PDE: ASSIGNMENT 2

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1. Problem 1

Consider the inner product defined as $< x, y >= x^*By$, where vectors $x, y \in \mathbb{C}^n$, and B is positive definite Hermitian $n \times n$ matrix.

1.1. **Part 1.** Show that $< x, y >= \overline{< y, x >}$.

$\overline{< y, x >} = \overline{y^*Bx} = \overline{Bx}y = x^*B^*y.$

Since B is Hermitian, $B^* = B$, so $\overline{< y, x >} = x^*By = < x, y >$.

1.2. **Part 2.** If M is an arbitrary $n \times n$ matrix, define the adjoint $M^\dagger$ by $< x, My >=< M^\dagger x, y >$. Give an explicit formula for $M^\dagger$ in terms of M and B.

$$
\begin{gathered}
< x, My >=< M^\dagger x, y > \\
x^*BMy = (M^\dagger x)^*By \\
x^*BM = (M^\dagger x)^*B \\
x^*BMB^{-1} = (M^\dagger x)^* = x^*M^{\dagger *} \\
BMB^{-1} = M^{\dagger *} \\
M^\dagger = (BMB^{-1})^*
\end{gathered}
$$

1.3. **Part 3.** Show that if for some $A = A^*$, and $M = B^{-1}A$, then $M = M^\dagger$.

$$
\begin{gathered}
M^\dagger = (BMB^{-1})^* \\
M^\dagger = (BB^{-1}AB^{-1})^* \\
M^\dagger = (AB^{-1})^* \\
M^\dagger = B^{-1*}A^*
\end{gathered}
$$

Since B is Hermitian and $A = A^*$,

$$M^\dagger = B^{-1}A = M$$

2. Problem 2

Consider the following system:

$$\hat{A}u = \frac{d}{dx}\left[c\frac{du}{dx}\right]$$

2.1. **Part 1.** Using center difference operations, approximate $\hat{A}u$ at $m\Delta x$.

Let u_m be the value of u at point m, and c_m be the value of c at point m. u' can then be approximated at point $m+0.5$ with center difference by

$$u'_m \approx \frac{u_{m+0.5} - u_{m-0.5}}{\Delta x}$$

where Δx is the difference between m_n and m_{n-1} The derivative of $c\frac{du}{dx}$ can then be approximated by

$$\frac{d}{dx}\left[c_m\frac{du}{dx}\right] \approx \frac{c_{m+0.5}u'_{m+0.5} - c_{m-0.5}u'_{m-0.5}}{\Delta x}$$

and by substituting $u'(m)$ in

$$\frac{d}{dx}\left[c_m\frac{du}{dx}\right] \approx \frac{c_{m+0.5}\left(\frac{u_{m+1}-u_m}{\Delta x}\right) - c_{m-0.5}\left(\frac{u_m-u_{m-1}}{\Delta x}\right)}{\Delta x}$$

2.2. **Part 2.** Show that the finite-difference estimation from Part 1 corresponds to approximating $\hat{A}u$ by $A\mathbf{u}$, where $\mathbf{u}$ is the column vector of M points u_m and A is the real-symmetric matrix $A = -D^TCD$.

$$D = \frac{1}{\Delta x}\begin{bmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \ddots & 0 \\ 0 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} c_{0.5} & 0 & \dots & 0 \\ 0 & c_{1.5} & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & c_{M+0.5} \end{bmatrix}$$

$$A\mathbf{u} = -D^TCD = \frac{1}{\Delta x^2}\begin{bmatrix} -c_{0.5} - c_{1.5} & c_{1.5} & 0 & \dots \\ c_{1.5} & -c_{1.5} - c_{2.5} & c_{2.5} & 0 \\ 0 & c_{2.5} & \ddots & \ddots \\ \vdots & 0 & \ddots & \ddots \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{M+1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{c_{1.5}(u_2-u_1)-c_{0.5}(u_1-u_0)}{\Delta x^2} \\ \frac{c_{2.5}(u_3-u_2)-c_{1.5}(u_2-u_1)}{\Delta x^2} \\ \vdots \\ \frac{c_{m+0.5}(u_{m+1}-u_m)-c_{m-0.5}(u_m-u_{m-1})}{\Delta x^2} \end{bmatrix}$$

2.3. **Part 3.** Let $c(x) = e^{3x}$. Use the above methods to attain the eigenvalues and eigenvectors. Plot the eigenvectors for the four smallest-magnitude eigenvalues. Verify that the first two eigenfunctions are orthogonal. Verify that you are getting second order convergence of the eigenvalues. ($L = 1$, $M = 100$)

see notebook

3. Problem 3

Consider a metal bar with length L, cross sectional area a, and varying temperature T along the rod. The rod is conceptually divided into N pieces of length $\Delta x = L/N$. Each piece has a uniform temperature T_n, giving a vector $\mathbf{T}$ of N temperatures. The rate at which heat flows from piece n to piece $n+1$ is given by $q = \frac{\kappa a}{\Delta x}(T_n - T_{n+1})$ (κ is the thermal conductivity of the rod). If an amount of heat ΔQ flows into a piece, the temperature changes by $\Delta T = \Delta Q/(c\rho a \Delta x)$ where c is the specific heat capacity, and ρ is the density of the metal. Assume the rod is ideally insulated.

3.1. Part 1.

Show that $\frac{dT_n}{dt} = \alpha(T_{n+1} - T_n) + \alpha(T_{n-1} - T_n)$. (This is Newton's Law of Cooling).

$$\frac{\Delta T_n}{\Delta t} = \frac{1}{c\rho a \Delta x}\frac{\Delta Q_n}{\Delta t}$$

$\frac{\Delta Q_n}{\Delta t}$ is the rate of heat flow into piece n, which is the sum of the heat flow from it's neighbors.

$$\frac{dT_n}{dt} = \frac{1}{c\rho a \Delta x}\left[\frac{\kappa a}{\Delta x}(T_{n+1} - T_n) + \frac{\kappa a}{\Delta x}(T_{n-1} - T_n)\right]$$

$$\frac{dT_n}{dt} = \frac{\kappa}{c\rho \Delta x^2}\left[(T_{n+1} - T_n) + (T_{n-1} - T_n)\right]$$

which matches Newton's Law of Cooling, with $\alpha = \frac{\kappa}{c\rho\Delta x}$. At the end points,

$$\frac{dT_1}{dt} = \frac{\kappa}{c\rho\Delta x^2}(T_2 - T_1) \quad , \quad \frac{dT_N}{dt} = \frac{\kappa}{c\rho\Delta x^2}(T_{N-1} - T_N)$$

3.2. Part 2.

Write the equation from Part 1 in matrix form: $\frac{d\mathbf{T}}{dt} = A\mathbf{T}$, for some matrix A.

$$\frac{d\mathbf{T}}{dt} = A\mathbf{T} = \frac{\kappa}{c\rho\Delta x^2}\begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{N-1} \\ T_N \end{bmatrix}$$

3.3. **Part 3.** Let $T(x,t)$ be the temperature along the rod. Suppose $T_n(t) = T([n-0.5]\Delta x, t)$ (the temperature at the center of the n^{th} piece). Take the limit $N \to \infty$ (with L fixed), and derive the PDE $\frac{\partial T}{\partial t} = \hat{A}T$. What is $\hat{A}$ (ignore the ends)?

As $N \to \infty$, Δx can be thought of as ∂x, and $T_{n+1} - T_n$ can be thought of as $\partial T(n,t)$.

$(T_{n+1} - T_n) + (T_{n-1} - T_n)$ can also be expressed as $(T_{n+1} - T_n) - (T_n - T_{n-1})$, which as $N \to \infty$ is $\partial^2 T$.

Thus

$$\lim_{N\to\infty} \left\{ \frac{dT_n}{dt} = \frac{\kappa}{c\rho\Delta x^2} \left[(T_{n+1} - T_n) + (T_{n-1} - T_n) \right] \right\}$$

becomes

$$\frac{\partial T}{\partial t} = \frac{\kappa}{c\rho} \frac{\partial^2 T}{\partial x^2}$$

So, ignoring the ends,

$$\hat{A} = \frac{\kappa}{c\rho} \frac{\partial^2}{\partial x^2}$$

3.4. **Part 4.** What are the boundary conditions on $T(x,t)$ at $x = 0$ and L? Check that if you go backwards and form a center-difference approximation of $\hat{A}$ with these boundary conditions, you will recover matrix A.

The boundary condition is the $\frac{\partial T}{\partial x} = 0$ at $x = 0, L$ since there is 0 heat flow at the ends.

The center-difference approximation of $\frac{\partial T}{\partial x}|_{n\Delta x} \approx T'_{n+0.5} = \frac{T_{n+1} - T_n}{\Delta x}$

The center difference approximation of $T''_n = \frac{T'_{n+0.5} - T'_{n-0.5}}{\Delta x} = \frac{T_{n+1} - T_n - (T_n - T_{n-1})}{\Delta x^2}$

This is equivalent to $\frac{T_{n+1} - 2T_n + T_{n-1}}{\Delta x^2}$, which is the same expression from Part 1 (with $\frac{\kappa}{c\rho} = 0$), and will thus result in the same A.

3.5. **Part 5.** How does $\hat{A}$ change in the $N \to \infty$ limit if the conductivity is a function $\kappa(x)$ of x?

$$\frac{\partial T}{\partial t} = \frac{1}{c\rho} \frac{\partial \kappa \partial T}{\partial x^2}$$

Thus,

$$A = \frac{1}{c\rho} \frac{\partial}{\partial x} \kappa \frac{\partial}{\partial x}$$

3.6. **Part 6.** Suppose that instead of a thin bar (1D), you have a metal plate (2D) with a temperature $T(x, y, t)$ and constant conductivity κ. If you go through the steps above and divide it into $N \times N$ squares of size $\Delta x \times \Delta y$, what PDE do you get for T in the limit $N \to \infty$.

The PDE will be the sum of the partial derivatives in the x and y directions.

$$\frac{dT_{m,n}}{dt} = \frac{\kappa}{c\rho}\left[\frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{\Delta x^2} + \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{\Delta y^2}\right]$$

And as $N \to \infty$,

$$\frac{\partial T}{\partial t} = \hat{A}T = \frac{1}{c\rho} \bigtriangledown \cdot \kappa \bigtriangledown T$$