

an Intensional Dependent Type Theory with Type-in-Type and Recursion

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Pre-syntax

Γ	$::=$	$\Diamond \mid \Gamma, x : \sigma$	var contexts
$\sigma, \tau, M, N, H_-, P$	$::=$	x	expressions: var
		\star	type universe
		$\Pi x : \sigma. \tau$	types
		$\text{fun } f : (x.\tau). x : \sigma. M \mid M N$	terms
v	$::=$	x	values
		$\star \mid \Pi x : \sigma. \tau$	
		$\text{fun } f : (x.\tau). x : \sigma. M$	

Judgment Forms

$\Gamma \vdash$	Γ context is well formed
$\Gamma \vdash M : \sigma$	M is a term of type σ
$\Gamma \vdash M \equiv N : \sigma$	Definitional Equality on terms
$\Gamma \vdash M \Rightarrow N : \sigma$	M parallel reduces to N
$\Gamma \vdash M \Rightarrow_* N : \sigma$	M parallel reduces to N after 0 or more steps
$M \rightsquigarrow N$	M CBV-reduces to N in 1 step

Judgments

The following judgments are mutually inductively defined.

Transitive reflexive closure

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M \Rightarrow_* M' : \sigma} \Rightarrow \text{*refl}$$

$$\frac{\Gamma \vdash M \Rightarrow_* M' : \sigma \quad \Gamma \vdash M' \Rightarrow M'' : \sigma}{\Gamma \vdash M \Rightarrow_* M'' : \sigma} \Rightarrow \text{*step}$$

Definitional Equality

$$\frac{\Gamma \vdash M \Rightarrow_* N : \sigma \quad \Gamma \vdash M' \Rightarrow_* N : \sigma}{\Gamma \vdash M \equiv M' : \sigma} \equiv\text{-Def}$$

Context Rules

$$\frac{}{\Diamond \vdash} \text{C-Emp}$$

$$\frac{\Gamma \vdash \sigma : \star}{\Gamma, x : \sigma \vdash} \text{C-Ext}$$

Conversion

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash \sigma \equiv \tau : \star}{\Gamma \vdash M : \tau} \text{Conv}$$

Variables

$$\frac{\Gamma, x : \sigma, \Gamma' \vdash}{\Gamma, x : \sigma, \Gamma' \vdash x : \sigma} \text{Var}$$

$$\frac{\Gamma \vdash x : \sigma}{\Gamma \vdash x \Rightarrow x : \sigma} \text{Var-}\Rightarrow$$

Type-in-Type

$$\frac{\Gamma \vdash}{\Gamma \vdash \star : \star} \star\text{-F}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \star \Rightarrow \star : \star} \star\text{-}\Rightarrow$$

Dependent Recursive Functions

$$\frac{\Gamma \vdash \sigma : \star \quad \Gamma, x : \sigma \vdash \tau : \star}{\Gamma \vdash \Pi x : \sigma. \tau : \star} \Pi\text{-F}$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \quad \Gamma, x : \sigma, f : \Pi x : \sigma. \tau \vdash M : \tau}{\Gamma \vdash \text{fun } f : (x. \tau). x : \sigma. M : \Pi x : \sigma. \tau} \Pi\text{-I}$$

$$\frac{\Gamma \vdash M : \Pi x : \sigma. \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M N : \tau[x := N]} \Pi\text{-E}$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \quad \Gamma, x : \sigma, f : \Pi x : \sigma. \tau \vdash M \Rightarrow M' : \tau \quad \Gamma \vdash N \Rightarrow N' : \sigma}{\Gamma \vdash (\text{fun } f : (x. \tau). x : \sigma. M) N \Rightarrow M[x := N', f := (\text{fun } f : (x. \tau). x : \sigma. M')] : \tau[x := N]} \Pi\text{-}\Rightarrow$$

structural rules,

$$\frac{\Gamma \vdash \sigma \Rightarrow \sigma' : \star \quad \Gamma, x : \sigma \vdash \tau \Rightarrow \tau' : \star}{\Gamma \vdash \Pi x : \sigma. \tau \Rightarrow \Pi x : \sigma'. \tau' : \star} \text{Pi-F-}\Rightarrow$$

$$\frac{\Gamma \vdash M \Rightarrow M' : \Pi x : \sigma. \tau \quad \Gamma \vdash N \Rightarrow N' : \sigma}{\Gamma \vdash M N \Rightarrow M' N' : \tau [x := N]} \text{Pi-E-}\Rightarrow$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \quad \Gamma, x : \sigma, f : \Pi x : \sigma. \tau \vdash M \Rightarrow M' : \tau}{\Gamma \vdash \text{fun } f : (x. \tau). x : \sigma. M \Rightarrow \text{fun } f : (x. \tau). x : \sigma. M' : \Pi x : \sigma. \tau} \text{Pi-I-}\Rightarrow$$

CBV,

$$\overline{(\text{fun } f : (x. \tau). x : \sigma. M) \ v \rightsquigarrow M [x := v, f := (\text{fun } f : (x. \tau). x : \sigma. M)]} \text{Pi-}\rightsquigarrow$$

$$\frac{M \rightsquigarrow M'}{M N \rightsquigarrow M' N} \text{Pi-E-}\rightsquigarrow\text{-1}$$

$$\frac{N \rightsquigarrow N'}{v N \rightsquigarrow v N'} \text{Pi-E-}\rightsquigarrow\text{-2}$$