PROBLEM SET 1

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1. Linear Algebra

1.1. **Problem 1.** Show that if $M \in \mathbb{R}^{n \times n}$ is a symmetric matrix and $d \leq n$, then $U \in \mathbb{R}^{n \times d}$ and $U^T U = I_{d \times d}$,

$$\max_{d} \operatorname{Tr}\left(U^T M U\right) = \sum_{k=1}^{d} \lambda_k^{(+)}$$

where $\lambda_k^{(+)}$ is the kth largest eigenvalue of M.

Since M is symmetric, we have the singular value decomposition (SVD)

$$M = UD^2U^T$$

where D is a diagonal matrix with values corresponding to the eigenvalues of M, in order of magnitude. Since $U^T U = I \Rightarrow U^T = U^{-1}$, we have

$$U^{-1}M(U^T)^{-1} = D^2$$

$$U^T M U = D^2$$

Therefore, $\operatorname{Tr}\left(U^T M U\right) = \operatorname{Tr}\left(D^2\right) = \sum_{k=1}^{n} \lambda_k^{(+)}$

In order to maximize this, we can set d such that $\lambda_{d+1}^{(+)}$ is the first occurrence of a complex eigenvalue in the $\lambda_k^{(+)}$ sequence who's magnitude is negative. If no such eigenvector exists, $d = n$.

2. Estimators

2.1. **Problem 2.** Given $x_1, \ldots x_n$ i.i.d samples from a distribution X with mean μ and covariance Σ, show that

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n} x_k \qquad , \qquad \Sigma_n = \frac{1}{n-1}(x_k - \mu)(x_k - \mu)^T$$

are unbiased estimators for μ and Σ. (Show that $\mathbf{E}[\mu_n] = \mu$ and $\mathbf{E}[\Sigma_n] = \Sigma$)

$$\mathbf{E}[\mu_n] = \mathbf{E}\left[\frac{x_1 + \cdots + x_n}{n}\right] = \frac{\mathbf{E}[x_1] + \cdots + \mathbf{E}[x_n]}{n} = \frac{n\mu}{n} = \mu$$

$$\mathbf{E}[\Sigma_n] = \frac{1}{n-1}\sum_{i=1}^{n}\mathbf{E}\left[(x_i - \mu_n)(x_i - \mu_n)^T\right] = \frac{1}{n-1}\sum_{i=1}^{n}\mathbf{E}\left[((x_i - \mu) - (\mu_n - \mu))((x_i - \mu) - (\mu_n - \mu))^T\right]$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\mathbf{E}\left[(x_i-\mu)(x_i-\mu)^T\right]+\mathbf{E}\left[(\mu_n-\mu)(\mu_n-\mu)^T\right]-\mathbf{E}\left[(x_i-\mu)(\mu_n-\mu)^T\right]-\mathbf{E}\left[(\mu_n-\mu)(x_i-\mu)^T\right]$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\mathbf{E}\left[(x_i-\mu)(x_i-\mu)^T\right] = \sum_{i=1}^{n}\mathbf{E}\left[\frac{(x_i-\mu)(x_i-\mu)^T}{n-1}\right] = \mathbf{E}\left[\sum_{i=1}^{n}\frac{(x_i-\mu)(x_i-\mu)^T}{n-1}\right] = \Sigma$$

3. Random Matrices

3.1. **Problem 3.** Let $W \in \mathbb{R}^{n\times n}$ be a Wigner Matrix, a symmetric random matrix whose diagonal and upper-diagonal entries are independent $W_{ii} \sim \mathbb{N}(0,2)$ and for $i < j$, $W_{ij} \sim N(0,1)$. Show that the distribution of the eigenvalues of $\frac{1}{\sqrt{n}}W$ converge to the semi-circle law of support $[-2,2]$

$$dSC(x) = \frac{1}{2\pi}\sqrt{4-x^2}1_{[-1,1]}(x)$$

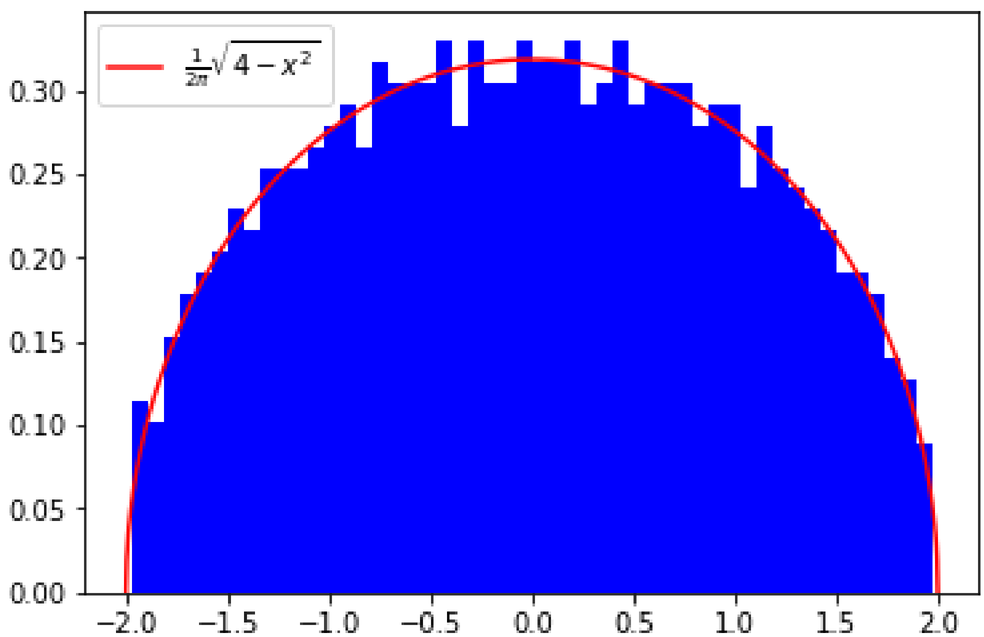


Figure 1. Histogram of eigenvalues for the $\frac{1}{\sqrt{n}}W$ ($n = 500$)

3.2. **Problem 4.** Use Slepian's Comparison Lemma to show that for a Wigner matrix $W \in \mathbb{R}^{n\times n}$, where $W_{ij|i\neq j} \sim N(0,1)$ and $W_{ii} \sim N(0,2)$.

$$\mathbf{E}\left[\lambda_{max}(W)\right] \leq 2\sqrt{n}$$

$$\lambda_{max}(W) = \max_v v^T W v \Rightarrow \mathbf{E}\left[\lambda_{max}(W)\right] = \mathbf{E}\left[\max_v v^t W v\right]$$

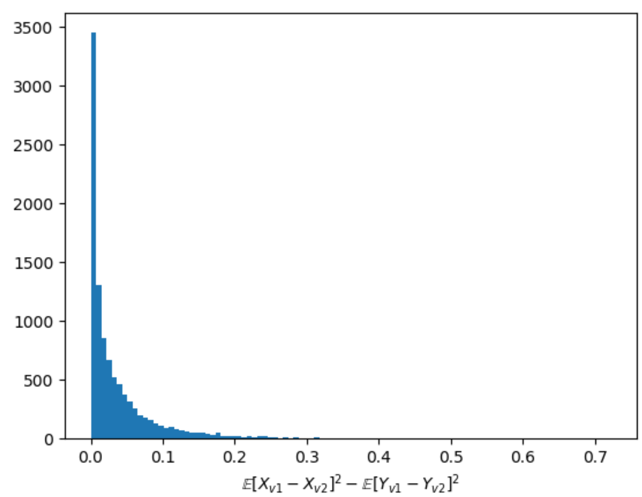
$$\text{Let } Y_v \doteqdot \max_v v^T W v$$

$$\text{Define } X_v = v^T g \qquad g \sim N(0, I_{n\times n})$$

Slepian's Comparison Lemma states that for random variables X_v and Y_v, if $\mathbf{E}[X_v] = \mathbf{E}[Y_v] = 0$, and $\forall v1, v2 \in V$ s.t. $v1 \neq v2$ $\mathbf{E}[X_{v1} - X_{v2}]^2 \geq \mathbf{E}[Y_{v1} - Y_{v2}]^2$, then

$$\mathbf{E}\left[\max_v Y_v\right] \leq \mathbf{E}\left[\max_v X_v\right]$$

Since X_v and Y_v are both related to Gaussians about 0, the first condition is satisfied. Monte Carlo simulation of $\mathbf{E}[2X_{v1} - 2X_{v2}]^2 - \mathbf{E}[Y_{v1} - Y_{v2}]^2$ are shown below; clearly the value is ≥ 0, thus satisfying the second condition. Therefore, it



is sufficient to say that

$$\mathbf{E}\left[\max_v Y_v\right] \leq \mathbf{E}\left[\max_v 2X_v\right]$$

By Jensen's inequality, we have that

$$\mathbf{E}\left[\max_v X_v\right]^2 \leq \mathbf{E}\left[\left(\max_v X_v\right)^2\right]$$

$$\mathbf{E}\left[\max_v v^T g\right]^2 \leq \mathbf{E}\left[\left(\max_v v^T g\right)^2\right]$$

Since $||v||_2 = 1$, $v^T g$ will be maximized when each element of v is equal to $\frac{1}{\sqrt{n}}$. Thus, $max_v v^T g = \frac{1}{\sqrt{n}} \sum_{i=1}^n g_i$. Since g is standard Gaussian, $\sum_{i=1}^n g \leq n$, thus $max_v v^T g \leq \sqrt{n}$. Therefore, we have

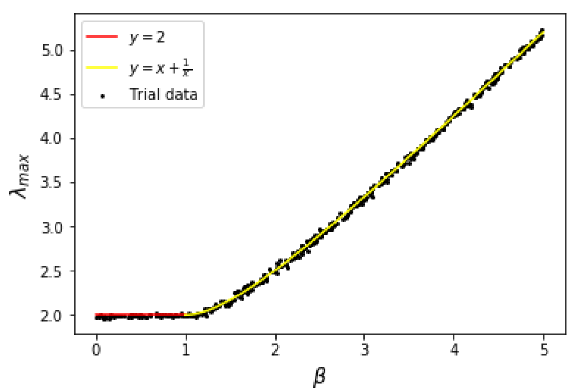
$$\mathbf{E}\left[\max_v v^T g\right]^2 \leq n$$
$$\mathbf{E}\left[\max_v v^T g\right] \leq \sqrt{n}$$

and finally

$$\mathbf{E}[\max_v Y_v] = \mathbf{E}[\lambda_{max}(W)] \leq 2\sqrt{n}$$

3.3. **Problem 5.** Consider the matrix $M = \frac{1}{\sqrt{n}} W + \beta v v^T$ for $||v||_2 = 1$ and W is a standard Gaussian Wigner Matrix. If $v = e_1$, this is a rank 1 perturbation of a Wigner matrix. Derive the limit of the largest eigenvalue.

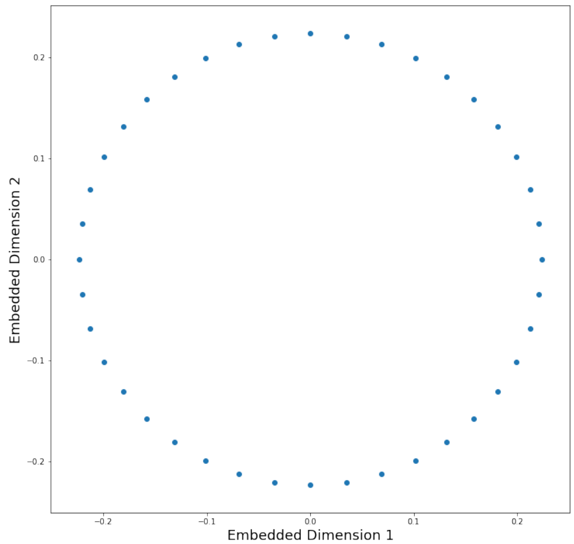
Rather than performing a rigorous proof, the limit can be derived empirically by generating 500 samples of M, with n set to a reasonably large number (1000), and β selected randomly each time from a uniform distribution $[0, 5]$.



We can see that when $\beta \leq 1$, $\lim_{n\to\infty} \lambda_{max} = 2$, and when $\beta > 1$, $\lim_{n\to\infty} \lambda_{max} = \beta + \frac{1}{\beta}$

4. Diffusion Maps

4.1. **Problem 6.** Derive the 2-D diffusion map embedding for the n-node ring graph. If the eigenvalues are complex, try creating real ones using multiplicity of eigenvalues. Is it a reasonable embedding of this graph in two dimensions?


0.2
0.1
0.0
−0.1
−0.2
Embedded Dimension 2
−0.2
−0.1
0.0
0.1
0.2
Embedded Dimension 1