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ECE478 Financial Signal Processing
Problem Set IV: Stochastic Calculus
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Notation here follows Shreve, *Stochastic Calculus for Finance, vols. I & II,* Springer, 2004.

Theoretical Problems

1. Let $\mathcal{F}_t$ be the filtration generated by a Wiener process $W(t)$. Let $R(t)$ be the interest rate process used to define the discount process $D(t)$. Assume there exists a unique risk-neutral measure, leading to the Wiener process $\tilde{W}(t)$ with respect to $\tilde{P}$. If $V(T)$ is a random variable that is $\mathcal{F}_T$-measurable, and $V(t)$ is defined via:

$$V(t) = \frac{1}{D(t)} \tilde{E}\left(D(T) V(T) | \mathcal{F}_t\right)$$

then $D(t) V(t)$ is a martingale.

   (a) Suppose $V(T) > 0$ a.s. Show that $V(t) > 0$ a.s. (from its definition above).

   (b) Show that there exists an adapted process $\tilde{\Gamma}(t)$ such that:

$$dV(t) = R(t) V(t) dt + \frac{\tilde{\Gamma}(t)}{D(t)} d\tilde{W}(t)$$

   Hint: Start with a formula for $d(D(t) V(t))$ as per the martingale representation theorem, then as you expand this out recognize that $dV dt = 0$.

   (c) Show that there exists an adapted process $\sigma(t)$ such that we can write:

$$dV(t) = R(t) V(t) dt + \sigma(t) V(t) d\tilde{W}(t)$$

   By the way, $\sigma(t)$ can be random and in particular it is fine if the formula for $\sigma(t)$ you derived involves $V(t)$. The point is there is SOME process you can put there that works! How did we use strict positivity? (Think of $D(t)$, $V(t)$ as continuous processes; $D(t)$ is intrinsically positive, but what happens if $V(t)$ can take on negative as well as positive values?) This shows that $V(t)$ is a generalized geometric Brownian motion process. The point of this problem is that every strictly positive asset is a generalized geometric Brownian motion.

2. Let $X(t), Y(t)$ be Itô processes given by:

$$\begin{aligned} dX(t) &= a(t) dt + b(t) dW(t) \\ dY(t) &= c(t) dt + d(t) dW(t) \end{aligned}$$

where a, b, c, d are adapted processes. Assume $Y(t) > 0$ a.s., and $V(t) = X(t) / Y(t)$. Obtain an SDE satisfied by $V(t)$, simplified so it has the above form (i.e., in the form of an Itô process). Note that $X(t), Y(t)$, but not $dX(t)$ or $dY(t)$, can appear in your final expression for $dV(t)$.

Simulating Stochastic Differential Equations

We are going to be viewing continuous-time deterministic functions and stochastic processes in discretized time. For simplicity, we will take equally spaced time intervals, say $t = n\delta$, $0 \leq n \leq N$, with final time $T = N\delta$. (Careful, including $t = 0$, this is $N + 1$ points). We will use the notation:

$$x[n] = x(n\delta) = x_{n\delta}$$

Consider a stochastic differential equation of the form:

$$dX(t) = \beta(t, X(t))\,dt + \gamma(t, X(t))\,dW(t)$$

where $\beta(\cdot), \gamma(\cdot)$ are deterministic functions, and as usual $W(t)$ is a Wiener process. The parameter δ controls the variance of the increment dW; specifically, in the discretization, let us use the notation:

$$d_W[n] = W[n+1] - W[n]$$

where $\{d_W[n]\}$ are iid $N(0, \delta)$. The SDE actually means:

$$X(t) = X(0) + \int_0^t \beta(u, X(u))\,du + \int_0^t \gamma(u, X(u))\,dW(u)$$

In discretized form, with $t = n\delta$ this becomes:

$$X[n] = X[0] + \sum_{m=0}^{n-1} \beta(m\delta, X[m])\,\delta + \sum_{m=0}^{n-1} \gamma(m\delta, X[m])\,d_W[m]$$

Let's default with $\delta = 0.01$ and $N = 250$ (which is approximately the number of days per year for a typical financial instrument).

1. Start with basic geometric Brownian motion:

   $$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

   with $S(0) = 1$, $\alpha = 0.1$, $\sigma = 0.2$. Also assume a constant underlying interest rate $r = 0.05$. Under the risk-neutral measure, dS_t satisfies a modified SDE involving dt and $d\tilde{W}_t$. **See Problem 3 below: If you ever detect $S_t \leq 0$ trap that condition, note that it occurred, and discard that path (don't use it).**

   (a) Write the modified SDE involving $d\tilde{W}_t$. Specifically, calculate the coefficients that appear in this SDE from the specific values of α, σ, r provided. Unless specified otherwise, however, grow your paths using the ORIGINAL formulation (i.e., the ACTUAL probabilities).

   (b) Generate 1000 paths of $S[n]$.

   (c) Use a Monte Carlo approach to estimate $E(S[N/2])$ and $E(S[N])$ directly.

   (d) We now want to connect this to the Black-Scholes model. Let $V(T) = (S(T) - K)^+$, the payout of a European call option with strike price K. In our discrete notation, $V[N] = (S[N] - K)^+$. Use your estimate for $E(S[N])$ as your value for K. Use the Black-Scholes model to obtain a formula for $V[N/2]$ in terms of $S[N/2]$, and graph it (for the specified parameters α, σ, r).

(e) Now take the first 10 paths you generated, $S^{(i)}[n]$, $1 \le i \le 10$. For each $S^{(i)}[N/2]$, you can compute $V^{(i)}[N/2]$ from the Black-Scholes formula. On the other hand, you can use a Monte Carlo approach to compute $V^{(i)}[N/2]$ using the martingale property of the discounted stock price. So, for each i, grow 1000 paths from $N/2$ to N and average to estimate $V^{(i)}[N/2]$. Compare these estimated values with the exact values in these cases. Report the results how you see fit: a table; you could superimpose two scatter plots ($S^{(i)}[N/2]$ versus actual $V^{(i)}[N/2]$, and $S^{(i)}[N/2]$ versus actual $V^{(i)}[N/2]$). **Comments:** Careful. First, you need to grow NEW paths, emanating from time $t = T/2$, towards $t = T$, taking $S^{(i)}[N/2]$ as an initial condition. Also, you are computing $V^{(i)}[N/2]$ as an expectation by virtue of the martingale property, BUT the question is **WHICH SDE FOR S_t DO YOU USE?**

2. **Cox-Ingersoll-Ross Interest Rate Model**

$$dR(t) = (a - \beta R(t))\,dt + \sigma\sqrt{R(t)}dW(t)$$

with α, β, σ positive, and let $R(0) = r > 0$. This model for interest rate $R(t)$ guarantees $R(t) > 0$. This model is typically used for short term interest rates, which tend to exhibit volatility. Although no closed form solution exists, it is possible to determine certain properties for it. In particular the mean and variance of $R(t)$ are given by:

$$\begin{aligned} E(R(t)) &= e^{-\beta t} r + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) \longrightarrow \frac{\alpha}{\beta} \\ var(R(t)) &= \frac{\sigma^2}{\beta} r\left(e^{-\beta t} - e^{-2\beta t}\right) + \frac{\alpha\sigma^2}{2\beta^2}\left(1 - 2e^{-\beta t} - e^{-2\beta t}\right) \longrightarrow \frac{\alpha\sigma^2}{2\beta^2} \end{aligned}$$

Discretizing this SDE for simulation purposes is very tricky because the property that $R(t) > 0$ does **not** carry over if you use the simple discretization as in the previous problem! This actually brings up a larger question– does the discretized system accurately reflect properties of the original SDE?

(a) Select: $\beta = 1$, $\alpha = 0.10\beta$, $r = 0.05$, $\sigma = 0.5$. Generate 1000 paths for $R(t)$ over a span $0 \le t \le 10$, using $\delta = 0.01$. In your code, trap the condition that $R \le 0$ (if this occurs, your code should display an exception, should halt that one path, but should continue computing the other paths). So the goal is to generate 1000 *valid* paths (with $R(t)$ never reaching ≤ 0). How many "bad" paths do you get in order to reach 1000?

(b) Graph the first 10 paths for $R(t)$, just to see what it looks like.

(c) Use a Monte Carlo approach to estimate the mean and variance of $R(t)$ at $t = 1$ and $t = 10$, and compare with the exact formulas given above. Here you should use the valid paths only.

3. Go back to Problem 1 again. We know the actual S_t (in continuous-time) is geometric Brownian motion so we necessarily have $S_t > 0$ a.s., and yet is that property *guaranteed* in your simulation? On the other hand, with all your simulations, did you ever encounter simulated $S_t \le 0$? I expect not. Why not? [Even if you got a negative answer, why did I guess that you wouldn't? The hint is that it is a guess on my part.]