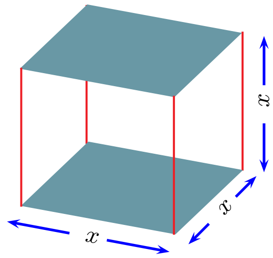
## Definition of Volume for a Cube

If one side of a cube measures $x$ units, we define the volume of such cube to be

$$V = x^3 \quad \text{cubic units}$$

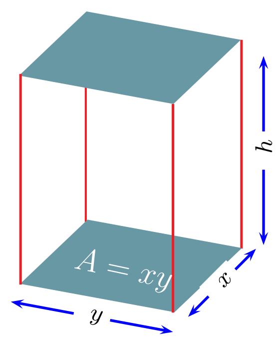



## Volume for a 'box'

We can generalize the above definition to volumes for shapes that are not necessarily cubes. Suppose, the hight or one of the sides of the above shape does not measure the same as the sides, $x$. Then the volume is given by:

If the bottom & all cross-sections are rectangles measuring $x$ units by $y$ units, and the height is $h$ units, the volume of such shape is

$$V = h \cdot xy \quad \text{cubic units}$$

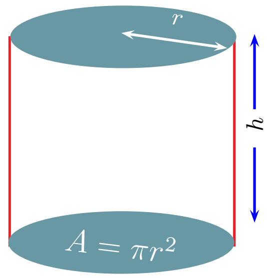



## Volume for a Cylinder

We can generalize the above definition to volumes for shapes that are not necessarily shaped like a box. Suppose, the hight is $h$ and the cross-sections are all circles. The Volume is given by:

If the bottom & all cross-sections are circles with constant radius $r$ units, and the height is $h$ units, the volume of such shape is

$$V = h \cdot \pi r^2 \quad \text{cubic units}$$

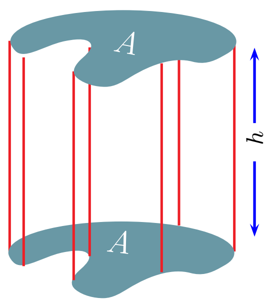



## Volume for a more general Cylinder

In fact, all formulas expressed above are just specific examples of *one* much broader principle. Generally, for shapes whose cross-sections are any finite area, so long as the cross sections contain exactly the same area, $A$, the volume is given by $V = h \cdot A$

If the bottom & all cross-sections all have a finite area, $A$ square units, and the height is $h$ units, the volume of such shape is

$$V = h \cdot A \quad \text{cubic units}$$

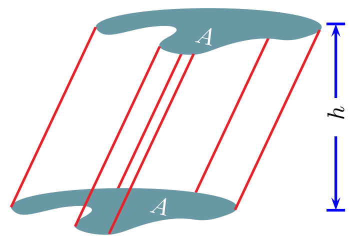



## Volume for *the* general Cylinder

In fact, *absolutely all* formulas expressed above are just specific examples of *one* even broader principle. Namely, the volume principle expressed above works even when the 'cylinder' is not upright. In other words, even if the 'cylinder' is 'slanted' the volume formula still works.

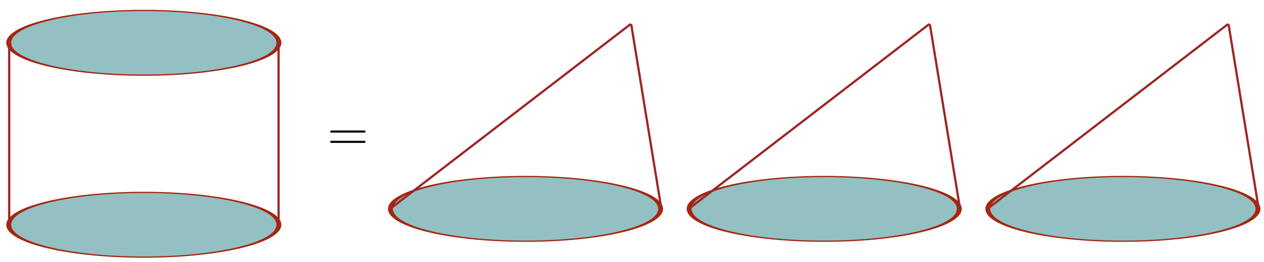
If the bottom & all cross-sections all have a finite area, $A$ square units, and the height is $h$ units, the volume of such shape is

$$V = h \cdot A \quad \text{cubic units}$$



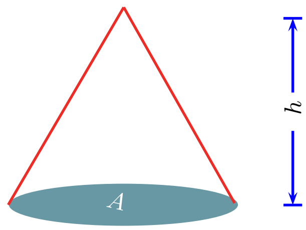

## Volume for Cone

Now we turn our attention to cones. While general cylinder has congruent [the same] area on each of the [appropriate] cross-sections, general cones have similar [same same but maybe not same size] cross-sections but they decrease in size until they reach one single point in 'size'. It is interesting that the volume for a general cone is always exactly 1/3 of the volume for the corresponding general cylinder. Moreover, just as with general cylinders, the formula for volumes of general cones follows this principle regardless of the shape of the cross-sections. In other words, the bottom [and all the cross-sections] can be a circles, squares, triangles, or just about any finite closed shape, the volume would still be 1/3 times the bottom area times the height, while the volume for the corresponding cylinder would be just the bottom area times the hight. We summarize these ideas for the usual cone and for a more generalized canonical shape.



Of course, the more interesting question here is *why?*. Why 1/3 for example, why not 1/2, or 3/5? The proof and the explanation are found in the corresponding lecture and/or in the homework exercises.

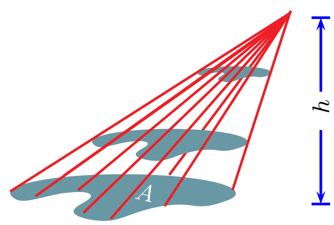
$$V = \frac{1}{3} \cdot Ah \quad \text{cubic units}$$




## Volume for *the* general Cone

Most generally,....

$$V = \frac{1}{3} \cdot Ah \quad \text{cubic units}$$

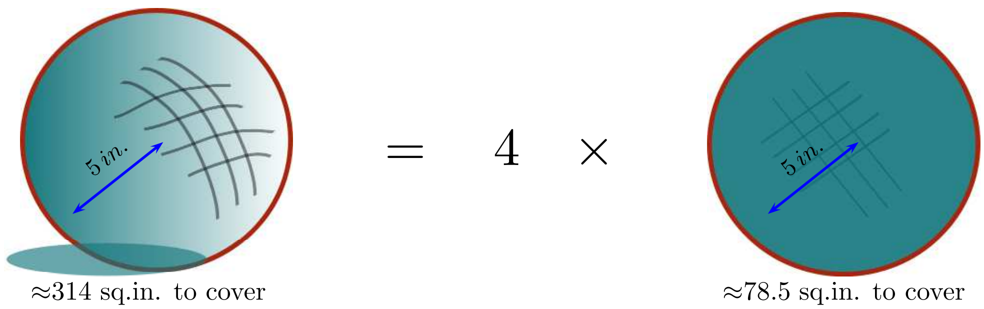



## Surface Area for a Sphere

Imagine a basketball with radius 5 inches. Now imagine covering the surface of the entire basketball with 1 square inch flexible tiles. Now imagine counting how many little 1 sq. inch tiles it would take to cover the entire basketball. The number of such square is exactly what we call the surface area of a sphere. The formula is well known. It is given by

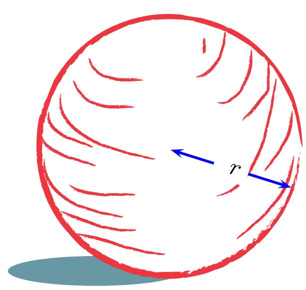
$$SA = 4\pi r^2$$

In other words, it would take $4 \cdot pi \cdot 5^2 \approx 314$ square inches to cover such a basketball. For another way to look at it, let us compare a basketball with radius 5 inches with a 5 inch radius pizza. The pizza has $\pi r^2$ square inches in area.. while the basketball was 4 times that much area, $4\pi r^2$, thus it would take the area of 4 pizzas to cover the surface area of the basketball.




Of course and as usual, any undertaker can memorize the surface area formula, $SA = 4\pi r^2$, but there is little glory in that. The real pleasure lies in understanding why this is the case. The reader is invited to set aside some quiet time for meditation and preparation to carry on the exercises which will prudently lead to a deep and profound understanding of exactly why the surface area of a sphere is $4\pi r^2$. For now we summarize:

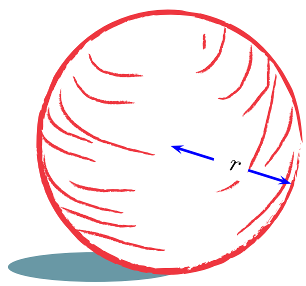
$SA = 4 \cdot \pi r^2$ square units




## Volume for a Sphere

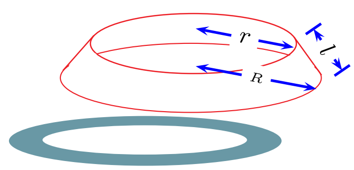
We leave the reader with two more timeless formulas, the formula for the volume of a sphere and the formula for the surface area of a frustum [link like shape]. As usual, the reader is invited to first be aware of such formulas then search long and hard as to why these are so. The exercises will lead you through these ideas.

$$V = \frac{4}{3} \cdot \pi r^3 \quad \text{cubic units}$$

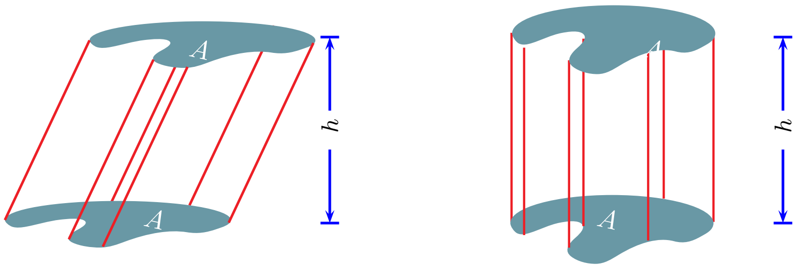



## Surface Area of a Frustum

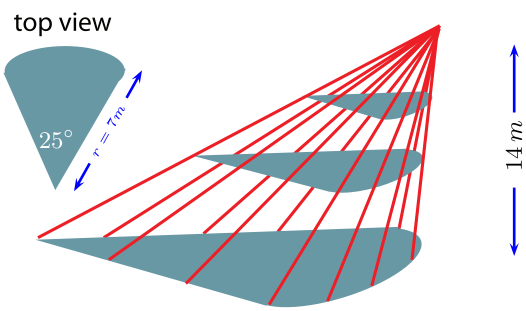
$$SA = 2\pi \cdot l \cdot \frac{r+R}{2} \quad \text{square units}$$



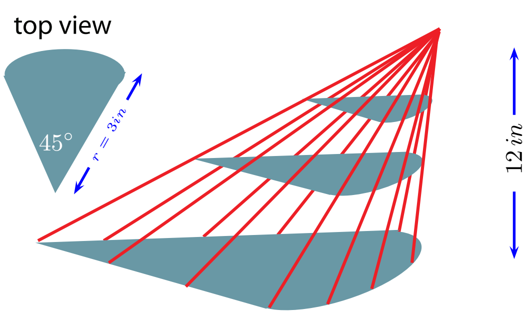
1. Explain why the following have the same volume. You may assume the cross sections are all identical in shape and size. You may assume the height is also the same.




2. Find the volume for the following general cylindrical shape. Assume the cross-sections are sectors of a circle with the bottom measuring radius $r = 7m$ and angle $\theta = 25°$

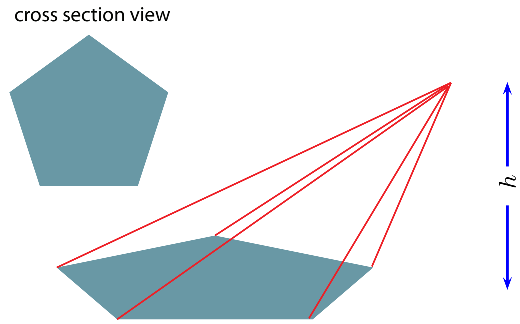



3. Find the volume for the following general cylindrical shape. Assume the cross-sections are sectors of a circle with the bottom measuring radius $r = 3in$ and angle $\theta = 45°$

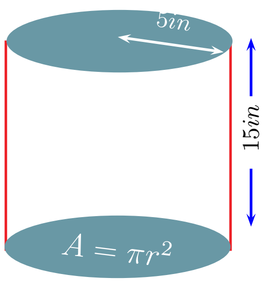



4. Assume, $A$ is the area of the bottom region. Explain why the volume of the following is given by

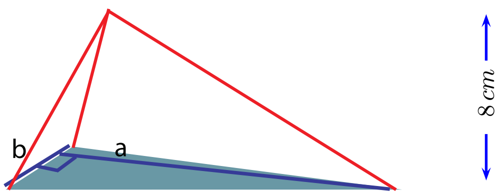
$$V = \frac{h}{3} \cdot A$$



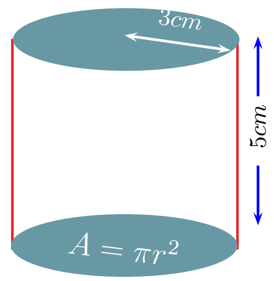

5. Find the volume for the following shape:



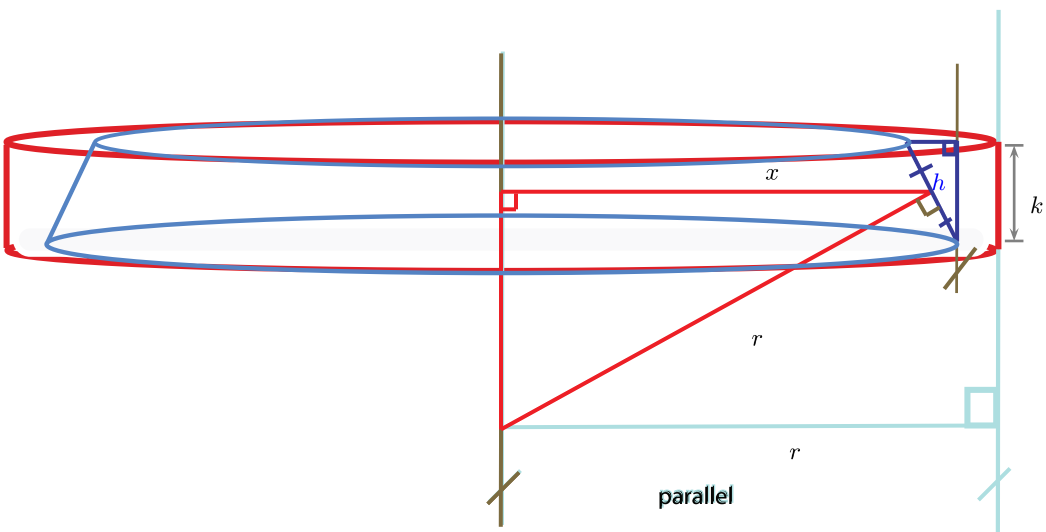

6. Find the volume for the following general cone shape: where $a = 2cm$ and $b = 15cm$




7. Find the volume for the following shape:




8. (don't even try this if you have a small soul)

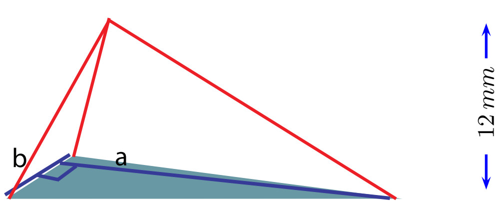



(a) Calculate the surface area of the frustum.

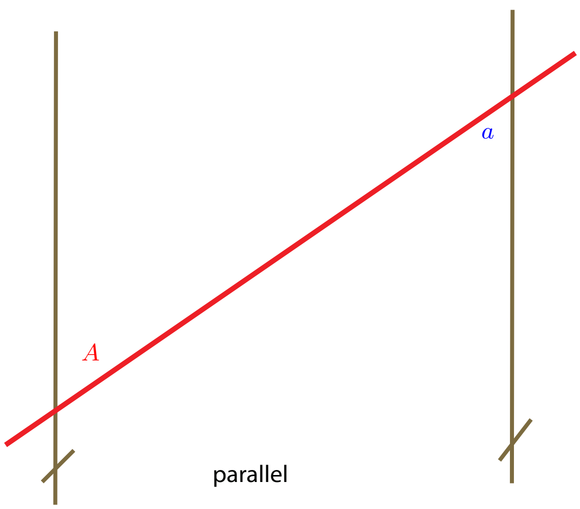
(b) Use similar triangles to show how $x, r, h$, and $k$ are related.

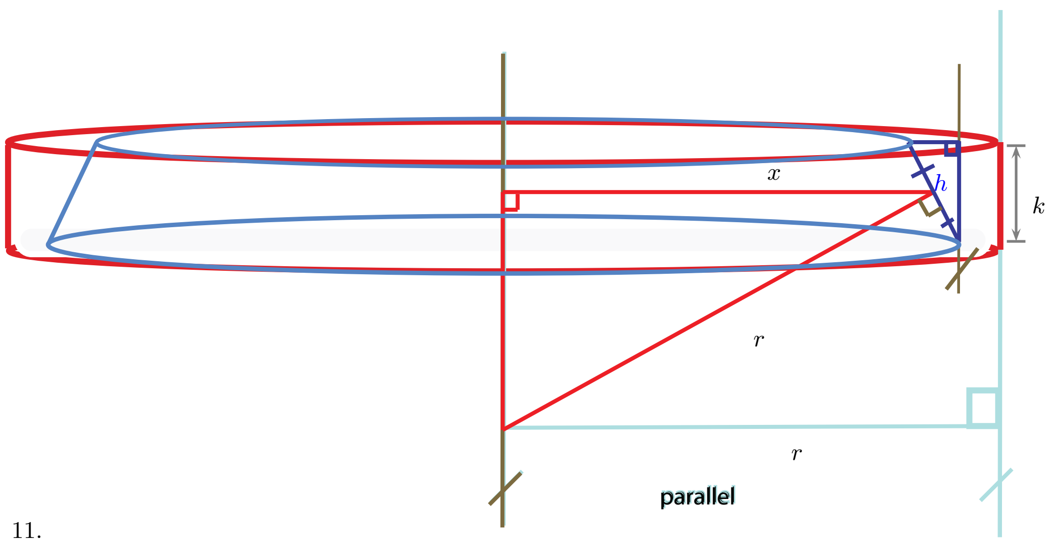
(c) Which area is larger, the cylinder or the frustum?

9. Find the volume for the following general cone shape: where $a = 3mm$ and $b = 6mm$



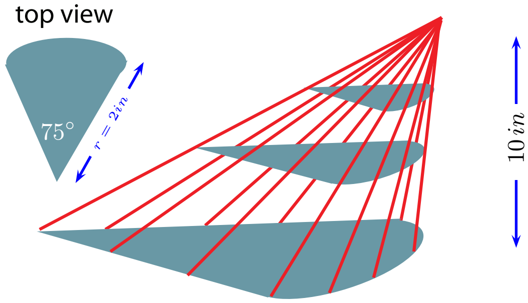
10. Explain why
angle $A$ measures the same as angle $a$

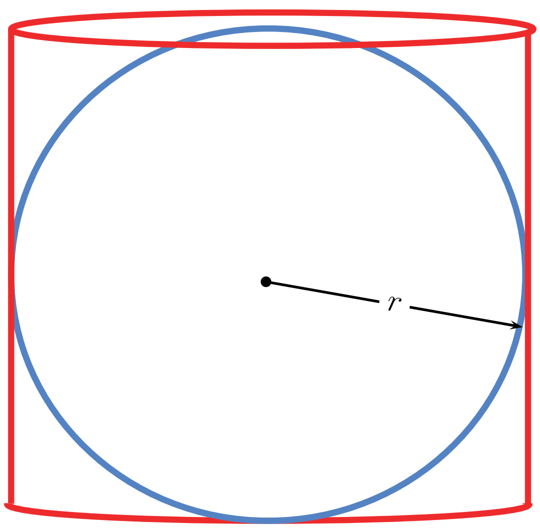



11.
   (a) Calculate the surface area of the frustum.
   (b) Calculate the surface area of the Cylinder.
   (c) Use similar triangles to show how $x, r, h$, and $k$ are related.
   (d) Which area is larger, the cylinder or the frustum?

12. Find the volume for the following general cylindrical shape. Assume the cross-sections are sectors of a circle with the bottom measuring radius $r = 2in$ and angle $\theta = 75°$



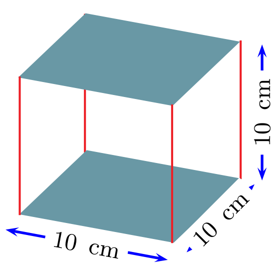

13.

(a) Which surface area is larger, the surface area for the sphere OR the surface area for the cylinder?

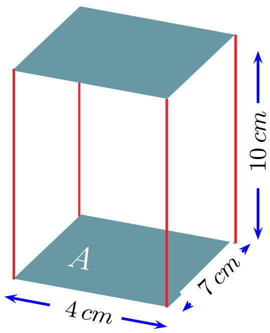
(b) Compute the surface area for the cylinder.

(c) What is the surface area for the sphere?

14. Find the volume for the following shape:

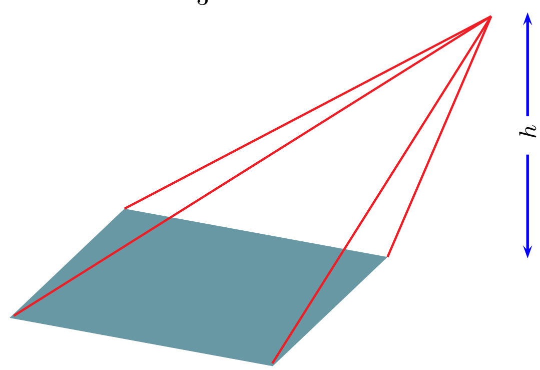


15. Find the volume for the following shape:

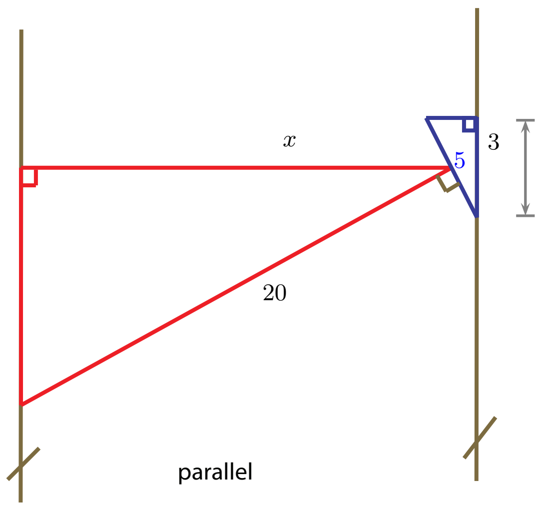



16. Assume, $A$ is the area of the bottom region. Explain why the volume of the following is given by

$$V = \frac{h}{3} \cdot A$$



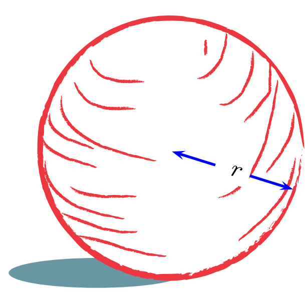

17. Use similarity to solve for $x$



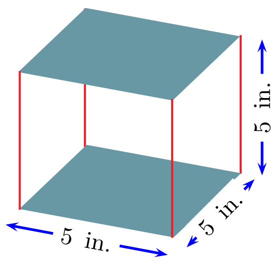

18. Explain why

$$V = \frac{4}{3} \cdot \pi r^3 \quad \text{cubic units}$$

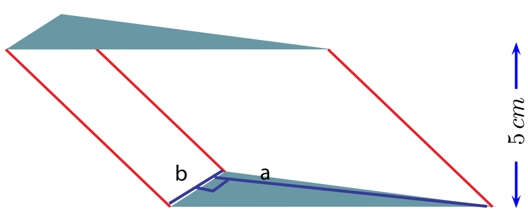
you may assume the surface area is knows as $SA = 4 \cdot \pi r^2$ square units



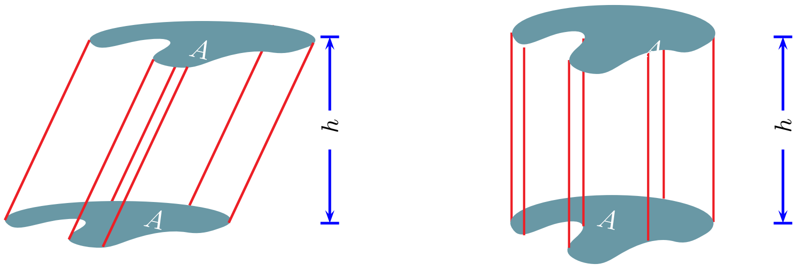
19. Find the volume for the following shape:



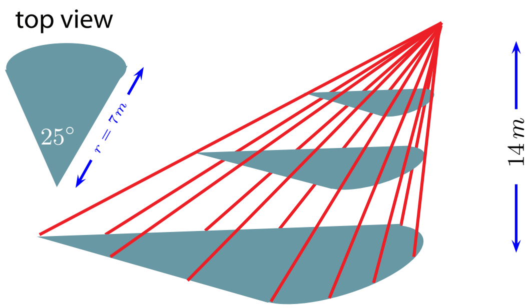

20. Find the volume for the following general cylindrical shape: where $a = 3cm$ and $b = 5cm$



1. Explain why the following have the same volume. You may assume the cross sections are all identical in shape and size. You may assume the height is also the same.




2. Find the volume for the following general cylindrical shape. Assume the cross-sections are sectors of a circle with the bottom measuring radius $r = 7m$ and angle $\theta = 25°$




**Solution:** The general cone principle applies here. The bottom & all the cross-sections are sectors. The bottom area is given [when $\theta$ is measured in degrees] by $A = \frac{\theta}{360} \cdot \pi r^2$ and height is given by $14m$, thus we use *the* general volume principle for cones, namely, height times area divided by 3:

$$V = \frac{h}{3} \cdot A$$

thus..

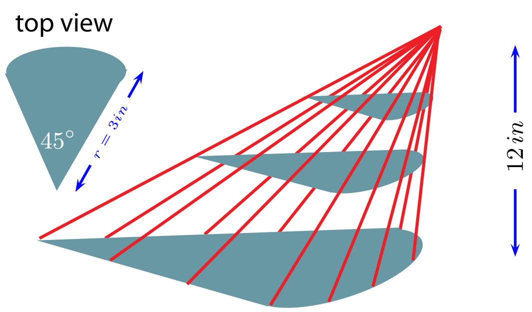
We first find the area of the bottom region:

$$\begin{aligned} A &= \frac{\theta}{360} \cdot \pi r^2 && \text{(sector formula in degrees)} \\ &= \frac{25}{360} \cdot \pi (7m)^2 && \text{(substitute)} \\ &\approx 10.69(m)^2 && \text{(approximate)} \end{aligned}$$

now we use the general principle for volume of cones:

$$\begin{aligned} V &= \frac{h}{3} \cdot A && \text{(given)} \\ &= \frac{(14m)}{3} \cdot \left[10.69(m)^2\right] && \text{(substitute)} \\ &\approx 49.89(m)^3 && \text{(BI)} \end{aligned}$$

3. Find the volume for the following general cylindrical shape. Assume the cross-sections are sectors of a circle with the bottom measuring radius $r = 3in$ and angle $\theta = 45°$




**Solution:** The general cone principle applies here. The bottom & all the cross-sections are sectors. The bottom area is given [when $\theta$ is measured in degrees] by $A = \frac{\theta}{360} \cdot \pi r^2$ and height is given by $12in$, thus we use *the* general volume principle for cones, namely, height times area divided by 3:

$$V = \frac{h}{3} \cdot A$$

thus..

We first find the area of the bottom region:

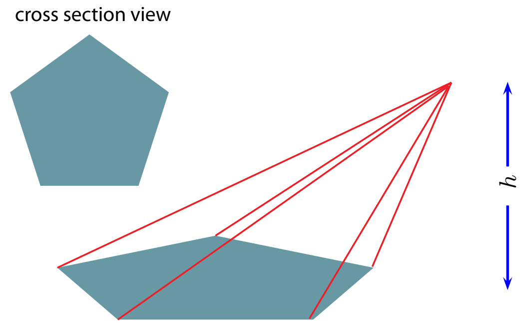
$$\begin{aligned} A &= \frac{\theta}{360} \cdot \pi r^2 && \text{(sector formula in degrees)} \\ &= \frac{45}{360} \cdot \pi(3in)^2 && \text{(substitute)} \\ &\approx 3.53(in)^2 && \text{(approximate)} \end{aligned}$$

now we use the general principle for volume of cones:

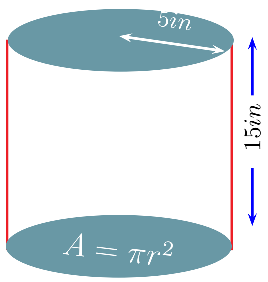
$$\begin{aligned} V &= \frac{h}{3} \cdot A && \text{(given)} \\ &= \frac{(12in)}{3} \cdot \left[3.53(in)^2\right] && \text{(substitute)} \\ &\approx 14.12(in)^3 && \text{(BI)} \end{aligned}$$

4. Assume, $A$ is the area of the bottom region. Explain why the volume of the following is given by

$$V = \frac{h}{3} \cdot A$$




5. Find the volume for the following shape:



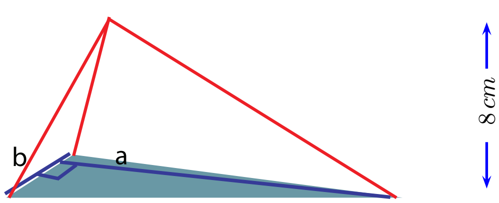

**Solution:** The general cylinder principle applies here since the bottom & all the cross-sections are circles with area $A = \pi r^2 = \pi(5in)^2 \approx 78.54(in)^2$ and with height $15in$, thus we use *the* general volume principle for cylinders, namely, height times area:

$$V = h \cdot A$$

thus..

$$\begin{aligned} V &= h \cdot A && \text{(given)} \\ &\approx (15in) \cdot \left[78.54(in)^2\right] && \text{(substitute)} \\ &\approx 1178.1(in)^3 && \text{(BI)} \end{aligned}$$

6. Find the volume for the following general cone shape: where $a = 2cm$ and $b = 15cm$



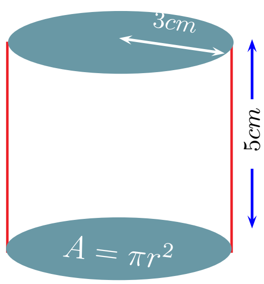
**Solution:** The general cone principle applies here. The bottom & all the cross-sections are triangles. The bottom area is given by $A = ba/2 = (2cm)(15cm)/2 = 15(cm)^2$ and height is given by $8cm$, thus we use *the* general volume principle for cones, namely, height times area divided by 3:

$$V = \frac{h}{3} \cdot A$$

thus..

$$\begin{aligned} V &= \frac{h}{3} \cdot A && \text{(given)} \\ &= \frac{(8cm)}{3} \cdot \left[15(cm)^2\right] && \text{(substitute)} \\ &\approx 40(cm)^3 && \text{(BI)} \end{aligned}$$

7. Find the volume for the following shape:



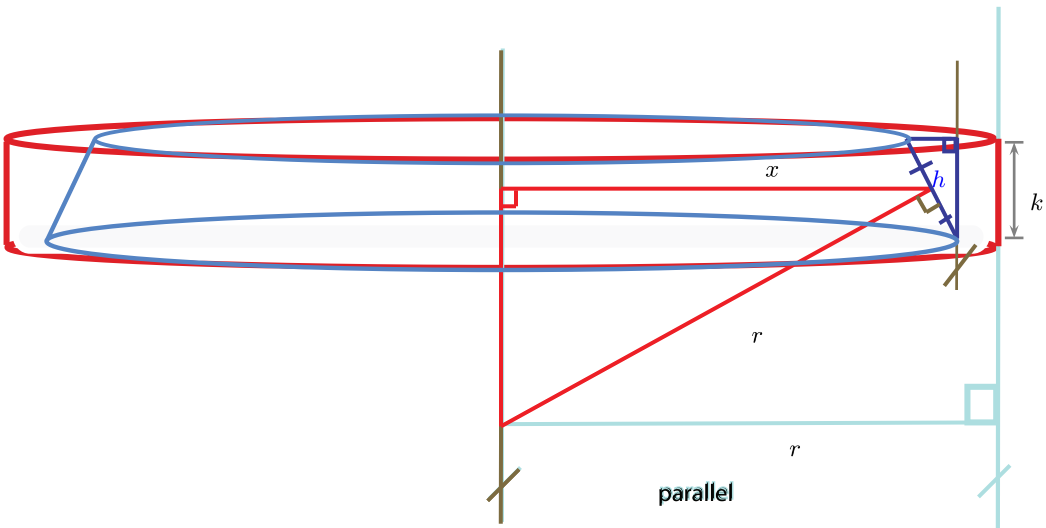

**Solution:** The general cylinder principle applies here since the bottom & all the cross-sections are circles with area $A = \pi r^2 = \pi(3cm)^2 \approx 28.27(cm)^2$ and with height $5cm$, thus we use *the* general volume principle for cylinders, namely, height times area:

$$V = h \cdot A$$

thus..

$$\begin{aligned} V &= h \cdot A && \text{(given)} \\ &\approx (5cm) \cdot \left[28.27(cm)^2\right] && \text{(substitute)} \\ &\approx 141.35(cm)^3 && \text{(BI)} \end{aligned}$$

8. (don't even try this if you have a small soul)




(a) Calculate the surface area of the frustum.

(b) Use similar triangles to show how $x, r, h$, and $k$ are related.

(c) Which area is larger, the cylinder or the frustum?

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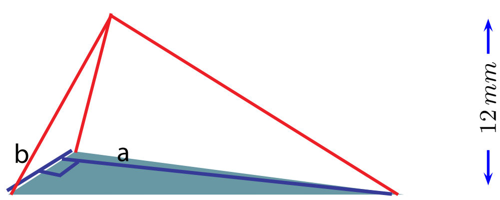
9. Find the volume for the following general cone shape: where $a = 3mm$ and $b = 6mm$




**Solution:** The general cone principle applies here. The bottom & all the cross-sections are triangles. The bottom area is given by $A = ba/2 = (3mm)(6mm)/2 = 9(mm)^2$ and height is given by $12mm$, thus we use *the* general volume principle for cones, namely, height times area divided by 3:

$$V = \frac{h}{3} \cdot A$$

thus..

$$\begin{aligned} V &= \frac{h}{3} \cdot A && \text{(given)} \\ &= \frac{(12mm)}{3} \cdot \left[9(mm)^2\right] && \text{(substitute)} \\ &\approx 36(mm)^3 && \text{(BI)} \end{aligned}$$

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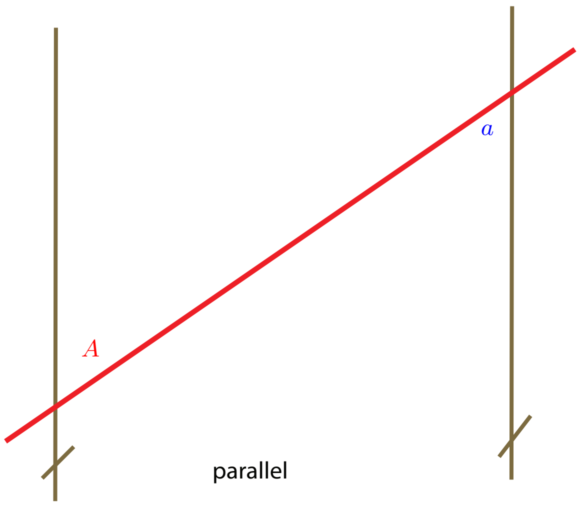
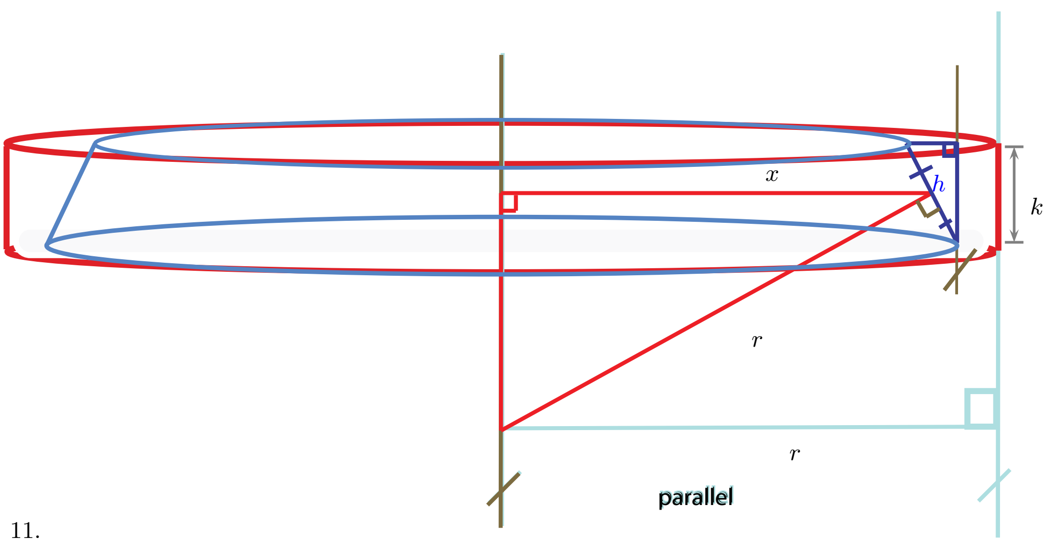
10. Explain why

angle $A$ measures the same as angle $a$




11.

(a) Calculate the surface area of the frustum.

**Solution:**

$$SA = 2\pi h \cdot x$$

(b) Calculate the surface area of the Cylinder.

**Solution:**

$$SA = 2\pi r \cdot k$$

(c) Use similar triangles to show how $x, r, h,$ and $k$ are related.

**Solution:**

$$\frac{x}{r} = \frac{k}{h}$$

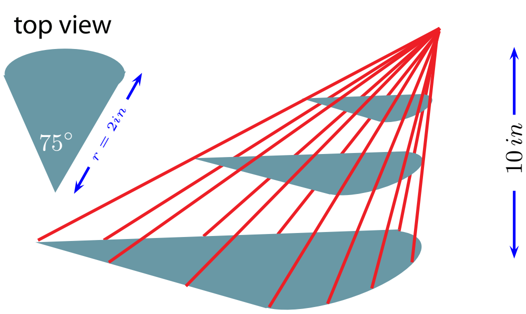
(d) Which area is larger, the cylinder or the frustum?

**Solution:**
SA for frustum is $SA = 2\pi \cdot hx$
SA for Cylinder is $SA = 2\pi \cdot rk$
BUT $\frac{x}{r} = \frac{k}{h}$ so $hx = rk$, which we substitute into the frustum area to obtain;

$$\begin{aligned} \text{SA for frustum} &= 2\pi \cdot hx && \text{(given)} \\ &= 2\pi \cdot rk && \text{(substitute, from similar triangles)} \\ &= \text{SA for cylinder} && \text{(yipi-kaey-yee)} \end{aligned}$$

Thus,
SA for frustum=SA for Cylinder

12. Find the volume for the following general cylindrical shape. Assume the cross-sections are sectors of a circle with the bottom measuring radius $r = 2in$ and angle $\theta = 75°$




**Solution:** The general cone principle applies here. The bottom & all the cross-sections are sectors. The bottom area is given [when $\theta$ is measured in degrees] by $A = \frac{\theta}{360} \cdot \pi r^2$ and height is given by $10in$, thus we use *the* general volume principle for cones, namely, height times area divided by 3:

$$V = \frac{h}{3} \cdot A$$

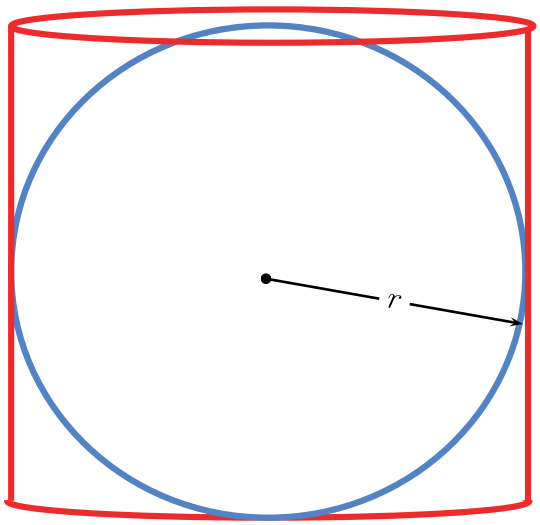
thus..

We first find the area of the bottom region:

$$\begin{aligned} A &= \frac{\theta}{360} \cdot \pi r^2 && \text{(sector formula in degrees)} \\ &= \frac{75}{360} \cdot \pi (2in)^2 && \text{(substitute)} \\ &\approx 2.62 (in)^2 && \text{(approximate)} \end{aligned}$$

now we use the general principle for volume of cones:

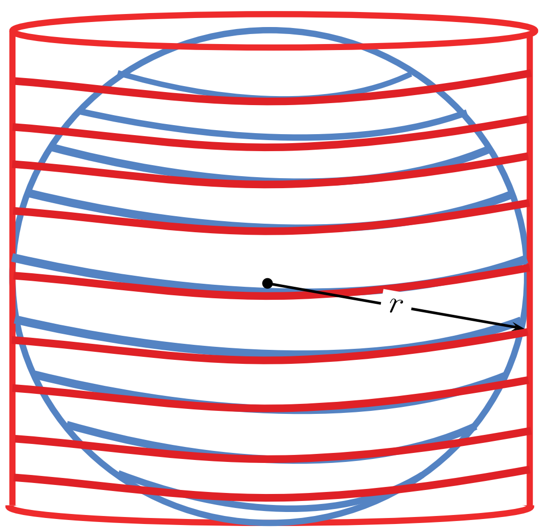
$$\begin{aligned} V &= \frac{h}{3} \cdot A && \text{(given)} \\ &= \frac{(10in)}{3} \cdot \left[2.62(in)^2\right] && \text{(substitute)} \\ &\approx 8.73(in)^3 && \text{(BI)} \end{aligned}$$




13.

(a) Which surface area is larger, the surface area for the sphere OR the surface area for the cylinder?

**Solution:** We can separate the surface area of the sphere, roughly speaking, into many consecutive and very small frustums, each of which is equal in area to the corresponding area of the pieces of the cylinder. Therefore, the two areas are equal.



(b) Compute the surface area for the cylinder.

> **Solution:** area for a cylinder is $SA = 2\pi r \cdot h$ for this cylinder $h = 2r$, thus
> $$SA = 4\pi r^2$$

(c) What is the surface area for the sphere?

> **Solution:** from above work $SA = 4\pi r^2$

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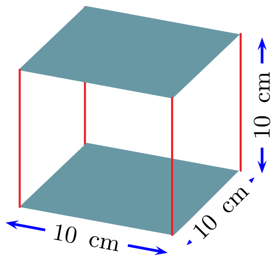
14. Find the volume for the following shape:




> **Solution:** For a cube, the volume is defined to be $V = x^3$ cubic units, thus the volume for this cube is
> $$V = (10\,\text{cm})^3 = 1000(\text{ cm})^3$$
>
> **alternative solution:**
>
> Note: the general cylinder principle applies here as well... the bottom & all the cross-sections are squares with area $10\,cm \times 10\,cm = 100(\text{ cm})^2$ and with height $10\,cm$, thus we use *the* general volume principle for cylinders, namely, height times area:
> $$V = h \cdot A$$
> thus..
> $$\begin{aligned} V &= h \cdot A && \text{(given)} \\ &= (10\,cm) \cdot (100(\text{ cm})^2) && \text{(substitute)} \\ &= 1000(\text{ cm})^3 && \text{(BI)} \end{aligned}$$
> this alternative point of view harnesses the great power of generalizing.

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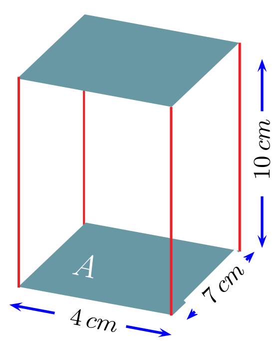
15. Find the volume for the following shape:



**Solution:** The general cylinder principle applies here since the bottom & all the cross-sections are rectangles with area $7\,cm \times 4\,cm = 28(\text{ cm})^2$ and with height $10\,cm$, thus we use *the* general volume principle for cylinders, namely, height times area:

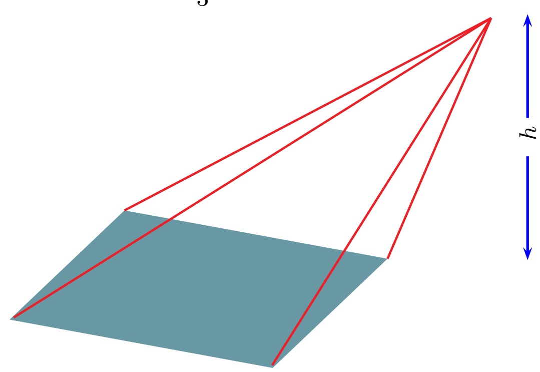
$$V = h \cdot A$$

thus..

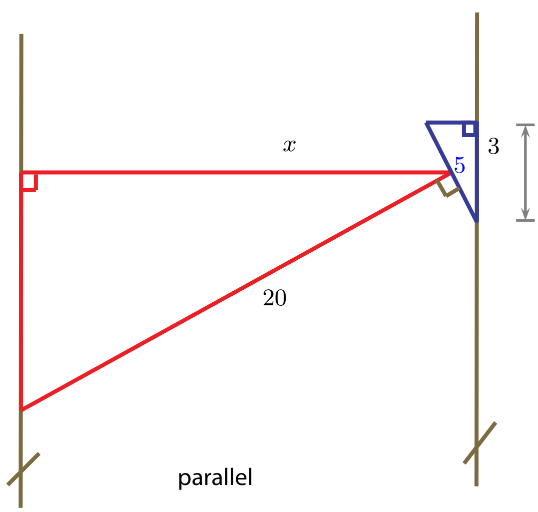
$$\begin{aligned} V &= h \cdot A && \text{(given)} \\ &= (10\,cm) \cdot (28(\text{ cm})^2) && \text{(substitute)} \\ &= 280(\text{ cm})^3 && \text{(BI)} \end{aligned}$$

16. Assume, $A$ is the area of the bottom region. Explain why the volume of the following is given by

$$V = \frac{h}{3} \cdot A$$




17. Use similarity to solve for $x$



**Solution:** using the fact that the triangles are similar;

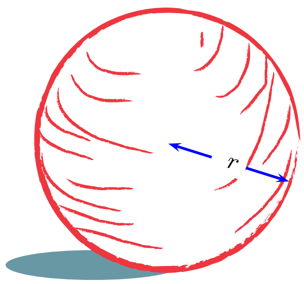
$$\frac{x}{20} = \frac{3}{5} \qquad \text{(similar triangles)}$$
$$x = \frac{60}{5} = 12 \qquad \text{(BI)}$$

---

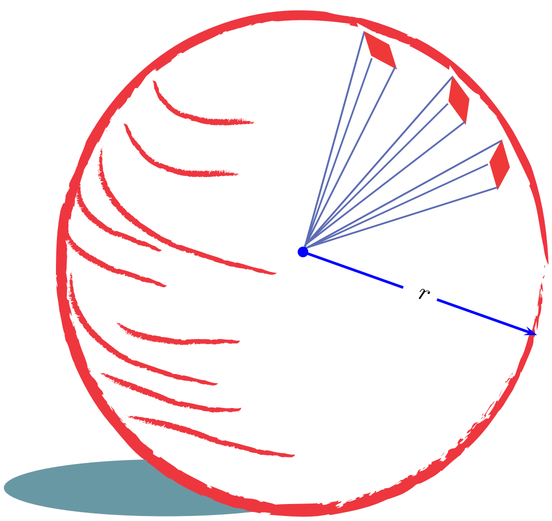
18. Explain why

$$V = \frac{4}{3} \cdot \pi r^3 \quad \text{cubic units}$$

you may assume the surface area is knows as $SA = 4 \cdot \pi r^2$ square units



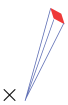

**Solution:**



We can compute the total volume, roughly speaking, by slicing up the sphere into many many small 'general cone' shapes all emitting from the enter outwards. The advantage in doing so is that we have already learned to compute volumes of 'general cone' shapes. Recall such volume is given by the area on the bottom region of the cone times the hight, times $\frac{1}{3}$. For all these cones, the height is $r$. While each individual area may be hard to compute since, it would depend on how small we make the pieces, we may be able to reason our way out of it. Another consideration is, how many pieces to use, the general principle being that the more pieces the better, since the sphere is round and we are approximating it by a 'tiled sphere' with square flat pieces [see diagram above].

Having said that we are ready to start computing. Let us consider a little warm up question, what would be the approximation of the volume if we were to use a 'tiled sphere' with 100 little cones, all of equal area? The computation will help us get ready to then slice it up into 1000 little cones, improving our approximation, then ultimately into $\infty$ little cones, turning the 'approximation' into the *exact* volume of the sphere. Note for each little cone, the $h$ is $r$ and the area is [assuming 100 equal little cones] $\frac{1}{100}$ of the total surface area of the sphere, which is $\frac{4\pi r^2}{100}$

**approximation using 100 little cones:**

$$\text{Volume of Sphere} \approx 100 \times$$ 

$$= 100 \times \left[\frac{1}{3} \cdot h \cdot \text{little Area}\right]$$

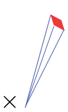
$$= 100 \times \left[\frac{1}{3} \cdot r \cdot \frac{4\pi r^2}{100}\right]$$

$$= \frac{4}{3} \cdot r \cdot \pi r^3 \qquad \text{(... therefore..)}$$

$$\text{Volume of Sphere} \approx \frac{4}{3} \cdot r \cdot \pi r^3$$

now if we used 1000 little cones, the approximation would be better, but the computation would be very similar, and so would the results:

**approximation using 1000 little cones:**

$$\text{Volume of Sphere} \approx 1000 \times$$ 

$$= 1000 \times \left[\frac{1}{3} \cdot h \cdot \text{little Area}\right]$$

$$= 1000 \times \left[\frac{1}{3} \cdot r \cdot \frac{4\pi r^2}{1000}\right]$$

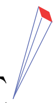
$$= \frac{4}{3} \cdot \pi r^3 \qquad \text{(... therefore..)}$$

$$\text{Volume of Sphere} \approx \frac{4}{3} \cdot r \cdot \pi r^3$$

Of course it took humans thousands of years to come to pieces with $\infty$, at lease in some level. But we did, and it can be shown rigorously that the ideas above extend to the unthinkable, remarkable, not-in-the-real-number-line scenario where we slice up the sphere into *infinite* many tiny pieces of infinitely small size, therefore computing NOT an approximation of the volume but the exact volume. Here is a sketch of the ideas.

**approximation using $\infty$ little cones:** Here we use $h_n$ and $A_n$ to denote the $n$th height and area respectively.

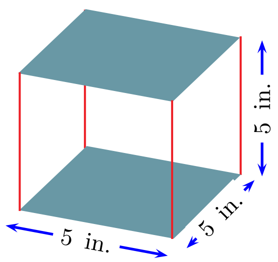
$$\text{Volume of Sphere} = \sum_{n=1}^{\infty}$$



$_n$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty}\left[\frac{1}{3}\cdot h_n \cdot \text{little Area}_n\right] \\
&= \sum_{n=1}^{\infty}\left[\frac{1}{3}\cdot r \cdot A_n\right] \\
&= \frac{r}{3}\sum_{n=1}^{\infty} A_n && \text{(pull constants)} \\
&= \frac{r}{3}\left[4\pi r^2\right] && \text{(sum of little areas is total SA of sphere)} \\
&= \frac{4}{3}\pi r^3 && \text{(BI.... therefore...)} \\
\text{Volume of Sphere} &= \frac{4}{3}\pi r^3 && \text{(yipe-kahey-yee)}
\end{aligned}
$$

19. Find the volume for the following shape:




**Solution:** For a cube, the volume is defined to be $V = x^3$ cubic units, thus the volume for this cube is

$$V = (5\,\text{in.})^3 = 125(\text{ in.})^3$$

**alternative solution:**

Note: the general cylinder principle applies here as well... the bottom & all the cross-sections are squares with area $5\,in. \times 5\,in. = 25(\text{ in.})^2$ and with height $5\,in.$, thus we use *the* general volume principle for cylinders, namely, height times area:

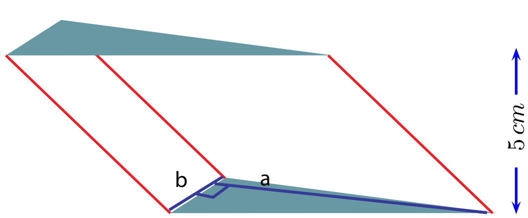
$$V = h \cdot A$$

thus..

$$\begin{aligned} V &= h \cdot A && \text{(given)} \\ &= (5\,in.) \cdot (25(\text{ in.})^2) && \text{(substitute)} \\ &= 125(\text{ in.})^3 && \text{(BI)} \end{aligned}$$

this alternative point of view harnesses the great power of generalizing.

20. Find the volume for the following general cylindrical shape: where $a = 3cm$ and $b = 5cm$




**Solution:** The general cylinder principle applies here since the bottom & all the cross-sections are triangles with area $A = ba/2 = (3cm)(5cm)/2 = 7.5(cm)^2$ and with height $5cm$, thus we use *the* general volume principle for cylinders, namely, height times area:

$$V = h \cdot A$$

thus..

$$
\begin{aligned}
V &= h \cdot A && \text{(given)} \\
&= (5cm) \cdot \left[7.5(cm)^2\right] && \text{(substitute)} \\
&= 37.5(cm)^3 && \text{(BI)}
\end{aligned}
$$