" SOLUTION OF DIRAC EQUATION "

" FREE PARTICLE SOLUTION " CW 2

For a free particle, $V = 0$. The Dirac equation is,

$$\frac{\partial \psi}{\partial t} + \frac{i}{\hbar} mc^2 \beta \psi + c\, \bar{\alpha} \cdot \bar{\nabla} \psi = 0 .$$

PARTICLE AT REST : $KE = 0 \Rightarrow p = 0 \Rightarrow \bar{\nabla} = 0$.

then, $$\frac{d\psi}{dt} + \frac{i}{\hbar} mc^2 \beta \psi = 0 \quad \text{--- (1)}$$

where, $$\psi^{(r)}(t) = e^{-\frac{i}{\hbar} \epsilon_r mc^2 t} N\, u^{(r)}(0) \quad \text{--- (2)}$$

here, $\epsilon_r = \begin{cases} +1 & ; \text{ +ve energy sol}^n \\ -1 & ; \text{ -ve energ sol}^n \end{cases}$

$u^r(0)$ is a (4×1) matrix.

(a) Positive energy sol^n : taking $N = 1$,

$$\psi(t) = e^{-\frac{i}{\hbar} mc^2 t} u(0) = e^{-\frac{i}{\hbar} mc^2 t} \begin{pmatrix} u_A \\ u_B \end{pmatrix} ;$$ u_A and u_B are (2×1) ~~mati~~ matrices.

Putting this to (1),

$$-\frac{i}{\hbar} mc^2 e^{-\frac{i}{\hbar} mc^2 t} \begin{pmatrix} u_A \\ u_B \end{pmatrix} + \frac{i}{\hbar} mc^2 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} e^{-\frac{i}{\hbar} mc^2 t} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} u_A \\ u_B \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} u_A \\ -u_B \end{pmatrix} ;$$ i.e., u_A is an arbitrary (2×1) matrix and $u_B = 0$.

4. The Dirac Equation

The choices for u_A can be $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, which are the basis for the space of (2×1) matrix.

Then, $u_A = \begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$

$u_B = \begin{pmatrix}0\\0\end{pmatrix}$

Therefore, the two linearly independent sol^{n}s are,

$$e^{-\frac{i}{\hbar}mc^2t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \text{ and } e^{-\frac{i}{\hbar}mc^2t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}. \quad \text{---} (3)$$

(b) Negative energy sol^n: taking $N=1$,

$\Psi(t) = e^{\frac{i}{\hbar}mc^2t}\, u(0) = e^{\frac{i}{\hbar}mc^2t}\begin{pmatrix}u_C\\u_D\end{pmatrix}$; u_C and u_D are (2×1) matrices

Putting this to (1),

$$-\begin{pmatrix}u_C\\u_D\end{pmatrix} = \begin{pmatrix}u_C\\-u_D\end{pmatrix} \Rightarrow \begin{pmatrix}u_C\\u_D\end{pmatrix} = \begin{pmatrix}-u_C\\u_D\end{pmatrix}; \text{ i.e,}$$

$u_C = \begin{pmatrix}0\\0\end{pmatrix}$, and u_D is an arbitrary (2×1) matrix.

We choose, $u_D = \begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.

Therefore, the two linearly independent sol^{n}s are,

$$e^{\frac{i}{\hbar}mc^2t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \text{ and } e^{\frac{i}{\hbar}mc^2t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \quad \text{---} (4)$$

Therefore, we get,

$$u^{(1)}(0) = e^{-\frac{i}{\hbar}mc^2t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\ u^{(2)}(0) = e^{-\frac{i}{\hbar}mc^2t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix};\ u^{(3)}(0) = e^{\frac{i}{\hbar}mc^2t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix};\ u^{(4)}(0) = e^{\frac{i}{\hbar}mc^2t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

2. **FREE MOVING PARTICLE** : In the absence of any external field or other interactions, the linear momentum will be a conserved quantity and so the total energy.

The Dirac eqn ; $\frac{\partial \psi}{\partial t} + \frac{i}{\hbar} mc^2 \beta \psi + c\,\bar{\alpha}\cdot\bar{\nabla}\psi = 0$ ——— ①

We take, $\psi(\bar{r},t) = N e^{-\frac{i}{\hbar}(Et - \bar{p}\cdot\bar{r})} u(\bar{p})$; where, $u(\bar{p})$ is a (4×1) matrix, ——— ②

and, $u(\bar{p}) = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$

Putting ② into ①,

$$\left(-\frac{i}{\hbar}E\right) e^{-\frac{i}{\hbar}(Et-\bar{p}\cdot\bar{r})} \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$+ \frac{i}{\hbar} mc^2 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} N e^{-\frac{i}{\hbar}(Et-\bar{p}\cdot\bar{r})} \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$+ cN \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \partial_i e^{-\frac{i}{\hbar}(Et - p^i x^i)} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$$-\frac{i}{\hbar} NE\, e^{-\frac{i}{\hbar}(Et-\bar{p}\cdot\bar{r})} \begin{pmatrix} u_A \\ u_B \end{pmatrix} + \frac{iN}{\hbar} mc^2 e^{-\frac{i}{\hbar}(Et-\bar{p}\cdot\bar{r})} \begin{pmatrix} u_A \\ -u_B \end{pmatrix}$$

$$+ cN\left(\frac{i}{\hbar}p^i\right) e^{-\frac{i}{\hbar}(Et-\bar{p}\cdot\bar{r})} \begin{pmatrix} \sigma^i u_B \\ \sigma^i u_A \end{pmatrix} = 0$$

$$\Rightarrow -E\begin{pmatrix} u_A \\ u_B \end{pmatrix} + mc^2 \begin{pmatrix} u_A \\ -u_B \end{pmatrix} + c p^i \sigma^i \begin{pmatrix} u_B \\ u_A \end{pmatrix} = 0$$

i.e,

$$\left.\begin{aligned} -(E - mc^2)\, u_A + c\,\sigma^i p^i\, u_B &= 0 \\ -(E + mc^2)\, u_B + c\,\sigma^i p^i\, u_A &= 0 \end{aligned}\right\}$$ ——— ③

they will have non-trivial soln when, the determinant of the coefficients of u_A and u_B ~~A~~ is zero.

then.

$$\begin{vmatrix} -(E-mc^2) & c\,\sigma^i p^i \\ c\,\sigma^i p^i & -(E+mc^2) \end{vmatrix} = (E-mc^2)(E+mc^2) - c^2(\sigma^i p^i)^2 = 0$$

$$\Rightarrow E^2 - m^2c^4 - c^2p^2 = 0$$

i.e. we must have +ve and -ve energy sol^{n}s.

$$\Rightarrow E^2 = p^2c^2 + m^2c^4$$

$$\Rightarrow E = \pm\sqrt{p^2c^2+m^2c^4} = \pm E_p$$

here, we used, $\sigma^i p^i = \sigma^1 p^1 + \sigma^2 p^2 + \sigma^3 p^3$

$$\Rightarrow \sigma^i p^i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p^1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p^2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p^3$$

$$\Rightarrow \sigma^i p^i = \begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix} = \begin{pmatrix} p^3 & P_- \\ P_+ & -p^3 \end{pmatrix}$$

$$\Rightarrow (\sigma^i p^i)^2 = \begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix}\begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix} = \begin{pmatrix} (p^3)^2 + P_+P_- & 0 \\ 0 & P_+P_- + (p^3)^2 \end{pmatrix}$$

$$\Rightarrow (\sigma^i p^i)^2 = \begin{pmatrix} \vec{p}^2 & 0 \\ 0 & \vec{p}^2 \end{pmatrix} = \vec{p}^2\, \mathbb{1}_{2\times 2}$$

for $E = E_p$,

from ⑤, $-(E_p - mc^2)\,u_A + c\,\sigma^i p^i\, u_B = 0$

$$\Rightarrow -(E_p - mc^2)\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} + c\begin{pmatrix} p^3 & P_- \\ P_+ & -p^3 \end{pmatrix}\begin{pmatrix} u^3 \\ u^4 \end{pmatrix} = 0$$

and,

$$-(E_p + mc^2)\begin{pmatrix} u^3 \\ u^4 \end{pmatrix} + c\begin{pmatrix} p^3 & P_- \\ P_+ & -p^3 \end{pmatrix}\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = 0$$

we get 4-eq^{n}s.,

$$\left.\begin{aligned} -(E_p-mc^2)u^1 + cp^3u^3 + cp_-u^4 &= 0 \\ -(E_p-mc^2)u^2 + cp_+u^3 - cp^3u^4 &= 0 \\ -(E_p+mc^2)u^3 + cp^3u^1 + cp_-u^2 &= 0 \\ -(E_p+mc^2)u^4 + cp_+u^1 - cp^3u^2 &= 0 \end{aligned}\right\} \quad -④$$

we'll take the values of the components such that the spinors match with the rest energy spinors at $\bar{p}=0$.

(a) +ve energy sol^{n}s :-

1st +ve energy sol^n : $u^1=1$, $u^2=0$

Using 1st 2-eq^{n}s of ④,

$$-(E_p-mc^2)+cp^3u^3+cp_-u^4=0$$

$$cp_+u^3-cp^3u^4=0$$

$$u^4=\frac{p_+}{p^3}u^3$$

$$(E_p-mc^2)=c\left(p^3+\frac{p_+p_-}{p^3}\right)u^3=c\frac{\bar{p}^2}{p^3}u^3$$

$$u^3=\frac{p^3}{c\bar{p}^2}\frac{(E_p-mc^2)(E_p+mc^2)}{(E_p+mc^2)}$$

$$u^3=\frac{p^3}{c\bar{p}^2}\frac{\bar{p}^2c^2}{E_p+mc^2}=\frac{cp^3}{E_p+mc^2}$$

and, $u^4=\dfrac{cp_+}{E_p+mc^2}$

$$\therefore u^{(1)}(p)=\begin{pmatrix}1\\0\\ \frac{cp^3}{E_p+mc^2}\\ \frac{cp_+}{E_p+mc^2}\end{pmatrix} \quad -⑤$$

2nd +ve energy sol^n : $u^1=0$, $u^2=1$

Using 1st 2-eq^{n}s of ④,

$$cp^3u^3+cp_-u^4=0$$

$$-(E_p-mc^2)+cp_+u^3-cp^3u^4=0$$

this gives, $u^3=\dfrac{cp_-}{E_p+mc^2}$

$$u^4=-\frac{cp^3}{E_p+mc^2}$$

$$\therefore u^{(2)}(p)=\begin{pmatrix}0\\1\\ \frac{cp_-}{E_p+mc^2}\\ \frac{-cp^3}{E_p+mc^2}\end{pmatrix} \quad -⑥$$

(b) -ve energy sol^{n}s :-

1st -ve energy sol^n: $u^3=1$, $u^4=0$

Using last 2-eq^{n}s of (4),

$$-(E_p+mc^2)+cp^3u^1+cp_-u^2=0$$

$$cp_+u^1-cp^3u^2=0$$

this gives, [putting $E_p=-E_p$]

$$u^{(3)}(p)=\begin{pmatrix}\frac{-cp^3}{E_p+mc^2}\\ \frac{-cp_+}{E_p+mc^2}\\ 1\\ 0\end{pmatrix}$$

We've, $u^2=\frac{p_+}{p^3}u^1$

$$(E_p+mc^2)=cp^3u^1+cp_-\left(\frac{p_+}{p^3}\right)u^1$$

Putting $E_p=-E_p$,

$$(-E_p+mc^2)\left(\frac{p^3}{c\bar{p}^2}\right)=u^1$$

$$\Rightarrow u^1=\frac{p^3}{c\bar{p}^2}\frac{(-E_p+mc^2)(E_p+mc^2)}{(E_p+mc^2)}$$

$$\Rightarrow u^1=-\frac{p^3c^2\bar{p}^2}{c\bar{p}^2(E_p+mc^2)}=\frac{-cp^3}{E_p+mc^2},$$

and,

$$u^2=\frac{-cp_+}{E_p+mc^2}$$

2nd -ve energy sol^n: $u^3=0$,

Using last 2-eq^{n}s of (4),

$$cp^3u^1+cp_-u^2=0$$

$$-(E_p+mc^2)+cp_+u^1-cp^3u^2=0$$

this gives, [putting $E_p=-E_p$]

$$u^{(4)}(p)=\begin{pmatrix}\frac{-cp_-}{E_p+mc^2}\\ \frac{cp^3}{E_p+mc^2}\\ 0\\ 1\end{pmatrix}$$

therefore, we get,

$$u^{(1)}(p)=\begin{pmatrix}1\\ 0\\ \frac{cp^3}{E_p+mc^2}\\ \frac{cp_+}{E_p+mc^2}\end{pmatrix}$$

$\vec{\sigma}\cdot\vec{p}$

$$u^{(2)}(p)=\begin{pmatrix}0\\ 1\\ \frac{cp_-}{E_p+mc^2}\\ \frac{-cp^3}{E_p+mc^2}\end{pmatrix}$$

$$u^{(3)}(p)=\begin{pmatrix}\frac{-cp^3}{E_p+mc^2}\\ \frac{-cp_+}{E_p+mc^2}\\ 1\\ 0\end{pmatrix};\quad \xleftarrow{-\vec{\sigma}\cdot\vec{p}}\quad u^{(4)}(p)=$$

$u^{(i)}$ — spinor ; u^i - component of spinor

NORMALIZATION OF DIRAC SPINORS :-

$$\psi^{(r)\dagger}\psi^{(r)} = N^2 u^{(r)\dagger} u^{(r)} = \frac{E_p}{mc^2} \quad \text{———} (1)$$

For $r=1$; $u^{(1)\dagger}u^{(1)} = \begin{pmatrix} 1 & 0 & \frac{cp^3}{E_p+mc^2} & \frac{cp_-}{E_p+mc^2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{cp^3}{E_p+mc^2} \\ \frac{cp_+}{E_p+mc^2} \end{pmatrix}$

$$= 1 + \frac{c^2}{(E_p+mc^2)^2}\left\{(p^3)^2 + p_-p_+\right\} = 1 + \frac{\bar{p}^2c^2}{(E_p+mc^2)^2}$$

$$\bar{\sigma}\cdot\bar{p} \quad \begin{pmatrix} p^3 & p_- \\ p_+ & -p^3 \end{pmatrix}$$

$$= 1 + \frac{E_p^2 - m^2c^4}{(E_p+mc^2)^2} = 1 + \frac{(E_p+mc^2)(E_p-mc^2)}{(E_p+mc^2)^2}$$

$$= 1 + \frac{E_p - mc^2}{E_p+mc^2} = \frac{E_p+mc^2+E_p-mc^2}{E_p+mc^2}$$

$$= \frac{2E_p}{E_p+mc^2}$$

Using (1),

$$N^2\left(\frac{2E_p}{E_p+mc^2}\right) = \frac{E_p}{mc^2} \Rightarrow N^2 = \frac{E_p+mc^2}{2mc^2}$$

$$\Rightarrow N = \sqrt{\frac{E_p+mc^2}{2mc^2}}$$

For free-particle, $E_p = mc^2$; then, $N = 1$.

" DIRAC EQUATION WITH EM COUPLING "

the free particle dirac eqⁿ, $\frac{\partial\psi}{\partial t} + \frac{i}{\hbar} mc^2 \beta\psi + c\,\bar{\alpha}\cdot\bar{\nabla}\psi = 0$

$$\Rightarrow i\hbar\frac{\partial\psi}{\partial t} = mc^2\beta\psi + c\,\bar{\alpha}\cdot\bar{p}\,\psi = \hat{H}\psi \quad \text{——} (1)$$

we've, $\psi \sim e^{-\frac{i}{\hbar}Et} \Rightarrow i\hbar\frac{\partial\psi}{\partial t} \sim E\psi$ (energy eigenvalue of the eqⁿ).

we take,

$$\psi = N e^{-\frac{i}{\hbar}[Et - p\cdot x]}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} \quad \text{——} (2)$$

here,

ψ_A and ψ_B have no space-time dependence.

We introduce em interaction through minimal coupling. then

$$p^\mu \rightarrow p^\mu - qA^\mu$$

$i\hbar\partial^i \rightarrow i\hbar\partial^i - qA^i$ and, $i\hbar\partial^0 \rightarrow i\hbar\partial^0 - qA^0$

$= -i\hbar\partial_i - qA^i$ $\Rightarrow i\hbar\frac{\partial}{\partial(ct)} \rightarrow i\hbar\frac{\partial}{\partial(ct)} - q\left(\frac{V}{c}\right)$

$= -i\hbar\bar{\nabla} - q\bar{A}$ $\Rightarrow i\hbar\frac{\partial}{\partial t} \rightarrow i\hbar\frac{\partial}{\partial t} - qV$.

Eqⁿ (1) becomes,

$$\left(i\hbar\frac{\partial}{\partial t} - qV\right)\psi = mc^2\beta\psi + c\,\bar{\alpha}\cdot(\bar{p} - q\bar{A})\psi$$

$$\Rightarrow \left(i\hbar\frac{\partial}{\partial t} - qV\right) N e^{-\frac{i}{\hbar}[Et - p\cdot x]}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = mc^2\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} N e^{-\frac{i}{\hbar}[Et - p\cdot x]}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$$

$$+ c\begin{pmatrix}0 & \bar{\sigma}\\ \bar{\sigma} & 0\end{pmatrix}\cdot(\bar{p} - q\bar{A}) N e^{-\frac{i}{\hbar}[Et - p\cdot x]}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$$

$$\Rightarrow (E - qV)\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = mc^2\begin{pmatrix}\psi_A\\ -\psi_B\end{pmatrix} + c\,\bar{\sigma}\cdot(\bar{p} - q\bar{A})\begin{pmatrix}\psi_B\\ \psi_A\end{pmatrix}$$

We get,

$$(E-qV)\psi_A = mc^2\psi_A + c\,\bar{\sigma}\cdot(\bar{p}-q\bar{A})\,\psi_B$$

$$\Rightarrow (E-qV-mc^2)\psi_A = c\,\bar{\sigma}\cdot(\bar{p}-q\bar{A})\psi_B = c\,\bar{\sigma}\cdot\bar{\pi}\,\psi_B \quad \text{③}$$

and,

$$(E-qV)\psi_B = -mc^2\psi_B + c\,\bar{\sigma}\cdot(\bar{p}-q\bar{A})\psi_A$$

$$\Rightarrow (E-qV+mc^2)\psi_B = c\,\bar{\sigma}\cdot\bar{\pi}\,\psi_A$$

$$\Rightarrow \psi_B = \frac{c\,\bar{\sigma}\cdot\bar{\pi}}{E-qV+mc^2}\psi_A \quad \text{④}$$

Putting ④ to ③, $(E-qV-mc^2)\psi_A = c^2(\bar{\sigma}\cdot\bar{\pi})\dfrac{(\bar{\sigma}\cdot\bar{\pi})}{E-qV+mc^2}\psi_A$

$$\Rightarrow (E'-qV)\psi_A = c^2(\bar{\sigma}\cdot\bar{\pi})\frac{(\bar{\sigma}\cdot\bar{\pi})}{(E'-qV+2mc^2)}\psi_A \quad ; \quad E' = E-mc^2$$

$$\Rightarrow E = E' + mc^2$$

(↳ KE ↳ rest mass energy ↳ total energy.)

$$\Rightarrow (E'-qV)\psi_A = c^2(\bar{\sigma}\cdot\bar{\pi})\frac{(\bar{\sigma}\cdot\bar{\pi})}{\left(\frac{E'-qV}{2mc^2}+1\right)2mc^2}\psi_A$$

$$\Rightarrow (E'-qV)\psi_A = c^2(\bar{\sigma}\cdot\bar{\pi})\frac{k}{2mc^2}(\bar{\sigma}\cdot\bar{\pi})\psi_A \quad ; \quad k = \frac{1}{1+\frac{E'-qV}{2mc^2}}$$

For small energy limit, $E' \ll mc^2$ i.e, $k \simeq 1$.

then, $(E'-qV)\psi_A \cong c^2(\bar{\sigma}\cdot\bar{\pi})\dfrac{(\bar{\sigma}\cdot\bar{\pi})}{2mc^2}\psi_A = \dfrac{1}{2m}(\bar{\sigma}\cdot\bar{\pi})(\bar{\sigma}\cdot\bar{\pi})\psi_A$

$$\Rightarrow (E-mc^2-qV)\psi_A = \frac{1}{2m}\{\bar{\pi}\cdot\bar{\pi} + i\,\bar{\sigma}\cdot(\bar{\pi}\times\bar{\pi})\}\psi_A$$

$$\Rightarrow E\psi_A = \{mc^2 + qV + \frac{1}{2m}\bar{\pi}\cdot\bar{\pi} + \frac{i}{2m}\bar{\sigma}\cdot(\bar{\pi}\times\bar{\pi})\}\psi_A \quad \text{⑤}$$

Now, $\bar{\pi} = \bar{p} - q\bar{A}$ $\therefore$ $\bar{\pi}\cdot\bar{\pi} = (\bar{p} - q\bar{A})^2$

and, $\bar{\pi}\times\bar{\pi} = (\bar{p} - q\bar{A}) \times (\bar{p} - q\bar{A})$

$$= \bar{p}\times\bar{p} - q(\bar{p}\times\bar{A} + \bar{A}\times\bar{p}) + q^2\,\bar{A}\times\bar{A}$$

$$\simeq +q\,i\hbar(\bar{\nabla}\times\bar{A} + \underbrace{\bar{A}\times\bar{\nabla}}_{\to 0}) = +i\hbar q\,\vec{B}$$

Putting the values to (5),

$$E\psi_A = \left\{mc^2 + qV + \frac{(\bar{p} - q\bar{A})^2}{2m} + \frac{i}{2m}(+i\hbar)q\,\bar{\sigma}\cdot\bar{B}\right\}\psi_A$$

$$= \left\{mc^2 + qV + \frac{(\bar{p} - q\bar{A})^2}{2m} - \frac{\hbar}{2m}q\,\bar{\sigma}\cdot\bar{B}\right\}\psi_A$$

the last term $= -\frac{\hbar}{2m}q\,\bar{\sigma}\cdot\bar{B}\,\psi_A$

$$\simeq -\frac{\hbar}{2m}q\,\frac{\bar{\sigma}\cdot\bar{B}}{2}\,g\,\psi_A \;;\quad g\text{-factor} \simeq 2.003$$

$$= -\frac{q}{2m}\,g\,\bar{S}\cdot\bar{B}\,\psi_A \;;\quad \bar{S} = \frac{\hbar}{2}\bar{\sigma}$$

$$= \bar{\mu}_S\cdot\bar{B}\,\psi_A\,.$$

Pauli's eqn : $i\hbar\frac{\partial}{\partial t}\psi_A = \left\{mc^2 + qV + \frac{(\bar{p} - q\bar{A})^2}{2m} - \frac{\hbar}{2m}q\,\bar{\sigma}\cdot\bar{B}\right\}\psi_A$.

"DIRAC EQUATION & SPIN"

DIRAC EQ^N REPRESENTS SPIN-1/2 PARTICLES

We consider a particle moving in the z-direction. Then, $\bar{p} = (0, 0, p^3)$

then,

$$u^{(1)}(\bar{p}) = \begin{pmatrix} 1 \\ 0 \\ \frac{cp^3}{E_p + mc^2} \\ 0 \end{pmatrix} ; \quad u^{(2)}(\bar{p}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-cp^3}{E_p + mc^2} \end{pmatrix}$$

$$u^{(3)}(\bar{p}) = \begin{pmatrix} \frac{-cp^3}{E_p + mc^2} \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \quad u^{(4)}(\bar{p}) = \begin{pmatrix} 0 \\ \frac{cp^3}{E_p + mc^2} \\ 0 \\ 1 \end{pmatrix}$$

in the z-direction,

$$P_+ = p^1 + ip^2 = 0$$

$$P_- = p^1 - ip^2 = 0$$

The generalization of spin in 4-D space, $\bar{S} = \frac{\hbar}{2} \bar{\Sigma}$

where, $\bar{\Sigma} = \begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{pmatrix}$

Along the direction of momentum, the projection of spin, $\bar{S}\cdot\bar{p} = \frac{\hbar}{2} \Sigma^3$ ——— ①

The spin-quantum eigenvalue is found from, $S^3 \psi = S\psi$.

where, S - spin quantum no.

Now,

$$\Sigma^3 u^{(1)}(\bar{p}) = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} u^{(1)}(p)$$

$$\Rightarrow \Sigma^3 u^{(1)}(\bar{p}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ cp^3/(E_p + mc^2) \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ cp^3/(E_p + mc^2) \\ 0 \end{pmatrix} = u^{(1)}(\bar{p})$$

similarly,

$$\Sigma^3 u^{(2)}(\bar{p}) = -u^{(2)}(p)$$

and, $\Sigma^3 u^{(3)}(\bar{p}) = u^{(3)}(\bar{p})$

$$\Sigma^3 u^{(4)}(\bar{p}) = -u^{(4)}(\bar{p})$$

i.e, the projection of spin along the direction of momentum $= \pm\hbar/2$, for both the positive and negative energy solutions.

"CONSERVATION OF ANGULAR MOMENTUM."

The Hamiltonian is the operator responsible for the time evo. Hence, a quantum operator is called conserved if it com with the Hamiltonian.

Dirac Hamiltonian, $\hat{H}_D = mc^2\beta\hat{I} + c\,\hat{\bar{\alpha}}\cdot\hat{\bar{p}}$ —— ①.

The angular momentum will be conserved ~~wh~~ if,

$$[\hat{H}_D, \hat{J}] = 0 \Rightarrow [\hat{H}_D + (\hat{L}^j + \hat{S}^j)] = 0$$

$$\Rightarrow [\hat{H}_D + \hat{L}^j] + [\hat{H}_D + \hat{S}^j] = 0$$ —— ②.

$$\bar{L} = \bar{r}\times\bar{p} \Rightarrow \hat{L}^j = \varepsilon_{jkl}\hat{x}_k\hat{p}_l$$

$$\hat{H}_D = mc^2\beta\hat{I} + c\,\hat{\bar{\alpha}}\cdot\hat{\bar{p}} \Rightarrow \hat{H}_D = mc^2\beta\hat{I} + c\,\hat{\alpha}^i\hat{p}^i$$

then, $[\hat{H}_D, \hat{L}^j] = [mc^2\beta\hat{I}, \varepsilon_{jkl}\hat{x}_k\hat{p}_l] + [c\hat{\alpha}^i\hat{p}^i, \varepsilon_{jkl}\hat{x}_k\hat{p}_l]$

$$= 0 + c\alpha^i\varepsilon_{jkl}[p^i, x_k p_l] = c\alpha^i\varepsilon_{jkl}[p^i, x^k p^l]$$

$$= c\alpha^i\varepsilon_{jkl}\{[p^i, x^k]p^l + x^k[p^i, p^l]\}$$

$$= c\alpha^i\varepsilon_{jkl}(-i\hbar\,\delta_{ik}p^l) = -i\hbar c\,\varepsilon_{jil}\alpha^i p^l = -i\hbar c(\hat{\bar{\alpha}}\times\hat{\bar{p}})_j$$

$\hat{S}^j = \frac{\hbar}{2} \xi^j \hat{I}$; then, $[\hat{H}_D, S^j] = [c\alpha^i p^i + mc^2\beta, \frac{\hbar}{2}\xi^j]$

$$= \frac{c\hbar}{2} p^i [\alpha^i, \xi^j] + \frac{\hbar}{2} mc^2 [\beta, \xi^j]$$

$$= \frac{c\hbar}{2} p^i \left[\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} \right] + \frac{\hbar}{2} mc^2 \left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} \right]$$

$$= \frac{c\hbar}{2} p^i \, 2i\varepsilon^{ijk} \sigma^k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 0$$

$$= i\hbar c\, \varepsilon_{ijk}\, p^i \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = i\hbar c\, \varepsilon_{ijk}\, p^i \alpha^k$$

$$= i\hbar c\, \varepsilon_{ijl}\, p^i \alpha^l = i\hbar c\, \varepsilon_{jli}\, \alpha^l p^i$$

thus,

$$= i\hbar c\, (\hat{\alpha} \times \hat{p})_j$$

$[\hat{H}_D, \hat{J}]$

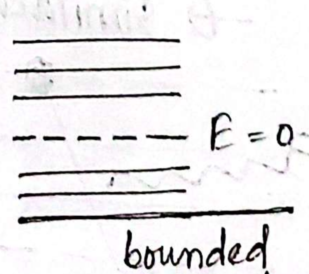
$= -i\hbar c\, (\hat{\alpha} \times \hat{p})_j + i\hbar c\, (\hat{\alpha} \times \hat{p})_j$

$= 0$.

"CHARGE CONJUGATION" (Hole theory)

Problem with negative energy sol^n : (Run-Down Problem)

The principle is that, for a stable system, the energy must be bounded from below. That implies, there ~~must~~ must be a lower limit of the −ve values of energy.



bounded ground state

If the system is not capable is radiating energy, it won't make any transition to lower energy level. Hence, for a meaningful Dirac theory, we cannot allow the Dirac

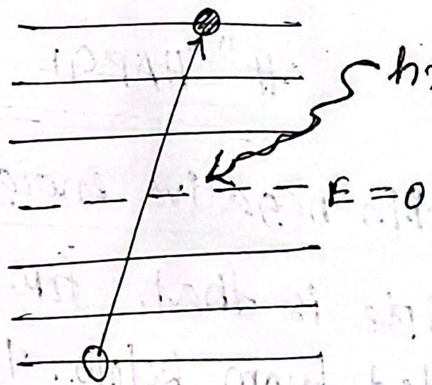
particle i.e, the electron to make transi.....
lower energy levels.

According to dirac hole theory,

- all the -ve energy states are occupied by electrons
- the electrons, being spin-1/2 particle, obeys Pauli's exclusion principle.

  hence, normally there will be no transition from +ve to -ve energy states.
- ~~positive energy states are then~~ an electron from -ve state may go to +ve energy states (which are genera not filled) by absorbing photon.
- a vacancy in created in the negative energy state due to the transition of electron of charge $-e$ and energy $-E$; the vacancy will appear as a particle of charge $+e$ and energy $+E$ ($h\nu = E+E = 2E$).

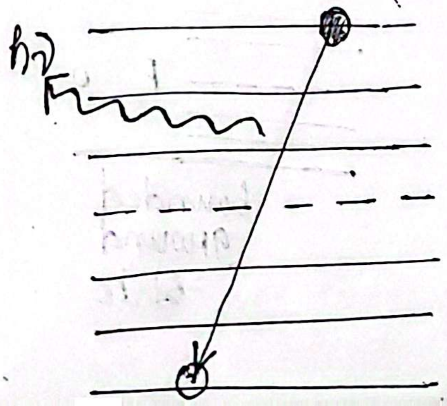
  this particle is called positron.



the above process is called ~~par~~ pair production

$$h\nu + N \rightarrow e^+ + e^- + N$$

- similarly, we can have



pair ~~annili~~ annihilation where an electron from the +ve state goes to a hole in the -ve energy sea.

"POSITRON"

Free particle Dirac eqn : $(i\hbar \not{\partial} - mc)\psi = 0$

$$\Rightarrow (i\hbar \gamma^\mu \partial_\mu - mc)\psi = 0$$

Minimal coupling to ~~ind~~ introduce em interaction :

$$p^\mu \to p^\mu - qA^\mu \text{ , then,}$$

Dirac eqn with em coupling : $(i\hbar \gamma^\mu \partial_\mu - q\gamma^\mu A_\mu - mc)\psi = 0$

therefore,

the wave eqn for electron : $(i\hbar \gamma^\mu \partial_\mu - q\gamma^\mu A_\mu - mc)\psi_e = 0$

the wave eqn for positron : $(i\hbar \gamma^\mu \partial_\mu + q\gamma^\mu A_\mu - mc)\psi_p = 0$

Positron wave ~~fn~~ eqn from electron wave eqn :-

$$(i\hbar \gamma^\mu \partial_\mu - q\gamma^\mu A_\mu - mc)\psi_e = 0$$

$$\Rightarrow \{(-i\hbar \partial_\mu - qA_\mu)\gamma^{\mu *} - mc\}\psi_e^* = 0 \quad \text{; (complex conjugate)}$$

$$\Rightarrow [-(i\hbar \partial_\mu + qA_\mu)\gamma^{\mu *} - mc]\psi_e^* = 0$$

We introduce a matrix C such that,

$$(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1} = -\gamma^\mu$$

then,

$$C\gamma^0[-(i\hbar \partial_\mu + qA_\mu)\gamma^{\mu *} - mc]\cancel{\psi_e^*}(C\gamma^0)^{-1}(C\gamma^0)\psi_e^* = 0$$

$$\Rightarrow [-(i\hbar \partial_\mu + qA_\mu)(-\gamma^\mu) - mc]\psi_p = 0 \quad \text{; here, } \psi_p = (C\gamma^0)\psi_e^*$$

$$\Rightarrow \{(i\hbar \partial_\mu + qA_\mu)\gamma^\mu - mc\}\psi_p = 0 \quad \text{; we used, } A^{\mu *} = A^\mu$$

$$\partial_\mu^* = \partial_\mu$$

In the standard representation, $C = i\gamma^2\gamma^0$

Commutation properties of C : (a) $[C,\gamma^1] = [C,\gamma^3] = 0$

(b) $\{C,\gamma^0\} = \{C,\gamma^2\} = 0$

We defined, $\psi_P = (C\gamma^0)\psi_e^* = C\bar{\psi}^T$

Properties of C : (a) $-C^\dagger = C$

(b) $C^T = -C$

(c) $-C^{-1} = C$

(d) $(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1} = -\gamma^\mu$

Proof of the properties of C :

(a) $C^\dagger = (i\gamma^2\gamma^0)^\dagger = -i(\gamma^0)^\dagger(\gamma^2)^\dagger$

$= -i\,\gamma^0\gamma^0\gamma^0\,\gamma^0\gamma^2\gamma^0 = -i\gamma^2\gamma^0 = -C$

(b) $C^{-1} = (i\gamma^2\gamma^0)^{-1} = -i(\gamma^0)^{-1}(\gamma^2)^{-1} = -i\gamma^0(-\gamma^2)$

$= i\gamma^0\gamma^2 = -i\gamma^2\gamma^0 = -C$

here, we know, $(\gamma^0)^2 = \beta^2 = \hat{I} \Rightarrow \gamma^0 = (\gamma^0)^{-1}$

$(\gamma^2)^{-1} = (\beta\alpha^2)^{-1} = (\alpha^2)^{-1}\beta^{-1} = (\alpha^2)^{-1}\beta$

$\Rightarrow (\gamma^2)^{-1} = \alpha^2\beta = -(\beta\alpha^2) = -\gamma^2$

as, $(\alpha^2)^2 = \hat{I} \Rightarrow \alpha^2 = (\alpha^2)^{-1}$

(c) $C^T = (C^{-1})^* = -C^* = -(i\gamma^2\gamma^0)^* = i(\gamma^2)^*(\gamma^0)^* = -i\gamma^2\gamma^0 = -C$

here, $\gamma_0^* = \gamma^{0*} = \gamma^0$

$$(\gamma^2)^* = (\alpha^* \beta \alpha^2)^* = (\beta^*)(\alpha^2)^* = \beta(-\alpha^2) = -\beta\alpha^2 = -\gamma^2$$

(d) $$(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1} = (C\gamma^0)(\gamma^{\mu +})^T(C\gamma^0)^{-1}$$

$$= C\gamma^0(\gamma^0\gamma^\mu\gamma^0)^T(C\gamma^0)^{-1} = C\gamma^0(\gamma^0)^T(\gamma^\mu)^T(\gamma^0)^T(C\gamma^0)^{-1}$$

$$= C\gamma^0\gamma^0(\gamma^\mu)^T\gamma^0\gamma^{0^{-1}}C^{-1} = C(\gamma^\mu)^T C^{-1}$$

For $\mu = 0$, $$C(\gamma^0)^T C^{-1} = (i\gamma^2\gamma^0)\gamma^0(-i\gamma^2\gamma^0)$$
$$= \gamma^2\gamma^2\gamma^0 = -\gamma^0$$

For $\mu = i$, $$C(\gamma^i)^T C^{-1} = (i\gamma^2\gamma^0)(\gamma^i)^T(-i\gamma^2\gamma^0)$$
$$= \gamma^2\gamma^0(-\gamma^i)\gamma^2\gamma^0$$
$$= \gamma^2\gamma^0\gamma^2\gamma^i\gamma^0$$
$$= -\gamma^2\gamma^2\gamma^0\gamma^i\gamma^0$$
$$= \gamma^0\gamma^i\gamma^0 = -\gamma^0\gamma^0\gamma^i = -\gamma^i \; ; \; i = 1,2,3.$$

$\therefore (C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1} = -\gamma^\mu$.

(e) We define, $\bar{\psi} = \psi^+\gamma^0 = (\psi^*)^T\gamma^0$

$$\Rightarrow (\bar{\psi})^T = ((\psi^*)^T\gamma^0)^T = (\gamma^0)^T\psi^* = \gamma^0\psi^*$$

$$\Rightarrow C\bar{\psi}^T = i\gamma^2\gamma^0\gamma^0\psi^* = i\gamma^2\psi^*$$

and, $$C\gamma^0\psi^* = i\gamma^2\gamma^0\gamma^0\psi^* = i\gamma^2\psi^*$$

$$\therefore \psi_P = C\gamma^0\psi^* = C\bar{\psi}^T$$

(*) Show that, $C\sigma_{\mu\nu}C^{-1} = -\sigma_{\mu\nu}^{T}$

$$C\sigma_{\mu\nu}C^{-1} = \left(\frac{i}{2}\right)(i)^2 \; \gamma^2\gamma^0(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^2\gamma^0$$

$$= +\frac{i}{2}\left(\gamma^2\gamma^0\gamma^\mu\gamma^\nu\gamma^2\gamma^0 - \gamma^2\gamma^0\gamma^\nu\gamma^\mu\gamma^2\gamma^0\right)$$

$$= +\frac{i}{2}\left(-\gamma^2\gamma^2\gamma^0\gamma^\mu\gamma^\nu\gamma^0 + \gamma^2\gamma^2\gamma^0\gamma^\nu\gamma^\mu\gamma^0\right)$$

$$= \frac{+i}{2}\left(\gamma^0\gamma^\mu\gamma^\nu\gamma^0 - \gamma^0\gamma^\nu\gamma^\mu\gamma^0\right)$$

$$= \frac{-i}{2}\left(\gamma^0\gamma^\nu\gamma^\mu\gamma^0 - \gamma^0\gamma^\mu\gamma^\nu\gamma^0\right)$$

$$= \frac{-i}{2}\left(\gamma^0\gamma^0\gamma^\nu\gamma^\mu - \gamma^0\gamma^0\gamma^\mu\gamma^\nu\right)$$

$$= \frac{-i}{2}\left(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu\right) = +\sigma_{\mu\nu}.$$

$$\sigma_{\mu\nu}^{T} = -\frac{i}{2}\left[C(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)C^{-1}\right]^T$$

$$= -\frac{i}{2}\left\{(C^{-1})^T(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)^T C^T\right\}$$

$$= \left(-\frac{i}{2}\right)\left\{i\gamma^0\gamma^2\left(\gamma^{\nu T}\gamma^{\mu T} - \gamma^{\mu T}\gamma^{\nu T}\right)\left(-i\gamma^2\gamma^0\right)\right\}$$

$$= -\frac{i}{2}\left\{\gamma^0\gamma^2\left(\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0 - \gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^\nu\gamma^0\right)\gamma^2\gamma^0\right\}$$

$$= \frac{-i}{2}\left\{\gamma^0\gamma^2\gamma^0\gamma^\nu\gamma^\mu\gamma^0\gamma^2\gamma^0 - \gamma^0\gamma^2\gamma^0\gamma^\mu\gamma^\nu\gamma^0\gamma^2\gamma^0\right\}$$

$$= -\frac{i}{2}\left\{+\gamma^2\gamma^\nu\gamma^\mu\gamma^2 - \gamma^2\gamma^\mu\gamma^\nu\gamma^2\right\}$$

$$= \frac{i}{2}\left\{\gamma^2\gamma^\mu\gamma^\nu\gamma^2 - \gamma^2\gamma^\nu\gamma^\mu\gamma^2\right\}$$

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$$= \frac{i}{2}\{\gamma^2\gamma^2\gamma^\mu\gamma^\nu - \gamma^2\gamma^2\gamma^\nu\gamma^\mu\} = \frac{i}{2}(-\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)$$

$$= -\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = -\sigma_{\mu\nu} \Rightarrow -\sigma_{\mu\nu}^T = \sigma_{\mu\nu}$$

$$\therefore \; C\sigma_{\mu\nu}C^{-1} = -\sigma_{\mu\nu}^T$$

<u>Is the state $\psi^c = \psi_p$ really a +ve energy state of a positron?</u>

Consider, for simplicity, one of the (-ve) energy states of dirac eq^n when the particle is at rest. We take,

$$\psi^{(4)}(0) = N e^{\frac{i}{\hbar}mc^2t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

The charge conjugate wave function corresponding to $\psi^{(4)}(0)$,

$$\psi^{(4)c} = (C\gamma^0)\psi^{(4)*} = (i\gamma^2\gamma^0)\gamma^0\psi^{(4)*} = i\gamma^2\psi^{(4)*}$$

$$\Rightarrow \psi^{(4)c} = i\begin{pmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{pmatrix}\psi^{(4)*}$$

$$\Rightarrow \psi^{(4)c} = i\begin{pmatrix}0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ 0 & i & 0 & 0\\ -i & 0 & 0 & 0\end{pmatrix} N e^{-\frac{i}{\hbar}mc^2t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\Rightarrow \psi_p^{(4)} = iNe^{-\frac{i}{\hbar}mc^2t}\begin{pmatrix}-i\\0\\0\\0\end{pmatrix} = Ne^{-\frac{i}{\hbar}mc^2t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \psi_e^{(1)}(0).$$

i.e., electronic wave function having negative energy and spin ~~prod~~ projection down will correspond to, upon charge conjugation, the positronic wave function having positive energy and spin projected down.