Physics 211 First HW Problem

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Distribution Theory Lecture 1

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Theory: The Test Function Space $\mathcal{D}$

Distributions act on a set of functions. This set of functions is called the *test function space*. The elements of the test function space $\mathcal{D}$ satisfy the following properties:

- They are continuously, infinitely differentiable in the entire real line. That means if $\phi(x)$ belongs $\mathcal{D}$, then the functions $\phi^{(k)}(x)$ where $k = 0.\ 1,\ 2...$ exist for all x. These functions are referred to as **infinitely smooth** functions. Elements of $\mathcal{D}$ are $C^\infty(\mathbb{R})$.
- They vanish outside some finite interval $\mathbb{I}$. That means, the elements do not have tails extending to infinity. Beyond some interval, they are already zero.

Fact: Any complex valued function, $f(t)$, that is continuous for all t and zero outside a finite interval can be approximated by an element of $\mathcal{D}$.

Problem 1

A canonical example of a test function is the following,

$$\xi(t) = \begin{cases} 0, & \text{for } |t| \geq 1 \\ \exp\left(-\frac{1}{1-t^2}\right), & \text{for } |t| < 1 \end{cases}$$

Statement: Given $\alpha > 0$, we define the function

$$\gamma_\alpha(t) = \frac{\xi(t/\alpha)}{\int_{-\infty}^{\infty} \xi(t/\alpha)dt}, \text{where} \quad \alpha > 0$$

which vanishes for $|t| \geq \alpha$, and is normalized over all space. Given some function $f(t)$ that is continuous for all t, we define a function $\phi_\alpha(t)$ as,

$$\phi_\alpha(t) = \int_{-\infty}^{\infty} f(\tau)\gamma_\alpha(t-\tau)d\tau$$

We must show that $\phi_\alpha(t)$ belongs to the test function space $\mathcal{D}$.

Attempt:
We start with our definition of the function $\phi_\alpha(t)$ and perform a test of its compactness and smoothness on the real line in order to prove that it belongs to the test function space $\mathcal{D}$,

$$\phi_\alpha(t) = \int_{-\infty}^{\infty} f(\tau)\gamma_\alpha(t-\tau)d\tau$$

Test for Smoothness

Let us first prove that all the derivatives of this convolution exists. To do this, we differentiate both sides wrt t, that is

$$\begin{aligned}\left(\frac{d}{dt}\right)^{(n)}[\phi_\alpha(t)] &= \left(\frac{d}{dt}\right)^{(n)}\int_{-\infty}^{\infty} f(\tau)\gamma_\alpha(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)\left(\frac{d}{dt}\right)^{(n)}[\gamma_\alpha(t-\tau)]d\tau\end{aligned}$$

Since $\gamma_\alpha(t-\tau)$ is bounded in the real line by a compact support, all its derivatives are also well-defined, and hence infinitely smooth. What is left for us to know is whether this integral will converge or not. Since $f(\tau)$ has a compact support and is piecewise continuous, the integrand is piecewise continuous altogether. Hence, the integral is absolutely convergent. This implies that $\phi_\alpha(t)$ is infinitely smooth.

Test for Compactness

Both $f(\tau)$ and $\gamma_\alpha(t-\tau)$ have a compact support on the real line. The convolution of the two functions is zero if the supports do not have an intersection, which is still part of the test function space. When we consider the intersection of the two supports, we can still see that ϕ_α is still compact since the functions in the integrand is non-zero only within γ_α's support. Hence, it is a universal result that ϕ_α has a compact support.

Since, the integral is infinitely smooth and has a compact support, ϕ_α also belongs to the test function space $\mathcal{D}$

Problem 2

We have to prove that

$$|f(t) - \phi_\alpha(t)| \leq \varepsilon$$

for some $\varepsilon > 0$. We now write the left hand side of the inequality as,

$$\begin{aligned}|f(t) - \phi_\alpha(t)| &= \left|f(t)\int_{-\infty}^{\infty}\gamma_\alpha(t-\tau)d\tau - \int_{-\infty}^{\infty} f(\tau)\gamma_\alpha(t-\tau)d\tau\right| \\ &= \left|\int_{-\infty}^{\infty}[f(t)-f(\tau)]\gamma_\alpha(t-\tau)d\tau\right|\end{aligned}$$

since $\int_{-\infty}^{\infty} \gamma_\alpha(t-\tau) = 1$. Via Schwarz inequality,

$$\left|\int_{-\infty}^{\infty} [f(t) - f(\tau)]\gamma_\alpha(t-\tau)d\tau\right| \leq \int_{-\infty}^{\infty} |f(t) - f(\tau)|\, \gamma_\alpha(t-\tau)d\tau$$

Now, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(t) - f(\tau)| < \epsilon$ for all $|t - \tau| < \delta$ since it was specified that $f(t)$ is continuous. Since γ_α is non-zero only for $|t - \tau| < \alpha$,

$$\begin{aligned} \int_{-\infty}^{\infty} |f(t) - f(\tau)|\, \gamma_\alpha(t-\tau)d\tau &\leq \int_{-\infty}^{\infty} \varepsilon\ \gamma_\alpha(t-\tau)d\tau \\ &= \varepsilon \end{aligned}$$

for $\alpha < \delta$.