FiveThirtyEight Riddler - August 28, 2020

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Prompt

Your friend's granddaughter claimed that she once won a game of War that lasted exactly 26 turns.

War is a two-player game in which a standard deck of cards is first shuffled and then divided into two piles with 26 cards each — one pile for each player. In every turn of the game, both players flip over and reveal the top card of their deck. The player whose card has a higher rank wins the turn and places both cards on the bottom of their pile. In the event that both cards have the same rank, the rules get a little more complicated, with each player flipping over additional cards to compare in a "mini War" showdown. Your friend's granddaughter said that for every turn of the game, she always drew the card of higher rank, with no "mini Wars."

Assuming a deck is randomly shuffled before every game, how many games of War would you expect to play until you had a game that lasted just 26 turns with no "mini Wars," like your friend's granddaughter?

Breaking Apart the Problem

The first step to solving this problem involves determining the odds of a given game of War ending in only 26 turns. To find those odds, let's break the problem down even further and consider only the first turn of a game. On that turn, one of three things can happen: Player 1 wins the first turn, Player 2 wins the first turn, or the two players reveal cards of equal rank and they tie. (Note that in a deck of 52 cards, we assume there are 13 such ranks: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A. Note also that there are $52/13 = 4$ cards of each rank).

To begin, we can show that the probability that the two players tie on the first turn is $3/51$. You can visualize this by imagining that Player 1 starts by revealing a card of rank x on the first turn. At this point, we know that the first turn can result in a tie only if Player 2 also reveals a card of rank x. Since Player 1 has already revealed one such card, there are only 3 remaining cards of rank x (of a total 51 cards remaining) that Player 2 could reveal to tie the turn. Thus, the probability of a tie on this turn is $3/51$.

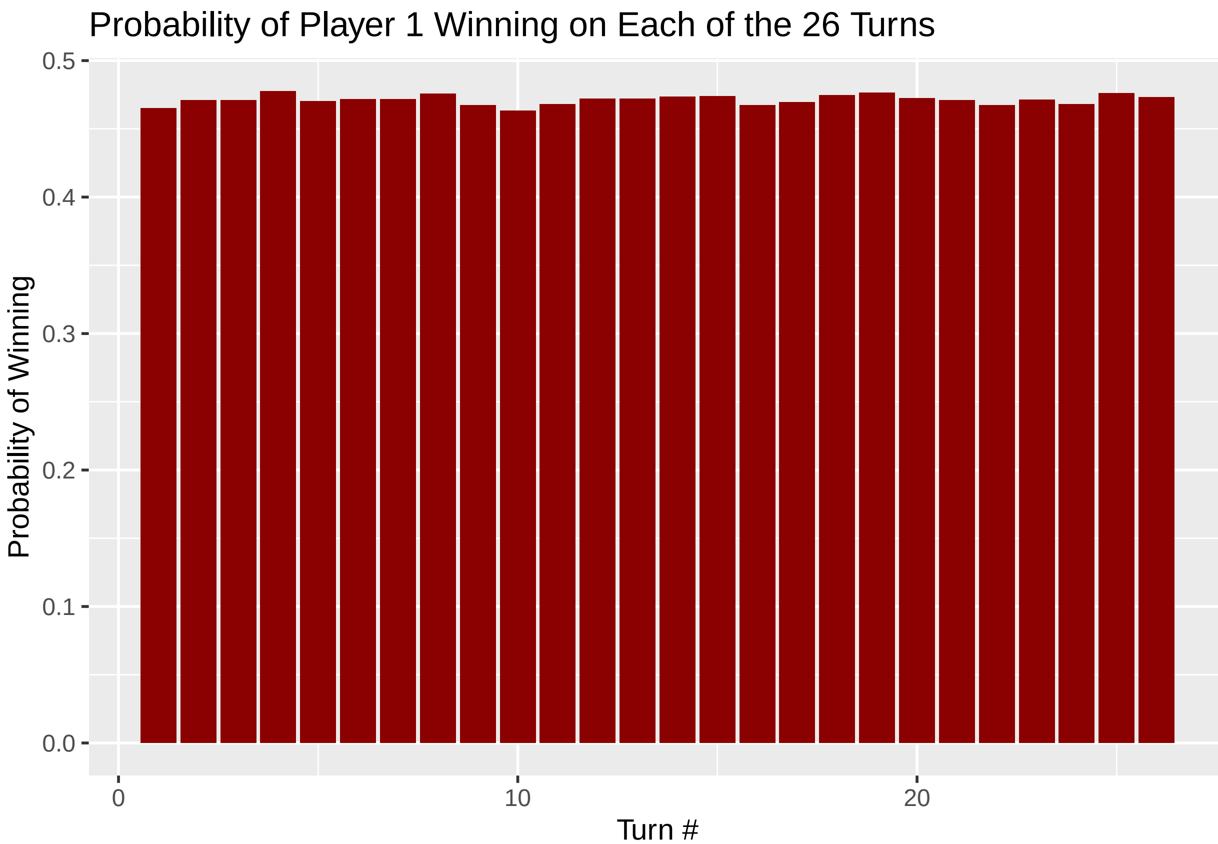
This computation also neatly reveals the respective probabilities that Player 1 or Player 2 will win the first turn. Given the symmetry of the problem (if the deck was shuffled fairly, neither player should have an advantage on the first turn), we can conclude, rather intuitively, that:

$$\Pr(\text{Player 1 Wins}) = \Pr(\text{Player 2 Wins}) = \frac{1 - (3/51)}{2} = 24/51$$

That is, the probability that each player wins the first turn is one-half of the total probability that *someone* wins (i.e. that the first turn doesn't end in a tie).

At this point, believe it or not, we have actually done most of the heavy lifting in this problem. This is due to the fact that the probability of winning *any* given turn is actually equal to the probability of winning the *first* turn. Intuitively, we know that this must be the case because we could have Player 1 and Player 2 start the game with any card in their piles. For example, they could both start the game by revealing the 14th card in their respective piles, thus making their 14th cards the first turn.

If you still aren't convinced, take a look at the simulation I've run below:



This graph shows the results of 15,000 simulated, 26-turns-long games of War. Note that when I say all the games represented here lasted 26 turns, I simply mean that I stopped the simulation after *all* of each player's 26 cards had been revealed (regardless of whether anybody had "won"). I also excluded any "mini Wars," choosing to just treat ties as ties and move on to the next turn.

Clearly, the simulation shows that probability of Player 1 winning a specific turn is flat across all 26 turns. Further, that probability appears to hover at right around 0.47, which is, conveniently, approximately 24/51.

Having determined that the probability of a player winning a given turn is 24/51, we can now say that the probability that a player wins all 26 turns (thus ending the game after 26 turns) is:

$$2 * (24/51)^{26} \approx 6.16 * 10^{-9}$$

Note that we multiply $(24/51)^{26}$ by 2 to account for the fact that the game would end if *either* player won in 26 turns.

Those are pretty slim odds! Your friend's granddaughter either got really lucky, or she did some funny business with the shuffling.

How long will it take?

Now we turn to the actual question posed in the prompt: how many games would you expect to play until you had a game that lasted just 26 turns? I'm going to interpret this as asking how many games we'd have to finish to have a greater than 50% chance of having played at least one game with only 26 turns. Equivalently, we can ask how many games we'd have to finish to have a less than 50% chance of having played *only* games with *more* than 26 turns.

We can go about finding the answer by solving the following inequality, which states that, after playing x games, the probability of having played *no* games with *more* than 26 turns is less than 50%:

$$[1 - (2 * (24/51)^{26})]^x < 0.5$$

If you plug this inequality into a calculator, it turns out you need x to be greater than $1.125 * 10^8$ (roughly). That means you'd have to play around 113 million games of War to have a better than 50% chance that at least one of them would end after only 26 turns. That's a heck of a lot of games.