HESSIAN-BASED IMAGE FILTERS

Contents

Summary

This manual documents a modular MATLAB package for Hessian-based image filtering, including multiscale detectors such as vesselness and ridge filters, and a faithful implementation of the single-scale neuriteness filter of Meijering et al. The document provides mathematical background, algorithmic design rationale, complete documentation of the software components, a validated test suite, and reproducible demos that generate figures for inclusion in this manual. The presentation follows a Tufte-style layout to emphasize clarity, provenance, and marginal commentary.

end

Introduction

Second-order differential structure plays a central role in the analysis of curvilinear features in images. By examining the Hessian matrix of an image, one can characterize local geometry—distinguishing between blobs, ridges, plates, and tubular structures. Hessian-based filters have become standard tools in biomedical image analysis, particularly for vessel and neurite detection.

This manual covers two related but conceptually distinct families of methods:

- *Multiscale Hessian detectors*, such as vesselness and ridge filters, which combine responses across scales using a max-over-scales strategy.
- *Neuriteness*, as introduced by Meijering et al., which is a single-scale, shape-based descriptor derived from a modified Hessian.

A key design principle of the accompanying software is to keep these families separate, sharing only the low-level numerical primitives.

Mathematical Background

The Hessian Matrix

Given an image $I(x,y)$, the Hessian matrix at scale σ is defined as

$$\mathbf{H} * \sigma(I) = \begin{pmatrix} \partial * xx I_\sigma & \partial_{xy} I_\sigma \, \partial_{xy} I_\sigma & \partial_{yy} I_\sigma \end{pmatrix}, \tag{1}$$

where I_σ denotes the image smoothed by a Gaussian of standard deviation σ.

Following Lindeberg, scale-normalized derivatives are used:

$$\partial_{xx} I_\sigma \leftarrow \sigma^2 \partial_{xx} I_\sigma, \tag{2}$$

and similarly for the other second derivatives. This normalization allows meaningful comparison of responses across scales.

Eigenvalues and Local Geometry

Let λ_1 and λ_2 denote the eigenvalues of the Hessian, ordered such that $|\lambda_1| \leq |\lambda_2|$. The signs and relative magnitudes of these eigenvalues characterize local structure:

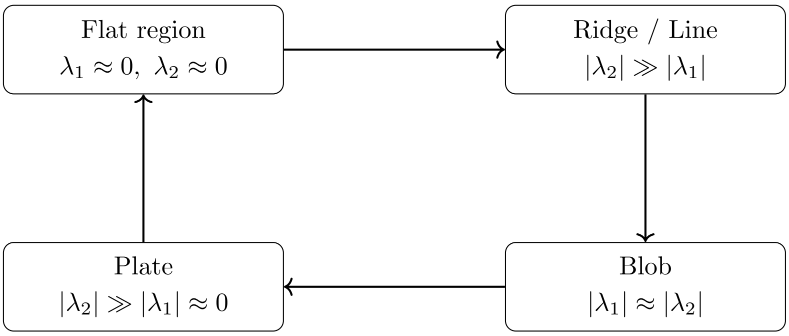

Figure 1: Interpretation of Hessian eigenvalues in 2D: flat regions, ridges, and blobs.

- $|\lambda_2| \gg |\lambda_1|$: line- or ridge-like structure.
- $|\lambda_1| \approx |\lambda_2|$: blob-like structure.
- both small: flat region.

Eigenvectors provide orientation information; the eigenvector corresponding to λ_2 is normal to a ridge, and the tangent direction is obtained by a 90° rotation.

Multiscale Hessian-Based Filters

Vesselness and Ridge Filters

Frangi's vesselness filter combines Hessian eigenvalues into a scalar response designed to enhance tubular structures while suppressing blobs and noise. At each scale σ, a response V_σ is computed and the final response is

$$V(x) = \max_{\sigma \in \Sigma} V_\sigma(x). \tag{3}$$

This multiscale max operation provides scale selection implicitly. Similar principles apply to ridge and plate filters.

Design Constraints

Multiscale filters assume:

- responses are comparable across scales,
- polarity (bright-on-dark vs dark-on-bright) is a semantic choice,
- detection confidence can be derived from response strength and scale agreement.

These assumptions do *not* hold for neuriteness, motivating a separate implementation.

Neuriteness (Meijering et al.)

Modified Hessian

Meijering et al. propose a modified Hessian to suppress blob-like responses:

$$\tilde{\lambda}_1 = \lambda_1 + \alpha\lambda_2,\ \tilde{\lambda}_2 = \lambda_2 + \alpha\lambda_1, \tag{4}$$

with $\alpha = -\frac{1}{3}$. The eigenvalues are then reordered so that $|\tilde{\lambda}_1| \geq |\tilde{\lambda}_2|$.

Neuriteness Measure

The neuriteness response is defined as

$$N(x) = \begin{cases} \dfrac{\tilde{\lambda}_1(x)}{\min(\tilde{\lambda}_1)}, & \tilde{\lambda}_1(x) < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

This yields a normalized, shape-based measure in $[0, 1]$.

Key Properties

- single-scale only,
- contrast-invariant,
- no max-over-scales,
- no detection confidence semantics.

Software Architecture

The package is organized into three layers: numerical core, multiscale engine, and neuriteness. Only the numerical core is shared.

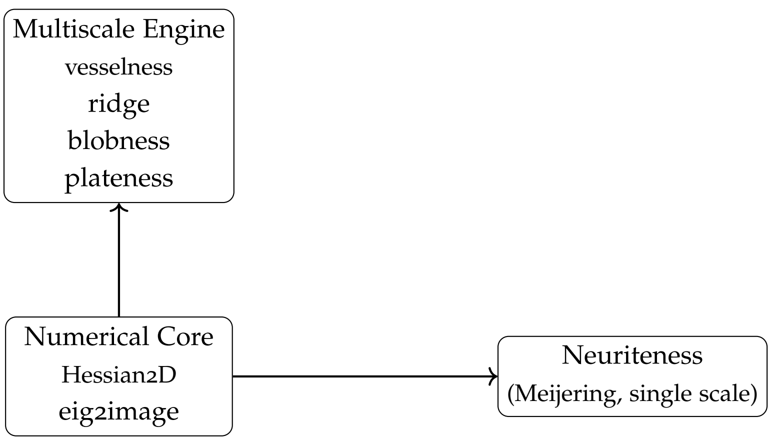

Figure 2: Software architecture showing shared numerical primitives and separation between multiscale Hessian detectors and single-scale neuriteness.

Core Functions

`Hessian2D.m` Computes second-order Gaussian derivatives.

`eig2image.m` Eigenvalues and eigenvectors of the Hessian.

Multiscale Engine

`hessian2DFilters.m` implements vesselness, ridge, blob, and plate filters using a max-over-scales strategy.

Neuriteness

`neuriteness2D.m` implements the Meijering neuriteness filter exactly as described in the literature.

Test Suite

The accompanying test suite is layered to match the architecture:

- numerical invariants (Hessian, eigenvalues),
- detector semantics (vesselness),
- shape invariants (neuriteness).

Each test enforces a documented mathematical or algorithmic guarantee; no test asserts behaviour not promised by the underlying theory.

Synthetic Benchmark Phantom

To illustrate the differing assumptions and response characteristics of multiscale Hessian-based detectors and single-scale neuriteness, we employ a synthetic benchmark image containing a mixture of geometric primitives:

- thin and thick curvilinear structures,
- intersections at multiple orientations,
- isotropic blob-like structures,
- plate-like regions with low curvature anisotropy.

This composite phantom intentionally violates the assumptions of any single structure model, making it suitable for qualitative comparison.

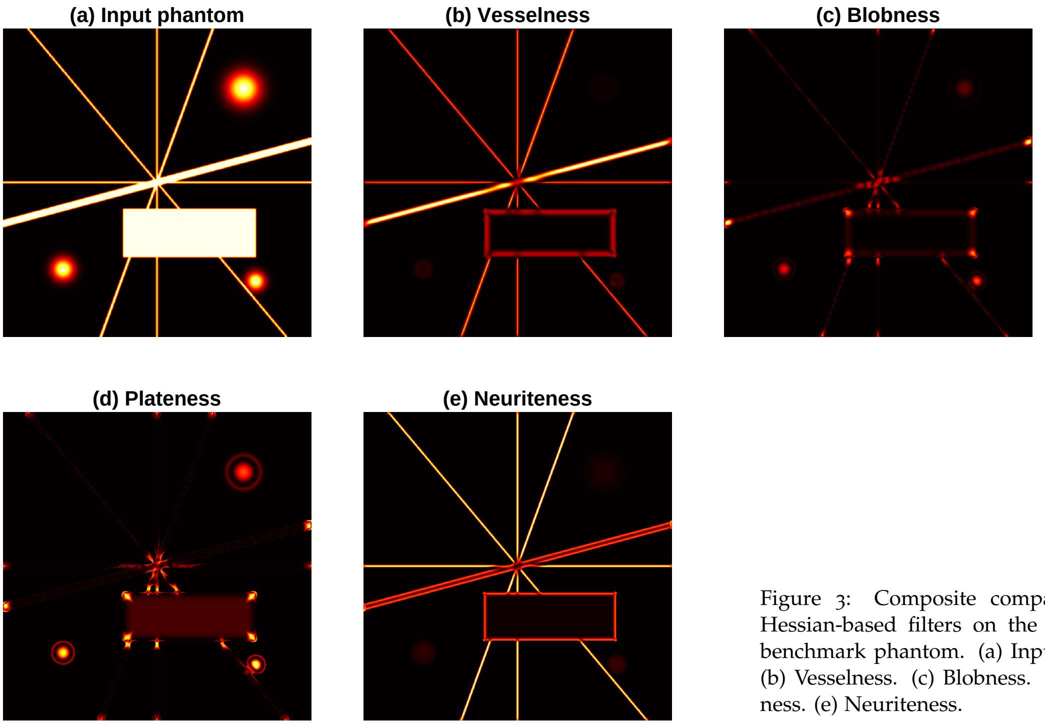

Figure 3: Composite comparison of Hessian-based filters on the synthetic benchmark phantom. (a) Input image. (b) Vesselness. (c) Blobness. (d) Plateness. (e) Neuriteness.

Demos and Figure Generation

Demo Script

The following MATLAB script generates figures used in this manual.

Listing 1: Demo script to generate figures for the manual

```
% demo_generate_figures.m
Iline = generateTestLineImage(256,30,3);
Iblob = generateTestBlobImage(256,10);

[V,~,D] = hessian2DFilters(Iline, ...
'FilterType','vesselness', ...
'Sigmas',1:4, ...
'Parameters',struct('beta',0.5,'c',15));

[N,DN] = neuriteness2D(Iline,2);

figure; imagesc(Iline); axis image off; colormap gray;
title('Input␣image');

figure; imagesc(V); axis image off; colormap hot;
title('Vesselness␣response');

figure; imagesc(N); axis image off; colormap hot;
title('Neuriteness␣response');
```

Figures produced by this script can be saved and included as PDF files in the `figures/` directory for use in this document.

In addition to simple line and blob examples, a more complex synthetic benchmark phantom is generated to highlight the complementary behavior of multiscale vesselness and single-scale neuriteness filters.

Repository Layout and Build Instructions

Folder Structure

```
hessian-neuriteness/
|-- core/
|   |-- Hessian2D.m
|   |-- eig2image.m
|   `-- hessianEigen2D.m
|
|-- engine/
|-- neuriteness/
|-- demos/
|-- tests/
`-- manual/
```

Building the Manual

```
cd manual
```

```
pdflatex manual.tex
bibtex manual
pdflatex manual.tex
pdflatex manual.tex
```

Test Suite

The test suite mirrors the architecture of the codebase:

- **Core tests**: numerical invariants of Hessian derivatives and eigenvalues.
- **Engine tests**: detection semantics for multiscale Hessian filters.
- **Neuriteness tests**: shape continuity and orientation invariants.

No test asserts behavior not guaranteed by the underlying theory.

Bibliography

A. F. Frangi, W. J. Niessen, K. L. Vincken, and M. A. Viergever. Multiscale vessel enhancement filtering. *Medical Image Computing and Computer-Assisted Intervention*, 1998.

E. Meijering, M. Jacob, J.-C. Sarria, P. Steiner, H. Hirling, and M. Unser. Design and validation of a tool for neurite tracing and analysis in fluorescence microscopy images. *Cytometry Part A*, 58A:167–176, 2004.

T. Lindeberg. Feature detection with automatic scale selection. *International Journal of Computer Vision*, 30(2):79–116, 1998.