Physics 242 LE 3 Problem Set

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1 Adiabatic approximation of the forced harmonic oscillator problem

According to the adiabatic theorem, a system that starts out in the nth eigenstate of the initial Hamiltonian will evolve as the nth eigenstate of the instantaneous Hamiltonian.(Griffiths 2018) The eigenstate picks up a phase factor called the dynamical phase,

$$\exp\left(-\frac{i}{\hbar}\int_0^t E_n(t')dt'\right) = e^{i\theta_n(t)} \tag{1}$$

and a geometric phase called Berry's phase, such that in the adiabatic limit, the state takes a form,

$$\Psi_n(t) = e^{i\theta_n(t)}e^{i\gamma_n(t)}\psi_n(t) \tag{2}$$

where $\psi_n(t)$ is the nth eigenstate of the instantaneous Hamiltonian. We can solve for these phase factors by first solving for the energies of the instantaneous Hamiltonian, $E_n(t)$. We consider a harmonic oscillator with natural frequency ω_o that is driven by a time-varying force $F(t)$,

$$F(t) = F_o e^{-t^2/\tau^2} \tag{3}$$

such that the instantaneous Hamiltonian at time t takes the form,

$$\hat{H}(t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_o^2 x^2 - xF(t) \tag{4}$$

in which we assume that the characteristic timescale of the driving force is much lower than the period of oscillation, such that

$$\omega_o << \frac{1}{\tau} \tag{5}$$

We start by completing the squares in the Hamiltonian, and effectively shifting the coordinate space

$$\begin{aligned}
\hat{H}(t) &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_o^2 x^2 - xF(t) && (6)\\
&= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_o^2\left(x^2 - \frac{2x}{m\omega_o^2}F(t)\right) && (7)\\
&= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_o^2\left(x^2 - \frac{2x}{m\omega_o^2}F(t) + \frac{F^2(t)}{m^2\omega_o^4} - \frac{F^2(t)}{m^2\omega_o^4}\right) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_o^2\left(x - \frac{F(t)}{m\omega_o^2}\right)^2 - \frac{F^2(t)}{m^2\omega_o^4} && (8)
\end{aligned}$$

We evaluate this Hamiltonian at a position $x' = x + f(t)$ where $f(t) = \frac{F(t)}{m\omega_o^2}$, such that we obtain a Hamiltonian akin to that of an unforced harmonic oscillator plus an additional energy term,

$$\begin{aligned}
\hat{H}'(t) &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2} + \frac{1}{2}m\omega_o^2\left(x' - \frac{F(t)}{m\omega_o^2}\right)^2 - \frac{F^2(t)}{m^2\omega_o^4} && (9)\\
&= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_o^2 x^2 - \frac{F^2(t)}{m^2\omega_o^4} && (10)
\end{aligned}$$

which when acted on the eigenstate $\psi_n(x+f)$ yields the energy eigenvalues E_n that are shifted as well,

$$\begin{aligned}
\hat{H}'(t)\psi_n(x+f) &= \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_o^2 x^2 - \frac{F^2(t)}{m^2\omega_o^4}\right)\psi_n(x+f) && (11)\\
&= E_n'\psi_n(x+f) && (12)
\end{aligned}$$

$$\longrightarrow \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_o^2 x^2\right)\psi_n(x+f) = \left(E_n' - \frac{F^2(t)}{m^2\omega_o^4}\right)\psi_n(x+f) \tag{13}$$

such that

$$E_n' = \hbar\omega_o\left(n + \frac{1}{2}\right) + \frac{F^2(t)}{m^2\omega_o^4} \tag{14}$$

which correspond to eigenstates whose position coordinates are also shifted,

$$\psi_n(x,t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega_o}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega_o(x+f)^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega_o}{\hbar}}(x+f)\right) \tag{15}$$

We proceed by evaluating the dynamical phase by evaluating the integral,

$$-\frac{i}{\hbar}\int_0^t E_n(t')dt' = -\frac{i}{\hbar}\int_0^t \left[\hbar\omega_o\left(n+\frac{1}{2}\right) + \frac{F^2(t')}{m^2\omega_o^4}\right] dt' \tag{16}$$

$$= -i\omega_o t\left(n+\frac{1}{2}\right) - i\frac{F_o^2}{\hbar m^2\omega_o^4}\int_0^t e^{-2t'^2/\tau^2}\; dt' \tag{17}$$

$$= -i\omega_o t\left(n+\frac{1}{2}\right) - i\sqrt{\frac{\pi\tau^2 F_o^4}{8\hbar^2 m^4\omega_o^8}}\mathrm{Erf}\left[\frac{\sqrt{2}t}{\tau}\right] \tag{18}$$

It was proven in Griffiths that when the eigenvalues are real, Berry's phase is zero. Therefore the eigenvectors Ψ_n are given as,

$$\Psi_n(x,t) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega_o}{\pi\hbar}\right)^{1/4} \exp\left[-i\omega_o t\left(n+\frac{1}{2}\right) - i\sqrt{\frac{\pi\tau^2F_o^4}{8\hbar^2m^4\omega_o^8}}\mathrm{Erf}\left[\frac{\sqrt{2}t}{\tau}\right] - \frac{m\omega_o(x+f)^2}{2\hbar}\right] H_n\left(\sqrt{\frac{m\omega_o}{\hbar}}(x+f)\right) \tag{19}$$

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2 N identical harmonic oscillators (LE 3 Item No. 4)

We are tasked to build the whole Hilbert space and the fermionic and bosonic subspaces of N identical harmonic oscillators. We know that the eigenkets of the harmonic oscillator Hamiltonian can be represented by the state vector $|n\rangle$, which alternatively, can be written in terms of its ground state,

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \tag{20}$$

Therefore, we can build the Hilbert space in two ways, one involving the ground state eigenket $|0\rangle$, and the other $|n\rangle$,

$$|n_1, n_2, ..., n_N\rangle = \bigotimes_{\ell=1}^{N} |n_\ell\rangle = |n_1\rangle \otimes |n_2\rangle \otimes ... \otimes |n_N\rangle \tag{21}$$

$$\bigotimes_{\ell=1}^{N} \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle = (a^\dagger)^{n_1} |0\rangle \otimes (a^\dagger)^{n_1} |0\rangle \otimes (a^\dagger)^{n_2} |0\rangle \otimes ... \otimes (a^\dagger)^{n_N} |0\rangle \left(\prod_{\ell=1}^{N} \frac{1}{\sqrt{n_\ell !}}\right) \tag{22}$$

We will now build the fermionic subspace from the basis that we have constructed. For the two-particle case, consider two states $|n_1\rangle$ and $|n_2\rangle$. The fermionic subspace is spanned by the completely anti-symmetric states. In particular, its elements are given by,

$$\begin{align} |n_1, n_2\rangle_- &= \frac{1}{\sqrt{2}} \left(P^{1\ 2}_{1\ 2} |n_1, n_2\rangle - P^{1\ 2}_{2\ 1} |n_1, n_2\rangle\right) \tag{23} \\ &= \frac{1}{\sqrt{2}} \left(|n_1, n_2\rangle - |n_2, n_1\rangle\right) \tag{24} \end{align}$$

For the three-particle case, we can apply the same principle,

$$\begin{align} |n_1, n_2, n_3\rangle_- = \frac{1}{\sqrt{6}} (&P^{1\ 2\ 3}_{1\ 2\ 3} |n_1, n_2, n_3\rangle - P^{1\ 2\ 3}_{2\ 1\ 3} |n_1, n_2, n_3\rangle - P^{1\ 2\ 3}_{1\ 3\ 2} |n_1, n_2, n_3\rangle - P^{1\ 2\ 3}_{3\ 2\ 1} |n_1, n_2, n_3\rangle \tag{25} \\ &+ P^{1\ 2\ 3}_{3\ 1\ 2} |n_1, n_2, n_3\rangle + P^{1\ 2\ 3}_{2\ 3\ 1} |n_1, n_2, n_3\rangle) \tag{26} \\ = \frac{1}{\sqrt{6}} (&|n_1, n_2, n_3\rangle - |n_2, n_1, n_3\rangle - |n_1, n_3, n_2\rangle - |n_3, n_2, n_1\rangle + |n_3, n_1, n_2\rangle + |n_2, n_3, n_1\rangle) \tag{27} \end{align}$$

We will now build the bosonic subspace from the basis that we have constructed. For the two-particle case, consider two states $|n_1\rangle$ and $|n_2\rangle$. The bosonic subspace is spanned by the completely symmetric states. For the case when the two energy states are equal, that is $n_1 = n_2 = n$, its elements are given by,

$$\begin{align} |n, n\rangle_+ &= \frac{1}{\sqrt{4}} \left(P^{1\ 2}_{1\ 2} |n_1, n_2\rangle + P^{1\ 2}_{2\ 1} |n_1, n_2\rangle\right) \tag{28} \\ &= \frac{1}{2} \left(|n, n\rangle + |n, n\rangle\right) = |n, n\rangle \tag{29} \end{align}$$

For the case when the two states are not equal,

$$\begin{align} |n_1, n_2\rangle_+ &= \frac{1}{\sqrt{2}} \left(P^{1\ 2}_{1\ 2} |n_1, n_2\rangle + P^{1\ 2}_{2\ 1} |n_1, n_2\rangle\right) \tag{30} \\ &= \frac{1}{\sqrt{2}} \left(|n_1, n_2\rangle + |n_2, n_1\rangle\right) \tag{31} \end{align}$$

For the three-particle case, we can apply the same principle and consider three cases. When the three states are distinct, the fermionic subspace basis is given by

$$\begin{align} |n_1, n_2, n_3\rangle_+ = \frac{1}{\sqrt{6}} (&P^{1\ 2\ 3}_{1\ 2\ 3} |n_1, n_2, n_3\rangle + P^{1\ 2\ 3}_{2\ 1\ 3} |n_1, n_2, n_3\rangle + P^{1\ 2\ 3}_{1\ 3\ 2} |n_1, n_2, n_3\rangle + P^{1\ 2\ 3}_{3\ 2\ 1} |n_1, n_2, n_3\rangle \tag{32} \\ &+ P^{1\ 2\ 3}_{3\ 1\ 2} |n_1, n_2, n_3\rangle + P^{1\ 2\ 3}_{2\ 3\ 1} |n_1, n_2, n_3\rangle) \tag{33} \\ = \frac{1}{\sqrt{6}} (&|n_1, n_2, n_3\rangle + |n_2, n_1, n_3\rangle + |n_1, n_3, n_2\rangle + |n_3, n_2, n_1\rangle + |n_3, n_1, n_2\rangle + |n_2, n_3, n_1\rangle) \tag{34} \end{align}$$

When two states are equal (e.g. $n_1 = n_2 = n$), whilst the third is distinct from the two, we obtain a subspace basis given by,

$$|n, n, n_3\rangle_+ = \frac{1}{\sqrt{12}}(P^{1\ 2\ 3}_{1\ 2\ 3}|n, n, n_3\rangle + P^{1\ 2\ 3}_{2\ 1\ 3}|n, n, n_3\rangle + P^{1\ 2\ 3}_{1\ 3\ 2}|n, n, n_3\rangle + P^{1\ 2\ 3}_{3\ 2\ 1}|n, n, n_3\rangle \tag{35}$$

$$+ P^{1\ 2\ 3}_{3\ 1\ 2}|n, n, n_3\rangle + P^{1\ 2\ 3}_{2\ 3\ 1}|n, n, n_3\rangle) \tag{36}$$

$$= \frac{1}{\sqrt{12}}(|n, n, n_3\rangle + |n, n, n_3\rangle + |n, n_3, n\rangle + |n_3, n, n\rangle + |n_3, n, n\rangle + |n, n_3, n\rangle) \tag{37}$$

$$= \frac{1}{\sqrt{3}}(|n, n, n_3\rangle + |n, n_3, n\rangle + |n_3, n, n\rangle) \tag{38}$$

Lastly, when the three states are equal, that is $n_1 = n_2 = n_3 = n$,

$$|n, n, n\rangle_+ = \frac{1}{\sqrt{36}}(P^{1\ 2\ 3}_{1\ 2\ 3}|n, n, n\rangle + P^{1\ 2\ 3}_{2\ 1\ 3}|n, n, n\rangle + P^{1\ 2\ 3}_{1\ 3\ 2}|n, n, n\rangle + P^{1\ 2\ 3}_{3\ 2\ 1}|n, n, n\rangle \tag{39}$$

$$+ P^{1\ 2\ 3}_{3\ 1\ 2}|n, n, n\rangle + P^{1\ 2\ 3}_{2\ 3\ 1}|n, n, n\rangle) \tag{40}$$

$$= |n, n, n\rangle \tag{41}$$

Deriving these basis is one thing. Verifying that these are indeed are eigenstates of the Hamiltonian in their respective subspace is another. The Hamiltonian for the two-particle case is given by,

$$\hat{H} = \hat{H}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \hat{H}_2 \tag{42}$$

while, for the three particle case, the Hamiltonian is given as

$$\hat{H} = \hat{H}_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 + \mathbb{1}_1 \otimes \hat{H}_2 \otimes \mathbb{1}_3 + \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes \hat{H}_3 \tag{43}$$

where

$$\hat{H}_i = \hbar\omega\left(\hat{N}_i + \frac{1}{2}\right) \qquad i = 1, 2, 3 \tag{44}$$

in which $\hat{N}_i$ is the number operator, ($\hat{N} = a^\dagger a$)

$$\hat{H}|n_1, n_2\rangle_- = (\hat{H}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \hat{H}_2)\left(\frac{1}{\sqrt{2}}(|n_1, n_2\rangle - |n_2, n_1\rangle)\right) \tag{45}$$

$$= \frac{\hbar\omega}{\sqrt{2}}[(n_1 + n_2 + 1)|n_1, n_2\rangle - (n_1 + n_2 + 1)|n_2, n_1\rangle] \tag{46}$$

$$= \frac{\hbar\omega}{\sqrt{2}}(n_1 + n_2 + 1)(|n_1, n_2\rangle - |n_2, n_1\rangle) \tag{47}$$

$$= \hbar\omega(n_1 + n_2 + 1)|n_1, n_2\rangle_- \tag{48}$$

$$\hat{H}|n_1, n_2\rangle_+ = (\hat{H}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \hat{H}_2)\left(\frac{1}{\sqrt{2}}(|n_1, n_2\rangle + |n_2, n_1\rangle)\right) \tag{49}$$

$$= \frac{\hbar\omega}{\sqrt{2}}[(n_1 + n_2 + 1)|n_1, n_2\rangle + (n_1 + n_2 + 1)|n_2, n_1\rangle] \tag{50}$$

$$= \frac{\hbar\omega}{\sqrt{2}}(n_1 + n_2 + 1)(|n_1, n_2\rangle + |n_2, n_1\rangle) \tag{51}$$

$$= \hbar\omega(n_1 + n_2 + 1)|n_1, n_2\rangle_+ \tag{52}$$

$$
\begin{aligned}
\hat{H}\left|n_1,n_2,n_3\right\rangle_- &= (\hat{H}_1\otimes\mathbb{1}_2\otimes\mathbb{1}_3+\mathbb{1}_1\otimes\hat{H}_2\otimes\mathbb{1}_3+\mathbb{1}_1\otimes\mathbb{1}_2\otimes\hat{H}_3) && (53)\\
&\qquad\left(\frac{1}{\sqrt{6}}(|n_1,n_2,n_3\rangle-|n_2,n_1,n_3\rangle-|n_1,n_3,n_2\rangle-|n_3,n_2,n_1\rangle+|n_3,n_1,n_2\rangle+|n_2,n_3,n_1\rangle)\right) && (54)\\
&=\frac{\hbar\omega}{\sqrt{6}}\left(n_1+n_2+n_3+\frac{3}{2}\right)(|n_1,n_2,n_3\rangle-|n_2,n_1,n_3\rangle-|n_1,n_3,n_2\rangle-|n_3,n_2,n_1\rangle+|n_3,n_1,n_2\rangle+|n_2,n_3,n_1\rangle) && (55)\\
&=\hbar\omega\left(n_1+n_2+n_3+\frac{3}{2}\right)\left|n_1,n_2,n_3\right\rangle_- && (56)\\
\hat{H}\left|n_1,n_2,n_3\right\rangle_+ &= (\hat{H}_1\otimes\mathbb{1}_2\otimes\mathbb{1}_3+\mathbb{1}_1\otimes\hat{H}_2\otimes\mathbb{1}_3+\mathbb{1}_1\otimes\mathbb{1}_2\otimes\hat{H}_3) && (57)\\
&\qquad\left(\frac{1}{\sqrt{6}}(|n_1,n_2,n_3\rangle+|n_2,n_1,n_3\rangle+|n_1,n_3,n_2\rangle+|n_3,n_2,n_1\rangle+|n_3,n_1,n_2\rangle+|n_2,n_3,n_1\rangle)\right) && (58)\\
&=\frac{\hbar\omega}{\sqrt{6}}\left(n_1+n_2+n_3+\frac{3}{2}\right)(|n_1,n_2,n_3\rangle+|n_2,n_1,n_3\rangle+|n_1,n_3,n_2\rangle+|n_3,n_2,n_1\rangle+|n_3,n_1,n_2\rangle+|n_2,n_3,n_1\rangle) && (59)\\
&=\hbar\omega\left(n_1+n_2+n_3+\frac{3}{2}\right)\left|n_1,n_2,n_3\right\rangle_+ && (60)\\
\hat{H}\left|n,n,n_3\right\rangle_+ &= (\hat{H}_1\otimes\mathbb{1}_2\otimes\mathbb{1}_3+\mathbb{1}_1\otimes\hat{H}_2\otimes\mathbb{1}_3+\mathbb{1}_1\otimes\mathbb{1}_2\otimes\hat{H}_3)\left(\frac{1}{\sqrt{3}}(|n,n,n_3\rangle+|n,n_3,n\rangle+|n_3,n,n\rangle)\right) && (61)\\
&=\frac{\hbar\omega}{\sqrt{3}}\left(2n+n_3+\frac{3}{2}\right)(|n,n,n_3\rangle+|n,n_3,n\rangle+|n_3,n,n\rangle) && (62)\\
&=\hbar\omega\left(2n+n_3+\frac{3}{2}\right)\left|n,n,n_3\right\rangle_+ && (63)
\end{aligned}
$$