Duality in Coherent Algebras

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Coherent Algebras

Definition

A coherent algebra $\mathcal{W}$ of order n is a subspace of $\mathrm{Mat}_{n\times n}(\mathbb{C})$ such that:

- MN and $M \circ N \in \mathcal{W}$ for all $M, N \in \mathcal{W}$.
- $M^T \in \mathcal{W}$ for all $M \in \mathcal{W}$.
- $I, J \in \mathcal{W}$.

Primitive Matrices

Definition

An element $M \in \mathcal{W}$ of a coherent algebra $\mathcal{W}$ is said to be primitive if $M^2 = M$ and $MN \in \mathrm{span}\{M\}$ for all $N \in \mathcal{W}$.

Schur-Primitive Matrices

Definition

Dually, an element $M \in \mathcal{W}$ of a coherent algebra $\mathcal{W}$ is said to be Schur-primitive if $M^{\circ 2} = M$ and $M \circ N \in \operatorname{span}\{M\}$ for all $N \in \mathcal{W}$.

Dual Bases

Every coherent algebra is spanned by its set of Schur-primitive matrices.

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Every coherent algebra is spanned by its set of Schur-primitive matrices.

Dually, every commutative coherent algebra is spanned by its set of primitive matrices as well.

Suppose that $\mathcal{W}$ is a commutative coherent algebra of order n.

Let $\Gamma(\mathcal{W})$ denote the set of Schur-primitive matrices in $\mathcal{W}$.

Dually, let $\Lambda(\mathcal{W})$ denote the set of primitive matrices in $\mathcal{W}$.

Suppose that $\{A_i : 1 \leq i \leq d\}$ and $\{E_i : 1 \leq i \leq d\}$ are orderings on $\Gamma(\mathcal{W})$ and $\Lambda(\mathcal{W})$, letting $d := \dim(\mathcal{W})$.

Character Tables and Dual Character Tables

The matrix $P \in \mathrm{Mat}_{d \times d}(\mathbb{C})$ such that $A_i = \sum_{j=1}^{d} P_{i,j} E_j$ for all $1 \leq i \leq d$ is said to be the character table of $\mathcal{W}$ (under these orderings).

Character Tables and Dual Character Tables

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Dually, the matrix $Q \in \mathrm{Mat}_{d\times d}(\mathbb{C})$ such that $E_i = \sum_{j=1}^{d} \left(\frac{1}{n} Q_{i,j}\right) A_j$ for all $1 \leq i \leq d$ is said to be the dual character table of $\mathcal{W}$ (under these orderings).

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We have that $P^{-1} = \frac{1}{n} Q$ and hence $PQ = nI$.

Dual Character Tables in Terms of Character Tables

Define the trace inner product $\langle \cdot, \cdot \rangle$ on $\mathsf{Mat}_{d\times d}(\mathbb{C})$:

$$\langle \cdot, \cdot \rangle : \mathsf{Mat}_{d\times d}(\mathbb{C}) \times \mathsf{Mat}_{d\times d}(\mathbb{C}) \to \mathbb{C}$$

$$\langle A, B \rangle := \mathrm{tr}\left(B^* A\right) = \mathrm{sum}\left(\overline{B} \circ A\right)$$

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Hence $Q = R \circ P^T$ where $R \in \mathrm{Mat}_{d \times d}(\mathbb{C})$ and $R_{i,j} := \frac{m_i}{v_j}$ for all $1 \leq i, j \leq d$.

Intersection Numbers and Dual Intersection Numbers

There is a set of numbers $\{p_{i,j}(k) : 1 \leq i, j, k \leq d\}$ such that $A_i A_j = \sum_{k=1}^{d} p_{i,j}(k) A_k$ for all $1 \leq i, j \leq d$.

These are said to be the intersection numbers of $\mathcal{W}$.

Intersection Numbers and Dual Intersection Numbers

Dually, there is a set of numbers $\{q_{i,j}(k) : 1 \leq i, j, k \leq d\}$ such that $E_i \circ E_j = \sum_{k=1}^{d} \left(\frac{1}{n} q_{i,j}(k)\right) E_k$ for all $1 \leq i, j \leq d$.

These are said to be the dual intersection numbers of $\mathcal{W}$ (or Krein numbers of $\mathcal{W}$).

Intersection Numbers and Dual Intersection Numbers

The intersection numbers of $\mathcal{W}$ are non-negative integers.

The dual intersection numbers of $\mathcal{W}$ are non-negative real numbers.

Duality Mappings

Definition

A duality mapping between (commutative) coherent algebras $\mathcal{W}$ and $\mathcal{X}$ of order n is a linear isomorphism $T : \mathcal{W} \to \mathcal{X}$ such that:

- $T(MN) = T(M) \circ T(N)$ for all $M, N \in \mathcal{W}$.
- $T(M \circ N) = \frac{1}{n} T(M) T(N)$ for all $M, N \in \mathcal{W}$.

Duality Mappings

There is a duality mapping between commutative coherent algebras $\mathcal{X}$ and $\mathcal{Y}$ of dimension d if and only if:

1. $p_{i,j}(k) = q_{i,j}(k)$ for all $1 \leq i, j, k \leq d$,

   where $\{p_{i,j}(k) : 1 \leq i, j, k \leq d\}$ are the intersection numbers of $\mathcal{X}$

   and $\{q_{i,j}(k) : 1 \leq i, j, k \leq d\}$ are the dual intersection numbers of $\mathcal{Y}$.
2. There is a character table P of $\mathcal{X}$ and dual character table Q of $\mathcal{Y}$ such that $\overline{P} = Q$.

Metric and Cometric Coherent Algebras

A (commutative) coherent algebra $\mathcal{W}$ is said to be metric if $\mathcal{W} = \mathrm{span}\{I, A, A^2, \cdots\}$ for some Schur-primitive matrix $A \in \mathcal{W}$.

Metric and Cometric Coherent Algebras

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Dually, a (commutative) coherent algebra $\mathcal{W}$ is said to be cometric if $\mathcal{W} = \text{span}\{J, E, E^{\circ 2}, \cdots\}$ for some primitive matrix $E \in \mathcal{W}$.

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Dually, a (commutative) coherent algebra $\mathcal{W}$ is said to be cometric if $\mathcal{W} = \text{span}\{J, E, E^{\circ 2}, \cdots\}$ for some primitive matrix $E \in \mathcal{W}$.

If there is a duality mapping on a coherent algebra $\mathcal{W}$, then $\mathcal{W}$ is metric if and only if it is cometric.

Metric and Cometric Coherent Algebras

Coherent algebras that are both metric and symmetric can be identified with the adjacency algebras of distance-regular graphs.

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Coherent algebras that are both metric and symmetric can be identified with the adjacency algebras of distance-regular graphs.

The eigenvalue multiplicities of a distance-regular graph are the valencies of its distance-graphs if there is a duality mapping on its adjacency algebra.

Self-Dual Coherent Algebras

Definition

A (commutative) coherent algebra $\mathcal{W}$ is said to be (formally) self-dual if there is a character table P for $\mathcal{W}$ with dual character table $\overline{P}$.

Finite Groups and Coherent Algebras

Suppose that G is a finite group of order n.

Let $L^2(G)$ denote the space of complex-valued functions on G.

Define $M_f \in \mathrm{Mat}_{G\times G}(\mathbb{C})$ as $(M_f)_{x,y} := f(x^{-1}y)$ for all $x, y \in G$.

We have that $M_f M_g = M_{f\star g}$ for all $f, g \in L^2(G)$ with convolution $f \star g$.

Finite Groups and Coherent Algebras

Additionally, define $A_g \in \mathrm{Mat}_{G\times G}(\mathbb{C})$ as $(A_g))_{x,y} = xy^{-1}$ for all $x, y \in G$ and $g \in G$.

The coherent algebra $\mathcal{W}_G := \mathrm{span}\{A_g : g \in G\}$

$= \{M_f : f \in L^2(G)\}$ can be identified with the group ring of G.

Refer to W_G as the group coherent algebra of G.

Self-Dual Coherent Algebras from Finite Abelian Groups

Assume that G is abelian.

Let $X(G) \subseteq L^2(G)$ denote the set of irreducible characters of G.

Then $W_G = \operatorname{span}\left\{\frac{1}{n}M_\chi : \chi \in X(G)\right\}$
where $\left\{\frac{1}{n}M_\chi : \chi \in X(G)\right\}$ is the set of primitive matrices in $\mathcal{W}_G$.

Self-Dual Coherent Algebras from Finite Abelian Groups

A character table for $\mathcal{W}_G$ can be identified with a character table for G in the usual sense.

There is an ordering $\{\chi_x : x \in G\}$ on $X(G)$ such that $\chi_x(y) = \chi_y(x)$ for all $x, y \in G$,

since G can be decomposed as a direct product of finite cyclic groups.

There is then a character table P for $\mathcal{W}_G$ such that $P\overline{P} = nI$,

since $X(G)$ forms an orthonormal basis for $L^2(G)$ as well.

Hence $\mathcal{W}_G$ is self-dual.

Fourier Transforms

Definition

A Fourier transform on a (commutative) coherent algebra $\mathcal{W}$ of order n is a duality mapping $T : \mathcal{W} \to \mathcal{W}$ such that:

- $T^{-1}(M) = \overline{\frac{1}{n} T(M)}$ for all Schur-primitive matrices $M \in \mathcal{W}$.

Fourier Transforms

A Fourier transform on the group coherent algebra $\mathcal{W}_G$ of a finite abelian group G can be identified with a Fourier transform on the space $L^2(G)$ of complex-valued functions on G.

Fourier Transforms and Self-Dual Coherent Algebras

There is a Fourier transform on a (commutative) coherent algebra $\mathcal{W}$ if and only if there is a character table P of $\mathcal{W}$ such that dual character table of $\mathcal{W}$ is $\overline{P}$.

Strongly-Regular Graphs

Suppose that G is a (connected) strongly-regular graph of order n and valency k.

$$\implies P := \begin{pmatrix} 1 & 1 & 1 \\ k & \lambda & \mu \\ n-1-k & -1-\lambda & -1-\mu \end{pmatrix}$$

is a character table for $\mathcal{A}$ for some $\lambda, \mu \in \mathbb{R}$

$$\text{and } Q := \begin{pmatrix} 1 & 1 & 1 \\ j & \left(\frac{j}{k}\right)\lambda & \left(\frac{j}{n-1-k}\right)(-1-\lambda) \\ n-1-j & \left(\frac{n-1-j}{k}\right)\mu & \left(\frac{n-1-j}{n-1-k}\right)-1-\mu \end{pmatrix}$$

is the dual character table of $\mathcal{A}$ relative to P for some $j \in \mathbb{N}$.

Strongly-Regular Graphs

Proposition

The adjacency algebra of G is self-dual if
G is an $\mathrm{SRG}(\cdot, \cdot, \lambda, \lambda + 1)$ for some $\lambda \geq 0$.

More Self-Dual Coherent Algebras

The adjacency algebras of (connected) self-complementary strongly-regular graphs are self-dual.

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The adjacency algebras of (connected) self-complementary strongly-regular graphs are self-dual.

The adjacency algebras of Hamming graphs are self-dual coherent algebras.

Is it possible to classify the self-dual distance-regular graphs?