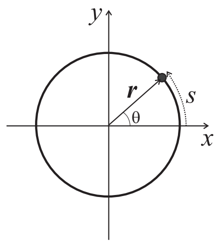
Rotational Motion

Motion in a Circle

The radian, θ is defined by the equation,



$$\theta = \frac{s}{r}$$

and the angular velocity, ω (units: rad s^{-1}), by the equation

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} \qquad \left(\mathrm{cf.}\quad v = \frac{\mathrm{d}r}{\mathrm{d}t}\right)$$

Similarly, we can also have an angular acceleration, α

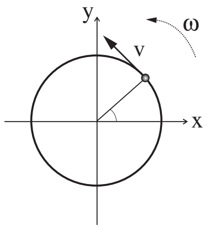
$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} \qquad \left(\mathrm{cf.}\quad a = \frac{\mathrm{d}v}{\mathrm{d}t}\right)$$

For uniform circular motion, we can define a rotational period, T, and rotational frequency, f related to ω by the equations

$$\omega = \frac{2\pi}{T} = 2\pi f$$

Vectors in Circular Motion

For a fixed angular velocity, ω, the velocity of a rotating particle, v, must be directly proportional to the radius of rotation, R.

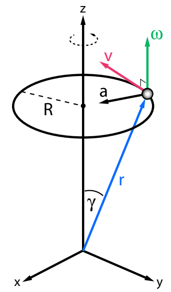


$$v = \omega R$$

However, both angular velocity and the particle position are vectors. To deal with this problem fully, we must be able to deal with the product of two vectors.

In the case shown above, angular velocity and particle position vectors were orthogonal, so the angular velocity vector is simply a vector of magnitude ωR in a direction perpendicular to the plane of motion (into the page for the diagram above).

To deal with the more general case, we must consider vectors from a point of reference that is not simply the centre of rotation.



$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

or

$$\boxed{\vec{v} = \hat{n}|\omega||r|\sin\gamma}$$

Remember that $\boldsymbol{n}$ describes a unit vector orthogonal to the plane containing the vectors $\boldsymbol{\omega}$ and $\boldsymbol{r}$ (as defined by the right hand rule).

Centripetal Acceleration

For uniform circular motion (constant ω) the centripetal acceleration is therefore

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{\omega} \times \frac{d\vec{r}}{dt} \qquad \text{or} \qquad \vec{a} = \vec{\omega} \times \vec{v}$$

The centripetal acceleration points radially inwards. It is constant in magnitude but not in direction. As $\boldsymbol{\omega}$ is perpendicular to $\boldsymbol{v}$ we can write $a = \omega v$. Remembering that $v = \omega R$ for this case

$$a = \omega^2 R = \frac{v^2}{R}$$

Using Newton's second law, the magnitude of the centripetal force is thus

$$\boxed{F = ma = \frac{mv^2}{R}}$$

An Example: Circular Motion of an Electron

Earlier, we learnt that the force of attraction between an electron and a proton in a hydrogen atom is approximately 8.2×10^{-8} N. Assuming that it is centripetal motion that prevents the electron from plummeting into the nucleus (it is not), we can calculate the velocity of the electron. The mass of the electron is 9.109×10^{-31} kg, and the radius of a hydrogen atom is 5.29×10^{-11} m. [$v=2.2 \times 10^{6}$ ms^{-1}]

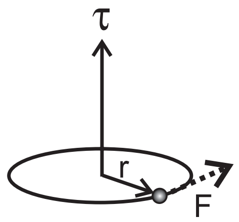
Torque

A torque can (roughly) be considered to be the rotational equivalent of a force. For

$$\tau = rF$$

a force applied perpendicular to r, the torque, τ, is $\tau = rF$ or more generally,

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$



Angular Momentum

Is it possible to define the torque in terms of a derivative of a momentum, like with linear forces? Let us define the angular momentum, $\boldsymbol{l}$

$$\boxed{\vec{l} = \vec{r} \times \vec{p}}$$

Then taking the time derivative we obtain

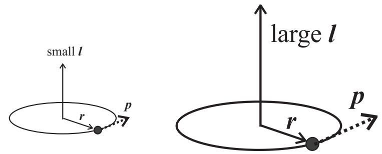
$$\frac{\mathrm{d}\vec{l}}{\mathrm{dt}} = \vec{r} \times \vec{F} = \vec{\tau}$$

i.e.

Properties of Gases & Classical Mechanics

$$\vec{\tau} = \frac{\mathrm{d}\vec{l}}{\mathrm{dt}} \qquad \left(\mathrm{cf.} \quad \vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right)$$

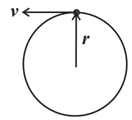
Angular momentum is a vector directed perpendicular to the plane of rotation (as defined by the right hand rule).



Angular momentum for uniform motion in a circle

For uniform motion in a circle (i.e. no angular acceleration) $\boldsymbol{p}$ and $\boldsymbol{v}$ are constant in magnitude and always directed perpendicular to $\boldsymbol{r}$, and the angular momentum has a constant magnitude

$$l = pr = mvr = mr^2\omega = I\omega$$



Conservation of Angular Momentum

If there is no external torque acting on a system, the total angular momentum is constant in magnitude and direction.

Angular momentum applies in almost all cases that we are interested in describing in Chemistry. Be sure you understand this concept, you will be seeing it a lot!

An Example: N_2 Rotation

The bond length of N_2 changes from 1.22 Å to 1.287 Å upon electronic excitation. Using conservation of angular momentum, calculate the fractional change in rotational period of the molecule. [11.3% increase in rotational period]

Rotational Kinetic Energy

The kinetic energy of a particle, i, rotating with a constant angular frequency ω about a fixed axis is (using $v = \omega R_i$ where R_i is the particle's distance from the axis of rotation).

$$K_{i,\text{ang}} = \tfrac{1}{2} m_i {v_i}^2 = \tfrac{1}{2} m_i \omega^2 {R_i}^2$$

For a system of particles all rotating with frequency ω, the rotational (angular) kinetic energy is therefore

$$K_{\text{ang}} = \sum_i K_{i,\text{ang}} = \tfrac{1}{2} \sum_i m_i \omega^2 {R_i}^2$$

Moments of Inertia

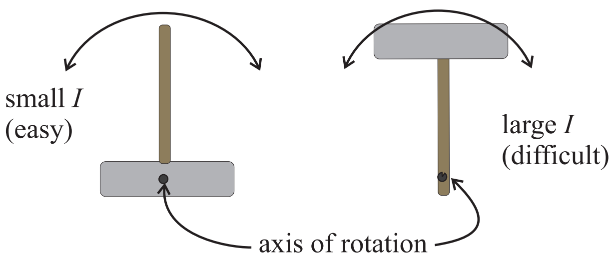
Defining the moment of inertia, I, as

$$\boxed{I = \sum_i m_i r_i^2}$$

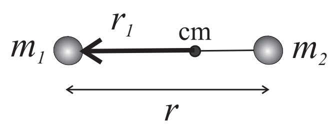
the rotational kinetic energy can be written

$$K_{\text{ang}} = \tfrac{1}{2} I \omega^2 \qquad \left(\text{cf.}\ \ K_{\text{lin}} = \tfrac{1}{2} m v^2\right)$$

The moment of inertia plays a similar role in rotational motion as mass does in linear motion. The magnitude of I depends on the axis of rotation.



Classical rotation of diatomic molecules



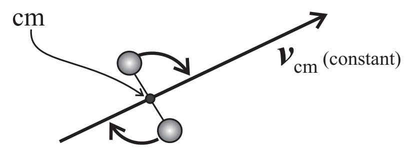
For a diatomic molecule with a bond length r, rotation must occur about the centre-of-mass, and the moment of inertia can be written

$$I = \sum_i m_i {r_i}^2 = \mu r^2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

μ is the reduced mass. Reduced mass is the \`effective\` inertial mass. Rather than considering the motion of the molecule, by using the reduced mass we can focus only on the motion of each atom relative to the centre of mass. We can prove this by considering the relative acceleration between the two atoms ($a = a_1 - a_2$) from Newton's third law.

In the absence of external forces on the molecule, the motion of the centre-of-mass is conserved, and the total kinetic energy of the molecule can be factored

$$K = K_{\text{CM}} + K_{\text{ang}}$$



Because both atoms rotate with the same frequency ω about the CM, the angular momentum and the angular kinetic energy of the molecule may be written as

$$l = \sum_i m_i {r_i}^2 \omega = I\omega \qquad K_{\text{ang}} = \tfrac{1}{2} \sum_i m_i r_i^2 \omega^2 = \frac{l^2}{2I}$$