20 minutes of entanglement

markwallace.org/teaching/entanglement

1 What is entanglement?

Entanglement quantifies the quantum mechanical observation that one can know everything about a system but nothing about its individual parts. It describes a situation where particles become connected in a way that the state of one particle instantly affects the state of another, no matter the distance between them.

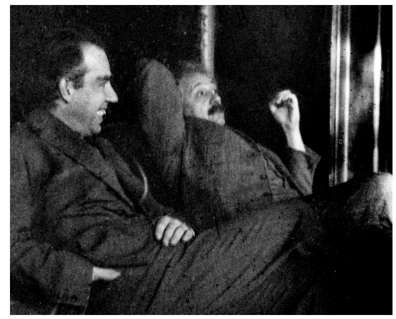

Figure 1: 'Einstein, stop telling God what to do.' - Niels Bohr

This fundamental feature of quantum mechanics sets it apart from what we might expect from classical intuition. The property of entanglement underpins developments in quantum computing and quantum cryptography. We can also find entanglement in biology, controlling processes as varied as photosynthesis, olfaction, and magnetoception.

Einstein referred to entanglement as "Spooky action at a distance". He argued for "hidden variables" that would eliminate the need for fundamental randomness in quantum mechanics, but experiments have since ruled out such hidden-variable theories.

2 Spin

Electron spin is an intrinsic angular momentum associated with the electron.

All possible spin states can be represented in a two-dimensional vector space. We can choose $|u\rangle$ and $|d\rangle$ as our two basis vectors and write all state as a linear superposition of these two states.

$$|\psi\rangle = \alpha|u\rangle + \beta|d\rangle$$

Recall, we can describe three operators σ_x,σ_y, σ_z that describe spin components in a specific direction.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Exercise: *Can you write out how σ_x, operates on $|u\rangle$ and $|d\rangle$? Can you do the same for σ_y and σ_z?*

3 Interacting spins

3.1 Product states

Now let us consider a system with not one, but two electrons. A state of this combined system is said to be entangled if it **cannot** be written in the form of a *product state*.

So considering two electrons, 1 and 2, we can write the overall product state as:

$$\begin{aligned} |\Psi\rangle &= |\psi_1\rangle \otimes |\psi_1\rangle \\ &= \alpha_1\alpha_2|uu\rangle + \alpha_1\beta_2|ud\rangle + \beta_1\alpha_2|du\rangle + \beta_1\beta_2|dd\rangle \end{aligned}$$

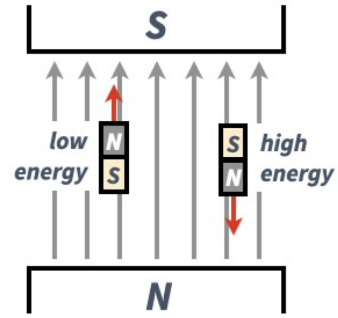

Figure 2: Spin alignment in an external field

The four complex numbers α_1, β_1,α_2, β_2 describe 8 parameters in total. Normalisation of the wavefunction for each individual electron removes two of these degrees of freedom. Phase invariance allows us to arbitrarily make two of these variables real (as any overall phase factor between different spin states cancels out when calculating probabilities the relative phase does not impact the measurable outcomes). That leaves 4 degrees of freedom.

3.2 Entangled states

Now let us consider the most general form of a linear superposiston of these four states, $|uu\rangle$, $|ud\rangle$, $|du\rangle$, and $|dd\rangle$:

$$|\psi\rangle = \phi_{uu}|uu\rangle + \phi_{ud}|ud\rangle + \phi_{du}|du\rangle + \phi_{dd}|dd\rangle$$

Again we have 4 complex numbers (ϕ_{uu}, ϕ_{ud}, ϕ_{du}, ϕ_{dd}), but now only one overall noramization condition, and one case of phase invaiance. That leaves 6 degrees of freedom. So this **entangled** system has more states than product states.

3.3 Singlet states

A singlet state is an example of a *maximally entangled state*. This state cannot be expressed as the product of two one-particle states.

$$|\psi_s\rangle = \frac{1}{\sqrt{2}}\Big\{|ud\rangle - |du\rangle\Big\}$$

Exercise: *Can you prove that* $|\psi_s\rangle$ *cannot be written as a product state?*

So what spins might we expect from such a state? Let's calculate the expectation values for σ_z of electron 1.

$$\begin{aligned}\langle\sigma_z\rangle &= \langle\psi_s|\sigma_z|\psi_s\rangle \\ &= \frac{1}{\sqrt{2}}\Big\{\langle ud| - \langle du|\Big\}\sigma_z\frac{1}{\sqrt{2}}\Big\{|ud\rangle - |du\rangle\Big\} \\ &= \frac{1}{2}\Big\{\langle ud| - \langle du|\Big\}\Big\{|ud\rangle + |du\rangle\Big\} \\ &= 0\end{aligned}$$

If the expectation value of a component of σ is zero, it means that the experimental outcome is equally as likely to be $+1$ or -1, thus the outcome is completely uncertain. Even though we know the exact state-vector, $|\Psi_s\rangle$, we know nothing at all about the outcome of any measurement of any component of either spin.

Exercise: *What about σ_x and σ_y?*

3.4 Spin products

Let's suppose we measure the spin state σ_z of both electrons. We need two operators that act on each electron, but they need different names. Let's call σ_z the operator that acts on the first electron and τ_z for second electron. Our observable is to measure both spins for electron pairs, and then they multiply the results, we measure the product $\tau_z\sigma_z$.

Let's apply this combined operator to our singlet state.

$$\begin{aligned}\tau_z\sigma_z|\Psi_s\rangle &= \tau_z\sigma_z\frac{1}{\sqrt{2}}\Big\{-|ud\rangle + |du\rangle\Big\} \\ &= -|\Psi_s\rangle\end{aligned}$$

So this tells us that for this singlet state electrons 1 and 2 always have opposite spins, even if the individual spins are random.

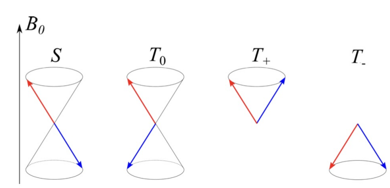

Figure 3: A vector model of spin alignment in an external field can be used to visualize the difference between singlet and triplet states.

Exercise: *Suppose that instead of measuring the z components of their spins, We measure the x components. To find out how their outcomes are correlated, calculate the observable $\tau_x\sigma_x$.*

This is interesting: $|\Psi_s\rangle$ is also an eigenvector of $\tau_x\sigma_x$ with eigenvalue -1. So regardless of the component, for $|\Psi_s\rangle$ the spins are always opposite. They are always **entangled**.

3.5 Other maximally entangled states

Given the general form of $|\Psi\rangle$, it is perhaps unsuprising that the singlet state $|\Psi_s\rangle$ is not the only maximally entangled state. There are three other *triplet* states to consider:

$$|\psi_{T,+}\rangle = \frac{1}{\sqrt{2}}\Big\{|uu\rangle + |dd\rangle\Big\}$$
$$|\psi_{T,0}\rangle = \frac{1}{\sqrt{2}}\Big\{|ud\rangle + |du\rangle\Big\}$$
$$|\psi_{T,-}\rangle = -\frac{1}{\sqrt{2}}\Big\{|uu\rangle - |dd\rangle\Big\}$$

4 Classical correlations

A classical analogue of two entangled spins might be to have two coins; flip one and set the other to the opposite configuration. The result of our observations here we can also call σ and τ, assigning heads as up and tails as down.

Figure 4: We can prepare two coins to provide a classically correlated two state system.

We can again calculate expectation values for $\langle\sigma_z\rangle$, $\langle\tau_z\rangle$ and $\langle\tau_z\sigma_z\rangle$. Notice that the average of the products does not equal to the product of the averages.

$$\langle\sigma_1\sigma_2\rangle - \langle\sigma_1\rangle\langle\sigma_2\rangle \neq 0$$

This is the *covariance* and is one way to quantify how two measurables are interrelated.

5 Summary

- **Non-locality**: Entangled particles exhibit instantaneous correlations regardless of distance, challenging classical notions of locality.
- **Measurement**: The act of measuring one entangled particle instantaneously determines the state of its counterpart, challenging classical causality.
- **Superposition**: Entangled particles encompass multiple states simultaneously, they are not like classical particles.
- Quantum mechanics implies **intrinsic randomness**.

6 Resources

L. Susskind and A. Friedman, **Quantum Mechanics: The theoretical minimum.** (Penguin, 2015, `ISBN:978-0465062904`) B. Simons **Part II: Advanced Quantum Mechanics Lecture Notes** `www.tcm.phy.cam.ac.uk/~bds10/aqp.html` D. Tong. **Part II: Applications of Quantum Mechanics Lecture Notes** `http://www.damtp.cam.ac.uk/user/tong/aqm/aqm.pdf` D.J. Griffiths. **Introduction to Quantum Mechanics.** Chapter 12. (CUP 2018, `ISBN:978-1107189638`) K. Konishi and G. Paffuti, **Quantum Mechanics: A New Introduction.** (OUP, 2009 `ISBN:978-0199560264`)