COMP9020 18s2 • Practice Questions 1

Fundamentals (Numbers, Sets, Words, Functions, and Relations)

Exercise 1. How many numbers are there between 100 and 1000 that are

(a) divisible by 3?

(b) divisible by 5?

(c) divisible by 15?

Solution: Using the formula $\lfloor \frac{m}{k} \rfloor - \lfloor \frac{n-1}{k} \rfloor$:

- $\lfloor \frac{1000}{3} \rfloor - \lfloor \frac{99}{3} \rfloor = 300$ numbers divisible by 3 $(102, 105, \ldots, 999)$;
- $\lfloor \frac{1000}{5} \rfloor - \lfloor \frac{99}{5} \rfloor = 181$ numbers divisible by 5 $(100, 105, \ldots, 1000)$;
- $\lfloor \frac{1000}{15} \rfloor - \lfloor \frac{99}{15} \rfloor = 60$ numbers divisible by 15 $(105, 120, \ldots, 990)$.

Exercise 2. Let $\Sigma = \{a, b, c\}$ and $\Phi = \{a, c, e\}$.

(a) How many words are in the set Σ^2?

(b) What are the elements of $\Sigma^2 \setminus \Phi^*$?

(c) Is it true that $\Sigma^* \setminus \Phi^* = (\Sigma \setminus \Phi)^*$? Why?

Solution: (a) $\Sigma^2 = \{aa, ab, ac, ba, \ldots, cc\}$, hence $|\Sigma^2| = 3 \cdot 3 = 9$.
(b) $\Sigma^2 \setminus \Phi^* = \{ab, ba, bb, bc, cb\}$, that is, all words in Σ^2 with the letter b.
(c) No; for example, $ab \in \Sigma^*$ and $ab \notin \Phi^*$, hence $ab \in \Sigma^* \setminus \Phi^*$; but $\Sigma \setminus \Phi = \{b\}$, hence $ab \notin (\Sigma \setminus \Phi)^*$.

Exercise 3. Prove that $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

Solution: We show in both directions that if an element belongs to $(A \setminus B) \cup (B \setminus A)$ then it also belongs to $(A \cup B) \setminus (A \cap B)$ and vice versa:

- Suppose that an element $x \in (A \setminus B) \cup (B \setminus A)$. Therefore, either $x \in A \setminus B$ or $x \in B \setminus A$. In either case, we conclude that $x \in A \cup B$ and (by the definition of set difference) $x \notin A \cap B$. Therefore, $x \in (A \cup B) \setminus (A \cap B)$.
- Suppose than $x \in (A \cup B) \setminus (A \cap B)$. This means that $x \in A \cup B$ (and, therefore, either $x \in A$ or $x \in B$), but $x \notin A \cap B$. If $x \in A$ and $x \notin A \cap B$, then $x \in A \setminus B$; alternatively, if $x \in B$ and $x \notin A \cap B$, then $x \in B \setminus A$. In either case, we conclude that $x \in (A \setminus B) \cup (B \setminus A)$.

Exercise 4. Consider the relation $R \subseteq \mathbb{R} \times \mathbb{R}$ defined by aRb if, and only if, $b + 0.5 \geq a \geq b - 0.5$. Is R

(a) reflexive?

(b) antireflexive?

(c) symmetric?

(d) antisymmetric?

(e) transitive?

Solution:

(a) Yes, since $a + 0.5 \geq a \geq a - 0.5$ for all $a \in \mathbb{R}$

(b) No; see (a)

(c) Yes, since $(b + 0.5 \geq a) \wedge (a \geq b - 0.5)$ implies $(b \geq a - 0.5) \wedge (a + 0.5 \geq b)$.

(d) No; e.g. $(0, 0.1) \in R$ and $(0.1, 0) \in R$.

(e) No; e.g. $(1.1, 1.5) \in R$ and $(1.5, 1.9) \in R$ but $(1.1, 1.9) \notin R$ since $1.9 - 0.5 > 1.1$

Exercise 5. For each of the following statements, provide a valid proof if it is true for all sets S and all relations $R_1 \subseteq S \times S$ and $R_2 \subseteq S \times S$. If the statement is not always true, provide a counterexample.

(a) If R_1 and R_2 are symmetric, then $R_1 \cap R_2$ is symmetric.

(b) If R_1 and R_2 are antisymmetric, then $R_1 \cup R_2$ is antisymmetric.

Solution:

(a) We will show that if R_1 and R_2 are symmetric, then $R_1 \cap R_2$ is symmetric. Suppose $(a, b) \in R_1 \cap R_2$. Then $(a, b) \in R_1$ and $(a, b) \in R_2$. Because R_1 is symmetric, $(b, a) \in R_1$. Because R_2 is symmetric, $(b, a) \in R_2$. Therefore $(b, a) \in R_1 \cap R_2$. Therefore $R_1 \cap R_2$ is symmetric.

(b) This is not the case. Consider relations on $\mathbb{N}$: $R_1 = \leq$ and $R_2 = \geq$. We have $1 \leq 2$, so $(1, 2) \in R_1$ and $(2, 1) \in R_2$. Therefore, $(1, 2), (2, 1) \in R_1 \cup R_2$ but $1 \neq 2$.