Physics 211 Fourth HW Problem

Mark Ivan G. Ugalino
Distribution Theory Lectures

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Problem 1

Show that

$$\frac{dt_+^\lambda}{dt} = \lambda t_+^{\lambda-1}$$

Solution: For $Re(\lambda) > -1$, $t+^\lambda$ is a regular distribution defined by the linear functional,

$$< t_+^\lambda, \phi(t) >= \int_0^\infty t^\lambda \phi(t) dt$$

For $Re(\lambda) < -1$, t_+^λ is a singular distribution which is defined by analytic continuation of t_+^λ for $Re(\lambda) > -1$. The action of the derivative can be obtained as follows,

$$\begin{aligned}
\left\langle \frac{d}{dt} t_+^\lambda, \phi(t) \right\rangle &= - < t_+^\lambda, \phi'(t) > \qquad \text{for } Re(\lambda) > 0 \\
&= - \int_0^\infty t^\lambda \phi'(t)\ dt \\
&= -t^\lambda \phi(t) \Big|_0^\infty + \int_0^\infty \lambda t^{\lambda-1} \phi(t)\ dt \\
&= \int_0^\infty \lambda t^{\lambda-1} \phi(t)\ dt \\
&= \lambda \int_{-\infty}^\infty t_+^{\lambda-1} \phi(t)\ dt \\
&=< \lambda t_+^{\lambda-1}, \phi(t) >
\end{aligned}$$

Thus,

$$\frac{dt_+^\lambda}{dt} = \lambda t_+^{\lambda-1}$$

By analytic continuation, this holds for all λ except at the simple poles, $\lambda = -1, -2, -3, ...$

Problem 2

Obtain

$$\int t_-^\lambda \; dt$$

Solution: We are tasked to obtain the indefinite integral of the distribution t_-^λ Let $f = t_-^\lambda$ such that

$$\begin{aligned} < f, \phi(t) > &= < t_-^\lambda, \phi(t) > \\ &= \int_{-\infty}^{0} |t|^\lambda \phi(t) \; dt, \qquad (Re[\lambda] > -1) \\ &= \int_0^\infty t^\lambda \phi(-t) \; dt \end{aligned}$$

We perform IBP to obtain,

$$\begin{aligned} \int_0^\infty t^\lambda \phi(-t) \; dt &= \frac{t^{\lambda+1}}{\lambda+1}\phi(-t)\bigg|_0^\infty + \int_0^\infty \frac{t^{\lambda+1}}{\lambda+1}\phi'(-t) \; dt \\ &= \int_0^\infty \frac{t^{\lambda+1}}{\lambda+1}\phi'(-t) \; dt \\ &= -\int_0^{-\infty} \frac{|t|^{\lambda+1}}{\lambda+1}\phi'(t) \; dt \\ &= \left\langle -\frac{|t|^{\lambda+1}}{\lambda+1}, -\phi'(t) \right\rangle \end{aligned}$$

By analytic continuation, this holds for all λ in the complex plane except at the simple poles at $\lambda = -1, -2, -3, ...$ Hence, the integral of the distribution t_-^λ is,

$$\int t_-^\lambda dt = -\frac{t_-^{\lambda+1}}{\lambda+1} + C_1(\lambda)$$

Problem 3

Show that the integral operation is linear.

Solution: Let us assume that there is some $f(t) = f_1(t) + f_2(t)$ and $h(t) = \alpha f(t)$ where $\alpha \in \mathbb{C}$, such that

$$\begin{aligned} < f(t), \phi(t) > &= \int_{-\infty}^{\infty} f(t)\phi(t)\ dt \\ &= \int_{-\infty}^{\infty} [f_1(t) + f_2(t)]\phi(t)\ dt \\ < g(t), \phi(t) > &= \int_{-\infty}^{\infty} \alpha f(t)\phi(t)\ dt \end{aligned}$$

where $\phi(t)$ is an arbitrary test function in $\mathcal{D}$. We also assume that the antiderivatives of $f(t)$, $f_1(t)$ and $f_2(t)$ are $g(t)$, $g_1(t)$ and $g_2(t)$ respectively.

$$\begin{aligned} < f(t), \phi(t) > &=< f_1(t) + f_2(t), \phi(t) > \\ &= \int_{-\infty}^{\infty} [f_1(t) + f_2(t)]\phi(t)\ dt \\ &= \int_{-\infty}^{\infty} f_1(t)\phi(t)\ dt + \int_{-\infty}^{\infty} f_2(t)\phi(t)\ dt \\ &= g_1(t)\phi(t)\Big|_{-\infty}^{\infty} + g_2(t)\phi(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g_1(t)\phi'(t)\ dt - \int_{-\infty}^{\infty} g_2(t)\phi'(t)\ dt \\ &= -\int_{-\infty}^{\infty} [g_1(t) + g_2(t)]\phi'(t)\ dt \\ &= -< g_1(t) + g_2(t), \phi'(t) >=< g_1(t) + g_2(t), -\phi'(t) > \checkmark \\ < g(t), \phi(t) > &=< \alpha f(t), \phi(t) > \\ &= \int_{-\infty}^{\infty} \alpha f(t)\phi(t)\ dt \\ &= \alpha \int_{-\infty}^{\infty} f(t)\phi(t)\ dt \\ &= \alpha \left[g(t)\phi(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g(t)\phi'(t)\ dt \right] \\ &= -\alpha \int_{-\infty}^{\infty} g(t)\phi'(t)\ dt \\ &=< \alpha g(t), -\phi'(t) > \checkmark \end{aligned}$$

Hence, the integral operation in the distribution space $\mathcal{D}'$ is linear.

Problem 4

Show that

$$\int |t|^\lambda \operatorname{sgn}(t)\, dt = \frac{|t|^{\lambda+1}}{\lambda+1} + C_2(\lambda)$$

Solution: We let $f = |t|^\lambda \operatorname{sgn}(t)$ such that,

$$\begin{aligned}
< f, \phi(t) > &= \left\langle |t|^\lambda \operatorname{sgn}(t), \phi(t) \right\rangle \\
&= \left\langle (t_+^\lambda + t_-^\lambda)[H(t) - H(-t)], \phi(t) \right\rangle \\
&= \int_0^\infty t^\lambda \phi(t)\, dt - \int_{-\infty}^0 |t|^\lambda \phi(t)\, dt \\
&= \int_0^\infty t^\lambda \phi(t)\, dt + \int_\infty^0 t^\lambda \phi(-t)\, dt \\
&= \frac{t^{\lambda+1}}{\lambda+1}\phi(t)\Bigg|_0^\infty - \frac{t^{\lambda+1}}{\lambda+1}\phi(-t)\Bigg|_0^\infty - \int_0^\infty \frac{t^{\lambda+1}}{\lambda+1}\phi'(t)\, dt + \int_0^\infty \frac{t^{\lambda+1}}{\lambda+1}\phi'(-t) d(-t) \\
&= -\int_0^\infty \frac{t^{\lambda+1}}{\lambda+1}\phi'(t)\, dt + \int_0^{-\infty} \frac{|t|^{\lambda+1}}{\lambda+1}\phi'(t)\, dt \\
&= \left\langle \frac{t_+^{\lambda+1} + t_-^{\lambda+1}}{\lambda+1}, -\phi'(t) \right\rangle \\
&= \left\langle \frac{|t|^{\lambda+1}}{\lambda+1}, -\phi'(t) \right\rangle
\end{aligned}$$

Hence, the integral of the distribution $|t|^\lambda \operatorname{sgn}(t)$ is,

$$\int |t|^\lambda \operatorname{sgn}(t)\, dt = \frac{|t|^{\lambda+1}}{\lambda+1} + C_2(\lambda)$$

Problem 5

Show that

$$\int \frac{1}{t}\, dt = \ln|t| + C_3(\lambda)$$

Solution: This problem is a special case to the general case in the previous item when $\lambda = -1$. Let us then obtain the action of the distribution

$$|t|^{\lambda}\mathrm{sgn}(t)$$

when $\lambda = -1$. We consider the functional,

$$\begin{aligned}
< |t|^{-1}\mathrm{sgn}(t), \phi(t) > &= \int_{-\infty}^{\infty} |t|^{-1}\mathrm{sgn}(t)\phi(t)\, dt \\
&= \int_{0}^{\infty} t^{-1}\phi(t) - \int_{-\infty}^{0} |t|^{-1}\phi(t)\, dt \\
&= \lim_{\epsilon\to 0^+}\left[\int_{\epsilon}^{\infty} t^{-1}\phi(t) - \int_{-\infty}^{-\epsilon} |t|^{-1}\phi(t)\, dt\right] = \lim_{\epsilon\to 0^+}\left[\int_{\epsilon}^{\infty} t^{-1}\phi(t) - \int_{\epsilon}^{\infty} t^{-1}\phi(-t)\, dt\right] \\
&= \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\infty} t^{-1}[\phi(t) - \phi(-t)]\, dt \\
&= \lim_{\epsilon\to 0^+}\left[\ln|t|[\phi(t) - \phi(-t)]\Big|_{\epsilon}^{\infty} - \int_{\epsilon}^{\infty} \ln|t|[\phi'(t) + \phi'(-t)]\, dt\right]
\end{aligned}$$

However, since $\phi(t)$ can be expanded in the form,

$$\phi(t) = \phi(0) + t\eta(t)$$

where $\eta(t)$ is continuous at $t = 0$, which then leaves us with

$$\begin{aligned}
< |t|^{-1}\mathrm{sgn}(t), \phi(t) > &= \lim_{\epsilon\to 0^+}\left[\phi(0)[\ln(\epsilon) - \ln(-\epsilon)] - \int_{\epsilon}^{\infty} \ln|t|[\phi'(t) + \phi'(-t)]\, dt\right] \\
&= -\lim_{\epsilon\to 0^+}\int_{\epsilon}^{\infty} \ln|t|[\phi'(t) + \phi'(-t)]\, dt \\
&= -\lim_{\epsilon\to 0^+}\left[\int_{\epsilon}^{\infty} \ln(t)\phi'(t)\, dt + \int_{\epsilon}^{\infty} \ln|t|\phi'(-t)\, dt\right] \\
&= -\lim_{\epsilon\to 0^+}\left[\int_{\epsilon}^{\infty} \ln(t)\phi'(t)\, dt - \int_{-\epsilon}^{-\infty} \ln(t)\phi'(t)\, dt\right] \\
&= -\int_{-\infty}^{\infty} \ln|t|[-\phi'(t)]\, dt \\
&= \langle \ln|t|, -\phi'(t)\rangle
\end{aligned}$$

Therefore, the integral of the distribution $\dfrac{1}{|t|}$ is given by,

$$\int \frac{1}{t}\, dt = \ln|t| + C_3(\lambda)$$

Problem 6

Obtain

$$\int \delta(t)\ dt$$

Solution: We let $f = \delta(t)$, such that

$$\begin{aligned}
< f, \phi(t) > &= \langle \delta(t), \phi(t) \rangle \\
&= \int_{-\infty}^{\infty} \delta(t)\phi(t)\ dt \\
&= \phi(0)
\end{aligned}$$

Since $\phi(t)$ has a compact support, we can express $\phi(0)$ as,

$$\begin{aligned}
\phi(0) &= -\int_{0}^{\infty} \phi'(t)dt \\
&= -\int_{-\infty}^{\infty} H(t)\phi'(t)dt \\
&= \langle H(t), -\phi'(t) \rangle
\end{aligned}$$

Hence, the integral of the distribution $\delta(t)$ is the step function $H(t)$,

$$\int \delta(t)\ dt = H(t) + C_4$$

Mark Ivan Ugelino
2013 70180

7.1

Define

$$PV\left(\frac{1}{t^n}\right) = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{dt^{n-1}} PV\left(\frac{1}{t}\right)$$

where the action of such a distribution is given by,

$$\left\langle PV\left(\frac{1}{t^n}\right), \phi(t)\right\rangle = \left\langle \frac{(-1)^n}{(n-1)!}\frac{d^{n-1}}{dt^{n-1}} PV\left(\frac{1}{t}\right), \phi(t)\right\rangle$$

$$= \frac{(-1)^n}{(n-1)!}\left\langle \frac{d^{n-1}}{dt^{n-1}} PV\left(\frac{1}{t}\right), \phi(t)\right\rangle$$

$$= \frac{(-1)^n}{(n-1)!}(-1)^{n-1}\left\langle PV\left(\frac{1}{t}\right), \phi^{(n-1)}(t)\right\rangle$$

$$= \frac{-1}{(n-1)!}\left\langle PV\left(\frac{1}{t}\right), \phi^{(n-1)}(t)\right\rangle$$

We must show that,

$$\frac{1}{(t \pm i0^+)^n} = PV\left(\frac{1}{t^n}\right) \pm (-1)^{n+1}\delta^{(n)}(t)$$

we can see further into our calculation that there is something wrong with the given problem. Let us consider the functional,

$$\left\langle \frac{1}{(t \pm i0^+)^n}, \phi(t)\right\rangle = \int_{-\infty}^{\infty} \frac{1}{(t \pm i0^+)^n}\phi(t)\,dt$$

where $\phi(t) \in \mathcal{L}$

$$= \lim_{\tau \to 0^+}\int_{-\infty}^{\infty} \frac{(\pm i)^n}{(\pm it - \tau)^n}\phi(t)\,dt$$

$$= (\pm i)^n \lim_{\tau\to 0^+}\int_{-\infty}^{\infty} \frac{1}{(-1)^n(\tau \mp it)^n}\phi(t)\,dt$$

$$= (\mp i)^n \lim_{\tau\to 0^+}\int_{-\infty}^{\infty} \frac{1}{(\tau \mp it)^n}\phi(t)\,dt$$

Mark Ivan Ugelino

2013 70180

7.2

$$(\mp i)^n \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} \frac{1}{(\tau \mp it)^n} \phi(t)\, dt$$

$$= (\mp i)^n \lim_{\tau \to 0^+} \lim_{\epsilon \to 0^+} \left[\int_{-\infty}^{-\epsilon} \frac{1}{(\tau \mp it)^n} \phi(t)\, dt + \int_{\epsilon}^{\infty} \frac{1}{(\tau \mp it)^n} \phi(t)\, dt \right]$$

$$\int_{-\infty}^{-\epsilon} \frac{1}{(\tau \mp it)^n} \phi(t)\, dt$$

$$= \int_{\infty}^{\epsilon} \frac{1}{(\tau \pm it)^n} \phi(-t)\, d(-t)$$

$$= \int_{\epsilon}^{\infty} \frac{1}{(\tau \pm it)^n} \phi(-t)\, dt \quad \text{I}$$

$$\int_{\epsilon}^{\infty} \frac{1}{(\tau \mp it)^n} \phi(t)\, dt \quad \text{II}$$

I. $$\int_{\epsilon}^{\infty} \frac{1}{(\tau \pm it)^n} \phi(-t)\, dt = \left(\frac{\pm 1}{i}\right) \phi(-t) \frac{(\tau \pm it)^{1-n}}{1-n} \Bigg|_{\epsilon}^{\infty}$$

$$+ \phi^{(1)}(-t) \frac{(\tau \pm it)^{2-n}}{(1-n)(2-n)} \left(\pm \frac{1}{i}\right)^2 \Bigg|_{\epsilon}^{\infty}$$

$$+ \phi^{(2)}(-t) \frac{(\tau \pm it)^{3-n}}{(1-n)(2-n)(3-n)} \left(\pm \frac{1}{i}\right)^3 \Bigg|_{\epsilon}^{\infty}$$

$$+ \phi^{(3)}(-t) \frac{(\tau \pm it)^{4-n}}{(1-n)(2-n)(4-n)(3-n)} \left(\pm \frac{1}{i}\right)^4 \Bigg|_{\epsilon}^{\infty}$$

$$+ \dots + \phi^{(n-2)}(t) \frac{(\tau \pm it)^{(n-1)-n}}{(1-n)(2-n)(3-n)(4-n)\dots[(n-1)-n]} \left(\pm \frac{1}{i}\right)^{n-1} \Bigg|_{\epsilon}^{\infty}$$

$$+ \int_{\epsilon}^{\infty} \phi^{(n-1)}(t) \left(\pm \frac{1}{i}\right)^{n-1} \frac{(t \pm it)^{(n-1)-n}}{(1-n)(2-n)(3-n)\dots[(n-1)-n]}$$

Merk Ivan Ugalino
2013 - 70180

7.8

I.

$$\int_\epsilon^\infty \frac{1}{(\tau \pm it)^n}\phi(-t)dt = \left(\pm\frac{1}{i}\right)\left[-\phi(-\epsilon)\right]\frac{(\tau \pm i\epsilon)^{1-n}(n-2)!}{(n-1)!}$$

$$+ \left(\pm\frac{1}{i}\right)^2\left[-\phi^{(1)}(-\epsilon)\right]\frac{(\tau \pm i\epsilon)^{2-n}(n-3)!}{(n-1)!}$$

$$+ \left(\pm\frac{1}{i}\right)^3\left[-\phi^{(2)}(-\epsilon)\right]\frac{(\tau \pm i\epsilon)^{3-n}(n-4)!}{(n-1)!}$$

$$+ \cdots + \left(\pm\frac{1}{i}\right)^{n-1}\left[-\phi^{(n-2)}(-\epsilon)\right]\frac{(\tau \pm i\epsilon)^{-1}(-1)^{n-1}}{(n-1)!}$$

$$+ \left(\pm\frac{1}{i}\right)^{n-1}\int_\epsilon^\infty \phi^{(n-1)}(-t)\frac{(\tau \pm it)^{-1}(-1)^{n-1}}{(n-1)!}dt$$

$$= \sum_{k=1}^{n-1}\left(\pm\frac{1}{i}\right)^k\left[-\phi^{(k-1)}(-\epsilon)\right]\frac{(\tau \pm i\epsilon)^{k-n}(n-k-1)!}{(n-1)!}$$

$$+ \left(\pm\frac{1}{i}\right)^{n-1}\frac{(-1)^{n-1}}{(n-1)!}\int_\epsilon^\infty \phi^{(n-1)}(-t)\,(\tau \pm it)^{-1}dt$$

$$\int_\epsilon^\infty \phi^{(n-1)}(-t)\,(\tau \pm it)^{-1}dt$$

$$= \phi^{(n-1)}(-t)\,\mathrm{Log}(\tau \pm it)(\mp i)\Big|_\epsilon^\infty$$

$$+ \int_\epsilon^\infty \phi^{(n)}(-t)\,\mathrm{Log}(\tau \pm it)(\mp i)$$

Merk Iwen Ugelino
2013 - 70180

7.4

I.

$$\int_{\epsilon}^{\infty} \frac{1}{(\tau \pm it)^n} \phi(-t)\, dt$$

$$= \sum_{k=1}^{n-1} (\mp i)^k \left[-\phi^{(k-1)}(-\epsilon)\right] \frac{(\tau \pm i\epsilon)^{k-n}(n-k-1)!}{(n-1)!}$$

$$+ \left(\pm \frac{1}{i}\right)^{n-1} \frac{(-1)^{n-1}}{(n-1)!} \left[\phi^{(n-1)}(-t) \operatorname{Log}(\tau \pm it)(\mp i) \Big|_{\epsilon}^{\infty} + \int_{\epsilon}^{\infty} dt\, \phi^{(n)}(-t) \operatorname{Log}(\tau \pm it)(\mp i) \right]$$

$-t \to t$

$$= \sum_{k=1}^{n-1} (\mp i)^k \left[-\phi^{(k-1)}(-\epsilon)\right] \frac{(\tau \pm i\epsilon)^{k-n}(n-k-1)!}{(n-1)!}$$

$$+ \left(\pm \frac{1}{i}\right)^{n-1} \frac{(-1)^{n-1}}{(n-1)!} \left[\phi^{(n-1)}(t) \operatorname{Log}(\tau \mp it)(\mp i) \Big|_{\epsilon} - \int_{-\infty}^{-\epsilon} \phi^{(n)}(t) \operatorname{Log}(\tau \mp it)(\mp i) \right]$$

II. $$\int_{\epsilon}^{\infty} \frac{1}{(\tau \mp it)^n} \phi(t)\, dt$$

$$= \phi(t) \frac{(\tau \mp it)^{1-n}}{1-n} \left(\frac{\mp 1}{i}\right) \Big|_{\epsilon}^{\infty} -$$

$$\phi^{(1)}(t) \frac{(\tau \mp it)^{2-n}}{(1-n)(2-n)} \left(\frac{\mp 1}{i}\right)^2 \Big|_{\epsilon}^{\infty} +$$

$$\phi^{(2)}(t) \frac{(\tau \mp it)^{3-n}}{(1-n)(2-n)(3-n)} \left(\frac{\mp 1}{i}\right)^3 \Big|_{\epsilon}^{\infty} + \ldots +$$ continued

Merk Iven G. Ugelino
2013 - 70180

7.5

II. $$+ \frac{(-1)^{n-2}\phi^{(n-2)}(t)}{(1-n)(2-n)(3-n)\cdots[(n-1)-n]}(\tau \mp it)^{(n-1)-n}\left(\frac{\mp 1}{i}\right)^{(n-1)}\Bigg|_{\epsilon}^{\infty}$$

$$+ \frac{(-1)^{n-1}\left(\frac{\mp 1}{i}\right)^{(n-1)}}{(1-n)(2-n)(3-n)\cdots[(n-1)-n]}\int_{\epsilon}^{\infty}\phi^{(n-1)}(t)\,(\tau \mp it)^{-1}$$

$$= \sum_{k=1}^{n-1}\left(\frac{\mp 1}{i}\right)^{k}\left[-\phi^{(k-1)}(\epsilon)\right]\frac{(\tau \mp i\epsilon)^{k-n}(n-k-1)!}{(n-1)!}$$

$$+ \frac{(-1)^{n-1}(-1)^{n-1}\left(\frac{\mp 1}{i}\right)^{(n-1)}}{(n-1)!}\left[\phi^{(n-1)}(t)\,\mathrm{Log}(\tau \mp it)(\mp i)\Big|_{\epsilon}^{\infty} - \int_{\epsilon}^{\infty} dt\,\phi^{(n)}(t)\,\mathrm{Log}(\tau \mp it)(\mp i)\right]$$

We have previously obtained the distributional limit of $\mathrm{Log}(\tau \mp it)$,

$$\lim_{\tau \to 0^+}\langle \mathrm{Log}(\tau \mp it), \phi(t)\rangle = \begin{cases} \left(\ln|t| \pm \frac{i\pi}{2}\right) & t<0 \\ \left(\ln|t| \mp \frac{i\pi}{2}\right) & t>0 \end{cases}$$

Hence, we are left with,

$$\int_{-\infty}^{\infty}\frac{1}{(t \pm i0^+)^n}\phi(t)\,dt =$$ contined

Merk Ivan G. Ugelino

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$$\int_{-\infty}^{\infty} \frac{1}{(t \pm i\, 0^+)^n} \phi(t)\, dt$$

$$= \lim_{\epsilon \to 0} \left[\frac{-1}{(n-1)!} \int_{-\infty}^{-\epsilon} \phi^{(n)}(t) \ln|t| - \frac{1}{(n-1)!} \int_{\epsilon}^{\infty} \phi^{(n)}(t) \ln(t)\, dt \mp \frac{i\pi\, \phi^{(n-1)}(0)}{(n-1)!} \right]$$

$$= \left\langle \frac{-1}{(n-1)!} PV\left(\frac{1}{t}\right), \phi^{(n)}(t) \right\rangle \mp \frac{i\pi}{(n-1)!} \left\langle \delta(t), \phi^{(n-1)}(t) \right\rangle$$

$$= \left\langle PV\left(\frac{1}{t^n}\right), \phi(t) \right\rangle \mp \frac{i\pi (-1)^{(n-1)}}{(n-1)!} \left\langle \delta^{(n-1)}(t), \phi(t) \right\rangle$$

$$= \left\langle PV\left(\frac{1}{t^n}\right), \phi(t) \right\rangle \pm \frac{i\pi (-1)^n}{(n-1)!} \left\langle \delta^{(n-1)}(t), \phi(t) \right\rangle$$

$$= \left\langle PV\left(\frac{1}{t^n}\right) \pm \frac{i\pi (-1)^n}{(n-1)!} \delta^{(n-1)}(t), \phi(t) \right\rangle$$

$\boxtimes$