Q1

1) Find the derivative of the following functions   10 points each
$f(x) = e^{\tan(2x)}$

$g(x) = \left(\dfrac{x^3-1}{x^2+1}\right)^5$

$h(x) = \sin(x)\ln(x^2+1)$

$k(x) = x\tan^{-1}(x)$

$m(x) = \ln(\sinh(x))$

2) Find the second derivative of $f(x) = e^x\cos(x)$ 10 points

3) Find $\dfrac{dy}{dx}$ for $y^5 + x^2y^3 = 1 + ye^x$ 10 points

4) Let $f(x) = \dfrac{e^x}{x^2+1}$. Find the equation of the tangent line at $(0,1)$.
10 points

4) Use linear approximation (or differentials) to estimate $\sqrt{15.8}$.   10 points

5) At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/hr and ship B is sailing north at 25 km/hr. How fast is the distance between the shops changing at 4:00 PM?   10 points

---

1) Find the derivative

$$f(x) = e^{\tan(2x)}$$

$$f'(x) = \frac{d}{dx}\left[e^{\tan(2x)}\right]$$

$$= \frac{de^u}{du}\left[e^{\tan(2x)}\right]\frac{d}{dx}\left[\tan(2x)\right]$$

$$= e^{\tan(2x)}\frac{d\tan(u)}{du}\left[\tan(2x)\right]\frac{d}{dx}(2x)$$

$$= e^{\tan(2x)}\sec^2(2x)(2)$$

$$\boxed{f'(x) = 2\sec^2(2x)\,e^{\tan(2x)}}$$

$$k(x) = x\tan^{-1}(x)$$

$$k'(x) = \frac{d}{dx}\left[x\tan^{-1}(x)\right]$$

$$= x\frac{d}{dx}\left[\tan^{-1}(x)\right] + \tan^{-1}(x)\frac{d}{dx}(x)$$

$$= x\left(\frac{1}{1+x^2}\right) + \tan^{-1}(x)(1)$$

$$\boxed{k'(x) = \frac{x}{1+x^2} + \tan^{-1}(x)}$$

$$g(x) = \left(\frac{x^3-1}{x^3+1}\right)^5$$

$$g'(x) = \frac{d}{dx}\left[\left(\frac{x^3-1}{x^3+1}\right)^5\right]$$

$$= \frac{du^5}{du}\left[\left(\frac{x^3-1}{x^3+1}\right)^5\right]\frac{d}{dx}\left(\frac{x^3-1}{x^3+1}\right)$$

$$= 5\left(\frac{x^3-1}{x^3+1}\right)^4\frac{d}{dx}\left(1+\frac{2}{x^3-1}\right)$$

$$= 5\left(\frac{x^3-1}{x^3+1}\right)^4\left[0+\frac{(x^3-1)\frac{d}{dx}(2)-2\frac{d}{dx}(x^3-1)}{(x^3-1)^2}\right]$$

$$= 5\left(\frac{x^3-1}{x^3+1}\right)^4\left[\frac{(x^3-1)-2(3x^2)}{(x^3-1)^2}\right]$$

$$\boxed{g'(x) = 5\left(\frac{x^3-1}{x^3+1}\right)^4\left[\frac{x^3-6x^2-1}{(x^3-1)^2}\right]}$$

Side work (polynomial division):

$$x^3-1\,\overline{)\,x^3+1}\qquad 1+\frac{2}{x^3-1}$$
$$\underline{-\;x^3-1}$$
$$0+2$$

$$h(x) = \sin(x)\ln(x^2+1)$$

$$h'(x) = \frac{d}{dx}\left[\sin(x)\ln(x^2+1)\right]$$

$$= \sin(x)\frac{d}{dx}\left[\ln(x^2+1)\right] + \ln(x^2+1)\frac{d}{dx}\left[\sin(x)\right]$$

$$= \sin(x)\frac{d\ln(u)}{du}\left[\ln(x^2+1)\right]\frac{d}{dx}(x^2+1) + \ln(x^2+1)\cos(x)$$

$$= \sin(x)\frac{1}{x^2+1}(2x) + \ln(x^2+1)\cos(x)$$

$$\boxed{h'(x) = \frac{2x\sin(x)}{x^2+1} + \cos(x)\ln(x^2+1)}$$

$$m(x) = \ln(\sinh(x))$$

$$= \frac{d}{dx}\left[\ln(\sinh(x))\right]$$

$$= \frac{d\ln(u)}{du}\left[\ln(\sinh(x))\right]\frac{d}{dx}\left[\sinh(x)\right]$$

$$= \frac{1}{\sinh(x)}\cosh(x)$$

$$\boxed{m(x) = \frac{\cosh(x)}{\sinh(x)}\ \text{ or } \coth(x)}$$

Q2

2) Find the second derivative of

$$f(x) = e^x \cos(x)$$

$$f'(x) = e^x \cos(x)$$

$$= \frac{d}{dx}\left[e^x \cos(x)\right]$$

$$= e^x \frac{d}{dx}\left[\cos(x)\right] + \cos(x)\frac{d}{dx}\left(e^x\right)$$

$$= e^x\left(-\sin(x)\right) + \cos(x)\left(e^x\right)$$

$$= -e^x \sin(x) + e^x \cos(x)$$

$$f'(x) = e^x\left(-\sin(x) + \cos(x)\right)$$

$$f''(x) = \frac{d}{dx}\left[e^x\left(-\sin(x) + \cos(x)\right)\right]$$

$$= e^x \frac{d}{dx}\left[\left(-\sin(x) + \cos(x)\right)\right] + \left(-\sin(x) + \cos(x)\right)\frac{d}{dx}\left(e^x\right)$$

$$= e^x\left(-\cos(x) - \sin(x)\right) + \left(-\sin(x) + \cos(x)\right)e^x$$

$$= -e^x \cos(x) - e^x \sin(x) - e^x \sin(x) + e^x \cos(x)$$

$$f''(x) = \boxed{-2e^x \sin(x)}$$

Q3

3) Find $\frac{dy}{dx}$ for $y^5 + x^2 y^2 = 1 + ye^x$

$$\frac{dy}{dx}\left(y^5 + x^2 y^2\right) = \frac{dy}{dx}\left(1 + ye^x\right)$$

$$5y^4 \frac{dy}{dx} + \left[x^2 \frac{d}{dy}\left(y^2\right) + y^2 \frac{d}{dy}\left(x^2\right)\right] = 0 + \left[y\frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dy}\left(y\right)\right]$$

$$5y^4 \frac{dy}{dx} + x^2\left(2y \frac{dy}{dx}\right) + y^2\left(2x\right) = y\left(e^x\right) + e^x \frac{dy}{dx}$$

$$5y\frac{dy}{dx} + 2x^2 y \frac{dy}{dx} - e^x \frac{dy}{dx} = e^x y - 2xy^2$$

$$\frac{dy}{dx}\left(5y + 2x^2 y - e^x\right) = y\left(e^x - 2xy\right)$$

$$\frac{\frac{dy}{dx}\left(5y + 2x^2 y - e^x\right)}{\left(5y + 2x^2 y - e^x\right)} = \frac{y\left(e^x - 2xy\right)}{\left(5y + 2x^2 y - e^x\right)}$$

$$\boxed{\frac{dy}{dx} = \frac{y\left(e^x - 2xy\right)}{5y + 2x^2 y - e^x}}$$

Q4

4) Let $f(x) = \dfrac{e^x}{x^2 + 1}$, find tangent line @ $(0,1)$

$f'(x) = \dfrac{d}{dx}\left(\dfrac{e^x}{x^2+1}\right)$

$= \dfrac{(x^2+1)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$

$= \dfrac{(x^2+1)e^x - e^x(2x)}{(x^2+1)^2}$

$= \dfrac{e^x x^2 + e^x - 2x e^x}{(x^2+1)^2}$

Let $x = 0$

$\rightarrow f'(0) = \dfrac{e^x(x^2 + 1 - 2x)}{(x^2+1)^2}$

$= \dfrac{e^0(0^2 + 1 - 2(0))}{(0^2+1)^2}$

$= \dfrac{1(0 + 1 - 0)}{(0+1)^2}$

$m = \dfrac{1}{1} = 1$

$y - y_1 = m(x - x_1)$, $(0,1)$

$y - (1) = 1(x - 0)$

$y - 1 = x$

$\boxed{y = x - 1} \longrightarrow y = x + 1$

5) Use linear approximation to estimate $\sqrt{15.8}$

$\sqrt{15.8} \approx \sqrt{16}$

$f(x) = \sqrt{x}$, $a = 16$

$f'(x) = \frac{d}{dx}\left(x^{1/2}\right)$

$\qquad = \frac{1}{2}x^{-1/2}$

$L(x) = f(a) + f'(a)(x - a)$

$\qquad = f(16) + f'(16)(x - a)$

$\qquad = \sqrt{16} + \frac{1}{2}(16)^{-1/2}(x - 16)$

$\qquad = 4 + \frac{1}{2}\left(\frac{1}{\sqrt{16}}\right)(x - 16)$

$\qquad = 4 + \frac{1}{8}(x - 16)$

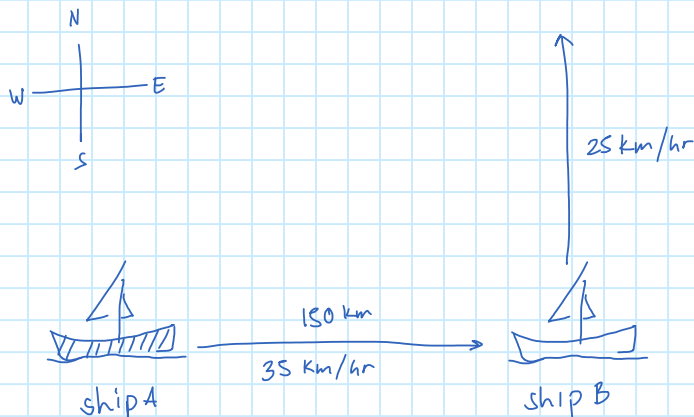$\qquad = 4 + \frac{1}{8}x - 2$

$L(x) = \frac{1}{8}x + 2$

$\sqrt{15.8} = L(15.8)$

$\qquad = \frac{1}{8}(15.8) + 2$

$\qquad = \boxed{3.975}$

Q6

6)

@ 12 00 nm

N

W ── ── E

S

25 km/hr

150 km

35 km/hr

ship A

ship B

@ 16:00 pm

ship A Distance traveled = 35 km (16 − 12)

$$= 140 \, km$$

ship B Distance traveled = 25 (16 − 12)

$$= 100 \, km$$

ship A Distance away from
initial position of ship B = 150 − 140

$$= 10 \, km$$

ship B

$$z^2 = x^2 + y^2 \longleftarrow$$

$$\frac{dz}{dt}(z^2) = \frac{dx}{dt}(x^2) + \frac{dy}{dt}(y^2)$$

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$z^2 = x^2 + y^2$$

$$z^2 = 10^2 + 100^2$$

$$= 100 + 10000$$

$$z = \sqrt{10100}$$

$$\frac{dz}{dt} = ?$$

$$z = \sqrt{10100}$$

$$y = 100 \, km$$

$$\frac{dy}{dt} = 25 \, km/hr$$

$$\frac{1}{2z}\left(2z \frac{dz}{dt}\right) = \left(2x \frac{dx}{dt} + 2y \frac{dy}{dt}\right) \frac{1}{2z}$$

$$\frac{dz}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{z}$$

Ship A

$$x = 10 km$$

$$\frac{dx}{dt} = 35 km/hr$$

$$\frac{dz}{dt} = \frac{10(35) + 100(25)}{\sqrt{10100}}$$

$$\boxed{\frac{dz}{dt} = \frac{2850}{\sqrt{10100}} \, km/hr}$$