# an Intensional Dependent Type Theory with Type-in-Type, Recursion and Data

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Owes much of it's presentation to http://www.cs.yale.edu/homes/vilhelm/papers/msfp12prog.pdf

## Pre-syntax

| | | | |
|---|---|---|---|
| $\Delta, \Theta$ | $::=$ | $. \mid x : \sigma, \Delta$ | telescope |
| $\Xi$ | $::=$ | $\Diamond$ | data contexts: empty |
| | $\mid$ | $\Xi, \mathsf{data}\, D\,\Delta\, \mathsf{where}\, \left\{ \overline{d\,\Theta \to D\overline{M} \mid} \right\}$ | data type |
| | $\mid$ | $\Xi, \mathsf{data}\, D\,\Delta$ | abstract data type |
| $\Gamma$ | $::=$ | $\Xi; \Delta$ | var contexts |
| $\sigma, \tau, M, N, H_-, P$ | $::=$ | $\mathrm{x}$ | expressions: var |
| | $\mid$ | $\star$ | type universe |
| | $\mid$ | $\Pi x : \sigma.\tau$ | types |
| | $\mid$ | $\mathsf{fun}\, f : (x.\tau)\,.\, x : \sigma.M \mid M\,N$ | terms |
| | $\mid$ | $D\,\overline{M}$ | data type constructor |
| | $\mid$ | $d\,\overline{M} \mid \mathsf{Case}_{x:D\,\overline{x}.\sigma}\, N\, \mathsf{of}\, \left\{ \overline{d\overline{x} \Rightarrow M \mid} \right\}$ | data |
| $\mathrm{v}$ | $::=$ | $\mathrm{x}$ | values |
| | $\mid$ | $\star$ | |
| | $\mid$ | $\Pi x : \sigma.\tau$ | |
| | $\mid$ | $\mathsf{fun}\, f : (x.\tau)\,.\, x : \sigma.M$ | |
| | $\mid$ | $D\,\overline{M} \mid d\,\overline{v}$ | |
| $\overline{\Box}$ | $::=$ | $\Diamond \mid \Box, \overline{\Box}$ | list of syntax |

## Judgment Forms

core judgments:

| | |
|---|---|
| $\Gamma \vdash$ | $\Gamma$ context is well formed |
| $\Gamma \vdash M : \sigma$ | $M$ is a term of type $\sigma$ |
| $\Gamma \vdash M \equiv N : \sigma$ | Definitional Equality on terms |
| $\Gamma \vdash M \Rrightarrow N : \sigma$ | $M$ parrelel-reduces to $N$ |
| $\Gamma \vdash M \Rrightarrow_* N : \sigma$ | $M$ parrelel-reduces to $N$ after 0 or more steps |
| $M \rightsquigarrow N$ | $M$ CBV-reduces to $N$ in 1 step |

generalized judgments:

$\Gamma \vdash \Delta : \overline{\ast}$        telescope only has types $\Delta$

$\Gamma \vdash \overline{M} : \Delta$        the list of terms matches the types of $\Delta$

$\Gamma \vdash \overline{M} \Rightarrow \overline{M'} : \Delta$        the list of terms parallel reduces to

$\overline{M} \rightsquigarrow \overline{N}$        $\overline{M}$ CBV-reduces to $\overline{N}$ in 1 step

# Judgments

The following judgments are mutually inductively defined.

### transitive reflexive closure

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M \Rightarrow_* M' : \sigma} \Rightarrow *\text{-refl}$$

$$\frac{\Gamma \vdash M \Rightarrow_* M' : \sigma \quad \Gamma \vdash M' \Rightarrow M'' : \sigma}{\Gamma \vdash M \Rightarrow_* M'; : \sigma} \Rightarrow *\text{-step}$$

### Definitional Equality

$$\frac{\Gamma \vdash M \Rightarrow_* N : \sigma \quad \Gamma \vdash M' \Rightarrow_* N : \sigma}{\Gamma \vdash M \equiv M' : \sigma} \equiv \text{-Def}$$

### Context Rules

$$\frac{}{\Diamond \vdash} \text{C-Emp}$$

$$\frac{\Xi \vdash \Delta : \overline{\ast} \quad \forall d. \left\{ \Xi, \mathsf{data}\, D\, \Delta \vdash \Theta : \overline{\ast} \quad \Xi, \Theta, \mathsf{data}\, D\, \Delta \vdash \overline{M} : \Theta \right\}}{\Xi, \mathsf{data}\, D\, \Delta\, \mathsf{where}\, \left\{ d\Theta \rightarrow D\overline{M}\, | \right\} \vdash} \text{C-def}$$

$$\frac{\Gamma \vdash \sigma : \star}{\Gamma, x : \sigma \vdash} \text{C-Ext}$$

### Conversion

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash \sigma \equiv \tau : \star}{\Gamma \vdash M : \tau} \text{Conv}$$

### Variables

$$\frac{\Gamma, x : \sigma, \Gamma' \vdash}{\Gamma, x : \sigma, \Gamma' \vdash x : \sigma} \text{Var}$$

$$\frac{\Gamma \vdash x : \sigma}{\Gamma \vdash x \Rightarrow x : \sigma} \text{Var-}\Rightarrow$$

# Type-in-Type

$$\frac{\Gamma \vdash}{\Gamma \vdash \star : \star} \; \star\text{-F}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \star \Rrightarrow \star : \star} \; \star\text{-}\Rrightarrow$$

# Dependent Recursive Functions

$$\frac{\Gamma \vdash \sigma : \star \qquad \Gamma, x : \sigma \vdash \tau : \star}{\Gamma \vdash \Pi x : \sigma.\tau \; : \star} \; \Pi\text{-F}$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \qquad \Gamma, x : \sigma, f : \Pi x : \sigma.\tau \vdash M : \tau}{\Gamma \vdash \mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M \; : \; \Pi x : \sigma.\tau} \; \Pi\text{-I}$$

$$\frac{\Gamma \vdash M : \Pi x : \sigma.\tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M\, N \; : \; \tau\,[x := N]} \; \Pi\text{-E}$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \qquad \Gamma, x : \sigma, f : \Pi x : \sigma.\tau \vdash M \Rrightarrow M' : \tau \qquad \Gamma \vdash N \Rrightarrow N' : \sigma}{\Gamma \vdash (\mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M)\; N \;\Rrightarrow\; M'\,[x := N', f := (\mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M')] \; : \; \tau\,[x := N]} \; \Pi\text{-}\Rrightarrow$$

structural rules,

$$\frac{\Gamma \vdash \sigma \Rrightarrow \sigma' : \star \qquad \Gamma, x : \sigma \vdash \tau \Rrightarrow \tau' : \star}{\Gamma \vdash \Pi x : \sigma.\tau \;\Rrightarrow\; \Pi x : \sigma'.\tau' : \star} \; \Pi\text{-F-}\Rrightarrow$$

$$\frac{\Gamma \vdash M \Rrightarrow M' \; : \; \Pi x : \sigma.\tau \qquad \Gamma \vdash N \Rrightarrow N' : \sigma}{\Gamma \vdash M\, N \;\Rrightarrow\; M'\, N' \; : \; \tau\,[x := N]} \; \Pi\text{-E-}\Rrightarrow$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \qquad \Gamma, x : \sigma, f : \Pi x : \sigma.\tau \vdash M \Rrightarrow M' : \tau}{\Gamma \vdash \mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M \;\Rrightarrow\; \mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M' \; : \; \Pi x : \sigma.\tau} \; \Pi\text{-I-}\Rrightarrow$$

CBV

$$\frac{}{(\mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M)\; v \;\rightsquigarrow\; M\,[x := v, f := (\mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M)]} \; \Pi\text{-} \rightsquigarrow$$

$$\frac{M \;\rightsquigarrow\; M'}{M\, N \;\rightsquigarrow\; M'\, N} \; \Pi\text{-E-}\rightsquigarrow\text{-1}$$

$$\frac{N \;\rightsquigarrow\; N'}{v\, N \;\rightsquigarrow\; v\, N'} \; \Pi\text{-E-}\rightsquigarrow\text{-2}$$

# Dependent Data

$$\frac{\begin{array}{c}\mathsf{data}\ D\,\Delta \in \Gamma \\ \Gamma \vdash \overline{M} : \Delta\end{array}}{\Gamma \vdash D\,\overline{M}\ :\ \star}\ D\text{-}\mathbf{F'}$$

$$\frac{\begin{array}{c}\mathsf{data}\ D\,\Delta\ \mathsf{where}\ \left\{\overline{d_i\,\Theta_i \to D\overline{M}_i\,|}\right\} \in \Gamma \\ \Gamma \vdash \overline{M} : \Delta\end{array}}{\Gamma \vdash D\,\overline{M}\ :\ \star}\ D\text{-}F$$

$$\frac{\begin{array}{c}\mathsf{data}\ D\,\Delta\ \mathsf{where}\ \{C\} \in \Gamma \\ d\,\Theta \to D\overline{M}^{\,'} \in C \\ \Gamma \vdash \overline{N} : \Theta\end{array}}{\Gamma \vdash d\,\overline{N}\ :\ D\,\overline{M}^{\,'}\left[\Theta := \overline{N}\right]}\ D\text{-}I$$

with some abuse of notation: $\overline{M}_i$ parameterized over $\Theta_i$ instead of $\overline{x}_i$

$$\frac{\begin{array}{c}\mathsf{data}\ D\,\Delta\ \mathsf{where}\ \left\{\overline{d_i\,\Theta_i \to D\overline{M}_i\,|}\right\} \in \Gamma \\ \Gamma, \overline{y} : \Delta, x : D\overline{y} \vdash \sigma : \star \\ \Gamma \vdash N : D\,\overline{P} \\ \forall i.\,\Gamma, \overline{x}_i : \Theta_i \vdash O_i : \sigma\left[x := d\overline{x}_i, \overline{y} := \overline{M}_i\right]\end{array}}{\Gamma \vdash \mathsf{Case}_{x:D\,\overline{y}.\sigma}\ N\ \mathsf{of}\ \left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\}\ :\ \sigma\left[x := N, \overline{y} := \overline{P}\right]}\ D\text{-}E$$

$$\frac{\begin{array}{c}\mathsf{data}\ D\,\Delta\ \mathsf{where}\ \left\{\overline{d_i\,\Theta_i \to D\overline{M}_i\,|}\right\} \in \Gamma \\ \Gamma, \overline{y} : \Delta, x : D\overline{y} \vdash \sigma : \star \\ \forall i.\,\Gamma, \overline{x}_i : \Theta_i \vdash O_i : \sigma\left[x := d\overline{x}_i, \overline{y} := \overline{M}_i\right] \\[4pt] d\,\Theta \to D\overline{M}^{\,'}\ \in\ \overline{d_i\,\Theta_i \to D\overline{M}_i\,|} \\[4pt] d\overline{x} \Rightarrow O\ \in\ \overline{d_i\overline{x}_i \Rightarrow O_i\,|} \\ \Gamma \vdash \overline{N} \Rightarrow \overline{N'} : \Theta \\[4pt] \Gamma \vdash O \Rightarrow O' : \sigma\left[x := d\overline{x}_i, \overline{y} := \overline{N}\right]\end{array}}{\begin{array}{c}\Gamma \vdash \mathsf{Case}_{x:D\,\overline{y}.\sigma}\ \left(d\,\overline{N}\right)\ \mathsf{of}\ \left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\} \\ \Rightarrow O'\left[\overline{x} := \overline{N'}\right] : \sigma\left[x := d\overline{x}_i, \overline{y} := \overline{N}\right]\end{array}}\ D\text{-}{\Rightarrow}$$

structural rules,

$$\frac{\begin{array}{c}\mathsf{data}\ D\,\Delta \in \Gamma \\ \Gamma \vdash \overline{M} \Rightarrow \overline{M'} : \Delta\end{array}}{\Gamma \vdash D\overline{M}\ \Rightarrow\ D\overline{M'} : \star}\ D\text{-}\mathbf{F'}\text{-}{\Rightarrow}$$

$$\dfrac{\begin{array}{c}\text{data } D\,\Delta \text{ where } \left\{\overline{d_i\,\Theta_i \to D\overline{M}_i\,|}\right\} \in \Gamma \\ \Gamma \vdash \overline{M} \Rrightarrow \overline{M'} : \Delta\end{array}}{\Gamma \vdash D\overline{M} \Rrightarrow D\overline{M'} : \star}\; D\text{-F-}\Rrightarrow$$

$$\dfrac{\begin{array}{c}\text{data } D\,\Delta \text{ where } \{C\} \in \Gamma \\ d : \Theta \to D\overline{M}^{'} \in C \\ \Gamma \vdash \overline{N} \Rrightarrow \overline{N'} : \Theta\end{array}}{\Gamma \vdash d\,\overline{N} \Rrightarrow d\,\overline{N'} : D\,\overline{M}^{'}\left[\Theta := \overline{N}\right]}\; D\text{-I}$$

$$\dfrac{\begin{array}{c}\text{data } D\,\Delta \text{ where } \left\{\overline{d_i\,\Theta_i \to D\overline{M}_i\,|}\right\} \in \Gamma \\ \Gamma, \overline{y} : \Delta, x : D\overline{y} \vdash \sigma : \star \\ \Gamma \vdash N \Rrightarrow N' : D\,\overline{P} \\ \forall i.\,\Gamma, \overline{x}_i : \Theta_i \vdash O_i \Rrightarrow O'_i : \sigma\left[x := d\overline{x}_i, \overline{y} := \overline{M}_i\right]\end{array}}{\Gamma \vdash \mathsf{Case}_{x:D\,\overline{y}.\sigma}\,N \text{ of } \left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\} \Rrightarrow \mathsf{Case}_{x:D\,\overline{y}.\sigma}\,N' \text{ of } \left\{\overline{d_i\overline{x}_i \Rightarrow O'_i\,|}\right\} : \sigma\left[x := N, \overline{y} := \overline{P}\right]}\; D\text{-E-}\Rrightarrow$$

CBV

$$\dfrac{d\overline{x} \Rightarrow O \ \in\ \overline{d_i\overline{x}_i \Rightarrow O_i\,|}}{\mathsf{Case}_{x:D\,\overline{y}.\sigma}\,(d\,\overline{v}) \text{ of } \left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\} \rightsquigarrow O\left[\overline{x} := \overline{v}\right]}\; D\text{-}\rightsquigarrow$$

$$\dfrac{M \rightsquigarrow M'}{\mathsf{Case}_{x:D\,\overline{y}.\sigma}\,(M) \text{ of } \left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\} \rightsquigarrow \mathsf{Case}_{x:D\,\overline{y}.\sigma}\,(M') \text{ of } \left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\}}\; D\text{-}\rightsquigarrow$$

$$\dfrac{\overline{M} \rightsquigarrow \overline{M'}}{d\overline{M} \rightsquigarrow d\overline{M'}}\; D\text{-}\rightsquigarrow$$

**Telescopes**

$$\dfrac{}{\Gamma \vdash .\,:\overline{\star}}\; \text{C-Emp}$$

$$\dfrac{\Gamma, x : \sigma \vdash \Delta : \overline{\star} \qquad \Gamma \vdash \sigma : \star}{\Gamma \vdash x : \sigma, \Delta\ : \overline{\star}}\; \Delta\text{-Ty-+}$$

$$\dfrac{}{\Gamma \vdash \Diamond : .}\; \Delta\text{-Trm-Emp}$$

$$\dfrac{\Gamma \vdash \overline{M} : \Delta\left[x := N\right] \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash N, \overline{M}\ : x : \sigma, \Delta}\; \Delta\text{-Trm-+}$$

**parallel reductions**

$$\frac{\Diamond}{\Gamma \vdash \Diamond \Rrightarrow \Diamond : .}$$

$$\frac{\Gamma \vdash \overline{M} \Rrightarrow \overline{M'} : \Delta \left[ x := N \right] \qquad \Gamma \vdash N \Rrightarrow N' : \sigma}{\Gamma \vdash N, \overline{M} \Rrightarrow N', \overline{M'} : x : \sigma, \Delta} \ \Delta\text{-Trm-}+$$

.

$$\frac{N \rightsquigarrow N'}{\overline{v}, N, \overline{M} \rightsquigarrow \overline{v}, N', \overline{M}} \ D\text{-}\rightsquigarrow$$