Q1

Tuesday, October 13, 2020 9:15 PM

Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem. (Enter your answers as a comma-separated list.)

$f(x) = 2x^2 - 4x + 9, \quad [-1, 3]$

$c =$ 1

Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$. ✓
2. f is differentiable on the open interval (a, b). ✓
3. $f(a) = f(b)$ ✓

Because the function is Polynomial

Then there is a number c in (a, b) such that $f'(c) = 0$.

$f(x) = 2x^2 - 4x + 9, \quad [-1, 3]$

$f(-1) = 2(-1)^2 - 4(-1) + 9$

$= 2 + 4 + 9$

$= 15$

$f(3) = 2(3)^2 - 4(3) + 9$

$= 18 - 12 + 9$

$= 15$

$f'(c) = 0$

$f'(x) = \frac{d}{dx}\left(2x^2 - 4x + 9\right)$

$= (2)2x - 4 + 0$

$f'(x) = 4x - 4$

$4x - 4 = 0$

$4x = 4$

$x = \boxed{1 = c}$

Q2

Tuesday, October 13, 2020 9:33 PM

Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem. (Enter your answers as a comma-separated list.)

$$f(x) = \sin\left(\frac{x}{2}\right), \quad \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

$c =$ π

Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$. ✓
2. f is differentiable on the open interval (a, b). ✓
3. $f(a) = f(b)$ ✓

Then there is a number c in (a, b) such that $f'(c) = 0$.

$$f(x) = \sin\left(\frac{x}{2}\right), \quad \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

$$f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\frac{\pi}{2}}{2}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f\left(\frac{3\pi}{2}\right) = \sin\left(\frac{\frac{3\pi}{2}}{2}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(c) = 0$$

$$f'(x) = \frac{d}{dx}\left[\sin\left(\frac{x}{2}\right)\right] = \cos\left(\frac{x}{2}\right)\frac{d}{dx}\left[\frac{x}{2}\right] = \cos\left(\frac{x}{2}\right)\frac{1}{2}$$

$$f'(x) = \frac{\cos\left(\frac{x}{2}\right)}{2}$$

$$(\cancel{2})\frac{\cos\left(\frac{x}{2}\right)}{\cancel{2}} = 0 \ (2)$$

$$\cos\left(\frac{x}{2}\right) = 0$$

$$\cos\left(\frac{\pi}{2}\right) = 0 \ \& \ \cos\left(\frac{3\pi}{2}\right) = 0$$

$$\frac{x}{2} = \frac{\pi}{2} + 2\pi n \qquad \frac{x}{2} = \frac{3\pi}{2} + 2\pi n$$

$$(2)\frac{x}{\cancel{2}} = \frac{\pi}{2} + 2\pi n \ (2) \qquad (2)\frac{x}{\cancel{2}} = \frac{3\pi}{2} + 2\pi n \ (2)$$

$$\underbrace{x = \pi + 4\pi n \qquad x = 3\pi + 4\pi n}_{\text{General solution}}$$

but $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$\therefore x = \pi$ Only $\rightarrow$ $\boxed{c = \pi}$

Q3

Wednesday, October 14, 2020 6:10 PM

Consider the following function.

$f(x) = 1 - x^{2/3}$

Find $f(-1)$ and $f(1)$.

$f(-1) =$ 0

$f(1) =$ 0

Find all values c in $(-1, 1)$ such that $f'(c) = 0$. (Enter your answers as a comma-separated list. If an answer does not exist, enter DNE.)

$c =$ DNE

Based off of this information, what conclusions can be made about Rolle's Theorem?

- ○ This contradicts Rolle's Theorem, since f is differentiable, $f(-1) = f(1)$, and $f'(c) = 0$ exists, but c is not in $(-1, 1)$.
- ○ This does not contradict Rolle's Theorem, since $f'(0) = 0$, and 0 is in the interval $(-1, 1)$.
- ○ This contradicts Rolle's Theorem, since $f(-1) = f(1)$, there should exist a number c in $(-1, 1)$ such that $f'(c) = 0$.
- ◉ This does not contradict Rolle's Theorem, since $f'(0)$ does not exist, and so f is not differentiable on $(-1, 1)$.
- ○ Nothing can be concluded.

$$f(x) = 1 - x^{2/3}$$

$$f(-1) = 1 - (-1)^{2/3}$$
$$= 1 - (\sqrt[3]{-1})^2$$
$$= 1 - (-1)^2$$
$$= 1 - 1$$
$$\boxed{f(-1) = 0}$$

$$f(1) = 1 - (1)^{2/3}$$
$$= 1 - (\sqrt[3]{1})^2$$
$$= 1 - (1)^2$$
$$= 1 - 1$$
$$\boxed{f(1) = 0}$$

$$f(x) = 1 - x^{2/3} \qquad f'(c) = 0$$

$$f'(x) = \frac{d}{dx}\left(1 - x^{2/3}\right)$$
$$f'(x) = \frac{2}{3}x^{-1/3}$$

$$\frac{2}{3}x^{-1/3} = 0$$
$$\frac{2}{3}\left(\frac{1}{\sqrt[3]{x}}\right) = 0$$
$$\frac{2}{3\sqrt[3]{x}} = 0$$

x cannot equals to zero unless the numerator is zero

$\therefore$ $\boxed{c = \text{Does not exist}}$

Q4

Wednesday, October 14, 2020 8:43 PM

Does the function satisfy the hypotheses of the Mean Value Theorem on the given interval?

$f(x) = 2x^2 - 5x + 1, \quad [0, 2]$

- ○ Yes, it does not matter if f is continuous or differentiable, every function satifies the Mean Value Theorem.
- ◉ Yes, f is continuous on [0, 2] and differentiable on (0, 2) since polynomials are continuous and differentiable on $\mathbb{R}$.
- ○ No, f is not continuous on [0, 2].
- ○ No, f is continuous on [0, 2] but not differentiable on (0, 2).
- ○ There is not enough information to verify if this function satifies the Mean Value Theorem.

If it satisfies the hypotheses, find all numbers c that satisfy the conclusion of the Mean Value Theorem. (Enter your answers as a comma-separated list. If it does not satisify the hypotheses, enter DNE).

$c =$ 1

$f(x) = 2x^2 - 5x + 1, \ [0, 2]$

$f'(x) = \frac{d}{dx}(2x^2 - 5x + 1)$

$= (2)2x - 5 + 0$

$f'(x) = 4x - 5 \longrightarrow f'(c) = 4c - 5$

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$f(0) = 2x^2 - 5x + 1$

$= 2(0)^2 - 5(0) + 1$

$= 1$

$f'(c) = \dfrac{f(2) - f(0)}{b - a}$

$= \dfrac{-1 - 1}{2 - 0}$

$= \dfrac{-2}{2}$

$= -1$

$f(2) = 2x^2 - 5x + 1$

$= 2(2)^2 - 5(2) + 1$

$= 8 - 10 + 1$

$= -1$

$4c - 5 = -1$

$4c = -1 + 5$

$4c = 4$

$c = \frac{4}{4}$

$\boxed{c = 1}$ on $(0, 2)$ domain

Q5

Wednesday, October 14, 2020 9:00 PM

Let $f(x) = (x-3)^{-2}$. Find all values of c in $(1, 7)$ such that $f(7) - f(1) = f'(c)(7-1)$. (Enter your answers as a comma-separated list. If an answer does not exist, enter DNE.)

$c =$ DNE ✓

Based off of this information, what conclusions can be made about the Mean Value Theorem?

- ○ This contradicts the Mean Value Theorem since f satisfies the hypotheses on the given interval but there does not exist any c on $(1, 7)$ such that $f'(c) = \frac{f(7) - f(1)}{7 - 1}$.
- ◉ This does not contradict the Mean Value Theorem since f is not continuous at $x = 3$.
- ○ This does not contradict the Mean Value Theorem since f is continuous on $(1, 7)$, and there exists a c on $(1, 7)$ such that $f'(c) = \frac{f(7) - f(1)}{7 - 1}$.
- ○ This contradicts the Mean Value Theorem since there exists a c on $(1, 7)$ such that $f'(c) = \frac{f(7) - f(1)}{7 - 1}$, but f is not continuous at $x = 3$.
- ○ Nothing can be concluded.

✓

$f(x) = (x-3)^{-2}$, $(1, 7)$, $f(7) - f(1) = f'(c)(7-1) \longrightarrow$ This is actualy $\frac{f(7) - f(1)}{7-1} = f'(c)$ which is also $\frac{f(b) - f(a)}{b-a} = f'(c)$

$$f(1) = (x-3)^{-2} = \frac{1}{(1-3)^2} = \frac{1}{(-2)^2} = \frac{1}{4}$$

$$f(7) = (x-3)^{-2} = \frac{1}{(7-3)^2} = \frac{1}{(4)^2} = \frac{1}{16}$$

$$f(7) - f(1) = f'(c)(7-1)$$

$$\frac{1}{16} - \frac{1}{4} = -2(c-3)^{-3}(7-1)$$

$$\frac{-3}{16} = -\frac{2}{(c-3)^3}(6)$$

$$\frac{-3}{16} = -\frac{12}{(c-3)^3}$$

$$-3(c-3)^3 = -12(16)$$

$$\frac{-3(c-3)^3}{-3} = \frac{-192}{-3}$$

$$(c-3)^3 = 64$$

$$\sqrt[3]{(c-3)^3} = \sqrt[3]{64}$$

$$c - 3 = 4$$

$$c = 4 + 3$$

$c = 7$ on $(1, 7)$ Domain

$$7 \notin (1, 7)$$

$$\therefore \boxed{c = DNE}$$

$$f(x) = (x-3)^{-2}$$

$$f'(x) = \frac{d}{dx}\left[(x-3)^{-2}\right]$$

$$= -2(x-3)^{-3}\frac{d}{dx}(x-3)$$

$$f'(x) = -2(x-3)^{-3}$$

$$f'(c) = -2(c-3)^{-3}$$

Q6

Wednesday, October 14, 2020 9:56 PM

If $f(4) = 3$ and $f'(x) \geq 3$ for $4 \leq x \leq 7$, how small can $f(7)$ possibly be?

12

$f(4) = 3$, $f'(x) \geq 3$ for $4 \leq x \leq 7$

By Mean Value Theorem
there exist c in $[4, 7]$
such that

$f(7) - f(4) = f'(c)(7-4)$, $f'(c) \geq 3$ on $[4,7]$

$f(7) - 3 = 3(7-4)$

$f(7) - 3 = 9$

$f(7) = 9 + 3$

$\boxed{f(7) = 12}$

The smallest possible value of $f(7)$ is 12.

Q7

Wednesday, October 14, 2020 10:16 PM

Does there exist a function f such that $f(0) = -8$, $f(2) = 9$, and $f'(x) \le 6$ for all x?

- Yes
- **No** ✓

$f(0) = -8$, $f(2) = 9$, and $f'(x) \le 6$ for all x?

$(0, 2)$

By the Mean Value Theorem there exist c in $(0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$$= \frac{9 - (-8)}{2 - 0}$$

$$f'(c) = \frac{17}{2}$$

$$\frac{17}{2} > 6$$

$\therefore$ $\boxed{\text{No such function exist}}$

Q8

Wednesday, October 14, 2020 10:22 PM

Let $f(x) = \frac{1}{x}$ and

$$g(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ 2 + \frac{1}{x} & \text{if } x < 0 \end{cases}$$

Show that $f'(x) = g'(x)$ for all x in their domains. Can we conclude from the corollary below that $f - g$ is constant?

If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

For $x > 0$, $f(x) = g(x)$, so $f'(x) = g'(x)$. For $x < 0$, $f'(x) =$ $-x^{-2}$ ✓ and $g'(x) =$ $-x^{-2}$ ✓, so $f'(x)$ [=] ✓ $g'(x)$. However, the domain of $g(x)$ is not an interval [it is $(-\infty, 0) \cup (0, \infty)$] so we cannot conclude that $f - g$ is constant (in fact it is not).

$f(x) = \frac{1}{x}$ and $g(x) \begin{cases} \frac{1}{x} & \text{if } x > 0 \longrightarrow \\ 2 + \frac{1}{x} & \text{if } x < 0 \longrightarrow \end{cases}$ $\boxed{f(x) = g(x) \therefore f'(x) = g'(x) \quad f'(x) = g'(x)}$

$f'(x) = \frac{d}{dx}\left(\frac{1}{x}\right)$

$= \frac{d}{dx}\left(x^{-1}\right)$

$\boxed{f'(x) = -x^{-2}}$ $=$ $\boxed{g'(x) = -x^{-2}}$

$g'(x) = \frac{d}{dx}\left(2 + \frac{1}{x}\right)$

$= 0 + \frac{d}{dx}\left(x^{-1}\right)$

Collary 7

If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is $f(x) = g(x) + c$, where c is a constant.

If $f(x)$ and $g(x) \longrightarrow f' = g'$, then same shape but may be shifted

$f(x)$ Domain $\mathbb{R}$

$g(x)$ $(-\infty, 0) \cup (0, \infty)$

so we cannot conclude that $f - g$ is constant