an Intensional Dependent Type Theory with Type-in-Type and Recursion

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Pre-syntax

Γ	$::=$	$\lozenge \mid \Gamma, x : \sigma$	var contexts
$\sigma, \tau, M, N, H_{-}, P$	$::=$	x	expressions: var
	$\mid$	$\star$	type universe
	$\mid$	$\Pi x : \sigma.\tau$	types
	$\mid$	$\mathsf{fun}\, f : (x.\tau) \,.\, x : \sigma.M \mid M\, N$	terms
v	$::=$	x	values
	$\mid$	$\star \mid \Pi x : \sigma.\tau$	
	$\mid$	$\mathsf{fun}\, f : (x.\tau) \,.\, x : \sigma.M$	

Judgment Forms

$\Gamma \vdash$	Γ context is well formed
$\Gamma \vdash M : \sigma$	M is a term of type σ
$\Gamma \vdash M \equiv N : \sigma$	Definitional Equality on terms
$\Gamma \vdash M \Rrightarrow N : \sigma$	M parallel reduces to N
$\Gamma \vdash M \Rrightarrow_{*} N : \sigma$	M parallel reduces to N after 0 or more steps
$M \rightsquigarrow N$	M CBV-reduces to N in 1 step

Judgments

The following judgments are mutually inductively defined.

Transitive reflexive closure

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M \Rrightarrow_{*} M' : \sigma} \Rrightarrow *\text{-refl}$$

$$\frac{\Gamma \vdash M \Rrightarrow_{*} M' : \sigma \quad \Gamma \vdash M' \Rrightarrow M'' : \sigma}{\Gamma \vdash M \Rrightarrow_{*} M';: \sigma} \Rrightarrow *\text{-step}$$

Definitional Equality

$$\frac{\Gamma \vdash M \Rrightarrow_{*} N : \sigma \quad \Gamma \vdash M' \Rrightarrow_{*} N : \sigma}{\Gamma \vdash M \equiv M' : \sigma} \equiv\text{-Def}$$

Context Rules

$$\frac{}{\Diamond \vdash} \text{C-Emp}$$

$$\frac{\Gamma \vdash \sigma : \star}{\Gamma, x : \sigma \vdash} \text{C-Ext}$$

Conversion

$$\frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash \sigma \equiv \tau : \star}{\Gamma \vdash M : \tau} \text{Conv}$$

Variables

$$\frac{\Gamma, x : \sigma, \Gamma' \vdash}{\Gamma, x : \sigma, \Gamma' \vdash x : \sigma} \text{Var}$$

$$\frac{\Gamma \vdash x : \sigma}{\Gamma \vdash x \Rrightarrow x : \sigma} \text{Var-}\Rrightarrow$$

Type-in-Type

$$\frac{\Gamma \vdash}{\Gamma \vdash \star : \star} \star\text{-F}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \star \Rrightarrow \star : \star} \star\text{-}\Rrightarrow$$

Dependent Recursive Functions

$$\frac{\Gamma \vdash \sigma : \star \qquad \Gamma, x : \sigma \vdash \tau : \star}{\Gamma \vdash \Pi x : \sigma.\tau \ : \star} \Pi\text{-F}$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \qquad \Gamma, x : \sigma, f : \Pi x : \sigma.\tau \vdash M : \tau}{\Gamma \vdash \mathsf{fun}\, f : (x.\tau)\,.\, x : \sigma.M \ : \ \mathit{\Pi} x : \sigma.\tau} \Pi\text{-I}$$

$$\frac{\Gamma \vdash M : \Pi x : \sigma.\tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M\, N \ : \ \tau\,[x \coloneqq N]} \Pi\text{-E}$$

$$\frac{\Gamma \vdash \sigma \Rrightarrow \sigma' : \star \qquad \Gamma, x : \sigma \vdash \tau \Rrightarrow \tau' : \star \qquad \Gamma, x : \sigma, f : \Pi x : \sigma.\tau \vdash M \Rrightarrow M' : \tau \qquad \Gamma \vdash N \Rrightarrow N' : \sigma}{\Gamma \vdash (\mathsf{fun}\, f : (x.\tau)\,.\, x : \sigma.M)\ N \ \Rrightarrow\ M'\,[x \coloneqq N', f \coloneqq (\mathsf{fun}\, f : (x.\tau')\,.\, x : \sigma'.M')] \ : \ \tau\,[x \coloneqq N]} \Pi\text{-}\Rrightarrow$$

structural rules,

$$\frac{\Gamma \vdash \sigma \Rrightarrow \sigma' \ : \star \qquad \Gamma, x : \sigma \vdash \tau \Rrightarrow \tau' \ : \star}{\Gamma \vdash \Pi x : \sigma . \tau \ \Rrightarrow \ \Pi x : \sigma' . \tau' \ : \star} \, \Pi\text{-F-}\Rrightarrow$$

$$\frac{\Gamma \vdash M \Rrightarrow M' \ : \ \Pi x : \sigma . \tau \qquad \Gamma \vdash N \Rrightarrow N' \ : \sigma}{\Gamma \vdash M \, N \ \Rrightarrow \ M' \, N' \ : \ \tau \, [x \coloneqq N]} \, \Pi\text{-E-}\Rrightarrow$$

$$\frac{\Gamma \vdash \sigma \Rrightarrow \sigma' : \star \qquad \Gamma, x : \sigma \vdash \tau \Rrightarrow \tau' : \star \qquad \Gamma, x : \sigma, f : \Pi x : \sigma . \tau \vdash M \Rrightarrow M' : \tau}{\Gamma \vdash \mathsf{fun}\, f : (x.\tau) \, . \, x : \sigma . M \ \Rrightarrow \ \mathsf{fun}\, f : (x.\tau') \, . \, x : \sigma' . M' \ : \ \mathit{\Pi} x : \sigma . \tau} \, \Pi\text{-I-}\Rrightarrow$$

CBV,

$$\frac{}{(\mathsf{fun}\, f : (x.\tau) \, . \, x : \sigma . M) \ v \ \rightsquigarrow \ M \, [x \coloneqq v, f \coloneqq (\mathsf{fun}\, f : (x.\tau) \, . \, x : \sigma . M)]} \, \Pi\text{-} \rightsquigarrow$$

$$\frac{M \ \rightsquigarrow \ M'}{M \, N \ \rightsquigarrow \ M' \, N} \, \Pi\text{-E-}\rightsquigarrow\text{-1}$$

$$\frac{N \ \rightsquigarrow \ N'}{v \, N \ \rightsquigarrow \ v \, N'} \, \Pi\text{-E-}\rightsquigarrow\text{-2}$$