

Chapter 3

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Part I

Introduction

We can now motivate one of the most fundamental problems with dependent type systems. Definitional equalities pervasive and unintuitive.

For instance, dependent types can prevent an out-of-bounds error when indexing into a length indexed list.

$$\begin{aligned} \mathbf{Vec} &: 1_c \rightarrow \mathbb{N}_c \rightarrow *, \\ \mathbf{rep} &: (X : *) \rightarrow X \rightarrow (y : \mathbb{N}_c) \rightarrow \mathbf{Vec} X y, \\ \mathbf{head} &: (X : *) \rightarrow (y : \mathbb{N}_c) \rightarrow \mathbf{Vec} X (1_c +_c y) \rightarrow X \\ \vdash \lambda x \Rightarrow \mathbf{head} \mathbb{B}_c x (\mathbf{rep} \mathbb{B}_c \mathit{true}_c (1_c +_c x)) &: \mathbb{N}_c \rightarrow \mathbb{B}_c \end{aligned}$$

\mathbf{head} is a function that expects a list of length $1_c +_c y$, making it impossible for \mathbf{head} to inspect an empty list. Luckily the \mathbf{rep} function will return a list of length $1_c +_c y$, exactly the type that is required.

Unfortunately, the following will not type check in the surface language,

$$\not\vdash \lambda x \Rightarrow \mathbf{head} \mathbb{B}_c x (\mathbf{rep} \mathbb{B}_c \mathit{true}_c (x +_c 1_c)) : \mathbb{N}_c \rightarrow \mathbb{B}_c$$

While “obviously” $1 + x = x + 1$, in the surface language definitional equality does not associate these two terms, $1_c +_c x \not\equiv x +_c 1_c$. Recall that **definitional equality** is the name for the conservative approximation of equality used internally by a dependent type system. Definitional equality prevents the use of a term of type $\mathbf{Vec} (1_c +_c x)$ where a term of type $\mathbf{Vec} (x +_c 1_c)$ is expected. Usually when dependent type systems encounter situations like this, they will give a type error and prevent further work. Especially frustrating from the programmer’s perspective, if the programmer made a mistake in the definition of addition, such that for some x , $1_c +_c x \not\equiv x +_c 1_c$, the system will not provide hints on which x break this equality. The lack of clear error messages and the requirement to prove obvious equalities is a problem in all dependent type systems, not just the surface language.

Why not sidestep definitional equality? A more convenient system can assume the equalities hold until a concrete witness of inequality is discovered. A good dependently typed programming language should

- Provisionally allow “obvious” but unproven equalities like $1_c +_c x = x +_c 1_c$ so programming is not blocked.
- Give static warnings about potential type inequalities; users can decide if they are worth verifying.
- Give runtime errors, with a concrete witness of inequality, if one of the provisional equalities is shown not to hold in practice.

For instance, if the $+_c$ function incorrectly computes $8_c +_c 1_c = 0_c$ the above function will “get stuck” on the input 8_c . If that application is encountered at runtime we can give an error stating $9_c \neq 0_c = 8_c +_c 1_c$.

should be turned into warnings. If a meaningful problem occurs at runtime a clear error message that pinpoints the exact source of errors should be provided. This way runtime errors and static warnings both inform the programmer. Since the user can gradually correct their program as errors surface, we call this workflow **gradual correctness**.

Though gradual correctness is an apparently simple idea, there are several subtle issues that must be dealt with.

- While it is easy to test when natural numbers are equal at runtime, testing that 2 functions are equal is in general undecidable.
- Some amount of type information must be maintained in the term so that good runtime error messages are possible.
- If equality checks are embedded into syntax, they may propagate into the type level.

We are unaware of research that directly handles all of these concerns. Furthermore, if we want the above, we cannot work in the surface language directly a new system is needed. This system is comprised of

- The **cast language**, a dependently typed language with embedded runtime checks, that have evaluation behavior.
- The **elaboration procedure** that transforms untyped surface syntax into checked cast language terms.

We show that a novel form of type soundness holds, that we call **cast soundness**. Instead of “well typed terms don’t get stuck”, we prove “well cast terms don’t get stuck without blame”.

Additionally, by construction, blame (in the sense of gradual typing) is reasonably handled. Several graduality properties hold for the system overall.

source location information,		
ℓ	$::=$	\dots
variable contexts,		
H	$::=$	$\Diamond \mid H, x : A$
expressions,		
a, b, A, B	$::=$	x
		$a ::_{A, \ell, o} B$ cast
		\star
		$(x : A) \rightarrow B$
		$\text{fun } f \ x \Rightarrow b$
		$b \ a$
observations,		
o	$::=$	$.$ current location
		$o.arg$ function type-arg
		$o.bod[a]$ function type-body

Figure 1: Cast Language Syntax

Part II

Cast Language

1 Syntax

The syntax for the cast language can be seen in figure 1. By design the cast language is almost identical to the surface language except for the cast construct.

The cast language can assume arbitrary equalities over types, $A = B$, with a cast, $a ::_{A, \ell, o} B$ given

- a source location ℓ where it was asserted,
- a concrete observation o that would witness inequality,
- the type of the underlying term A ,
- and the expected type of the term B .

Every time there is a type mismatch between the type of the term and the type expected from the usage, the elaboration procedure will produce a cast.

Observations allow indexing into terms to pinpoint errors For instance if we want to highlight the C sub expression in $(x : A) \rightarrow (y : (x : B) \rightarrow \underline{C}) \rightarrow D$ we can use the observation $..Bod[x].arg.Bod[y]$. Since, in general, the C may specifically depend on x and y they are tracked as part of the observation. For instance, given the type $(X : \star) \rightarrow X$ we might want to point out A when $X = A \rightarrow B$ resulting in the type $(X : \star) \rightarrow (\underline{A} \rightarrow B)$, the observation would then read $..Bod[A \rightarrow B].arg$ recording the specific input that allows an argument to be inspected.

2 How Casts Should Reduce

How does the cast construct interact with the existing constructs? These are all the interactions that could cause a term to be stuck in evaluation or block type checking

$$\begin{array}{l} \star :: B \\ ((x : A) \rightarrow B) :: C \\ (b :: C) a \end{array}$$

we can account for these as follows, by realizing obvious casts should evaporate, freeing up the underling term

$\star :: \star$	\rightsquigarrow	\star		
$\star :: B$	\rightsquigarrow	$\star :: B'$	when	$B \rightsquigarrow B'$
$\star ::_{\ell, o} B$	blame	ℓ, o	when	B cannot be \star
$((x : A) \rightarrow B) :: \star$	\rightsquigarrow	\star		
$((x : A) \rightarrow B) :: C$	\rightsquigarrow	$((x : A) \rightarrow B) :: C'$	when	$C \rightsquigarrow C'$
$((x : A) \rightarrow B) ::_{\ell, o} C$	blame	ℓ, o	when	C cannot be \star
$(b ::_{(x:A') \rightarrow B'} (x : A) \rightarrow B) a$	\rightsquigarrow	$(b (a :: A')) ::_{B'[x:=a::A']} B[x := A]$		
$(b :: C) a$	\rightsquigarrow	$(b :: C') a$	when	$C \rightsquigarrow C'$
$(b ::_{\ell, o} C) a$	blame	ℓ, o	when	C cannot be $(x : A)$

The fully formal reduction rules are listed later, but their complete detail can be distracting. Type universes live in the type universe, so any cast that contradicts this should be blamed. Similarly for function types. Terms that take input must be functions, so any cast that contradicts this should blame the source location. The most interesting case is when a cast confirms that the applied term is a function, then we use the function cast to determine the correct cast over the argument, and maintain the appropriate cast over the body. This distribution is similar to the way higher order contracts invert the blame of arguments [FF02]¹.

Note that the rules outlined here are not deterministic since there are cases when we might blame or continue reducing the argument. One of the subtle innovations in this chapter is to completely separate blame from reduction. This sidesteps many of the complexities of having a reduction relevant **abort** term in a dependent type theory. As far as reduction is concerned, bad terms simply “get stuck” as it might on an unsatisfiable term in the typing context. Otherwise the reduction behavior is well behaved and deterministic. Terms may be blamable by the rules outlined in this chapter, but it is easy to imagine more sophisticated extensions without causing theoretical issues.

This outlines the minimum requirements for cast reductions, there are plausibly many additional reductions that could be considered. Some tempting reductions are

$$\begin{array}{l} a ::_C C \rightsquigarrow_{=} a \\ a ::_{C'} C \rightsquigarrow_{=} a \quad \text{when } C' \equiv C \end{array}$$

however these rules preclude extracting blame that may be embedded within the casts themselves. These rules also seem to complicate the type theory.

¹and also found in gradual type systems [WF09]

Despite this we will use these reductions in examples to keep the book keeping to a minimum.

3 Examples

We can reexamine some of the examples terms from Chapter 2, but this time with equality assumptions

3.1 Higher Order Functions

We deal with higher order functions by distributing function casts around applications. If an application happens to a cast of function type, the argument and body cast is separated and the argument cast is swapped. For instance in

$$\begin{aligned}
& ((\lambda x \Rightarrow x \& \& x) ::_{Bool \rightarrow Bool, \ell, .} Nat \rightarrow Nat) \ 7 \\
& \rightsquigarrow ((\lambda x \Rightarrow x \& \& x) (7 ::_{Nat, \ell, .arg} Bool)) ::_{Bool, \ell, .bod[7]} Nat \\
& \rightsquigarrow ((7 ::_{Nat, \ell, .arg} Bool) \& \& (7 ::_{Nat, \ell, .arg} Bool)) \\
& \quad ::_{Bool, \ell, .bod[7]} Nat
\end{aligned}$$

if evaluation gets stuck on $\&\&$ and we can blame the argument of the cast for equating Nat and $Bool$. The body observation records the argument the function is called with. For instance in the $.bod[7]$ observation. In a dependently typed function the exact argument of use may be important to give a good error. Because casts can be embedded inside of casts, types themselves need to normalize and casts need to simplify. Since our system has one universe of types, type casts only need to simplify themselves when a term of type \star is cast to \star . For instance,

$$\begin{aligned}
& ((\lambda x \Rightarrow x) ::_{(Bool \rightarrow Bool) ::_{\star, \ell, .arg} \star, \ell, .} Nat \rightarrow Nat) \ 7 \\
& \rightsquigarrow ((\lambda x \Rightarrow x) ::_{Bool \rightarrow Bool} Nat \rightarrow Nat) \ 7 \\
& \rightsquigarrow ((\lambda x \Rightarrow x) (7 ::_{Nat, \ell, .arg} Bool)) ::_{Bool, \ell, .bod[7]} Nat \\
& \rightsquigarrow ((7 ::_{Nat, \ell, .arg} Bool)) ::_{Bool, \ell, .bod[7]} Nat
\end{aligned}$$

3.2 Pretending $true = false$

Recall that we proved $\neg true_c \doteq_{\mathbb{B}_c} false_c$ what happens if we assume it anyway? Every type equality assumption needs an underlying term, here we can choose $refl_{true_c : \mathbb{B}_c} : true_c \doteq_{\mathbb{B}_c} true_c$, and cast that term to $true_c \doteq_{\mathbb{B}_c} false_c$ resulting in $refl_{true_c : \mathbb{B}_c} ::_{true_c \doteq_{\mathbb{B}_c} true_c} true_c \doteq_{\mathbb{B}_c} false_c$. Recall that $\neg true_c \doteq_{\mathbb{B}_c} false_c$ is a short hand for $true_c \doteq_{\mathbb{B}_c} false_c \rightarrow \perp$. What if we try to use our term of type $true_c \doteq_{\mathbb{B}_c} false_c$ to get a term of type \perp ?

$$\begin{array}{l}
\rightsquigarrow \\
true_c \dot{=}_{\mathbb{B}_c} false_c := (C : (\mathbb{B}_c \rightarrow \star)) \rightarrow C true_c \rightarrow C false_c \\
true_c \dot{=}_{\mathbb{B}_c} true_c := (C : (\mathbb{B}_c \rightarrow \star)) \rightarrow C true_c \rightarrow C true_c \\
\rightsquigarrow = \\
refl_{true_c : \mathbb{B}_c} := \lambda - cx \Rightarrow cx \\
\rightsquigarrow \\
\rightsquigarrow = \\
\rightsquigarrow \\
toLogic := \lambda b \Rightarrow b \star Unit_c \perp_c \\
\rightsquigarrow \\
true_c := \lambda - x - \Rightarrow x \\
\rightsquigarrow \\
\rightsquigarrow \\
\rightsquigarrow \\
toLogic := \lambda b \Rightarrow b \star Unit_c \perp_c \\
\rightsquigarrow \\
false_c := \lambda - -y \Rightarrow y \\
\rightsquigarrow \rightsquigarrow \rightsquigarrow \rightsquigarrow
\end{array}$$

$$\begin{array}{l}
(\lambda pr \Rightarrow pr toLogic tt_c) (refl_{true_c : \mathbb{B}_c}) \\
\left(refl_{true_c : \mathbb{B}_c} ::_{true_c \dot{=}_{\mathbb{B}_c} true_c} (C : (\mathbb{B}_c \rightarrow \star)) \rightarrow C true_c \rightarrow C false_c \right) \\
\left(refl_{true_c : \mathbb{B}_c} ::_{(C : (\mathbb{B}_c \rightarrow \star)) \rightarrow C true_c \rightarrow C true_c} (C : (\mathbb{B}_c \rightarrow \star)) \rightarrow C true_c \rightarrow C true_c \right) \\
(refl_{true_c : \mathbb{B}_c} toLogic :: toLogic true_c \rightarrow toLogic true_c) \\
((\lambda - cx \Rightarrow cx) toLogic :: toLogic true_c \rightarrow toLogic true_c) \\
((\lambda cx \Rightarrow cx) :: toLogic true_c \rightarrow toLogic true_c) \\
((\lambda cx \Rightarrow cx) tt_c :: toLogic true_c \rightarrow toLogic true_c) \\
(tt_c :: toLogic true_c \rightarrow toLogic true_c) \\
(tt_c :: (\lambda b \Rightarrow b \star Unit_c \perp_c) \rightarrow toLogic true_c) \\
(tt_c :: (true_c \star Unit_c \perp_c) \rightarrow toLogic true_c) \\
(tt_c :: ((\lambda - x - \Rightarrow x) \star Unit_c \perp_c) \rightarrow toLogic true_c) \\
(tt_c :: ((\lambda x - \Rightarrow x) U) \rightarrow toLogic true_c) \\
(tt_c :: ((\lambda - \Rightarrow Unit_c) U) \rightarrow toLogic true_c) \\
(tt_c :: Unit_c) :: (\lambda b \Rightarrow b \star Unit_c \perp_c) \rightarrow toLogic true_c \\
(tt_c :: Unit_c) :: (\lambda b \Rightarrow b \star Unit_c \perp_c) \rightarrow toLogic true_c \\
(tt_c :: Unit_c) :: ((\lambda - -y \Rightarrow y) \rightarrow toLogic true_c) \rightarrow toLogic true_c \\
(tt_c :: Unit_c) :: ((\lambda - -y \Rightarrow y) \rightarrow toLogic true_c) \rightarrow toLogic true_c
\end{array}$$

As in the above the example, the term $(tt_c :: Unit_c) :: \perp_c$ has not yet “gotten stuck”. Applying \star will uncover the error

$$\begin{array}{l}
\perp_c := (X : \star) \rightarrow X \quad ((tt_c :: Unit_c) :: \perp_c) \star \\
\rightsquigarrow = \quad ((tt_c :: Unit_c) :: (X : \star) \rightarrow X) \star \\
Unit_c := (X : \star) \rightarrow X \rightarrow X \quad ((tt_c :: ((X : \star) \rightarrow X \rightarrow X)) \star) :: \star \\
\rightsquigarrow = \quad (tt_c \star) :: (\star \rightarrow \star) :: \star \\
tt_c := \lambda - x \Rightarrow x \quad ((\lambda - x \Rightarrow x) \star) :: (\star \rightarrow \star) :: \star \\
\rightsquigarrow \quad (\lambda x \Rightarrow x) :: (\star \rightarrow \star) :: \star \\
Blame! \quad (\lambda x \Rightarrow x) :: (\star \rightarrow \star) :: \star
\end{array}$$

The location and observation information that was left out of the computation could generate an error message like

$$(C : (\mathbb{B}_c \rightarrow \star)). C true_c \rightarrow \underline{C true_c} \neq (C : (\mathbb{B}_c \rightarrow \star)). C true_c \rightarrow \underline{C false_c}$$

when

$$C := \lambda b : \mathbb{B}_c. b \star \star \perp$$

$$C true_c = \perp \neq \star = C false_c$$

Reminding the programmer that they should not confuse true with false.

4 Cast System

Type soundness, proving some undesirable behaviors are unreachable from a well type term, is the most important property for a typed programming language to exhibit. How should this apply to the cast language, where bad behaviors are intended to be reachable? We allow bad behavior, but require that when a

$$\begin{array}{c}
\frac{x : A \in H}{H \vdash x : A} \text{cast-var} \\
\frac{H \vdash a : A \quad H \vdash A : \star \quad H \vdash B : \star}{H \vdash a ::_{A,\ell,o} B : B} \text{cast-::} \\
\frac{}{H \vdash \star : \star} \text{cast-}\star \\
\frac{H \vdash A : \star \quad H, x : A \vdash B : \star}{H \vdash (x : A) \rightarrow B : \star} \text{cast-fun-ty} \\
\frac{H, f : (x : A) \rightarrow B, x : A \vdash b : B}{H \vdash \text{fun } f x \Rightarrow b : (x : A) \rightarrow B} \text{cast-fun} \\
\frac{H \vdash b : (x : A) \rightarrow B \quad H \vdash a : A}{H \vdash b a : B[x := a]} \text{cast-fun-app} \\
\frac{H \vdash a : A \quad A \equiv A'}{H \vdash a : A'} \text{cast-conv}
\end{array}$$

TODO: remove regularity stuff

Figure 2: Cast Language Type Assignment Rules

bad state is reached we blame the original faulty type annotations. Where the slogan for type soundness is “well typed terms don’t get stuck”, the slogan for cast soundness is “well typed terms don’t get stuck without blame”.

In Chapter 2 we proved type soundness for a minimal dependently typed language, with a progress and preservation style proof given a suitable definition of term equivalence. We can extend that proof to support cast soundness with only a few modifications.

The cast language supports its own type assignment system (figure 2). This system ensures that computations will not get stuck without enough information for good runtime error messages. Specifically computations will not get stuck without a source location and a witness of inequality. TODO: highlight differences from CH2

As before we need a suitable evaluation relation to generate our proof of equality.

TODO: review par reductions, highlight differences from CH2

4.1 Definitional Equality

As in Chapter 2, \Rightarrow_* can be shown to be confluent. The proofs follow the same structure, but since observations can contain terms, \Rightarrow and max must be extended to observations. Proofs must be extended to mutually induct on observations.

$$\frac{b \Rightarrow b' \quad a \Rightarrow a'}{(\text{fun } f \ x \Rightarrow b) \ a \Rightarrow b' [f := \text{fun } f \ x \Rightarrow b', x := a']} \Rightarrow\text{-fun-app-red}$$

$$\frac{b \Rightarrow b' \quad a \Rightarrow a' \quad A_1 \Rightarrow A'_1 \quad A_2 \Rightarrow A'_2 \quad B_1 \Rightarrow B'_1 \quad B_2 \Rightarrow B'_2 \quad o \Rightarrow o'}{(b ::_{(x:A_1) \rightarrow B_1, \ell, o} (x : A_2) \rightarrow B_2) \ a \Rightarrow (b' \ a' ::_{A'_2, \ell, o.\text{arg}} A'_1) ::_{B'_1 [x:=a' ::_{A'_2, \ell, o.\text{arg}} A'_1], \ell, o'.\text{bod}[a']} B'_2 [x := a'])} \Rightarrow\text{-fun-::red}$$

$$\frac{a \Rightarrow a'}{a ::_{\star, \ell, o} \star \Rightarrow a'} \Rightarrow\text{-::red}$$

$$\frac{\frac{}{x \Rightarrow x} \Rightarrow\text{-var}}{a \Rightarrow a' \quad A_1 \Rightarrow A'_1 \quad A_2 \Rightarrow A'_2 \quad o \Rightarrow o' \Rightarrow\text{-::} \frac{a ::_{A_1, \ell, o} A_2 \Rightarrow a' ::_{A'_1, \ell, o'} A'_2}}{a ::_{A_1, \ell, o} A_2 \Rightarrow a' ::_{A'_1, \ell, o'} A'_2} \Rightarrow\text{-::}}$$

$$\frac{}{\star \Rightarrow \star} \Rightarrow\text{-}\star$$

$$\frac{A \Rightarrow A' \quad B \Rightarrow B'}{(x : A) \rightarrow B \Rightarrow (x : A') \rightarrow B'} \Rightarrow\text{-fun-ty}$$

$$\frac{b \Rightarrow b'}{\text{fun } f \ x \Rightarrow b \Rightarrow \text{fun } f \ x \Rightarrow b'} \Rightarrow\text{-fun}$$

$$\frac{b \Rightarrow b' \quad a \Rightarrow a'}{b \ a \Rightarrow b' \ a'} \Rightarrow\text{-fun-app}$$

$$\frac{}{. \Rightarrow .} \Rightarrow\text{-obs-emp}$$

$$\frac{o \Rightarrow o'}{o.\text{arg} \Rightarrow o'.\text{arg}} \Rightarrow\text{-obs-arg}$$

$$\frac{o \Rightarrow o' \quad a \Rightarrow a'}{o.\text{bod}[a] \Rightarrow o'.\text{bod}[a']} \Rightarrow\text{-obs-bod}$$

$$\frac{}{a \Rightarrow_* a} \Rightarrow_*\text{-refl}$$

$$\frac{a \Rightarrow_* a' \quad a' \Rightarrow a''}{a \Rightarrow_* a''} \Rightarrow_*\text{-trans}$$

$$\frac{a \Rightarrow_* a'' \quad a' \Rightarrow_* a''}{a \equiv a'} \equiv\text{-def}$$

$$\begin{array}{lcl}
\text{max}(\text{fun } f \ x \Rightarrow b) \ a & = & \text{max}(b) [f := \text{fun } f \ x \Rightarrow \text{max}(b), x] \\
\text{max}(b ::_{(x:A_1) \rightarrow B_1, \ell, o} (x : A_2) \rightarrow B_2) \ a & = & (\text{max}(b) (\text{max}(a) ::_{\text{max}(A_2), \ell, \text{max}(o).a} \text{max}(A_1) \rightarrow B_2) a) \\
& & ::_{\text{max}(B_2) [x := \text{max}(a) ::_{\text{max}(A_2), \ell, \text{max}(o).arg} \text{max}(A_1)], \ell, \text{max}(o).bod} \text{max}(B_2) [x := \text{max}(a) ::_{\text{max}(A_2), \ell, \text{max}(o).arg} \text{max}(A_1)] \ell, \text{max}(o).bod} \\
\text{max}(b ::_{B_1, \ell, o} B_2) & = & \text{max}(b) ::_{\text{max}(B_1), \ell, \text{max}(o)} \text{max}(B_2) \\
\text{max}(\dots) & = & \dots \\
\text{max}(\cdot) & = & \cdot \\
\text{max}(o.arg) & = & \text{max}(o).arg \\
\text{max}(o.bod[a]) & = & \text{max}(o).bod[\text{max}(a)]
\end{array}$$

If $a \Rightarrow a'$ then $a' \Rightarrow \text{max}(a)$. If $o \Rightarrow o'$ then $o' \Rightarrow \text{max}(o)$
 by mutual induction on the derivations of $m \Rightarrow m'$ and $o \Rightarrow o'$.
 The diamond property follows directly, if $a \Rightarrow a'$, $a \Rightarrow a''$, implies $a' \Rightarrow a''$.
 The diamond property implies the confluence of \Rightarrow_* .
 It follows that \equiv is transitive.

4.1.1 Stability

$\forall A, B, C. (x : A) \rightarrow B \Rightarrow_* C \Rightarrow \exists A', B'. C = (x : A') \rightarrow B' \wedge A \Rightarrow_* A' \wedge B \Rightarrow_* B'$
 by induction on \Rightarrow_*
 Therefore the following rule is admissible

$$\frac{(x : A) \rightarrow B \equiv (x : A') \rightarrow B'}{A \equiv A' \quad B \equiv B'}$$

4.2 Preservation

4.2.1 Context Weakening

The following rule is admissible

$$\frac{H \vdash a : A}{H, H' \vdash a : A}$$

by induction on typing derivations

4.2.2 Substitution Preservation

The following rule is admissible

$$\frac{H \vdash c : C \quad H, x : C, H' \vdash a : A}{H, H' [x := c] \vdash a [x := c] : A [x := c]}$$

by induction over typing derivations

cast-: : $H, x : C, H' \vdash a : A, H, x : C, H' \vdash A : \star, H, x : C, H' \vdash B : \star$, wellformed ℓ, o
 $H, H' [x := c] \vdash a [x := c] : A [x := c]$ by induction
 $H, H' [x := c] \vdash A [x := c] : \star$ by induction
 $H, H' [x := c] \vdash B [x := c] : \star$ by induction
 $H, H' [x := c] \vdash a [x := c] ::_{A[x:=c], \ell, o[x:=c]} B [x := c] : B [x := c]$ cast-: :
 other rules ... correspond to the inductive

$$\frac{}{\Diamond \equiv \Diamond} \equiv\text{-ctx-empty}$$

$$\frac{H \equiv H' \quad A \equiv A'}{H, x : A \equiv H', x : A'} \equiv\text{-ctx-ext}$$

Figure 3: Contextual Equivalence

4.2.3 Context Preservation

As before the notion of definitional equality can be extended to cast contexts in 3.

the following rule is admissible

$$\frac{\Gamma \vdash n : N \quad \Gamma \equiv \Gamma'}{\Gamma' \vdash n : N}$$

by induction over typing derivations

$$\begin{array}{ll} \text{cast-:} & H \vdash a : A, H \vdash A : \star, H \vdash B : \star, \text{wellformed } \ell, o \\ & H' \vdash a : A \quad \text{by induction} \\ & H' \vdash A : \star \quad \text{by induction} \\ & H' \vdash B : \star \quad \text{by induction} \\ & H' \vdash a ::_{A, \ell, o} B : B \quad \text{cast-:} \end{array}$$

other rules ... correspond to the inductive cases in Chapter 2

4.2.4 Inversions

As before we show inversions on the term syntaxes, generalizing the induction hypothesis up to equality, when needed.

$$\frac{H \vdash \text{fun } f x \Rightarrow a : C \quad C \equiv (x : A) \rightarrow B}{H, f : (x : A) \rightarrow B, x : A \vdash b : B}$$

by induction on the cast derivation

$$\begin{array}{ll} \text{cast-fun} & H, f : (x : A') \rightarrow B', x : A' \vdash b : B', (x : A') \rightarrow B' \equiv (x : A) \rightarrow B \\ & A' \equiv A, \quad B' \equiv B \quad \text{by stability of fun-ty} \\ & H, f : (x : A') \rightarrow B', x : A' \equiv H, f : (x : A) \rightarrow B, x : A \quad \text{by reflexivity of } \equiv, \text{ extended with} \\ & H, f : (x : A) \rightarrow B, x : A \vdash b : B' \quad \text{by context preservation} \\ & H, f : (x : A) \rightarrow B, x : A \vdash b : B \quad \text{ty-conv} \\ \text{cast-conv} & H \vdash \text{fun } f x \Rightarrow b : C', C' \equiv C, C \equiv (x : A) \rightarrow B \\ & C' \equiv (x : A) \rightarrow B \quad \text{by transitivity} \\ & \dots \quad \text{by induction} \\ \text{other rules} & \text{impossible} \quad \text{the term position has the form fun } \end{array}$$

This allows us to conclude the corollary

$$\frac{H \vdash \text{fun } f x \Rightarrow a : (x : A) \rightarrow B}{H, f : (x : A) \rightarrow B, x : A \vdash b : B}$$

The following rule is admissible

$$\frac{H \vdash (x : A) \rightarrow B : C \quad C \equiv \star}{H \vdash A : \star \quad H, x : A \vdash B : \star}$$

By induction on the typing derivations

cast-fun-ty $H \vdash (x : A) \rightarrow B : C$

follows directly

cast-conv $H \vdash (x : A) \rightarrow B : C', C' \equiv C$

$C' \equiv \star$ by transitivity

... by induction

other rules impossible since the term position has the form $(x : A) \rightarrow B$ leading to the corollary

$$\frac{H \vdash (x : A) \rightarrow B : \star}{H \vdash A : \star \quad H, x : A \vdash B : \star}$$

The following rule is admissible

$$\frac{H \vdash a ::_{A,\ell,o} B : C}{H \vdash a : A \quad H \vdash A : \star \quad H \vdash B : \star}$$

By induction on the typing derivations

cast-:: $H \vdash a : A \quad H \vdash A : \star \quad H \vdash B : \star$

follows directly

cast-conv $H \vdash a ::_{A,\ell,o} B : C', C' \equiv C$

... by induction

other rules impossible the term position has the form $a ::_{A,\ell,o} B$

Note that each of the output judgments are smaller then the input judgments, this allows other proofs to use induction on the output of this derivation.

4.2.5 \Rightarrow -Preservation

The following rule is admissible

$$\frac{a \Rightarrow a' \quad H \vdash a : A}{H \vdash a' : A}$$

by induction

cast- \star	$\Rightarrow\text{-}\star$	$H \vdash \star : \star, \star \Rightarrow \star$	fo
cast-var	$\Rightarrow\text{-var}$	$H \vdash x : A, x \Rightarrow x$	fo
ty-conv		$H \vdash a : A, A \equiv A'$	
	all \Rightarrow	$a \Rightarrow a'$ $H \vdash a' : A$ $H \vdash a' : A'$	by ca
cast- $::$		well formed ℓ, o	
	$\Rightarrow\text{-}::\text{-red}$	$H \vdash a : \star, a \Rightarrow a'$ $H \vdash a' : \star$	d
	$\Rightarrow\text{-}::$	$H \vdash a : A_1, H \vdash A_2 : \star, a \Rightarrow a', A_1 \Rightarrow A'_1, A_2 \Rightarrow A'_2, o \Rightarrow o'$ $H \vdash a' : A_1$ $H \vdash a' : A'_1$ $H \vdash A'_2 : \star$ $H \vdash a' ::_{A'_1, \ell, o'} A'_2 : A'_2$ $H \vdash a' ::_{A'_1, \ell, o'} A'_2 : A_2$	by ca by ca ca
cast-fun-ty		$H \vdash A : \star, H, x : A \vdash B : \star$	
	$\Rightarrow\text{-fun-ty}$	$A \Rightarrow A', B \Rightarrow B'$ $H \vdash A' : \star$ $H, x : A \vdash B' : \star$ $H, x : A \equiv H, x : A'$ $H, x : A' \vdash B' : \star$ $H \vdash (x : A') \rightarrow B' : \star$	by by by by ca
cast-fun		$H, f : (x : A) \rightarrow B, x : A \vdash a : A$	
	$\Rightarrow\text{-fun}$	$a \Rightarrow a'$ $H, f : (x : A) \rightarrow B, x : A \vdash a' : A$ $H \vdash \text{fun } f x \Rightarrow a' : (x : A) \rightarrow B$	by ca
cast-fun-app			
	$\Rightarrow\text{-fun-app-red}$	$H \vdash \text{fun } f x \Rightarrow b : (x : A) \rightarrow B, H \vdash a : A, a \Rightarrow a', b \Rightarrow b'$ $\text{fun } f x \Rightarrow b \Rightarrow \text{fun } f x \Rightarrow b'$ $H \vdash \text{fun } f x \Rightarrow b' : (x : A) \rightarrow B$ $H, f : (x : A) \rightarrow B, x : A \vdash a'$ $H \vdash a' : A$ $H \vdash b' [f := \text{fun } f x \Rightarrow b', x := a'] : B [x := a']$ $B [x := a'] \equiv M [x := a]$ $H \vdash b' [f := \text{fun } f x \Rightarrow b', x := a'] : B [x := A]$	\Rightarrow by by by by ca
	$\Rightarrow\text{-fun-app}$	$H \vdash a : (x : A) \rightarrow B, H \vdash a : A, a \Rightarrow a', b \Rightarrow b'$ $H \vdash b' : (x : B) \rightarrow A$ $H \vdash A' : A$ $H \vdash b' a' : B [x := a']$ $B [x := a'] \equiv B [x := a]$ $H \vdash b' [f := \text{fun } f x \Rightarrow b', x := a'] : B [x := a]$	by by by ca by ca
	$\Rightarrow\text{-fun-}::\text{-red}$	$H \vdash (b ::_{(x:A_1) \rightarrow B_1, \ell, o} (x : A_2) \rightarrow B_2) : (x : A_2) \rightarrow B_2, H \vdash a : A_2,$ $b \Rightarrow b', a \Rightarrow a', A_1 \Rightarrow A'_1, A_2 \Rightarrow A'_2, B_1 \Rightarrow B'_1, B_2 \Rightarrow B'_2, o \Rightarrow o',$ $H \vdash a' : A_2$ $H \vdash a' : A'_2$ $H \vdash b : (x : A_1) \rightarrow B_1, H \vdash (x : A_1) \rightarrow B_1 : \star, H \vdash (x : A_2) \rightarrow B_2 : \star$ $(x : A_2) \rightarrow B_2 \Rightarrow (x : A'_2) \rightarrow B'_2$ $H \vdash (x : A'_2) \rightarrow B'_2 : \star$ $H \vdash A'_2 : \star, H, x : A'_2 \vdash B'_2 : \star$ $(x : A_1) \rightarrow B_1 \Rightarrow (x : A'_1) \rightarrow B'_1$ $H \vdash (x : A'_1) \rightarrow B'_1 : \star$ $H \vdash A'_1 : \star, H, x : A'_1 \vdash B'_1 : \star$ let $ac = a' ::_{A'_2, \ell, o'.arg} A'_1$ $H \vdash ac : A'_1$ $H \vdash b' : (x : A_1) \rightarrow B_1$ $H \vdash b' : (x : A'_1) \rightarrow B'_1$	by ca ca \Rightarrow by fu \Rightarrow by fu

$$\begin{array}{c}
\frac{}{\star \mathbf{Val}} \text{Val-}\star \\
\frac{}{(x : A) \rightarrow B \mathbf{Val}} \text{Val-fun-ty} \\
\frac{}{\text{fun } f \ x \Rightarrow b \mathbf{Val}} \text{Val-fun} \\
\frac{a \mathbf{Val} \quad A \mathbf{Val} \quad B \mathbf{Val}}{a \not\approx \star} \\
\frac{a \not\approx \star \quad (x : C) \rightarrow C'}{a ::_{A, \ell o} B \mathbf{Val}} \text{Val-::}
\end{array}$$

Figure 4: Cast Language Values

4.3 Progress

Unlike the surface language, it is not longer practical to characterize values syntactically. Values are specified by judgments in figure 4. They are standard except for the Val-:: , which states that a type (\star or function type) under a cast is not a value.

Small steps are listed in figure 4.3. They are standard for call-by-value except that casts can distribute over application, and casts can reduce when both types are \star .

In addition to small step and values we also specify blame judgments in figure 4.3. Blame tracks the information needed to create a good error message and is inspired by the many systems that use blame tracking [FF02, WF09, Wad15] and universes, only inequalities of the form $\star \not\approx A \rightarrow B$ can be witnessed. The first 2 rules of the blame judgment witness these concrete type inequalities. The rest of the blame rules will recursively extract concrete witnesses from larger terms.

\rightsquigarrow preserves types since the following rule is admissible

$$\frac{m \rightsquigarrow m'}{m \Rightarrow m'}$$

4.3.1 Canonical forms

As in Chapter 2 we will need technical lemmas that determine the form of a value of a type in the empty context. However, canonical function values look different because they must account for the possibility of blame arising from a stuck term.

If $\vdash a : A$, $a \mathbf{Val}$, and $A \equiv \star$ then either

- $a = \star$,

³well founded since cast-inversion produces smaller judgments

⁴well founded since cast-inversion produces smaller judgments

$$\begin{array}{c}
\frac{a \text{ Val}}{(\text{fun } f \ x \Rightarrow b) \ a \rightsquigarrow b [f := \text{fun } f \ x \Rightarrow b, x := a]} \\
\frac{b \text{ Val} \quad a \text{ Val}}{(\text{fun } f \ x \Rightarrow b) \ a \rightsquigarrow b [f := \text{fun } f \ x \Rightarrow b, x := a]} \\
\hline
\frac{(b ::_{(x:A_1) \rightarrow B_1, \ell, o} (x : A_2) \rightarrow B_2) \ a \rightsquigarrow (b \ a ::_{A_2, \ell, o. \text{arg}} A_1) ::_{B_1[x := a ::_{A_2, \ell, o. \text{arg}} A_1], \ell, o. \text{bod}[a]} B_2[x := a]}{a \text{ Val}} \\
\frac{a ::_{\star, \ell, o} \star \rightsquigarrow a}{a \rightsquigarrow a'} \\
\frac{a \rightsquigarrow a'}{a ::_{A, \ell, o} B \rightsquigarrow a' ::_{A, \ell, o} B} \\
\frac{a \text{ Val} \quad A \rightsquigarrow A'}{a ::_{A, \ell, o} B \rightsquigarrow a ::_{A', \ell, o} B} \\
\frac{a \text{ Val} \quad A \text{ Val} \quad B \rightsquigarrow B'}{a ::_{A, \ell, o} B \rightsquigarrow a ::_{A, \ell, o} B'} \\
\frac{b \rightsquigarrow b'}{b \ a \rightsquigarrow b' \ a} \\
\frac{b \text{ Val} \quad a \rightsquigarrow a'}{b \ a \rightsquigarrow b \ a'}
\end{array}$$

$$\begin{array}{c}
\frac{}{\text{Blame } \ell \ o \ (a ::_{(x:A) \rightarrow B, \ell, o} \star)} \\
\hline
\text{Blame } \ell \ o \ (a ::_{\star, \ell, o} (x : A) \rightarrow B) \\
\frac{\text{Blame } \ell \ o \ a}{\text{Blame } \ell \ o \ (a ::_{A, \ell', o'} B)} \\
\frac{\text{Blame } \ell \ o \ A}{\text{Blame } \ell \ o \ (a ::_{A, \ell', o'} B)} \\
\frac{\text{Blame } \ell \ o \ B}{\text{Blame } \ell \ o \ (a ::_{A, \ell', o'} B)} \\
\frac{\text{Blame } \ell \ o \ b}{\text{Blame } \ell \ o \ (b \ a)} \\
\frac{\text{Blame } \ell \ o \ a}{\text{Blame } \ell \ o \ (b \ a)}
\end{array}$$

- or there exists C, B , such that $a = (x : C) \rightarrow B$

by induction on the cast derivation

cast- \star	$\vdash \star : \star$	follows since $a = \star$
cast-fun-ty	$\vdash (x : A) \rightarrow B : \star$	follows since $a = (x : A) \rightarrow B$
ty-conv	$\vdash a : A, a \mathbf{Val}, A \equiv A', A' \equiv \star$ $A' \equiv \star$ $a = \star$ or there exists C, B , such that $a = (x : C) \rightarrow B$	which concluded $\vdash a : A'$ by transitivity, symmetry by induction
cast- $::$	$\vdash a : A_1, a ::_{A_1, \ell, o} A_2 \mathbf{Val}, \vdash A_1 : \star, \vdash A_2 : \star, A_2 \equiv \star$ $a \not\equiv \star, a \not\equiv (x : C) \rightarrow B, a \mathbf{Val}$ but $a = \star$ or there exists C, B , such that $a = (x : C) \rightarrow B$!	since it must have been a value by induction so cast- $::$ case was impossible
cast-fun	$f : (x : A) \rightarrow B, x : A \vdash b : B, (x : A) \rightarrow B \equiv \star$ $(x : A) \rightarrow B \not\equiv \star$!	by the stability of \equiv so cast-fun case was impossible

other rules impossible

Leading to the corollary (by reflexivity of \equiv),

If $\vdash A : \star$, and $A \mathbf{Val}$ then either $A = \star$, or there exists C, B , such that

$A = (x : C) \rightarrow B$

Likewise

If $\vdash a : A$, $a \mathbf{Val}$, and $A \equiv (x : C) \rightarrow B$ then either

- $a = \text{fun } f \ x \Rightarrow b$
- or $a = d ::_{D, \ell, o} (x : C') \rightarrow B', d \mathbf{Val}, D \mathbf{Val}, C' \equiv C, B' \equiv B$

by induction on the cast derivation

cast- \star	$\star \equiv (x : C) \rightarrow B$ $\star \not\equiv (x : C) \rightarrow B$!	by the stability of \equiv
cast-fun-ty	$\star \equiv (x : C) \rightarrow B$ $\star \not\equiv (x : C) \rightarrow B$!	by the stability of \equiv
cast-fun	$f : (x : C') \rightarrow B', x : C' \vdash b : B', (x : C') \rightarrow B' \equiv (x : C) \rightarrow B$	follows directly
cast- $::$	$\vdash a : A_1, \vdash A_1 : \star, \vdash A_2 : \star, A_2 \equiv (x : C) \rightarrow B$ $a \not\equiv \star, a \not\equiv (x : C) \rightarrow B, a \mathbf{Val}, A_1 \mathbf{Val}, A_2 \mathbf{Val}$ $A_2 = (x : C') \rightarrow B'$ $C' \equiv C, B' \equiv B$	since it must have been a value by Val- $::$ by the stability of \equiv , $A_2 \mathbf{Val}$ ⁵ by the stability of \equiv
ty-conv	$\vdash a : A, A \equiv A', A' \equiv (x : C) \rightarrow B$ $A \equiv (x : C) \rightarrow B$	by transitivity by induction

other rules impossible

As a corollary (by reflexivity of \equiv)

If $\vdash a : (x : C) \rightarrow B$, and $a \mathbf{Val}$

- $a = \text{fun } f \ x \Rightarrow b$
- or $a = d ::_{D, \ell, o} (x : C') \rightarrow B', d \mathbf{Val}, D \mathbf{Val}, C' \equiv C, B' \equiv B$

which further implies $a \not\equiv \star, a \not\equiv (x : C) \rightarrow C'$

⁵TODO: make an explicit lemma?

4.3.2 Progress

If $\vdash a : A$ then either

- a **Val**
- there exists a' such that $a \rightsquigarrow a'$
- or there exists ℓ, o such that **Blame** $\ell o a$

As usual this follows from induction on the typing derivation

cast-var	$\vdash \star : \star$ $\star \mathbf{Val}$	
cast-var	$\vdash x : A$	
cast-conv	$\vdash a : A', A' \equiv A$ $a \mathbf{Val}$, there exists a' such that $a \rightsquigarrow a'$, or there exists ℓ, o such that $\mathbf{Blame} \ell o a$	
cast-::	$\vdash a : A, \vdash A : \star, \vdash B : \star$ $a \mathbf{Val}$, there exists a' such that $a \rightsquigarrow a'$, or there exists ℓ_a, o_a such that $\mathbf{Blame} \ell_a o_a a$ $A \mathbf{Val}$, there exists A' such that $A \rightsquigarrow A'$, or there exists ℓ_A, o_A such that $\mathbf{Blame} \ell_A o_A A$ $B \mathbf{Val}$, there exists B' such that $B \rightsquigarrow B'$, or there exists ℓ_B, o_B such that $\mathbf{Blame} \ell_B o_B B$ if $a \rightsquigarrow a'$, $a ::_{A, \ell, o} B \rightsquigarrow a' ::_{A, \ell, o} B$ if $a \mathbf{Val}, A \rightsquigarrow A'$, $a ::_{A, \ell, o} B \rightsquigarrow a ::_{A', \ell, o} B$ if $a \mathbf{Val}, A \mathbf{Val}, B \rightsquigarrow B'$, $a ::_{A, \ell, o} B \rightsquigarrow a ::_{A, \ell, o} B'$ if $a \mathbf{Val}, A \mathbf{Val}, B \mathbf{Val}$, $A = \star$ or $A = (x : C_A) \rightarrow D_A$ $B = \star$ or $B = (x : C_B) \rightarrow D_B$ if $A = \star, B = \star$ $a ::_{\star, \ell, o} \star \rightsquigarrow a$ if $A = (x : C_A) \rightarrow D_A, B = (x : C_B) \rightarrow D_B$ $a \not\rightsquigarrow \star, a \not\rightsquigarrow (x : C_B) \rightarrow D_B$ if $A = \star, B = (x : C_B) \rightarrow D_B$ $a ::_{A, \ell, o} B \mathbf{Val}$ if $A = (x : C_A) \rightarrow D_A, B = \star$ $\mathbf{Blame} \ell o (a \rightarrow D_B)$ if there exists ℓ_a, o_a such that $\mathbf{Blame} \ell_a o_a a$ $\mathbf{Blame} \ell_a o_a a$ if there exists ℓ_A, o_A such that $\mathbf{Blame} \ell_A o_A A$ $\mathbf{Blame} \ell_A o_A A$ if there exists ℓ_B, o_B such that $\mathbf{Blame} \ell_B o_B B$ $\mathbf{Blame} \ell_B o_B B$	
cast-fun-ty	$\vdash (x : A) \rightarrow B : \star$ $(x : A) \rightarrow B \mathbf{Val}$	
cast-fun-app	$\vdash \text{fun } f x \Rightarrow b : (x : A) \rightarrow B$ $\text{fun } f x \Rightarrow b \mathbf{Val}$	
cast-fun-app	$\vdash b : (x : A) \rightarrow B, \Gamma \vdash a : A$ $a \mathbf{Val}$, there exists a' such that $a \rightsquigarrow a'$, or there exists ℓ_a, o_a such that $\mathbf{Blame} \ell_a o_a a$ $b \mathbf{Val}$, there exists b' such that $b \rightsquigarrow b'$, or there exists ℓ_b, o_b such that $\mathbf{Blame} \ell_b o_b b$ if $b \rightsquigarrow b'$, $b a \rightsquigarrow b' a$ if $b \mathbf{Val}, a \rightsquigarrow a'$, $b a \rightsquigarrow b a'$ if $b \mathbf{Val}, a \mathbf{Val}$, $b = \text{fun } f x \Rightarrow c$ or $b = d_b ::_{D_b, \ell_b, o_b} (x : A') \rightarrow B', d_b \mathbf{Val}, D_b \mathbf{Val}$, $(\text{fun } f x \Rightarrow c)$ if $b = \text{fun } f x \Rightarrow c$ $(\text{fun } f x \Rightarrow c)$ $b = d_b ::_{D_b, \ell_b, o_b} (x : A') \rightarrow B', d_b \mathbf{Val}, D_b \mathbf{Val}, A' \equiv A, B' \equiv B$ $\vdash D_b : \star$ $D_b = \star$ or $D_b = (x : C_{D_b}) \rightarrow D_{D_b}$ if $D_b = \star$, $\mathbf{Blame} \ell_b o_b b$ if $D_b = D_b = (x : C_{D_b}) \rightarrow D_{D_b}$, let $a_c = a ::_{A, \ell, o} B$ $(d_b ::_{(x : A_{D_b})} (x : A_{D_b}) \rightarrow D_{D_b})$ $(d_b a_c) ::_{B_{D_b} [a_c]} B_{D_b} [a_c]$ $\mathbf{Blame} \ell_b o_b b$ $\mathbf{Blame} \ell_a o_a a$	
	if there exists ℓ_b, o_b such that $\mathbf{Blame} \ell_b o_b b$ if there exists ℓ_a, o_a such that $\mathbf{Blame} \ell_a o_a a$	

⁶todo clean up extra conclusions

$$\frac{\overline{\diamond \vdash} \text{ wf-empty}}{\frac{H \vdash \quad H \vdash A : \star}{H, x : A \vdash} \text{ wf-ext}}$$

Figure 5: context well formedness

4.4 Cast Soundness

We can now prove cast soundness.

For any $\diamond \vdash c : C$, $c', c \rightsquigarrow^* c'$, if **Stuck** c' then **Blame** $\ell o c'$, where **Stuck** c' means c' is not a value and c' does not step. This follows by iteration the progress and preservation lemmas.

4.5 Further Properties

Because of the conversion rule and non-termination, type-checking is undecidable. All previous arguments from chapter 2 apply to why this may be acceptable, but the most relevant is that the system should only be used through the Elaboration procedure described in the next section.

As in the surface languages, the cast language is logically unsound, by design.

Just as there are many different flavors of definitional equality that could have been used in chapter 2, there are also many possible degrees that runtime equality can be enforced. The **Blame** relation in 4.3 outlines the minimal possible checking to support cast soundness. For instance⁷, **head** $\mathbb{B} 1$ (**rep** $\mathbb{B} true 0$) will result in blame since 1 and 0 have different head constructors. But **head** $\mathbb{B} 1$ (**rep** $\mathbb{B} true 9$) will not result in blame since 1 and 9 have the same head constructor and the computation can reduce to *true*.

It is likely that more aggressive checking is preferable in practice, especially in the presence of data types. That is why in our implementation we check equalities up to call-by-value.

This behavior is consistent with the conjectured partial correctness of logically unsound Call-by-Value execution for dependent types in [JZSW10].

Unlike static type-checking, these runtime checks have runtime costs. Since the language allows nontermination, checks can take forever to resolve. We don't expect this to be an issue in practice, since we could limit the number of steps allowed. Additionally, our implementation avoids casts when it knows that the types are equal.

4.5.1 Regularity

We will often want to limit ourselves to well formed contexts, which are defined in 5. This definition excludes nonsensical contexts like $x : 3_c \vdash$. Sensible contexts will be required for some further proofs.

⁷assuming the data types of chapter 3

Because the derivations were chosen to simplify the type soundness proof, the \equiv relation is untyped and thus untyped terms can be associated with typed terms. For instance, $\star \equiv (\lambda - \Rightarrow \star)(\star\star)$, (you can replace $(\star\star)$ with any untypeable expression). Which leads to junk typing judgments like $\vdash \star : (\lambda - \Rightarrow \star)(\star\star)$, while not $\vdash (\lambda - \Rightarrow \star)(\star\star) : \star$. Because of this we will sometimes need to assume $H \vdash A : \star$, in addition $H \vdash a : A$.

Part III

Elaboration

5 Elaboration

Even though the cast language allows us to optimistically assert equalities, manually noting every cast would be cumbersome. This can be solved with an elaboration procedure that translates (untyped) terms from the surface language into the cast language. If the term is well typed in the surface language elaboration will produce a term without blamable errors. Terms with unproven equality in types are mapped to a cast with enough information to point out the original source when an inequality is witnessed. Elaboration serves a similar role as the bidirectional type system did in chapter 2.

First we enrich the surface language with location information, ℓ , at every position that could result in a type mismatch⁶. Not that the location tags match exactly with the check annotations of the bidirectional system. We also insist that the set of locations is nonempty, and designate a specific null location `.that` that can be used when we need to generate fresh terms, but have no sensible location information⁸. All the meta theory from chapter 2 goes through assuming that all locations are equivalent. We will avoid writing these annotations when they are unneeded (explicitly in 7).

5.1 Examples

For example,

$\vdash (\lambda x \Rightarrow 7) ::_{\ell} \mathbb{B} \rightarrow \mathbb{B}$ elaborates to $\vdash (\lambda x \Rightarrow 7 ::_{\mathbb{N}, \ell, \text{bod}[x]} \mathbb{B})$
 $f : \mathbb{B} \rightarrow \mathbb{B} \vdash f_{\ell} 7 : \mathbb{B}$ elaborates to $f : \mathbb{B} \rightarrow \mathbb{B} \vdash f (7 ::_{\mathbb{N}, \ell, \text{arg}} \mathbb{B}) : \mathbb{B}$
 $f : \mathbb{N} \rightarrow \mathbb{B} \rightarrow \mathbb{B} \vdash f_{\ell} 7_{\ell'} 3 : \mathbb{B}$ elaborates to $f : \mathbb{N} \rightarrow \mathbb{B} \rightarrow \mathbb{B} \vdash f (7 ::_{\mathbb{N}, \ell, \text{arg}} \mathbb{N}) (3 ::_{\mathbb{N}, \ell', \text{arg}} \mathbb{B}) : \mathbb{B}$

⁸For instance, the parallel reduction relation will associate all locations, $\frac{M \Rightarrow M' \quad N \Rightarrow N'}{(x : M) \rightarrow N \Rightarrow (x : M') \rightarrow N'} \Rightarrow\text{-fun-ty}$, so that the relation does not discriminate over syntaxes that come from different locations. While the *max* function will map terms into the null location, $\text{max}((x : M_{\ell}) \rightarrow N_{\ell'}) = \text{max}((x : \text{max}(M)) \rightarrow \text{max}(N))$ so that the output is unique.

source labels,		
ℓ	$::=$	\dots
	$ $	$.$
		no source label
expressions,		
m, n, M, N	$::=$	x
	$ $	$m ::_{\ell} M^{\ell'}$
	$ $	\star
	$ $	$(x : M_{\ell}) \rightarrow N_{\ell'}$
	$ $	$\text{fun } f \ x \Rightarrow m$
	$ $	$m_{\ell} \ n$
		variable
		annotation
		type universe
		function type
		function
		application

Figure 6: Surface Language Syntax, With Locations

$m ::_{\ell} M^{\ell'}$	written	$m ::_{\ell} M$	when	ℓ' is irrelevant
$m ::_{\ell} M$	written	$m :: M$	when	ℓ is irrelevant
$(x : M_{\ell}) \rightarrow N_{\ell'}$	written	$(x : M) \rightarrow N$	when	ℓ, ℓ' are irrelevant
$m_{\ell} \ n$	written	$m \ n$	when	ℓ is irrelevant

Figure 7: Surface Language Abbreviations

5.2 Elaboration

The rules for elaboration are presented in figure 8. Elaboration rules are written in a style of bidirectional type checking. However, unlike bidirectional type checking, when checking an inference elaboration adds a cast assertion that the two types are equal. Thus any conversion checking can be suspended until runtime.

There are several desirable properties of elaboration,

- elaborated terms preserve the erasure (defined in 9)

1. if $H \vdash m \mathbf{Elab} \ a \xrightarrow{\cdot} A$ then $|m| = |a|$
2. if $H \vdash m \ a \xleftarrow{\ell, o} A \mathbf{Elab} \ a$ then $|m| = |a|$

by mutual induction on the **Elab** derivations

$$\begin{array}{c}
\frac{x : A \in H}{H \vdash x \mathbf{Elab} \, x \xrightarrow{\cdot} A} \overrightarrow{\mathbf{Elab}\text{-var}} \\
\\
\frac{}{H \vdash \star \mathbf{Elab} \, \star \xrightarrow{\cdot} \star} \overrightarrow{\mathbf{Elab}\text{-}\star} \\
\\
\frac{H \vdash M \xleftarrow{\cdot} \star \mathbf{Elab} \, A \quad H, x : A \vdash N \xleftarrow{\cdot} \star \mathbf{Elab} \, B}{H \vdash ((x : M_\ell) \rightarrow N_{\ell'}) \mathbf{Elab} \, ((x : A) \rightarrow B) \xrightarrow{\cdot} \star} \overrightarrow{\mathbf{Elab}\text{-fun-ty}} \\
\\
\frac{H \vdash m \mathbf{Elab} \, b \xrightarrow{\cdot} (x : A) \rightarrow B \quad H \vdash n \xleftarrow{\cdot} A \mathbf{Elab} \, a}{H \vdash (m_\ell n) \mathbf{Elab} \, (b a) \xrightarrow{\cdot} B[x := a]} \overrightarrow{\mathbf{Elab}\text{-fun-app}} \\
\\
\frac{H \vdash n \mathbf{Elab} \, a \xrightarrow{\cdot} (x : A) \rightarrow B \quad H \vdash n \xleftarrow{\cdot} A \mathbf{Elab} \, a}{H \vdash (m_\ell n) \mathbf{Elab} \, (b a) \xrightarrow{\cdot} B[x := a]} \overrightarrow{\mathbf{Elab}\text{-fun-app}} \\
\\
\frac{H \vdash M \xleftarrow{\cdot} \star \mathbf{Elab} \, A \quad H \vdash m \xleftarrow{\cdot} A \mathbf{Elab} \, a}{H \vdash (m ::_\ell M^{\ell'}) \mathbf{Elab} \, a \xrightarrow{\cdot} A} \overrightarrow{\mathbf{Elab}\text{-::}} \\
\\
\frac{H, f : (x : A) \rightarrow B, x : A \vdash m \xleftarrow{\cdot} B \mathbf{Elab} \, b}{H \vdash (\text{fun } f x \Rightarrow m) \xleftarrow{\cdot} (x : A) \rightarrow B \mathbf{Elab} \, (\text{fun } f x \Rightarrow b)} \overleftarrow{\mathbf{Elab}\text{-fun}} \\
\\
\frac{H \vdash m \mathbf{Elab} \, a \xrightarrow{\cdot} A}{H \vdash m \xleftarrow{\cdot} B \mathbf{Elab} \, (a ::_{A, \ell, o} B)} \overleftarrow{\mathbf{Elab}\text{-cast}} \\
\\
\frac{H \vdash m \mathbf{Elab} \, a \xrightarrow{\cdot} \star}{H \vdash m \xleftarrow{\cdot} \star \mathbf{Elab} \, a} \overleftarrow{\mathbf{Elab}\text{-conv-}\star}
\end{array}$$

Figure 8: Elaboration

$$\begin{array}{lcl}
|x| & = & x \\
|\star| & = & \star \\
|m ::_\ell M| & = & |m| \\
|(x : M_\ell) \rightarrow N_{\ell'}| & = & (x : |M|) \rightarrow |N| \\
|m_\ell n| & = & |m| |n| \\
|\text{fun } f x \Rightarrow m| & = & \text{fun } f x \Rightarrow |m| \\
|\Diamond| & = & \Diamond \\
|\Gamma, x : A| & = & |\Gamma|, x : |A| \\
|a ::_{A, \ell, o} B| & = & |a| \\
|(x : A) \rightarrow B| & = & (x : |A|) \rightarrow |B| \\
|\text{fun } f x \Rightarrow b| & = & \text{fun } f x \Rightarrow |b| \\
|ba| & = & |b| |a| \\
|H, x : M| & = & |H|, x : |M|
\end{array}$$

Figure 9: Erasure

$\overrightarrow{\mathbf{Elab}}\text{-var}$	$ x = x = x $	
$\overrightarrow{\mathbf{Elab}}\text{-}\star$	$ \star = \star = \star $	
$\overrightarrow{\mathbf{Elab}}\text{-fun-ty}$	$H \vdash M \xleftarrow{\ell, \cdot} \star \mathbf{Elab} A \quad H, x : A \vdash N \xleftarrow{\ell', \cdot} \star \mathbf{Elab} B$	
	$ M = A $	by induction
	$ N = B $	by induction
	$ (x : M_\ell) \rightarrow N_{\ell'} = (x : M) \rightarrow N = (x : A) \rightarrow B = (x : A) \rightarrow B $	
$\overrightarrow{\mathbf{Elab}}\text{-fun-app}$	$H \vdash m \mathbf{Elab} b \xrightarrow{\cdot} C \quad C \Rightarrow_* (x : A) \rightarrow B \quad H \vdash n \xleftarrow{\ell, \arg} A \mathbf{Elab} a$	
	$ m = b $	by induction
	$ n = a $	by induction
	$ m_\ell n = m n = b a = b a $	
$\overleftarrow{\mathbf{Elab}}\text{-fun}$	$H, f : (x : A) \rightarrow B, x : A \vdash m \xleftarrow{\ell, o, bod[x]} B \mathbf{Elab} b$	
	$ m = b $	by induction
	$ \text{fun } f x \Rightarrow m = \text{fun } f x \Rightarrow m = \text{fun } f x \Rightarrow b = \text{fun } f x \Rightarrow b $	
$\overrightarrow{\mathbf{Elab}}\text{-::}$	$H \vdash M \xleftarrow{\ell, \cdot} \star \mathbf{Elab} A \quad H \vdash m \xleftarrow{\ell, \cdot} A \mathbf{Elab} a$	
	$ m = a $	by induction
	$ m ::_\ell M = m = a $	
$\overleftarrow{\mathbf{Elab}}\text{-conv}$	$H \vdash m \mathbf{Elab} a \xrightarrow{\cdot} A, \dots$	
	$ m = a $	by induction
	$ m = a = a ::_{A, \ell, o} A' $	

- It follows that whenever an elaborated cast term evaluates, the corresponding surface term evaluates consistently
 1. if $H \vdash m \mathbf{Elab} a \xrightarrow{\cdot} A$, and $a \rightsquigarrow_* \star$ then $m \rightsquigarrow_* \star$
 2. if $H \vdash \mathbf{Elab} m a \xrightarrow{\cdot} A$, and $a \rightsquigarrow_* (x : A) \rightarrow B$ then there exists N and M such that $m \rightsquigarrow_* (x : N) \rightarrow M$

Since $a \rightsquigarrow_* a'$ implies $|a| \rightsquigarrow_* |a'|$ and $m \rightsquigarrow_* m'$ implies $|m| \rightsquigarrow_* |m'|$.

- elaborated terms are well-cast in a well formed ctx.
 1. for any $H \vdash a \mathbf{Elab} m \xrightarrow{\cdot} A$ then $H \vdash a : A, H \vdash A : \star$
 2. for any $H \vdash, H \vdash A : \star, H \vdash a \xleftarrow{\ell, o} A \mathbf{Elab} m$ then $H \vdash a : A$

by mutual induction on \mathbf{Elab} derivations

$\overrightarrow{\mathbf{Elab}}\text{-var}$	$x : A \in H$ $H \vdash A : \star$ $H \vdash x : A$	by $H \vdash$ cast-var
$\overrightarrow{\mathbf{Elab}}\text{-fun-ty}$	$H \vdash M^{\overleftarrow{\ell}} \star \mathbf{Elab} A \quad H, x : A \vdash N^{\overleftarrow{\ell'}} \star \mathbf{Elab} B$ $H \vdash A : \star$ $H, x : A \vdash B : \star$	by induction by induction
$\overrightarrow{\mathbf{Elab}}\text{-fun-app}$	$H \vdash m \mathbf{Elab} b \overrightarrow{\rightarrow} C \quad C \Rightarrow_* (x : A) \rightarrow B \quad H \vdash n^{\overleftarrow{\ell, arg}} A \mathbf{Elab} a$ $H \vdash b : C, H \vdash C : \star$ $H \vdash (x : A) \rightarrow B : \star$ $H \vdash A : \star, H, x : A \vdash B : \star$ $H \vdash b : (x : A) \rightarrow B$ $H \vdash a : A$ $H \vdash B[x := a] : \star$ $H \vdash ba : B[x := a]$	by induction by \Rightarrow_* preservation by fun-ty inversion cast-conv by induction by subst preservation cast-fun-app
$\overleftarrow{\mathbf{Elab}}\text{-fun}$	$H, f : (x : A) \rightarrow B, x : A \vdash, H, f : (x : A) \rightarrow B, x : A \vdash B : \star$ $H, f : (x : A) \rightarrow B, x : A \vdash m^{\overleftarrow{\ell, o, bod[x]}} B \mathbf{Elab} b$	TODO revise! by reg by removing free vars similarly with fun-ty inversion by induction cast-fun
$\overrightarrow{\mathbf{Elab}}\text{-:}$	$H \vdash M^{\overleftarrow{\ell}} \star \mathbf{Elab} A \quad H \vdash m^{\overleftarrow{\ell}} A \mathbf{Elab} a$ $H \vdash A : \star$ $H \vdash a : A$	by induction by induction
$\overleftarrow{\mathbf{Elab}}\text{-conv}$	$H \vdash, H \vdash A' : \star, H \vdash m \mathbf{Elab} a \overrightarrow{\rightarrow} A, \text{ with some } \ell, o$ $H \vdash a : A, H \vdash A : \star$ $H \vdash a ::_{A, \ell, o} B : B$	by induction cast-::

- We conjecture that every term well typed in the bidirectional surface language elaborates

1. if $\Gamma \vdash$, then there exists H such that $\Gamma \mathbf{Elab} H$
2. $\Gamma \vdash m \overrightarrow{\rightarrow} M$ then there exists H, a and A such that $\Gamma \mathbf{Elab} H, H \vdash m \mathbf{Elab} a \overrightarrow{\rightarrow} A$
3. $\Gamma \vdash m^{\overleftarrow{\ell}} M$ and given ℓ, o then there exists H, a and A such that $\Gamma \mathbf{Elab} H, H \vdash \mathbf{Elab} a m^{\overleftarrow{\ell, o}} A$

- We conjecture blame never points to something that checked in the bidirectional system

1. if $\vdash m \overrightarrow{\rightarrow} M$, and $\vdash \mathbf{Elab} m a \overrightarrow{\rightarrow} A$, then for no $a \rightsquigarrow^* a'$ will $\mathbf{Blame} \ell o a'$ occur

The last three guarantees are similar to the gradual guarantee [SVCB15] for gradual typing.

As formulated here the elaboration procedure is total. However in a realistic implementation the function application would reduce the function type to weak head normal form (to match realistic bidirectional implementations) and will be undecidable for some pathological terms. Because the application rule needs to determine if a type level computation results in a function type. If that computation “runs forever”, elaboration will “run forever”. If we did not allow general recursion (and the non-termination allowable by type-in-type), it seems elaboration would always terminate.

Unlike in gradual typing, we cannot elaborate arbitrary untyped syntax. The underlying type of a cast needs to be known so that a function type can swap its argument type at application. For instance, $\lambda x \Rightarrow x$ will not elaborate since the intended type is not known. Fortunately, our experimental testing suggests that a majority of randomly generated terms can be elaborated, while only a small minority of terms would type-check in the surface language. The programmer can make any term elaborate if they annotate the intended type. For instance, $(\lambda x \Rightarrow x) :: * \rightarrow *$ will elaborate.

Part IV

Suitable Warnings

Part V

Related Work

5.3 Dependent types and equality

have motivated many research projects [Pro13, SW15, CTW21]. However, these impressive efforts are still require a prohibitively high level of expertise from programmers. Further, since program equivalence is undecidable in general, no system will be able to statically verify every “obvious” equality for arbitrary user defined data types and functions. The above work is important, but it is not yet usable for non-experts. In the meantime systems should trust the programmer when they use an unverified equality.

5.4 Contract Systems, Gradual Types, and Blame

This paper has been influenced by the large amount of work done on higher order contracts [FF02], gradual types [SVCB15, GCT16], and especially blame [WF09, Wad15, AJSW17]. Most work in those areas focuses on simply typed languages that are not necessarily pure.

The implementation also takes inspiration from “Abstracting gradual typing” [GCT16], where static evidence annotations become runtime checks. A system for gradual dependent types has been presented in [ETG19]. That paper is

largely concerned with establishing decidable type checking via an approximate term normalization. However, that system retains the definitional style of equality, so that it is possible, in principle, to get $vec(1+x) \neq vec(x+1)$ as a runtime error.

While the gradual typing goals of mixing static certainty with runtime checks are similar to our work here, the approach and details are different. Instead of trying to strengthen untyped languages by adding types, we take a dependent type system and allow more permissive equalities. This leads to different trade-offs in the design space. For instance, we cannot support completely unannotated code, but we do not need to complicate the type language with wildcards for uncertainty. Further we assume someone using a dependent type system feels positively about types in general and will not want fragments of completely typed code.

5.5 Refinement Style Approaches

In this paper we have described a full-spectrum dependently typed language. This means computation can appear uniformly in both term and type position. An alternative approach to dependent types is found in refinement type systems. Refinement type systems restrict type dependency, possibly to specific base types such as `int` or `bool`. It is straightforward to check these decidable equalities at runtime in the “Hybrid Type Checking” methodology [Fla06]. A notable example is [OTMW04] which describes a refinement system that limits predicates to base types. A refinement type system with higher order features is gradualized in [ZMMW20].

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Part VI

TODO

6 Alternative Elab formalization

$$\begin{array}{c}
\frac{x : A \in H}{H \vdash x \mathbf{Elab} x \vec{\cdot} A} \overrightarrow{\mathbf{Elab}\text{-var}} \\
\\
\frac{}{H \vdash \star \mathbf{Elab} \star \vec{\cdot} \star} \overrightarrow{\mathbf{Elab}\text{-}\star} \\
\\
\frac{H \vdash M \overleftarrow{\cdot}_{\ell, \cdot} \star \mathbf{Elab} A \quad H, x : A \vdash N \overleftarrow{\cdot}_{\ell', \cdot} \star \mathbf{Elab} B}{H \vdash ((x : M_\ell) \rightarrow N_{\ell'}) \mathbf{Elab} ((x : A) \rightarrow B) \vec{\cdot} \star} \overrightarrow{\mathbf{Elab}\text{-fun-ty}} \\
\\
\frac{H \vdash m \mathbf{Elab} b \vec{\cdot} (x : A) \rightarrow B \quad H \vdash n \overleftarrow{\cdot}_{\ell, \text{arg}} A \mathbf{Elab} a}{H \vdash (m_\ell n) \mathbf{Elab} (b a) \vec{\cdot} B[x := a]} \overrightarrow{\mathbf{Elab}\text{-fun-app}} \\
\\
\frac{H \vdash n \mathbf{Elab} a \vec{\cdot} C \quad C \Rightarrow (x : A) \rightarrow B \quad H \vdash n \overleftarrow{\cdot}_{\ell, \text{arg}} A \mathbf{Elab} a}{H \vdash (m_\ell n) \mathbf{Elab} (b a) \vec{\cdot} B[x := a]} \overrightarrow{\mathbf{Elab}\text{-fun-app}} \\
\\
\frac{H \vdash M \overleftarrow{\cdot}_{\ell', \cdot} \star \mathbf{Elab} A \quad H \vdash m \overleftarrow{\cdot}_{\ell, \cdot} A \mathbf{Elab} a}{H \vdash (m ::_\ell M^{\ell'}) \mathbf{Elab} a \vec{\cdot} A} \overrightarrow{\mathbf{Elab}\text{-}\cdot\cdot}
\end{array}$$

$$\begin{array}{c}
\frac{C \Rightarrow (x : A) \rightarrow B \quad H, f : (x : A) \rightarrow B, x : A \vdash m \overleftarrow{\ell, o, bod[x]} B \mathbf{Elab} b \overleftarrow{\quad} \mathbf{Elab-fun}}{H \vdash (\text{fun } f \ x \Rightarrow m) \overleftarrow{\ell, o} C \mathbf{Elab} (\text{fun } f \ x \Rightarrow b)} \\
\\
\frac{H \vdash m \mathbf{Elab} a \overrightarrow{\cdot} A}{H \vdash m \overleftarrow{\ell, o} B \mathbf{Elab} (a ::_{A, \ell, o} B) \overleftarrow{\quad} \mathbf{Elab-cast}}
\end{array}$$

The $\overleftarrow{\mathbf{Elab}}\text{-conv-}\star$ rule is a bit of a hack, and does not reflect the implementation. This formalization allows us to remove the special case of the conversion rule. The above is more accurate, but harder to prove. and proofs are very sensitive to the exact formalization of bidirectional type checking. seems more complicated to prove, though such a proof might be more enlightening.

The elaboration relation is mostly well behaved, but as presented here, it is undecidable for some pathological terms. Because the application rule follows the bidirectional style, we may need to determine if a type level computation results in a function type. If that computation “runs forever”, elaboration will “run forever”. If we did not allow general recursion (and the non-termination allowable by type-in-type), we suspect elaboration would always terminate.

7 Other

- Salvage dictation notes?
- where should I talk about error msgs?
- TODO relate to <https://hal.archives-ouvertes.fr/hal-01849166v3/document> ?

In every popular type system users are allowed to assume unsafe equalities. Thus

Part VII

scratch

8 error msgs

If programmers found dependent type systems easier to learn and use, software could become more reliable. Unfortunately, dependent type systems have yet to see widespread use in industry. One source of difficulty is the conservative equality checking required by most dependent type systems. This conservative equality is a source of some of the confusing error messages dependent type systems are known for [ESH19].

This error will help the programmer fix the bug in `add`. There is evidence that specific examples like this can help clarify the type error messages in OCaml [SJW16] and there has been an effort to make refinement type error messages more concrete in other systems like Liquid Haskell [HXB⁺19].

Part VIII

unused

TODO can $\Gamma \mathbf{Elab} H$ be made simpler?

by mutual induction on the bi-directional typing derivations and the well formedness of $\Gamma \vdash$

$\xrightarrow{ty-var}$	$x : M \in \Gamma$ \dots $x : A \in H$ $H \vdash x \mathbf{Elab} x \xrightarrow{\cdot} A$	mutual induction $\xrightarrow{\mathbf{Elab-var}}$
$\xrightarrow{ty-\star}$	\dots $H \vdash \star \mathbf{Elab} \star \xrightarrow{\cdot} \star$	mutual induction $\xrightarrow{\mathbf{Elab-\star}}$
$\xrightarrow{ty-::}$	$\Gamma \vdash m \xleftarrow{\cdot} M, \Gamma \vdash M \xleftarrow{\cdot} \star$, for some ℓ $H \vdash m \xleftarrow{\ell, \cdot} A \mathbf{Elab} a$ $\Gamma \vdash M \xleftarrow{\cdot} \star$ $H \vdash M \xleftarrow{\ell, \cdot} \star \mathbf{Elab} A'$ $m \sim a, M \sim A, m : M \text{ then } a : A$ $H \vdash (m ::_{\ell} M) \mathbf{Elab} a \xrightarrow{\cdot} A$	$(\ell \text{ comes from the syntactic extension})$ by induction, $o = .$ by regularity by induction, $o = .$ TODO $\xrightarrow{\mathbf{Elab-::}}$
$\xrightarrow{ty-fun-ty}$	$\Gamma \vdash M \xleftarrow{\cdot} \star \quad \Gamma, x : M \vdash N \xleftarrow{\cdot} \star$, with ℓ, ℓ' $H \vdash M \xleftarrow{\ell, \cdot} \star \mathbf{Elab} A$ $H, x : A \vdash N \xleftarrow{\ell', \cdot} \star \mathbf{Elab} B$	by induction, $o = .$ $\dots M \sim A$ $\xrightarrow{\mathbf{Elab-fun-ty}}$
$\xrightarrow{ty-fun-app}$	$H \vdash ((x : M_{\ell}) \rightarrow N_{\ell'}) \mathbf{Elab} ((x : A) \rightarrow B) \xrightarrow{\cdot} \star$ $\Gamma \vdash m \xrightarrow{\cdot} (x : N) \rightarrow M \quad \Gamma \vdash n \xleftarrow{\cdot} N$, with ℓ $H \vdash m \mathbf{Elab} b \xrightarrow{\cdot} C$ $C \Rightarrow_{\star} (x : A)$ $H \vdash n \xleftarrow{\ell, arg} A \mathbf{Elab} a$ $H \vdash (m_{\ell} n) \mathbf{Elab} (b a) \xrightarrow{\cdot} B [x := a]$	by induction ??? by induction $\xrightarrow{\mathbf{Elab-fun-app}}$

TODO: Mutual cases

The cast language records all the potential type conflicts, and uses runtime annotations to monitor these discrepancies. If the types do not conflict at runtime, evaluation will continue, otherwise the system will correctly blame the source location of the conflict with a concrete inequality. Terms that type according to the cast language type system will be referred to as **well casts**.

$C := \lambda b : \mathbb{B}_c. b \star \star \perp$
 $C \text{ true}_c = \perp \neq \star = C \text{ false}_c$
 $(\lambda pr : (\Pi C : (\mathbb{B}_c \rightarrow \star)). C \text{ true}_c \rightarrow C \text{ false}_c). pr (\lambda b : \mathbb{B}_c. b \star \star \perp) \perp \quad : \neg \text{true}_c =_{\mathbb{B}_c} \text{false}_c) \text{ refl}_{\text{true}_c : \mathbb{B}_c}$
 is elaborated to

8.1 Pretending $\star =_\star \perp$

Recall that we proved $\neg \star =_\star \perp$ what happens if we none the less assume it?
 Every type equality assumption needs an underlying term, here we can choose
 $\text{refl}_{\star : \star} : \star =_\star \star$, and cast that term to $\star =_\star \perp$ resulting in $\text{refl}_{\star : \star} :: \star =_\star \star =_\star \perp$.
 Recall that $\neg \star =_\star \perp$ is a short hand for $\star =_\star \perp \rightarrow \perp$

spoofing an equality
 $(\lambda pr : (\star =_\star \perp). pr (\lambda x. x) \perp \quad : \neg \star =_\star \perp) \text{ refl}_{\star : \star}$
 elaborates to
 $(\lambda pr : (\star =_\star \perp). pr (\lambda x. x) \perp \quad : \neg \star =_\star \perp) (\text{refl}_{\star : \star} :: (\star =_\star \star) =_l (\star =_\star \perp))$
 $\text{refl}_{\star : \star} :: (\star =_\star \star) =_l (\star =_\star \perp) (\lambda x. x) \perp \quad : \perp$
 $(\lambda C : (\star \rightarrow \star). \lambda x : C \star. x :: (\Pi C : (\star \rightarrow \star). C \star \rightarrow C \star) =_l (\Pi C : (\star \rightarrow \star). C \star \rightarrow C \perp)) (\lambda x. x) \perp$
 $: \perp$
 $(\lambda x : \star. x :: (\star \rightarrow \star) =_{l, bod} (\star \rightarrow \perp)) \perp \quad : \perp$
 $(\perp :: \star =_{l, bod. bod} \perp) \quad : \perp$

note that the program has not yet “gotten stuck”. to exercise this error,
 \perp must be eliminated, this can be done by tying to summon another type by
 applying it to \perp

$((\perp :: \star =_{l, bod. bod} \perp) \quad : \perp) \star$
 $((\Pi x : \star. x) :: \star =_{l, bod. bod} (\Pi x : \star. x)) \star$
 the computation is stuck, and the original application can be blamed on
 account that the “proof” has a discoverable type error at the point of application
 l

$\Pi C : (\star \rightarrow \star). C \star \rightarrow \underline{C \star} \neq \Pi C : (\star \rightarrow \star). C \star \rightarrow \underline{C \perp}$
 when
 $C := \lambda x. x$
 $C \perp = \perp \neq \star = C \star$

9 old proof

- Every term well typed in the surface language elaborates
 1. if $\Gamma \vdash$, then there exists H such that $\Gamma \mathbf{Elab} H$
 2. if $\Gamma \vdash, \Gamma \vdash m \xrightarrow{?} M$ then there exists H, a and A such that $\Gamma \mathbf{Elab} H,$
 $H \vdash m \mathbf{Elab} a \xrightarrow{?} A$
 3. if $\Gamma \vdash, \Gamma \vdash m \xleftarrow{?} M$ and given ℓ , othen there exists H, a and A such
 that $\Gamma \mathbf{Elab} H, H \vdash \mathbf{Elab} a m \xleftarrow{\ell_o} A$

TODO can $\Gamma \mathbf{Elab} H$ be made simpler?

by mutual induction on the bi-directional typing derivations and the well formedness of $\Gamma \vdash$

$$\begin{array}{c}
\frac{\Gamma \vdash m \overset{\leftarrow}{\vdash} M}{\Gamma \vdash m :: M \overset{\rightarrow}{\vdash} M} \text{ty-}:: \\
\frac{\Gamma \vdash M \overset{\leftarrow}{\vdash} \star \quad \Gamma, x : M \vdash N \overset{\leftarrow}{\vdash} \star}{\Gamma \vdash (x : M) \rightarrow N \overset{\rightarrow}{\vdash} \star} \text{ty-fun-ty} \\
\frac{\Gamma \vdash m \overset{\rightarrow}{\vdash} (x : N) \rightarrow M \quad \Gamma \vdash n \overset{\leftarrow}{\vdash} N}{\Gamma \vdash m n \overset{\rightarrow}{\vdash} M[x := n]} \text{ty-fun-app} \\
\frac{\Gamma, f : (x : N) \rightarrow M, x : N \vdash m \overset{\leftarrow}{\vdash} M \quad \Gamma \vdash M \overset{\leftarrow}{\vdash} \star \quad \Gamma, x : M \vdash N \overset{\leftarrow}{\vdash} \star}{\Gamma \vdash \text{fun } f x \Rightarrow m \overset{\leftarrow}{\vdash} (x : N) \rightarrow M} \overleftarrow{\text{ty-fun}} \\
\frac{\Gamma \vdash m \overset{\rightarrow}{\vdash} M \quad M \equiv M' \quad \Gamma \vdash M' \overset{\leftarrow}{\vdash} \star}{\Gamma \vdash m \overset{\leftarrow}{\vdash} M'} \overleftarrow{\text{ty-conv}} \\
\frac{H \vdash M \overset{\leftarrow}{\vdash}_{\ell, \cdot} \star \mathbf{Elab} A \quad H, x : A \vdash N \overset{\leftarrow}{\vdash}_{\ell', \cdot} \star \mathbf{Elab} B}{H \vdash ((x : M_\ell) \rightarrow N_{\ell'}) \mathbf{Elab} ((x : A) \rightarrow B) \overset{\rightarrow}{\vdash} \star} \mathbf{Elab-fun-ty} \\
\frac{H \vdash m \mathbf{Elab} b \overset{\rightarrow}{\vdash} C \quad C \Rightarrow_* (x : A) \rightarrow B \quad H \vdash n \overset{\leftarrow}{\vdash}_{\ell, \text{arg}} A \mathbf{Elab} a}{H \vdash (m_\ell n) \mathbf{Elab} (b a) \overset{\rightarrow}{\vdash} B[x := a]} \mathbf{Elab-fun-app} \\
\frac{H, f : (x : A) \rightarrow B, x : A \vdash m \overset{\leftarrow}{\vdash}_{\ell, o, \text{bod}[x]} B \mathbf{Elab} b}{H \vdash (\text{fun } f x \Rightarrow m) \overset{\leftarrow}{\vdash}_{\ell, o} (x : A) \rightarrow B \mathbf{Elab} (\text{fun } f x \Rightarrow b)} \overleftarrow{\mathbf{Elab-fun}} \\
\frac{H \vdash n \mathbf{Elab} a \overset{\rightarrow}{\vdash} (x : A) \rightarrow B \quad H \vdash n \overset{\leftarrow}{\vdash}_{\ell, \text{arg}} A \mathbf{Elab} a}{H \vdash (m_\ell n) \mathbf{Elab} (b a) \overset{\rightarrow}{\vdash} B[x := a]} \mathbf{Elab-fun-app} \\
\frac{H \vdash M \overset{\leftarrow}{\vdash}_{\ell, \cdot} \star \mathbf{Elab} A \quad H \vdash m \overset{\leftarrow}{\vdash}_{\ell, \cdot} A \mathbf{Elab} a}{H \vdash (m ::_\ell M) \mathbf{Elab} a \overset{\rightarrow}{\vdash} A} \mathbf{Elab-}:: \\
\frac{H \vdash m \mathbf{Elab} a \overset{\rightarrow}{\vdash} A}{H \vdash m \overset{\leftarrow}{\vdash}_{\ell, o} A' \mathbf{Elab} (a ::_{A, \ell, o} A')} \overleftarrow{\mathbf{Elab-conv}}
\end{array}$$

$\vec{ty}\text{-var}$	$x : M \in \Gamma$ \dots $x : A \in H$ $H \vdash x \mathbf{Elab} x \vec{\rightarrow} A$	mutual induction $\vec{\mathbf{Elab}}\text{-var}$
$\vec{ty}\text{-}\star$	\dots $H \vdash \star \mathbf{Elab} \star \vec{\rightarrow} \star$	mutual induction $\vec{\mathbf{Elab}}\text{-}\star$
$\vec{ty}\text{-}::$	$\Gamma \vdash m \stackrel{\leftarrow}{\vdash} M$, for some ℓ $H \vdash m \stackrel{\leftarrow}{\vdash}_{\ell, \cdot} A \mathbf{Elab} a$ $\Gamma \vdash M \stackrel{\leftarrow}{\vdash} \star$ $H \vdash M \stackrel{\leftarrow}{\vdash}_{\ell, \cdot} \star \mathbf{Elab} A'$ $m \sim a, M \sim A, m:M \text{ then } a:A$ $H \vdash (m ::_{\ell} M) \mathbf{Elab} a \vec{\rightarrow} A$	(ℓ comes from the syntactic extension to th by induction, $o = \cdot$. by regularity by induction ⁹ TODO $\vec{\mathbf{Elab}}\text{-}::$
$\vec{ty}\text{-fun-ty}$	$\Gamma \vdash M \stackrel{\leftarrow}{\vdash} \star \quad \Gamma, x : M \vdash N \stackrel{\leftarrow}{\vdash} \star$, with ℓ, ℓ' $H \vdash M \stackrel{\leftarrow}{\vdash}_{\ell, \cdot} \star \mathbf{Elab} A$ $H, x : A \vdash N \stackrel{\leftarrow}{\vdash}_{\ell', \cdot} \star \mathbf{Elab} B$ $H \vdash ((x : M_{\ell}) \rightarrow N_{\ell'}) \mathbf{Elab} ((x : A) \rightarrow B) \vec{\rightarrow} \star$	by induction, $o = \cdot$. $\dots M \sim A$ $\vec{\mathbf{Elab}}\text{-fun-ty}$
$\vec{ty}\text{-fun-app}$	$\Gamma \vdash m \vec{\rightarrow} (x : N) \rightarrow M \quad \Gamma \vdash n \stackrel{\leftarrow}{\vdash} N$, with ℓ $H \vdash m \mathbf{Elab} b \vec{\rightarrow} C$ $C \Rightarrow_* (x : A)$ $H \vdash n \stackrel{\leftarrow}{\vdash}_{\ell, \text{arg}} A \mathbf{Elab} a$ $H \vdash (m_{\ell} n) \mathbf{Elab} (b a) \vec{\rightarrow} B[x := a]$	by induction ??? by induction $\vec{\mathbf{Elab}}\text{-fun-app}$

1. Blame never points to something that checked in the bidirectional system

- (a) if $\vdash m \vec{\rightarrow} M$, and $\vdash \mathbf{Elab} m a \vec{\rightarrow} A$, then for no $a \rightsquigarrow^* a'$ will $\mathbf{Blame} \ell o a'$ occur

⁹not well founded?