an Intensional Dependent Type Theory with Type-in-Type and Recursion

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Surface Language

l				position identifier
Γ	$::=$		$\Diamond \mid \Gamma, x : M$	var contexts
m, n, h, M, N, H, P	$::=$		x	expressions: var
		$\mid$	$m ::_l M$	type annotation
		$\mid$	$\star$	type universe
		$\mid$	$\Pi x : M_l . N_{l'}$	
		$\mid$	$\mathsf{fun}\, f.\, x.\, m \mid m\,{}_l n$	
v	$::=$		x	values
		$\mid$	$\star \mid \Pi x : M.N$	
		$\mid$	$\mathsf{fun}\, f.\, x.\, n$	

Cast Language

H	$::=$		$\Diamond \mid H, x : A$	var contexts
a_h	$::=$		x	
		$\mid$	$\star$	
		$\mid$	$\Pi x : A.B$	
		$\mid$	$\mathsf{fun}\, f.\, x.a \mid b\, a$	
e	$::=$		$A \mid e =_{l,o} A$	type equality chain
a, b, c, A, B, C	$::=$		$\star \mid \Pi x : A.B$	
		$\mid$	$a_h :: e$	
o	$::=$		$. \mid o.arg$	observation
		$\mid$	$o.bod[x := a]$	

Substitution

$$
\begin{aligned}
\star\,[x \coloneqq a] &= \star \\
(\Pi x : A.B)\,[x \coloneqq a] &= \Pi x : A\,[x \coloneqq a]\,.B\,[x \coloneqq a] \\
(x :: e)\,[x \coloneqq a_h :: e' =_{l,o} A] &= a_h :: e' =_{l,o} e \\
(a_h :: e)\,[x \coloneqq a] &= a_h\,[x \coloneqq a] :: e\,[x \coloneqq a] \\
\star\,[x \coloneqq a] &= \star \\
(\Pi x : A.B)\,[x \coloneqq a] &= \Pi x : A\,[x \coloneqq a]\,.B\,[x \coloneqq a] \\
(\mathsf{fun}\, f.\, y.b)\,[x \coloneqq a] &= \mathsf{fun}\, f.\, y.b\,[x \coloneqq a] \\
(b\,c)\,[x \coloneqq a] &= b\,[x \coloneqq a]\ c\,[x \coloneqq a] \\
(e =_{l,o} A)\,[x \coloneqq a] &\quad e\,[x \coloneqq a] =_{l,o[x \coloneqq a]} A\,[x \coloneqq a] \\
.\,[x \coloneqq a] &\quad . \\
(o.arg)\,[x \coloneqq a] &\quad o\,[x \coloneqq a]\,.arg \\
(o.bod[y \coloneqq b])\,[x \coloneqq a] &\quad o\,[x \coloneqq a]\,.bod[y \coloneqq b\,[x \coloneqq a]]
\end{aligned}
$$

lookup

$$
\begin{aligned}
A\uparrow &= A \\
e =_{l,o} A\uparrow &= A \\
A\downarrow &= A \\
e =_{l,o} A\downarrow &= e\downarrow
\end{aligned}
$$

casts

TODO

Judgments

$$
\begin{aligned}
\Gamma \sim H \vdash n\, Elab\, a & \\
\Gamma \sim H \vdash n\, Elab_M\, a & \\
H \vdash a : A & \quad \text{apparent type} \\
H \vdash a_h : A & \quad \text{head type} \\
H \vdash a_h \Rrightarrow a : A & \\
H \vdash e \Rrightarrow e' : \overline{\star} & \\
H \vdash a \Rrightarrow a : A &
\end{aligned}
$$

Elaboration

infer

$$\frac{x : A \in H}{\Gamma \sim H \vdash x\, Elab\, x :: A}$$

$$\frac{\Gamma \sim H \vdash M\, Elab_{\star,l}\, C \quad \Gamma \sim H \vdash m\, Elab_{C,l}\, a}{\Gamma \sim H \vdash m ::_l M\, Elab\, a}$$

$$\frac{...}{\Gamma \sim H \vdash \star \, Elab \, \star}$$

$$\frac{\Gamma \sim H \vdash M \, Elab_{\star,l} \, A \quad \Gamma, x : M \sim H, x : A \vdash N \, Elab_{\star,l'} \, B}{\Gamma \vdash \Pi x : M_l . N_{l'} \, Elab \, (\Pi x : A.B) :: \star}$$

$$\frac{\Gamma \sim H \vdash m \, Elab \, b :: e \quad (\Pi x : A.B) :: e' \in e \quad \Gamma \sim H \vdash n \, Elab_{A,l} \, a}{\Gamma \sim H \vdash m \, {}_l n \, Elab \, b \, a}$$

nondeterminism ok here? need to make ok with equality

check

$$\frac{...}{H \vdash \star \, Elab_{\star,l} \, \star}$$

$$\frac{H, f : (\Pi x : A.B) :: e, x : A \vdash m \, Elab_{B,??} \, b}{H \vdash \mathsf{fun} \, f. \, x. \, m \, Elab_{(\Pi x:A.B)::e,l} \, \mathsf{fun} \, f. \, x.b}$$

$$\frac{H \vdash m \, Elab \, a_h :: e}{H \vdash m \, Elab_{A,l} \, a_h :: e =_{l,.} A}$$

1 Typing

1.1 Cast Typing

$$\frac{H \vdash A : \star}{H \vdash A : \overline{\star}} eq - ty - 1$$

$$\frac{H \vdash e : \overline{\star} \quad H \vdash A : \star}{H \vdash e =_{l,o} A : \overline{\star}} eq - ty - 2$$

1.2 Head Typing

$$\frac{...}{H \vdash \star : \star} \star - ty$$

$$\frac{x : A \in H}{H \vdash x : A} var - ty$$

$$\frac{H \vdash A : \star \quad H, x : A \vdash B : \star}{H \vdash \Pi x : A.B \; : \star} \Pi - ty$$

allow typing to uncast heads

$$\frac{H, f : \Pi x : A.B, x : A \vdash b : B}{H \vdash \mathsf{fun} \, f. \, x.b \; : \; \Pi x : A.B} \Pi - \mathsf{fun} - ty$$

$$\frac{H \vdash b : \Pi x : A.B \quad H \vdash a : A}{H \vdash b \, a \; : \; B \, [x := a]} \Pi - app - ty$$

1.3 Cast Term Typing

$$\frac{\ldots}{H \vdash \star : \star}\,\star - ty$$

$$\frac{H \vdash A : \star \quad H, x : A \vdash B : \star}{H \vdash \Pi x : A.B \ : \star}\,\Pi - ty$$

$$\frac{H \vdash a : A \quad H \vdash A \equiv A' : \star}{H \vdash a : A'}\,conv$$

$$\frac{H \vdash e : \bar{\star} \quad H \vdash a_h : B \downarrow}{H \vdash a_h :: e \quad : \quad e \uparrow}\,apparent$$

(TODO: may regret these as functions instead of relations)

2 Definitional Equality

$$\frac{H \vdash A \Rrightarrow_* B : \star \quad H \vdash A' \Rrightarrow_* B : \star}{H \vdash A \equiv A' : \star}$$

3 Par reductions

$$\frac{H \vdash a : A}{H \vdash a \Rrightarrow_* a : A}$$

$$\frac{H \vdash a \Rrightarrow_* b : A \quad H \vdash b \Rrightarrow c : A}{H \vdash a \Rrightarrow_* c : A}$$

4 Par reduction

4.1 Cast Par reduction

$$\frac{H \vdash A \Rrightarrow A' \ : \star}{H \vdash A \Rrightarrow A' \ : \bar{\star}}$$

$$\frac{H \vdash e \Rrightarrow e' : \bar{\star} \quad H \vdash A \Rrightarrow A' \ : \star}{H \vdash e =_{l,o} A \Rrightarrow e' =_{l,o} A' \ : \bar{\star}}$$

4.2 Head Par reduction

$$\frac{H, f : \Pi x : A.B, x : A \vdash b \Rrightarrow b' : B \quad H \vdash a \Rrightarrow a' : A \quad e\, Elim_{\Pi}\, x, -, -, e_B \quad H \vdash e_B \Rrightarrow e' : \bar{\star}}{H \vdash (\mathsf{fun}\, f.\, x.b) :: e\, a \Rrightarrow (b' :: e')\, [f := (\mathsf{fun}\, f.\, x.b')\, , x := a'] \ : \ B\, [x := a]}$$

TODO: fix this

TODO: justify the shorthand in the remaining rules, by prompting a typing judgment to an innocuous cast

$$\frac{x : A \in H}{H \vdash x \Rrightarrow x : A}$$

$$\frac{H \vdash}{H \vdash \star \Rrightarrow \star : \star}$$

$$\frac{H \vdash A \Rrightarrow A' : \star \quad H, x : A \vdash B \Rrightarrow B' : \star}{H \vdash \Pi x : A.B \Rrightarrow \Pi x : A'.B' \,:\, \star}$$

$$\frac{H, f : \Pi x : A.B, x : A \vdash b \Rrightarrow b' \,:\, B}{H \vdash \mathsf{fun}\, f.\, x.b \Rrightarrow \mathsf{fun}\, f.\, x.b' \,:\, \Pi x : A.B}$$

$$\frac{H \vdash b \Rrightarrow b' : \Pi x : A.B \quad H \vdash a \Rrightarrow b' : A}{H \vdash b\, a \Rrightarrow b'\, a' \,:\, B\, [x \coloneqq a]}$$

4.3 Cast Term Par reduction

$$\frac{H \vdash}{H \vdash \star \Rrightarrow \star : \star}$$

$$\frac{H \vdash A \Rrightarrow A' : \star \quad H, x : A \vdash B \Rrightarrow B' : \star}{H \vdash \Pi x : A.B \Rrightarrow \Pi x : A'.B' \,:\, \star}$$

$$\frac{H \vdash e \Rrightarrow e' : \star \quad H \vdash a \Rrightarrow a' : e\downarrow}{H \vdash a :: e \Rrightarrow a' :: e' \,:\, e\uparrow}$$

Cbv Small Step

$$\begin{array}{lcl} v & ::= & \star \\ & | & x :: v_{eq} \\ & | & (\Pi x : A.B) :: v_{eq} \\ & | & (\mathsf{fun}\, f.\, x.a) :: v_{eq} \\ v_{eq} & ::= & v \\ & | & v_{eq} =_{l,o} v \end{array}$$

$$\frac{A \rightsquigarrow A'}{v_{eq} =_{l,o} A \rightsquigarrow v_{eq} =_{l,o} A'}$$

$$\frac{e \rightsquigarrow e'}{e =_{l,o} A \rightsquigarrow e' =_{l,o} A}$$

$$\frac{e \rightsquigarrow e'}{a_h :: e \rightsquigarrow a_h :: e'}$$

$$\frac{a_h \rightsquigarrow a'_h}{a_h :: v_{eq} \rightsquigarrow a_h :: v_{eq}}$$

$$\frac{b \rightsquigarrow b'}{b\, a \rightsquigarrow b'\, a}$$

$$\frac{a \rightsquigarrow a'}{v\, a \rightsquigarrow v\, a'}$$

...

$$\frac{v_{eq}\,Elim_{\Pi}\,x,-,-,e_B}{((\mathsf{fun}\,f.\,x.b)::v_{eq}\,v)::v'_{eq}\rightsquigarrow(b::e_B)\,[f:=(\mathsf{fun}\,f.\,x.b)::v_{eq},x:=v]::v'_{eq}}$$

apparent ty of bod should be *, think that might follow from the typing judgment

abuse of notation, $(b :: e_B)$, b already has casts

...

Dynamic Check

$$\frac{}{\star\,Elim_{\star}}$$

$$\frac{Elim_{\star}e}{e =_{l,o} \star\,Elim_{\star}}$$

...

$$\frac{}{\Pi x:A.B\,Elim_{\Pi}\,x,A,A,B}$$

...

$$\frac{e\,Elim_{\Pi}\,y,A',e_A,e_B}{e =_{l,o} \Pi x:A.B\,Elim_{\Pi}\,y,A,(A =_{l,o.arg} e_A)\,,((e_B\,[x:=y::A' =_{l,o.arg} A]) =_{l,o.bod} B)}$$

Alt rules/ notation

$$\frac{\forall C\in B =_{l,o} e =_{l',o'} A,\ H\vdash C:\star \quad H\vdash a_h:B}{H\vdash a_h::B =_{l,o} e =_{l',o'} A\ :A}\,apparent$$

$$\frac{v_{eq}Elim_{\Pi}\,v_{eq\,A},e_B \quad v_{eq\,A}\,Elim_{\star} \quad v_{eq\,A}\,Elim_{\star}}{((\mathsf{fun}\,f.\,x.b)::v_{eq}\,v)::v'_{eq}\rightsquigarrow(\mathsf{fun}\,f.\,x.b)::v_{eq}\,v}$$

.....

$$\frac{v_{eq}Elim_{\Pi}\,v_{eq\,A},e_B \quad v_{eq\,A}\,Elim_{\star} \quad v_{eq\,A}\,Elim_{\star}}{(\mathsf{fun}\,f.\,x.b)::v_{eq}\,v\rightsquigarrow b\,[f:=(\mathsf{fun}\,f.\,x.b)\,,x:=v]}$$

...

$$\frac{v_{eq}\,Elim_{\Pi}\,x,-,v_{eq\,A},e_B \quad v_{eq\,A}\,Elim_{\star}}{((\mathsf{fun}\,f.\,x.b)::v_{eq}\,v)::v'_{eq}\rightsquigarrow(b::e_B)\,[f:=(\mathsf{fun}\,f.\,x.b)::v_{eq},x:=v]::v'_{eq}}$$

$$\frac{H,f:\Pi x:A.B,x:A\vdash b\Rrightarrow b'\ :B}{H\vdash((\mathsf{fun}\,f.\,x.b)::e)\ a\Rrightarrow((\mathsf{fun}\,f.\,x.b)::e)\ a\ :\ B\,[x:=a]}$$

$$\begin{aligned}\underrightarrow{A} &= A\\ \underrightarrow{e =_{l,o} A} &= A\end{aligned}$$

a