an Intensional Dependent Type Theory with Type-in-Type and Recursion

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1 Pre-syntax

Γ	$::=$	$\lozenge \mid \Gamma, x : M$	var contexts
m,n,h,p,M, N, H, P	$::=$	x	expressions: var
	$\mid$	$m :: M$	annotation
	$\mid$	$\star$	type universe
	$\mid$	$\Pi x : M.N$	types
	$\mid$	$\mathsf{fun}\, f.\, x.m \mid m\, n$	terms
v	$::=$	x	values
	$\mid$	$\star \mid \Pi x : M.N$	
	$\mid$	$\mathsf{fun}\, f.\, x.m$	

Supports type casts in the form of $m :: M$. However unlike the implementation the meta-theory is not bidirectional.

2 Judgment Forms

$\Gamma \vdash$	Γ context is well formed
$\Gamma \vdash m : M$	m checks as a term of type M
$\Gamma \vdash m \equiv m' : M$	Definitional Equality on terms
$\Gamma \vdash m \Rrightarrow m' : M$	m parallel reduces to m'
$\Gamma \vdash m \Rrightarrow_* m' : M$	m parallel reduces to m' after 0 or more steps
$m \rightsquigarrow m'$	m CBV-reduces to m' in 1 step

3 Judgments

The following judgments are mutually inductively defined.

3.1 Context Rules

$$\frac{}{\lozenge \vdash}\ \text{C-Emp}$$

$$\frac{\Gamma \vdash M : \star}{\Gamma, x : M \vdash}\ \text{C-Ext}$$

3.2 Definitional Equality

$$\frac{\Gamma \vdash m \Rrightarrow_* n : M \quad \Gamma \vdash m' \Rrightarrow_* n : M}{\Gamma \vdash m \equiv m' : M} \equiv\text{-Def}$$

3.3 Conversion

$$\frac{\Gamma \vdash m : M \qquad \Gamma \vdash M \equiv M' : \star}{\Gamma \vdash m : M'} \text{Conv}$$

3.4 Variables

$$\frac{\Gamma, x : M, \Gamma' \vdash}{\Gamma, x : M, \Gamma' \vdash x : M} \text{Var}$$

$$\frac{\Gamma, x : M, \Gamma' \vdash}{\Gamma, x : M, \Gamma' \vdash x \Rrightarrow x : M} \text{Var-}\Rrightarrow$$

3.5 Annotation

$$\frac{\Gamma \vdash m : M}{\Gamma \vdash m :: M : M} ::$$

$$\frac{\Gamma \vdash m \Rrightarrow m' : M}{\Gamma \vdash m :: M \Rrightarrow m' : M} :: \Rrightarrow$$

$$\frac{\Gamma \vdash m \Rrightarrow m' : M \quad \Gamma \vdash M \Rrightarrow M' : \star}{\Gamma \vdash m :: M \Rrightarrow m' :: M' : M} ::\text{-S-}\Rrightarrow$$

3.5.1 Call-by-Value

$$\frac{m \rightsquigarrow m'}{m :: M \rightsquigarrow m' :: M} ::\text{-S-}\rightsquigarrow\text{-1}$$

$$\frac{}{v :: M \rightsquigarrow v} ::\text{-S-}\rightsquigarrow\text{-2}$$

3.6 Type-in-Type

$$\frac{\Gamma \vdash}{\Gamma \vdash \star : \star} \star\text{-F}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \star \Rrightarrow \star : \star} \star\text{-}\Rrightarrow$$

3.7 Dependent Recursive Functions

$$\frac{\Gamma \vdash M : \star \qquad \Gamma, x : M \vdash N : \star}{\Gamma \vdash \Pi x : M.N \ : \star}\ \Pi\text{-F}$$

$$\frac{\Gamma \vdash M : \star \qquad \Gamma, x : M \vdash N : \star \qquad \Gamma, f : \Pi x : M.N, x : M \vdash n : N}{\Gamma \vdash \mathsf{fun}\, f.\, x.n \ : \ \mathit{\Pi} x : M.N}\ \Pi\text{-I}$$

$$\frac{\Gamma \vdash n : \ \Pi x : M.N \qquad \Gamma \vdash m : M}{\Gamma \vdash n\, m \ : \ N\, [x \coloneqq m]}\ \Pi\text{-E}$$

$$\frac{\Gamma \vdash M : \star \qquad \Gamma, x : M \vdash N : \star \qquad \Gamma, f : \Pi x : M.N, x : M \vdash n \Rrightarrow n' : N \qquad \Gamma \vdash m \Rrightarrow m' : M}{\Gamma \vdash (\mathsf{fun}\, f.\, x.n)\ m \ \Rrightarrow \ n'\, [x \coloneqq m', f \coloneqq (\mathsf{fun}\, f.\, x.n')] \ : \ M\, [x \coloneqq m]}\ \Pi\text{-}\Rrightarrow$$

3.7.1 Structural Rules

$$\frac{\Gamma \vdash M \Rrightarrow M' \ : \star \qquad \Gamma, x : M \vdash N \Rrightarrow N' \ : \star}{\Gamma \vdash \Pi x : M.N \ \Rrightarrow \ \Pi x : M'.N' \ : \star}\ \Pi\text{-F-}\Rrightarrow$$

$$\frac{\Gamma \vdash n \Rrightarrow n' \ : \ \Pi x : M.N \qquad \Gamma \vdash m \Rrightarrow m' \ : M}{\Gamma \vdash n\, m \ \Rrightarrow \ n'\, m' \ : \ N\, [x \coloneqq m]}\ \Pi\text{-E-}\Rrightarrow$$

$$\frac{\Gamma \vdash M : \star \qquad \Gamma, x : M \vdash N : \star \qquad \Gamma, f : \Pi x : M.N, x : M \vdash n \Rrightarrow n' : N \qquad \Gamma \vdash m \Rrightarrow m' : M}{\Gamma \vdash (\mathsf{fun}\, f.\, x.n)\ m \ \Rrightarrow \ (\mathsf{fun}\, f.\, x.n')\ m' \ : \ M\, [x \coloneqq m]}\ \Pi\text{-I-}\Rrightarrow$$

3.7.2 Call-by-Value

$$\frac{}{(\mathsf{fun}\, f.\, x.m)\ v \ \rightsquigarrow \ m\, [x \coloneqq v, f \coloneqq (\mathsf{fun}\, f.\, x.m)]}\ \Pi\text{-} \rightsquigarrow$$

$$\frac{m \ \rightsquigarrow \ m'}{m\, n \ \rightsquigarrow \ m'\, n}\ \Pi\text{-E-}\rightsquigarrow\text{-1}$$

$$\frac{n \ \rightsquigarrow \ n'}{v\, n \ \rightsquigarrow \ v\, n'}\ \Pi\text{-E-}\rightsquigarrow\text{-2}$$

3.8 Transitive reflexive closure of Parallel Reductions

$$\frac{\Gamma \vdash m : M}{\Gamma \vdash m \Rrightarrow_* m : M}\ \Rrightarrow *\text{-refl}$$

$$\frac{\Gamma \vdash m \Rrightarrow_* \ m' : M \quad \Gamma \vdash m' \Rrightarrow m'' : M}{\Gamma \vdash m \Rrightarrow_* \ m'' : M}\ \Rrightarrow *\text{-step}$$