

Duality in Coherent Algebras

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Coherent Algebras

Definition

A coherent algebra \mathcal{W} of order n is a subspace of $\text{Mat}_{n \times n}(\mathbb{C})$ such that:

- MN and $M \circ N \in \mathcal{W}$ for all $M, N \in \mathcal{W}$.
- $M^T \in \mathcal{W}$ for all $M \in \mathcal{W}$.
- $I, J \in \mathcal{W}$.

Primitive Matrices

Definition

An element $M \in \mathcal{W}$ of a coherent algebra \mathcal{W} is said to be primitive if $M^2 = M$ and $MN \in \text{span}\{M\}$ for all $N \in \mathcal{W}$.

Schur-Primitive Matrices

Definition

Dually, an element $M \in \mathcal{W}$ of a coherent algebra \mathcal{W} is said to be Schur-primitive if $M^{\circ 2} = M$ and $M \circ N \in \text{span}\{M\}$ for all $N \in \mathcal{W}$.

Dual Bases

Theorem

Every commutative coherent algebra is spanned by its set of primitive matrices relative to both Schur-product and ordinary matrix multiplication.

Suppose that \mathcal{W} is a commutative coherent algebra of order n .

Let $\Gamma(\mathcal{W})$ denote the set of Schur-primitive matrices in \mathcal{W} .

Dually, let $\Lambda(\mathcal{W})$ denote the set of primitive matrices in \mathcal{W} .

Suppose that $\{A_i : 1 \leq i \leq d\}$ and $\{E_i : 1 \leq i \leq d\}$ are orderings on $\Gamma(\mathcal{W})$ and $\Lambda(\mathcal{W})$, letting $d := \dim(\mathcal{W})$.

Character Tables and Dual Character Tables

The matrix $P \in \text{Mat}_{d \times d}(\mathbb{C})$ such that $A_i = \sum_{j=1}^d P_{i,j} E_j$ for all $1 \leq i \leq d$ is said to be the character table of \mathcal{W} (under these orderings).

Character Tables and Dual Character Tables

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Dually, the matrix $Q \in \text{Mat}_{d \times d}(\mathbb{C})$ such that $E_i = \sum_{j=1}^d \left(\frac{1}{n} Q_{i,j}\right) A_j$ for all $1 \leq i \leq d$ is said to be the dual character table of \mathcal{W} (under these orderings).

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We have that $P^{-1} = \frac{1}{n} Q$ and hence $PQ = nI$.

Dual Character Tables in Terms of Character Tables

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$$\langle \cdot, \cdot \rangle : \text{Mat}_{d \times d}(\mathbb{C}) \times \text{Mat}_{d \times d}(\mathbb{C}) \rightarrow \mathbb{C}$$

$$\langle A, B \rangle := \text{tr}(B^* A) = \text{sum}(\overline{B} \circ A)$$

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Hence $Q = R \circ P^T$ where $R \in \text{Mat}_{d \times d}(\mathbb{C})$ and $R_{i,j} := \frac{m_i}{v_j}$ for all $1 \leq i, j \leq d$.

Intersection Numbers and Dual Intersection Numbers

There is a set of numbers $\{p_{i,j}(k) : 1 \leq i, j, k \leq d\}$ such that

$$A_i A_j = \sum_{k=1}^d p_{i,j}(k) A_k \text{ for all } 1 \leq i, j \leq d.$$

These are said to be the intersection numbers of \mathcal{W} .

Intersection Numbers and Dual Intersection Numbers

Dually, there is a set of numbers $\{q_{i,j}(k) : 1 \leq i, j, k \leq d\}$ such that $E_i \circ E_j = \sum_{k=1}^d \left(\frac{1}{n} q_{i,j}(k)\right) E_k$ for all $1 \leq i, j \leq d$.

These are said to be the dual intersection numbers of \mathcal{W} (or Krein numbers of \mathcal{W}).

Intersection Numbers and Dual Intersection Numbers

The intersection numbers of \mathcal{W} are non-negative integers.

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Duality Mappings

Definition

A duality mapping between (commutative) coherent algebras \mathcal{W} and \mathcal{X} of order n is a linear isomorphism $T : \mathcal{W} \rightarrow \mathcal{X}$ such that:

- $T(MN) = T(M) \circ T(N)$ for all $M, N \in \mathcal{W}$.
- $T(M \circ N) = \frac{1}{n} T(M) T(N)$ for all $M, N \in \mathcal{W}$.

Dual Coherent Algebras

There is a duality mapping between coherent algebras \mathcal{X} and \mathcal{Y} of dimension d if and only if:

- 1 There is a character table P of \mathcal{X} and dual character table Q of \mathcal{Y} such that $\overline{P} = Q$.

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- 1 There is a character table P of \mathcal{X} and dual character table Q of \mathcal{Y} such that $\overline{P} = Q$.
- 2 $p_{i,j}(k) = q_{i,j}(k)$ for all $1 \leq i, j, k \leq d$,

where $\{p_{i,j}(k) : 1 \leq i, j, k \leq d\}$ are the intersection numbers of \mathcal{X}

and $\{q_{i,j}(k) : 1 \leq i, j, k \leq d\}$ are the dual intersection numbers of \mathcal{Y} .

Metric and Cometric Coherent Algebras

A (commutative) coherent algebra \mathcal{W} is said to be metric if $\mathcal{W} = \text{span}\{I, A, A^2, \dots\}$ for some Schur-primitive matrix $A \in \mathcal{W}$.

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Dually, a (commutative) coherent algebra \mathcal{W} is said to be cometric if $\mathcal{W} = \text{span}\{J, E, E^{\circ 2}, \dots\}$ for some primitive matrix $E \in \mathcal{W}$.

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Dually, a (commutative) coherent algebra \mathcal{W} is said to be cometric if $\mathcal{W} = \text{span}\{J, E, E^{\circ 2}, \dots\}$ for some primitive matrix $E \in \mathcal{W}$.

If there is a duality mapping on a coherent algebra \mathcal{W} , then \mathcal{W} is metric if and only if it is cometric.

Metric and Cometric Coherent Algebras

Coherent algebras that are both metric and symmetric can be identified with the adjacency algebras of distance-regular graphs.

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Coherent algebras that are both metric and symmetric can be identified with the adjacency algebras of distance-regular graphs.

The eigenvalue multiplicities of a distance-regular graph are the valencies of its distance-graphs if there is a duality mapping on its adjacency algebra.

Self-Dual Coherent Algebras

Definition

A coherent algebra \mathcal{W} of order n is said to be self-dual if there is a duality mapping $T : \mathcal{W} \rightarrow \mathcal{W}$ on \mathcal{W} such that:

- $T^{-1}(M) = \frac{1}{n} T(M)$ for all $M \in \mathcal{W}$.
- $T(\lambda M) = \bar{\lambda} T(M)$ for all $\lambda \in \mathbb{C}$ and $M \in \mathcal{W}$.

Self-Dual Coherent Algebras

A coherent algebra \mathcal{W} is self-dual if and only if there is a character table P for \mathcal{W} with dual character table \overline{P} .

Strongly-Regular Graphs

Suppose that G is a (connected) strongly-regular graph of order n and valency k .

$$\implies P := \begin{pmatrix} 1 & 1 & 1 \\ k & \lambda & \mu \\ n-1-k & -1-\lambda & -1-\mu \end{pmatrix}$$

is a character table for \mathcal{A} for some $\lambda, \mu \in \mathbb{R}$

$$\text{and } Q := \begin{pmatrix} 1 & 1 & 1 \\ j & \left(\frac{j}{k}\right) \lambda & \left(\frac{j}{n-1-k}\right) (-1-\lambda) \\ n-1-j & \left(\frac{n-1-j}{k}\right) \mu & \left(\frac{n-1-j}{n-1-k}\right) (-1-\mu) \end{pmatrix}$$

is the dual character table of \mathcal{A} relative to P for some $j \in \mathbb{N}$.

Self-Dual Strongly-Regular Graphs

Theorem

Suppose that G is a strongly-regular graph of order n and valency k . Then G is self-dual if and only if:

$$(\lambda - \mu)^2 = n$$

with $\{k, \lambda, \mu\}$ the distinct eigenvalues of G .

Self-Dual Strongly-Regular Graphs

Theorem

A strongly-regular graph G is self-dual if and only if G is either:

- 1 A conference graph.
- 2 An SRG $(m^2, (m-1)\gamma, (\gamma-1)(\gamma-2) + (m-2), \gamma(\gamma-1))$ for some $m, \gamma \in \mathbb{N}$.

Self-Dual Strongly-Regular Graphs

Theorem

A strongly-regular graph is self-dual if and only if the multiplicities of its eigenvalues are the valencies of its distance-graphs.

Self-Dual Distance-Transitive Graphs

Example

The Hamming graphs are self-dual.

To be continued ...