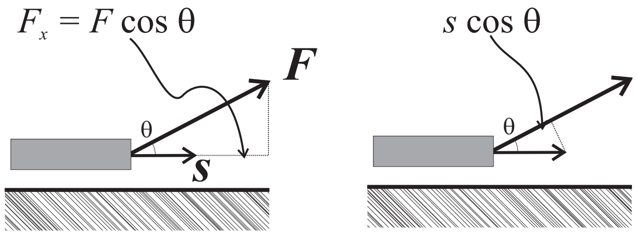
Translational Motion II

Work

The concept of mechanical work provides the link between **force** and **energy**. Work is done on an object when a force acts on it in the direction of motion.



The mechanical work, W, done by a constant force, F, is simply the force times the total displacement, s, in the direction of the force. This is most generally described using vector notation as a scalar product.

$$\boxed{W = \vec{F} \cdot \vec{s}}$$

Notice that the resultant work done is a **scalar** quantity not a vector. Also notice that if there is no displacement in the direction of the force, no work is done. Conversely, if the displacement is **in** the direction of the force the work done is simply ***W=Fs.***

The work done can also be written as

$$W = |F||s|\cos\theta$$

where $|F|$ and $|s|$ are the magnitudes of the Force and displacement, and θ is the angle between the two vectors. Again, notice that **work done is a scalar**.

For the above definitions, we have assumed the force applied is constant. If the force is not constant we need to extend our definition of work. The work done by a force acting on an object moving in a fixed direction is

Properties of Gases & Classical Mechanics

$$W = \int_{x_1}^{x_2} F_x \, dx$$

where dx is the infinitesimal displacement along x.

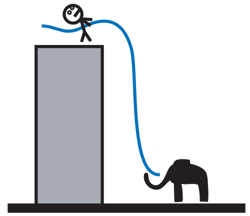
If the movement is not in a constant direction, again we need to extend our mathematical description to integrate over the path taken during the motion.

$$W = \int_C \vec{F} \cdot d\vec{s}$$

Here C describes the path taken during the motion. We also need to include the scalar product as now we must treat both force and displacement as vectors. **We will not deal with path integrals in this lecture course.**

An Example: Lifting Elephants

Your lecturer decides to haul an Asian Bull Elephant to the roof of the Chemistry Research Laboratory, using a steel cable weighing 500 g per metre. Assuming the CRL is 100 m high, and that the average weight of an Asian Bull Elephant is 2300 kg, calculate the work done.



Dealing with the elephant first:

$$W \;=\; F\,s \;=\; (mg)s = 2300 \times 9.8 \times 100 = 2.25 \times 10^6 \quad J$$

Now the cable:

$$W \;=\; \int_0^{100} F\,ds \;=\; \int_0^{100} (mg)s\,ds = 0.5 \times 9.8 \left[\frac{s^2}{2}\right]_0^{100} = 2.45 \times 10^4 \quad J$$

So in total:

$$W \; = 2.28 \times 10^6 \quad J$$

Kinetic Energy

The kinetic energy, K, of a particle is the energy a particle possesses by virtue of its motion. For a particle of mass m moving along x with velocity v_x

$$\boxed{K = \tfrac{1}{2}mv_x^2}$$

Returning to the equation for the work done on a particle

$$W = \int F_x\, dx$$

We can use Newton's 2nd Law to rewrite

$$F_x\, dx = ma_x\, dx = m\frac{dv_x}{dt}\, dx = mv_x\, \mathrm{d}v_x$$

which gives

$$W = \int_{v_1}^{v_2} mv_x\, \mathrm{d}v_x = \tfrac{1}{2}m(v_2^2 - v_1^2)$$

The work done on the particle is equal to its change in kinetic energy.

$$\boxed{W = \Delta K}$$

An Example: The Cathode Ray Tube

An electron accelerated in a TV tube reaches the screen with a kinetic energy of 10000 eV. Find the velocity of the electron.

We must first convert from eV into Joules.

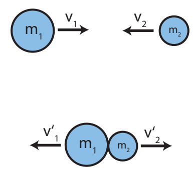
$$K = 10^4\, eV = 10^4 \times 1.6 \times 10^{-19}\, J = 1.6 \times 10^{-15}\, J$$

Before calculating the velocity

$$v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2 \times 1.6 \times 10^{-15}}{9.109 \times 10^{-31}}} = 5.93 \times 10^7\, ms^{-1}$$

Pretty fast!

An Example: Elastic Collisions



In an elastic collision, **kinetic energy is conserved**, along with total energy and momentum which are always conserved.

Consider the head-on collision between two atoms. Derive an expression for the final velocity of both particles in terms of their masses and initial velocities assuming the collision is elastic.

First let's set up what we know about the system.

Initial momentum of system:

$$p_i = m_1v_{i1} + m_2v_{i2}$$

Final momentum of the system:

$$p_f = m_1v_{f1} + m_2v_{f2}$$

Initial kinetic energy of system:

$$K_i = \tfrac{1}{2}m_1v_{i1}^2 + \tfrac{1}{2}m_2v_{i2}^2$$

Final kinetic energy of the system:

$$K_f = \tfrac{1}{2}m_1v_{f1}^2 + \tfrac{1}{2}m_2v_{f2}^2$$

Now let's apply conservation of momentum,

$$\begin{aligned} m_1v_{i1} + m_2v_{i2} &= m_1v_{f1} + m_2v_{f2} \\ m_1\left(v_{i1} - v_{f1}\right) &= m_2\left(v_{f2} - v_{i2}\right) \qquad \text{—❶} \end{aligned}$$

and similarly, conservation of kinetic energy, as it is an elastic collision.

$$\begin{aligned} m_1v_{i1}^2 + m_2v_{i2}^2 &= m_1v_{f1}^2 + m_2v_{f2}^2 \\ m_1\left(v_{i1}^2 - v_{f1}^2\right) &= m_2\left(v_{f2}^2 - v_{i2}^2\right) \qquad \text{—❷} \end{aligned}$$

We can solve these two simultaneous equations (❶ and ❷) to determine the final velocities, however it may not be immediately obvious how to do so. One route is to note that a^2-b^2=$(a+b)(a-b)$ and substitute into ❷.

$$m_1\left(v_{i1}+v_{f1}\right)\left(v_{i1}-v_{f1}\right) \quad = \quad m_2\left(v_{f2}+v_{i2}\right)\left(v_{f2}-v_{i2}\right) \qquad \text{—❸}$$

❸ / ❶ then gives us

$$v_{i1}+v_{f1} \;=\; v_{f2}+v_{i2} \qquad \text{—❹}$$

Thus the difference in initial velocities is equal to the difference in final velocities. We can the sub this expression into ❶ or ❷ to retrieve the final velocities in terms of the initial conditions:

For example, take ❹ and multiply by m_1

$$m_1 v_{i1}+m_1 v_{f1} \;=\; m_1 v_{f2}+m_1 v_{i2}$$

Add this to ❶ giving

$$\begin{aligned} m_1 v_{i1}+m_1 v_{f1}+m_1 v_{i1}-m_1 v_{f1} &= m_1 v_{f2}+m_1 v_{i2}+m_2 v_{f2}-m_2 v_{i2} \\ 2 m_1 v_{i1} &= \left(m_1+m_2\right) v_{f2}+\left(m_1-m_2\right) v_{i2} \end{aligned}$$

So our expression for the final velocity of particle 2 is

$$v_{f2}=\frac{2 m_1}{m_1+m_2} v_{i1}+\frac{m_2-m_1}{m_1+m_2} v_{i2}$$

The equivalent expression for particle 1 is straightforward to derive.

Potential Energy

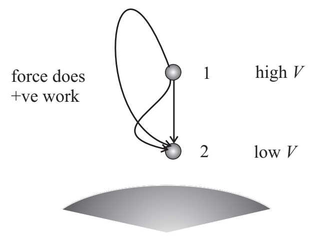
The potential energy, V, is the energy associated with the position of a particle. Potential energy may be thought of as stored energy, or the capacity to do work.

The Link Between Force, Work and Potential Energy.

For some forces the work done by the force is **independent of the path** taken. Such forces are called **conservative forces** and include gravity and electrostatic forces.

Conservative forces can be represented by potential energy functions because they depend solely on position. For non-conservative forces, such as friction, the work done and the force depends on the path taken.

Consider gravity as an example of a conservative force:



The work done dW by the gravitational force F is independent of the path. In moving the particle from position 1 to 2 its capacity to do work (i.e. potential energy) is reduced. The fixed amount of work is therefore minus the change in potential energy:

$$\boxed{\mathrm{d}V=-\mathrm{d}W}$$

As we already know that $W = \int F_x \, dx$, we can combine these equations to yield

$$\boxed{F=-\frac{\mathrm{d}V}{\mathrm{d}x}}$$

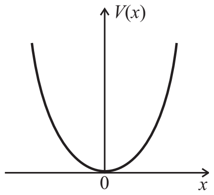
Therefore, the (finite) change in potential energy between points x_1 and x_2 is

$$\boxed{V(x_2)-V(x_1) \;=\; \Delta V \;=\; -\int_1^2 F_x \, \mathrm{d}x \;=\; -W}$$

An Example: The Harmonic Spring

$$V(x) = \frac{1}{2}kx^2$$

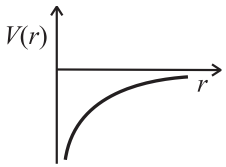
$$F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x} = -kx$$



An Example: Gravitational Potential Energy

$$V(r) = -\frac{Gm_1m_2}{r}$$

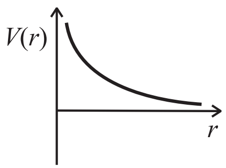
$$F(r) = -\frac{\mathrm{d}V}{\mathrm{d}r} = -\frac{Gm_1m_2}{r^2}$$



An Example: Electrostatic Potential Energy

$$V(r) = \frac{q_1q_2}{4\pi\epsilon_0 r}$$

$$F(r) = -\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{q_1q_2}{4\pi\epsilon_0 r^2}$$



Here q_1 and q_2 have the same sign.

Conservation of Energy

The total energy in a closed system of particles is constant.

As we showed above, the work done by a force is related to changes in both the kinetic and potential energy. Note that again we focus exclusively on conservative forces.

$$\boxed{W = \Delta K = -\Delta V}$$

Rearranging $\Delta K = -\Delta V$ yields $\Delta K + \Delta V = 0$. Therefore, the sum of these energies, called the total energy, E, must be constant:

$$\boxed{E = K + V}$$

Frames of Reference

Take the previous example of an elastic collision between two particles, In applying conservation of momentum we had to pick a frame of reference. We measured the velocity of m_1 and m_2 relative to a fixed frame of reference. We can link velocity of an object in one frame of reference (v) with motion in another (v') by simply subtracting the relative velocity between the two frames (v_{rel}).

$$\vec{v}' = \vec{v} - \vec{v}_{rel}$$

Conservation of momentum can only be applied within one frame of reference. This concept is often useful for simplifying collisions; in particular in the **centre of mass frame of reference** the total momentum is zero.

Centre of Mass

The centre of mass of a system is the point where the system responds as if it were a point with a mass equal to the sum of the masses of its constituent parts. In one-dimension for N particles in a system this is described by

$$\boxed{Mx_{CM} = \sum_{i=1}^{N} m_i x_i} \qquad \boxed{M = \sum_{i=1}^{N} m_i}$$

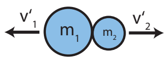
where x_{CM} is the distance to the centre of mass, m_i is a particles mass and x_i is its position. For two particles (such as in a diatomic molecule) this simplifies to

$$(m_1 + m_2)\, x_{CM} = m_1 x_1 + m_2 x_2$$

An Example: COM Frame in Collisions

Take the previous example of an elastic collision between two particles, we could've simplified our calculation by working in the centre of mass frame.

In the centre of mass frame, the velocity of the frame is zero, as is the total momentum. Let's denote the velocities in the new frame as v'



Initial momentum of system:

$$0 = m_1 v'_{i1} + m_2 v'_{i2}$$

Final momentum of the system:

$$0 = m_1 v'_{f1} + m_2 v'_{f2}$$

Initial kinetic energy of system:

$$K_i = \tfrac{1}{2} m_1 v'^2_{i1} + \tfrac{1}{2} m_2 v'^2_{i2}$$

Final kinetic energy of the system:

$$K_f = \tfrac{1}{2} m_1 v'^2_{f1} + \tfrac{1}{2} m_2 v'^2_{f2}$$

Now it's much more straightforward to show

$$\begin{aligned}
m_1 {v'_{i1}}^2 + m_2 {v'_{i2}}^2 &= m_1 {v'_{f1}}^2 + m_2 {v'_{f2}}^2 \\
m_1 {v'_{i1}}^2 + m_2 \left(-\frac{m_1}{m_2} v_{i1}\right)^2 &= m_1 {v'_{f1}}^2 + m_2 \left(-\frac{m_1}{m_2} v_{f1}\right)^2 \\
{v'_{i1}}^2 \left(m_1 + \frac{m_1^2}{m_2}\right) &= {v'_{f1}}^2 \left(m_1 + \frac{m_1^2}{m_2}\right) \\
{v'_{i1}}^2 &= {v'_{f1}}^2
\end{aligned}$$

This equation has two solutions. Either $v'_{i1} = v'_{f1}$ which is pretty boring; nothing's happened. Or $v'_{i1} = -v'_{f1}$ (and $v'_{i2} = -v'_{f2}$) which tells us that after the collision the particles have reversed their velocities.