Physics 211 Second HW Problem

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Distribution Theory Lecture 3

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Theory: Distributions

We let $f(t)$ be locally integrable in $\mathbb{R}$. We define the functional,

$$\begin{aligned}< f, \phi > &=< f(t), \phi(t) > \\ &= \int_{-\infty}^{\infty} f(t)\phi(t)dt\end{aligned}$$

for all $\phi(t)$ in $\mathcal{D}$. Since the integral exists for all $f(t)$ in $\mathcal{D}$, the functional is defined all over $\mathcal{D}$. $f(t)$ defines a distribution over $\mathcal{D}$ such that

- $f(t)$ defines a functional over $\mathcal{D}$.
- The functional is linear; that is, for every $\phi_1(t)$ and $\phi_2(t)$ in $\mathcal{D}$ and some complex number λ,

$$\begin{aligned}< f, \phi_1 + \phi_2 > &= \int_{-\infty}^{\infty} f(t)(\phi_1 + \phi_2)(t)dt \\ &= \int_{-\infty}^{\infty} f(t)[\phi_1(t) + \phi_2(t)]dt \\ &= \int_{-\infty}^{\infty} f(t)\phi_1(t)d + \int_{-\infty}^{\infty} f(t)\phi_2(t)dt \\ &=< f, \phi_1 > + < f, \phi_2 > \\ < f, \lambda\phi_1 > &= \int_{-\infty}^{\infty} f(\lambda\phi_1)(t)dt \\ &= \lambda \int_{-\infty}^{\infty} f(t)\phi_1(t)dt \\ &= \lambda < f, \phi_1 >\end{aligned}$$

- The functional is continuous; that is, if we let $\{\phi_\nu(t)\}$ be a sequence of test functions in $\mathcal{D}$ converging to $\phi(t)$ in $\mathcal{D}$. That $\phi_\nu(t)$ converges to $\phi(t)$ as $\nu \to \infty$ implies that for every $\varepsilon > 0$,

$$|\phi_\nu(t) - \phi(t)| < \varepsilon$$

for some $\nu \geq N_0$. For the functional to be continuous, it must be that the sequence of numbers $\{< f, \phi_\nu >\}_{\nu=1}^{\infty}$ converges to the number $< f, \phi >$. A sequence of numbers $\{a_k\}_{k=1}^{\infty}$ converges to a for every $\varepsilon > 0$, there exists some integer K_0 such that

$$|a_k - a| < \varepsilon$$

for all $k \geq K_0$. Hence, for a sequence $\{< f, \phi_\nu >\}_{=1}^{\infty}$ that converges to $< f, \phi >$ for every $\epsilon > 0$, there exists some integer N_0 such that

$$| < f, \phi_\nu > - < f, \phi > | < \epsilon$$

for all $\nu \geq N_0$.

$$\begin{aligned}
| < f, \phi_\nu > - < f, \phi > | &= \left| \int_{-\infty}^{\infty} f(t)\phi_\nu(t)dt - \int_{-\infty}^{\infty} f(t)\phi(t)dt \right| \\
&= \left| \int_{\Delta} f(t)[\phi_\nu(t) - \phi(t)]dt \right| \\
&\leq \int_{\Delta} |f(t)| \, |\phi_\nu(t) - \phi(t)| \, dt \\
&\leq \varepsilon \int_{\Delta} |f(t)| dt, && (\nu \geq N_0) \\
&= A\varepsilon, && (\nu \geq N_0)
\end{aligned}$$

We assume there is some $\epsilon = A\varepsilon$, such that

$$| < f, \phi_\nu > - < f, \phi > | < \epsilon$$

which completes our definition.

Problem 1

Define

$$t_+^{-n} = \begin{cases} 0, & \text{for } t < 0 \\ \dfrac{1}{t^n}, & \text{for } t > 0 \end{cases}$$

as a distribution, where $n = 1, 2, 3, ...$

Proof. We first assume that t_+^{-n} defines a regular distribution, such that the functional,

$$< t_+^{-n}, \phi > = \int_{-\infty}^{\infty} t_+^{-n}(t')\phi(t')dt' \tag{1}$$

$$= \int_0^{\infty} \frac{1}{t'^n}\phi(t')dt' \tag{2}$$

must exist for all $\phi(t)$ in $\mathcal{D}$. If it happens that $\phi(0) \neq 0$, the integral does not exist; hence, it cannot define a regular distribution in $\mathcal{D}$.

Let $\epsilon > 0$, and note that $\phi(t)$ is nonzero for some interval $|t'| < a$, then consider the integral,

$$\int_\epsilon^a \frac{1}{t'^n}\phi(t')dt' \tag{3}$$

which we can integrate via IBP. Doing the procedure, we obtain

$$\int_\epsilon^a \frac{1}{t'^n}\phi(t')dt' = \frac{1}{n-1}\frac{1}{t'^{n-1}}\phi(t')\Bigg|_\epsilon^a - \int_\epsilon^a \frac{1}{n-1}\frac{1}{t'^{n-1}}\phi'(t')dt' \tag{4}$$

$$= -\frac{1}{n-1}\left[\cancel{\frac{1}{a^{n-1}}\phi(a)} - \frac{1}{\epsilon^{n-1}}\phi(\epsilon)\right] + \int_\epsilon^a \frac{1}{n-1}\frac{1}{t'^{n-1}}\phi'(t')dt' \tag{5}$$

Applying the IBP one more time yields,

$$\int_\epsilon^a \frac{1}{t'^n}\phi(t')dt' = -\frac{1}{n-1}\left[\cancel{\frac{1}{a^{n-1}}\phi(a)} - \frac{1}{\epsilon^{n-1}}\phi(\epsilon)\right] - \frac{1}{(n-1)(n-2)}\left[\cancel{\frac{1}{a^{n-2}}\phi(a)} - \frac{1}{\epsilon^{n-2}}\phi(\epsilon)\right] \tag{6}$$
$$+ \frac{1}{(n-1)(n-2)}\int_\epsilon^a \frac{1}{t'^{n-2}}\phi'(t')dt'$$

in which we can see that the IBP integral does not flip its sign for every IBP iteration because a negative sign from the integral of the $1/t^n$ term is always neutralized by the negative sign outside the integral.

However, notice that as we let $\epsilon \to 0$, the first non-zero term diverges. If we perform IBP n times, we obtain,

$$\int_\epsilon^a \frac{1}{t'^n}\phi(t')dt' = \underbrace{\left[\frac{\epsilon^{1-n}}{n-1} + \frac{\epsilon^{2-n}}{(n-2)(n-1)} + ... + \frac{\ln(\epsilon)}{(n-1)(n-2)(n-3)...(2)(1)}\right]\phi(\epsilon)}_{\text{diverges as } \epsilon \to 0} \tag{7}$$

$$- \frac{1}{(n-1)(n-2)(n-3)...(2)(1)}\underbrace{\int_\epsilon^a \ln(t')D_{t'}^{(n)}\phi(t')dt'}_{\text{converges for } \epsilon \to 0} \tag{8}$$

$$= \text{Divergent terms} - \frac{1}{(n-1)!}\int_\epsilon^a \ln(t')D_{t'}^{(n)}\phi(t')dt' \tag{9}$$

Hence, we shall define the functional as,

$$< t_+^{-n}, \phi > = \frac{-1}{(n-1)!} \int_0^\infty \ln(t') D_{t'}^{(n)} \phi(t') dt' \tag{10}$$

$$= \text{FP} \left[\int_0^\infty \frac{1}{t'^n} \phi(t') dt' \right] \tag{11}$$

Now, let us test the linearity of this functional. Given some test functions ϕ_1 and ϕ_2, and some complex number λ,

$$\begin{aligned}
< t_+^{-n}, \phi_1 + \phi_2 > &= -\frac{1}{(n-1)!} \int_0^\infty \ln(t') D_{t'}^{(n)} [\phi_1 + \phi_2](t') dt' \\
&= -\frac{1}{(n-1)!} \int_0^\infty \ln(t') \left[D_{t'}^{(n)} \phi_1(t') + D_{t'}^{(n)} \phi_2(t') \right] dt' \\
&= -\frac{1}{(n-1)!} \int_0^\infty \ln(t') D_{t'}^{(n)} \phi_1(t') dt' + \frac{1}{(n-1)!} \int_0^\infty \ln(t') D_{t'}^{(n)} \phi_2(t') dt' \\
&=< t_+^{-n}, \phi_1 > + < t_+^{-n}, \phi_2 > \\
< t_+^{-n}, \lambda\phi_1 > &= -\frac{1}{(n-1)!} \int_0^\infty \lambda \ln(t') D_{t'}^{(n)} \phi_1(t') dt' \\
&= -\lambda \frac{1}{(n-1)!} \int_0^\infty \ln(t') D_{t'}^{(n)} \phi_1(t') dt' \\
&= \lambda < t_+^{-n}, \phi_1 >
\end{aligned}$$

Lastly, let us confirm whether the functional is continuous or not. Say we have a test function ϕ_ν which is a part of a sequence of test functions that converges to ϕ. Note that all derivatives of ϕ has the same compact support.

$$\begin{aligned}
| < t_+^{-n}, \phi_\nu > - < t_+^{-n}, \phi > | &= \left| \frac{1}{(n-1)!} \int_0^\infty \ln(t') D_{t'}^{(n)} \phi_\nu(t') dt' - \frac{1}{(n-1)!} \int_0^\infty \ln(t') D_{t'}^{(n)} \phi(t') dt' \right| \\
&= \left| \frac{1}{(n-1)!} \int_\Delta \ln(t') D_{t'}^{(n)} [\phi_\nu(t') - \phi(t')] dt' \right| \\
&\leq \frac{1}{(n-1)!} \int_\Delta | \ln(t') | \left| D_{t'}^{(n)} [\phi_\nu(t') - \phi(t')] \right| dt'
\end{aligned}$$

Since we know that the sequence of test functions $\{D_{t'}^{(n)} \phi_\nu\}_{\nu=1}^\infty$ converges to $D_{t'}^{(n)} \phi$, there is some ϵ such that,

$$\left| D_{t'}^{(n)} [\phi_\nu(t') - \phi(t')] \right| < \epsilon$$

Hence,

$$| < t_+^{-n}, \phi_\nu > - < t_+^{-n}, \phi > | \leq \epsilon \frac{1}{(n-1)!} \int_\Delta | \ln(t') | dt', \qquad (\nu \geq N_0) \tag{12}$$

$$= A\epsilon, \qquad (\nu \geq N_0) \tag{13}$$

where A is some number, such that for some $\varepsilon = A\epsilon$,

$$| < t_+^{-n}, \phi_\nu > - < t_+^{-n}, \phi > | \leq \varepsilon$$

Therefore, the functional is continuous. Hence, we can define a singular distribution t_+^{-n} whose functional is defined by a regular distribution $\ln(t)$.

Problem 2

Define

$$t_+^{-\lambda} = \begin{cases} 0, & \text{for } t < 0 \\ \dfrac{1}{t^\lambda}, & \text{for } t > 0 \end{cases}$$

as a distribution, where $\lambda > 1$.

Proof. We first assume that $t_+^{-\lambda}$ defines a regular distribution, such that the functional,

$$\begin{aligned} < t_+^{-\lambda}, \phi > &= \int_{-\infty}^{\infty} t_+^{-\lambda}(t')\phi(t')dt' && (14) \\ &= \int_0^{\infty} \frac{1}{t'^\lambda}\phi(t')dt' && (15) \end{aligned}$$

must exist for all $\phi(t)$ in $\mathcal{D}$. If it happens that $\phi(0) \neq 0$, the integral does not exist; hence, it cannot define a regular distribution in $\mathcal{D}$.

Let $\epsilon > 0$, and note that $\phi(t)$ is nonzero for some interval $|t'| < a$, then consider the integral,

$$\int_\epsilon^a \frac{1}{t'^\lambda}\phi(t')dt' \tag{16}$$

which we can integrate via IBP. Doing the procedure, we obtain

$$\begin{aligned} \int_\epsilon^a \frac{1}{t'^\lambda}\phi(t')dt' &= \frac{1}{\lambda-1}\frac{1}{t'^{\lambda-1}}\phi(t')\Bigg|_\epsilon^a - \int_\epsilon^a \frac{1}{\lambda-1}\frac{1}{t'^{\lambda-1}}\phi'(t')dt' && (17) \\ &= -\frac{1}{\lambda-1}\left[\cancel{\frac{1}{a^{\lambda-1}}\phi(a)} - \frac{1}{\epsilon^{\lambda-1}}\phi(\epsilon)\right] + \int_\epsilon^a \frac{1}{\lambda-1}\frac{1}{t'^{\lambda-1}}\phi'(t')dt' && (18) \end{aligned}$$

However, notice that as we let $\epsilon \to 0$, the first non-zero term diverges. If we perform IBP n times, we obtain,

$$\begin{aligned} \int_\epsilon^a \frac{1}{t'^\lambda}\phi(t')dt' &= \underbrace{\left[\frac{\epsilon^{1-\lambda}}{\lambda-1} + \frac{\epsilon^{2-\lambda}}{(\lambda-1)(\lambda-2)} + ... + \frac{\epsilon^{\lfloor\lambda\rfloor-\lambda}}{(\lambda-1)(\lambda-2)(\lambda-3)...(\lambda-\lfloor\lambda\rfloor)}\right]\phi(\epsilon)}_{\text{diverges as } \epsilon\to 0} && (19) \\ &\quad + \frac{1}{(\lambda-1)(\lambda-2)(\lambda-3)...(\lambda-\lfloor\lambda\rfloor)}\int_\epsilon^a \frac{1}{t'^{\lambda-\lfloor\lambda\rfloor}}D_{t'}^{(\lfloor\lambda\rfloor)}\phi(t')dt' && (20) \\ &= \text{Divergent terms} + \frac{1}{(\lambda-1)(\lambda-2)(\lambda-3)...(\lambda-\lfloor\lambda\rfloor)}\int_\epsilon^a \frac{1}{t'^{\lambda-\lfloor\lambda\rfloor}}D_{t'}^{(\lfloor\lambda\rfloor)}\phi(t')dt' && (21) \\ &= \text{Divergent terms} + \frac{1}{(\lambda-1)(\lambda-2)(\lambda-3)...(\lambda-\lfloor\lambda\rfloor)}\int_\epsilon^a \frac{1}{t'^{\lambda-\lfloor\lambda\rfloor}}D_{t'}^{(\lfloor\lambda\rfloor)}\phi(t')dt' && (22) \\ &= \text{Divergent terms} + \underbrace{\frac{\Gamma(\lambda-\lfloor\lambda\rfloor)}{\Gamma(\lambda)}}_{\text{since } \lambda>1 \text{ and } \lambda\in\mathbb{R}} \underbrace{\int_\epsilon^a \frac{1}{t'^{\lambda-\lfloor\lambda\rfloor}}D_{t'}^{(\lfloor\lambda\rfloor)}\phi(t')dt'}_{\text{Converges as } \epsilon\to 0 \text{ since } 0<\lambda-\lfloor\lambda\rfloor<1} && (23) \\ & && (24) \end{aligned}$$

Hence, we shall define the functional as,

$$\begin{aligned} < t_+^{-\lambda}, \phi > &= \frac{\Gamma(\lambda-\lfloor\lambda\rfloor)}{\Gamma(\lambda)}\int_0^{\infty} \frac{1}{t'^{\lambda-\lfloor\lambda\rfloor}}D_{t'}^{(\lfloor\lambda\rfloor)}\phi(t')dt' && (25) \\ &= \text{FP}\left[\int_0^{\infty} \frac{1}{t'^\lambda}\phi(t')dt'\right] && (26) \end{aligned}$$

Now, let us test the linearity of this functional. Given some test functions ϕ_1 and ϕ_2, and some complex number λ,

$$
\begin{aligned}
< t_+^{-\lambda}, \phi_1 + \phi_2 > &= \frac{\Gamma(\lambda - \lfloor\lambda\rfloor)}{\Gamma(\lambda)} \int_0^\infty \frac{1}{t'^{\lambda - \lfloor\lambda\rfloor}} D_{t'}^{(\lfloor\lambda\rfloor)} [\phi_1 + \phi_2](t') dt' \\
&= \frac{\Gamma(\lambda - \lfloor\lambda\rfloor)}{\Gamma(\lambda)} \int_0^\infty \frac{1}{t'^{\lambda - \lfloor\lambda\rfloor}} \left[D_{t'}^{(\lfloor\lambda\rfloor)} \phi_1(t') + D_{t'}^{(\lfloor\lambda\rfloor)} \phi_2(t') \right] dt' \\
&= \frac{\Gamma(\lambda - \lfloor\lambda\rfloor)}{\Gamma(\lambda)} \int_0^\infty \frac{1}{t'^{\lambda - \lfloor\lambda\rfloor}} D_{t'}^{(\lfloor\lambda\rfloor)} \phi_1(t') dt' + \frac{\Gamma(\lambda - \lfloor\lambda\rfloor)}{\Gamma(\lambda)} \int_0^\infty \frac{1}{t'^{\lambda - \lfloor\lambda\rfloor}} D_{t'}^{(\lfloor\lambda\rfloor)} \phi_2(t') dt' \\
&= < t_+^{-\lambda}, \phi_1 > + < t_+^{-\lambda}, \phi_2 > \\
< t_+^{-\lambda}, \lambda\phi_1 > &= \frac{\Gamma(\lambda - \lfloor\lambda\rfloor)}{\Gamma(\lambda)} \int_0^\infty \lambda \frac{1}{t'^{\lambda - \lfloor\lambda\rfloor}} D_{t'}^{(\lfloor\lambda\rfloor)} \phi_1(t') dt' \\
&= \frac{\Gamma(\lambda - \lfloor\lambda\rfloor)}{\Gamma(\lambda)} \int_0^\infty \frac{1}{t'^{\lambda - \lfloor\lambda\rfloor}} D_{t'}^{(\lfloor\lambda\rfloor)} \phi_1(t') dt' \\
&= \lambda < t_+^{-\lambda}, \phi_1 >
\end{aligned}
$$

Lastly, let us confirm whether the functional is continuous or not. Say we have a test function ϕ_ν which is a part of a sequence of test functions that converges to ϕ. Note that all derivatives of ϕ has the same compact support.

$$
\begin{aligned}
| < t_+^{-\lambda}, \phi_\nu > - < t_+^{-\lambda}, \phi > | &= \left| \frac{\Gamma(\lambda - \lfloor\lambda\rfloor)}{\Gamma(\lambda)} \int_0^\infty \frac{1}{t'^{\lambda - \lfloor\lambda\rfloor}} D_{t'}^{(\lfloor\lambda\rfloor)} \phi_\nu(t') dt' - \frac{\Gamma(\lambda - \lfloor\lambda\rfloor)}{\Gamma(\lambda)} \int_0^\infty \frac{1}{t'^{\lambda - \lfloor\lambda\rfloor}} D_{t'}^{(\lfloor\lambda\rfloor)} \phi(t') dt' \right| \\
&= \left| \frac{\Gamma(\lambda - \lfloor\lambda\rfloor)}{\Gamma(\lambda)} \int_\Delta \frac{1}{t'^{\lambda - \lfloor\lambda\rfloor}} D_{t'}^{(\lfloor\lambda\rfloor)} [\phi_\nu(t') - \phi(t')] dt' \right| \\
&\leq \frac{\Gamma(\lambda - \lfloor\lambda\rfloor)}{\Gamma(\lambda)} \int_\Delta \left| \frac{1}{t'^{\lambda - \lfloor\lambda\rfloor}} \right| \left| D_{t'}^{(\lfloor\lambda\rfloor)} [\phi_\nu(t') - \phi(t')] \right| dt'
\end{aligned}
$$

Since we know that the sequence of test functions $\{D_{t'}^{(\lfloor\lambda\rfloor)} \phi_\nu\}_{\nu=1}^\infty$ converges to $D_{t'}^{(\lfloor\lambda\rfloor)} \phi$, there is some ϵ such that,

$$
\left| D_{t'}^{(\lfloor\lambda\rfloor)} [\phi_\nu(t') - \phi(t')] \right| < \epsilon
$$

Hence,

$$
\begin{aligned}
| < t_+^{-\lambda}, \phi_\nu > - < t_+^{-\lambda}, \phi > | &\leq \epsilon \frac{\Gamma(\lambda - \lfloor\lambda\rfloor)}{\Gamma(\lambda)} \int_\Delta \left| \frac{1}{t'^{\lambda - \lfloor\lambda\rfloor}} \right| dt', \qquad (\nu \geq N_0) \qquad (27) \\
&= A\epsilon, \qquad (\nu \geq N_0) \qquad (28)
\end{aligned}
$$

where A is some number, such that for some $\varepsilon = A\epsilon$,

$$
| < t_+^{-\lambda}, \phi_\nu > - < t_+^{-\lambda}, \phi > | \leq \varepsilon
$$

Therefore, the functional is continuous. Hence, we can define a singular distribution t_+^{-n} whose functional is defined by a regular distribution $\frac{1}{t^{\lambda - \lfloor\lambda\rfloor}}$.