Physics 231 Problem Set 3 v2

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Some pertinent formulas/definitions:

$$V(r) = \int \frac{kdq}{|\vec{r'} - \vec{r}|} \tag{1}$$

$$\vec{E}(r) = -\vec{\nabla}V(r) \tag{2}$$

$$\nabla^2 V(r) = -\frac{\rho}{\epsilon_o} \tag{3}$$

1 Potential due to a charged spherical shell

Previously, we were able to identify the separation vector between a source point and a field point ($\vec{r'} - \vec{r}$) by means of geometric construction for this charge configuration. We shall use that in this problem to compute for the electric potential at a point P that is z units away from the center of the sphere of radius R and surface charge density σ.

Solution. The magnitude of the separation vector ($\vec{r} - \vec{r'}$) is given by

$$|\vec{r} - \vec{r'}| = \sqrt{R^2 + z^2 - 2Rz\cos\theta}$$

which when substituted to (1) yields,

$$\begin{aligned} V(r) &= \int \frac{k\sigma(R^2 \sin\theta' d\theta' d\phi')}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}} \\ &= k\sigma R^2 \int_0^{2\pi} d\phi' \int_0^{\pi} \frac{\sin\theta' d\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}} \\ &= 2\pi k\sigma R^2 \int_0^{\pi} \frac{\sin\theta' d\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}} \end{aligned} \tag{4}$$

By doing a u-substitution of the form,

$$u = R^2 + z^2 - 2Rz\cos\theta', \quad du = 2Rz\sin\theta' d\theta'$$

integral (4) effectively becomes,

$$\begin{aligned} V(z) = \frac{\pi k\sigma R}{z} \int_{(R-z)^2}^{(R+z)^2} u^{-1/2} du &= \frac{2\pi k\sigma R}{z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right] \\ &= \frac{2\pi k\sigma R}{z}[(R+z) - \text{abs}(R-z)] \end{aligned}$$

where

$$\text{abs}(R-z) = \begin{cases} R-z, & \text{for } R > z \\ z-R, & \text{for } R < z \end{cases}$$

For cases when $z < R$, that is when $R - z > 0$,

$$V(z) = \frac{2\pi k\sigma R}{z}[(R+z)+(R-z)] = \frac{4\pi k\sigma R^2}{z}\boxed{= \frac{kQ}{z}} \longrightarrow \vec{E} = -\vec{\nabla}V = -\partial_z\left(\frac{kQ}{z}\right)\boxed{= \frac{kQ}{z^2}}$$

where $Q = \sigma(4\pi R^2)$.
For cases when $z > R$, that is when $R - z < 0$,

$$V(z) = \frac{2\pi k\sigma R}{z}[(R+z)-(R-z)]\boxed{= 4\pi k\sigma R} \longrightarrow \vec{E} = -\vec{\nabla}V = -\partial_z(4\pi k\sigma R) = \boxed{= 0}$$

which implies that the potential inside the spherical shell is constant.

2 Screened Coulomb potential

A screened couloumb potential is the usual potential due to some charge q that is damped by an exponential function $\exp(-\lambda r)$. In this problem, we are asked to calculate for the **electric field**, **charge density**, ρ, and **total charge** Q, due to a screened Coulomb potential of the form,

$$V(r) = \frac{A\exp(-\lambda r)}{r}$$

Solution. (a) The electric field due to this potential is given by,

$$\begin{aligned}
\vec{E}(r) &= -\vec{\nabla}V(r) \\
&= -\partial_r V(r)\hat{r} = -\partial_r\left(\frac{A\exp(-\lambda r)}{r}\right)\hat{r} \\
&= -A\left(-\lambda\frac{\exp(-\lambda r)}{r} - \frac{\exp(-\lambda r)}{r^2}\right)\hat{r} = \boxed{A\exp(-\lambda r)\left(\frac{\lambda r + 1}{r^2}\right)\hat{r}}
\end{aligned}$$

(b) The charge density ρ can be calculated by using Laplace's equation. Doing so yields,

$$\begin{aligned}
\frac{\rho}{\epsilon_o} &= -\nabla^2 V(r) \\
&= -\vec{\nabla}\cdot\vec{\nabla}V(r) = \vec{\nabla}\cdot\vec{E} \\
&= \vec{\nabla}\cdot\left[A\exp(-\lambda r)(\lambda r + 1)\left(\frac{\hat{r}}{r^2}\right)\right] \\
&= \vec{\nabla}[A\exp(-\lambda r)(\lambda r + 1)]\cdot\frac{\hat{r}}{r^2} + A\exp(-\lambda r)(\lambda r + 1)\left[\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)\right] \\
&\boxed{= \frac{-A\lambda^2\exp(-\lambda r)}{r} + 4\pi A\exp(-\lambda r)(\lambda r + 1)\delta^{(3)}(r)}
\end{aligned}$$

(c) We can calculate for the total charge by integrating the charge density over all space,

$$
\begin{aligned}
Q_{tot} &= \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \int_0^{\infty} r'^2 dr' \left[\frac{-A\lambda^2 \exp(-\lambda r')}{r'} + 4\pi A \exp(-\lambda r')(\lambda r' + 1)\delta^{(3)}(r')\right] \\
&= 4\pi A \left\{-\lambda^2 \int_0^{\infty} r' \exp(-\lambda r') + \int_0^{\infty} \exp(-\lambda r')(\lambda r' + 1)\delta(r')\right\} \\
&= 4\pi A \left\{-\lambda^2 \left(\frac{1}{\lambda^2}\right) + 1\right\} \\
&\boxed{= 0.}
\end{aligned}
$$

3 Potential due to a charged cone

We are asked to calculate for the potential difference between two points, i.e. at the apex of a charged cone of diameter and height h, and at the center of its base.

Solution. We start by finding the potential due to the configuration at the apex of the cone. We set up our potential by identifying our infinitesimal charge element dq, which by definition would be equal to $dq = \sigma(2\pi z)dh'$, where dh' corresponds to the slanted length along the surface of the cone. By construction, $dh' = \frac{\sqrt{5}}{2}dz$, where z is the coordinate along the axis through the center of the cone.

$$
V(z=a) = \int_0^h \frac{kdq}{r} = \int_0^h \frac{\sqrt{5}\pi k\sigma z dz}{\frac{\sqrt{5}}{2}z} = 2\pi\sigma k h
$$

For the potential at the center of the base of the cone, we shall use the same differential charge dq but a different separation distance r. By construction, we can see that our separation distance only depends on the height of the cone, and some perpendicular distance z of the differential loop that we are considering away from the base of the cone.

$$
\begin{aligned}
V(z=b) = \int_0^h \frac{kdq}{r} &= \int_0^h \frac{\sqrt{5}\pi k\sigma z dz}{\sqrt{\frac{z^2}{4} + (h-z)^2}} \\
&= \sqrt{5}\pi k\sigma \int_0^h dz \frac{z}{\sqrt{\frac{z^2}{4} + (h-z)^2}} \\
&= \sqrt{5}\pi k\sigma \int_0^h dz \frac{z}{\sqrt{\frac{5}{4}(z^2 - \frac{8}{5}hz + \frac{16}{25}h^2 - \frac{16}{25}h^2) + h^2}} \\
&= 2\pi k\sigma \left[\int_0^h dz \frac{z}{\sqrt{(z - \frac{4h}{5})^2 + \frac{4h^2}{25}}}\right] \\
&\text{(by completing the squares)}
\end{aligned}
$$

We can perform a u-substitution of the form,

$$
u = z - \frac{4h}{5}, \quad du = dz
$$

and a trigonometric substitution of the form,

$$\frac{5}{2h}u = \tan\theta, \quad \frac{5}{2h}du = \sec^2\theta d\theta$$

and yield an integral,

$$\begin{aligned} V(z=b) &= \frac{4\pi kh\sigma}{5}\int_{\theta_1}^{\theta_2} d\theta \sec\theta(\tan\theta + 2) \\ &= \frac{4\pi kh\sigma}{5}\left[\sec\theta + 2\ln|\sec\theta + \tan\theta|\right]\Big|_{\theta_1}^{\theta_2} \end{aligned}$$

Recall that

$$\tan\theta = \frac{5z-4h}{2h}$$

such that

$$\sec\theta = \frac{\sqrt{(5z-4h)^2+4h^2}}{2h}.$$

This yields,

$$\begin{aligned} V(z=b) &= \frac{4\pi kh\sigma}{5}\left\{\frac{\sqrt{(5z-4h)^2+4h^2}}{2h} + 2\ln\left|\frac{\sqrt{(5z-4h)^2+4h^2}+(5z-4h)}{2h}\right|\right\}\Bigg|_0^h \\ &= \frac{4\pi kh\sigma}{5}\left\{\left(\frac{\sqrt{5}}{2} - \sqrt{5}\right) + 2\ln\left|\frac{(\sqrt{5}+1)}{(2\sqrt{5}-4)}\right|\right\} \\ &= \frac{4\pi kh\sigma}{5}\left[-\frac{\sqrt{5}}{2} + 2\ln\left|\frac{7+3\sqrt{5}}{2}\right|\right] \end{aligned}$$

The potential difference between the apex and the center of the base would then be,

$$V(z=b) - V(z=a) \boxed{= 2\pi kh\sigma\left\{\frac{2}{5}\left[-\frac{\sqrt{5}}{2} + 2\ln\left|\frac{7+3\sqrt{5}}{2}\right|\right] - 1\right\}}$$

4 Potential due to an infinite line charge

We are asked to calculate for the potential due to an infinite line charge of charge density λ at some point P that is at a perpendicular distance s units away from the line charge.

Solution. We will solve this problem by calculating for the potential due to a finite line charge and take the limit as its length approaches infinity.

$$V(s) = \int_{-L}^{L} \frac{k\lambda dx'}{\sqrt{x'^2+s^2}}$$

This integral can be solved by performing a trigonometric substitution of the form,

$$\tan\theta = \frac{x'}{s}, \qquad \sec^2\theta d\theta = \frac{dx'}{s}$$

which yields,

$$
\begin{aligned}
V(s) &= k\lambda \int_{\theta_1}^{\theta_2} \frac{\sec^2\theta d\theta}{\sqrt{\tan^2\theta + 1}} = k\lambda \int_{\theta_1}^{\theta_2} \sec\theta d\theta \\
&= k\lambda \ln\left[\frac{\sqrt{L^2+s^2}+L}{\sqrt{L^2+s^2}-L}\right]
\end{aligned}
$$

Taking the limit as L approaches infinity would allow us to make approximations on our potential expression such that,

$$
\begin{aligned}
V(s) = k\lambda \ln\left[\frac{\sqrt{L^2+s^2}+L}{\sqrt{L^2+s^2}-L}\right] &= k\lambda \ln\left[\frac{\sqrt{1+(\frac{s}{L})^2}+1}{\sqrt{1+(\frac{s}{L})^2}-1}\right] \\
&\approx k\lambda \ln\left[\frac{1+\frac{1}{4}(\frac{s}{L})^2}{\frac{1}{4}(\frac{s}{L})^2}\right] \\
&\text{(via binomial approximation, } (1+x)^\alpha \approx 1+\alpha x) \\
&= k\lambda\left[\ln\left(1+\frac{s^2}{4L^2}\right) - \ln\left(\frac{s^2}{4L^2}\right)\right] \\
&\approx k\lambda\left[\frac{s^2}{4L^2} - \ln\left(\frac{s^2}{4L^2}\right)\right] \\
&\text{(via } \ln(1+x) \approx x)
\end{aligned}
$$

As $L \longrightarrow \infty$, the expression

$$V(s) \approx -2k\lambda \ln\left(\frac{s}{2L}\right)$$

will diverge. Therefore, it is intuitive that we take some reference point that is a perpendicular distance c away from the line charge, and take the potential difference between an arbitrary point c and at a point s from the line charge such that,

$$
\begin{aligned}
V(s) - V(c) &= -2k\lambda \ln\left(\frac{s}{2L}\right) + 2k\lambda \ln\left(\frac{c}{2L}\right) \\
&= -2k\lambda[\ln(s) - \ln(c)] \\
&\boxed{= -2k\lambda\left[\ln\left(\frac{s}{c}\right)\right]}
\end{aligned}
$$

$$\vec{E} = -\vec{\nabla}V = -\partial_s\left[-2k\lambda\left[\ln\left(\frac{s}{c}\right)\right]\right]\hat{s} \boxed{= \frac{2k\lambda}{s}\hat{s}}$$