

Chapter 4 (draft): Data and Pattern Matching

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Part I

Introduction

clean up!

The encoded data presented in Chapter 2 is unrealistically inconvenient and is especially implausible for a dependently typed programming language intended to be easy to use.

User defined data is an essential feature of a realistic programming language. Simple data types like `Nat` and `Bool` are essential for organizing readable programs. Dependent data like `Id` can represent the mathematical predicate of equality. Dependent data can also be used to preserve invariants like the length in `Vec`.

We have opted for data definitions like those found in systems like Agda and Coq. A data definition is formed by a type constructor indexed by value arguments, and a set of constructors that tag data and characterize their arguments. See Figure 1 for the definitions of several standard data types. Data is easy to build and reason about, since data can only be created from its constructors. Unfortunately data elimination is more murky.

How should data be used? One option is a direct eliminator scheme, like Coq uses in its core language. It is worth formalizing that example.

Part II

Data and elimination

A minimal accounting of data can be given by extending the surface language syntax in ...

differentiate identifiers with font

motive should not need to insist on the type info of the binder?

Grey out things that are surface syntax but not needed for theory

Make identifiers consistent with chapter 2, and locations in chapter 3

Make a module syntax?

The slightly awkward eliminator syntax is designed to be forward compatible with the pattern matching system defined in the rest of this section. Mutual patterns. Recursion does the work of induction in the MLTT style

abbreviate away dumb arrows, unstated separator is a space, also usual syntax `(:*)`? also shorthands for telescopes

relevant motives!

ebnf? if r
underline
pat or p?
alternativ

```

data Bool : * {
| True  : Bool
| False : Bool
};

data Nat : * {
| Z : Nat
| S : Nat -> Nat
};
-- Syntactic sugar expands decimal numbers
-- into their unary representation.

data Vec : (A : *) -> Nat -> * {
| Nil  : (A : *) -> Vec A Z
| Cons : (A : *) -> A -> (x : Nat)
        -> Vec A x -> Vec A (S x)
};

data Id : (A : *) -> A -> A -> * {
| refl : (A : *) -> (a : A) -> Id A a a
};

```

Figure 1: Definitions of Common Data Types

telescope,		
Δ, Θ	$::=$	empty telescope
	$ \quad x : M, \Delta$	extend telescope
list of O , separated with s		
$\overline{sO}, \overline{Os}$	$::=$	empty list
	$ \quad sO\overline{sO}$	extend list
data type identifier,		
D	$::=$...
data constructor identifier,		
d	$::=$...
contexts,		
Γ	$::=$...
	$ \quad \Gamma, \text{data } D : \Delta \rightarrow * \left\{ \overline{ d : \Theta \rightarrow D\overline{m} } \right\}$	data def
	$ \quad \Gamma, \text{data } D : \Delta \rightarrow *$	abstract data
m, n, M, N	$::=$...
	$ \quad D\overline{m}$	type cons.
	$ \quad d\overline{m}$	data cons.
	$ \quad \text{case } \overline{N}, m \left\{ \overline{ pat \Rightarrow m } \right\}$	data elim. without motive
	$ \quad \text{case } \overline{N}, m \langle \overline{x \Rightarrow y : D \overline{x} \Rightarrow M} \rangle \left\{ \overline{ pat \Rightarrow m } \right\}$	data elim. with motive
(minimal) patterns,		
pat	$::=$	$\Rightarrow \overline{x} \Rightarrow (d\overline{y})$
pat	$::=$	$x \Rightarrow pat$
	$ \quad (d\overline{x})$	match a variable
		match a constructor
values,		
v	$::=$...
	$ \quad D\overline{v}$	
	$ \quad d\overline{v}$	

Constructors build application, drop this with applications

Figure 2: Surface Language Data

- 1 Incomplete Eliminations
- 2 (non) Strict Positivity
- 3 Specification

$$\frac{\overline{\Gamma \vdash \cdot} \dots}{\Gamma \vdash M : \star \quad \Gamma, x : M \vdash \Delta \dots} \frac{}{\Gamma \vdash x : M, \Delta} \dots$$

suspect this also hinges on regularity

$$\frac{\Gamma \vdash}{\Gamma \vdash \Diamond : \cdot} \dots$$

$$\frac{\Gamma, x : M \vdash \Delta \quad \Gamma \vdash m : M \quad \Gamma \vdash \bar{n} [x := m] : \Delta [x := m]}{\Gamma \vdash m, \bar{n} : x : M, \Delta} \dots$$

$$\frac{\text{data } D \Delta \in \Gamma}{\Gamma \vdash D : \Delta \rightarrow \star} \dots$$

$$\frac{d : \Theta \rightarrow D\bar{m} \in \Gamma}{\Gamma \vdash d : \Theta \rightarrow D\bar{m}} \dots$$

define these \in

$$\frac{\begin{array}{c} \text{data } D \Delta \in \Gamma \\ \Gamma \vdash \bar{N} : \Delta \quad \Gamma \vdash n : D\bar{N} \\ \Gamma, \bar{x} : \Delta, z : D\bar{x} \vdash M : \star \\ \forall d : \Theta \rightarrow D\bar{m} \in \Gamma. \quad \Gamma, \bar{x} : \Delta, \bar{y}_d : \Theta \vdash m_d : M \end{array}}{\Gamma \vdash \text{case } \bar{N}, n \left\{ \bar{x} \Rightarrow (d \bar{y}_d) \Rightarrow m_d \right\} : M [\bar{x} := \bar{N}, z := n]} \dots$$

don't need $\Gamma \vdash \bar{N} : \Delta$?

$$\frac{\begin{array}{c} \text{data } D \Delta \in \Gamma \\ \Gamma \vdash \bar{N} : \Delta \quad \Gamma \vdash n : D\bar{N} \\ \Gamma, \bar{x} : \Delta, z : D\bar{x} \vdash M : \star \\ \forall d : \Theta \rightarrow D\bar{m} \in \Gamma. \quad \Gamma, \bar{x} : \Delta, \bar{y}_d : \Theta \vdash m_d : M \end{array}}{\Gamma \vdash \text{case } \bar{N}, n \left\{ \bar{x} \Rightarrow z : D\bar{x} \Rightarrow M \right\} \left\{ \bar{x} \Rightarrow (d \bar{y}_d) \Rightarrow m_d \right\} : M [\bar{x} := \bar{N}, z := n]} \dots$$

may not need scrut wf check? oh but what about the empty types!

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \text{data } D \Delta} \dots$$

$$\frac{\Gamma \vdash \text{data } D \Delta \quad \forall d. \Gamma, \text{data } D \Delta \vdash \Theta_d \quad \forall d. \Gamma, \text{data } D \Delta, \Theta_d \vdash \bar{m}_d : \Delta}{\Gamma \vdash \text{data } D : \Delta \left\{ \bar{d} : \Theta_d \rightarrow D\bar{m}_d \right\}} \dots$$

to ensure regularity

$$\frac{\Gamma \vdash \text{data } D \Delta}{\Gamma, \text{data } D \Delta \vdash} \dots$$

$$\frac{\Gamma \vdash \text{data } D : \Delta \left\{ \overline{d : \Theta \rightarrow D\overline{m}} \right\}}{\Gamma, \text{data } D : \Delta \left\{ \overline{d : \Theta \rightarrow D\overline{m}} \right\} \vdash \dots}$$

red

$$\frac{\begin{array}{c} \overline{N} \Rightarrow \overline{N'} \quad \overline{m} \Rightarrow \overline{m'} \\ \forall \overline{x} \Rightarrow (d \overline{y}_d) \Rightarrow m_d \in \left\{ \overline{|\Rightarrow \overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} . m_d \Rightarrow m'_d \\ m_d \Rightarrow m'_d \end{array}}{\text{case } \overline{N}, d\overline{m} \langle \dots \rangle \left\{ \overline{|\Rightarrow \overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \Rightarrow \text{case } \overline{N'}, d\overline{m'} \left\{ \overline{|\Rightarrow \overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m'_{d'}} \right\}} \Rightarrow \text{-case} \langle \rangle \text{-red}$$

it's actually kind of fine discriminating between non converting motives?

$$\frac{\begin{array}{c} \overline{N} \Rightarrow \overline{N'} \quad \overline{m} \Rightarrow \overline{m'} \\ \exists \overline{x} \Rightarrow (d \overline{y}_d) \Rightarrow m_d \in \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \\ m_d \Rightarrow m'_d \end{array}}{\text{case } \overline{N}, d\overline{m} \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \Rightarrow m'_d [\overline{x} := \overline{N'}, \overline{y}_d := \overline{m'}]} \Rightarrow \text{-case-red}$$

structural reductions

$$\frac{\begin{array}{c} \overline{N} \Rightarrow \overline{N'} \quad m \Rightarrow m' \\ M \Rightarrow M' \\ \forall d. \Rightarrow \overline{x} \Rightarrow (d \overline{y}_d) \Rightarrow m_d \in \left\{ \overline{|\Rightarrow \overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \\ m_d \Rightarrow m'_d \end{array}}{\begin{array}{c} \text{case } \overline{N}, m \langle \overline{x} \Rightarrow z : D \overline{x} \Rightarrow M \rangle \left\{ \overline{|\Rightarrow \overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \Rightarrow \\ \text{case } \overline{N}, m' \langle \overline{x} \Rightarrow z : D \overline{x} \Rightarrow M' \rangle \left\{ \overline{|\Rightarrow \overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \end{array}} \Rightarrow \text{-case} \langle \rangle$$

$$\frac{\begin{array}{c} \overline{N} \Rightarrow \overline{N'} \quad m \Rightarrow m' \\ M \Rightarrow M' \\ \forall d. \Rightarrow \overline{x} \Rightarrow (d \overline{y}_d) \Rightarrow m_d \in \left\{ \overline{|\Rightarrow \overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \\ m_d \Rightarrow m'_d \end{array}}{\begin{array}{c} \text{case } \overline{N}, m \left\{ \overline{|\Rightarrow \overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \Rightarrow \\ \text{case } \overline{N}, m' \left\{ \overline{|\Rightarrow \overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \end{array}} \Rightarrow \text{-case} \langle \rangle$$

$$\frac{\overline{m} \Rightarrow \overline{m'}}{D\overline{m} \Rightarrow D\overline{m'}} \dots$$

$$\frac{\overline{m} \Rightarrow \overline{m'}}{d\overline{m} \Rightarrow d\overline{m'}} \dots$$

extend reductions over lists

cbv

$$\frac{\begin{array}{c} \text{case } \overline{N}, n \langle \dots \rangle \left\{ \overline{|\Rightarrow \overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \rightsquigarrow \text{case } \overline{N}, n \left\{ \overline{|\Rightarrow \overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \dots \\ \exists \overline{x} \Rightarrow (d \overline{y}_d) \Rightarrow m_d \in \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \end{array}}{\text{case } \overline{V}, d\overline{v} \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \rightsquigarrow m_d [\overline{x} := \overline{V}, \overline{y}_d := \overline{v}]} \Rightarrow \text{-case-red}$$

$$\frac{\overline{n} \rightsquigarrow \overline{n'}}{\text{case } \overline{V}, d\overline{n} \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\} \rightsquigarrow \text{case } \overline{V}, d\overline{n'} \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}} \right\}}$$

$$\frac{\overline{N} \rightsquigarrow \overline{N'}}{\text{case } \overline{N}, d\overline{n} \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}|} \right\} \rightsquigarrow \text{case } \overline{N'}, d\overline{n} \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}|} \right\}}$$

structural reductions

$$\frac{\begin{array}{c} \overline{N} \Rightarrow \overline{N'} \quad m \Rightarrow m' \\ M \Rightarrow M' \\ \forall d. \overline{x} \Rightarrow (d \overline{y}_d) \Rightarrow m_d \in \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}|} \right\} \\ m_d \Rightarrow m'_d \end{array}}{\begin{array}{c} \text{case } \overline{N}, m \langle \overline{x} \Rightarrow z : D \overline{x} \Rightarrow M \rangle \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}|} \right\} \Rightarrow \\ \text{case } \overline{N}, m' \langle \overline{x} \Rightarrow z : D \overline{x} \Rightarrow M' \rangle \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}|} \right\} \end{array}} \Rightarrow \text{-case} <> \\ \frac{\begin{array}{c} \overline{N} \Rightarrow \overline{N'} \quad m \Rightarrow m' \\ M \Rightarrow M' \\ \forall d. \overline{x} \Rightarrow (d \overline{y}_d) \Rightarrow m_d \in \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}|} \right\} \\ m_d \Rightarrow m'_d \end{array}}{\begin{array}{c} \text{case } \overline{N}, m \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}|} \right\} \Rightarrow \\ \text{case } \overline{N}, m' \left\{ \overline{|\overline{x} \Rightarrow (d' \overline{y}_{d'}) \Rightarrow m_{d'}|} \right\} \end{array}} \Rightarrow \text{-case} <> \\ \frac{\overline{m} \rightsquigarrow \overline{m'}}{D\overline{m} \rightsquigarrow D\overline{m'}} \dots$$

what about D? how much of a value should it be?

$$\frac{\overline{m} \rightsquigarrow \overline{m'}}{D\overline{m} \rightsquigarrow D\overline{m'}} \dots$$

extend step over lists

4 Bidirectional extension

a convenient bidirectional interpretation

$$\frac{\text{data } D \Delta \in \Gamma}{\Gamma \vdash D \overrightarrow{\cdot} \Delta \rightarrow * \dots}$$

$$\frac{d : \Theta \rightarrow D\overline{m} \in \Gamma}{\Gamma \vdash d \overrightarrow{\cdot} \Theta \rightarrow D\overline{m} \dots}$$

bidirectional non dependent elimination

$$\frac{\begin{array}{c} \text{data } D \Delta \in \Gamma \\ \Gamma \vdash n \overrightarrow{\cdot} D\overline{N} \\ \Gamma, \overline{x} : \Delta, z : D \overline{x} \vdash M \overleftarrow{\cdot} \star \\ \forall d : \Theta \rightarrow D\overline{m} \in \Gamma. \quad \Gamma, \overline{x} : \Delta, \overline{y}_d : \Theta \vdash m_d \overleftarrow{\cdot} M \end{array}}{\Gamma \vdash \text{case } n \left\{ \overline{|\overline{y}_d \Rightarrow m_d|} \right\} \overleftarrow{\cdot} M} \dots$$

bidirectional dependent elimination

$$\frac{\begin{array}{c} \text{data } D \Delta \in \Gamma \\ \Gamma \vdash \overline{N} \overleftarrow{\cdot} \Delta \quad \Gamma \vdash n \overleftarrow{\cdot} D\overline{N} \\ \Gamma, \overline{x} : \Delta, z : D \overline{x} \vdash M \overleftarrow{\cdot} \star \\ \forall d : \Theta \rightarrow D\overline{m} \in \Gamma. \quad \Gamma, \overline{x} : \Delta, \overline{y}_d : \Theta \vdash m_d \overleftarrow{\cdot} M \end{array}}{\Gamma \vdash \text{case } \overline{N}, n \langle \overline{x} \Rightarrow z : D \overline{x} \Rightarrow M \rangle \left\{ \overline{|\overline{x} \Rightarrow (d \overline{y}_d) \Rightarrow m_d|} \right\} \overleftarrow{\cdot} M} \dots$$

$$: M [\overline{x} := \overline{N}, z := n]$$

```

-- eliminator style
head' : (A : *) -> (n : Nat) ->
  Vec A (S n) ->
  A ;
head' A n v =
  case A, (S n), v <
  A' => n' => _ : Vec A' n' =>
    case n' < _ => * > {
      | (Z _) => Unit
      | (S _) => A'
    }
  >{
  | _ => (Z)      => (Nil _      ) => tt
  | _ => (S _) => (Cons _ a _ _) => a
  } ;

-- pattern match style
head : (A : *) -> (n : Nat) ->
  Vec A (S n) ->
  A ;
head A n v =
  case v < _ => A > {
    | (Cons _ a _ _) => a
  } ;

```

clean when I get motive inference working

syntax highlighting would be bomb

Figure 3: Eliminators vs. Pattern Matching

may not need scrut wf check? oh but what about the empty types!

$$\frac{\Gamma \vdash \Delta \overleftarrow{wf}}{\Gamma \vdash \text{data } D \Delta \overleftarrow{wf} \dots}$$

abuse of notation...

ok? $\Gamma \vdash \Delta \overleftarrow{\vdash} \star$, perhaps $\Gamma \vdash \Delta \overleftarrow{wf}$ and $\Gamma \vdash \Delta \overleftarrow{wf}$. or $\Gamma \vdash \Delta \text{ok}$...or $\Gamma \vdash \Delta \vdash \dots$ or $\Gamma \text{context}$, $\Delta \text{telescope}$ as in [CD18]

abuse of notation...

$$\frac{\Gamma \vdash \text{data } D \Delta \quad \forall d. \Gamma, \text{data } D \Delta \vdash \Theta_d \quad \forall d. \Gamma, \text{data } D \Delta, \Theta_d \vdash \overline{m}_d : \Delta}{\Gamma \vdash \text{data } D : \Delta \left\{ \overline{d} : \Theta_d \rightarrow D \overline{m}_d \right\}} \dots$$

Part III

Pattern Matching

Unfortunately, eliminators are cumbersome for programmers to deal with directly. For instance, in figure 3 we show how `Vec` data can be directly eliminated in the definition of `head'`. The `head'` function needs to redirect impossible inputs to a dummy type and requires several copies of the same variable that cannot be identified automatically. Pattern matching is much more ergonomic than a direct eliminator, where variables will be assigned their definitions as needed, and unreachable branches can be omitted from code. For this reason, pattern matching has been considered an “essential” feature for dependently typed languages since [Coq92] and is implemented in Agda and the user facing language of Coq.

$m...$	$::=$	\dots	
		$\text{case } \bar{n}, \left\{ \overline{ \text{pat} \Rightarrow m } \right\}$	data elim. without motive
		$\text{case } \bar{n}, \langle \bar{x} \Rightarrow M \rangle \left\{ \overline{ \text{pat} \Rightarrow m } \right\}$	data elim. with motive
(minimal) patterns,			
pat	$::=$	x	match a variable
		$(d \text{ pat})$	match a constructor

Figure 4: Surface Language Data

Figure 4 shows the extensions to the surface language for data and pattern matching. The syntax of data constructors and data type constructors is standard. Our case eliminators match a tuple of expressions, allowing us to be very precise about the typing of branches.

While pattern matching is an extremely practical feature, theoretical accounts tend to be messy. To implement standard pattern matching, a unification procedure is needed to resolve the equational constraints that arise. There are many different strategies to handle these equational constraints. Several options are explored in [CD18]. Worse, the constraints are undecidable in general, since arbitrary computation can be embedded in the type of a constructor.

there is a lot of jenckyness about unification in general, but I think the additional points lose focus?

Parenttheses are used to distinguish between matching a single variable and a constructor that takes no arguments
TAS pattern matching

$$\begin{array}{c}
 \Gamma \vdash \bar{n} : \Delta' \\
 \Gamma, \bar{x} : \Delta' \vdash M : \star \\
 \forall i. (\forall \Gamma', \sigma, \bar{n}'. \Gamma, \bar{x} : \Delta' \sim \sigma \Gamma' \wedge \Gamma' \vdash \overline{\text{pat}_i} \sim_{\sigma'} \bar{n}' : \sigma \Delta' \supset \sigma' \Gamma' \vdash \sigma' m_i : \sigma' \sigma M) \\
 \forall \Gamma', \sigma, \bar{n}'. \Gamma \sim \sigma \Gamma' \wedge \Gamma' \vdash \bar{n}' : \sigma \Delta' \supset \exists i. \sigma' \Gamma' \vdash \overline{\text{pat}_i} \sim_{\sigma'} \bar{n}' : \sigma \Delta'
 \end{array}
 \begin{array}{l}
 (\text{scrutinees type check}) \\
 (\text{motive exists and is well formed}) \\
 (\text{every branch is well typed over all possible}) \\
 (\text{every input is handled})
 \end{array}$$

$$\Gamma \vdash \text{case } \bar{n}, \left\{ \overline{|\text{pat} \Rightarrow_i m_i|} \right\} : M [\bar{x} := \bar{n}]$$

last condition is incorrect, since a loop will type but not match

should delta be well formed?

last condition is optional if you're willing to modify type soundness to allow pattern match errors (they are no worse then the non-termination already allowed, and much better behaved). allows for repeated patterns.

matching

$$\begin{array}{c}
 \overline{x \sim_{\{x:=m\}} m} \dots \\
 \frac{\overline{\text{pat} \sim_{\sigma} \bar{m}}}{d \text{ pat} \sim_{\sigma} d \bar{m}} \dots \\
 \frac{\text{pat}' \sim_{\sigma} n \quad \overline{\text{pat} \sim_{\sigma'} \bar{m}}}{\text{pat}', \text{pat} \sim_{\sigma \cup \sigma'} n, \bar{m}} \dots \\
 \overline{\cdot \sim_{\emptyset} \cdot} \dots
 \end{array}$$

bidirectional
aproxomations
 $\Gamma \vdash \overline{\text{pat}} :_E ? \Delta$

$$\begin{array}{c}
 \overline{\Gamma \vdash x :_{\{x:M\}} M} \dots \\
 \frac{d : \Theta \rightarrow D \bar{n} \in \Gamma \quad \Gamma \vdash \bar{m} :_c ? \Theta}{\Gamma \vdash d \bar{m} :_{\{M \sim D \bar{n} [\Theta := \bar{m}]\} \cup c} ? M} \dots \\
 \overline{\Gamma \vdash \cdot :_{\emptyset} ?} \dots
 \end{array}$$

$$\frac{d : \Theta \rightarrow D\bar{n} \in \Gamma \quad \Gamma \vdash \bar{m} :_c ?\Theta}{\Gamma \vdash d\bar{m} : \{M \sim D\bar{n}[\Theta := \bar{m}]\} \cup_c ?M} \dots$$

abuse of notation

unification

$U(pat = M, \sigma +)$

not quite a sub propely a ctx thing

$$\begin{aligned} & \overline{U(\emptyset, \emptyset)} \dots \\ & \frac{U(E, a) \quad m \equiv m'}{U(\{m \sim m'\} \cup E, a)} \dots \\ & \frac{U(E[x := m], a[x := m])}{U(\{x \sim m\} \cup E, \{a, x := m\})} \dots \\ & \frac{U(E[x := m], a[x := m])}{U(\{m \sim x\} \cup E, a \cup \{x := m\})} \dots \\ & \frac{U(\bar{m} \sim \bar{m}' \cup E, a) \quad n \equiv d\bar{m} \quad n' \equiv d\bar{m}'}{U(\{n \sim n'\} \cup E, a)} \dots \\ & \frac{U(\bar{m} \sim \bar{m}' \cup E, a) \quad N \equiv D\bar{m} \quad N' \equiv D\bar{m}'}{U(\{N \sim N'\} \cup E, a)} \dots \end{aligned}$$

there is no inherent reason to limit vars to those that appear in the pattern
non termination of unification (cyclic)

$$\begin{aligned} & \frac{\begin{array}{c} \Gamma \vdash \bar{n} : \overrightarrow{\Delta?} \Delta \\ \Gamma, \Delta \vdash M \stackrel{\leftarrow}{\star} \\ \forall i \ (\Gamma \vdash \overline{pat}_i :_E ?\Delta \quad U(E, \sigma) \quad \sigma(\Gamma, |\overline{pat}_i|) \vdash \sigma m \stackrel{\leftarrow}{\star} \sigma(M[\Delta := \overline{pat}_i])) \\ \Gamma \vdash \overline{pat} : \Delta \text{ complete} \end{array}}{\Gamma \vdash \text{case } \bar{n}, \langle \Delta? \Rightarrow M \rangle \left\{ \overline{pat} \Rightarrow m \right\} \stackrel{\rightarrow}{\rightarrow} M[\Delta? := \bar{n}]} \dots \\ & \frac{\begin{array}{c} \Gamma \vdash \bar{n} \stackrel{\rightarrow}{\rightarrow} \Delta \\ \forall i \ (\Gamma \vdash \overline{pat}_i :_E ?\Delta \quad U(E, \sigma) \quad \sigma(\Gamma, |\overline{pat}_i|) \vdash \sigma m \stackrel{\leftarrow}{\star} \sigma(M)) \\ \Gamma \vdash \overline{pat} : \Delta \text{ complete} \end{array}}{\Gamma \vdash \text{case } \bar{n}, \left\{ \overline{pat} \Rightarrow m \right\} \stackrel{\leftarrow}{\star} M} \dots \end{aligned}$$

This complicates the simple story from chapter 2, where the bidirectional system made the TAS system tractable by only adding annotations (and having annotatability). It seems plausible that the bidirectional rules here can be considered an extension over the TAS rules outlines above, but the exact formulation would probably be cumbersome involving non-trivial transformations of case expressions, along with an appropriate equivalence associating those equivalences.

Part IV

Cast Language Data

Surprisingly, the cast language can be extended with a pattern matching construct without unification. This becomes clear when considering the normal forms of data terms. While in standard dependent type theory a normal form of data must have the data constructor in head position (justifying the pattern syntax), in the cast language the normal form of data will have a stack of casts applied to it. Casts will be wrapped around constructors during the

elaboration procedure, and will accumulate during evaluation. If the cast language is extended with a path variable that can represent the stack of equalities then that stack can be matched and used in the body of the pattern. Since the type constructor is known, it is possible to check the coverage of the patterns. If every constructor is accounted for, only blameable data remains. Quantifying over casts allows blame to be redirected, so if the program gets stuck in a pattern branch it can blame the malformed input. We conjecture that this extension preserves cast soundness.

to stylize consistently, should use math font, or like a nice image

It makes sense to further extend the cast syntax so that blame can be invoked directly by a path. For example, in 5. Branches set up in this way prove their “unreachability” by using their input to generate blame. Once direct blame is added it makes sense to allow operations on paths, such as concatenation, and reverse so that more proofs of inequality are possible. Proofs of inequality can be further helped by inspecting the arguments of type constructors and data constructors. See 6 for the extended path syntax.

Extending the syntax in this way complicates the type system and the reduction rules. We include a sample of our reduction rules in

fig

. We conjecture that these extensions support all of our expected properties.

technically speaking, telescopes should generalize to the different syntactic classes

~ is a bit of a hack

old style red rules

$$\overline{rev (pp') \rightsquigarrow (rev p') (rev p)}$$

$$\overline{inTC_i (pp') \rightsquigarrow (inTC_i p') (inTC_i p)}$$

$$\overline{inC_i (pp') \rightsquigarrow (inC_i p') (inC_i p)}$$

$$\overline{inTC_i refl \rightsquigarrow refl}$$

$$\overline{inC_i refl \rightsquigarrow refl}$$

$$\frac{\bar{a}_i = a' \ \bar{c}_i = c' \ \bar{b}_i = b'}{\overline{inTC_i (D \bar{a}_{\ell.D} \bar{c} \bar{b}) \rightsquigarrow a'_{\ell.c'} b'}}$$

$$\overline{inC_i ((a :: A)_{\ell.c} b) \rightsquigarrow inC_i (a_{\ell.c} b)}$$

$$\overline{inC_i (a_{\ell.c} (b :: B)) \rightsquigarrow inC_i (a_{\ell.c} b)}$$

$$\overline{inC_i (a_{\ell.(c::C)} b) \rightsquigarrow inC_i (a_{\ell.c} b)}$$

$$\frac{\bar{a}_i = a' \ \bar{c}_i = c' \ \bar{b}_i = b'}{\overline{inTC_i (d \bar{a}_{\ell.d} \bar{c} \bar{b}) \rightsquigarrow a'_{\ell.c'} b'}}$$

$$\overline{a ::_{A,p,refl,x,C} B \rightsquigarrow a ::_{A,p,x,C} B}$$

$$\frac{a ::_{A,p} A'_{\ell.C''} B',x.C \rightsquigarrow}{a ::_{A,p,x,C} C [x := A'] ::_{\ell.C[x:=C'']} C [x := B']}^c$$

c?

$$\overline{(a ::_{A,p,x,C} C) \sim_{\ell_o} b \rightsquigarrow a \sim_{\ell_o} b}$$

```

-- standard data in normal form, 3
S (S (S 0))

-- cast data in normal form
S (S (S 0) :: Nat ) :: Nat :: Nat :: Nat
S (S (S 0) :: Nat ) :: Bool :: Nat
True :: Nat

-- cast pattern matching
case x <_ => Bool> {
| (Z :: _) => True
| (S (Z :: _) :: _) => True
| (S (S :: _) :: _) => False
}

-- extract specific blame,
-- c is a path from Bool~Nat
case x <_ => Nat> {
| (S ((true::c)::_) :: _) =>
  add (false :: c) 2
}

-- can reconstitute any term,
-- not always possible with unification
-- based pattern matching
case x <_:Nat => Nat> {
| (Z :: c) => Z :: c
| (S x :: c) => S x :: c
}

-- direct blame
case x <_ => Nat> {
| (S (true::c) :: _) => Bool /=c Nat
}

peek x =
case x <_: Id Nat 0 1 => Nat> {
| (refl x :: _) => x
}

peek (refl 4 :: Id Nat 0 1) = 4

```

Figure 5: Cast Pattern Matching

path var,	
x_p	
path exp.,	
$p, p' ::=$	$refl$
	$A_{\ell.C} B$
	concrete cast
	x_p
	pp'
	$rev p$
	$inTC_i p$
	$inC_i p$
cast pattern,	
$patc ::=$	$x \mid d \overline{pat} ::_{x_p}$
cast expression,	
$a... ::=$	\dots
	$D \bar{a}$
	type cons.
	$d \bar{a}$
	data cons.
	$case \bar{n}, \langle \overline{\Delta} \Rightarrow M \rangle \{ \overline{patc} \Rightarrow n \}$
	data elim.
	$A \neq_p B$
	force blame
	$a ::_{A, \ell.C} B$
	concrete cast
	$a ::_{A, p, x.C} B$
	symbolic cast
	$a \sim_{\ell_o} b$
observations,	
$o ::=$	\dots
	$o.App[a]$
	application
	$o.TCon[i]$
	type cons. arg.
	$o.DCon[i]$
	data cons. arg.

Figure 6: Cast Language Data

$$\overline{a \sim_{\ell_o} (b ::_{B, p, x.C} C) \rightsquigarrow a \sim_{\ell_o} b}$$

Part V

Elaborating Eliminations

To make the overall system behave as expected we do not want to expose users to equality patterns, or force them to manually do the blame bookkeeping. To work around this we extend a standard unification algorithm to cast patterns with instrumentation to remember paths that were required for the solution. Then if pattern matching is satisfiable, compile additional casts into the branch based on its assignments. Unlisted patterns can be checked to confirm they are unsatisfiable. If the pattern is unsatisfiable then elaboration can use the proof of unsatisfiability to construct explicit blame. If an unlisted pattern cannot be proven “unreachable” then we could warn the user, and like most functional programming languages, blame the incomplete match if that pattern ever occurs.

"can", "could", weasel word until implemented

add rules of unification? the rules as implemented are standard, needing extended normalization is weird

We found that our normalization procedure (Figure ??) unexpectedly blocked unification. This is because the normalization is conservative about throwing away casts that could later lead to errors. For instance, with normalization, unification will get stuck on terms like $x ::_{\mathbb{N}} \mathbb{N} = 5$. Thus we employ a more optimistic normalization procedure that will collapse casts that are equivalent. Thus $x ::_{\mathbb{N}} \mathbb{N} \rightsquigarrow x$ allows us to solve the unification problem and assign $x := 5$. This approach has worked well so far.

In order to make the additional casts generated from unification we further extend the cast construct so it supports congruence. For instance, if x has type $Vec y z$ and the unification procedure established $y := Bool$ and $z := 5$ then we should cast $x :: Vec Bool 5$. Unlike in Section 3 this requires that we use evidence to inject a cast

the exact
unprincip
But all of
tern matc
ciple and
actually c
tion wher
an issue, l
needs to b

within a type. This can be done inherently by adding an "untyped" component to the cast annotation. The syntax that achieves this can be seen in 6.

Part VI

Related work

5 Systems with Data

Minimal data with Sigma and Unit

ML W types

UTT[Luo90, Luo94]

I am unaware of any clear, complete account of CIC in English. A bidirectional account of CIC is given in [LB21], though it uses a different style of bidirectionality then discussed here to maintain compatibility with the existing Galina grammar.

6 Pattern matching

Early work by Coq92 [Coq92]

with a lot of follow up from McBride [MM04]

reiterated in [Nor07]

A tutorial implementation of dynamic pattern unification Adam Gundry and Conor McBride (2012)
<http://adam.gundry.co.uk/pub/pattern-unify/> (this links give you the choice to read a more detailed chapter of Adam Gundry's thesis instead)

with substantial follow up in [CD18]

<https://research.chalmers.se/en/publication/519011> ?

https://sozeau.gitlabpages.inria.fr/www/research/publications/Equations:_A_Dependent_Pattern-Matching_Compiler.pdf ?

<http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.99.1405&rep=rep1&type=pdf> ?

talk about normalization

References

- [CD18] Jesper Cockx and Dominique Devriese. Proof-relevant unification: Dependent pattern matching with only the axioms of your type theory. *Journal of Functional Programming*, 28:e12, 2018.
- [Coq92] Thierry Coquand. Pattern matching with dependent types. In *Proceedings of the Workshop on Types for Proofs and Programs*, pages 71–83. Citeseer, 1992.
- [LB21] Meven Lennon-Bertrand. Complete Bidirectional Typing for the Calculus of Inductive Constructions. In Liron Cohen and Cezary Kaliszyk, editors, *12th International Conference on Interactive Theorem Proving (ITP 2021)*, volume 193 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 24:1–24:19, Dagstuhl, Germany, 2021. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.
- [Luo90] Zhaohui Luo. *An Extended Calculus of Constructions*. PhD thesis, University of Edinburgh, 1990.
- [Luo94] Zhaohui Luo. *Computation and Reasoning: A Type Theory for Computer Science*. 1994.
- [MM04] Conor McBride and James Mckinna. The view from the left. *Journal of Functional Programming*, 14(1):69–111, 2004.

[Nor07] Ulf Norell. *Towards a practical programming language based on dependent type theory*. PhD thesis, Department of Computer Science and Engineering, Chalmers University of Technology, SE-412 96 Göteborg, Sweden, September 2007.

Part VII

TODO

- review chapter 9 of file:///Users/stephaniesavir/Downloads/Computation-and-Reasoning.dependently

Todo list

clean up!	1
ebnf? if reinventing it underline	1
pat or p?	1
alternatively	1
differentiate identifiers with font	1
motive should not need to insist on the type info of the binder?	1
Grey out things that are surface syntax but not needed for theory	1
Make identifiers consistent with chapter 2, and locations in chapter 3	1
Make a module syntax?	1
abbreviate away dumb arrows, unstated separator is a space, also usual syntax ($:\ast$)? also shorthands for telescopes	1
Constructors build application, drop this with applications	2
suspect this also hinges on regularity	3
define these \in	3
don't need $\Gamma \vdash \overline{N} : \Delta$?	3
may not need scrut wf check? oh but what about the empty types!	3
to ensure regularity	3
it's actually kind of fine discriminating between non converting motives?	4
extend reductions over lists	4
what about D? how much of a value should it be?	5
extend step over lists	5
clean when I get motive inference working	6
syntax highlighting would be bomb	6
may not need scrut wf check? oh but what about the empty types!	6
abuse of notation...	6
abuse of notation...	6
type annotations are allowed	7
pat or p?	7
there is a lot of jenyness about unification in general, but I think the additional points lose focus?	7
last condition is incorrect, since a loop will type but not match	7
should delta be well formed?	7
abuse of notation	8
not quite a sub propely a ctx thing	8
to stylize consistently, should use math font, or like a nice image	9
fig	9
technically speaking, telescopes should generalize to the different syntactic classes	9
\sim is a bit of a hack	9
c?	9
"can", "could", weasel word until implemented	11
add rules of unification? the rules as implemented are standard, needing extended normalization is weird	11
the exact equivalence is unprincipled and hacky. But all of dependent pattern matching is unprincipled and hacky. I can't actually contrive a situation where this could be an issue, but feel like it needs to be mentioned.	11

Minimal data with Sigma and Unit	12
A tutorial implementation of dynamic pattern unification Adam Gundry and Conor McBride (2012) http://adam.gundry.co.uk/publications/unify/ (this links give you the choice to read a more detailed chapter of Adam Gundry's thesis instead)	12
https://research.chalmers.se/en/publication/519011 ?	12
https://sozeau.gitlabpages.inria.fr/www/research/publications/Equations:_A_Dependent_Pattern-Matching_Compiler.pdf ?	12
http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.99.1405&rep=rep1&type=pdf ?	12
talk about normalization	12

7 notes

there are several simpler systems that can be worked through: eliminator style patterns, cast patterns, but to bring it all together we need congruence over functions.

adding paths and path variables means that constructs can still fail at runtime, but they can blame the actually problematic components

validating the K axiom, not that equalities are unique, merely that we don't care which one of the unique equalities is used.

Other extensions to the Calculus of Constructions that are primarily concerned with data (UCC, CIC) will be reviewed in chapter 4.

Coq and Lean trace their core theory back to the Calculus of Constructions.

8 unused

...