an Intensional Dependent Type Theory with Type-in-Type, Recursion and Data

February 5, 2021

Owes much of it's presentation to http://www.cs.yale.edu/homes/vilhelm/papers/msfp12prog.pdf

Pre-syntax

Δ, Θ	$::=$	$. \mid x : \sigma, \Delta$	telescope
Ξ	$::=$	$\Diamond \mid \Xi, \mathsf{data}\, D\, \Delta\, \mathsf{where} \left\{ \overline{d : \Theta \to D\overline{M} \mid} \right\}$	data contexts
Γ	$::=$	$\Xi; \Delta$	var contexts
$\sigma, \tau, M, N, H_{-}, P$	$::=$	x	expressions: var
	$\mid$	$\star$	type universe
	$\mid$	$\Pi x : \sigma . \tau$	types
	$\mid$	$\mathsf{fun}\, f : (x.\tau)\, .\, x : \sigma . M \mid M\, N$	terms
	$\mid$	$D\, \overline{M}$	data type constructor
	$\mid$	$d\, \overline{M} \mid \mathsf{Case}_{x : D\, \overline{x} . \sigma}\, N\, \mathsf{of} \left\{ \overline{d\overline{x} \Rightarrow M \mid} \right\}$	data
v	$::=$	x	values
	$\mid$	$\star$	
	$\mid$	$\Pi x : \sigma . \tau$	
	$\mid$	$\mathsf{fun}\, f : (x.\tau)\, .\, x : \sigma . M$	
	$\mid$	$D\, \overline{M} \mid d\, \overline{v}$	

Judgment Forms

core judgments:

$\Gamma \vdash$	Γ context is well formed
$\Gamma \vdash M : \sigma$	M is a term of type σ
$\Gamma \vdash M \equiv N : \sigma$	Definitional Equality on terms
$\Gamma \vdash M \Rrightarrow N : \sigma$	M parrelel-reduces to N
$\Gamma \vdash M \Rrightarrow_{*} N : \sigma$	M parrelel-reduces to N after 0 or more steps
$M \rightsquigarrow N$	M CBV-reduces to N in 1 step

generalized judgments:

$\Gamma \vdash \Delta : \overline{*}$
$\Gamma \vdash \overline{M} : \Delta$ the list of terms matches the types of Δ
$\Gamma \vdash \overline{M} \Rrightarrow \overline{M'} : \Delta$

$\overline{M} \rightsquigarrow \overline{N}$ $\overline{M}$ CBV-reduces to $\overline{N}$ in 1 step

Judgments

The following judgments are mutually inductively defined.

transitive reflexive closure

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M \Rrightarrow_* M' : \sigma} \Rrightarrow *\text{-refl}$$

$$\frac{\Gamma \vdash M \Rrightarrow_* M' : \sigma \quad \Gamma \vdash M' \Rrightarrow M'' : \sigma}{\Gamma \vdash M \Rrightarrow_* M';: \sigma} \Rrightarrow *\text{-step}$$

Definitional Equality

$$\frac{\Gamma \vdash M \Rrightarrow_* N : \sigma \quad \Gamma \vdash M' \Rrightarrow_* N : \sigma}{\Gamma \vdash M \equiv M' : \sigma} \equiv\text{-Def}$$

Context Rules

$$\frac{}{\lozenge \vdash} \text{C-Emp}$$

$$\frac{\Xi \vdash \Delta : \overline{*} \qquad \forall d. \left\{\Xi, \mathsf{data}\ D\,\Delta\ \mathsf{where}\ \{\,\} \vdash \mathit{\Theta} : \overline{*} \qquad \Xi, \mathit{\Theta}, \mathsf{data}\ D\,\Delta\ \mathsf{where}\ \{\,\} \vdash \overline{M} : \mathit{\Theta}\right\}}{\Xi, \mathsf{data}\ D\,\Delta\ \mathsf{where}\ \left\{\overline{d : \mathit{\Theta} \to D\overline{M}\,|}\right\} \vdash} \text{C-def}$$

$$\frac{\Gamma \vdash \sigma : \star}{\Gamma, x : \sigma \vdash} \text{C-Ext}$$

Conversion

$$\frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash \sigma \equiv \tau : \star}{\Gamma \vdash M : \tau} \text{Conv}$$

Variables

$$\frac{\Gamma, x : \sigma, \Gamma' \vdash}{\Gamma, x : \sigma, \Gamma' \vdash x : \sigma} \text{Var}$$

$$\frac{\Gamma \vdash x : \sigma}{\Gamma \vdash x \Rrightarrow x\ : \sigma} \text{Var-}\Rrightarrow$$

Type-in-Type

$$\frac{\Gamma \vdash}{\Gamma \vdash \star : \star}\ \star\text{-F}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \star \Rrightarrow \star : \star}\ \star\text{-}\Rrightarrow$$

Dependent Recursive Functions

$$\frac{\Gamma \vdash \sigma : \star \qquad \Gamma, x : \sigma \vdash \tau : \star}{\Gamma \vdash \Pi x : \sigma.\tau\ : \star}\ \Pi\text{-F}$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \qquad \Gamma, x : \sigma, f : \Pi x : \sigma.\tau \vdash M : \tau}{\Gamma \vdash \mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M\ :\ \mathit{\Pi} x : \sigma.\tau}\ \Pi\text{-I}$$

$$\frac{\Gamma \vdash M : \Pi x : \sigma.\tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M\, N\ :\ \tau\, [x \coloneqq N]}\ \Pi\text{-E}$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \qquad \Gamma, x : \sigma, f : \Pi x : \sigma.\tau \vdash M \Rrightarrow M' : \tau \qquad \Gamma \vdash N \Rrightarrow N' : \sigma}{\Gamma \vdash (\mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M)\ N\ \Rrightarrow\ M'\, [x \coloneqq N', f \coloneqq (\mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M')]\ :\ \tau\, [x \coloneqq N]}\ \Pi\text{-}\Rrightarrow$$

structural rules,

$$\frac{\Gamma \vdash \sigma \Rrightarrow \sigma'\ : \star \qquad \Gamma, x : \sigma \vdash \tau \Rrightarrow \tau'\ : \star}{\Gamma \vdash \Pi x : \sigma.\tau\ \Rrightarrow\ \Pi x : \sigma'.\tau'\ : \star}\ \Pi\text{-F-}\Rrightarrow$$

$$\frac{\Gamma \vdash M \Rrightarrow M'\ :\ \Pi x : \sigma.\tau \qquad \Gamma \vdash N \Rrightarrow N'\ : \sigma}{\Gamma \vdash M\, N\ \Rrightarrow\ M'\, N'\ :\ \tau\, [x \coloneqq N]}\ \Pi\text{-E-}\Rrightarrow$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \qquad \Gamma, x : \sigma, f : \Pi x : \sigma.\tau \vdash M \Rrightarrow M' : \tau}{\Gamma \vdash \mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M\ \Rrightarrow\ \mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M'\ :\ \mathit{\Pi} x : \sigma.\tau}\ \Pi\text{-I-}\Rrightarrow$$

CBV

$$\frac{}{(\mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M)\ v\ \rightsquigarrow\ M\, [x \coloneqq v, f \coloneqq (\mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M)]}\ \Pi\text{-}\rightsquigarrow$$

$$\frac{M\ \rightsquigarrow\ M'}{M\, N\ \rightsquigarrow\ M'\, N}\ \Pi\text{-E-}\rightsquigarrow\text{-1}$$

$$\frac{N\ \rightsquigarrow\ N'}{v\, N\ \rightsquigarrow\ v\, N'}\ \Pi\text{-E-}\rightsquigarrow\text{-2}$$

Dependent Data

with some abuse of notation: lists become applications

$$\frac{\begin{array}{c}\mathsf{data}\ D\,\Delta\ \mathsf{where}\ \left\{\overline{d_i : \Theta_i \to D\overline{M}_i}\right\} \in \Gamma \\ \Gamma \vdash \overline{M} : \Delta\end{array}}{\Gamma \vdash D\,\overline{M}\ : \star}\ D\text{-F}$$

$$\frac{\begin{array}{c}\mathsf{data}\ D\,\Delta\ \mathsf{where}\ \{C\} \in \Gamma \\ d : \Theta \to D\overline{M}' \in C \\ \Gamma \vdash \overline{N} : \Theta\end{array}}{\Gamma \vdash d\,\overline{N}\ :\ D\,\overline{M}'\left[\Theta := \overline{N}\right]}\ D\text{-I}$$

with some abuse of notation: $\overline{M}_i$ parameterized over Θ_i instead of $\overline{x}_i$

$$\frac{\begin{array}{c}\mathsf{data}\ D\,\Delta\ \mathsf{where}\ \left\{\overline{d_i : \Theta_i \to D\overline{M}_i}\right\} \in \Gamma \\ \Gamma, \overline{y} : \Delta, x : D\overline{y} \vdash \sigma : \star \\ \Gamma \vdash N : D\,\overline{P} \\ \forall i.\, \Gamma, \overline{x}_i : \Theta_i \vdash O_i : \sigma\left[x := d\overline{x}_i, \overline{y} := \overline{M}_i\right]\end{array}}{\Gamma \vdash \mathsf{Case}_{x:D\,\overline{y}.\sigma}\ N\ \mathsf{of}\left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\}\ :\ \sigma\left[x := N, \overline{y} := \overline{P}\right]}\ D\text{-E}$$

$$\frac{\begin{array}{c}\mathsf{data}\ D\,\Delta\ \mathsf{where}\ \left\{\overline{d_i : \Theta_i \to D\overline{M}_i}\right\} \in \Gamma \\ \Gamma, \overline{y} : \Delta, x : D\overline{y} \vdash \sigma : \star \\ \forall i.\, \Gamma, \overline{x}_i : \Theta_i \vdash O_i : \sigma\left[x := d\overline{x}_i, \overline{y} := \overline{M}_i\right] \\ \\ d : \Theta \to D\overline{M}'\ \in\ \overline{d_i : \Theta_i \to D\overline{M}_i} \\ \\ d\overline{x} \Rightarrow O\ \in\ \overline{d_i\overline{x}_i \Rightarrow O_i\,|} \\ \Gamma \vdash \overline{N} \Rrightarrow \overline{N'} : \Theta \\ \\ \Gamma \vdash O \Rrightarrow O' : \sigma\left[x := d\overline{x}_i, \overline{y} := \overline{N}\right]\end{array}}{\begin{array}{c}\Gamma \vdash \mathsf{Case}_{x:D\,\overline{y}.\sigma}\ \left(d\,\overline{N}\right)\ \mathsf{of}\left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\} \\ \Rrightarrow O'\left[\overline{x} := \overline{N'}\right] : \sigma\left[x := d\overline{x}_i, \overline{y} := \overline{N}\right]\end{array}}\ D\text{-}\Rrightarrow$$

structural rules,

$$\frac{\begin{array}{c}\mathsf{data}\ D\,\Delta\ \mathsf{where}\ \left\{\overline{d_i : \Theta_i \to D\overline{M}_i}\right\} \in \Gamma \\ \Gamma \vdash \overline{M} \Rrightarrow \overline{M'} : \Delta\end{array}}{\Gamma \vdash D\overline{M}\ \Rrightarrow D\overline{M'} : \star}\ D\text{-F-}\Rrightarrow$$

$$\frac{\begin{array}{c}\mathsf{data}\ D\,\Delta\ \mathsf{where}\ \{C\} \in \Gamma \\ d : \Theta \to D\overline{M}' \in C \\ \Gamma \vdash \overline{N} \Rrightarrow \overline{N'} : \Theta\end{array}}{\Gamma \vdash d\,\overline{N}\ \Rrightarrow d\,\overline{N'} :\ D\,\overline{M}'\left[\Theta := \overline{N}\right]}\ D\text{-I}$$

$$\frac{\begin{array}{c}\mathsf{data}\ D\,\Delta\ \mathsf{where}\ \left\{\overline{d_i : \Theta_i \to D\overline{M}_i}\right\} \in \Gamma \\ \Gamma, \overline{y} : \Delta, x : D\overline{y} \vdash \sigma : \star \\ \Gamma \vdash N \Rrightarrow N' : D\,\overline{P} \\ \forall i.\, \Gamma, \overline{x}_i : \Theta_i \vdash O_i \Rrightarrow O_i' : \sigma\left[x \coloneqq d\overline{x}_i, \overline{y} \coloneqq \overline{M}_i\right]\end{array}}{\Gamma \vdash \mathsf{Case}_{x:D\,\overline{y}.\sigma}\, N\ \mathsf{of}\left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\} \Rrightarrow \mathsf{Case}_{x:D\,\overline{y}.\sigma}\, N'\ \mathsf{of}\left\{\overline{d_i\overline{x}_i \Rightarrow O_i'\,|}\right\} : \sigma\left[x \coloneqq N, \overline{y} \coloneqq \overline{P}\right]}\ D\text{-E-}\Rrightarrow$$

CBV

$$\frac{d\overline{x} \Rightarrow O \in \overline{d_i\overline{x}_i \Rightarrow O_i\,|}}{\mathsf{Case}_{x:D\,\overline{y}.\sigma}\ (d\,\overline{v})\ \mathsf{of}\left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\} \rightsquigarrow O\,[\overline{x} \coloneqq \overline{v}]}\ D\text{-}\rightsquigarrow$$

$$\frac{M \rightsquigarrow M'}{\mathsf{Case}_{x:D\,\overline{y}.\sigma}\ (M)\ \mathsf{of}\left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\} \rightsquigarrow \mathsf{Case}_{x:D\,\overline{y}.\sigma}\ (M')\ \mathsf{of}\left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\}}\ D\text{-}\rightsquigarrow$$

$$\frac{\overline{M} \rightsquigarrow \overline{M'}}{d\overline{M} \rightsquigarrow d\overline{M'}}\ D\text{-}\rightsquigarrow$$

Telescopes

$$\frac{}{\Gamma \vdash\ .\ : \overline{\star}}\ \text{C-Emp}$$

$$\frac{\Gamma, x : \sigma \vdash \Delta : \overline{\star} \qquad \Gamma \vdash \sigma : \star}{\Gamma \vdash x : \sigma, \Delta\ : \overline{\star}}\ \Delta\text{-Ty-}+$$

$$\frac{}{\Gamma \vdash : .}\ \Delta\text{-Trm-Emp}$$

$$\frac{\Gamma \vdash \overline{M} : \Delta\,[x \coloneqq N] \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash N, \overline{M}\ :\ x : \sigma, \Delta}\ \Delta\text{-Trm-}+$$

parallel reductions

$$\frac{}{\Gamma \vdash \Rrightarrow : .}$$

$$\frac{\Gamma \vdash \overline{M} \Rrightarrow \overline{M'} : \Delta\,[x \coloneqq N] \qquad \Gamma \vdash \overline{N} \Rrightarrow \overline{N'} : \sigma}{\Gamma \vdash N, \overline{M}\ \Rrightarrow N', \overline{M'}\ :\ x : \sigma, \Delta}\ \Delta\text{-Trm-}+$$

.

$$\frac{N \rightsquigarrow N'}{\overline{v}, N, \overline{M} \rightsquigarrow \overline{v}, N', \overline{M}}\ D\text{-}\rightsquigarrow$$