Physics 211 Third HW Problem

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Distribution Theory Lecture 3

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Theory: Operations on Distributions

We define $f(g(x))$ where $f(t)$ is a regular distribution such that the integral

$$\int_{-\infty}^{\infty} f(t)\phi(t)dt$$

exists. For some function, $g(t)$ consider the integral

$$\int_{-\infty}^{\infty} f(g(t))\phi(t)dt$$

We perform a change of variables, $x = g(t)$, such that

$$\begin{aligned} t &= g^{-1}(x) \\ &= h(x) \\ dt &= \frac{dh}{dx}dx \end{aligned}$$

which yields,

$$\int_{-\infty}^{\infty} f(g(t))\phi(t)dt = \int_{-\infty}^{\infty} f(x)\phi(h(x))\frac{dh}{dx}dx$$

For this identity to hold for all distributions, the following must be satisfied:

1. $g(t)$ and $h(x)$ must be infinitely smooth.

2. $g(t),\ h(x) \neq 0$ over the interval $a < t < b$ and $c < x < d$, and that g and h map these intervals onto one another.

Problem

Show that

$$\delta(\sin(t)) = \sum_{k=-\infty}^{\infty} \delta(t - \pi k)$$

(*Original question was: $\delta(t - 2\pi k)$; however, the zeroes of $\sin(t)$ are π units apart.)

Solution: Let us consider the functional involving the composition of the Dirac delta $\delta(t)$ and $\sin(t)$,

$$< \delta(\sin(t)), \phi(t) >= \int_{-\infty}^{\infty} \delta(\sin(t))\phi(t)dt \tag{1}$$

and take every zero of $\sin(t)$, $t_k = \pi k$, in some neighborhood $\Delta_k : [\pi k - \epsilon, \pi k + \epsilon]$, such that we can recast the integral to

$$\int_{-\infty}^{\infty} \delta(\sin(t))\phi(t)dt = \sum_{k=-\infty}^{\infty} \int_{\Delta_k} \delta(\sin(t))\phi(t)dt \tag{2}$$

Take note that this is valid since both $\sin(t)$ and its inverse are smooth. Then, we perform a change of variables in each neighborhood, $x_k = \sin(t)$, such that $t = \sin^{-1}(x_k)$.

$$\begin{aligned} dx_k &= \cos(t)dt &(3)\\ &= \cos(\sin^{-1}(x_k))dt &(4) \end{aligned}$$

This recasts our integral into the form,

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(\sin(t))\phi(t)dt &= \sum_{k=-\infty}^{\infty} \int_{\sin(\Delta_k)} \delta(x_k)\frac{\phi(\sin^{-1}(x_k))}{\cos(\sin^{-1}(x_k))}dx &(5)\\ &= \sum_{k=-\infty}^{\infty} \frac{\phi(\sin^{-1}(0))}{\cos(\sin^{-1}(0))} &(6)\\ &= \sum_{k=-\infty}^{\infty} \frac{\phi(\pi k)}{\cos(\pi k)} &(7)\\ &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t+\pi k)\delta(t)dt &(8) \end{aligned}$$

Via the sifting property of distributions, we obtain

$$\int_{-\infty}^{\infty} \delta(\sin(t))\phi(t)dt = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t)\delta(t - 2\pi k)dt \tag{9}$$

which then reveals that

$$\delta(\sin(t)) = \sum_{k=-\infty}^{\infty} \delta(t - \pi k)$$

since $\phi(t)$ is an arbitrary test function in $\mathcal{D}$.