

# an Intensional Dependent Type Theory with Type-in-Type, Recursion and Data

February 11, 2021

Owes much of it's presentation to <http://www.cs.yale.edu/homes/vilhelm/papers/msfp12prog.pdf>

## Pre-syntax

$\Delta, \Theta$	$::=$	$\cdot \mid x : \sigma, \Delta$	telescope
$\Xi$	$::=$	$\diamond$	data contexts: empty
		$\mid \Xi, \text{data } D \Delta \text{ where } \{\overline{d \Theta \rightarrow D \overline{M}}\}$	data type
		$\mid \Xi, \text{data } D \Delta$	abstract data type
$\Gamma$	$::=$	$\Xi; \Delta$	var contexts
$\sigma, \tau, M, N, H_-, P$	$::=$	$x$	expressions: var
		$\mid \star$	type universe
		$\mid \Pi x : \sigma. \tau$	types
		$\mid \text{fun } f : (x. \tau) . x : \sigma. M \mid M N$	terms
		$\mid D \overline{M}$	data type constructor
		$\mid d \overline{M} \mid \text{Case}_{x:D \overline{x}. \sigma} N \text{ of } \{\overline{d \overline{x} \Rightarrow M}\}$	data
$v$	$::=$	$x$	values
		$\mid \star$	
		$\mid \Pi x : \sigma. \tau$	
		$\mid \text{fun } f : (x. \tau) . x : \sigma. M$	
		$\mid D \overline{M} \mid d \overline{v}$	
$\overline{\square}$	$::=$	$\diamond \mid \square, \overline{\square}$	list of syntax

## Judgment Forms

core judgments:

$\Gamma \vdash$	$\Gamma$ context is well formed
$\Gamma \vdash M : \sigma$	$M$ is a term of type $\sigma$
$\Gamma \vdash M \equiv N : \sigma$	Definitional Equality on terms
$\Gamma \vdash M \Rightarrow N : \sigma$	$M$ parrelel-reduces to $N$
$\Gamma \vdash M \Rightarrow_* N : \sigma$	$M$ parrelel-reduces to $N$ after 0 or more steps
$M \rightsquigarrow N$	$M$ CBV-reduces to $N$ in 1 step

generalized judgments:

$$\begin{array}{ll}
\Gamma \vdash \Delta : \bar{*} & \text{telescope only has types } \Delta \\
\Gamma \vdash \overline{M} : \Delta & \text{the list of terms matches the types of } \Delta \\
\Gamma \vdash \overline{M} \Rightarrow \overline{M'} : \Delta & \text{the list of terms parallel reduces to} \\
\overline{M} \rightsquigarrow \overline{N} & \overline{M} \text{ CBV-reduces to } \overline{N} \text{ in 1 step}
\end{array}$$

## Judgments

The following judgments are mutually inductively defined.

### transitive reflexive closure

$$\begin{array}{c}
\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M \Rightarrow_* M' : \sigma} \Rightarrow \text{*refl} \\
\frac{\Gamma \vdash M \Rightarrow_* M' : \sigma \quad \Gamma \vdash M' \Rightarrow M'' : \sigma}{\Gamma \vdash M \Rightarrow_* M'' : \sigma} \Rightarrow \text{*step}
\end{array}$$

### Definitional Equality

$$\frac{\Gamma \vdash M \Rightarrow_* N : \sigma \quad \Gamma \vdash M' \Rightarrow_* N : \sigma}{\Gamma \vdash M \equiv M' : \sigma} \equiv \text{-Def}$$

### Context Rules

$$\begin{array}{c}
\overline{\Diamond} \vdash \text{C-Emp} \\
\frac{\Xi \vdash \Delta : \bar{*} \quad \forall d. \{ \Xi, \text{data } D \Delta \vdash \Theta : \bar{*} \quad \Xi, \Theta, \text{data } D \Delta \vdash \overline{M} : \Theta \}}{\Xi, \text{data } D \Delta \text{ where } \{ \overline{d\Theta} \rightarrow D\overline{M} \} \vdash} \text{C-def} \\
\frac{\Gamma \vdash \sigma : \star}{\Gamma, x : \sigma \vdash} \text{C-Ext}
\end{array}$$

### Conversion

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash \sigma \equiv \tau : \star}{\Gamma \vdash M : \tau} \text{Conv}$$

### Variables

$$\begin{array}{c}
\frac{\Gamma, x : \sigma, \Gamma' \vdash}{\Gamma, x : \sigma, \Gamma' \vdash x : \sigma} \text{Var} \\
\frac{\Gamma \vdash x : \sigma}{\Gamma \vdash x \Rightarrow x : \sigma} \text{Var-}\Rightarrow
\end{array}$$

## Type-in-Type

$$\frac{\Gamma \vdash}{\Gamma \vdash \star : \star} \star\text{-F}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \star \Rightarrow \star : \star} \star\text{-}\Rightarrow$$

## Dependent Recursive Functions

$$\frac{\Gamma \vdash \sigma : \star \quad \Gamma, x : \sigma \vdash \tau : \star}{\Gamma \vdash \Pi x : \sigma. \tau : \star} \Pi\text{-F}$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \quad \Gamma, x : \sigma, f : \Pi x : \sigma. \tau \vdash M : \tau}{\Gamma \vdash \text{fun } f : (x.\tau). x : \sigma. M : \Pi x : \sigma. \tau} \Pi\text{-I}$$

$$\frac{\Gamma \vdash M : \Pi x : \sigma. \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M N : \tau[x := N]} \Pi\text{-E}$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \quad \Gamma, x : \sigma, f : \Pi x : \sigma. \tau \vdash M \Rightarrow M' : \tau \quad \Gamma \vdash N \Rightarrow N' : \sigma}{\Gamma \vdash (\text{fun } f : (x.\tau). x : \sigma. M) N \Rightarrow M'[x := N', f := (\text{fun } f : (x.\tau). x : \sigma. M')]} \Pi\text{-}\Rightarrow$$

structural rules,

$$\frac{\Gamma \vdash \sigma \Rightarrow \sigma' : \star \quad \Gamma, x : \sigma \vdash \tau \Rightarrow \tau' : \star}{\Gamma \vdash \Pi x : \sigma. \tau \Rightarrow \Pi x : \sigma'. \tau' : \star} \Pi\text{-F-}\Rightarrow$$

$$\frac{\Gamma \vdash M \Rightarrow M' : \Pi x : \sigma. \tau \quad \Gamma \vdash N \Rightarrow N' : \sigma}{\Gamma \vdash M N \Rightarrow M' N' : \tau[x := N]} \Pi\text{-E-}\Rightarrow$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \quad \Gamma, x : \sigma, f : \Pi x : \sigma. \tau \vdash M \Rightarrow M' : \tau}{\Gamma \vdash \text{fun } f : (x.\tau). x : \sigma. M \Rightarrow \text{fun } f : (x.\tau). x : \sigma. M' : \Pi x : \sigma. \tau} \Pi\text{-I-}\Rightarrow$$

CBV

$$\overline{(\text{fun } f : (x.\tau). x : \sigma. M) v \rightsquigarrow M[x := v, f := (\text{fun } f : (x.\tau). x : \sigma. M)]} \Pi\text{-}\rightsquigarrow$$

$$\frac{M \rightsquigarrow M'}{M N \rightsquigarrow M' N} \Pi\text{-E-}\rightsquigarrow\text{-1}$$

$$\frac{N \rightsquigarrow N'}{v N \rightsquigarrow v N'} \Pi\text{-E-}\rightsquigarrow\text{-2}$$

## Dependent Data

$$\frac{\text{data } D \Delta \in \Gamma \quad \Gamma \vdash \overline{M} : \Delta}{\Gamma \vdash D \overline{M} : \star} D\text{-F}'$$

$$\frac{\text{data } D \Delta \text{ where } \left\{ \overline{d_i \Theta_i \rightarrow D \overline{M}_i} \right\} \in \Gamma \quad \Gamma \vdash \overline{M} : \Delta}{\Gamma \vdash D \overline{M} : \star} D\text{-F}$$

$$\frac{\text{data } D \Delta \text{ where } \{C\} \in \Gamma \quad d\Theta \rightarrow D\overline{M}' \in C \quad \Gamma \vdash \overline{N} : \Theta}{\Gamma \vdash d\overline{N} : D\overline{M}' [\Theta := \overline{N}]} D\text{-I}$$

with some abuse of notation:  $\overline{M}_i$  parameterized over  $\Theta_i$  instead of  $\overline{x}_i$

$$\frac{\text{data } D \Delta \text{ where } \left\{ \overline{d_i \Theta_i \rightarrow D \overline{M}_i} \right\} \in \Gamma \quad \Gamma, \overline{y} : \Delta, x : D\overline{y} \vdash \sigma : \star \quad \Gamma \vdash N : D\overline{P} \quad \forall i. \Gamma, \overline{x}_i : \Theta_i \vdash O_i : \sigma [x := d\overline{x}_i, \overline{y} := \overline{M}_i]}{\Gamma \vdash \text{Case}_{x:D\overline{y}. \sigma} N \text{ of } \left\{ \overline{d_i \overline{x}_i \Rightarrow O_i} \right\} : \sigma [x := N, \overline{y} := \overline{P}]} D\text{-E}$$

$$\text{data } D \Delta \text{ where } \left\{ \overline{d_i \Theta_i \rightarrow D \overline{M}_i} \right\} \in \Gamma \quad \Gamma, \overline{y} : \Delta, x : D\overline{y} \vdash \sigma : \star \quad \forall i. \Gamma, \overline{x}_i : \Theta_i \vdash O_i : \sigma [x := d\overline{x}_i, \overline{y} := \overline{M}_i]$$

$$d\Theta \rightarrow D\overline{M}' \in \overline{d_i \Theta_i \rightarrow D \overline{M}_i}$$

$$d\overline{x} \Rightarrow O \in \overline{d_i \overline{x}_i \Rightarrow O_i} \quad \Gamma \vdash \overline{N} \Rightarrow \overline{N}' : \Theta$$

$$\frac{\Gamma \vdash O \Rightarrow O' : \sigma [x := d_i \overline{x}_i, \overline{y} := \overline{N}]}{\Gamma \vdash \text{Case}_{x:D\overline{y}. \sigma} (d\overline{N}) \text{ of } \left\{ \overline{d_i \overline{x}_i \Rightarrow O_i} \right\} \Rightarrow O' [\overline{x} := \overline{N}'] : \sigma [x := d\overline{x}_i, \overline{y} := \overline{N}]} D\text{-}\Rightarrow$$

structural rules,

$$\frac{\text{data } D \Delta \in \Gamma \quad \Gamma \vdash \overline{M} \Rightarrow \overline{M}' : \Delta}{\Gamma \vdash D\overline{M} \Rightarrow D\overline{M}' : \star} D\text{-F}'\text{-}\Rightarrow$$

$$\begin{array}{c}
\frac{\text{data } D \Delta \text{ where } \left\{ \overline{d_i \Theta_i \rightarrow D\overline{M}_i} \right\} \in \Gamma \quad \Gamma \vdash \overline{M} \Rightarrow \overline{M'} : \Delta}{\Gamma \vdash D\overline{M} \Rightarrow D\overline{M'} : \star} D\text{-F-}\Rightarrow \\
\\
\frac{\text{data } D \Delta \text{ where } \{C\} \in \Gamma \quad d : \Theta \rightarrow D\overline{M'} \in C \quad \Gamma \vdash \overline{N} \Rightarrow \overline{N'} : \Theta}{\Gamma \vdash d\overline{N} \Rightarrow d\overline{N'} : D\overline{M'} [\Theta := \overline{N}]} D\text{-I} \\
\\
\frac{\text{data } D \Delta \text{ where } \left\{ \overline{d_i \Theta_i \rightarrow D\overline{M}_i} \right\} \in \Gamma \quad \Gamma, \overline{y} : \Delta, x : D\overline{y} \vdash \sigma : \star \quad \Gamma \vdash N \Rightarrow N' : D\overline{P} \quad \forall i. \Gamma, \overline{x}_i : \Theta_i \vdash O_i \Rightarrow O'_i : \sigma [x := d\overline{x}_i, \overline{y} := \overline{M}_i]}{\Gamma \vdash \text{Case}_{x:D\overline{y}.\sigma} N \text{ of } \left\{ \overline{d_i \overline{x}_i \Rightarrow O_i} \right\} \Rightarrow \text{Case}_{x:D\overline{y}.\sigma} N' \text{ of } \left\{ \overline{d_i \overline{x}_i \Rightarrow O'_i} \right\} : \sigma [x := N, \overline{y} := \overline{P}]} D\text{-E-}\Rightarrow \\
\\
\text{CBV} \\
\\
\frac{d\overline{x} \Rightarrow O \in \overline{d_i \overline{x}_i \Rightarrow O_i}}{\text{Case}_{x:D\overline{y}.\sigma} (d\overline{v}) \text{ of } \left\{ \overline{d_i \overline{x}_i \Rightarrow O_i} \right\} \rightsquigarrow O [\overline{x} := \overline{v}]} D\text{-}\rightsquigarrow \\
\\
\frac{M \rightsquigarrow M'}{\text{Case}_{x:D\overline{y}.\sigma} (M) \text{ of } \left\{ \overline{d_i \overline{x}_i \Rightarrow O_i} \right\} \rightsquigarrow \text{Case}_{x:D\overline{y}.\sigma} (M') \text{ of } \left\{ \overline{d_i \overline{x}_i \Rightarrow O_i} \right\}} D\text{-}\rightsquigarrow \\
\\
\frac{\overline{M} \rightsquigarrow \overline{M'}}{d\overline{M} \rightsquigarrow d\overline{M'}} D\text{-}\rightsquigarrow
\end{array}$$

## Telescopes

$$\begin{array}{c}
\overline{\Gamma \vdash . : \overline{\star}} \text{C-Emp} \\
\\
\frac{\Gamma, x : \sigma \vdash \Delta : \overline{\star} \quad \Gamma \vdash \sigma : \star}{\Gamma \vdash x : \sigma, \Delta : \overline{\star}} \Delta\text{-Ty-+} \\
\\
\overline{\Gamma \vdash \Diamond : .} \Delta\text{-Trm-Emp} \\
\\
\frac{\Gamma \vdash \overline{M} : \Delta [x := N] \quad \Gamma \vdash N : \sigma}{\Gamma \vdash N, \overline{M} : x : \sigma, \Delta} \Delta\text{-Trm-+}
\end{array}$$

parallel reductions

$$\frac{\diamond}{\Gamma \vdash \diamond \Rightarrow \diamond : .}$$

$$\frac{\Gamma \vdash \overline{M} \Rightarrow \overline{M'} : \Delta [x := N] \quad \Gamma \vdash N \Rightarrow N' : \sigma}{\Gamma \vdash N, \overline{M} \Rightarrow N', \overline{M'} : x : \sigma, \Delta} \Delta\text{-Trm-+}$$

$$\frac{N \rightsquigarrow N'}{\overline{v}, N, \overline{M} \rightsquigarrow \overline{v}, N', \overline{M}} D\text{-}\rightsquigarrow$$