Physics 232 Third Long Exam

Mark Ivan G. Ugalino

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1 Photons with spin and orbital angular momentum

1.1 Zangwill Part 1

a. We show that we can express the angular momentum of an electromagnetic field in empty space as a sum of spin and orbital angular momentum. The angular momentum of an electromagnetic field is defined as,

$$\mathbf{L} = \int d^3r \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) = \int d^3r \mathbf{r} \times (\mathbf{E} \times (\nabla \times \mathbf{A})) \tag{1}$$

$$L_i = \epsilon_0 \int dr \left\{\epsilon_{ijk}\epsilon_{kmn}\epsilon_{npq} r_j E_m \partial_p A_q\right\} \tag{2}$$

Via product rule,

$$\begin{aligned} \partial_p(r_j E_m A_q) &= E_m A_q \partial_p r_j + r_j \partial_p (A_q E_m) &(3)\\ &= E_m A_q \partial_p r_j + r_j E_m \partial_p A_q + r_j A_q \partial_p E_m &(4) \end{aligned}$$

and the identity from Zangwill (1.39), we can rewrite the cross products in index notation,

$$\begin{aligned} L_i &= \epsilon_0 \epsilon_{ijk} \int dr \left(\delta_{kp}\delta_{mq} - \delta_{kq}\delta_{mp}\right) \left\{\partial_p \left(r_j E_m A_q\right) - A_q E_m \delta_{pj} - A_q r_j \partial_p E_m\right\} &(5)\\ &= \epsilon_0 \epsilon_{ijk} \int dr \left(\delta_{kp}\delta_{mq} - \delta_{kq}\delta_{mp}\right) \partial_p \left(r_j E_m A_q\right) - \left(\delta_{kp}\delta_{mq} - \delta_{kq}\delta_{mp}\right) A_q r_j \partial_p E_m - \left(\delta_{kp}\delta_{mq} - \delta_{kq}\delta_{mp}\right) A_q E_m \delta_{pj} &(6)\\ &= \epsilon_0 \epsilon_{ijk} \int dr (\delta_{kq}\delta_{mj} - \delta_{kj}\delta_{mq}) A_q E_m - \left(\delta_{kp}\delta_{mq} - \delta_{kq}\delta_{mp}\right) A_q r_j \partial_p E_m &(7)\\ &= \epsilon_0 \epsilon_{ijk} \int d^3r \ E_j A_k - \delta_{kj} E_q A_q + A_k r_j \partial_m E_m - A_m r_j \partial_k E_q &(8) \end{aligned}$$

By acting the Levi-Civita symbol on the Dirac delta δ_{jk} inside the integrand, and invoking Gauss's law for free source space, we obtain an expression,

$$L_i = \epsilon_0 \epsilon_{ijk} \int d^3r \ E_j A_k - A_m r_j \partial_k E_q \tag{9}$$

Obviously, the first term already gives us the cross product between $\mathbf{E}$ and $\mathbf{A}$. We deal with the second term by doing a second product rule to obtain the other term,

$$\begin{aligned} L_i &= \epsilon_0 \epsilon_{ijk} \int d^3r \ E_j A_k - \partial_k (r_j E_q A_m) + E_q A_m \partial_k r_j + r_j E_q \partial_k A_m &(10)\\ &= \epsilon_0 \int d^3r \mathbf{E} \times \mathbf{A} + \epsilon_0 \int d^3r E_q (\mathbf{r} \times \nabla) A_m &(11)\\ &= \mathbf{L}_{\text{spin}} + \mathbf{L}_{\text{orbital}} &(12) \end{aligned}$$

b. The decomposition is not gauge invariant, since a shift in the gauge, say $\mathbf{A} \longrightarrow \mathbf{A} + \nabla\phi$ yields,

$$\mathbf{L}'_{\text{spin}} = \epsilon_0 \int d^3r \mathbf{E} \times (\mathbf{A} + \nabla\phi) = \mathbf{L}_{\text{spin}} + \epsilon_0 \int d^3r \mathbf{E} \times (\nabla\phi) \neq \mathbf{L}_{\text{spin}} \tag{13}$$

c. In the Coulomb gauge, (16.17) relates the vector potential and electric field as

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \tag{14}$$

Hence, given that the formula we can obtain the vector potential by integration the electric field in time,

$$\begin{aligned} \mathbf{A}_\pm &= \int dt \mathbf{E}_\pm &(15)\\ &= \int dt \ E_0 \frac{\hat{x} \pm i\hat{y}}{\sqrt{2}} \exp[i(kz - \omega t)] &(16)\\ &= -\frac{i}{\omega} \mathbf{E}_\pm &(17) \end{aligned}$$

The time-averaged angular momentum is given by

$$< L_{\text{spin}} > = \frac{\epsilon_0}{2}\text{Re}\left[\int d^3r\mathbf{E}\times\left(-\frac{i}{\omega}\mathbf{E}_\pm\right)^*\right] \tag{18}$$

$$= \frac{\epsilon_0}{2}\text{Re}\left[\int d^3rE_0^2\exp[i(kz-\omega t)]\exp[-i(kz-\omega t)]\left(\frac{\hat{x}\pm i\hat{y}}{\sqrt{2}}\right)\times\left(\frac{\hat{x}\mp i\hat{y}}{\sqrt{2}}\right)\frac{i}{\omega}\right] \tag{19}$$

$$= \frac{\epsilon_0}{2}\text{Re}\left[\int d^3r\ E_0^2\frac{\pm i\hat{z}\pm i\hat{z}}{2\omega}\right] = \frac{\epsilon_0}{2\omega}(\pm\hat{z})E_0^2V \tag{20}$$

where V is some arbitrary volume. The time-averaged total energy is given by,

$$< U_{\text{EM}} > = \frac{\epsilon_0}{2}\int d^3r\ \mathbf{E}\cdot\mathbf{E}^* \tag{21}$$

$$= \frac{\epsilon_0}{2}\int d^3rE_0^2\exp[i(kz-\omega t)]\exp[-i(kz-\omega t)]\left(\frac{\hat{x}\pm i\hat{y}}{\sqrt{2}}\right)\cdot\left(\frac{\hat{x}\mp i\hat{y}}{\sqrt{2}}\right) \tag{22}$$

$$= \frac{\epsilon_0}{2}E_0^2V \tag{23}$$

where V is some arbitrary volume. The two quantities are related by,

$$< L_{\text{spin}} > \cdot\hat{z} = \pm\frac{1}{\omega} < U_{\text{EM}} > \tag{24}$$

Therefore, if $< U_{\text{EM}} >$, then $< L_{\text{spin}} > \cdot\hat{z} = \pm\hbar$ which corresponds to the spin of a photon.

1.2 Hermosa Part 2

Here, we define a Lorenz gauge vector potential of the form,

$$\mathbf{A}(x,y,z) = (\alpha\hat{x}+\beta\hat{y})u(x,y,z)\exp(ikz) \tag{25}$$

under paraxial approximation where x and y are the transverse coordinates and the wave is travelling in the z direction.
a. We redefine the Lorenz gauge to have a time-dependent piece, in order for the electric field to be non-zero under the Lorenz gauge condition. A sinusoidally varying time-dependent piece, $e^{-i\omega t}$ can be attached to the Lorenz condition, while keeping it valid,

$$\mathbf{A}(x,y,z;t) = (\alpha\hat{x}+\beta\hat{y})u(x,y)\exp(i(kz-\omega t)) \tag{26}$$

From this gauge vector potential, we can evaluate the scalar potential ϕ by using the Lorenz gauge condition,

$$\frac{\partial\varphi}{\partial t} = -c^2\nabla\cdot\mathbf{A} \tag{27}$$

$$= -c^2\left[\alpha\frac{\partial u}{\partial x}+\beta\frac{\partial u}{\partial y}\right]\exp(i(kz-\omega t)) \tag{28}$$

$$\longrightarrow\varphi(x,y,z,t) = \frac{ic^2}{\omega}\left[\alpha\frac{\partial u}{\partial x}+\beta\frac{\partial u}{\partial y}\right]\exp(i(kz-\omega t)) \tag{29}$$

We can calculate the electric field by substituting the expression we obtained for the scalar potential and the vector potential while doing paraxial approximation,

$$\mathbf{E} = -\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t} \tag{30}$$

$$= -\nabla\left[\frac{ic^2}{\omega}\left[\alpha\frac{\partial u}{\partial x}+\beta\frac{\partial u}{\partial y}\right]\exp(i(kz-\omega t))\right] + (i\omega)(\alpha\hat{x}+\beta\hat{y})u(x,y)\exp(i(kz-\omega t)) \tag{31}$$

$$= -\frac{ic^2}{\omega}\left[\alpha\nabla\left(\frac{\partial u}{\partial x}\right)\exp(i(kz-\omega t)) + \alpha\frac{\partial u}{\partial x}\nabla(\exp(i(kz-\omega t))) + \beta\nabla\left(\frac{\partial u}{\partial y}\right)\exp(i(kz-\omega t)) + \beta\frac{\partial u}{\partial y}\nabla(\exp(i(kz-\omega t)))\right]$$
$$+ (i\omega)(\alpha\hat{x}+\beta\hat{y})u(x,y)\exp(i(kz-\omega t)) \tag{32}$$

Since we want a physical solution, we would want $u(x,y)$ not to change very fast. Hence, second derivatives in x and y should vanish.

$$\mathbf{E} = -\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t} \tag{33}$$

$$= -\nabla\left[\frac{ic^2}{\omega}\left[\alpha\frac{\partial u}{\partial x}+\beta\frac{\partial u}{\partial y}\right]\exp(i(kz-\omega t))\right] + (i\omega)(\alpha\hat{x}+\beta\hat{y})u(x,y)\exp(i(kz-\omega t)) \tag{34}$$

$$= -\frac{ic^2}{\omega}\left[\alpha\frac{\partial u}{\partial x}\nabla(\exp(i(kz-\omega t))) + \beta\frac{\partial u}{\partial y}\nabla(\exp(i(kz-\omega t)))\right] + (i\omega)(\alpha\hat{x}+\beta\hat{y})u(x,y)\exp(i(kz-\omega t)) \tag{35}$$

$$\mathbf{E} = -\hat{z}\frac{ic^2}{\omega}(ik)\left[\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y}\right]\exp(i(kz-\omega t)) + (i\omega)(\alpha\hat{x}+\beta\hat{y})u(x,y)\exp(i(kz-\omega t)) \tag{36}$$

$$= \hat{z}\frac{c^2}{\omega}(k)\left[\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right]\hat{u}(x,y,z,t) + (i\omega)\hat{u}(x,y,z,t)(\alpha\hat{x}+\beta\hat{y}) \tag{37}$$

which is similar in form to,

$$\mathbf{E} = \frac{i}{k}\frac{\partial u}{\partial x}\hat{z} + u\hat{x} \tag{38}$$

hence we can use the formula for the magnetic field.

b. The magnetic field can be obtained by taking the curl of the vector potential,

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{39}$$

$$= -\frac{\partial A_y}{\partial z}\hat{x} + \frac{\partial A_x}{\partial z}\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z} \tag{40}$$

$$= -\left(\beta u(x,y)(ik)\exp(i(kz-\omega t))\right)\hat{x} + \left(\alpha u(x,y)(ik)\exp(i(kz-\omega t))\right)\hat{y} + \hat{z}\exp(i(kz-\omega t))\left(\beta\frac{\partial u}{\partial x} - \alpha\frac{\partial u}{\partial y}\right) \tag{41}$$

$$= ik(-\beta\hat{x}+\alpha\hat{y})\hat{u}(x,y,z,t) + \hat{z}\left(\beta\frac{\partial}{\partial x} - \alpha\frac{\partial}{\partial y}\right)\hat{u}(x,y,z,t) \tag{42}$$

2 Photonic Band Gap Material

a. We are required to derive the generalized wave equation satisfied by the electric field $\mathbf{E}(\mathbf{r},t)$ in non-magnetic matter when the permittivity is a function of position $\epsilon(\mathbf{r})$. Maxwell's equations in matter when there are no sources present, that is $\rho_f = \mathbf{j}_f = 0$, are

$$\nabla \cdot \mathbf{D} = 0 \qquad \nabla \cdot \mathbf{B} = 0 \tag{43}$$

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{44}$$

When $\epsilon = \epsilon(\mathbf{r})$, the divergence of $\mathbf{D}$ can be expressed as,

$$\nabla \cdot \mathbf{D} = \nabla \cdot [\epsilon(\mathbf{r})\mathbf{E}] \tag{45}$$

$$= \epsilon(\mathbf{r})\nabla \cdot \mathbf{E} + \mathbf{E}\cdot\nabla\epsilon(\mathbf{r}) \tag{46}$$

$$= 0. \tag{47}$$

which gives us the divergence of the electric field $\mathbf{E}$ as,

$$\nabla \cdot \mathbf{E} = -\frac{\mathbf{E}\cdot\nabla\epsilon(\mathbf{r})}{\epsilon(\mathbf{r})} \tag{48}$$

We set this aside and work with other Maxwell equations. The wave equation has a second derivative in space and time. Hence, it is useful for us to massage the expression involving the curl of $\mathbf{E}$. Taking the curl of both sides yields,

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) \tag{49}$$

$$= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\frac{\partial}{\partial t}\left(\mu_0\frac{\partial \mathbf{D}}{\partial t}\right) \tag{50}$$

$$\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\mu_0\epsilon(\mathbf{r})\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{51}$$

Substituting the divergence of the electric field that we have obtained earlier, we get a generalized wave equation of the following form,

$$\boxed{\nabla^2\mathbf{E} - \mu_0\epsilon(\mathbf{r})\frac{\partial^2\mathbf{E}}{\partial t^2} = -\nabla\left(\frac{\mathbf{E}\cdot\nabla\epsilon(\mathbf{r})}{\epsilon(\mathbf{r})}\right)} \tag{52}$$

When the permittivity is only a function of z, its gradient will only have a value along the $\hat{z}$ direction. Since our electric field changes in z and points in the $\hat{x}$ direction, the dot product between $\mathbf{E}$ and $\nabla\epsilon(\mathbf{r})$ is equal to zero. Hence, the equation reduces to,

$$\frac{\partial^2\mathbf{E}}{\partial z^2} - \mu_0\epsilon(z)\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \tag{53}$$

where $\mathbf{E}(z,t) = E(z,t)\hat{x}$.

b. We have to show that the Fourier components $\hat{\mathbf{E}}(k)$ of the electric field,

$$E(z,t) = E(z)\exp(-i\omega t) \tag{54}$$

while letting $\epsilon(z) = \epsilon_0[1+\alpha\cos(2k_0 z)]$, satisfy the coupled linear equations

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\hat{E}(k) = \frac{\omega^2\alpha}{2c^2}\{\hat{E}(k-2k_0)+\hat{E}(k+2k_0)\}. \tag{55}$$

We shall use our earlier result for when the permittivity and the electric field are z-dependent. Doing so yields,

$$\begin{aligned}
\frac{\mathrm{d}^2E(z)}{\mathrm{d}z^2}\exp(-i\omega t) - \mu_0\epsilon(z)E(z)\frac{\mathrm{d}^2(\exp(-i\omega t))}{\mathrm{d}t^2} &= \exp(-i\omega t)\frac{\mathrm{d}^2E(z)}{\mathrm{d}z^2} - \mu_0\epsilon(z)E(z)(-i\omega)^2\exp(-i\omega t) && (56)\\
&= \left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \mu_0\epsilon(z)\omega^2\right]E(z,t) && (57)\\
&= 0 && (58)
\end{aligned}$$

which reads as,

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \mu_0\epsilon(z)\omega^2\right]E(z,t) = 0 \tag{59}$$

We shall look at the k-space counterpart of this equation by expressing $E(z)$ as a Fourier pair of $\hat{E}$,

$$E(z) = \int_{-\infty}^{\infty} dk\hat{E}(k)\cos(kz) \tag{60}$$

and substituting it back, which yields

$$\begin{aligned}
\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \mu_0\epsilon(z)\omega^2\right]\int_{-\infty}^{\infty} dk\hat{E}(k)\cos(kz) &= \int_{-\infty}^{\infty} dk\ \left\{-k^2\hat{E}(k)\cos(kz) + \mu_0\epsilon_0\omega^2[\cos(kz)+\alpha\cos(2k_0z)\cos(kz)]\hat{E}(k)\right\} && (61)\\
&= \int_{-\infty}^{\infty} dk\ \hat{E}(k)\cos(kz)\left\{-k^2 + \frac{\omega^2}{c^2}[1+\alpha\cos(2k_0z)]\right\} && (62)\\
&= \int_{-\infty}^{\infty} dk\ \hat{E}(k)\cos(kz)\left(\frac{\omega^2}{c^2} - k^2\right) + \int_{-\infty}^{\infty} dk\ \frac{\omega^2\alpha}{c^2}\hat{E}(k)\cos(kz)\cos(2k_0z) && (63)\\
&= 0, && (64)
\end{aligned}$$

We can rearrange the terms in order to find the relationship between the integrands,

$$\begin{aligned}
\int_{-\infty}^{\infty} dk\ \hat{E}(k)\cos(kz)\left(k^2 - \frac{\omega^2}{c^2}\right) &= \int_{-\infty}^{\infty} dk\ \frac{\omega^2\alpha}{c^2}\hat{E}(k)\cos(kz)\cos(2k_0z) && (65)\\
\text{Since}\cos(\alpha-\beta)+\cos(\alpha+\beta) = 2\cos(\alpha)\cos(\beta) \qquad &= \int_{-\infty}^{\infty} dk\ \frac{\omega^2\alpha}{2c^2}\hat{E}(k)[\cos((k-2k_0)z)+\cos((k+2k_0)z)] && (66)
\end{aligned}$$

By performing a change of variables in the integration of the right hand side, that is

$$\begin{aligned}
\int_{-\infty}^{\infty} dk\ \frac{\omega^2\alpha}{2c^2}\hat{E}(k)\cos((k-2k_0)z) + \int_{-\infty}^{\infty} dk\ \frac{\omega^2\alpha}{2c^2}\hat{E}(k)\cos((k+2k_0)z) &= \int_{-\infty}^{\infty} dk'\ \frac{\omega^2\alpha}{2c^2}\hat{E}(k+2k_0)\cos(k'z)\\
&\quad + \int_{-\infty}^{\infty} dk'\ \frac{\omega^2\alpha}{2c^2}\hat{E}(k-2k_0)\cos(k'z) && (67)\\
&= \int_{-\infty}^{\infty} dk'\ \frac{\omega^2\alpha}{2c^2}\{\hat{E}(k-2k_0)+\hat{E}(k+2k_0)\}\cos(k'z) && (68)
\end{aligned}$$

Hence,

$$\int_{-\infty}^{\infty} dk\ \hat{E}(k)\cos(kz)\left(k^2 - \frac{\omega^2}{c^2}\right) = \int_{-\infty}^{\infty} dk'\ \frac{\omega^2\alpha}{2c^2}\{\hat{E}(k-2k_0)+\hat{E}(k+2k_0)\}\cos(k'z) \tag{69}$$

which eventually leads us to,

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\hat{E}(k) = \frac{\omega^2\alpha}{2c^2}\{\hat{E}(k-2k_0)+\hat{E}(k+2k_0)\}. \qquad \text{qed} \tag{70}$$

c. We suppose that the factor α is very small and focus on values of k that is within the neighborhood of k_0, like $k = k_0 + q$, where $|q| << k_0$. Doing so will give us,

$$\begin{aligned}
\left[(k_0+q)^2 - \frac{\omega^2}{c^2}\right]\hat{E}(k_0+q) &= \frac{\omega^2\alpha}{2c^2}\{\hat{E}(k_0+q-2k_0)+\hat{E}(k_0+q+2k_0)\} && (71)\\
&= \frac{\omega^2\alpha}{2c^2}\{\hat{E}(q-k_0)+\hat{E}(q+3k_0)\} && (72)
\end{aligned}$$

Since $\alpha << 1$, it could be the case that either $\left[(k_0+q)^2 - \frac{\omega^2}{c^2}\right]$ or $\hat{E}$ is small. We can test this by testing values in the neighborhood of k. In the approximation that α is very small,

$$\left[k_0^2(1+\frac{2q}{k_0}) - \frac{\omega^2}{c^2}\right]\hat{E}(k_0+q) - \mathcal{O}(\alpha) = 0 \tag{73}$$

Since it is trivial to consider that the Fourier component is equal to zero, we obtain the relation

$$\omega^2 \approx c^2(k_0^2 + 2qk_0) \tag{74}$$

which when taken the square root of becomes,

$$\omega \approx c(k_0^2 + 2qk_0)^{1/2} \tag{75}$$
$$\approx ck_0(1+\frac{2q}{k_0})^{1/2} \tag{76}$$
$$\approx ck_0(1+\frac{q}{k_0}) = c(k_0+q) \tag{77}$$

Doing the same procedure for $k = q - k_0$ gives us,

$$\omega \approx c(k_0^2 - 2qk_0)^{1/2} \tag{78}$$
$$\approx ck_0(1-\frac{2q}{k_0})^{1/2} \tag{79}$$
$$\approx ck_0(1-\frac{q}{k_0}) = c(k_0-q) \tag{80}$$

Doing the same procedure for $k = q + 3k_0$ gives us,

$$\left[9k_0^2(1+\frac{2q}{3k_0}) - \frac{\omega^2}{c^2}\right]\hat{E}(3k_0+q) - \mathcal{O}(\alpha) = 0 \tag{81}$$

$$\omega^2 = 9k_0^2c^2(1+\frac{2q}{3k_0}) \tag{82}$$

which when taken the square root of becomes,

$$\omega = 3k_0c(1+\frac{2q}{3k_0})^{1/2} \tag{83}$$
$$\approx 3k_0c(1+\frac{q}{3k_0}) = c(3k_0+q) \tag{84}$$

which leads us to believe that the terms in the square bracket is not small compared to the Fourier component when $k = 3k_0 + q$. Since the wave frequency cannot differ greatly from the $\alpha = 0$ case (according to the hint), we must drop other Fourier components, that is they are very small compared to the $k = q \pm k_0$ case. Hence, we are just left with two Fourier components, $\hat{E}(q \pm k_0)$, or

$$\left[(k_0+q)^2 - \frac{\omega^2}{c^2}\right]\hat{E}(k_0+q) = \frac{\omega^2\alpha}{2c^2}\hat{E}(q-k_0) \tag{85}$$
$$\left[(q-k_0)^2 - \frac{\omega^2}{c^2}\right]\hat{E}(k_0-q) = \frac{\omega^2\alpha}{2c^2}\hat{E}(q+k_0) \tag{86}$$

which can be rewritten as a 2×2 matrix equation,

$$\begin{bmatrix} \left[(k_0+q)^2 - \frac{\omega^2}{c^2}\right] & -\frac{\omega^2\alpha}{2c^2} \\ -\frac{\omega^2\alpha}{2c^2} & \left[(q-k_0)^2 - \frac{\omega^2}{c^2}\right] \end{bmatrix} \begin{bmatrix} \hat{E}(k_0+q) \\ \hat{E}(k_0-q) \end{bmatrix} = 0 \tag{87}$$
$$\mathbf{M}\begin{bmatrix} \hat{E}(k_0+q) \\ \hat{E}(k_0-q) \end{bmatrix} = 0 \tag{88}$$

d. We can solve this eigenvalue problem by taking the determinant of the matrix $\mathbf{M}$ to be equal to zero, that is

$$\left[(k_0+q)^2 - \frac{\omega^2}{c^2}\right]\left[(q-k_0)^2 - \frac{\omega^2}{c^2}\right] - \frac{\omega^4\alpha^2}{4c^4} = (q^2-k_0^2)^2 - \frac{\omega^2}{c^2}\left[(q-k_0)^2 + (q+k_0)^2\right] + \frac{\omega^4}{c^4}\left(1-\frac{\alpha^2}{4}\right) \tag{89}$$
$$= 0 \tag{90}$$

We can write this in a prettier form, that is

$$\left(1-\frac{\alpha^2}{4}\right)x^2 + [(q-k_0)^2 + (q+k_0)^2]x - (q^2-k_0^2)^2 = 0. \tag{91}$$

where $x = \frac{\omega^2}{c^2}$. Since it doesn't seem like it can be factored out, we use the quadratic equation to solve for its roots. Doing so yields,

$$x_\pm = \frac{-[(q-k_0)^2 + (q+k_0)^2] \pm \sqrt{[(q-k_0)^2 + (q+k_0)^2]^2 - 4\left(1 - \frac{\alpha^2}{4}\right)(q^2 - k_0^2)^2}}{2\left(1 - \frac{\alpha^2}{4}\right)} \tag{92}$$

When $q = 0$,

$$x_\pm = \frac{-2k_0^2 \pm \sqrt{4k_0^4 - 4\left(1 - \frac{\alpha^2}{4}\right)k_0^4}}{2\left(1 - \frac{\alpha^2}{4}\right)} = \frac{-2k_0^2 \pm k_0^2\alpha}{2 - \frac{\alpha^2}{2}} \tag{93}$$

$$= k_0^2 \frac{2 \mp \alpha}{2 - \frac{\alpha^2}{2}} \tag{94}$$

which yields,

$$\omega_\pm^2 = -c^2 k_0^2 \frac{2 \mp \alpha}{2 - \frac{\alpha^2}{2}} \longrightarrow \omega_\pm = \mp ck_0 \sqrt{\frac{1 \mp \frac{\alpha}{2}}{1 - \frac{\alpha^2}{4}}} \tag{95}$$

When $\alpha << 1$,

$$\omega_\pm \approx \mp ck_0 \sqrt{1 \mp \frac{\alpha}{2}} \approx \mp ck_0 \left(1 \mp \frac{\alpha}{4}\right) = ck_0 \left(\frac{\alpha}{4} \mp 1\right) \tag{96}$$

3 A Magnetic Lorentz Model

We consider a lattice of filamentary wires that carries current in the opposite direction from the rows above and below it. Each row is displaced by a distance $a/2$ from the rows just above and below it. Each wire feels a restoring force $-ku$ if it moves a distance from its equilibrium position. Each wire also feels a damping force $-m\gamma\dot{u}$ when it is in motion.

a. When the wires are arranged in a square lattice, the symmetry implies a zero magnetic force for each wire. To prove this, we can consider one 3×3 square lattice and find that the wire at the center experiences zero magnetic force. When we move a row by a small distance u, we end up with a change in force, δF proportional to u, $\delta F = I\ell B(u)$. Hence, if the displacement u is small, it is as if there is no net force experienced by the wires in the material.

b. We can obtain a solution for $u_x(t)$ by setting up Newton's second law, noting that we are dealing with a force per unit length. Doing so yields,

$$m\ddot{u}_x = -ku_x - m\gamma\dot{u}_x - IB\exp(-i\omega t) \tag{97}$$

Since the right hand side of the equation involves an exponential function in time, it is worth considering an ansatz for the solution $u_x(t) = A\exp(-i\omega t)$. Substituting this ansatz back to the equation, we obtain

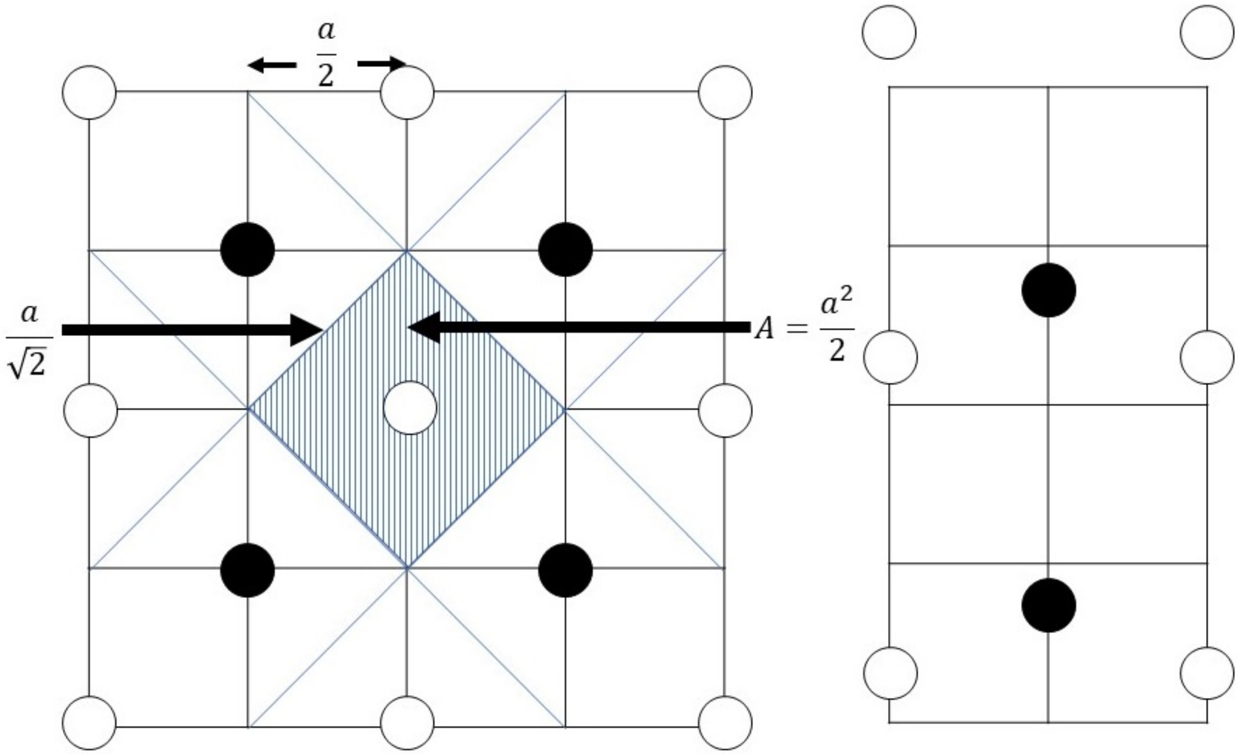
$$\omega^2 mA\exp(-i\omega t) = -kA\exp(-i\omega t) + im\gamma\omega\exp(-i\omega t) - IB\exp(-i\omega t) \tag{98}$$

$$\omega^2 mA = -kA + im\gamma\omega A - IB \longrightarrow A = \frac{IB}{\omega^2 m - k + im\gamma\omega} \tag{99}$$

Thus, the solution for $u_x(t)$ is given by,

$$\boxed{u_x(t) = \frac{IB}{\omega^2 m - k + im\gamma\omega}\exp(-i\omega t)} \tag{100}$$

c. The displacements expose current sheets at the top and bottom faces. To find the total field inside the sample, we must look at the current density per cross-section of the sample and use this to find the magnetic field inside the sample. Since the currents are located within alternating lattice sites, we must find a unit cell that completely encloses one filamentary current in the cross-section.



Shown in the figure is a three sheet diagram in which wires are in alternating position. If we draw imaginary lines connecting each corner, we obtain unit cells completely enclosing each current element. The current density of the top sheet that was exposed after immersing the material to an external magnetic field is given by,

$$K = u_x \left(\frac{I}{A}\right) = u_x \left(\frac{2I}{a^2}\right) \tag{101}$$

while the exposed sheet at the bottom is just $-K$ since it goes in the opposite direction. The magnetic field in between the two sheets is just given by Ampere's law, which leads to

$$\mathbf{B}_{\text{induced}} = 2\left(-\mu_0 \frac{\mathbf{K}}{2}\right) = -\mu_0 u_x \left(\frac{2I}{a^2}\right)\hat{y} \tag{102}$$

$$= -\mu_0 \left(\frac{2I}{a^2}\right) \frac{IB}{\omega^2 m - k + im\gamma\omega} \exp(-i\omega t))\hat{y} \tag{103}$$

$$= -\mu_0 \left(\frac{2I^2}{ma^2}\right) \frac{B}{\omega^2 - \frac{k}{m} + i\gamma\omega} \exp(-i\omega t)\hat{y} \tag{104}$$

$$= -\frac{\Omega_P^2}{\omega^2 - \omega_0^2 + i\gamma\omega}\mathbf{B} \tag{105}$$

and eventually proves that the total field inside satisfies the self-consistency equation,

$$\mathbf{B} = \mathbf{B}_{\text{ext}} + \mathbf{B}_{\text{induced}} \tag{106}$$

$$= \mathbf{B}_{\text{ext}} - \frac{\Omega_P^2}{\omega^2 - \omega_0^2 + i\gamma\omega}\mathbf{B} \tag{107}$$

d. Our boundary condition for the auxilliary field tells us that the tangential component must be continuous at the boundary. Hence we know that,

$$\mu_0 \mathbf{B} = \mu(\omega)\mathbf{B}_{\text{ext}} \tag{108}$$

Substituting the expression that we have obtained earlier for $\mathbf{B}$, we obtain an expression for the magnetic permeability $\mu(\omega)$.

$$\mu(\omega) = \mu_o \frac{\mathbf{B}}{\mathbf{B}_{\text{ext}}} \tag{109}$$

$$= \mu_0 \frac{\mathbf{B}_{\text{ext}} - \frac{\Omega_P^2}{\omega^2 - \omega_0^2 + i\gamma\omega}\mathbf{B}}{\mathbf{B}_{\text{ext}}} \tag{110}$$

$$= \mu_0 \left[1 - \frac{\Omega_P^2}{\omega^2 - \omega_0^2 + i\gamma\omega} \frac{\mu(\omega)}{\mu_0}\right] \tag{111}$$

We can massage this expression a little to isolate our expression for $\mu(\omega)$,

$$\mu_0 - \mu(\omega)\frac{\Omega_P^2}{\omega^2-\omega_0^2+i\gamma\omega} = \mu(\omega) \longrightarrow \mu_0 = \mu(\omega)\left[1+\frac{\Omega_P^2}{\omega^2-\omega_0^2+i\gamma\omega}\right] \tag{112}$$

which gives us,

$$\boxed{\mu(\omega) = \frac{\mu_0}{1+\frac{\Omega_P^2}{\omega^2-\omega_0^2+i\gamma\omega}}} \tag{113}$$

4 Goos-Hänchen with lossy materials

a. The Fresnel coefficients for s-polarized and p-polarized light are given as,

$$r_{\mathrm{s}} = \frac{Z_2\cos\theta_{\mathrm{i}} - Z_1\cos\theta_{\mathrm{t}}}{Z_2\cos\theta_{\mathrm{i}} + Z_1\cos\theta_{\mathrm{t}}} \tag{114}$$

$$r_{\mathrm{p}} = \frac{Z_2\cos\theta_{\mathrm{t}} - Z_1\cos\theta_{\mathrm{i}}}{Z_2\cos\theta_{\mathrm{t}} + Z_1\cos\theta_{\mathrm{i}}} \tag{115}$$

where Z_1 and Z_2 are impedances of materials 1 and 2.

$$r_{\mathrm{s}} = \frac{n_1\cos\theta_{\mathrm{i}} - n_2\cos\theta_{\mathrm{t}}}{n_1\cos\theta_{\mathrm{i}} + n_2\cos\theta_{\mathrm{t}}} = \frac{n_1\cos\theta_{\mathrm{i}} - n_2\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_{\mathrm{i}}\right)^2}}{n_1\cos\theta_{\mathrm{i}} + n_2\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_{\mathrm{i}}\right)^2}} \tag{116}$$

$$= \frac{\cos\theta_{\mathrm{i}} - \sqrt{n_2^2-\sin^2\theta_{\mathrm{i}}}}{\cos\theta_{\mathrm{i}} + \sqrt{n_2^2-\sin^2\theta_{\mathrm{i}}}} = \frac{\cos\theta_{\mathrm{i}} - \sqrt{\epsilon-\sin^2\theta_{\mathrm{i}}}}{\cos\theta_{\mathrm{i}} + \sqrt{\epsilon-\sin^2\theta_{\mathrm{i}}}} \tag{117}$$

$$r_{\mathrm{p}} = \frac{n_1\cos\theta_t - n_2\cos\theta_{\mathrm{i}}}{n_1\cos\theta_{\mathrm{t}} + n_2\cos\theta_{\mathrm{i}}} = \frac{n_1\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_{\mathrm{i}}\right)^2} - n_2\cos\theta_{\mathrm{i}}}{n_1\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_{\mathrm{i}}\right)^2} + n_2\cos\theta_{\mathrm{i}}} \tag{118}$$

$$= \frac{\frac{1}{n_2}\sqrt{n_2^2-\sin^2\theta_{\mathrm{i}}} - n_2\cos\theta_{\mathrm{i}}}{\frac{1}{n_2}\sqrt{n_2^2-\sin^2\theta_{\mathrm{i}}} + n_2\cos\theta_{\mathrm{i}}} = \frac{\sqrt{\epsilon-\sin^2\theta_{\mathrm{i}}} - \epsilon\cos\theta_{\mathrm{i}}}{\sqrt{\epsilon-\sin^2\theta_{\mathrm{i}}} + \epsilon\cos\theta_{\mathrm{i}}} \tag{119}$$

b. Artmann's formula is given by

$$D_\mu = -\frac{\lambda}{2\pi}\frac{\partial\varphi_\mu}{\partial\theta} \tag{120}$$

where ϕ_μ is the phase if the complex Fresnel coefficient is written in the Euler form, $r_\mu = R_\mu\exp(i\varphi_\mu)$. Let us rewrite the expression for r_μ and isolate the phase term.

$$\ln\frac{r_\mu}{R_\mu} = i\varphi_\mu \tag{121}$$

and take its derivative with respect to θ,

$$\frac{R_\mu}{r_\mu}\left(\frac{R_\mu r'_\mu - R'_\mu r_\mu}{R_\mu^2}\right) = i\frac{\partial\varphi_\mu}{\partial\theta} \longrightarrow \frac{r'_\mu}{r_\mu} - \frac{R'_\mu}{R_\mu} = i\frac{\partial\varphi_\mu}{\partial\theta} \longrightarrow \frac{r'_\mu}{r_\mu} = \frac{R'_\mu}{R_\mu} + i\frac{\partial\varphi_\mu}{\partial\theta} \tag{122}$$

Hence we can write Artmann's formula as,

$$D_\mu = -\frac{\lambda}{2\pi}\mathrm{Im}\frac{r'_\mu}{r_\mu} \tag{123}$$

c.Using the elegant form of the Artmann formula that was derived previously, we can obtain the Goos-Hänchen shift.

$$\frac{\partial r_s}{\partial \theta} = \frac{\left(-\sin\theta_i + \frac{(\sin\theta_i\cos\theta_i)}{\sqrt{\epsilon-\sin^2\theta_i}}\right)(\cos\theta_\mathrm{i} + \sqrt{\epsilon-\sin^2\theta_\mathrm{i}}) - (\cos\theta_i - \sqrt{\epsilon-\sin^2\theta_i})\left(-\sin\theta_i - \frac{(\sin\theta_i\cos\theta_i)}{\sqrt{\epsilon-\sin^2\theta_i}}\right)}{(\cos\theta_i + \sqrt{\epsilon-\sin^2\theta_i})^2} \tag{124}$$

$$= \frac{-\sin\theta_i\cos\theta_i + \sin\theta_i\cos\theta_i + 2\frac{\sin\theta_i\cos^2\theta_i}{\sqrt{\epsilon-\sin^2\theta_i}} - 2\sin\theta_i\sqrt{\epsilon-\sin^2\theta_i} - \sin\theta_i\cos\theta_i + \sin\theta_i\cos\theta_i}{(\cos\theta_i + \sqrt{\epsilon-\sin^2\theta_i})^2} \tag{125}$$

$$= \frac{2\left(\frac{\sin\theta_i\cos^2\theta_i}{\sqrt{\epsilon-\sin^2\theta_i}} - \sin\theta_i\sqrt{\epsilon-\sin^2\theta_i}\right)}{(\cos\theta_i + \sqrt{\epsilon-\sin^2\theta_i})^2} \tag{126}$$

$$= \frac{2\sin\theta_i(\cos^2\theta_i - \epsilon + \sin^2\theta_i)}{\sqrt{\epsilon-\sin^2\theta_i}(\cos\theta_i + \sqrt{\epsilon-\sin^2\theta_i})^2} \tag{127}$$

$$\frac{r_s'}{r_s} = \frac{2\sin\theta_i(1-\epsilon)}{\sqrt{\epsilon-\sin^2\theta_i}(\cos\theta_i + \sqrt{\epsilon-\sin^2\theta_i})^2}\frac{\cos\theta_\mathrm{i} + \sqrt{\epsilon-\sin^2\theta_\mathrm{i}}}{\cos\theta_\mathrm{i} - \sqrt{\epsilon-\sin^2\theta_\mathrm{i}}} \tag{128}$$

$$= \frac{2\sin\theta_i(1-\epsilon)}{\sqrt{\epsilon-\sin^2\theta_i}(\cos^2\theta_i - \epsilon + \sin^2\theta_i)} = \frac{2\sin\theta_i}{\sqrt{\epsilon-\sin^2\theta_i}} \tag{129}$$

$$= \frac{2\sin\theta_i\sqrt{\epsilon-\sin^2\theta_i}}{\epsilon-\sin^2\theta_i} = \frac{2\sin\theta_i\sqrt{\epsilon_r + i\epsilon_i - \sin^2\theta_i}}{\epsilon_r + i\epsilon_i - \sin^2\theta_i} \tag{130}$$

$$= \frac{2\sin\theta_i\sqrt{\epsilon_r + i\epsilon_i - \sin^2\theta_i}[\epsilon_r - \sin^2\theta_i - i\epsilon_i]}{(\epsilon_r - \sin^2\theta_i)^2 + \epsilon_i^2} \tag{131}$$

The Goos-Hänchen shift for s-polarized waves is given by,

$$D_s = -\frac{\lambda}{2\pi}\mathrm{Im}\frac{r_s'}{r_s} \tag{132}$$

Using the identity,

$$\sqrt{x+iy} = \frac{1}{2}\sqrt{2}\left[\sqrt{\sqrt{x^2+y^2}+x} + i\,\mathrm{sgn}(y)\sqrt{\sqrt{x^2+y^2}-x}\right] \tag{133}$$

we find that

$$D_S = \frac{\lambda}{\sqrt{2}\pi}\frac{\sin\theta\varepsilon_i}{\left|\varepsilon-\sin^2\theta\right|\sqrt{\left|\varepsilon-\sin^2\theta\right| + \varepsilon_r - \sin^2\theta}} \tag{134}$$

whose form we obtained from a paper by Götte, Aiello, and Woerdman.

We do the same procedure for p-polarized light, and obtain

$$\frac{r_\mathrm{p}'}{r_p} = \frac{1}{r_p}\frac{\left(-\frac{(\sin\theta_i\cos\theta_i)}{\sqrt{\epsilon-\sin^2\theta_i}} + \epsilon\sin\theta_i\right)(\sqrt{\epsilon-\sin^2\theta_\mathrm{i}} + \epsilon\cos\theta_\mathrm{i}) + (\sqrt{\epsilon-\sin^2\theta_\mathrm{i}} - \epsilon\cos\theta_\mathrm{i})\left(\frac{(\sin\theta_i\cos\theta_i)}{\sqrt{\epsilon-\sin^2\theta_i}} + \epsilon\sin\theta_i\right)}{(\sqrt{\epsilon-\sin^2\theta_\mathrm{i}} + \epsilon\cos\theta_\mathrm{i})^2} \tag{135}$$

$$= \frac{1}{r_p}\frac{-\sin\theta_i\cos\theta_i + \sin\theta_i\cos\theta_i + \epsilon^2\sin\theta_i\cos\theta_i - \epsilon^2\sin\theta_i\cos\theta_i - 2\epsilon\frac{\sin\theta_i\cos^2\theta_i}{\sqrt{\epsilon-\sin^2\theta_i}} + 2\epsilon\sin\theta_i\sqrt{\epsilon-\sin^2\theta_i}}{(\sqrt{\epsilon-\sin^2\theta_\mathrm{i}} + \epsilon\cos\theta_\mathrm{i})^2} \tag{136}$$

$$= \frac{2\epsilon}{r_p}\frac{\sin\theta_i\sqrt{\epsilon-\sin^2\theta_i} - \frac{\sin\theta_i\cos^2\theta_i}{\sqrt{\epsilon-\sin^2\theta_i}}}{(\sqrt{\epsilon-\sin^2\theta_\mathrm{i}} + \epsilon\cos\theta_\mathrm{i})^2} = \frac{2\epsilon}{r_p}\frac{\sin\theta_i(\epsilon-1)}{\sqrt{\epsilon-\sin^2\theta_i}(\sqrt{\epsilon-\sin^2\theta_\mathrm{i}} + \epsilon\cos\theta_\mathrm{i})^2} \tag{137}$$

$$= 2\epsilon(\epsilon-1)\sin\theta_i\frac{\sqrt{\epsilon-\sin^2\theta_\mathrm{i}} + \epsilon\cos\theta_\mathrm{i}}{\sqrt{\epsilon-\sin^2\theta_\mathrm{i}} - \epsilon\cos\theta_\mathrm{i}}\frac{1}{\sqrt{\epsilon-\sin^2\theta_i}(\sqrt{\epsilon-\sin^2\theta_\mathrm{i}} + \epsilon\cos\theta_\mathrm{i})^2} \tag{138}$$

$$= 2\epsilon(\epsilon-1)\frac{\sin\theta_i}{\sqrt{\epsilon-\sin^2\theta_i}(\epsilon-\sin^2\theta - \epsilon^2\cos^2\theta_i)} \tag{139}$$

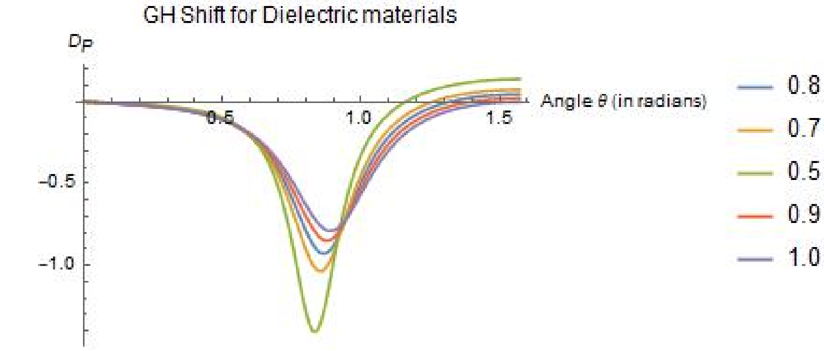
The Goos-Hänchen shift for p-polarized waves is given by,

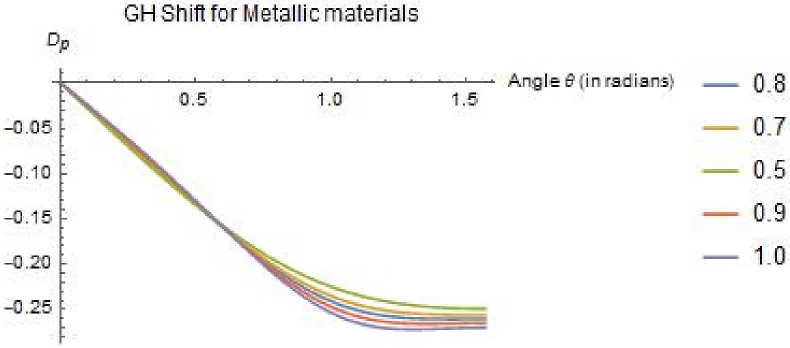
$$D_p = -\frac{\lambda}{2\pi}\mathrm{Im}\frac{r_p'}{r_p} \tag{140}$$

I tried calculating for the imaginary part of r_p'/r_p but I failed in doing so. Therefore, I will refer to the result of Götte, Aiello, and Woerdman, that is,

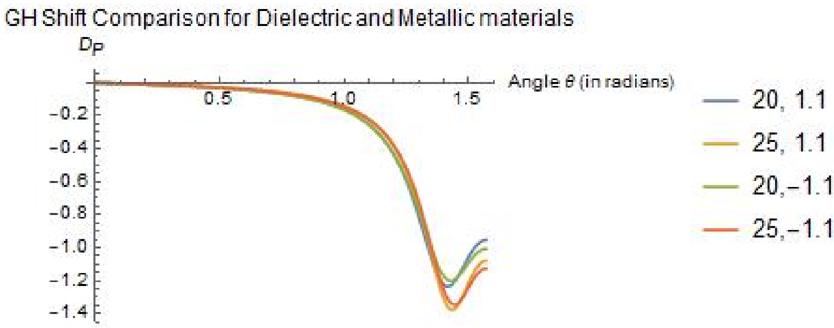
$$D_p = -\frac{\lambda}{\sqrt{2}\pi}\sin(\theta_i)\varepsilon_i\frac{|\varepsilon|^2\cos^2(\theta_i) + \sin^2(\theta_i)\left(\left|\varepsilon-\sin^2(\theta_i)\right| - \sin^2(\theta_i)\right)}{\left|\sin^2(\theta_i) - \varepsilon\cos^2(\theta_i)\right|^2\left|\varepsilon-\sin^2(\theta_i)\right|\sqrt{\left|\varepsilon-\sin^2(\theta_i)\right| + \varepsilon_r - \sin^2(\theta_i)}} \tag{141}$$

c. and d.





I plotted the expressions from the previous items for the GH shift that is scaled by the wavelength λ. The plot legends represent the ϵ_i used for the plots. For the plot for low-loss materials (dielectrics), I used $\epsilon_r = 1.1$ and for high-loss materials, $\epsilon_r = -1.1$. We can see that the maximum shift is not located for when the angle of incidence is equal to 90, rather it is shifted to around 0.8 radians at around 45 degrees. We can see as well that when we compare the plots for the dielectric and metals, the maximum shift for dielectrics approach that for metals as ϵ_i increases. What we can do is compare it at values when $\epsilon_i > \epsilon_r$ for dielectrics.



We can see that when the ϵ_i is sufficiently larger than ϵ_r, the GH shift is the same for dielectric and metallic materials.