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Preconditioners for the indefinite linear system arising from the *hp* discretization of Maxwell's equations

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SUMMARY

This paper describes a preconditioner for the iterative solution to the indefinite linear equation system that is obtained when the vector wave equation is discretized by higher-order **H**(curl)-conforming finite elements. The preconditioner is based on a decomposition of the **H**(curl)-conforming functions into blocks comprising lower-order functions and higher-order functions. Fast convergence is obtained when the grid spacing is sufficiently small. A method for selecting suitable grid spacings is proposed and a series of numerical examples are included to demonstrate the effectiveness of the approach. Copyright © 2008 John Wiley & Sons, Ltd.

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KEY WORDS: higher-order edge finite elements; iterative solution techniques; preconditioners; indefinite linear systems

1. INTRODUCTION

The use of **H**(curl)-conforming finite elements, or edge finite elements as they are better known within the engineering community, is widely accepted as a highly effective method for approximating the time harmonic vector wave equation:

$$\operatorname{curl}\mu^{-1}\operatorname{curl}\mathbf{A}+\kappa\mathbf{A}=\mathbf{J} \tag{1}$$

where $\mathbf{A}$ is the vector potential, μ is the permeability and $\mathbf{J}$ is a source term. The parameter κ influences the problem type: $\kappa=0$ corresponds to the magnetostatic problem, $\kappa=\mathrm{i}\omega\sigma$ to quasi-static fields, $\kappa=-\varepsilon\omega^2$ to high-frequency applications and $\kappa=\mathrm{i}\omega\sigma-\varepsilon\omega^2$ to general time harmonic fields. Depending on the application, it is observed that κ is a function of the conductivity σ, the permittivity ε, the frequency ω and the complex unit $\mathrm{i}=\sqrt{-1}$.

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H(curl)-conforming finite elements can be traced back to the pioneering work of Nédélec [1, 2]. The last two decades has been a period of extensive activity in this area and a large variety of different discretization schemes each with varying degrees of accuracy have been proposed. A recent review by Ledger and Morgan [3] summarizes some of the key developments in the field. In this review it was highlighted how, until recently, schemes that proposed higher-order edge element approximations were often fragmented and it was unclear as to which elements could be combined together. A substantial development was made by Demkowicz and his coworkers who proposed an approach capable of generating **H**(curl)-conforming finite elements of arbitrary order on hybrid meshes of quadrilaterals and triangles [4, 5] and later on structured meshes of hexahedra [6].

Further progress was made by Ainsworth and Coyle [7] who developed an alternative **H**(curl)-conforming finite element basis on hybrid meshes of quadrilaterals and triangles, which is better conditioned than the basis functions proposed by Demkowicz and his coworkers. Ainsworth and Coyle later extended these ideas to arbitrary-order **H**(curl)-conforming elements on tetrahedral meshes [8]. In [9] Ainsworth and Coyle derive theoretical bounds for the condition number of the mass, curl–curl and system matrices for quadrilateral and hexahedral elements when $\kappa>0$. These bounds are useful for investigating the performance of iterative solution techniques applied to the linear system obtained when $\kappa>0$ as, in this case, the number of iterations required to reach a specified tolerance depends on the condition number alone [10]. Ainsworth and Coyle [9] show that preconditioning in the form of diagonal scaling has a positive effect on the growth of the condition number (for higher-order elements) for the mass and system matrices and thus will lead to reduced number of iterations for problems where $\kappa>0$. For $\kappa<0$ the performance of iterative solution techniques depends on the distribution of the eigenvalues in the complex plane. As in the case where $\kappa>0$, it is desirable to use preconditioning techniques in an attempt to improve the speed of convergence of the iterative algorithm.

A new set of higher-order **H**(curl)-conforming finite elements has been developed by Schöberl and Zaglmayr [11] and Zaglmayr [12], which lends itself to the efficient iterative solution to the linear equation system. To discuss their strategy, the definitions of the spaces **H**(curl) and **H**(div) are first introduced:

$$\mathbf{H}(\mathrm{curl}) = \{\mathbf{v}|\mathbf{v}\in(L_2)^d, \mathrm{curl}\,\mathbf{v}\in(L_2)^{2d-3}\}$$
$$\mathbf{H}(\mathrm{div}) = \{\mathbf{v}|\mathbf{v}\in(L_2)^d, \mathrm{div}\,\mathbf{v}\in L_2\}$$

where $d=2,3$ is the number of space dimensions of the problem. It is well known that these spaces satisfy the sequence

$$H^1 \xrightarrow{\nabla} \mathbf{H}(\mathrm{curl}) \xrightarrow{\mathrm{curl}} \mathbf{H}(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2$$

in three space dimensions [13]. The spaces **H**(curl) and **H**(div) also satisfy a similar set of two shortened sequences in two space dimensions [12]. By interpreting the above diagram it can be observed that the gradient of a H^1-conforming function is a **H**(curl)-conforming function. The curl of a **H**(curl)-conforming function is a **H**(div)-conforming function and, finally, the divergence of a **H**(div)-conforming function is an L_2-conforming function. The idea of Schöberl and Zaglmayr was to construct finite element bases that satisfy this sequence *exactly* on a discrete level.

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The result is the De Rham complex

$$\begin{array}{ccccccc} H^1 & \overset{\nabla}{\longrightarrow} & \mathbf{H}(\mathrm{curl}) & \overset{\mathrm{curl}}{\longrightarrow} & \mathbf{H}(\mathrm{div}) & \overset{\mathrm{div}}{\longrightarrow} & L^2 \\ \cup & & \cup & & \cup & & \cup \\ W_{h,p+1} & \overset{\nabla}{\longrightarrow} & V_{h,p} & \overset{\mathrm{curl}}{\longrightarrow} & Q_{h,p-1} & \overset{\mathrm{div}}{\longrightarrow} & S_{h,p-2} \end{array}$$

for problems set in three space dimensions (two similar diagrams exist for the two-dimensional case [12]). In the above, $W_{h,p+1}$ is the space of discrete H^1-conforming finite element basis functions of degree $p+1$, $V_{h,p}$ is the space of $\mathbf{H}(\mathrm{curl})$-conforming finite element basis functions of degree p, $Q_{h,p-1}$ is the space of $\mathbf{H}(\mathrm{div})$-conforming finite element basis functions of degree $p-1$ and $S_{h,p-2}$ is the space of L_2-conforming finite element basis functions of degree $p-2$. In addition to the new sets of basis functions, Zaglmayr [12] also proposed an efficient preconditioner for solving linear systems resulting from the discretization of the weak form of (1) for problems where $\kappa>0$. Such a preconditioner is a generalization of the one proposed by Schöberl *et al.* [14] for H^1-conforming problems.

The purpose of this work is to examine how to construct a preconditioner for the indefinite linear equation system that is obtained when a higher-order $\mathbf{H}(\mathrm{curl})$ discretization of (1) is adopted for the case where $\kappa<0$. The proposed preconditioner is a modification of that proposed by Zaglmayr for the positive-definite case. There are many possible engineering applications that correspond to the case where $\kappa<0$; one such example is the electromagnetic scattering problem, which is important for the aerospace industry (e.g. [15]).

This paper is structured as follows: In Section 2 the key ingredients of the Zaglmayr preconditioner are examined. In Section 3 it is shown how the preconditioner proposed by Zaglmayr can be modified to solve problems where the system matrix is indefinite. Numerical experiments in Sections 4 and 5 justify the effectiveness of the approach. A conclusion completes the work.

2. INGREDIENTS OF THE PRECONDITIONER FOR $\kappa>0$

In order to describe the ingredients of the preconditioner for $\kappa>0$, the following model problem is introduced: Find $\mathbf{u}$ such that

$$\begin{aligned} \mathrm{curl}\,\mu^{-1}\mathrm{curl}\,\mathbf{u}+\kappa\mathbf{u}&=\mathbf{f} \quad \text{in } \Omega \\ \mathbf{n}\times\mathbf{u}&=\mathbf{n}\times\mathbf{g} \quad \text{on } \partial\Omega \end{aligned} \tag{2}$$

with $\kappa, \mu>0$ homogeneous and isotropic in Ω. For simplicity, only Dirichlet boundary conditions have been specified on $\partial\Omega$ although Neumann boundary conditions could equally be included. The weak form of this problem is: Find $\mathbf{u}\in\mathbf{H}_{\mathrm{D}}(\mathrm{curl})$ such that

$$(\mu^{-1}\mathrm{curl}\,\mathbf{u},\mathrm{curl}\,\mathbf{w})_\Omega+(\kappa\mathbf{u},\mathbf{w})_\Omega=(\mathbf{f},\mathbf{w})_\Omega \quad \forall\mathbf{w}\in\mathbf{H}_0(\mathrm{curl}) \tag{3}$$

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where $(\mathbf{u},\mathbf{v})_\Omega = \int_\Omega \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\Omega$ and

$$\mathbf{H}_\mathrm{D}(\mathrm{curl}) = \{\mathbf{v}|\mathbf{v}\in\mathbf{H}(\mathrm{curl}), \mathbf{n}\times\mathbf{v}=\mathbf{n}\times\mathbf{g} \text{ on } \partial\Omega\}$$

$$\mathbf{H}_0(\mathrm{curl}) = \{\mathbf{v}|\mathbf{v}\in\mathbf{H}(\mathrm{curl}), \mathbf{n}\times\mathbf{v}=\mathbf{0} \text{ on } \partial\Omega\}$$

It is well known that the Galerkin finite element discretization of this variational statement yields the linear system of equations

$$Kx = b \tag{4}$$

where K is a sparse $n\times n$ symmetric system matrix, x is a column vector of unknowns and b is a known column vector. Matrix K has the property that $x^\mathrm{T}Kx>0$ for all non-zero vectors x, which means that it is positive definite and consequently has positive-real eigenvalues.

Consideration is restricted to two-dimensional problems discretized by uniform degree p elements on meshes of quadrilateral elements. The extension of the approach to triangular elements in two dimensions and tetrahedral and hexahedral elements in three dimensions is postponed until future work. On this type of mesh, the global finite element space for Schöberl and Zaglmayr's H^1-conforming discretization [11, 12] corresponds to the decomposition

$$W_{p+1} = W_V + \sum_{E_i} W_{E_i} + \sum_{I_i} W_{I_i} \subset H^1$$

where the subscripts V, E and I reflect the blocks associated with the vertices, edges and element interiors, respectively. In addition, the global finite element space for their $\mathbf{H}(\mathrm{curl})$-conforming discretization can also be decomposed according to

$$V_p = V_{\mathscr{N}_0} + \sum_{E_i} V_{E_i} + \sum_{I_i} V_{I_i} \subset \mathbf{H}(\mathrm{curl})$$

where the subscripts $\mathscr{N}_0$, E and I reflect the blocks associated with the lowest-order ($p=0$) edge functions, higher-order ($p>0$) edge functions and interior functions, respectively. The De Rham complex is then fulfilled by choosing

$$\nabla W_V \subset V_{\mathscr{N}_0}, \quad \nabla W_{E_i}^{p+1} = V_{E_i}^p, \quad W_{I_i}^{p+1} \subset V_{I_i}^p$$

If a discretization based on Schöberl and Zaglmayr's basis functions is used, the linear system in (4) may be partitioned as

$$K = \begin{pmatrix} K_{\mathscr{N}_0\mathscr{N}_0} & K_{\mathscr{N}_0 E} & K_{\mathscr{N}_0 I} \\ K_{E\mathscr{N}_0} & K_{EE} & K_{EI} \\ K_{I\mathscr{N}_0} & K_{IE} & K_{II} \end{pmatrix}, \quad x = \begin{pmatrix} x_{\mathscr{N}_0} \\ x_E \\ x_I \end{pmatrix}, \quad b = \begin{pmatrix} b_{\mathscr{N}_0} \\ b_E \\ b_I \end{pmatrix}$$

where each of the blocks in K is sparse. For high p the size of the interior block grows rapidly ($O(p^2)$ in two dimensions). However, as the interior degrees of freedom are independent between elements, static condensation can be used to eliminate them, yielding

$$\underbrace{\begin{pmatrix} \hat{K}_{\mathscr{N}_0\mathscr{N}_0} & \hat{K}_{\mathscr{N}_0 E} \\ \hat{K}_{E\mathscr{N}_0} & \hat{K}_{EE} \end{pmatrix}}_{\hat{K}} \underbrace{\begin{pmatrix} x_{\mathscr{N}_0} \\ x_E \end{pmatrix}}_{\hat{x}} = \underbrace{\begin{pmatrix} \hat{b}_{\mathscr{N}_0} \\ \hat{b}_E \end{pmatrix}}_{\hat{b}}$$

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where

$$\begin{pmatrix} \hat{K}_{\mathcal{N}_0\mathcal{N}_0} & \hat{K}_{\mathcal{N}_0 E} \\ \hat{K}_{E\mathcal{N}_0} & \hat{K}_{EE} \end{pmatrix} = \begin{pmatrix} K_{\mathcal{N}_0\mathcal{N}_0} & K_{\mathcal{N}_0 E} \\ K_{E\mathcal{N}_0} & K_{EE} \end{pmatrix} - \begin{pmatrix} K_{\mathcal{N}_0 I} \\ K_{EI} \end{pmatrix} K_{II}^{-1} (K_{I\mathcal{N}_0} \quad K_{IE})$$

$$\begin{pmatrix} \hat{b}_{\mathcal{N}_0} \\ \hat{b}_E \end{pmatrix} = \begin{pmatrix} b_{\mathcal{N}_0} \\ b_E \end{pmatrix} - \begin{pmatrix} K_{\mathcal{N}_0 I} \\ K_{EI} \end{pmatrix} K_{II}^{-1} b_I$$

In reality static condensation is performed on an element level for better efficiency. Static condensation has been reported to improve the condition number of the system matrix for **H**(curl)-conforming problems [7].

Zaglmayr [12] proposed the block Jacobi preconditioner C^{-1} with

$$C = \begin{pmatrix} \hat{K}_{\mathcal{N}_0\mathcal{N}_0} & 0 \\ 0 & \hat{K}_{EE} \end{pmatrix}$$

for the iterative solution to $\hat{K}\hat{x} = \hat{b}$. In the iterative scheme, the application of the preconditioner requires operations of the form $C^{-1}\hat{K}y = C^{-1}z$ with known y (and hence known z) to be evaluated at least once every iteration. In reality the inverse is never formed and instead the linear system $Cw = z$ is solved for w. Owing to the block structure of C the application of C^{-1} can be found by applying $\hat{K}_{\mathcal{N}_0\mathcal{N}_0}^{-1}$ and $\hat{K}_{EE}^{-1}$ separately. In Zaglmayr's approach, the application of $\hat{K}_{\mathcal{N}_0\mathcal{N}_0}^{-1}$ is achieved through the use of a direct solver and the application of $\hat{K}_{EE}^{-1}$ is achieved through an iterative solver. Numerical experiments that show the effectiveness of this preconditioner have been reported by Zaglmayr [12].

3. INGREDIENTS OF THE PRECONDITIONER FOR $\kappa<0$

In order to describe the ingredients of the preconditioner for $\kappa<0$, the model problem (2) with $\kappa<0$, $\mu>0$ homogeneous and isotropic in Ω is now considered. The linear system resulting from using **H**(curl)-conforming elements is

$$Kx = b \tag{5}$$

where the symmetric matrix K is such that $x^{\mathrm{T}}Kx<0$ or $x^{\mathrm{T}}Kx>0$ depending on x. This means that K is a symmetric indefinite matrix with both positive and negative eigenvalues. By adopting Schöberl and Zaglmayr's elements and performing static condensation, this system becomes

$$\underbrace{\begin{pmatrix} \hat{K}_{\mathcal{N}_0\mathcal{N}_0} & \hat{K}_{\mathcal{N}_0 E} \\ \hat{K}_{E\mathcal{N}_0} & \hat{K}_{EE} \end{pmatrix}}_{\hat{K}} \underbrace{\begin{pmatrix} x_{\mathcal{N}_0} \\ x_E \end{pmatrix}}_{\hat{x}} = \underbrace{\begin{pmatrix} \hat{b}_{\mathcal{N}_0} \\ \hat{b}_E \end{pmatrix}}_{\hat{b}}$$

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As a first approach for developing an iterative strategy for this equation, one might be tempted to apply the same preconditioner that was used for the case where $\kappa>0$, namely C^{-1} with

$$C=\begin{pmatrix}\hat{K}_{\mathcal{N}_0\mathcal{N}_0} & 0\\ 0 & \hat{K}_{EE}\end{pmatrix}$$

In this case the blocks $\hat{K}_{\mathcal{N}_0\mathcal{N}_0}$ and $\hat{K}_{EE}$ are indefinite. If the approach outlined for the case where $\kappa>0$ is followed, the application of $\hat{K}^{-1}_{\mathcal{N}_0\mathcal{N}_0}$ is to be performed by a direct solver. Despite $\hat{K}_{\mathcal{N}_0\mathcal{N}_0}$ being indefinite, this does not pose any difficulty; however, the application of $\hat{K}^{-1}_{EE}$ using an iterative solver is made difficult as $\hat{K}_{EE}$ is indefinite. In particular, one must then consider an iterative approach that is suited to indefinite linear equation systems such as the generalized minimum residual (GMRES) method or the quasi-minimal residual (QMR) method [10] for the application of this part of the preconditioner. In addition, depending on the distribution of the eigenvalues of $\hat{K}_{EE}$, convergence might be slow. To overcome these restrictions, the modified preconditioner D^{-1} with

$$D=\begin{pmatrix}\hat{K}_{\mathcal{N}_0\mathcal{N}_0} & 0\\ 0 & -\hat{\tilde{K}}_{EE}\end{pmatrix} \tag{6}$$

is proposed. In this preconditioner, the application of $\hat{K}^{-1}_{\mathcal{N}_0\mathcal{N}_0}$ is accomplished using a direct solver. The block $\hat{\tilde{K}}_{EE}$ is chosen so that it is positive definite, which means that the application of $\hat{\tilde{K}}^{-1}_{EE}$ is guaranteed to converge in $\dim(\hat{\tilde{K}}_{EE})$ iterations for exact arithmetic if a Krylov subspace iterative solver such as conjugate gradient (CG) is used [10].

To construct a suitable $\hat{\tilde{K}}_{EE}$ it is first recalled that Schöberl and Zaglmayr's higher-order $\mathbf{H}(\mathrm{curl})$ edge basis functions are constructed as the gradients of their corresponding H^1 edge functions and have vanishing curl. By considering typical entries in the blocks K_{EE}, K_{EI} and K_{IE} of K for $\kappa<0$ it is possible to express

$$(K_{EE})_{ij}=(\mu^{-1}\mathrm{curl}\,\boldsymbol{\phi}_i^E,\mathrm{curl}\,\boldsymbol{\phi}_j^E)_\Omega+(\kappa\boldsymbol{\phi}_i^E,\boldsymbol{\phi}_j^E)_\Omega=-(|\kappa|\boldsymbol{\phi}_i^E,\boldsymbol{\phi}_j^E)_\Omega=-(\tilde{K}_{EE})_{ij} \tag{7}$$

$$(K_{EI})_{ij}=(\mu^{-1}\mathrm{curl}\,\boldsymbol{\phi}_i^E,\mathrm{curl}\,\boldsymbol{\phi}_j^I)_\Omega+(\kappa\boldsymbol{\phi}_i^E,\boldsymbol{\phi}_j^I)_\Omega=-(|\kappa|\boldsymbol{\phi}_i^E,\boldsymbol{\phi}_j^I)_\Omega=-(\tilde{K}_{EI})_{ij} \tag{8}$$

$$(K_{IE})_{ij}=(\mu^{-1}\mathrm{curl}\,\boldsymbol{\phi}_i^I,\mathrm{curl}\,\boldsymbol{\phi}_j^E)_\Omega+(\kappa\boldsymbol{\phi}_i^I,\boldsymbol{\phi}_j^E)_\Omega=-(|\kappa|\boldsymbol{\phi}_i^I,\boldsymbol{\phi}_j^E)_\Omega=-(\tilde{K}_{IE})_{ij} \tag{9}$$

where $\boldsymbol{\phi}_i^E$ and $\boldsymbol{\phi}_i^I$ are typical higher-order edge and interior basis functions, respectively. The interior functions include functions with non-zero curl and this prompts the definition of

$$\hat{\tilde{K}}_{EE}=\tilde{K}_{EE}-\tilde{K}_{EI}\tilde{K}_{II}^{-1}\tilde{K}_{IE} \tag{10}$$

where typical entries in $\tilde{K}_{II}$ are defined as

$$(\tilde{K}_{II})_{ij}=(\mu^{-1}\mathrm{curl}\,\boldsymbol{\phi}_i^I,\mathrm{curl}\,\boldsymbol{\phi}_j^I)_\Omega+(|\kappa|\boldsymbol{\phi}_i^I,\boldsymbol{\phi}_j^I)_\Omega \tag{11}$$

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The definition of $\hat{\tilde{K}}_{EE}$ is in fact a modification of the static condensation procedure for determining $\hat{K}_{EE}$ where κ has been replaced by $|\kappa|$. To check whether $\hat{\tilde{K}}_{EE}$ has the required positive-definite property, it is first noted that the matrices $\tilde{K}_{EE}$, $\tilde{K}_{EI}$, $\tilde{K}_{II}$ and $\tilde{K}_{IE}$ are derived from the matrix

$$\tilde{K}=\begin{pmatrix} \tilde{K}_{\mathcal{N}_0\mathcal{N}_0} & \tilde{K}_{\mathcal{N}_0 E} & \tilde{K}_{\mathcal{N}_0 I} \\ \tilde{K}_{E\mathcal{N}_0} & \tilde{K}_{EE} & \tilde{K}_{EI} \\ \tilde{K}_{I\mathcal{N}_0} & \tilde{K}_{IE} & \tilde{K}_{II} \end{pmatrix}$$

which in turn is identical to matrix K obtained for the case when $\kappa>0$. This matrix is symmetric positive definite and it can be shown (e.g. [10]) that its Schur complement $\hat{\tilde{K}}_{EE}$ also has the positive-definite property. Finally, as only some of the interior functions have non-zero curl, $-\hat{\tilde{K}}_{EE}$ is an approximation of $\hat{K}_{EE}$ and therefore $\hat{\tilde{K}}_{EE}$ is multiplied by -1 in the definition of D.

4. RESULTS FOR $\kappa<0$

To test the preconditioner proposed in Section 3, the model problem in which the field

$$\mathbf{u}=\begin{pmatrix} k_y \\ -k_x \end{pmatrix}\exp\{\mathrm{i}(k_x x+k_y y)\} \tag{12}$$

with $\kappa=-\omega^2=-(k_x^2+k_y^2)$ satisfies $\operatorname{curl}\operatorname{curl}\mathbf{u}+\kappa\mathbf{u}=\mathbf{0}$ in the domain $\Omega=(0,1)^2$ is considered. Dirichlet boundary conditions are used although Neumann boundary conditions could equally be applied. This problem corresponds to the propagation of a plane wave across Ω where $\omega=2\pi/\lambda$ is the wave number and λ is the wavelength of the wave. The QMR [10] technique with preconditioner D^{-1} is used to solve the linear system $\hat{K}\hat{x}=\hat{b}$. It is noted that other iterative schemes such as BICG (biconjugate gradients) or GMRES approach could also be used, but QMR was found to perform well for the problems considered. A direct solver is used for application of $K^{-1}_{\mathcal{N}_0\mathcal{N}_0}$ and CG is used for the application of $\hat{\tilde{K}}^{-1}_{EE}$.

The convergence of the iterative technique is investigated for a mesh of uniformly sized quadrilaterals with mesh spacing $h=\frac{1}{4}$. On this mesh the polynomial degrees $p=1$, 3 and 5 are considered in turn. Consideration is restricted to waves that propagate diagonally across the domain (i.e. $k_x=k_y$). In particular the cases $\kappa=-10^{-6}$, -10^{-3}, -1 and -10^3 are considered. For each case, the Euclidean norm ($\|\cdot\|_2$) of the relative residual given by $\|r\|_2/\|D^{-1}\hat{b}\|_2$ where $r=(D^{-1}\hat{K}\hat{x}_i-D^{-1}\hat{b})$ is plotted against the number of iterations. The iterative scheme was deemed to converge when the convergence criteria $\|r\|_2/\|D^{-1}\hat{b}\|_2<10^{-9}$ were met. If the number of iterations exceeds the dimension of the matrix the iterative procedure was terminated. The results are shown in Figure 1. In this figure rapid convergence is observed for $\kappa=-10^{-3}$ and -1 for each of the polynomial degrees considered. For $\kappa=-10^{-6}$ the convergence is rapid until a relative residual of approximately 10^{-8}; thereafter the iterations stagnate. It is believed that this behaviour is due to the round-off error associated with matrix $\hat{K}$ being close to singular, as the condition numbers of $\hat{K}$ measured in the Euclidean norm for $p=1,3,5$ are 1.4185e+09, 6.8085e+09 and 1.7419e+10, respectively. It can also be observed that there is a marked increase in the number of iterations

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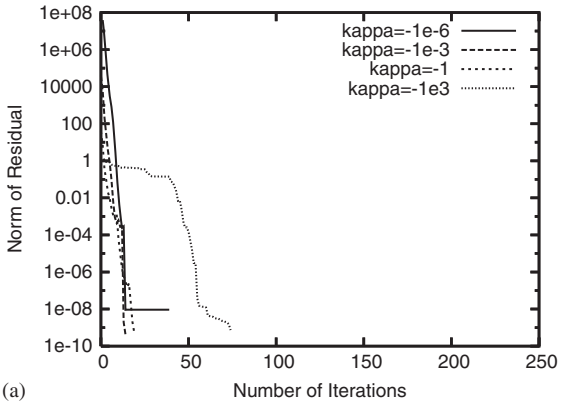
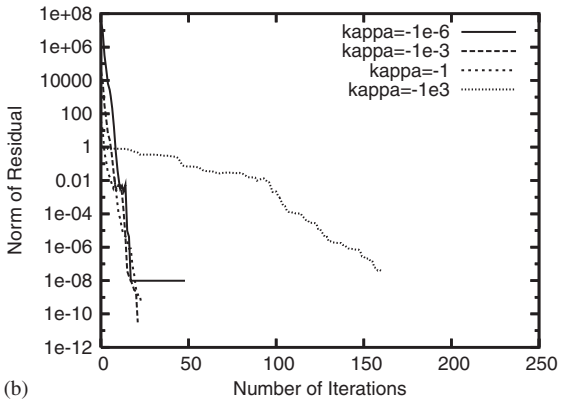


(a)



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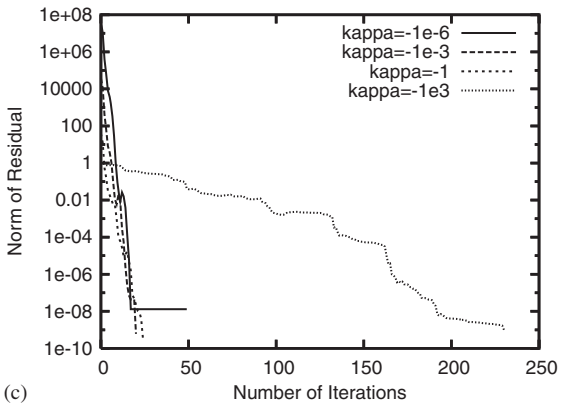

(c)

Figure 1. Convergence of relative residual for QMR with the preconditioner D^{-1} for a range of polynomial orders and negative values of κ: (a) $p=1$; (b) $p=3$; and (c) $p=5$.

for the case where $\kappa=-10^3$ for each of the polynomial degrees considered. In order to interpret this behaviour it is first remarked that the $\hat{K}_{\mathcal{N}_0\mathcal{N}_0}$ block plays a special role in the preconditioner and carries global information [12]. In particular this block corresponds to a discretization of the

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problem using the lowest-order **H**(curl)-conforming elements. For the mesh spacing $h=\frac{1}{4}$ and $\kappa=-10^3$, the lowest-order discretization corresponds to an approximation in which there is only one sampling point inside each element and approximately only one element per wavelength. The Nyquest sampling theorem dictates that there must be at least two sampling points per wavelength in order to sample a sine wave [16]. The larger iteration counts would therefore seem to suggest that insufficient global information is being carried by the preconditioner for this particular case. For other values of κ considered a similar interpretation leads to the conclusion that there are more than two elements per wave length and hence more than two sampling points.

To further illustrate that the mesh spacing in conjunction with the lowest-order discretization plays a crucial role in the performance of the the preconditioner for $\kappa<0$, $\kappa=-10^3$ is fixed and a sequence of meshes with decreasing spacing is considered. Polynomial degrees $p=1,3$ and 5 are considered and meshes are constructed in which the spacing is such that there are two, four, six and eight elements per wavelength. The results of applying the iterative technique to these discretizations is shown in Figure 2. In this figure it is observed how reducing the mesh spacing decreases the number of iterations required to achieve the convergence criteria. For the cases considered, the spacing is such that there are at least two elements per wavelength and the procedure is seen to converge to the required tolerance within the specified number of iterations. The memory constraints of the small desktop computer used for the experiments prevent the cases of $p=3$ with eight elements per wavelength and $p=5$ with six and eight elements per wavelength from being computed. However, it is expected that the results would follow a similar pattern to those shown.

To investigate the impact that the reduction in the spacing has on the preconditioner, the convergence behaviour of QMR is investigated. This can be described (e.g. Malhotra *et al.* [17]) by

$$\|d-B\hat{x}_j\|_2 \leqslant \sqrt{j+1}\|U\|_2\|U^{-1}\|_2\|d-B\hat{x}_0\|_2 \min_{\psi\in P_j:\psi(0)=1} \max_{\lambda\in\lambda(\hat{A})} |\psi(\lambda)| \tag{13}$$

where $d=D^{-1}b$ and $B=D^{-1}\hat{K}=U\Lambda U^{-1}$ is assumed to be diagonalizable and have eigenvalues $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$. The dominating factor in (13) is $\min_{\psi\in P_j:\psi(0)=1}\max_{\lambda\in\lambda(\hat{A})}|\psi(\lambda)|$. This is small if and only if there exists a polynomial ψ of degree at most j and with normalization $\psi(0)=1$ that is small for all eigenvalues of B. Thus the smaller the degree j of such a polynomial $\psi(0)=1$ that is small on the eigenspectrum of B, the faster the convergence will be.

Motivated by these considerations, Malhotra *et al.* suggest that a preconditioner should be chosen such that the eigenspectrum of B has eigenvalues that are clustered and bounded away from zero. Indeed, the more clustered the eigenvalues, the faster the convergence will be. Figure 3 shows the eigenspectrum of B for $\kappa=-10^3$ when $p=1$ is fixed and meshes with spacings two, four, six and eight elements per wavelength are used. In this figure it is observed that as the mesh spacing is reduced, the eigenvalues become more clustered. This directly corresponds to the case shown in Figure 2 where the number of iterations required to achieve convergence of QMR falls as the mesh spacing is reduced. In contrast, the eigenspectrum of the unpreconditioned matrix $\hat{K}$, for the same set of discretizations, does not exhibit this behaviour, as illustrated in Figure 4. Unlike $B=D^{-1}\hat{k}$, which has complex eigenvalues, $\hat{K}$ is symmetric indefinite and therefore its eigenvalues are all real. Not only are the magnitude of the eigenvalues $\hat{K}$ much larger than those of B but their spectrum also does not exhibit clustering as the mesh spacing is reduced. Therefore, without preconditioning, we cannot necessarily expect that the number of iterations will fall as the mesh spacing is reduced.

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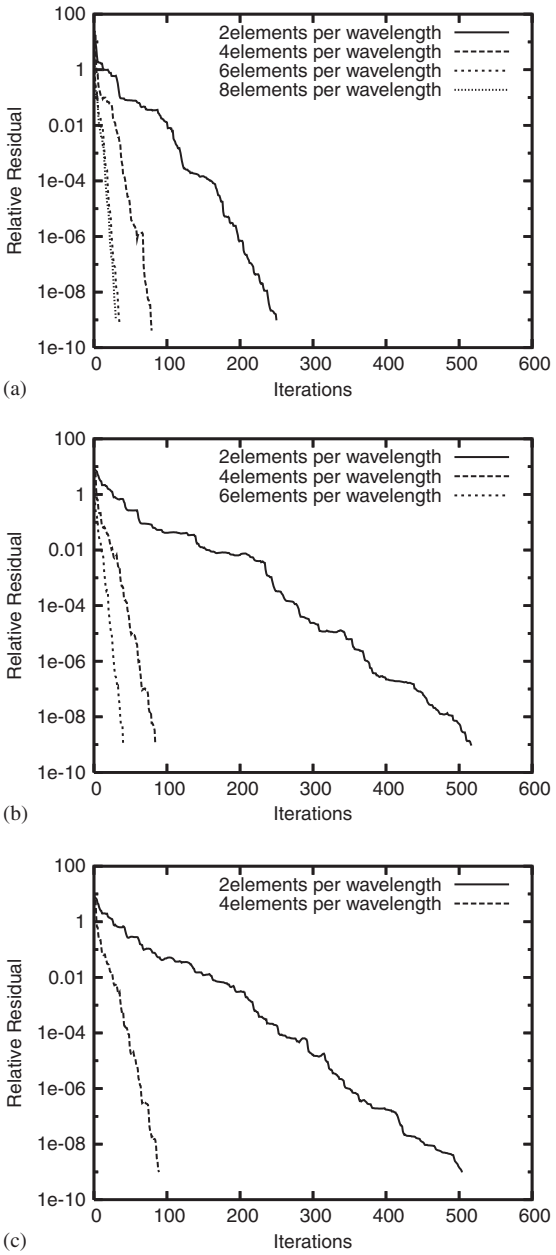

Figure 2. Convergence of relative residual for QMR with the preconditioner D^{-1} for $\kappa = -10^3$ using a range of polynomial orders and decreasing mesh spacing: (a) $p = 1$; (b) $p = 3$; and (c) $p = 5$.

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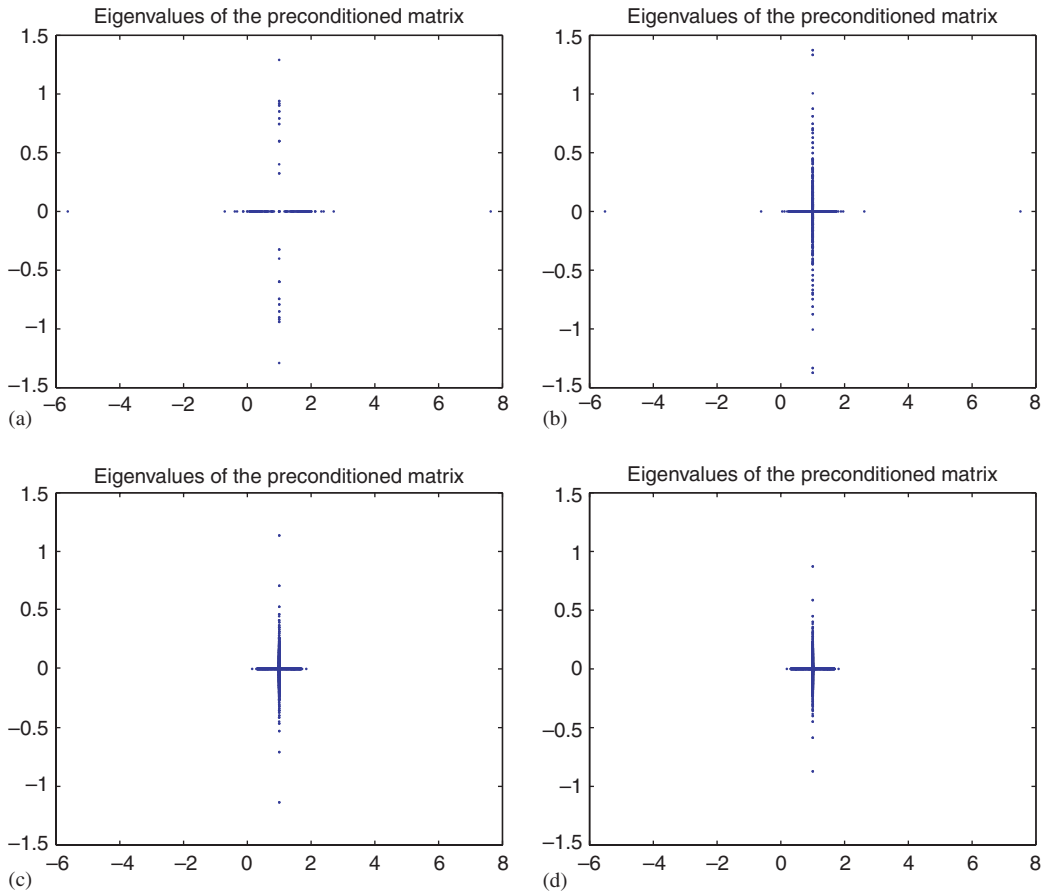

Figure 3. Eigenspectrum of $B = D^{-1}\hat{K}$ for the case when $\kappa = -10^3$ and $p = 1$ with varying mesh spacings: (a) two elements per wavelength; (b) four elements per wavelength; (c) six elements per wavelength; and (d) eight elements per wavelength.

5. CHOICE OF MESH SPACING FOR $\kappa < 0$

For problems with smooth solutions, such as the one considered in the previous section, it is well known that the finite element refinement strategy that results in the fewest number of unknowns is obtained by performing p refinement on a coarse mesh (e.g. [7, 15]). Although one must balance this strategy against the ability to solve the linear system, it is certainly true that when using a direct solution strategy, one would prefer to perform p refinement on a coarse mesh. However, for the preconditioned iterative solution strategy presented previously, it was observed how the indefinite nature of the system matrix leads to an additional constraint on the mesh spacing. In particular, it was seen how the spacing must be selected to be small enough so that the iterative strategy will converge; thereafter p refinement (on the same mesh) can be performed while maintaining the same rapid convergence behaviour of the iterative solver. This somewhat surprising behaviour is also supported by the theory presented relatively recently by Gopalakrishnan *et al.* [18] who

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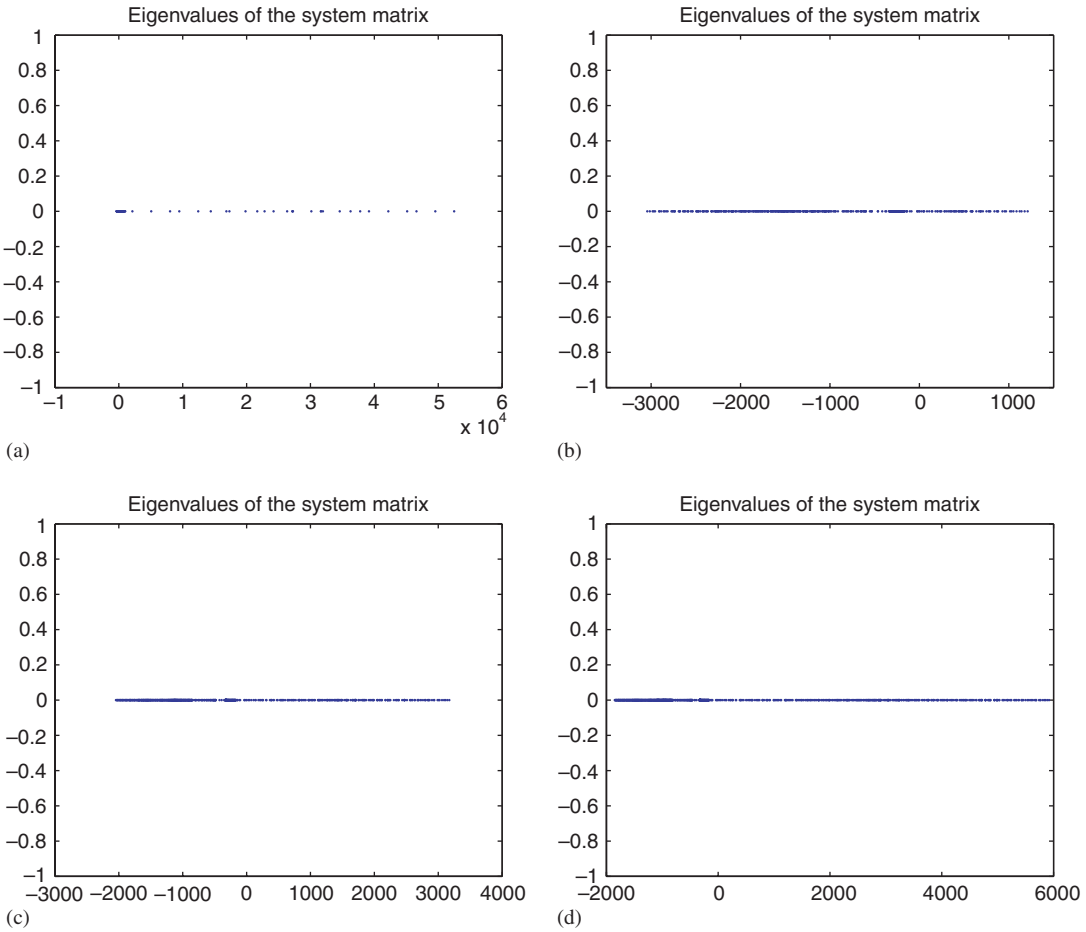

Figure 4. Eigenspectrum of $\hat{K}$ for the case when $\kappa = -10^3$ and $p = 1$ with varying mesh spacings: (a) two elements per wavelength; (b) four elements per wavelength; (c) six elements per wavelength; and (d) eight elements per wavelength.

show, in the context of multigrid for indefinite problems, that there exists a critical mesh spacing that was required in order to guarantee convergence for lowest-order elements. Specifically, in the appendix of [18], it is shown that there also exists a critical mesh spacing for the p-version of extension of **H**(curl)-conforming elements.

Beck *et al.* [19] have previously proposed a heuristic argument that the coarse mesh spacing for multigrid based on low-order **H**(curl)-conforming elements for indefinite problems should not exceed two elements per wavelength. Further work by Gopalakrishnan and Pasciak [20] on Schwartz preconditioners for low-order **H**(curl)-conforming elements indicates that this might not be sufficient for large ω. Rather than using a heuristic argument based on elements per wavelength, it is proposed that the spacing be chosen on the basis of controlling the pollution error. For wave propagation problems, it is well known that the error is made up of two parts: the pollution error

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and the interpolation error. Previously engineers had adopted rule of thumb of using 10–12 points per wavelength in order to obtain a *good* solution with the lowest-order nodal finite elements. However, it is now widely accepted that this no longer holds as ω is increased. A dispersion relationship has been derived, which shows that for a fixed p, the number of elements required per wavelength to control the pollution error increases as ω is increased [21, 22].

It is proposed that the mesh spacing h, for high wave numbers ω, be chosen such that $3-\omega h-(\omega h)^{1/3}>0$, independent of the polynomial degree p. This choice is made purely to ensure the rapid convergence of the preconditioned iterative solution technique. Specifically, it represents the choice of mesh size for the lowest-order elements such that subsequent p refinement results in super exponential convergence of the error for the finite element solution. The criteria are derived from the relationship proposed by Ainsworth [21, 22] for selecting the polynomial degree and mesh spacing so as to properly control the pollution error. Once this critical mesh spacing has been selected, p refinement is recommended to best capture the propagated wave.

Table I shows, for a range of values of $\kappa=-\omega^2$, the critical values of h (h_{crit}) obtained from the criteria $3-\omega h-(\omega h)^{1/3}>0$. Alongside, the corresponding number of iterations required in order to achieve the convergence for the model problem that was discussed in Section 4 is quoted. Results are shown for discretizations where the mesh spacing is uniform and uniform polynomial degrees $p=1, 3$ and 5 are used. It is observed that although the number of iterations increases for higher ω^2, this is at the same time as a corresponding growth in the size of the linear system.

To illustrate that the approach is still competitive, it is first remarked that a discretization based on using the lowest-order edge elements on a mesh with spacing h_{crit} would lead to a rather inaccurate solution. However, by selecting the initial mesh spacing to be h_{crit} and increasing p, the proposed preconditioner works well and leads to rapid convergence of the iterative solution to

Table I. Number of iterations required to reach the prescribed tolerance when the mesh spacing is chosen such that $3-\omega h-(\omega h)^{1/3}>0$.

ω^2	h_{crit}	$p=1$ iterations	$p=3$ iterations	$p=5$ iterations
1	1	5	7	19
10	$\frac{1}{2}$	11	17	18
50	$\frac{1}{4}$	32	37	38
100	$\frac{1}{6}$	33	36	40
200	$\frac{1}{8}$	39	46	50
400	$\frac{1}{11}$	54	63	78
600	$\frac{1}{14}$	56	68	77
800	$\frac{1}{16}$	67	74	81
1000	$\frac{1}{18}$	78	92	93
1200	$\frac{1}{19}$	85	128	137
1400	$\frac{1}{21}$	110	118	128
1600	$\frac{1}{22}$	133	179	183
1800	$\frac{1}{24}$	132	158	168
2000	$\frac{1}{25}$	161	162	170

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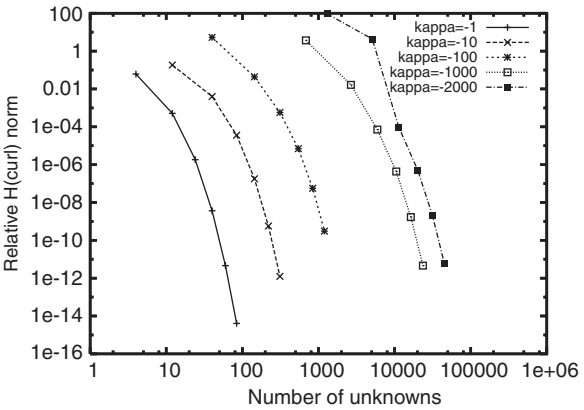

Figure 5. Error measured in the **H**(curl) norm against the number of unknowns for discretizations based the mesh spacing h_{crit} with uniform polynomial refinement from $p=0$ to 5 in increments of 1.

the linear system. At the same time p refinement will lead to exponential convergence of the finite element solution. As an illustration of this, convergence curves for the error in the finite element solution corresponding to performing p refinement on meshes where the spacing is selected as h_{crit} for $\kappa=-1,-10,-100,-1000$ and -2000 is shown in Figure 5. The error in this figure is quoted in the usual relative **H**(curl) norm (e.g. [13]). For each curve, the set of points reflects increasing polynomial degree from $p=0$ to 5 in increments of 1. The first point in each case corresponds to using a discretization based on the lowest-order edge element on a mesh with spacing h_{crit}. Although the resulting solution is inaccurate, this discretization carries enough global information to allow the rapid convergence of the iterative solver for the higher polynomial degrees on the same mesh. The corresponding iterations counts for QMR for $p=1,3$ and 5 can be found in Table I. The shape of the error curves reflects that exponential convergence of the finite element solution in the **H**(curl) norm is being obtained for p refinement for the values of κ considered.

6. CONCLUSION

In this paper the preconditioner proposed by Zaglmayr [12] has been modified to enable indefinite linear systems of equations to be solved iteratively. The new preconditioner enables systems of equations resulting from the discretization of the vector wave equation for $\kappa<0$ by higher-order quadrilateral edge elements to be solved effectively. The application of the preconditioner requires a direct solution of the block made up of the lowest-order edge functions, irrespective of the polynomial degree in the mesh. The higher-order block is always solved iteratively. The mesh spacing was seen to play a crucial role in the convergence of the scheme. By choosing the spacing using the dispersion relationship of Ainsworth [21, 22], it was observed that, for the numerical examples presented, rapid convergence of the iterative solver could be obtained. For discretizations based on this spacing, the preconditioner was seen to lead to rapid convergence of the iterative solver and p refinement was seen to yield the usual exponential rate of convergence for the finite element solution. A detailed theoretical investigation of the approach, including the extension to other element types, is planed for future work.

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