

# Duality in Coherent Algebras

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# Coherent Algebras

## Definition

A coherent algebra  $\mathcal{W}$  of order  $n$  is a subspace of  $\text{Mat}_{n \times n}(\mathbb{C})$  such that:

- $MN$  and  $M \circ N \in \mathcal{W}$  for all  $M, N \in \mathcal{W}$ .
- $M^T \in \mathcal{W}$  for all  $M \in \mathcal{W}$ .
- $I, J \in \mathcal{W}$ .

# Duality Mappings

## Definition

A duality mapping between (commutative) coherent algebras  $\mathcal{W}$  and  $\mathcal{X}$  of order  $n$  is a linear isomorphism  $T : \mathcal{W} \rightarrow \mathcal{X}$  such that:

- $T(MN) = T(M) \circ T(N)$  for all  $M, N \in \mathcal{W}$ .
- $T(M \circ N) = \frac{1}{n} T(M) T(N)$  for all  $M, N \in \mathcal{W}$ .

# Primitive Matrices

## Definition

An element  $M \in \mathcal{W}$  of a coherent algebra  $\mathcal{W}$  is said to be primitive if  $M^2 = M$  and  $MN \in \text{span}\{M\}$  for all  $N \in \mathcal{W}$ .

# Schur-Primitive Matrices

## Definition

Dually, an element  $M \in \mathcal{W}$  of a coherent algebra  $\mathcal{W}$  is said to be Schur-primitive if  $M^{\circ 2} = M$  and  $M \circ N \in \text{span}\{M\}$  for all  $N \in \mathcal{W}$ .

# Dual Bases

Every coherent algebra is spanned by its set of Schur-primitive matrices.

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Every coherent algebra is spanned by its set of Schur-primitive matrices.

Dually, every commutative coherent algebra is spanned by its set of primitive matrices as well.

Suppose that  $\mathcal{W}$  is a commutative coherent algebra of order  $n$ .

Let  $\Gamma(\mathcal{W})$  denote the set of Schur-primitive matrices in  $\mathcal{W}$ .

Dually, let  $\Lambda(\mathcal{W})$  denote the set of primitive matrices in  $\mathcal{W}$ .

Suppose that  $\{A_i : 1 \leq i \leq d\}$  and  $\{E_i : 1 \leq i \leq d\}$  are orderings on  $\Gamma(\mathcal{W})$  and  $\Lambda(\mathcal{W})$ , letting  $d := \dim(\mathcal{W})$ .



# Character Tables and Dual Character Tables

The matrix  $P \in \text{Mat}_{d \times d}(\mathbb{C})$  such that  $A_i = \sum_{j=1}^d P_{i,j} E_j$  is said to be the character table of  $\mathcal{W}$  (under these orderings).

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Dually, the matrix  $Q \in \text{Mat}_{d \times d}(\mathbb{C})$  such that  $E_i = \sum_{j=1}^d \left(\frac{1}{n} Q_{i,j}\right) A_j$  is said to be the dual character table of  $\mathcal{W}$  (under these orderings).

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We have that  $P^{-1} = \frac{1}{n} Q$  and hence  $PQ = nI$ .

# Intersection Numbers and Dual Intersection Numbers

There is a set of numbers  $\{p_{i,j}(k) : 1 \leq i, j, k \leq d\}$  such that

$$A_i A_j = \sum_{k=1}^d p_{i,j}(k) A_k \text{ for all } 1 \leq i, j \leq d.$$

These are said to be the intersection numbers of  $\mathcal{W}$ .

# Intersection Numbers and Dual Intersection Numbers

Dually, there is a set of numbers  $\{q_{i,j}(k) : 1 \leq i, j, k \leq d\}$  such that  $E_i \circ E_j = \sum_{k=1}^d \left(\frac{1}{n} q_{i,j}(k)\right) E_k$  for all  $1 \leq i, j \leq d$ .

These are said to be the dual intersection numbers of  $\mathcal{W}$  (or Krein numbers of  $\mathcal{W}$ ).

# Intersection Numbers and Dual Intersection Numbers

The intersection numbers of  $\mathcal{W}$  are non-negative integers.

The dual intersection numbers of  $\mathcal{W}$  are non-negative real numbers.

# Duality Mappings

There is a duality mapping between commutative coherent algebras  $\mathcal{X}$  and  $\mathcal{Y}$  of dimension  $d$  if and only if:

1  $p_{i,j}(k) = q_{i,j}(k)$  for all  $1 \leq i, j, k \leq d$ ,

where  $\{p_{i,j}(k) : 1 \leq i, j, k \leq d\}$  are the intersection numbers of  $\mathcal{X}$

and  $\{q_{i,j}(k) : 1 \leq i, j, k \leq d\}$  are the dual intersection numbers of  $\mathcal{Y}$ .

2 There is a character table  $P$  of  $\mathcal{X}$  and dual character table  $Q$  of  $\mathcal{Y}$  such that  $\bar{P} = Q$ .

# Finite Groups

Suppose that  $G$  is a finite group of order  $n$ .

Let  $L^2(G)$  denote the space of complex-valued functions on  $G$ .

Define  $M_f \in \text{Mat}_{G \times G}(\mathbb{C})$  as  $(M_f)_{x,y} := f(x^{-1}y)$  for all  $x, y \in G$ .

We have that  $M_f M_g = M_{f \star g}$  for all  $f, g \in L^2(G)$  with convolution  $f \star g$ .



# Finite Groups

Additionally, define  $A_g \in \text{Mat}_{G \times G}(\mathbb{C})$  as  $(A_g)_{x,y} = xy^{-1}$  for all  $x, y \in G$  and  $g \in G$ .

The coherent algebra  $\mathcal{W}_G := \text{span}\{A_g : g \in G\}$   
 $= \{M_f : f \in L^2(G)\}$  can be identified with the group ring of  $G$ .

Refer to  $\mathcal{W}_G$  as the group coherent algebra of  $G$ .

# Finite Abelian Groups

Assume that  $G$  is abelian.

Let  $X(G) \subseteq L^2(G)$  denote the set of irreducible characters of  $G$ .

Then  $W_G = \text{span} \left\{ \frac{1}{n} M_\chi : \chi \in X(G) \right\}$

where  $\left\{ \frac{1}{n} M_\chi : \chi \in X(G) \right\}$  is the set of primitive matrices in  $\mathcal{W}_G$ .

# Finite Abelian Groups

A character table for  $\mathcal{W}_G$  can be identified with a character table for  $G$  in the usual sense.

There is an ordering  $\{\chi_x : x \in G\}$  on  $X(G)$  such that  $\chi_x(y) = \chi_y(x)$  for all  $x, y \in G$ ,

since  $G$  can be decomposed as a direct product of finite cyclic groups.

There is then a character table  $P$  for  $\mathcal{W}_G$  such that  $P\overline{P} = nI$ , since  $X(G)$  forms an orthonormal basis for  $L^2(G)$  as well.

# Fourier Transforms

## Definition

A Fourier transform on a (commutative) coherent algebra  $\mathcal{W}$  of order  $n$  is a duality mapping  $T : \mathcal{W} \rightarrow \mathcal{W}$  such that:

- $T^{-1}(M) = \overline{\frac{1}{n} T(M)}$  for all Schur-primitive matrices  $M \in \mathcal{W}$ .

# Fourier Transforms

A Fourier transform on the group coherent algebra  $\mathcal{W}_G$  of a finite abelian group  $G$  can be identified with a Fourier transform on the space  $L^2(G)$  of complex-valued functions on  $G$ .

# Self-Dual Coherent Algebras

There is a Fourier transform on a (commutative) coherent algebra  $\mathcal{W}$  if and only if there is a character table  $P$  of  $\mathcal{W}$  such that dual character table of  $\mathcal{W}$  is  $\overline{P}$ .

A (commutative) coherent algebra is said to be self-dual if there is character table  $P$  of  $\mathcal{W}$  such that  $\overline{P}$  is the dual character table of  $\mathcal{W}$ .

# More Self-Dual Coherent Algebras

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There is a duality mapping on the adjacency algebra of every self-complementary strongly-regular graph.

# Metric and Cometric Coherent Algebras

A (commutative) coherent algebra  $\mathcal{W}$  is said to be metric if  $\mathcal{W} = \text{span}\{I, A, A^2, \dots\}$  for some Schur-primitive metric  $A \in \mathcal{W}$ .

# Metric and Cometric Coherent Algebras

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Dually, a (commutative) coherent algebra is said to be cometric if  $\mathcal{W} = \text{span}\{J, E, E^2, \dots\}$  for some primitive matrix  $E \in \mathcal{W}$ .

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Dually, a (commutative) coherent algebra is said to be cometric if  $\mathcal{W} = \text{span}\{J, E, E^2, \dots\}$  for some primitive matrix  $E \in \mathcal{W}$ .

A self-dual coherent algebra is metric if and only if it is cometric.

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Coherent algebras that are both metric and symmetric can be identified with the adjacency algebras of distance-regular graphs.

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Coherent algebras that are both metric and symmetric can be identified with the adjacency algebras of distance-regular graphs.

The eigenvalue multiplicities of a self-dual distance-regular graph are the valencies of its distance-graphs.

Is it possible to classify the self-dual distance-regular graphs?