Due: 16th of September 2018 at 11:59pm

COMP 9020 – Assignment 2

Note: In your assignment, *how* you arrived at your solution is as important (if not more so) than the solution itself and will be assessed accordingly. There may be more than one way to find a solution, and your approach should contain enough detail to justify its correctness. Lecture content can be assumed to be common knowledge.

1. If $R_1 \subseteq S \times T$ and $R_2 \subseteq T \times U$ are binary relations, the *composition* of R_1 and R_2 is the relation $R_1; R_2$ defined as:

   $R_1; R_2 := \{(a, c) : \text{There exists } b \in T \text{ such that } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$

   (a) If $f : S \to T$ and $g : T \to U$ are functions is $f; g$ a function?

   (b) If $R \subseteq S \times S$ is transitive, show that $R = R \cup (R; R)$. (*Hint: One way to show $A = B$ is to show $A \subseteq B$ and $B \subseteq A$. One of these directions is trivial.*)

   Let $R \subseteq S \times S$ be any binary relation on a set S. Consider the sequence of relations $R^0, R^1, R^2, \ldots$, defined as follows:

   $$\begin{aligned} R^0 &:= R, \text{ and} \\ R^{i+1} &:= R^i \cup (R^i; R) \text{ for } i \geq 0 \end{aligned}$$

   (c) Prove that if $R^i = R^{i+1}$ for some i, then $R^i = R^j$ for all $j \geq i$.

   (d) Prove that if $R^i = R^{i+1}$ for some i, then $R^k \subseteq R^i$ for all $k \geq 0$.

   (e) If $|S| = n$, explain why $R^n = R^{n+1}$. (*Hint: Show that if $(a, b) \in R^{n+1}$ then $(a, b) \in R^i$ for some $i < n + 1$.*)

   In the above sequence, R^n is defined to be the *transitive closure* of R, denoted R^* (closely related to the * operator used to describe the set of all words over an alphabet).

   (f) Show that R^* is transitive.

(20 marks)

2. The following table describes several subjects and the students taking them:

Potions	Charms	Herbology	Astronomy	Transfiguration
Harry	Ron	Harry	Hermione	Hermione
Ron	Luna	George	Neville	Fred
Malfoy	Ginny	Neville	Seamus	Luna

You have been tasked to create an examination timetable for these subjects, and your goal is to find the *smallest number* of timeslots needed so that all subjects can be examined, without any conflicts occurring (i.e. no students having to take two or more exams at the same time).

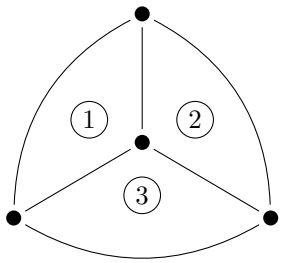
(a) Explain how this can be formulated as a graph-based problem. That is, describe what the vertices and edges would be, and how to relate the given problem to a common graph problem.

(b) For this problem in particular determine the minimum number of timeslots required.

(c) Suppose instead your goal was to determine the *largest number* of subjects that can be examined at the same time without conflicts. How do your answers to (a) and (b) change?

(10 marks)

3. Given a plane-drawing (i.e. no crossing edges) of a *connected* planar graph G, a *face* is a region that is enclosed by edges. For example, the following plane-drawing of K_4 has 3 faces (labelled 1,2,3):



(a) How many edges must a connected graph with n vertices and 1 face have?

(b) By examining several planar graphs, come up with an equation that relates the number of vertices (n), the number of edges (m) and the number of faces (f) of a plane-drawing of a planar graph.

(c) Prove, by induction on f or otherwise, that your formula is correct. *Hint: What happens if you delete an edge of a plane-drawing that doesn't disconnect the graph?*

(10 marks)

4. Extend the syntactical definition of propositional formulae to include the connective $\circ$:

   - If φ and ψ are propositional formulae, then $(\varphi \circ \psi)$ is a propositional formula.

   Given a truth valuation $v : Prop \to \mathbb{B}$, define the semantics for $\circ$ as

   $$v(\varphi \circ \psi) \ = !\big(v(\varphi) \,\&\, v(\psi)\big)$$

   (a) Draw the truth table for $(p \circ q) \circ (p \circ q)$. Give a logically equivalent formula.

   (b) For each of the following formulae, give a logically equivalent formula that only uses $\circ$ and propositional variables. Justify your answer.

      i. $\neg p$
      ii. $p \vee q$
      iii. $p \to q$
      iv. $p \leftrightarrow q$

(10 marks)

Advice on how to do the assignment

All submitted work must be done individually without consulting someone else's solutions in accordance with the University's "Academic Dishonesty and Plagiarism" policies.

- Assignments are to be submitted via WebCMS (or give) as a single pdf (max size 2Mb). In Linux, the following command

  ```
  pdfjoin --outfile output.pdf input1.pdf input2.pdf ...
  ```

  can be used to combine multiple pdf files. The command

  ```
  convert -density 150x150 -compress jpeg input.pdf output.pdf
  ```

  can be used to reduce the filesize of a pdf (change 150 to reduce/improve quality/filesize). Please ensure your files are legible before submitting.

- Be careful with giving multiple or alternative answers. If you give multiple answers, then we will give you marks only for "your worst answer", as this indicates how well you understood the question.

- Some of the questions are very easy (with the help of the lecture notes or book). You can use the material presented in the lecture or book (without proving it). You do not need to write more than necessary (see comment above).

- When giving answers to questions, we always would like you to prove/explain/motivate your answers.

- If you use further resources (books, scientific papers, the internet,...) to formulate your answers, then add references to your sources.