d^2}{dx^2}$ now? Will Poisson's equations have unique solutions? Under what conditions on $f(x)$ would a solution exist?

Since the boundary is periodic, the eigenfunctions will be $\sin(kx)$, and $\cos(kx)$, where $k = \frac{2\pi n}{L}$ with $n = 1, 2, \ldots$ for sine and $n = 0, 1, 2, \ldots$ for cosine. 0 is ignored for sine because we do not allow the zero function as an eigenfunction, and negative n's since they are linearly dependent on sine and cosine.

$\sin(\phi + kx) = \cos(\phi)\sin(kx) + \sin(\phi)\cos(kx)$, and is therefore linearly dependent on sine and cosine. And exponential functions are only periodic for $e^{ikx} = \cos(kx) + i\sin(kx)$, which is linearly dependent on sine and cosine.

The equation will not have unique solutions since the vector space is spanned by $u(x) = \alpha 1 \forall \alpha$ Thus, it can have infinite solutions.

To solve $\frac{d^2}{dx^2}u(x) = f(x)$, we would divide each term in the Fourier series by it's eigenvalue $(\frac{2\pi n}{L})^2$, which is only defined for $n > 0$. This implies the $c_0 = 0$, or equivalently $\int_0^L f(x) = 0$. Under this condition, the equation is solvable.

4.2. **Part 2.** If instead we considered $\frac{d^2}{dx^2}v(x) = g(x)$ with the boundary condition $v(0) = v(L) + 1$, do these functions form a vector space?

A vector space must include the zero function, but if $v(x) = 0$, then $v(0) \neq v(L)+1$. Thus, the functions do not form a vector space.

4.3. **Part 3.** How can we transform $v(x)$ from Part 2 back into the original $\frac{d^2}{dx^2}u(x) = f(x)$ problem with $u(x) = u(L)$ by writing $u(x) = v(x)+q(x)$ and $f(x) = g(x)+r(x)$ for some function q and r?

Let q be a twice differential function with $q(L) - q(1) = -1$. Then $u(L) - u(0) = (v(L) - v(0)) + (q(L) - q(0)) = -1 + 1 = 0$ and u is periodic. Then plug $v = u - q$ back into $\frac{d^2}{dx^2}v(x) = g(x)$:

$$\frac{d^2}{dx^2}(u - q) = g(x)$$

$$\frac{d^2}{dx^2}u(x) = g(x) + \frac{d^2}{dx^2}q$$

Take $\frac{d^2}{dx^2}q$ as r and

$$\frac{d^2}{dx^2}u(x) = g(x) + r(x) = f(x)$$

5. Problem 5

Consider a finite difference approximation as shown below.

$$u'(x) \approx \frac{-u(x + 2\Delta x) + c \cdot u(x + \Delta x) - c \cdot u(x - \Delta x) + u(x - 2\Delta x)}{d\Delta x}$$

5.1. **Part 1.** Substitute the Taylor Series for $u(x + \Delta x)$ to show that an appropriate choice of c and d will make this approximation 4^{th} order accurate (errors are proportional to $(\Delta x)^2$).

$$u(x + \Delta x) = \sum_{n=0}^{\infty} \frac{u^{(n)}(x)(\Delta x)^n}{n!}$$

Denote $\hat{u}'(x)$ as the finite difference approximation of $u'(x)$.

$$\hat{u}'(x) = \frac{-\sum_{n=0}^{\infty} \frac{u^{(n)}(x)(2\Delta x)^n}{n!} + c\sum_{n=0}^{\infty} \frac{u^{(n)}(x)(\Delta x)^n}{n!} - c\sum_{n=0}^{\infty} \frac{u^{(n)}(x)(-\Delta x)^n}{n!} + -\sum_{n=0}^{\infty} \frac{u^{(n)}(x)(-2\Delta x)^n}{n!}}{d\Delta x}$$

$$\hat{u}'(x) = \frac{-2\sum_{n=0}^{\infty} \frac{u^{(2n+1)}(x)(2\Delta x)^{(2n+1)}}{(2n+1)!} + 2c\sum_{n=0}^{\infty} \frac{u^{(2n+1)}(x)(\Delta x)^{(2n+1)}}{(2n+1)!}}{d\Delta x}$$

$$\hat{u}'(x) = \frac{2\sum_{n=0}^{\infty} \frac{u^{(2n+1)}(x)(\Delta x)^{(2n+1)}}{(2n+1)!}\left(c - 2^{(2n+1)}\right)}{d\Delta x}$$

$$\hat{u}'(x) = \frac{2}{d}\sum_{n=0}^{\infty} \frac{u^{(2n+1)}(x)(\Delta x)^{2n}}{(2n+1)!}\left(c - 2^{2n+1}\right)$$

$$\frac{d}{2}\hat{u}'(x) = u'(x)(c-2) + \frac{1}{3!}u^{(3)}(x)(\Delta x)^2(c-8) + \frac{1}{5!}u^{(5)}(x)(\Delta x)^4(c-32) + \dots$$

$$\frac{d}{2}\hat{u}'(x) - u'(x)(c-2) = \frac{1}{3!}u^{(3)}(x)(\Delta x)^2(c-8) + \frac{1}{5!}u^{(5)}(x)(\Delta x)^4(c-32) + \dots$$

Let $c = 8$ and $d = 12$.

$$6\hat{u}'(x) - 6u'(x) = \frac{1}{5!}u^{(5)}(x)(\Delta x)^4(-24) + \dots$$

$$\hat{u}'(x) - u'(x) = \frac{-u^{(5)}(x)}{30}(\Delta x)^4$$

$$\therefore \hat{u}'(x) - u'(x) \propto (\Delta x)^4$$

5.2. **Part 2.** Check Part 1 by numerically computing $u'(1)$ for $u(x) = \sin(x)$, as a function of Δx. Verify the approximation is $4^t h$ order accurate using a loglog plot.

see notebook.