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you can find the original Mathematica-notebook under https://github.com/marketdesignresearch/BO-CA/blob/main/Theory/MVNN_initialization7_2.nb.

You can execute this notebook and vary the parameters in the interactive plot for free without any license with <https://www.wolfram.com/player/> or you can manipulate and the extend the code with a Mathematica license.

```
In[1]:= EofW[A_, B_, p_] := (1 - p) * A / 2 + p * B / 2
```

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In[2]:= VofW[A_, B_, p_] := (1 - p) * A ^ 2 / 3 + p * B ^ 2 / 3 - EofW[A, B, p] ^ 2
```

We solve the original problem (w.l.o.g. we assume $A \leq B$):

```
In[3]:= Solve[EofW[A, B, p] == M / d && VofW[A, B, p] == V / d && 1 - p ≥ 0 && B ≥ A,
  {A, B, p}, NonNegativeReals]
```

 **Solve:** Equations may not give solutions for all "solve" variables.

$$\left\{ \left\{ A \rightarrow \frac{2 M - \frac{B d (-M^2 + 3 d V)}{B^2 d^2 - 4 B d M + 3 M^2 + 3 d V}}{d - \frac{d (-M^2 + 3 d V)}{B^2 d^2 - 4 B d M + 3 M^2 + 3 d V}} \text{ if } V > 0 \text{ \&\& } M > 0 \text{ \&\& } d > \frac{M^2}{3 V} \text{ \&\& } B \geq \frac{3 M^2 + 3 d V}{2 d M}, \right. \right.$$

$$\left. p \rightarrow \frac{-M^2 + 3 d V}{B^2 d^2 - 4 B d M + 3 M^2 + 3 d V} \text{ if } V > 0 \text{ \&\& } M > 0 \text{ \&\& } d > \frac{M^2}{3 V} \text{ \&\& } B \geq \frac{3 M^2 + 3 d V}{2 d M} \right\}$$

We see that a solution only exists if $d > \frac{M^2}{3 V}$. Further we see that, if a solution exists, we have infinitely

many solutions for B (every $B \geq \frac{3 M^2 + 3 d V}{2 d M}$ is a possible choice). For every possible choice of B there a unique solution for A and p given above.

In the next step we see that if we additionally want $A > 0$ to hold, we just need to choose a value for B which is at least by an arbitrarily small but positive epsilon larger than the minimal possible choice of B.

We see this by observing that when we add $A > 0$ as an constraint, the only difference in the solution is that B is given by a strict inequality $B > \dots$ instead of $B \geq \dots$:

```
In[4]:= Solve[EofW[A, B, p] == M / d && VofW[A, B, p] == V / d && 1 - p ≥ 0 && B ≥ A && A > 0,
  {A, B, p}, NonNegativeReals]
```

 **Solve:** Equations may not give solutions for all "solve" variables.

$$\left\{ \left\{ A \rightarrow \frac{2M - \frac{Bd(-M^2+3dV)}{B^2d^2-4BdM+3M^2+3dV}}{d - \frac{d(-M^2+3dV)}{B^2d^2-4BdM+3M^2+3dV}} \text{ if } V > 0 \ \&\& \ M > 0 \ \&\& \ d > \frac{M^2}{3V} \ \&\& \ B > \frac{3M^2+3dV}{2dM}, \right. \right.$$

$$\left. p \rightarrow \frac{-M^2+3dV}{B^2d^2-4BdM+3M^2+3dV} \text{ if } V > 0 \ \&\& \ M > 0 \ \&\& \ d > \frac{M^2}{3V} \ \&\& \ B > \frac{3M^2+3dV}{2dM} \right\}$$

Our choice of B is:

$$\text{In[5]:= Bchoice} := \text{If}\left[d > \frac{M^2}{3V}, \text{Max}\left[\frac{3M^2+3dV}{2Md} + \text{epsilon}/d, \text{Binit}\right], 2*M/d\right]$$

, where the first case is motivated by the arguments above and the second case guarantees the correct expectation as we will prove later.

This uniquely defines p and A as in the paper by plugging B into the formulas for A and p above (in the case a solution exists). In the other cases we also make specific choices.

$$\text{In[6]:= pchoice} := \text{With}\left[\{B = \text{Bchoice}\}, \text{If}\left[d > \frac{M^2}{3V}, \frac{-M^2+3dV}{B^2d^2-4BdM+3M^2+3dV}, 1\right]\right]$$

We see that in the formula for A given above the expression for p appears, so we can use an easier expression for A:

$$\text{In[7]:= Achoice} := \text{With}\left[\{B = \text{Bchoice}, p = \text{pchoice}\}, \text{If}\left[d > \frac{M^2}{3V}, \frac{(2M-Bdp)}{d(1-p)}, 0\right]\right]$$

Now we can check if our chosen solution fulfills all the desired properties.

Proof of item 1:

$$\text{In[8]:= Simplify[EofW[Achoice, Bchoice, pchoice]]}$$

$$\text{Out[8]= } \frac{M}{d}$$

Proof of item 2 and 3:

We use Mathematica for this proof since the terms can get quite complicated:

$$\text{In[9]:= VofW[Achoice, Bchoice, pchoice]}$$

$$\text{Out[9]= } \frac{1}{3} \left(1 - \text{If}\left[d > \frac{M^2}{3V}, (-M^2+3dV) / \left(\text{If}\left[d > \frac{M^2}{3V}, \text{Max}\left[\frac{3M^2+3dV}{2Md} + \frac{\text{epsilon}}{d}, \text{Binit}\right], \frac{2M}{d}\right]^2 d^2 - \right. \right.$$

$$\begin{aligned}
& d^2 - 4 \operatorname{If}\left[d > \frac{M^2}{3V}, \operatorname{Max}\left[\frac{3M^2 + 3dV}{2Md} + \frac{\text{epsilon}}{d}, \text{Binit}\right], \frac{2M}{d}\right] \\
& dM + 3M^2 + 3dV \Bigg), 1] \Bigg), 0] + \\
& \frac{1}{2} \operatorname{If}\left[d > \frac{M^2}{3V}, (-M^2 + 3dV) / \left(\operatorname{If}\left[d > \frac{M^2}{3V}, \operatorname{Max}\left[\frac{3M^2 + 3dV}{2Md} + \frac{\text{epsilon}}{d}, \text{Binit}\right], \frac{2M}{d}\right]^2 d^2 - \right. \right. \\
& \left. \left. 4 \operatorname{If}\left[d > \frac{M^2}{3V}, \operatorname{Max}\left[\frac{3M^2 + 3dV}{2Md} + \frac{\text{epsilon}}{d}, \text{Binit}\right], \frac{2M}{d}\right] dM + 3M^2 + 3dV \right), 1] \times \right. \\
& \left. \operatorname{If}\left[d > \frac{M^2}{3V}, \operatorname{Max}\left[\frac{3M^2 + 3dV}{2Md} + \frac{\text{epsilon}}{d}, \text{Binit}\right], \frac{2M}{d}\right] \right)^2
\end{aligned}$$

But Mathematica can easily simplify complicated terms such as this one:

In[10]:= **Simplify[VofW[Achoice, Bchoice, pchoice]]**

$$\text{Out[10]= } \begin{cases} \frac{M^2}{3d^2} & 3dV^2 \leq M^2V \\ \frac{V}{d} & \text{True} \end{cases}$$

This is even true for $V \geq 0$, but it can be slightly further simplified if $V > 0$:

In[11]:= **Assuming[V > 0, Simplify[VofW[Achoice, Bchoice, pchoice]]]**

$$\text{Out[11]= } \begin{cases} \frac{M^2}{3d^2} & 3dV \leq M^2 \\ \frac{V}{d} & 3dV > M^2 \\ \text{Indeterminate} & \text{True} \end{cases}$$

We see that the $\frac{V}{d} \leq \text{VofW}$ holds always true (for $d > 0$) and $\frac{V}{d} = \text{VofW}$ if $d > \frac{M^2}{3V}$ or we can check it in the next line:

In[12]:= **Reduce[\frac{V}{d} \leq Simplify[VofW[Achoice, Bchoice, pchoice]] && V > 0 && d > 0 && M > 0]**

Out[12]= $V > 0 \ \&\& \ M > 0 \ \&\& \ d > 0$

Proof of item 4:

In[13]:= **Reduce[0 < Simplify[Achoice, d > \frac{M^2}{3V}] && V > 0 && d > 0 && M > 0 && epsilon > 0 && d > \frac{M^2}{3V}]**

Out[13]= $\text{Binit} \in \mathbb{R} \ \&\& \ V > 0 \ \&\& \ M > 0 \ \&\& \ d > \frac{M^2}{3V} \ \&\& \ \text{epsilon} > 0$

Proof of item 5:

$$\text{In[14]} := \text{Reduce}\left[0 == \text{Simplify}\left[\text{Achoice}, d > \frac{M^2}{3V}\right] \&\& V > 0 \&\& d > 0 \&\&\right.$$

$$\left. M > 0 \&\& \text{epsilon} == 0 \&\& \text{Binit} \leq \text{Simplify}\left[\text{Bchoice}, d > \frac{M^2}{3V}\right] \&\& d > \frac{M^2}{3V}\right]$$

$$\text{Out[14]} = V > 0 \&\& M > 0 \&\& d > \frac{M^2}{3V} \&\& \text{Binit} \leq \frac{3M^2 + 3dV}{2dM} \&\& \text{epsilon} == 0$$

(In the case of $d \leq \frac{M^2}{3V}$, Achoice is zero anyway. Note that $\text{Binit} < \text{Bchoice}$ implies $\text{Binit} \leq \frac{3M^2 + 3dV}{2dM}$ for $\text{epsilon} == 0$.)

Proof of item 6:

$$\text{In[15]} := \text{Reduce}\left[0 \leq \text{Simplify}\left[\text{Achoice}, d > \frac{M^2}{3V}\right] \&\&\right.$$

$$\left. V > 0 \&\& d > 0 \&\& M > 0 \&\& \text{epsilon} \geq 0 \&\& \text{Binit} \geq 0 \&\& d > \frac{M^2}{3V}\right]$$

$$\text{Out[15]} = V > 0 \&\& M > 0 \&\& d > \frac{M^2}{3V} \&\& \text{Binit} \geq 0 \&\& \text{epsilon} \geq 0$$

$$\text{In[16]} := \text{Reduce}\left[0 \leq \text{Simplify}\left[\text{Bchoice}, d > \frac{M^2}{3V}\right] \&\&\right.$$

$$\left. V > 0 \&\& d > 0 \&\& M > 0 \&\& \text{epsilon} \geq 0 \&\& \text{Binit} \geq 0 \&\& d > \frac{M^2}{3V}\right]$$

$$\text{Out[16]} = V > 0 \&\& M > 0 \&\& d > \frac{M^2}{3V} \&\& \text{epsilon} \geq 0 \&\& \text{Binit} \geq 0$$

Proof of item 7: $(A \leq B)$

$$\text{In[17]} := \text{Reduce}\left[\text{Simplify}\left[A_{\text{choice}}, d > \frac{M^2}{3V}\right] \leq \text{Simplify}\left[B_{\text{choice}}, d > \frac{M^2}{3V}\right] \&\&\right.$$

$$\left. V > 0 \&\& d > 0 \&\& M > 0 \&\& \text{epsilon} \geq 0 \&\& B_{\text{init}} \geq 0 \&\& d > \frac{M^2}{3V}\right]$$

$$\text{Out[17]} = V > 0 \&\& M > 0 \&\& d > \frac{M^2}{3V} \&\& \text{epsilon} \geq 0 \&\& B_{\text{init}} \geq 0$$

Proof of item 8:

$$\text{In[18]} := \text{Reduce}\left[\frac{3V}{2M} \leq \text{Simplify}\left[B_{\text{choice}}, d > \frac{M^2}{3V}\right] \&\&\right.$$

$$\left. V > 0 \&\& d > 0 \&\& M > 0 \&\& \text{epsilon} \geq 0 \&\& B_{\text{init}} \geq 0 \&\& d > \frac{M^2}{3V}\right]$$

$$\text{Out[18]} = V > 0 \&\& M > 0 \&\& d > \frac{M^2}{3V} \&\& B_{\text{init}} \geq 0 \&\& \text{epsilon} \geq 0$$

$$\text{In[19]} := \text{Reduce}\left[\frac{3V}{2M} \leq \text{Simplify}\left[B_{\text{choice}}, d \leq \frac{M^2}{3V}\right] \&\&\right.$$

$$\left. V > 0 \&\& d > 0 \&\& M > 0 \&\& \text{epsilon} \geq 0 \&\& B_{\text{init}} \geq 0 \&\& d \leq \frac{M^2}{3V}\right]$$

$$\text{Out[19]} = \text{epsilon} \geq 0 \&\& B_{\text{init}} \geq 0 \&\& V > 0 \&\& M > 0 \&\& 0 < d \leq \frac{M^2}{3V}$$

Proof of item 9:

$$\text{In[20]} := \text{Assuming}\left[V > 0 \&\& M > 0 \&\& \text{epsilon} \geq 0 \&\& B_{\text{init}} \geq 0, \text{Limit}[B_{\text{choice}}, d \rightarrow \infty]\right]$$

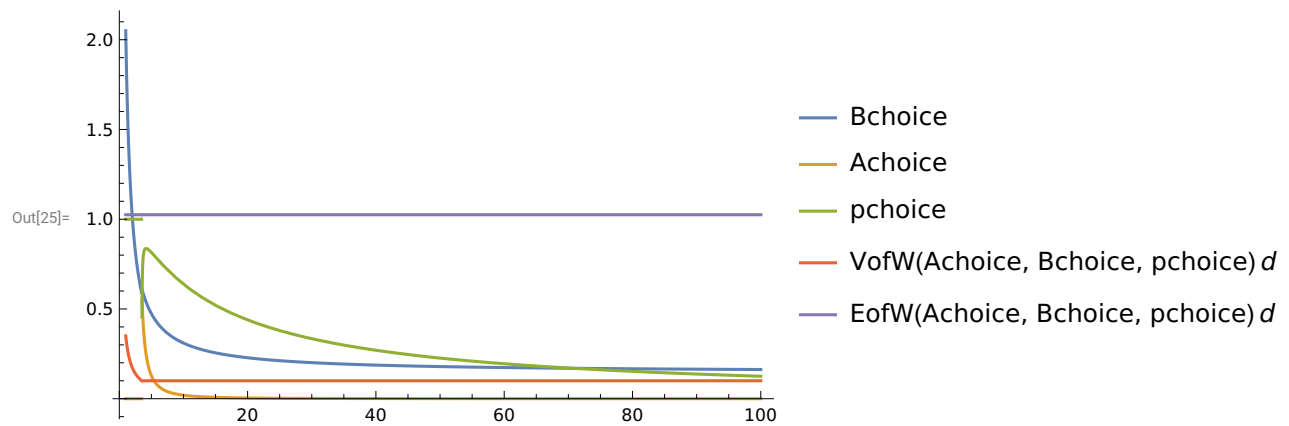
$$\text{Out[20]} = \begin{cases} B_{\text{init}} & 2 B_{\text{init}} M > 3 V \\ \frac{3V}{2M} & \text{True} \end{cases}$$

Visualizations:

```

In[21]:= M := 1.025
V := 1/10
Binit := 0.05
epsilon := 0.1
Plot[{Bchoice, Achoice, pchoice, VofW[Achoice, Bchoice, pchoice]*d,
      EofW[Achoice, Bchoice, pchoice]*d}, {d, 1, 100}, PlotLegends → "Expressions"]
Clear[M, V, Binit, epsilon]

```



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In[353]:= Manipulate[Plot[{If[d >  $\frac{M^2}{3 V}$ , Max[ $\frac{3 M^2 + 3 d V}{2 M d}$  + epsilon/d, Binit], 2 * M/d],
  ReplaceAll[If[d >  $\frac{M^2}{3 V}$ ,  $\frac{-M^2 + 3 d V}{B^2 d^2 - 4 B d M + 3 M^2 + 3 d V}$ , 1],
  B → If[d >  $\frac{M^2}{3 V}$ , Max[ $\frac{3 M^2 + 3 d V}{2 M d}$  + epsilon/d, Binit], 2 * M/d]],
  ReplaceAll[If[d >  $\frac{M^2}{3 V}$ ,  $\frac{2 M - \frac{B d (-M^2 + 3 d V)}{B^2 d^2 - 4 B d M + 3 M^2 + 3 d V}}{d - \frac{d (-M^2 + 3 d V)}{B^2 d^2 - 4 B d M + 3 M^2 + 3 d V}}$ , 0],
  B → If[d >  $\frac{M^2}{3 V}$ , Max[ $\frac{3 M^2 + 3 d V}{2 M d}$  + epsilon/d, Binit], 2 * M/d]],
  {
     $\begin{cases} \frac{M^2}{3 d^2} & 3 d V \leq M^2 \\ \frac{V}{d} & 3 d V > M^2 * d, \frac{M}{d} * d \end{cases}$ , {d, 1, dMax},
    Indeterminate True
  ],
  PlotLegends → {"B", "p", "A", "V*d", "E*d"}], {M, 1, 2},
  {V, 1/50, 1}, {Binit, 0, 0.1}, {epsilon, 0, 0.2}, {dMax, 10, 1000}]

```


Out[353]=

