Type Soundness in an Intensional Dependent Type Theory with Type-in-Type and Recursion

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1 Examples

logical unsoundness:

$\mathsf{fun}\, f : (x.x)\,.\, x : \star . f\, x \qquad : \Pi x : \star . x$

1.1 some constructs

While logically unsound, the language is extremely expressive. The following calculus of Constructions constructs are expressible,

$a_1 =_A a_2 \coloneqq \lambda A : \star . \lambda a_1 : A . \lambda a_2 : A . \Pi C : (A \to \star)\,. C\, a_1 \to C\, a_2$

$refl_{a:A} \coloneqq \lambda C : (A \to \star)\,. \lambda x : C\, a . x \qquad : a =_A a$

$Unit \coloneqq \Pi A : \star . A \to A$

$tt \coloneqq \lambda A : \star . \lambda a : A . a$

$\bot \coloneqq \Pi x : \star . x$

$\neg A \coloneqq \Pi A : \star .. A \to \bot$

1.1.1 Church Booleans

$\mathbb{B}_c \coloneqq \Pi A : \star . A \to A \to A$

$true_c \coloneqq \lambda A : \star . \lambda then : A . \lambda else : A . then$

$false_c \coloneqq \lambda A : \star . \lambda then : A . \lambda else : A . else$

1.1.2 Church $\mathbb{N}$

$\mathbb{N}_c \coloneqq \Pi A : \star . (A \to A) \to A \to A$

$0_c \coloneqq \lambda A : \star . \lambda s : (A \to A) . \lambda z : A . z$

$1_c \coloneqq \lambda A : \star . \lambda s : (A \to A) . \lambda z : A . s\, z$

$2_c \coloneqq \lambda A : \star . \lambda s : (A \to A) . \lambda z : A . s\, (s\, z)$

...

1.1.3 Large Elimination

Since there is type-in-type, a kind of large elimination is possible

$\lambda b : \mathbb{B}_c.b \star Unit\ \perp$

$\lambda n : \mathbb{N}_c.n \star (\lambda - .Unit)\ \perp$

thus $\neg 1_c =_{\mathbb{N}_c} 0_c$ is provable (in a non trivial way):

$\lambda pr : (\mathit{\Pi} C : (\mathbb{N}_c \to \star) .C\, 1_c \to C\, 0_c) .pr\ (\lambda n : \mathbb{N}_c.n \star (\lambda - .Unit)\ \perp)\ tt \qquad : \neg 1_c =_{\mathbb{N}_c} 0_c$

$\neg true_c =_{\mathbb{B}_c} false_c$ is provable:

$\lambda pr : (\mathit{\Pi} C : (\mathbb{B}_c \to \star) .C\, true_c \to C\, false_c) .pr\ (\lambda b : \mathbb{B}_c.b \star Unit\ \perp)\ tt \qquad : \neg true_c =_{\mathbb{B}_c} false_c$

$\neg Unit =_\star \perp$ is provable:

$\lambda pr : (\mathit{\Pi} C : (\star \to \star) .C\, Unit \to C\ \perp) .pr\ (\lambda x.x)\, tt \qquad : \neg Unit =_\star \perp$

$\neg \star =_\star \perp$ is provable:

$\lambda pr : (\mathit{\Pi} C : (\star \to \star) .C \star \to C\ \perp) .pr\ (\lambda x.x)\ \perp \qquad : \neg \star =_\star \perp$

There are more examples in [1] where Cardelli has studied a similar system.

2 Properties

2.1 Contexts

2.1.1 Sub-Contexts are well formed

The following rules are admissible:

$$\frac{\Gamma, \Gamma' \vdash}{\Gamma \vdash}$$

$$\frac{\Gamma, \Gamma' \vdash M : \sigma}{\Gamma \vdash}$$

$$\frac{\Gamma, \Gamma' \vdash M \Rrightarrow M' : \sigma}{\Gamma \vdash}$$

$$\frac{\Gamma, \Gamma' \vdash M \Rrightarrow_* M' : \sigma}{\Gamma \vdash}$$

$$\frac{\Gamma, \Gamma' \vdash M \equiv M' : \sigma}{\Gamma \vdash}$$

by mutual induction on the derivations.

2.1.2 Context weakening

For any derivation of $\Gamma \vdash \sigma : \star$, the following rules are admissible:

$$\frac{\Gamma, \Gamma' \vdash}{\Gamma, x : \sigma, \Gamma' \vdash}$$

$$\frac{\Gamma, \Gamma' \vdash M : \tau}{\Gamma, x : \sigma, \Gamma' \vdash M : \tau}$$

$$\frac{\Gamma, \Gamma' \vdash M \Rrightarrow M' : \sigma}{\Gamma, x : \sigma, \Gamma' \vdash M \Rrightarrow M' : \sigma}$$

$$\frac{\Gamma, \Gamma' \vdash M \Rrightarrow_* M' : \sigma}{\Gamma, x : \sigma, \Gamma' \vdash M \Rrightarrow_* M' : \sigma}$$

$$\frac{\Gamma, \Gamma' \vdash M \equiv M' : \tau}{\Gamma, x : \sigma, \Gamma' \vdash M \equiv M' : \tau}$$

by mutual induction on the derivations.

2.1.3 $\Rrightarrow$ is reflexive

The following rule is admissible:

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M \Rrightarrow M : \sigma} \Rrightarrow\text{-refl}$$

by induction

2.1.4 $\equiv$ is reflexive

The following rule is admissible:

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M \equiv M : \sigma} \equiv\text{-refl}$$

by $\Rrightarrow *$-refl

2.1.5 Context substitution

For any derivation of $\Gamma \vdash N : \tau$ the following rules are admissible:

$$\frac{\Gamma, x : \tau, \Gamma' \vdash}{\Gamma, \Gamma' [x \coloneqq N] \vdash}$$

$$\frac{\Gamma, x : \tau, \Gamma' \vdash M : \sigma}{\Gamma, \Gamma' [x \coloneqq N] \vdash M [x \coloneqq N] : \sigma [x \coloneqq N]}$$

$$\frac{\Gamma, x : \tau, \Gamma' \vdash M \Rrightarrow M' : \sigma}{\Gamma, \Gamma' [x \coloneqq N] \vdash M [x \coloneqq N] \Rrightarrow M' [x \coloneqq N] : \sigma [x \coloneqq N]}$$

$$\frac{\Gamma, x : \tau, \Gamma' \vdash M \Rrightarrow_* M' : \sigma}{\Gamma, \Gamma' [x \coloneqq N] \vdash M [x \coloneqq N] \Rrightarrow_* M' [x \coloneqq N] : \sigma}$$

$$\frac{\Gamma, x : \tau, \Gamma' \vdash M \equiv M' : \sigma}{\Gamma, \Gamma' [x \coloneqq N] \vdash M [x \coloneqq N] \equiv M' [x \coloneqq N] : \sigma [x \coloneqq N]}$$

by mutual induction on the derivations. Specifically, at every usage of x from the var rule in the original derivation, replace the usage of the var rule with the derivation of $\Gamma \vdash N : \tau$ weakened to the context of $\Gamma, \Gamma' [x \coloneqq N] \vdash N : \tau$, and apply $\Rrightarrow$-refl , $\Rrightarrow_*$-refl or $\equiv$-refl when needed.

2.2 Computation

2.2.1 $\Rrightarrow$ preserves type of source

The following rules are admissible:

$$\frac{\Gamma \vdash N \Rrightarrow N' : \tau}{\Gamma \vdash N : \tau}$$

by induction

2.2.2 $\Rrightarrow$-substitution

The following rule is admissible:

$$\frac{\Gamma, x : \sigma, \Gamma' \vdash M \Rrightarrow M' : \tau \quad \Gamma \vdash N \Rrightarrow N' : \sigma}{\Gamma, \Gamma' [x \coloneqq N] \vdash M [x \coloneqq N] \Rrightarrow M' [x \coloneqq N'] : \tau [x \coloneqq N]}$$

by induction on the $\Rrightarrow$ derivations

2.2.3 $\Rrightarrow$ is confluent

if $\Gamma \vdash M \Rrightarrow N : \sigma$ and $\Gamma \vdash M \Rrightarrow N' : \sigma$ then there exists P such that
$\Gamma \vdash N \Rrightarrow P : \sigma$ and $\Gamma \vdash N' \Rrightarrow P : \sigma$
by standard techniques

2.3 $\Rrightarrow_*$

2.3.1 $\Rrightarrow_*$ is transitive

The following rule is admissible:

$$\frac{\Gamma \vdash M \Rrightarrow_* M' : \sigma \quad \Gamma \vdash M' \Rrightarrow_* M'' : \sigma}{\Gamma \vdash M \Rrightarrow_* M' : \sigma} \Rrightarrow *\text{-trans}$$

by induction

2.3.2 $\Rrightarrow$ preserves type in destination

$$\frac{\Gamma \vdash N \Rrightarrow N' : \tau}{\Gamma \vdash N' : \tau}$$

By induction on the $\Rrightarrow$ derivation with the help of the substitution lemma.

- Π-$\Rrightarrow$

- $M'\,[x := N', f := (\mathsf{fun}\, f : (x.\tau')\,.\,x : \sigma'.M')] \,:\, \tau'\,[x := N']$ by the substitution lemma used on the inductive hypotheses
- $\tau\,[x := N] \Rrightarrow \tau'\,[x := N']$ by $\Rrightarrow$-substitution , so $\tau\,[x := N] \equiv \tau'\,[x := N']$
- by the conversion rule $M'\,[x := N', f := (\mathsf{fun}\, f : (x.\tau')\,.\,x : \sigma'.M')] \,:\, \tau\,[x := N]$

- Π-E-$\Rrightarrow$
  - $M'\,N' \,:\, \tau\,[x := N']$, by $\Rrightarrow$-substitution and reflexivity, $\tau\,[x := N] \Rrightarrow \tau\,[x := N']$, so $\tau\,[x := N] \equiv \tau\,[x := N']$
  - by the conversion rule $M'\,N' \,:\, \tau\,[x := N]$
- Π-I-$\Rrightarrow$
  - $\mathsf{fun}\, f \,:\, (x.\tau')\,.\,x \,:\, \sigma'.M' \,:\, \mathit{\Pi} x \,:\, \sigma'.\tau'$, $\mathit{\Pi} x \,:\, \sigma.\tau \Rrightarrow \mathit{\Pi} x \,:\, \sigma'.\tau'$, so $\mathit{\Pi} x : \sigma.\tau \equiv \mathit{\Pi} x : \sigma'.\tau'$
  - by the conversion rule $\mathsf{fun}\, f : (x.\tau')\,.\,x : \sigma'.M' \,:\, \mathit{\Pi} x : \sigma.\tau$
- all other cases are trivial

2.3.3 $\Rrightarrow_*$ preserves type

The following rule is admissible:

$$\frac{\Gamma \vdash M \Rrightarrow_* M' : \sigma}{\Gamma \vdash M : \sigma}$$

by induction

$$\frac{\Gamma \vdash M \Rrightarrow_* M' : \sigma}{\Gamma \vdash M' : \sigma}$$

by induction

2.3.4 $\Rrightarrow_*$ is confluent

if $\Gamma \vdash M \Rrightarrow_* N : \sigma$ and $\Gamma \vdash M \Rrightarrow_* N' : \sigma$ then there exists P such that $\Gamma \vdash N \Rrightarrow_* P : \sigma$ and $\Gamma \vdash N' \Rrightarrow_* P : \sigma$

Follows from $\Rrightarrow *$-trans and the confluence of $\Rrightarrow$ using standard techniques

2.3.5 $\equiv$ is symmetric

The following rule is admissible:

$$\frac{\Gamma \vdash M \equiv N : \sigma}{\Gamma \vdash N \equiv M : \sigma}\ \equiv\text{-sym}$$

trivial

2.3.6 $\equiv$ is transitive

$$\frac{\Gamma \vdash M \equiv N : \sigma \qquad \Gamma \vdash N \equiv P : \sigma}{\Gamma \vdash M \equiv P : \sigma} \equiv\text{-trans}$$

by the confluence of $\Rrightarrow_*$

2.3.7 $\equiv$ preserves type

The following rules are admissible:

$$\frac{\Gamma \vdash M \equiv M' : \sigma}{\Gamma \vdash M : \sigma}$$

$$\frac{\Gamma \vdash M \equiv M' : \sigma}{\Gamma \vdash M' : \sigma}$$

by the def of $\Rrightarrow_*$

2.3.8 Regularity

The following rule is admissible:

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \sigma : \star}$$

by induction with $\equiv$-preservation for the Conv case

2.3.9 $\rightsquigarrow$ implies $\Rrightarrow$

For any derivations of $\Gamma \vdash M : \sigma$, $M \rightsquigarrow M'$

$$\Gamma \vdash M \Rrightarrow M' : \sigma$$

by induction on $\rightsquigarrow$

2.3.10 $\rightsquigarrow$ preserves type

For any derivations of $\Gamma \vdash M : \sigma$, $M \rightsquigarrow M'$,

$$\Gamma \vdash M' : \sigma$$

since $\rightsquigarrow$ implies $\Rrightarrow$ and $\Rrightarrow$ preserves types

2.4 Type constructors

2.4.1 Type constructors are stable

- if $\Gamma \vdash * \Rrightarrow M : \sigma$ then M is $*$
- if $\Gamma \vdash * \Rrightarrow_* M : \sigma$ then M is $*$
- if $\Gamma \vdash \Pi x : \sigma.\tau \Rrightarrow M : \sigma$ then M is $\Pi x : \sigma'.\tau'$ for some σ', τ'

- if $\Gamma \vdash \Pi x : \sigma.\tau \Rrightarrow_{*} M : \sigma$ then M is $\Pi x : \sigma'.\tau'$ for some σ', τ'

by induction on the respective relations

2.4.2 Type constructors definitionaly unique

There is no derivation of $\Gamma \vdash * \equiv \Pi x : \sigma.\tau \ : \sigma'$ for any $\Gamma, \sigma, \tau, \sigma'$
from $\equiv$ -Def and constructor stability

2.5 Canonical forms

If $\Diamond \vdash v : \sigma$ then

- if σ is $\star$ then v is $\star$ or $\Pi x : \sigma.\tau$
- if σ is $\Pi x : \sigma'.\tau$ for some σ', τ then v is $\mathsf{fun}\, f : (x.\tau') \,.\, x : \sigma''.P'$ for some τ', σ'', P'

By induction on the typing derivation

- Conv,
  - if σ is $\star$ then eventually, it was typed with type-in-type, or Π-F. it could not have been typed by Π-I since constructors are definitionaly unique
  - if σ is $\Pi x : \sigma'.\tau$ then eventually, it was typed with Π-I. it could not have been typed by type-in-type, or Π-F since constructors are definitionaly unique
- type-in-type, $\Diamond \vdash v : \sigma$ is $\Diamond \vdash \star : \star$
- Π-F, $\Diamond \vdash v : \sigma$ is $\Diamond \vdash \Pi x : \sigma.\tau \ : \star$
- Π-I, $\Diamond \vdash v : \sigma$ is $\Diamond \vdash \mathsf{fun}\, f : (x.\tau) \,.\, x : \sigma.M \ : \ \mathit{\Pi} x : \sigma.\tau$
- no other typing rules are applicable

2.6 Progress

$\Diamond \vdash M : \sigma$ implies that M is a value or there exists N such that $M \rightsquigarrow N$.

By direct induction on the typing derivation with the help of the canonical forms lemma

Explicitly:

- M is typed by the conversion rule, then by **induction**, M is a value or there exists N such that $M \rightsquigarrow N$
- M cannot be typed by the variable rule in the empty context
- M is typed by type-in-type. M is $\star$, a value

- M is typed by Π-F. M is $\Pi x : \sigma.\tau$, a value
- M is typed by Π-I. M is $\mathsf{fun}\, f : (x.\tau) \,.\, x : \sigma.M'$, a value
- M is typed by Π-E. M is $P\,N$ then exist some σ, τ for $\Diamond \vdash P : \Pi x : \sigma.\tau$ and $\Diamond \vdash N : \sigma$. By **induction** (on the P branch of the derivation) P is a value or there exists P' such that $P \rightsquigarrow P'$. By **induction** (on the N branch of the derivation) N is a value or there exists N' such that $N \rightsquigarrow N'$
  - if P is a value then by **canonical forms**, P is$\mathsf{fun}\, f : (x.\tau) \,.\, x : \sigma.P'$ and
    * if N is a value then the one step reduction is $(\mathsf{fun}\, f : (x.\tau) \,.\, x : \sigma.P')\ N \rightsquigarrow P' \left[x \coloneqq N, f \coloneqq \mathsf{fun}\, f : (x.\tau) \,.\, x : \sigma.M\right]$
    * otherwise there exists N' such that $N \rightsquigarrow N'$, and the one step reduction is $(\mathsf{fun}\, f : (x.\tau) \,.\, x : \sigma.P')\ N \rightsquigarrow (\mathsf{fun}\, f : (x.\tau) \,.\, x : \sigma.P')\ N'$
  - otherwise, there exists P' such that $P \rightsquigarrow P'$ and the one step reduction is $P\,N \rightsquigarrow P'\,N$

2.7 Type Soundness

For any well typed term in an empty context, no sequence of small step reductions will cause result in a computation to "get stuck". Either a final value will be reached or further reductions can be taken. This follows by iterating the progress and preservation lemmas.

3 Non-Properties

- decidable type checking
- normalization/logical soundness

Definitional Equality does not preserve type constructors on the nose

If $\Gamma \vdash \sigma \equiv \sigma' : \star$ then
if σ is $\Pi x : \sigma''.\tau$ for some σ'', τ then σ' is $\Pi x : \sigma'''.\tau'$ for some σ''', τ'
counter example $\vdash \Pi x : \star.\star \equiv (\lambda x : \star.x)(\Pi x : \star.\star) : \star$
this implies the additional work in the Canonical forms lemma

4 differences from implementation

differences from Agda development

- in both presentations standard properties of variables binding and substitution are assumed

- In Agda the parallel reduction relation does not track the original typing judgment. This should not matter for the proof of confluence.
- only proved the function part of the canonical forms lemma (all that is needed for the proof)

differences from prototype

- bidirectional, type annotations are not always needed on functions
- toplevel recursion in addition to function recursion
- type annotations are not relevant for definitional equality

5 Proof improvements

- Clarify what unsoundness means
- proof outline at the top of document
- better function notation
- transition away from Greek letters

References

[1] Luca Cardelli. *A Polymorphic [lambda]-calculus with Type: Type*. Technical Report, DEC SRC, 130 Lytton Avenue, Palo Alto, CA 94301. May. SRC Research Report, 1986.