an Intensional Dependent Type Theory with Type-in-Type, Recursion and Data

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There is an error in the parallel reduction that has not yet been corrected

Owes much of it's presentation to http://www.cs.yale.edu/homes/vilhelm/papers/msfp12prog.pdf

Pre-syntax

Δ, Θ	::=	$. \mid x : \sigma, \Delta$	telescope
Ξ	::=	$\Diamond$	data contexts: empty
	$\mid$	$\Xi, \mathsf{data}\, D\, \Delta\, \mathsf{where}\, \left\{ \overline{d\, \Theta \to D\overline{M} \mid} \right\}$	data type
	$\mid$	$\Xi, \mathsf{data}\, D\, \Delta$	abstract data type
Γ	::=	$\Xi; \Delta$	var contexts
$\sigma, \tau, M, N, H_-, P$	::=	x	expressions: var
	$\mid$	$\star$	type universe
	$\mid$	$\Pi x : \sigma.\tau$	types
	$\mid$	$\mathsf{fun}\, f : (x.\tau)\,.\, x : \sigma.M \mid M\, N$	terms
	$\mid$	$D\, \overline{M}$	data type constructor
	$\mid$	$d\, \overline{M} \mid \mathsf{Case}_{x:D\, \overline{x}.\sigma}\, N\, \mathsf{of}\, \left\{ \overline{d\overline{x} \Rightarrow M \mid} \right\}$	data
v	::=	x	values
	$\mid$	$\star$	
	$\mid$	$\Pi x : \sigma.\tau$	
	$\mid$	$\mathsf{fun}\, f : (x.\tau)\,.\, x : \sigma.M$	
	$\mid$	$D\, \overline{M} \mid d\, \overline{v}$	
$\overline{\Box}$	::=	$\Diamond \mid \Box, \overline{\Box}$	list of syntax

Judgment Forms

core judgments:

$$
\begin{array}{ll}
\Gamma \vdash & \Gamma \text{ context is well formed} \\
\Gamma \vdash M : \sigma & M \text{ is a term of type } \sigma \\
\Gamma \vdash M \equiv N : \sigma & \text{Definitional Equality on terms} \\
\Gamma \vdash M \Rightarrow N : \sigma & M \text{ parallel-reduces to } N \\
\Gamma \vdash M \Rightarrow_* N : \sigma & M \text{ parallel-reduces to } N \text{ after 0 or more steps} \\
M \rightsquigarrow N & M \text{ CBV-reduces to } N \text{ in 1 step} \\
\text{generalized judgments:} & \\
\Gamma \vdash \Delta : \overline{\ast} & \text{telescope only has types } \Delta \\
\Gamma \vdash \overline{M} : \Delta & \text{the list of terms matches the types of } \Delta \\
\Gamma \vdash \overline{M} \Rightarrow \overline{M'} : \Delta & \text{the list of terms parallel reduces to} \\
\overline{M} \rightsquigarrow \overline{N} & \overline{M} \text{ CBV-reduces to } \overline{N} \text{ in 1 step}
\end{array}
$$

Judgments

The following judgments are mutually inductively defined.

transitive reflexive closure

$$
\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M \Rightarrow_* M' : \sigma} \Rightarrow \ast\text{-refl}
$$

$$
\frac{\Gamma \vdash M \Rightarrow_* M' : \sigma \quad \Gamma \vdash M' \Rightarrow M'' : \sigma}{\Gamma \vdash M \Rightarrow_* M' : \sigma} \Rightarrow \ast\text{-step}
$$

Definitional Equality

$$
\frac{\Gamma \vdash M \Rightarrow_* N : \sigma \quad \Gamma \vdash M' \Rightarrow_* N : \sigma}{\Gamma \vdash M \equiv M' : \sigma} \equiv\text{-Def}
$$

Context Rules

$$
\frac{}{\Diamond \vdash} \text{C-Emp}
$$

$$
\frac{\Xi \vdash \Delta : \overline{\ast} \quad \forall d. \left\{ \Xi, \mathsf{data}\, D\, \Delta \vdash \Theta : \overline{\ast} \quad \Xi, \Theta, \mathsf{data}\, D\, \Delta \vdash \overline{M} : \Theta \right\}}{\Xi, \mathsf{data}\, D\, \Delta\ \mathsf{where}\ \left\{ \overline{d\Theta \to D\overline{M} \mid} \right\} \vdash} \text{C-def}
$$

$$
\frac{\Gamma \vdash \sigma : \star}{\Gamma, x : \sigma \vdash} \text{C-Ext}
$$

Conversion

$$
\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash \sigma \equiv \tau : \star}{\Gamma \vdash M : \tau} \text{Conv}
$$

Variables

$$\frac{\Gamma, x : \sigma, \Gamma' \vdash}{\Gamma, x : \sigma, \Gamma' \vdash x : \sigma} \text{ Var}$$

$$\frac{\Gamma \vdash x : \sigma}{\Gamma \vdash x \Rightarrow x : \sigma} \text{ Var-}\Rightarrow$$

Type-in-Type

$$\frac{\Gamma \vdash}{\Gamma \vdash \star : \star} \, \star\text{-F}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \star \Rightarrow \star : \star} \, \star\text{-}\Rightarrow$$

Dependent Recursive Functions

$$\frac{\Gamma \vdash \sigma : \star \qquad \Gamma, x : \sigma \vdash \tau : \star}{\Gamma \vdash \Pi x : \sigma.\tau \, : \star} \, \Pi\text{-F}$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \qquad \Gamma, x : \sigma, f : \Pi x : \sigma.\tau \vdash M : \tau}{\Gamma \vdash \mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M \, : \, \Pi x : \sigma.\tau} \, \Pi\text{-I}$$

$$\frac{\Gamma \vdash M : \Pi x : \sigma.\tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M\,N \, : \, \tau\,[x := N]} \, \Pi\text{-E}$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \qquad \Gamma, x : \sigma, f : \Pi x : \sigma.\tau \vdash M \Rightarrow M' : \tau \qquad \Gamma \vdash N \Rightarrow N' : \sigma}{\Gamma \vdash (\mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M)\,N \, \Rightarrow \, M'\,[x := N', f := (\mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M')] \, : \, \tau\,[x := N]} \, \Pi\text{-}\Rightarrow$$

structural rules,

$$\frac{\Gamma \vdash \sigma \Rightarrow \sigma' : \star \qquad \Gamma, x : \sigma \vdash \tau \Rightarrow \tau' : \star}{\Gamma \vdash \Pi x : \sigma.\tau \, \Rightarrow \, \Pi x : \sigma'.\tau' : \star} \, \Pi\text{-F-}\Rightarrow$$

$$\frac{\Gamma \vdash M \Rightarrow M' : \Pi x : \sigma.\tau \qquad \Gamma \vdash N \Rightarrow N' : \sigma}{\Gamma \vdash M\,N \, \Rightarrow \, M'\,N' : \tau\,[x := N]} \, \Pi\text{-E-}\Rightarrow$$

$$\frac{\Gamma, x : \sigma \vdash \tau : \star \qquad \Gamma, x : \sigma, f : \Pi x : \sigma.\tau \vdash M \Rightarrow M' : \tau}{\Gamma \vdash \mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M \, \Rightarrow \, \mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M' \, : \, \Pi x : \sigma.\tau} \, \Pi\text{-I-}\Rightarrow$$

CBV

$$\frac{}{(\mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M)\,v \, \leadsto \, M\,[x := v, f := (\mathsf{fun}\, f : (x.\tau)\,.\,x : \sigma.M)]} \, \Pi\text{-}\leadsto$$

$$\frac{M \, \leadsto \, M'}{M\,N \, \leadsto \, M'\,N} \, \Pi\text{-E-}\leadsto\text{-1}$$

$$\frac{N \, \leadsto \, N'}{v\,N \, \leadsto \, v\,N'} \, \Pi\text{-E-}\leadsto\text{-2}$$

Dependent Data

$$\dfrac{\begin{array}{c} \mathsf{data}\ D\,\Delta \in \Gamma \\ \Gamma \vdash \overline{M} : \Delta \end{array}}{\Gamma \vdash D\,\overline{M}\ :\star}\ D\textbf{-F'}$$

$$\dfrac{\begin{array}{c} \mathsf{data}\ D\,\Delta\ \mathsf{where}\ \left\{\overline{d_i\,\Theta_i \to D\overline{M}_i\,|}\right\} \in \Gamma \\ \Gamma \vdash \overline{M} : \Delta \end{array}}{\Gamma \vdash D\,\overline{M}\ :\star}\ D\text{-F}$$

$$\dfrac{\begin{array}{c} \mathsf{data}\ D\,\Delta\ \mathsf{where}\ \{C\} \in \Gamma \\ d\,\Theta \to D\overline{M}' \in C \\ \Gamma \vdash \overline{N} : \Theta \end{array}}{\Gamma \vdash d\,\overline{N}\ :\ D\,\overline{M}'\left[\Theta := \overline{N}\right]}\ D\text{-I}$$

with some abuse of notation: $\overline{M}_i$ parameterized over Θ_i instead of $\overline{x}_i$

$$\dfrac{\begin{array}{c} \mathsf{data}\ D\,\Delta\ \mathsf{where}\ \left\{\overline{d_i\,\Theta_i \to D\overline{M}_i\,|}\right\} \in \Gamma \\ \Gamma, \overline{y} : \Delta, x : D\overline{y} \vdash \sigma : \star \\ \Gamma \vdash N : D\,\overline{P} \\ \forall i.\,\Gamma, \overline{x}_i : \Theta_i \vdash O_i : \sigma\left[x := d\overline{x}_i, \overline{y} := \overline{M}_i\right] \end{array}}{\Gamma \vdash \mathsf{Case}_{x:D\,\overline{y}.\sigma}\ N\ \mathsf{of}\ \left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\}\ :\ \sigma\left[x := N, \overline{y} := \overline{P}\right]}\ D\text{-E}$$

$$\dfrac{\begin{array}{c} \mathsf{data}\ D\,\Delta\ \mathsf{where}\ \left\{\overline{d_i\,\Theta_i \to D\overline{M}_i\,|}\right\} \in \Gamma \\ \Gamma, \overline{y} : \Delta, x : D\overline{y} \vdash \sigma : \star \\ \forall i.\,\Gamma, \overline{x}_i : \Theta_i \vdash O_i : \sigma\left[x := d\overline{x}_i, \overline{y} := \overline{M}_i\right] \\[2mm] d\,\Theta \to D\overline{M}' \in \overline{d_i\,\Theta_i \to D\overline{M}_i\,|} \\[2mm] d\overline{x} \Rightarrow O \in \overline{d_i\overline{x}_i \Rightarrow O_i\,|} \\ \Gamma \vdash \overline{N} \Rightarrow \overline{N'} : \Theta \\[2mm] \Gamma \vdash O \Rightarrow O' : \sigma\left[x := d_i\overline{x}_i, \overline{y} := \overline{N}\right] \end{array}}{\begin{array}{c}\Gamma \vdash \mathsf{Case}_{x:D\,\overline{y}.\sigma}\ \left(d\,\overline{N}\right)\ \mathsf{of}\ \left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\} \\ \Rightarrow O'\left[\overline{x} := \overline{N'}\right] : \sigma\left[x := d\overline{x}_i, \overline{y} := \overline{N}\right]\end{array}}\ D\text{-}\Rightarrow$$

structural rules,

$$\dfrac{\begin{array}{c} \mathsf{data}\ D\,\Delta \in \Gamma \\ \Gamma \vdash \overline{M} \Rightarrow \overline{M'} : \Delta \end{array}}{\Gamma \vdash D\overline{M} \Rightarrow D\overline{M'} : \star}\ D\text{-F'-}\Rightarrow$$

$$\frac{\begin{array}{c}\text{data } D\,\Delta \text{ where } \left\{\overline{d_i\,\Theta_i \to D\overline{M}_i\,|}\right\} \in \Gamma \\ \Gamma \vdash \overline{M} \Rightarrow \overline{M}' : \Delta\end{array}}{\Gamma \vdash D\overline{M} \Rightarrow D\overline{M}' : \star}\ D\text{-}F\text{-}\Rightarrow$$

$$\frac{\begin{array}{c}\text{data } D\,\Delta \text{ where } \{C\} \in \Gamma \\ d : \Theta \to D\overline{M}' \in C \\ \Gamma \vdash \overline{N} \Rightarrow \overline{N}' : \Theta\end{array}}{\Gamma \vdash d\,\overline{N} \Rightarrow d\,\overline{N}' : D\,\overline{M}'\left[\Theta \coloneqq \overline{N}\right]}\ D\text{-}I$$

$$\frac{\begin{array}{c}\text{data } D\,\Delta \text{ where } \left\{\overline{d_i\,\Theta_i \to D\overline{M}_i\,|}\right\} \in \Gamma \\ \Gamma, \overline{y} : \Delta, x : D\overline{y} \vdash \sigma : \star \\ \Gamma \vdash N \Rightarrow N' : D\,\overline{P} \\ \forall i.\,\Gamma, \overline{x}_i : \Theta_i \vdash O_i \Rightarrow O_i' : \sigma\left[x \coloneqq d\overline{x}_i, \overline{y} \coloneqq \overline{M}_i\right]\end{array}}{\Gamma \vdash \mathsf{Case}_{x:D\,\overline{y}.\sigma}\ N \text{ of } \left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\} \Rightarrow \mathsf{Case}_{x:D\,\overline{y}.\sigma}\ N' \text{ of } \left\{\overline{d_i\overline{x}_i \Rightarrow O_i'\,|}\right\} : \sigma\left[x \coloneqq N, \overline{y} \coloneqq \overline{P}\right]}\ D\text{-}E\text{-}\Rightarrow$$

CBV

$$\frac{d\overline{x} \Rightarrow O \ \in \ \overline{d_i\overline{x}_i \Rightarrow O_i\,|}}{\mathsf{Case}_{x:D\,\overline{y}.\sigma}\ (d\,\overline{v}) \text{ of } \left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\} \rightsquigarrow O\left[\overline{x} \coloneqq \overline{v}\right]}\ D\text{-}\rightsquigarrow$$

$$\frac{M \rightsquigarrow M'}{\mathsf{Case}_{x:D\,\overline{y}.\sigma}\ (M) \text{ of } \left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\} \rightsquigarrow \mathsf{Case}_{x:D\,\overline{y}.\sigma}\ (M') \text{ of } \left\{\overline{d_i\overline{x}_i \Rightarrow O_i\,|}\right\}}\ D\text{-}\rightsquigarrow$$

$$\frac{\overline{M} \rightsquigarrow \overline{M}'}{d\overline{M} \rightsquigarrow d\overline{M}'}\ D\text{-}\rightsquigarrow$$

Telescopes

$$\frac{}{\Gamma \vdash .\ :\ \overline{\star}}\ \text{C-Emp}$$

$$\frac{\Gamma, x : \sigma \vdash \Delta : \overline{\star} \qquad \Gamma \vdash \sigma : \star}{\Gamma \vdash x : \sigma, \Delta\ :\ \overline{\star}}\ \Delta\text{-Ty-}+$$

$$\frac{}{\Gamma \vdash \Diamond : .}\ \Delta\text{-Trm-Emp}$$

$$\frac{\Gamma \vdash \overline{M} : \Delta\left[x \coloneqq N\right] \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash N, \overline{M}\ :\ x : \sigma, \Delta}\ \Delta\text{-Trm-}+$$

parallel reductions

$$\frac{\Diamond}{\Gamma \vdash \Diamond \Rrightarrow \Diamond : .}$$

$$\frac{\Gamma \vdash \overline{M} \Rrightarrow \overline{M'} : \Delta\,[x := N] \qquad \Gamma \vdash N \Rrightarrow N' : \sigma}{\Gamma \vdash N, \overline{M} \Rrightarrow N', \overline{M'} \,:\, x : \sigma, \Delta} \; \Delta\text{-Trm-+}$$

.

$$\frac{N \rightsquigarrow N'}{\overline{v}, N, \overline{M} \rightsquigarrow \overline{v}, N', \overline{M}} \; D\text{-}\rightsquigarrow$$