A Dynamic Dependent Type Theory with Type-in-Type and Recursion

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1 Language

1.1 Surface Language

l			position identifier
Γ	$::=$	$\Diamond \mid \Gamma, x : M$	var contexts
m, n, h, M, N, H, P	$::=$	x	expressions: variable
	$\mid$	$m ::_l M$	type annotation
	$\mid$	$\star$	type universe
	$\mid$	$\Pi x : M_l.N_{l'}$	function type
	$\mid$	$\mathsf{fun}\, f.\, x.\, m \mid m\,_l n$	function constructor, eliminator
v	$::=$	x	values
	$\mid$	$\star \mid \Pi x : M.N$	type values
	$\mid$	$\mathsf{fun}\, f.\, x.\, n$	function values

1.2 Cast Language

H	$::=$	$\Diamond \mid H, x : A$	var contexts
a_h, b_h, c_h	$::=$	x	Term Head
	$\mid$	$\star$	
	$\mid$	$\Pi x : A.B$	
	$\mid$	$\mathsf{fun}\, f.\, x.a \mid b\, a$	
e	$::=$	$A \mid e =_{l,o} A$	type equality chain
a, b, c, A, B, C	$::=$	$\star \mid \Pi x : A.B$	Term
	$\mid$	$a_h :: e$	
o	$::=$	$. \mid o.arg$	observation
	$\mid$	$o.bod[a]$	

There is syntactic ambiguity at $\star$ and Π which are both a Term Head and a Term. When rules apply equally to both forms they may not be restated. Similarly for A and e.

2 Definitions

2.1 Substitution

$$
\begin{array}{rll}
\star\,[x \coloneqq a] & = \star & b\,[x \coloneqq a] \to c \\
(\Pi x : A.B)\,[x \coloneqq a] & = \Pi x : A\,[x \coloneqq a]\,.B\,[x \coloneqq a] & \\
(x :: A =_{l,o} e)\,[x \coloneqq a_h :: e'] & = a_h :: e' =_{l,o} e\,[x \coloneqq a_h :: e'] & \\
(y :: e)\,[x \coloneqq a] & = y :: e\,[x \coloneqq a_h :: e'] & \\
(b_h :: e)\,[x \coloneqq a] & = b_h\,[x \coloneqq a] :: e\,[x \coloneqq a] & \\
\star\,[x \coloneqq a] & = \star & b_h\,[x \coloneqq a] \dashrightarrow c_h \\
(\Pi x : A.B)\,[x \coloneqq a] & = \Pi x : A\,[x \coloneqq a]\,.B\,[x \coloneqq a] & \\
(\mathsf{fun}\, f.\, y.b)\,[x \coloneqq a] & = \mathsf{fun}\, f.\, y.b\,[x \coloneqq a] & \\
(b\, c)\,[x \coloneqq a] & = b\,[x \coloneqq a]\ c\,[x \coloneqq a] & \\
(e =_{l,o} A)\,[x \coloneqq a] & e\,[x \coloneqq a] =_{l,o[x \coloneqq a]} A\,[x \coloneqq a] & e\,[x \coloneqq a] \to e' \\
B\,[x \coloneqq a] & B\,[x \coloneqq a] & \\
.\,[x \coloneqq a] & . & o\,[x \coloneqq a] \to o' \\
(o.arg)\,[x \coloneqq a] & o\,[x \coloneqq a]\,.arg & \\
(o.bod[b])\,[x \coloneqq a] & o\,[x \coloneqq a]\,.bod[b\,[x \coloneqq a]] &
\end{array}
$$

and for contexts.

2.2 lookup

$$
\begin{array}{rll}
A\uparrow & = A & \text{apparent type} \\
e =_{l,o} A\uparrow & = A & \\
A\downarrow & = A & \text{raw type} \\
e =_{l,o} A\downarrow & = e\downarrow &
\end{array}
$$

2.3 Casts

Occasionally we will use the shorthand $A :: e$ to inject additional casts into A,

$$
\begin{array}{rl}
(A :: B) :: (B' =_{l,o} e) & = A :: B =_{l,o} e \\
(A :: e' =_{l,o} B) :: (B' =_{l,o} e) & = A :: e' =_{l,o} e
\end{array}
$$

3 Judgments

$H \vdash n\, Elab\, a$	Infer cast
$H \vdash n\, Elab_{A,l}\, a$	Check cast*
$H \vdash$	well formed context (not presented)
$H \vdash a : A$	apparent type
$H \vdash e : \overline{\star}$	well formed casts
$H \vdash a \equiv a' : A$	
$H \vdash a \Rrightarrow_* a' : A$	typed transitive closure of par reductions
$H \vdash a \Rrightarrow a' : A$	par reductions
$H \vdash e \sim e' : \overline{\star}$	
$H \vdash o \Rrightarrow o'$	
$H \vdash b \sim b' : B$	same except for observations and evidence
$H \vdash e \sim e' : \overline{\star}$	
$H \vdash e\, Elim_\star$	concrete elimination
$H \vdash e\, Elim_\Pi x : e_A.e_B$	

3.1 Judgment Variants

There are variants of judgments that reduce book keeping. All of these helper judgments could be expanded into the top level judgments above, but would make the presentation significantly messier.

3.1.1 Head Judgments

It is helpful to present some judgments that only consider head form, this avoids some bookkeeping with casts.

$H \vdash a_h : A$	head type
$a_h \Rrightarrow a$	
$a_h \sim a'_h$	

3.1.2 Untyped versions of Judgments

Some Judgments do not rely on type contexts, but are almost always used in a typed setting, so these compound judgments are used.

$H \vdash b \sim b' : B$	$= b \sim b'$	$H \vdash b : B$		
$H \vdash e \sim e' : \overline{\star}$	$= e \sim e'$	$H \vdash e : \overline{\star}$		
$H \vdash e \Rrightarrow e' : \overline{\star}$	$= e \Rrightarrow e'$	$H \vdash e : \overline{\star}$		
$H \vdash o \Rrightarrow o'$	$= o \Rrightarrow o'$	$H \vdash$		
$H \vdash e\, Elim_\star$	$= e\, Elim_\star$	$H \vdash e : \overline{\star}$		
$H \vdash e\, Elim_\Pi x : e_A.e_B$	$= e\, Elim_\Pi x : e_A.e_b$	$H \vdash e : \overline{\star}$	$H \vdash e_A : \overline{\star}$	$H \vdash e_B : \overline{\star}$

and likewise for head judgment

$H \vdash a_h \sim a'_h : A \quad = a_h \sim a'_h \qquad H \vdash a_h : A$

Unfortunately, for the purposes of induction, the following judgments need to be spelled out explicitly to avoid the conversion rule:

$H \vdash a \Rrightarrow a' : A$ typed par reduction

and for its head judgment

$H \vdash a_h \Rrightarrow a : A$

3.1.3 Typed versions of Judgments

3.2 Elaboration

3.2.1 Infer

$$\frac{x : A \in H}{H \vdash x\, Elab\, x :: A}$$

$$\frac{H \vdash M\, Elab_{\star,l}\, C \quad H \vdash m\, Elab_{C,l}\, a}{H \vdash m ::_l M\, Elab\, a}$$

$$\frac{H \vdash}{H \vdash \star\, Elab\, \star}$$

$$\frac{H \vdash M\, Elab_{\star,l}\, A \quad H, x : A \vdash N\, Elab_{\star,l'}\, B}{H \vdash \Pi x : M_l.N_{l'}\, Elab\, \Pi x : A.B}$$

$$\frac{H \vdash m\, Elab\, b_h :: e \quad \Pi x : A.B = e \uparrow \quad H \vdash n\, Elab_{A,l}\, a}{H \vdash m\, {}_l n\, Elab\ (b_h :: e)\ a}$$

3.2.2 Check

$$\frac{H \vdash}{H \vdash \star\, Elab_{\star,l}\, \star}$$

$$\frac{H, f : \Pi x : A.B,\ x : A \vdash m\, Elab_{B,l}\, b}{H \vdash \mathsf{fun}\, f.\, x.\, m\, Elab_{\Pi x:A.B,l}\, \mathsf{fun}\, f.\, x.b}$$

$$\frac{H \vdash m\, Elab\, a_h :: e}{H \vdash m\, Elab_{A,l}\, a_h :: e =_{l,.} A}$$

3.3 Typing

$$\frac{H \vdash a : A \quad H \vdash A \equiv A' : \star}{H \vdash a : A'}\, conv$$

3.3.1 Term Typing

$$\frac{H \vdash}{H \vdash \star : \star}\, \star - ty$$

$$\frac{H \vdash A : \star \quad H, x : A \vdash B : \star}{H \vdash \Pi x : A.B\ :\ \star}\, \Pi - ty$$

$$\frac{H \vdash e : \overline{\star} \quad H \vdash a_h : e \downarrow}{H \vdash a_h :: e \quad : \quad e \uparrow}\, apparent$$

3.3.2 Head Typing

$$\frac{x : A \in H}{H \vdash x \ : \ A} var - ty$$

$$\frac{H, f : \Pi x : A.B, x : A \vdash b : B}{H \vdash \mathsf{fun}\, f.\, x.b \ : \ \Pi x : A.B} \Pi - \mathsf{fun} - ty$$

$$\frac{H \vdash b : \Pi x : A.B \quad H \vdash a : A}{H \vdash b\, a \ : \ B\,[x \coloneqq a]} \Pi - app - ty$$

3.3.3 Cast Typing

$$\frac{H \vdash A : \star}{H \vdash A : \overline{\star}} eq - ty - 1$$

$$\frac{H \vdash e : \overline{\star} \quad H \vdash A : \star}{H \vdash e =_{l,o} A : \overline{\star}} eq - ty - 2$$

3.4 Definitional Equality

$$\frac{H \vdash a \Rrightarrow_{*} b : A \quad H \vdash a' \Rrightarrow_{*} b' \ : A \quad H \vdash b \sim b' : A}{H \vdash a \equiv a' : A}$$

3.5 Consistent

A relation that equates terms except for source location and observation information

$$\frac{}{\star \sim \star}$$

$$\frac{A \sim A' \quad B \sim B'}{\Pi x : A.B \sim \Pi x : A'.B'}$$

$$\frac{a_h \sim a'_h \quad e \sim e'}{a_h :: e \sim a'_h :: e'}$$

$$\frac{e \sim e' \quad A \sim A'}{e =_{l,o} A \sim e' =_{l',o'} A'}$$

$$\frac{a \sim a'}{\mathsf{fun}\, f.\, x.a \sim \mathsf{fun}\, f.\, x.a'}$$

$$\frac{b \sim b' \quad a \sim a'}{b\, a \sim b'\, a'}$$

3.6 Parallel Reductions

$$\frac{H \vdash a : A}{H \vdash a \Rrightarrow_* a : A}$$

$$\frac{H \vdash a \Rrightarrow_* b : A \quad H \vdash b \Rrightarrow c : A}{H \vdash a \Rrightarrow_* c : A}$$

3.7 Parallel Reduction

3.7.1 Term Par reduction

$$\frac{}{\star \Rrightarrow \star}$$

$$\frac{A \Rrightarrow A' \quad B \Rrightarrow B'}{\Pi x : A.B \Rrightarrow \Pi x : A'.B'}$$

$$\frac{a_h \Rrightarrow a'_h \quad e \Rrightarrow e'}{a_h :: e \Rrightarrow a'_h :: e'}$$

3.7.2 Head Par reduction

$$\frac{b \Rrightarrow b' \quad a \Rrightarrow a' \quad e\, Elim_{\Pi}\, x : e_A.e_B \quad e_A \Rrightarrow e'_A \quad e_B \Rrightarrow e'_B}{(\mathsf{fun}\, f.\, x.b) :: e\, a \Rrightarrow (b' \left[f \coloneqq (\mathsf{fun}\, f.\, x.b'), x \coloneqq a' :: e'_A\right] :: e'_B \left[x \coloneqq a'\right])} \Pi C \Rrightarrow$$

$$\frac{}{x \Rrightarrow x}$$

$$\frac{b \Rrightarrow b'}{\mathsf{fun}\, f.\, x.b \Rrightarrow \mathsf{fun}\, f.\, x.b'} \Pi I \Rrightarrow$$

$$\frac{b \Rrightarrow b' \quad a \Rrightarrow a'}{b\, a \Rrightarrow b'\, a'} \Pi E \Rrightarrow$$

3.7.3 Cast Par reduction

$$\frac{e \Rrightarrow e' \quad A \Rrightarrow A' \quad o \Rrightarrow o'}{e =_{l,o} A \Rrightarrow e' =_{l,o'} A'}$$

annoyingly need to support observation reductions, to allow a substitution lemma to simplify the proof

3.7.4 Observation Par reduction

$$\frac{}{. \Rrightarrow .}$$

$$\frac{o \Rrightarrow o'}{o.arg \Rrightarrow o'.arg}$$

$$\frac{o \Rrightarrow o' \quad a \Rrightarrow a'}{o.bod[a] \Rrightarrow o'.bod[a']}$$

3.7.5 Typed Term Par reduction

$$\frac{H \vdash}{H \vdash \star \Rrightarrow \star : \star}$$

$$\frac{H \vdash A \Rrightarrow A' : \star \quad H \vdash B \Rrightarrow B' : \star}{H \vdash \Pi x : A.B \Rrightarrow \Pi x : A'.B' : \star}$$

$$\frac{H \vdash a_h \Rrightarrow a'_h : e \downarrow \quad H \vdash e \Rrightarrow e' : \bar{\star}}{H \vdash a_h :: e \Rrightarrow a'_h :: e \uparrow}$$

3.7.6 Typed Head Par reduction

$$\frac{H, f : \Pi x : A.B, x : B \vdash b \Rrightarrow b' : B \quad H \vdash a \Rrightarrow a' : A \quad e\, Elim_{\Pi}\, x : e_A.e_B \quad H \vdash e_A \Rrightarrow e'_A : \bar{\star} \quad H, x : B \vdash e_B \Rrightarrow e}{H \vdash (\mathsf{fun}\, f.\, x.b) :: e\, a \Rrightarrow (b' \left[f := (\mathsf{fun}\, f.\, x.b')\, , x := a' :: e'_A\right] :: e'_B \left[x := a'\right]) \ : \ B \left[x := a\right]}$$

$$\frac{x : A \in H}{H \vdash x \Rrightarrow x : A}$$

$$\frac{H, f : \Pi x : A.B, x : B \vdash b \Rrightarrow b' : B}{H \vdash \mathsf{fun}\, f.\, x.b \Rrightarrow \mathsf{fun}\, f.\, x.b' \ : \ \Pi x : A.B}\Pi I \Rrightarrow$$

$$\frac{H \vdash b \Rrightarrow b' : \Pi x : A.B \quad H \vdash a \Rrightarrow a' : A}{H \vdash b\, a \Rrightarrow b'\, a' : B \left[x := a\right]}\Pi E \Rrightarrow$$

3.8 Dynamic Check

$$\frac{}{\star\, Elim_{\star}}$$

$$\frac{}{\star :: \star\, Elim_{\star}}$$

$$\frac{e\, Elim_{\star} \quad A\, Elim_{\star}}{e =_{l,o} A\, Elim_{\star}}$$

$$\frac{}{\Pi x : A.B\, Elim_{\Pi}\, x : A.B}$$

$$\frac{e\, Elim_{\star}}{\Pi x : A.B :: e\, Elim_{\Pi}\, x : A.B}$$

$$\frac{e\, Elim_{\Pi}\, x : e_A.e_B}{\Pi x : A.B =_{l,o} e\, Elim_{\Pi}\, x : (A =_{l,o.arg} e_A)\, .e_B \left[x := x :: A =_{l,o.arg} A'\right] =_{l,o.bod[x]} B}$$

$$\frac{e\, Elim_{\Pi}\, x : e_A.e_B \quad e''\, Elim_{\star}}{(\Pi x : A.B :: e'') =_{l,o} e\, Elim_{\Pi}\, x : (A =_{l,o.arg} e_A)\, .e_B \left[x := x :: A =_{l,o.arg} A'\right] =_{l,o.bod[x]} B}$$

4 Call-by-Value Small Step

$$
\begin{array}{lcll}
v & ::= & & \star \mid \Pi x : A.B \\
 & & & v_h :: v_{eq} \\
v_h & ::= & \mid & x \\
 & & \mid & \star \\
 & & \mid & \Pi x : A.B \\
 & & \mid & \mathsf{fun}\, f.\, x.a \\
v_{eq} & ::= & & v \\
 & & \mid & v_{eq} =_{l,o} v \\
v_{obs} & ::= & & . \\
 & & \mid & v_{obs}.arg \\
 & & \mid & v_{obs}.bod[v]
\end{array}
$$

$$\frac{A \rightsquigarrow A'}{v_{obs}.bod[A] \rightsquigarrow v_{obs}.bod[A']}$$

$$\frac{o \rightsquigarrow o'}{v_{eq} =_{l,o} A \rightsquigarrow v_{eq} =_{l,o'} A}$$

$$\frac{A \rightsquigarrow A'}{v_{eq} =_{l,v_{obs}} A \rightsquigarrow v_{eq} =_{l,v_{obs}} A'}$$

$$\frac{e \rightsquigarrow e'}{e =_{l,o} A \rightsquigarrow e' =_{l,o} A}$$

$$\frac{e \rightsquigarrow e'}{a_h :: e \rightsquigarrow a_h :: e'}$$

$$\frac{a_h \rightsquigarrow a_h'}{a_h :: v_{eq} \rightsquigarrow a_h :: v_{eq}}$$

$$\frac{b \rightsquigarrow b'}{b\, a \rightsquigarrow b'\, a}$$

$$\frac{a \rightsquigarrow a'}{v\, a \rightsquigarrow v\, a'}$$

$$\frac{v_{eq}\, Elim_{\Pi}\, x : e_A.e_B}{(\mathsf{fun}\, f.\, x.b) :: v_{eq}\, v :: v_{eq}' \rightsquigarrow (b\, [f \coloneqq (\mathsf{fun}\, f.\, x.b)\, , x \coloneqq v :: e_A] :: e_B'\, [x \coloneqq v])}$$

(this substitutes non-value casts into values, which is a little awkward but doesn't break anything)