Duality in Coherent Algebras

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June 2018

Coherent Algebras

Definition

A coherent algebra $\mathcal{W}$ of order n is a subspace of $\mathrm{Mat}_{n\times n}(\mathbb{C})$ such that:

- MN and $M \circ N \in \mathcal{W}$ for all $M, N \in \mathcal{W}$.
- $M^T \in \mathcal{W}$ for all $M \in \mathcal{W}$.
- $I, J \in \mathcal{W}$.

Primitive Matrices

Definition

An element $M \in \mathcal{W}$ of a coherent algebra $\mathcal{W}$ is said to be primitive if $M^2 = M$ and $MN \in \text{span}\{M\}$ for all $N \in \mathcal{W}$.

Schur-Primitive Matrices

Definition

Dually, an element $M \in \mathcal{W}$ of a coherent algebra $\mathcal{W}$ is said to be Schur-primitive if $M^{\circ 2} = M$ and $M \circ N \in \operatorname{span}\{M\}$ for all $N \in \mathcal{W}$.

Dual Bases

Theorem

Every commutative coherent algebra is spanned by its set of primitive matrices relative to both Schur-product and ordinary matrix multiplication.

Suppose that $\mathcal{W}$ is a commutative coherent algebra of order n.

Let $\Gamma(\mathcal{W})$ denote the set of Schur-primitive matrices in $\mathcal{W}$.

Dually, let $\Lambda(\mathcal{W})$ denote the set of primitive matrices in $\mathcal{W}$.

Suppose that $\{A_i : 1 \leq i \leq d\}$ and $\{E_i : 1 \leq i \leq d\}$ are orderings on $\Gamma(\mathcal{W})$ and $\Lambda(\mathcal{W})$, letting $d := \dim(\mathcal{W})$.

Character Tables and Dual Character Tables

The matrix $P \in \mathrm{Mat}_{d \times d}(\mathbb{C})$ such that $A_i = \sum_{j=1}^{d} P_{i,j} E_j$ for all $1 \leq i \leq d$ is said to be the character table of $\mathcal{W}$ (under these orderings).

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Dually, the matrix $Q \in \mathrm{Mat}_{d \times d}(\mathbb{C})$ such that $E_i = \sum_{j=1}^{d} \left(\frac{1}{n} Q_{i,j} \right) A_j$ for all $1 \leq i \leq d$ is said to be the dual character table of $\mathcal{W}$ (under these orderings).

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We have that $P^{-1} = \frac{1}{n} Q$ and hence $PQ = nI$.

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$$\langle \cdot, \cdot \rangle : \mathsf{Mat}_{d \times d}(\mathbb{C}) \times \mathsf{Mat}_{d \times d}(\mathbb{C}) \to \mathbb{C}$$

$$\langle A, B \rangle := \mathrm{tr}\left(B^* A\right) = \mathrm{sum}\left(\overline{B} \circ A\right)$$

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$\implies v_j Q_{i,j} = \langle E_i, A_j \rangle = m_i P_{j,i}$ for all $1 \leq i, j \leq d$.

Hence $Q = R \circ P^T$ where $R \in \mathrm{Mat}_{d \times d}(\mathbb{C})$ and $R_{i,j} := \frac{m_i}{v_j}$ for all $1 \leq i, j \leq d$.

Intersection Numbers and Dual Intersection Numbers

There is a set of numbers $\{p_{i,j}(k) : 1 \leq i, j, k \leq d\}$ such that $A_i A_j = \sum_{k=1}^{d} p_{i,j}(k) A_k$ for all $1 \leq i, j \leq d$.

These are said to be the intersection numbers of $\mathcal{W}$.

Intersection Numbers and Dual Intersection Numbers

Dually, there is a set of numbers $\{q_{i,j}(k) : 1 \leq i, j, k \leq d\}$ such that $E_i \circ E_j = \sum_{k=1}^{d} \left(\frac{1}{n} q_{i,j}(k)\right) E_k$ for all $1 \leq i, j \leq d$.

These are said to be the dual intersection numbers of $\mathcal{W}$ (or Krein numbers of $\mathcal{W}$).

Intersection Numbers and Dual Intersection Numbers

The intersection numbers of $\mathcal{W}$ are non-negative integers.

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Duality Mappings

Definition

A duality mapping between (commutative) coherent algebras $\mathcal{W}$ and $\mathcal{X}$ of order n is a linear isomorphism $T : \mathcal{W} \to \mathcal{X}$ such that:

- $T(MN) = T(M) \circ T(N)$ for all $M, N \in \mathcal{W}$.
- $T(M \circ N) = \frac{1}{n} T(M) T(N)$ for all $M, N \in \mathcal{W}$.

Dual Coherent Algebras

There is a duality mapping between coherent algebras $\mathcal{X}$ and $\mathcal{Y}$ of dimension d if and only if:

1. There is a character table P of $\mathcal{X}$ and dual character table Q of $\mathcal{Y}$ such that $\overline{P} = Q$.

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1. There is a character table P of $\mathcal{X}$ and dual character table Q of $\mathcal{Y}$ such that $\overline{P} = Q$.
2. $p_{i,j}(k) = q_{i,j}(k)$ for all $1 \leq i, j, k \leq d$,

   where $\{p_{i,j}(k) : 1 \leq i, j, k \leq d\}$ are the intersection numbers of $\mathcal{X}$

   and $\{q_{i,j}(k) : 1 \leq i, j, k \leq d\}$ are the dual intersection numbers of $\mathcal{Y}$.

Metric and Cometric Coherent Algebras

A (commutative) coherent algebra $\mathcal{W}$ is said to be metric if $\mathcal{W} = \text{span}\{I, A, A^2, \cdots\}$ for some Schur-primitive matrix $A \in \mathcal{W}$.

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Dually, a (commutative) coherent algebra $\mathcal{W}$ is said to be cometric if $\mathcal{W} = \mathrm{span}\{J, E, E^{\circ 2}, \cdots\}$ for some primitive matrix $E \in \mathcal{W}$.

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Dually, a (commutative) coherent algebra $\mathcal{W}$ is said to be cometric if $\mathcal{W} = \text{span}\{J, E, E^{\circ 2}, \cdots\}$ for some primitive matrix $E \in \mathcal{W}$.

If there is a duality mapping on a coherent algebra $\mathcal{W}$, then $\mathcal{W}$ is metric if and only if it is cometric.

Metric and Cometric Coherent Algebras

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Coherent algebras that are both metric and symmetric can be identified with the adjacency algebras of distance-regular graphs.

The eigenvalue multiplicities of a distance-regular graph are the valencies of its distance-graphs if there is a duality mapping on its adjacency algebra.

Self-Dual Coherent Algebras

Definition

A coherent algebra $\mathcal{W}$ of order n is said to be self-dual if there is a duality mapping $T : \mathcal{W} \to \mathcal{W}$ on $\mathcal{W}$ such that:

- $T^{-1}(M) = \frac{1}{n} T(M)$ for all $M \in \mathcal{W}$.
- $T(\lambda M) = \overline{\lambda}\ T(M)$ for all $\lambda \in \mathbb{C}$ and $M \in \mathcal{W}$.

Self-Dual Coherent Algebras

A coherent algebra $\mathcal{W}$ is self-dual if and only if there is a character table P for $\mathcal{W}$ with dual character table $\overline{P}$.

Strongly-Regular Graphs

Suppose that G is a (connected) strongly-regular graph of order n and valency k.

$$\implies P := \begin{pmatrix} 1 & 1 & 1 \\ k & \lambda & \mu \\ n-1-k & -1-\lambda & -1-\mu \end{pmatrix}$$

is a character table for $\mathcal{A}$ for some $\lambda, \mu \in \mathbb{R}$

$$\text{and } Q := \begin{pmatrix} 1 & 1 & 1 \\ j & \left(\frac{j}{k}\right)\lambda & \left(\frac{j}{n-1-k}\right)(-1-\lambda) \\ n-1-j & \left(\frac{n-1-j}{k}\right)\mu & \left(\frac{n-1-j}{n-1-k}\right)(-1-\mu) \end{pmatrix}$$

is the dual character table of $\mathcal{A}$ relative to P for some $j \in \mathbb{N}$.

Self-Dual Strongly-Regular Graphs

Theorem

Suppose that G is a strongly-regular graph of order n and valency k. Then G is self-dual if and only if:

$$(\lambda - \mu)^2 = n$$

with $\{k, \lambda, \mu\}$ the distinct eigenvalues of G.

Self-Dual Strongly-Regular Graphs

Theorem

A strongly-regular graph G is self-dual if and only if G is either:

1. A conference graph.
2. An SRG $\left(m^2, (m-1)\gamma, (\gamma-1)(\gamma-2)+(m-2), \gamma(\gamma-1)\right)$ for some $m, \gamma \in \mathbb{N}$.

Self-Dual Strongly-Regular Graphs

Theorem

A strongly-regular graph G is self-dual if and only if the multiplicities of its eigenvalues are the valencies of G and $\overline{G}$.

Self-Dual Distance-Transitive Graphs

Example

The Hamming graphs are self-dual.

To be continued ...