PDE: ASSIGNMENT 4

MARK DITSWORTH

1. Problem 1

Consider the operator $\hat{A} = -c(x)\nabla^2$ in a 2d region $\Omega \subset \mathbb{R}^2$ with Dirichlet boundaries where $c(x) > 0$. Suppose the eigenfunctions of $\hat{A}$ are $u_n(x)$ with eigenvalues λ_n numbered in order $(\lambda_1 < \lambda_2 < \dots)$.Let $G(x, x')$ be the Green's function of $\hat{A}$.

1.1. **Part 1.**

If $f(x) = \sum_n \alpha_n u_n(x)$, show α_n in terms of f and u_n. Then find $\int_\Omega G(x, x')f(x')d^2x'$ in terms of α_n and u_n.

1.2. **Part 2.**

For any possible value of $u(x)$, find

$$\frac{\int_\Omega \int_\Omega \frac{1}{c(x)} \overline{u(x)} G(x, x')u(x')d^2x d^2x'}{\int_\Omega \frac{|u(x'')|^2}{c(x'')} d^2x''}$$

2. Problem 2

Solve the Laplacian eigenproblem $-\nabla^2 u = \lambda u$ in a 2d radius-1 cylinder, $r <= 1$ with Dirichlet boundary conditions $u|_{r=1,\Omega} = 0$ by "brute force" in Julia and compare to the analytical Bessel solutions.

3. Problem 3

Recall that the displacement $u(x, t)$ of a string (with fixed ends $u(0, t) = u(L, t) = 0$) satisfies the wave equation $\frac{\partial^2 u}{\partial x^2} + f(x, t) = \frac{\partial^2 u}{\partial t^2}$, where $f(x, t)$ is an external force density (pressure) on the string.