# PDE: ASSIGNMENT 3

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## 1. Problem 1

Consider a three-component vector field $\mathbf{u}(\mathbf{x})$ on some finite volume domain $\Omega \in \mathbb{R}^3$. Define the inner product of two vectors $\mathbf{u}$ and $\mathbf{v}$ by the volume integral: $\langle \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \overline{\mathbf{u}} \cdot \mathbf{v}$. Consider the curl operator $\nabla \times$.

### 1.1. Part 1.

Derive the identity: $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot (\nabla \times \mathbf{v})$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \nabla \cdot \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} = \frac{\partial}{\partial x}(u_2 v_3 - u_3 v_2) + \frac{\partial}{\partial y}(u_3 v_1 - u_1 v_3) + \frac{\partial}{\partial z}(u_1 v_2 - u_2 v_1)$$

$$(\nabla \times \mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot (\nabla \times \mathbf{v}) = \begin{bmatrix} \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} & \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} & \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \end{bmatrix} \cdot \mathbf{v} - \mathbf{u} \cdot \begin{bmatrix} \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{bmatrix}$$

$$= \frac{\partial u_3}{\partial y} v_1 - \frac{\partial u_2}{\partial z} v_1 + \frac{\partial u_1}{\partial z} v_2 - \frac{\partial u_3}{\partial x} v_2 + \frac{\partial u_2}{\partial x} v_3 - \frac{\partial u_1}{\partial y} v_3 - \frac{\partial v_3}{\partial y} u_1 + \frac{\partial v_2}{\partial z} u_1 - \frac{\partial v_1}{\partial z} u_2 + \frac{\partial v_3}{\partial x} u_2 - \frac{\partial v_2}{\partial x} u_3 + \frac{\partial v_1}{\partial y} u_3$$

$$= \frac{\partial u_2}{\partial x} v_3 + \frac{\partial v_3}{\partial x} u_2 - \left( \frac{\partial u_3}{\partial x} v_2 + \frac{\partial v_2}{\partial x} u_3 \right) + \frac{\partial u_3}{\partial y} v_1 + \frac{\partial v_1}{\partial y} u_3 - \left( \frac{\partial u_1}{\partial y} v_3 + \frac{\partial v_3}{\partial y} u_1 \right) + \frac{\partial u_1}{\partial z} v_2 + \frac{\partial v_2}{\partial z} u_1$$

$$+ \left( \frac{\partial u_2}{\partial z} v_1 + \frac{\partial v_1}{\partial z} u_2 \right)$$

by the chain rule,

$$= \frac{\partial}{\partial x}(u_2 v_3 - u_3 v_2) + \frac{\partial}{\partial y}(u_3 v_1 - u_1 v_3) + \frac{\partial}{\partial z}(u_1 v_2 - u_2 v_1)$$

$$\therefore \nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot (\nabla \times \mathbf{v})$$

1.2. **Part 2.** Show that $\langle \mathbf{u}, \nabla \times \mathbf{v} \rangle = \langle \nabla \times \mathbf{u}, \mathbf{v} \rangle + \iint_{\partial\Omega} \mathbf{w} \cdot dS$ for some $\mathbf{w}$.

From Part 1, $\overline{\mathbf{u}} \cdot (\nabla \times \mathbf{v}) = (\nabla \times \overline{\mathbf{u}}) \cdot \mathbf{v} - \nabla \cdot (\overline{\mathbf{u}} \times \mathbf{v})$

$$\int_\Omega \overline{\mathbf{u}} \cdot (\nabla \times \mathbf{v}) = \int_\Omega (\nabla \times \overline{\mathbf{u}}) \cdot \mathbf{v} - \int_\Omega \nabla \cdot (\overline{\mathbf{u}} \times \mathbf{v})$$

From the definition of the inner product,

$$\langle \mathbf{u}, \nabla \times \mathbf{v} \rangle = \langle \nabla \times \mathbf{u}, \mathbf{v} \rangle - \int_\Omega \nabla \cdot (\overline{\mathbf{u}} \times \mathbf{v})$$

Using the divergence theorem,

$$\langle \mathbf{u}, \nabla \times \mathbf{v} \rangle = \langle \nabla \times \mathbf{u}, \mathbf{v} \rangle + \iint_{\partial\Omega} (\overline{\mathbf{u}} \times \mathbf{v}) \cdot dS.$$

Thus, $\mathbf{w} = \overline{\mathbf{u}} \times \mathbf{v}$, and we are done.

1.3. **Part 3.** Give a possible boundary condition on the vector space such that $\nabla\times$ is self-adjoint with this inner product.

If $\langle \mathbf{u}, \nabla \times \mathbf{v} \rangle = \langle \nabla \times \mathbf{u}, \mathbf{v} \rangle$ then $\nabla\times$ is self-adjoint. For this to be true, $\iint_{\partial\Omega} (\overline{\mathbf{u}} \times \mathbf{v}) \cdot dS = 0$.

Thus, with the boundary condition $\overline{\mathbf{u}} \times \mathbf{v} = 0$, the above equality will be true and $\nabla\times$ will be self-adjoint.

1.4. **Part 4.** Show that $\nabla \times \nabla\times$ is self-adjoint for this inner product under *either* some boundary condition on $\mathbf{u}$ or some boundary condition on the *derivatives* of $\mathbf{u}$. Is it positive or negative definite or semi-definite?

If $\langle \mathbf{u}, \nabla \times \nabla \times \mathbf{v} \rangle = \langle \nabla \times \nabla \times \mathbf{u}, \mathbf{v} \rangle$ then $\nabla \times \nabla\times$ is self-adjoint.

$$\langle \mathbf{u}, \nabla \times \nabla \times \mathbf{v} \rangle = \langle \nabla \times \mathbf{u}, \nabla \times \mathbf{v} \rangle + \iint_{\partial\Omega} (\overline{\mathbf{u}} \times \nabla \times \mathbf{v}) \cdot dS$$

$$\langle \mathbf{u}, \nabla \times \nabla \times \mathbf{v} \rangle = \langle \nabla \times \nabla \times \mathbf{u}, \mathbf{v} \rangle + \iint_{\partial\Omega} (\nabla \times \overline{\mathbf{u}} \times \mathbf{v}) \cdot dS + \iint_{\partial\Omega} (\overline{\mathbf{u}} \times \nabla \times \mathbf{v}) \cdot dS$$

If $\mathbf{u} \times \mathbf{n}|_{\partial\Omega} = 0$ or $\nabla \times \mathbf{u} \times \mathbf{n}|_{\partial\Omega} = 0$, then $\nabla \times \nabla\times$ is self-adjoint.

$\langle \mathbf{u}, \nabla \times \nabla \times \mathbf{u} \rangle = \langle \nabla \times \overline{\mathbf{u}}, \nabla \times \mathbf{u} \rangle = \int_\Omega |\nabla \times \mathbf{u}|^2 \geq 0$, thus it is positive semi-definite.

1.5. **Part 5.** Two of Maxwell's equations are $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and $\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$. Take the curl of both sides of the first equation to obtain a PDE in $\mathbf{E}$ alone. Suppose that $\Omega$ is the interior of a hollow metal container, where the boundary conditions are that $\mathbf{E}$ is perpendicular to the metal at the surface. Combining these facts with the previous parts, explain why one would expect *oscillating* solutions to Maxwell's equations.

$$\nabla \times \nabla \times \mathbf{E} = \nabla \times -\frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \hat{A}\mathbf{E}$$

where $\hat{A} = -c^2 \nabla \times \nabla \times$. Since $\mathbf{E}$ is normal to the surface, Part 4 gives that $\hat{A}$ is self-adjoint and negative semi-definite. Thus, the PDE is a Hyperbolic equation with eigenvalues $\lambda \leq 0$, and thus, oscillating solutions.

## 2. Problem 2

Solve for the 2-D eigenfunctions of $\nabla^2$ in an annular region $\Omega$ that *does not contain the origin*, so that you will need both $J_m$ and $Y_m$ solutions to Bessel's equation. The separation of variables $u(r, \Theta) = \rho(r)\tau(\Theta)$ leads to functions $\tau(\Theta)$ spanned by $\sin(m\Theta)$ and $\cos(m\Theta)$ and functions $\rho(r)$ that satisfy Bessel's equation. That is, the eigenfunctions are of the form:

$$u(r, \Theta) = [\alpha J_m(kr) + \beta Y_m(kr)] \times [A\cos(m\Theta) + B\sin(m\Theta)$$

for arbitrary constants $A$ and $B$, integers $m = 1, 2, \ldots$, and constants $\alpha$, $\beta$ and $k$, which must be determined.
The boundary conditions are Neumann boundary condition $\frac{\partial u}{\partial r} = 0$ at R1 and R2.

2.1. **Part 1.** Using the boundary conditions, write two equations for $\alpha$, $\beta$, and $k$, of the form $E\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 0$ for some $2 \times 2$ matrix $E$. This only has a solution when the determinant is 0. Use this fact to obtain a single equation for $k$ of the form $f_m(k) = 0$ for some function $f_m$ that depends on $m$. In terms of $k$, write down a possible expression for $\alpha$, $\beta$.

$$\frac{\partial u}{\partial r} = \alpha J'_m(kr) + \beta Y'_m(kr) = 0|_{r=R_1,R_2}$$

$$E\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \begin{bmatrix}J'_m(kR_1) & Y'_m(kR_1) \\ J'_m(kR_2) & Y'_m(kR_2)\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \mathbf{0}$$

$$f_m(k) = \det E = J'_m(kR_1)Y'_m(kR_2) - J'_m(kR_2)Y'_m(kR_1) = 0$$

$$\frac{\partial u}{\partial r}|_{r=R_1} = \frac{\partial u}{\partial r}|_{r=R_2} = 0$$

$$\alpha J'_m(kR_1) + \beta Y'_m(kR_1) = \alpha J'_m(kR_2) + \beta Y'_m(kR_2) = 0$$

$$\beta = \alpha \frac{J'_m(kR_1)}{Y'_m(kR_1)} = \alpha \frac{J'_m(kR_2)}{Y'_m(kR_2)}$$

2.2. **Part 2.** Assuming $R1 = 1, R2 = 2$, plot $f_m(k)$ vs $k \in [0, 20]$ for $m = 0, 1, 2$.
*see notebook.*

2.3. **Part 3.** For $m = 0$, find the first three (smallest $k > 0$) solutions to $f_0(k) = 0$.
*see notebook.*

2.4. **Part 4.** Since $\nabla^2$ is self-adjoint, the eigenfunctions must be orthogonal. Check that the solutions from Part 3 are orthogonal.
*see notebook.*

2.5. **Part 5.** Let the operator $\hat{A}$ now be $c(r)\nabla^2$ with $c(r) = 2$ for $r < R_1$ and $c(r) = 1$ for $r \geq R_1$. Impose Dirichlet boundary conditions $u(R_2) = 0$. What is the form of the eigenfunctions? If we solve for eigenfunctions $\hat{A}u = \lambda u$, with $u$ finite everywhere, what conditions must $u$ satisfy at $r = R_1$ for $\hat{A}u$ to be well-defined and finite? Write down a condition $f_m(k) = 0$ that must be satisfied in order for the above equation to have a solution. The roots of this function give the eigenvalues.

The eigenfunctions are of the form:

$$u(r, \theta) = [A\cos(m\theta) + B\sin(m\theta)] \times \begin{cases} \alpha J_m(k_1 r) & r < R_1 \\ \beta J_m(k_2 r) + \gamma Y_m(k_2 r) & r > R_1 \end{cases}$$

with $k_1 = k_2/\sqrt{2}$.

If $\hat{A}$ is to be well defined and finite, then

$$\alpha J_m(kR_1/\sqrt{2}) = \beta J_m(kR_1) + \gamma Y_m(kR_1)$$

and

$$\alpha J'_m(kR_1/\sqrt{2}) = \beta J'_m(kR_1) + \gamma Y'_m(kR_1)$$

paired with the Dirichlet boundary conditions, this gives the system,

$$\begin{bmatrix} -J_m(kR_1/\sqrt{2}) & J_m(kR_1) & Y_m(kR_1) \\ -J'_m(kR_1/\sqrt{2}) & J'_m(kR_1) & Y_m(kR_1) \\ 0 & J_m(kR_2) & Y_m(kR_2) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = E_m(k) \begin{bmatrix} \alpha \\ \beta \\ \gamma] \end{bmatrix} = \mathbf{0}$$

$f_m(k)$ is thus $\det E_m(k)$.

## 3. Problem 3

The Bessel functions $u(x) = J_m(kx)$, solve the eigenproblem:

$$\hat{A}u = u'' + \frac{u'}{r} - \frac{m^2}{r^2}u = -k^2 = \lambda u$$

on $[0, R]$ where $u(R) = 0$ and $u(0) = 0$ for $m > 0$.

### 3.1. Part 1.

Show that $\hat{A}$ is of the form of a Sturm-Louville operator and is therefor self-adjoint for a particular inner product.

### 3.2. Part 2.

Show that $\hat{A}$ is negative definite.

### 3.3. Part 3.

Write a center-difference discretization of the operator $\hat{A}$ for $u_n = u(n\Delta x)$ with $m = 1, \ldots, R\Delta x = \frac{R}{N+1}$.

### 3.4. Part 4.

In Julia, for the matrix approximation $A$ of $\hat{A}$ for $m = 1$ (with $N = 100$, $R = 1$). Compare its smallest-magnitude eigenfunction to $J_1(k_{1,1}r/R)$ where $k_{1,1}$ is the first root of $J_1$.