You can follow this proof directly from this pdf or
you can find the original Mathematica-notebook under https://github.com/marketdesignresearch/BO-CA/blob/main/Theory/MVNN_initialization7_2.nb.
You can execute this notebook and vary the parameters in the interactive plot for free without any license with https://www.wolfram.com/player/ or you can manipulate and the extend the code with a Mathematica license.

```
In[1]:= EofW[A_, B_, p_] := (1 - p) * A / 2 + p * B / 2
```

```
In[2]:= VofW[A_, B_, p_] := (1 - p) * A ^ 2 / 3 + p * B ^ 2 / 3 - EofW[A, B, p] ^ 2
```

We solve the original problem (w.l.o.g. we assume A<=B):

```
In[3]:= Solve[EofW[A, B, p] == M / d && VofW[A, B, p] == V / d && 1 - p ≥ 0 && B ≥ A,
  {A, B, p}, NonNegativeReals]
```

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{A \to \boxed{\frac{2M - \frac{B\,d\,(-M^2+3\,d\,V)}{B^2 d^2 - 4\,B\,d\,M + 3\,M^2 + 3\,d\,V}}{d - \frac{d\,(-M^2+3\,d\,V)}{B^2 d^2 - 4\,B\,d\,M + 3\,M^2 + 3\,d\,V}} \quad \text{if } V > 0 \,\&\&\, M > 0 \,\&\&\, d > \frac{M^2}{3\,V} \,\&\&\, B \geq \frac{3\,M^2 + 3\,d\,V}{2\,d\,M}},\right.\right.$$

$$\left.\left. p \to \boxed{\frac{-M^2 + 3\,d\,V}{B^2 d^2 - 4\,B\,d\,M + 3\,M^2 + 3\,d\,V} \quad \text{if } V > 0 \,\&\&\, M > 0 \,\&\&\, d > \frac{M^2}{3\,V} \,\&\&\, B \geq \frac{3\,M^2 + 3\,d\,V}{2\,d\,M}}\right\}\right\}$$

We see that a solution only exists if $d > \frac{M^2}{3\,V}$. Further we see that, if a solution exists, we have infinitely many solutions for B (every $B \geq \frac{3\,M^2+3\,d\,V}{2\,d\,M}$ is a possible choice). For every possible choice of B there a unique solution for A and p given above.
In the next step we see that if we additionally want A>0 to hold, we just need to choose a value for B which is at least by an arbitrarily small but positive epsilon larger than the minimal possible choice of B. We see this by observing that when we add A>0 as an constraint, the only difference in the solution is that B is given by a strict inequality B>... instead of B≥... :

```
In[4]:= Solve[EofW[A, B, p] == M / d && VofW[A, B, p] == V / d && 1 - p ≥ 0 && B ≥ A && A > 0,
  {A, B, p}, NonNegativeReals]
```

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{A \to \boxed{\frac{2\,M - \frac{B\,d\,(-M^2+3\,d\,V)}{B^2\,d^2-4\,B\,d\,M+3\,M^2+3\,d\,V}}{d - \frac{d\,(-M^2+3\,d\,V)}{B^2\,d^2-4\,B\,d\,M+3\,M^2+3\,d\,V}} \text{ if } V > 0\,\&\&\,M > 0\,\&\&\,d > \frac{M^2}{3\,V}\,\&\&\,B > \frac{3\,M^2+3\,d\,V}{2\,d\,M}},\right.\right.$$

$$\left.\left. p \to \boxed{\frac{-M^2+3\,d\,V}{B^2\,d^2-4\,B\,d\,M+3\,M^2+3\,d\,V} \text{ if } V > 0\,\&\&\,M > 0\,\&\&\,d > \frac{M^2}{3\,V}\,\&\&\,B > \frac{3\,M^2+3\,d\,V}{2\,d\,M}}\right\}\right\}$$

Our choice of B is:

In[5]:= $\text{Bchoice} := \text{If}\left[d > \frac{M^2}{3\,V},\ \text{Max}\left[\frac{3\,M^2+3\,d\,V}{2\,M\,d} + \text{epsilon}/d,\ \text{Binit}\right],\ 2 * M/d\right]$

, where the first case is motivated by the arguments above and the second case guarantees the correct expectation as we will prove later.

This uniquely defines p and A as in the paper by plugging B into the formulas for A and p above (in the case a solution exists). In the other cases we also make specific choices.

In[6]:= $\text{pchoice} := \text{With}\left[\{B = \text{Bchoice}\},\ \text{If}\left[d > \frac{M^2}{3\,V},\ \frac{-M^2+3\,d\,V}{B^2\,d^2-4\,B\,d\,M+3\,M^2+3\,d\,V},\ 1\right]\right]$

We see that in the formula for A given above the expression for p appears, so we can use an easier expression for A:

In[7]:= $\text{Achoice} := \text{With}\left[\{B = \text{Bchoice},\ p = \text{pchoice}\},\ \text{If}\left[d > \frac{M^2}{3\,V},\ \frac{(2\,M - B\,d\,p)}{d\,(1-p)},\ 0\right]\right]$

Now we can check if our chosen solution fulfills all the desired properties.

Proof of item 1:

In[8]:= `Simplify[EofW[Achoice, Bchoice, pchoice]]`

Out[8]= $\frac{M}{d}$

Proof of item 2 and 3:

We use Mathematica for this proof since the terms can get quite complicated:

In[9]:= `VofW[Achoice, Bchoice, pchoice]`

Out[9]= $\frac{1}{3}\left(1 - \text{If}\left[d > \frac{M^2}{3\,V},\ (-M^2+3\,d\,V)\middle/\left(\text{If}\left[d > \frac{M^2}{3\,V},\ \text{Max}\left[\frac{3\,M^2+3\,d\,V}{2\,M\,d} + \frac{\text{epsilon}}{d},\ \text{Binit}\right],\ \frac{2\,M}{d}\right]^2 d^2 - \right.\right.\right.$

$$4\,\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]\,d\,M+3\,M^2+3\,d\,V\Big),\ 1\Big]\Big)$$

$$\text{If}\Big[d>\frac{M^2}{3\,V},\ \Big(2\,M-\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]\,d$$

$$\text{If}\Big[d>\frac{M^2}{3\,V},\ \big(-M^2+3\,d\,V\big)\Big/\Big(\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]^2\,d^2-$$

$$4\,\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]\,d\,M+3\,M^2+3\,d\,V\Big),\ 1\Big]\Big)\Big/$$

$$\Big(d\,\Big(1-\text{If}\Big[d>\frac{M^2}{3\,V},\ \big(-M^2+3\,d\,V\big)\Big/\Big(\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]^2\,d^2-$$

$$4\,\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]\,d\,M+3\,M^2+3\,d\,V\Big),\ 1\Big]\Big)\Big),$$

$$0\Big]^2+\frac{1}{3}\,\text{If}\Big[d>\frac{M^2}{3\,V},\ \big(-M^2+3\,d\,V\big)\Big/\Big(\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]^2\,d^2-$$

$$4\,\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]\,d\,M+3\,M^2+3\,d\,V\Big),\ 1\Big]$$

$$\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]^2-$$

$$\Big(\frac{1}{2}\,\Big(1-\text{If}\Big[d>\frac{M^2}{3\,V},\ \big(-M^2+3\,d\,V\big)\Big/\Big(\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]^2\,d^2-$$

$$4\,\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]\,d\,M+3\,M^2+3\,d\,V\Big),\ 1\Big]\Big)$$

$$\text{If}\Big[d>\frac{M^2}{3\,V},\ \Big(2\,M-\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]\,d$$

$$\text{If}\Big[d>\frac{M^2}{3\,V},\ \big(-M^2+3\,d\,V\big)\Big/\Big(\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]^2\,d^2-$$

$$4\,\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]\,d\,M+3\,M^2+3\,d\,V\Big),\ 1\Big]\Big)\Big/$$

$$\Big(d\,\Big(1-\text{If}\Big[d>\frac{M^2}{3\,V},\ \big(-M^2+3\,d\,V\big)\Big/\Big(\text{If}\Big[d>\frac{M^2}{3\,V},\ \text{Max}\Big[\frac{3\,M^2+3\,d\,V}{2\,M\,d}+\frac{\text{epsilon}}{d},\ \text{Binit}\Big],\ \frac{2\,M}{d}\Big]^2$$

$$d^2 - 4\,\mathrm{If}\Big[d > \frac{M^2}{3\,V},\ \mathrm{Max}\Big[\frac{3\,M^2 + 3\,d\,V}{2\,M\,d} + \frac{\mathrm{epsilon}}{d},\ \mathrm{Binit}\Big],\ \frac{2\,M}{d}\Big]$$

$$d\,M + 3\,M^2 + 3\,d\,V\Big),\ 1\Big]\Big)\Big),\ 0\Big] +$$

$$\frac{1}{2}\,\mathrm{If}\Big[d > \frac{M^2}{3\,V},\ \left(-M^2 + 3\,d\,V\right) \Big/ \Big(\mathrm{If}\Big[d > \frac{M^2}{3\,V},\ \mathrm{Max}\Big[\frac{3\,M^2 + 3\,d\,V}{2\,M\,d} + \frac{\mathrm{epsilon}}{d},\ \mathrm{Binit}\Big],\ \frac{2\,M}{d}\Big]^2 d^2 -$$

$$4\,\mathrm{If}\Big[d > \frac{M^2}{3\,V},\ \mathrm{Max}\Big[\frac{3\,M^2 + 3\,d\,V}{2\,M\,d} + \frac{\mathrm{epsilon}}{d},\ \mathrm{Binit}\Big],\ \frac{2\,M}{d}\Big]\,d\,M + 3\,M^2 + 3\,d\,V\Big),\ 1\Big] \times$$

$$\mathrm{If}\Big[d > \frac{M^2}{3\,V},\ \mathrm{Max}\Big[\frac{3\,M^2 + 3\,d\,V}{2\,M\,d} + \frac{\mathrm{epsilon}}{d},\ \mathrm{Binit}\Big],\ \frac{2\,M}{d}\Big]\Big)^2$$

But Mathematica can easily simplify complicated terms such as this one:

In[10]:=
```
Simplify[VofW[Achoice, Bchoice, pchoice]]
```

Out[10]=

$$\begin{cases} \frac{M^2}{3\,d^2} & 3\,d\,V^2 \leq M^2\,V \\ \frac{V}{d} & \mathrm{True} \end{cases}$$

This is even true for V≥0, but it can be slightly further simplified if V>0:

In[11]:=
```
Assuming[V > 0, Simplify[VofW[Achoice, Bchoice, pchoice]]]
```

Out[11]=

$$\begin{cases} \frac{M^2}{3\,d^2} & 3\,d\,V \leq M^2 \\ \frac{V}{d} & 3\,d\,V > M^2 \\ \mathrm{Indeterminate} & \mathrm{True} \end{cases}$$

We see that the $\frac{V}{d} \leq \mathrm{VofW}$ holds always true (for d>0) and $\frac{V}{d} = \mathrm{VofW}$ if $d > \frac{M^2}{3\,V}$ or we can check it in the next line:

In[12]:=

$$\mathrm{Reduce}\Big[\frac{V}{d} \leq \mathrm{Simplify[VofW[Achoice,\ Bchoice,\ pchoice]]}\ \&\&\ V > 0\ \&\&\ d > 0\ \&\&\ M > 0\Big]$$

Out[12]= $V > 0\ \&\&\ M > 0\ \&\&\ d > 0$

Proof of item 4:

In[13]:=

$$\mathrm{Reduce}\Big[0 < \mathrm{Simplify}\Big[\mathrm{Achoice},\ d > \frac{M^2}{3\,V}\Big]\ \&\&\ V > 0\ \&\&\ d > 0\ \&\&\ M > 0\ \&\&\ \mathrm{epsilon} > 0\ \&\&\ d > \frac{M^2}{3\,V}\Big]$$

Out[13]=

$$\mathrm{Binit} \in \mathbb{R}\ \&\&\ V > 0\ \&\&\ M > 0\ \&\&\ d > \frac{M^2}{3\,V}\ \&\&\ \mathrm{epsilon} > 0$$

Proof of item 5:

In[14]:=
```
Reduce[0 == Simplify[Achoice, d > M^2/(3 V)] && V > 0 && d > 0 &&
  M > 0 && epsilon == 0 && Binit ≤ Simplify[Bchoice, d > M^2/(3 V)] && d > M^2/(3 V)]
```

Out[14]= $V > 0 \;\&\&\; M > 0 \;\&\&\; d > \frac{M^2}{3\,V} \;\&\&\; \text{Binit} \le \frac{3\,M^2 + 3\,d\,V}{2\,d\,M} \;\&\&\; \text{epsilon} == 0$

(In the case of $d \le \frac{M^2}{3\,V}$, Achoice is zero anyway. Note that Binit<Bchoice implies $\text{Binit} \le \frac{3\,M^2+3\,d\,V}{2\,d\,M}$ for epsilon==0.)

Proof of item 6:

In[15]:=
```
Reduce[0 ≤ Simplify[Achoice, d > M^2/(3 V)] &&
  V > 0 && d > 0 && M > 0 && epsilon ≥ 0 && Binit ≥ 0 && d > M^2/(3 V)]
```

Out[15]= $V > 0 \;\&\&\; M > 0 \;\&\&\; d > \frac{M^2}{3\,V} \;\&\&\; \text{Binit} \ge 0 \;\&\&\; \text{epsilon} \ge 0$

In[16]:=
```
Reduce[0 ≤ Simplify[Bchoice, d > M^2/(3 V)] &&
  V > 0 && d > 0 && M > 0 && epsilon ≥ 0 && Binit ≥ 0 && d > M^2/(3 V)]
```

Out[16]= $V > 0 \;\&\&\; M > 0 \;\&\&\; d > \frac{M^2}{3\,V} \;\&\&\; \text{epsilon} \ge 0 \;\&\&\; \text{Binit} \ge 0$

Proof of item 7: (A ≤B)

In[17]:= $\text{Reduce}\left[\text{Simplify}\left[\text{Achoice}, d > \frac{M^2}{3\,V}\right] \leq \text{Simplify}\left[\text{Bchoice}, d > \frac{M^2}{3\,V}\right] \&\& \; V > 0 \&\& d > 0 \&\& M > 0 \&\& \text{epsilon} \geq 0 \&\& \text{Binit} \geq 0 \&\& d > \frac{M^2}{3\,V}\right]$

Out[17]= $V > 0 \&\& M > 0 \&\& d > \frac{M^2}{3\,V} \&\& \text{epsilon} \geq 0 \&\& \text{Binit} \geq 0$

Proof of item 8:

In[18]:= $\text{Reduce}\left[\frac{3\,V}{2\,M} \leq \text{Simplify}\left[\text{Bchoice}, d > \frac{M^2}{3\,V}\right] \&\& \; V > 0 \&\& d > 0 \&\& M > 0 \&\& \text{epsilon} \geq 0 \&\& \text{Binit} \geq 0 \&\& d > \frac{M^2}{3\,V}\right]$

Out[18]= $V > 0 \&\& M > 0 \&\& d > \frac{M^2}{3\,V} \&\& \text{Binit} \geq 0 \&\& \text{epsilon} \geq 0$

In[19]:= $\text{Reduce}\left[\frac{3\,V}{2\,M} \leq \text{Simplify}\left[\text{Bchoice}, d \leq \frac{M^2}{3\,V}\right] \&\& \; V > 0 \&\& d > 0 \&\& M > 0 \&\& \text{epsilon} \geq 0 \&\& \text{Binit} \geq 0 \&\& d \leq \frac{M^2}{3\,V}\right]$

Out[19]= $\text{epsilon} \geq 0 \&\& \text{Binit} \geq 0 \&\& V > 0 \&\& M > 0 \&\& 0 < d \leq \frac{M^2}{3\,V}$

Proof of item 9:

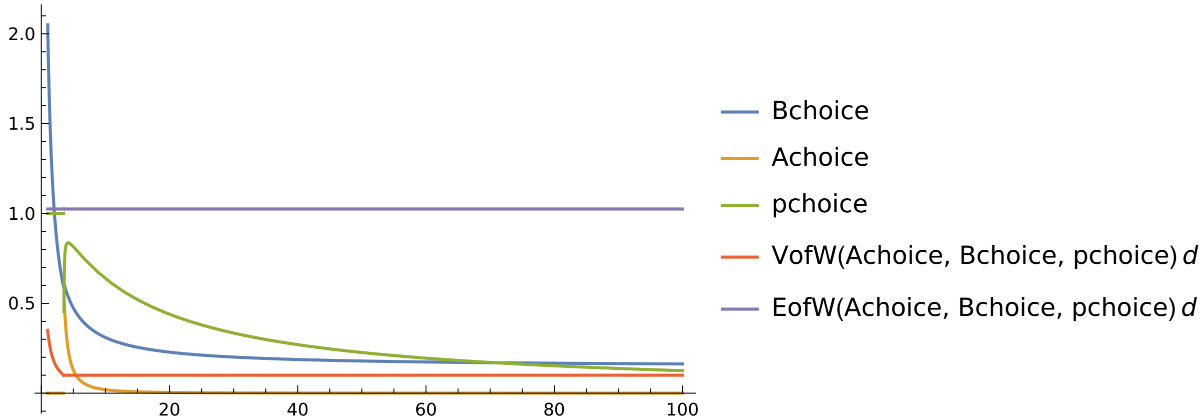
In[20]:= $\text{Assuming}\left[V > 0 \&\& M > 0 \&\& \text{epsilon} \geq 0 \&\& \text{Binit} \geq 0, \text{Limit}[\text{Bchoice}, d \to \infty]\right]$

Out[20]= $\begin{cases} \text{Binit} & 2\,\text{Binit}\,M > 3\,V \\ \frac{3\,V}{2\,M} & \text{True} \end{cases}$

Visualizations:

In[21]:=
```
M := 1.025
V := 1/10
Binit := 0.05
epsilon := 0.1
Plot[{Bchoice, Achoice, pchoice, VofW[Achoice, Bchoice, pchoice] * d,
  EofW[Achoice, Bchoice, pchoice] * d}, {d, 1, 100}, PlotLegends → "Expressions"]
Clear[M, V, Binit, epsilon]
```

Out[25]=



In[353]:=
$$\text{Manipulate}\Big[\text{Plot}\Big[\Big\{\text{If}\Big[d > \frac{M^2}{3 V}, \text{Max}\Big[\frac{3 M^2 + 3 d V}{2 M d} + \text{epsilon}/d, \text{Binit}\Big], 2 * M/d\Big],$$

$$\text{ReplaceAll}\Big[\text{If}\Big[d > \frac{M^2}{3 V}, \frac{-M^2 + 3 d V}{B^2 d^2 - 4 B d M + 3 M^2 + 3 d V}, 1\Big],$$

$$B \to \text{If}\Big[d > \frac{M^2}{3 V}, \text{Max}\Big[\frac{3 M^2 + 3 d V}{2 M d} + \text{epsilon}/d, \text{Binit}\Big], 2 * M/d\Big]\Big],$$

$$\text{ReplaceAll}\Big[\text{If}\Big[d > \frac{M^2}{3 V}, \frac{2 M - \frac{B d (-M^2+3 d V)}{B^2 d^2-4 B d M+3 M^2+3 d V}}{d - \frac{d (-M^2+3 d V)}{B^2 d^2-4 B d M+3 M^2+3 d V}}, 0\Big],$$

$$B \to \text{If}\Big[d > \frac{M^2}{3 V}, \text{Max}\Big[\frac{3 M^2 + 3 d V}{2 M d} + \text{epsilon}/d, \text{Binit}\Big], 2 * M/d\Big]\Big],$$

$$\begin{cases} \frac{M^2}{3 d^2} & 3 d V \le M^2 \\ \frac{V}{d} & 3 d V > M^2 \\ \text{Indeterminate} & \text{True} \end{cases} * d, \frac{M}{d} * d\Big\}, \{d, 1, \text{dMax}\},$$

PlotLegends → {"B", "p", "A", "V*d", "E*d"}], {M, 1, 2},

{V, 1/50, 1}, {Binit, 0, 0.1}, {epsilon, 0, 0.2}, {dMax, 10, 1000}]

Out[353]=