Duality in Coherent Algebras

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February 2018

Coherent Algebras

Definition

A coherent algebra $\mathcal{W}$ of order n is a subspace of $\mathrm{Mat}_{n\times n}(\mathbb{C})$ such that:

- MN and $M \circ N \in \mathcal{W}$ for all $M, N \in \mathcal{W}$.
- $M^T \in \mathcal{W}$ for all $M \in \mathcal{W}$.
- $I, J \in \mathcal{W}$.

Duality Mappings

Definition

A duality mapping between (commutative) coherent algebras $\mathcal{W}$ and $\mathcal{X}$ of order n is a linear isomorphism $T : \mathcal{W} \to \mathcal{X}$ such that:

- $T(MN) = T(M) \circ T(N)$ for all $M, N \in \mathcal{W}$.
- $T(M \circ N) = \frac{1}{n} T(M)T(N)$ for all $M, N \in \mathcal{W}$.

Primitive Matrices

Definition

An element $M \in \mathcal{W}$ of a coherent algebra $\mathcal{W}$ is said to be primitive if $M^2 = M$ and $MN \in \text{span}\{M\}$ for all $N \in \mathcal{W}$.

Schur-Primitive Matrices

Definition

Dually, an element $M \in \mathcal{W}$ of a coherent algebra $\mathcal{W}$ is said to be Schur-primitive if $M^{\circ 2} = M$ and $M \circ N \in \text{span}\{M\}$ for all $N \in \mathcal{W}$.

Dual Bases

Every coherent algebra is spanned by its set of Schur-primitive matrices.

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Every coherent algebra is spanned by its set of Schur-primitive matrices.

Dually, every commutative coherent algebra is spanned by its set of primitive matrices as well.

Suppose that $\mathcal{W}$ is a commutative coherent algebra of order n.

Let $\Gamma(\mathcal{W})$ denote the set of Schur-primitive matrices in $\mathcal{W}$.

Dually, let $\Lambda(\mathcal{W})$ denote the set of primitive matrices in $\mathcal{W}$.

Suppose that $\{A_i : 1 \leq i \leq d\}$ and $\{E_i : 1 \leq i \leq d\}$ are orderings on $\Gamma(\mathcal{W})$ and $\Lambda(\mathcal{W})$, letting $d := \dim(\mathcal{W})$.

Character Tables and Dual Character Tables

The matrix $P \in \mathrm{Mat}_{d\times d}(\mathbb{C})$ such that $A_i = \sum_{j=1}^{d} P_{i,j}E_j$ is said to be the character table of $\mathcal{W}$ (under these orderings).

Character Tables and Dual Character Tables

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Dually, the matrix $Q \in \mathrm{Mat}_{d\times d}(\mathbb{C})$ such that $E_i = \sum_{j=1}^{d} \left(\frac{1}{n} Q_{i,j}\right) A_j$ is said to be the dual character table of $\mathcal{W}$ (under these orderings).

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We have that $P^{-1} = \frac{1}{n} Q$ and hence $PQ = nI$.

Intersection Numbers and Dual Intersection Numbers

There is a set of numbers $\{p_{i,j}(k) : 1 \leq i, j, k \leq d\}$ such that $A_i A_j = \sum_{k=1}^{d} p_{i,j}(k) A_k$ for all $1 \leq i, j \leq d$.

These are said to be the intersection numbers of $\mathcal{W}$.

Intersection Numbers and Dual Intersection Numbers

Dually, there is a set of numbers $\{q_{i,j}(k) : 1 \leq i, j, k \leq d\}$ such that $E_i \circ E_j = \sum_{k=1}^{d} q_{i,j}(k) E_k$ for all $1 \leq i, j \leq d$.

These are said to be the dual intersection numbers of $\mathcal{W}$ (or Krein numbers of $\mathcal{W}$).

Intersection Numbers and Dual Intersection Numbers

The intersection numbers of $\mathcal{W}$ are non-negative integers.

The dual intersection numbers of $\mathcal{W}$ are non-negative real numbers.

Duality Mappings

There is a duality mapping between commutative coherent algebras $\mathcal{X}$ and $\mathcal{Y}$ of dimension d if and only if:

1. $p_{i,j}(k) = q_{i,j}(k)$ for all $1 \leq i, j, k \leq d$,

   where $\{p_{i,j}(k) : 1 \leq i, j, k \leq d\}$ are the intersection numbers of $\mathcal{X}$

   and $\{q_{i,j}(k) : 1 \leq i, j, k \leq d\}$ are the dual intersection numbers of $\mathcal{Y}$.
2. There is a character table P of $\mathcal{X}$ and dual character table Q of $\mathcal{Y}$ such that $\overline{P} = Q$.

Finite Groups

Suppose that G is a finite group of order n.

Let $L^2(G)$ denote the space of complex-valued functions on G.

Define $M_f \in \mathrm{Mat}_{G\times G}(\mathbb{C})$ as $(M_f)_{x,y} := f(x^{-1}y)$ for all $x, y \in G$.

We have that $M_f M_g = M_{f\star g}$ for all $f, g \in L^2(G)$ with convolution $f \star g$.

Finite Groups

Additionally, define $A_g \in \mathrm{Mat}_{G\times G}(\mathbb{C})$ as $(A_g))_{x,y} = xy^{-1}$ for all $x, y \in G$ and $g \in G$.

The coherent algebra $\mathcal{W}_G := \mathrm{span}\{A_g : g \in G\}$

$= \{M_f : f \in L^2(G)\}$ can be identified with the group ring of G.

Refer to W_G as the group coherent algebra of G.

Finite Abelian Groups

Assume that G is abelian.

Let $X(G) \subseteq L^2(G)$ denote the set of irreducible characters of G.

Then $W_G = \operatorname{span}\left\{\frac{1}{n}M_\chi : \chi \in X(G)\right\}$
where $\left\{\frac{1}{n}M_\chi : \chi \in X(G)\right\}$ is the set of primitive matrices in $\mathcal{W}_G$.

Finite Abelian Groups

A character table for $\mathcal{W}_G$ can be identified with a character table for G in the usual sense.

There is an ordering $\{\chi_x : x \in G\}$ on $X(G)$ such that
$\chi_x(y) = \chi_y(x)$ for all $x, y \in G$,

since G can be decomposed as a direct product of finite cyclic groups.

There is then a character table P for $\mathcal{W}_G$ such that $P\overline{P} = nI$,

since $X(G)$ forms an orthonormal basis for $L^2(G)$ as well.

Fourier Transforms

Definition

A Fourier transform on a (commutative) coherent algebra $\mathcal{W}$ of order n is a duality mapping $T : \mathcal{W} \to \mathcal{W}$ such that:

- $T^{-1}(M) = \overline{\frac{1}{n} T(M)}$ for all Schur-primitive matrices $M \in \mathcal{W}$.

Fourier Transforms

A Fourier transform on the group coherent algebra $\mathcal{W}_G$ of a finite abelian group G can be identified with a Fourier transform on the space $L^2(G)$ of complex-valued functions on G.

Self-Dual Coherent Algebras

There is a Fourier transform on a (commutative) coherent algebra $\mathcal{W}$ if and only if there is a character table P of $\mathcal{W}$ such that dual character table of $\mathcal{W}$ is $\overline{P}$.

A (commutative) coherent algebra is said to be self-dual if there is character table P of $\mathcal{W}$ such that $\overline{P}$ is the dual character table of $\mathcal{W}$.

More Self-Dual Coherent Algebras

The adjacency algebras of Hamming graphs are self-dual coherent algebras.

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of coherent algebras $\mathcal{X}$ and $\mathcal{Y}$ is self-dual if there is a duality mapping betwen them.

There is a duality mapping on the adjacency algebra of every self-complementary strongly-regular graph.

Metric and Cometric Coherent Algebras

A (commutative) coherent algebra $\mathcal{W}$ is said to be metric if $\mathcal{W} = \text{span}\{I, A, A^2, \cdots\}$ for some Schur-primitive metric $A \in \mathcal{W}$.

Metric and Cometric Coherent Algebras

A (commutative) coherent algebra $\mathcal{W}$ is said to be metric if $\mathcal{W} = \text{span}\{I, A, A^2, \cdots\}$ for some Schur-primitive metric $A \in \mathcal{W}$.

Dually, a (commutative) coherent algebra is said to be cometric if $\mathcal{W} = \text{span}\{J, E, E^2, \cdots\}$ for some primitive matrix $E \in \mathcal{W}$.

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Dually, a (commutative) coherent algebra is said to be cometric if $\mathcal{W} = \mathrm{span}\{J, E, E^2, \cdots\}$ for some primitive matrix $E \in \mathcal{W}$.
A self-dual coherent algebra is metric if and only if it is cometric.

Metric and Cometric Coherent Algebras

Coherent algebras that are both metric and symmetric can be identified with the adjacency algebras of distance-regular graphs.

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Coherent algebras that are both metric and symmetric can be identified with the adjacency algebras of distance-regular graphs.

The eigenvalue multiplicities of a self-dual distance-regular graph are the valencies of its distance-graphs.

Is it possible to classify the self-dual distance-regular graphs?