

Graphs

Exercise 1. True or false?

- (a) The complete bipartite graph $K_{5,5}$ has no cycle of length five.
- (b) If T is a tree with at least four edges, then $\chi(T) = 3$.
- (c) Let C_n denote a cycle on n vertices. For all $n \geq 5$ it holds $\chi(C_n) \neq \chi(C_{n-1})$.
- (d) It is possible to remove two edges from K_6 so that the resulting graph has a clique number of 4.

Solution:

- (a) True: $K_{5,5}$ is 2-colourable but a cycle of length five requires 3 colours.
- (b) False: Trees are 2-colourable, so $\chi(T) \leq 2$ for all trees T .
- (c) True: Odd-length cycles have chromatic number 3, whereas even-length cycles have chromatic number 2.
- (d) True: Consider two edges that do not have a vertex in common. Any collection of 5 vertices will necessarily contain both endpoints of at least one of these edges. If we remove these edges, then the resulting graph cannot have a 5-clique. It will have a 4-clique, which can be found by taking any four vertices that do not include both endpoints of one of the removed edges.

Exercise 2. What is the minimum number of edges that need to be removed from K_5 so that the resulting graph has a chromatic number of

- (a) 3?
- (b) 2?
- (c) 1?

Solution:

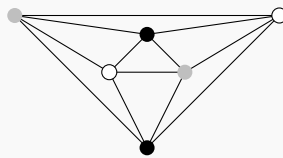
- (a) 2 edges. To achieve $\chi = 3$, one needs (at least) to avoid having any 4-cliques. Removing one edge leaves a 4-clique (actually two such cliques — including any one but not both of that edge's endpoints, plus the remaining three vertices). Removing two edges suffices — remove any pair of edges which do not share a common vertex; the remaining graph can then be coloured with 3 colours.
- (b) 4 edges. A chromatic number of 2 means that the graph is bipartite, with two groups of nodes where each group can be painted with one colour. To minimise the number of *removed* edges, we want to have as many edges as possible in the remaining bipartite graph. We therefore look at *complete bipartite* graphs with a total of 5 vertices. As $K_{1,4}$ has four edges and $K_{2,3}$ has six edges, the latter is the better choice. To reach it we need to remove 4 of the edges in K_5 .
- (c) 10 edges. A chromatic number of 1 means a fully disconnected graph, with no edges at all. Therefore all 10 edges of the original graph must be removed.

Exercise 3. Consider the complete 3-partite graphs $K_{4,1,1}$, $K_{3,2,1}$, $K_{2,2,2}$.

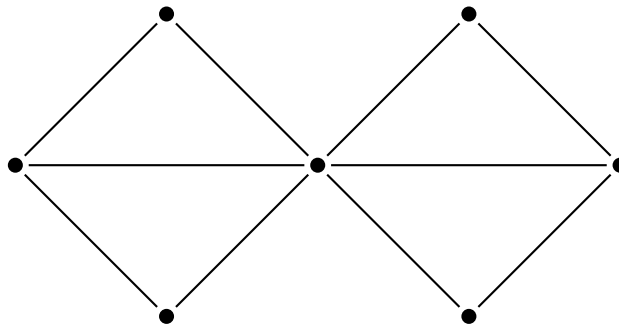
- (a) What is the chromatic number of each of these graph?
- (b) Which of these graphs are planar?

Solution:

- (a) For every 3-partite graph three colours suffice: use a different colour for each of the groups of vertices. Three colours are also necessary: any three vertices selected from three different groups will form a clique.
- (b) $K_{4,1,1}$ can be easily drawn without intersections. $K_{3,2,1}$ contains $K_{3,3}$ hence is not planar. A planar drawing of $K_{2,2,2}$ is:



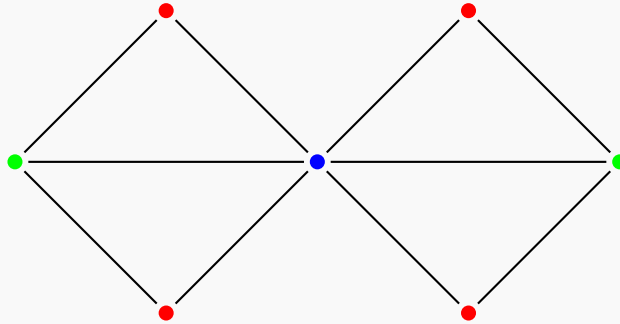
Exercise 4. Consider the following graph, G :



- (a) What is the chromatic number of G ?
- (b) What is the clique number of G ?
- (c) Does G have a Hamiltonian path and/or a Hamiltonian cycle?
- (d) Does G have an Eulerian path and/or an Eulerian cycle?

Solution:

(a) Here is a 3-colouring of the graph.



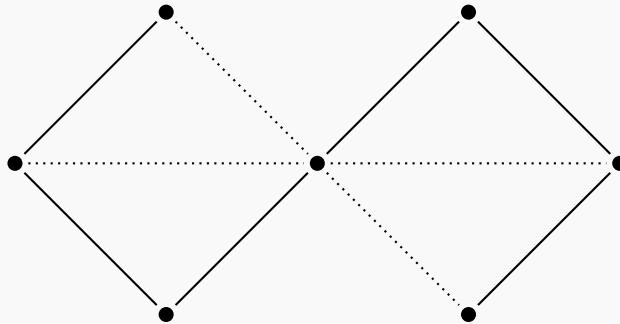
We cannot do any better because the graph contains a 3-clique, so $\chi(G) = 3$.

(b) As observed the graph has a 3-clique, so $3 \leq \kappa(G)$. Since

$$3 \leq \kappa(G) \leq \chi(G) \leq 3,$$

it follows that the clique-number of G is 3.

(c) Here is a Hamiltonian path:



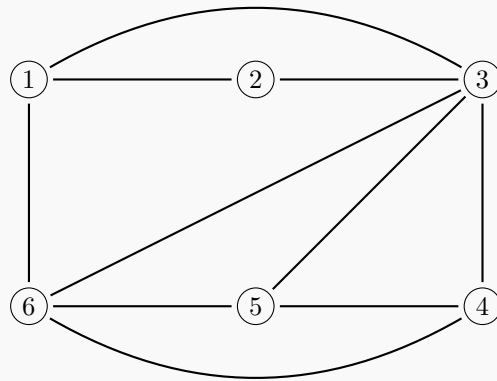
It does not contain a Hamiltonian circuit because there is only one vertex disjoint path from the left-side to the right-side.

(d) The graph has two vertices of odd degree, so it contains an Eulerian path, but not an Eulerian circuit.

Exercise 5. Draw a single graph with 6 vertices and 10 edges that satisfies each of the following:

- (a) is planar,
- (b) contains a Hamiltonian circuit, and
- (c) does not contain an Eulerian path.

Solution: Here is a graph with 6 vertices and 10 edges:



- (a) It is planar because it has been drawn in the plane with no crossing edges.
- (b) 1-2-3-4-5-6-1 is a Hamiltonian circuit.
- (c) It has four vertices of odd degree (1,3,4,5), therefore it cannot have an Eulerian path.