Generalized Linear Model

PSYC 573

University of Southern California
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Regression for Prediction

One outcome Y, one or more predictors X_1, X_2, $\ldots$

E.g.,

- What will a student's college GPA be given an SAT score of x ?
- How long will a person live if the person adopts diet x?
- What will the earth's global temperature be if the carbon emission level is x?

Keep These in Mind

1. Likelihood function is defined for the outcome Y

2. Prediction is probabilistic (i.e., uncertain) and contains error

Generalized Linear Models (GLM)

GLM

Three components:

- Conditional distribution of Y
- Link function
- Linear predictor

Some Examples

Outcome type	Support	Distributions	Link
continuous	$[-\infty, \infty]$	Normal	Identity
count (fixed duration)	$\{0, 1, \ldots\}$	Poisson	Log
count (known # of trials)	$\{0, 1, \ldots, N\}$	Binomial	Logit
binary	$\{0, 1\}$	Bernoulli	Logit
ordinal	$\{0, 1, \ldots, K\}$	categorical	Logit
nominal	K-vector of $\{0, 1\}$	categorical	Logit
multinomial	K-vector of $\{0, 1, \ldots, K\}$	categorical	Logit

Mathematical Form (One Predictor)

$$
\begin{aligned}
Y_i &\sim \mathrm{Dist}(\mu_i, \tau) \\
g(\mu_i) &= \eta_i \\
\eta_i &= \beta_0 + \beta_1 X_i
\end{aligned}
$$

- Dist: conditional distribution of $Y \mid X$ (e.g., normal, Bernoulli, $\ldots$)
  - I.e., distribution of **prediction error**; not the marginal distribution of Y
- μ_i: mean parameter for the ith observation
- η_i: linear predictor
- $g(\cdot)$: link function
- (τ: dispersion parameter)

Illustration

Next few slides contain example GLMs, with the same predictor X

```
num_obs <- 100
x <- runif(num_obs, min = 1, max = 5)  # uniform x
beta0 <- 0.2; beta1 <- 0.5
```

Normal, Identity Link

aka linear regression

Model Simulation

$$
\begin{aligned}
Y_i &\sim N(\mu_i, \sigma) \\
\mu_i &= \eta_i \\
\eta_i &= \beta_0 + \beta_1 X_i
\end{aligned}
$$

Poisson, Log Link

aka poisson regression

Model Simulation

$$\begin{aligned} Y_i &\sim \text{Pois}(\mu_i) \\ \log(\mu_i) &= \eta_i \\ \eta_i &= \beta_0 + \beta_1 X_i \end{aligned}$$

Bernoulli, Logit Link

aka binary logistic regression

Model Simulation

$$
\begin{aligned}
Y_i &\sim \mathrm{Bern}(\mu_i) \\
\log\left(\frac{\mu_i}{1-\mu_i}\right) &= \eta_i \\
\eta_i &= \beta_0 + \beta_1 X_i
\end{aligned}
$$

Binomial, Logit Link

aka binomial logistic regression

Model Simulation

$$
\begin{aligned}
Y_i &\sim \mathrm{Bin}(N, \mu_i) \\
\log\left(\frac{\mu_i}{1-\mu_i}\right) &= \eta_i \\
\eta_i &= \beta_0 + \beta_1 X_i
\end{aligned}
$$

Remarks

Different link functions can be used

- E.g., identity link or probit link for Bernoulli variables

Linearity is a strong assumption

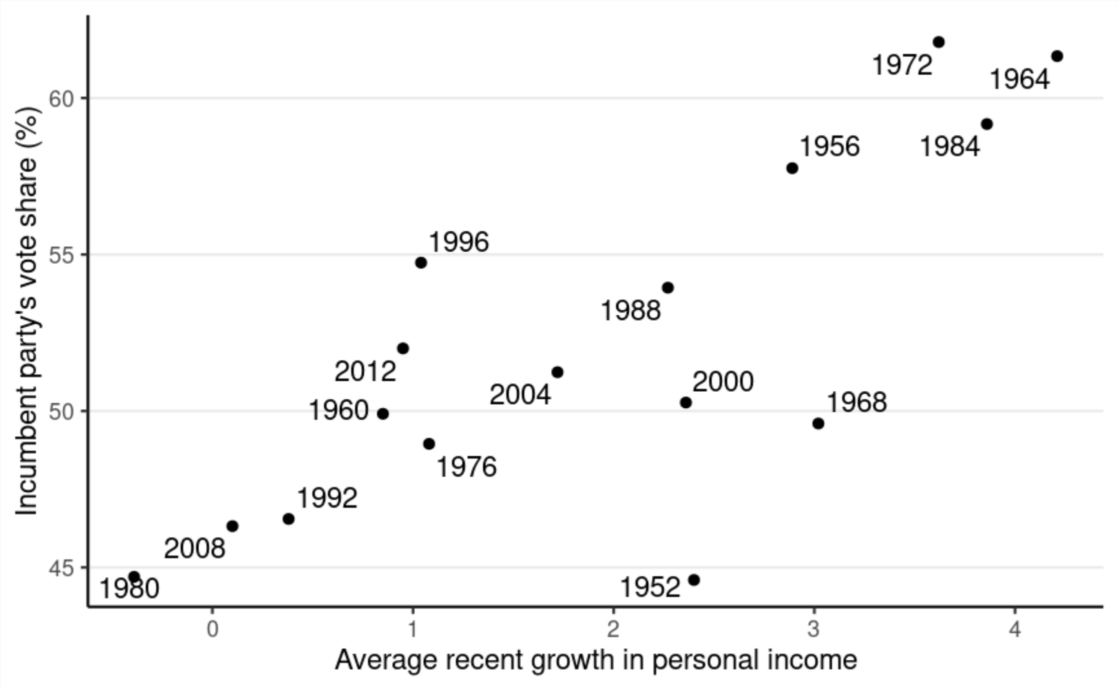
- GLM can allow η and X to be nonlinearly related, as long as it's linear in the coefficients
  - E.g., $\eta_i = \beta_0 + \beta_1 \log(X_i)$
  - E.g., $\eta_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2$
  - But not something like $\eta_i = \beta_0 \log(\beta_1 + x_i)$

Linear Regression

Many relations can be approximated as linear

But many relations cannot be approximated as linear

Example: "Bread and peace" model



Linear Regression Model

Model:

$$\begin{aligned}\mathbf{vote}_i &\sim N(\mu_i, \sigma)\\ \mu_i &= \beta_0 + \beta_1 \mathbf{growth}_i\end{aligned}$$

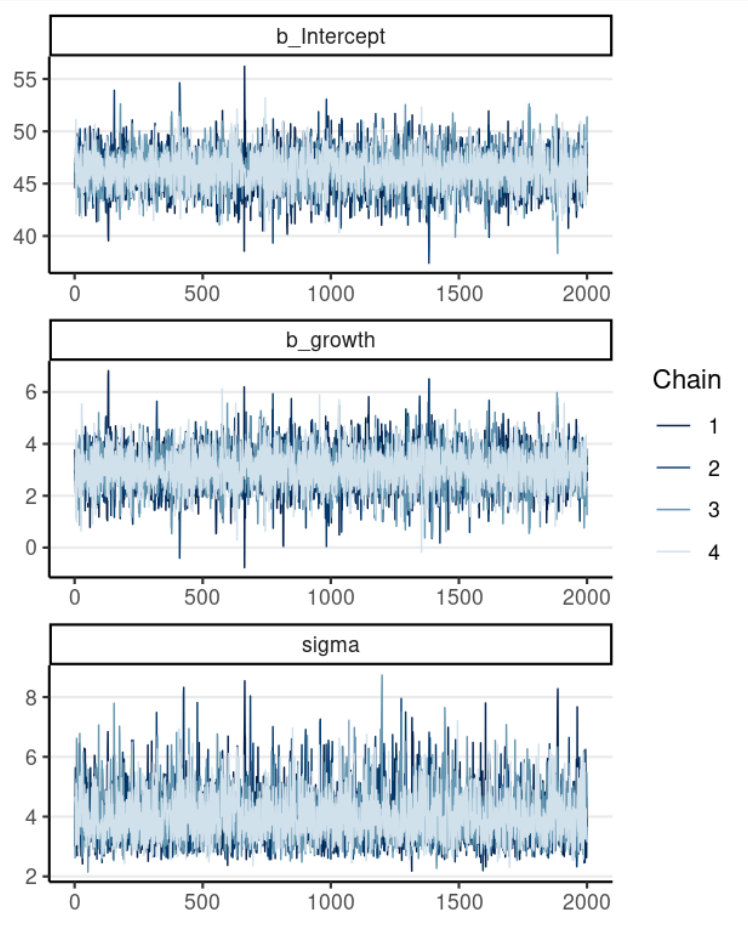
σ: SD (margin) of prediction error

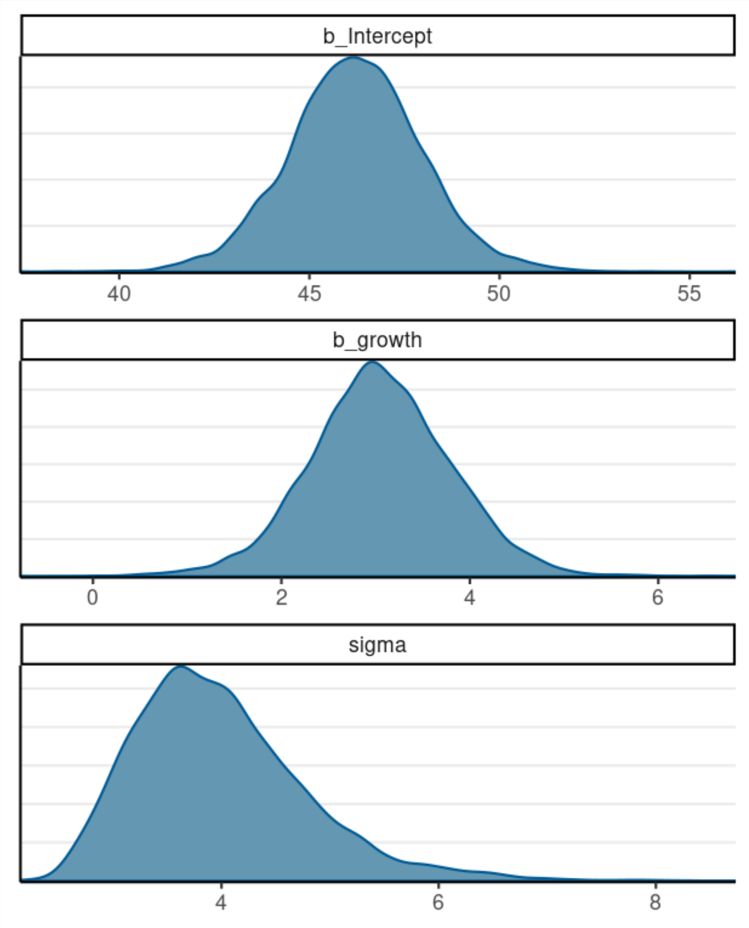
Prior:

$$\begin{aligned}\beta_0 &\sim N(45, 10)\\ \beta_1 &\sim N(0, 10)\\ \sigma &\sim t_4^+(0, 5)\end{aligned}$$

Stan | brms | brms results

```
data {
  int<lower=0> N;  // number of observations
  vector[N] y;  // outcome;
  vector[N] x;  // predictor;
}
parameters {
  real beta0;  // regression intercept
  real beta1;  // regression coefficient
  real<lower=0> sigma;  // SD of prediction error
}
model {
  // model
  y ~ normal(beta0 + beta1 * x, sigma);
  // prior
  beta0 ~ normal(45, 10);
  beta1 ~ normal(0, 10);
  sigma ~ student_t(4, 0, 5);
}
generated quantities {
  vector[N] y_rep;  // place holder
  for (n in 1:N)
    y_rep[n] = normal_rng(beta0 + beta1 * x[n], sigma);
}
```

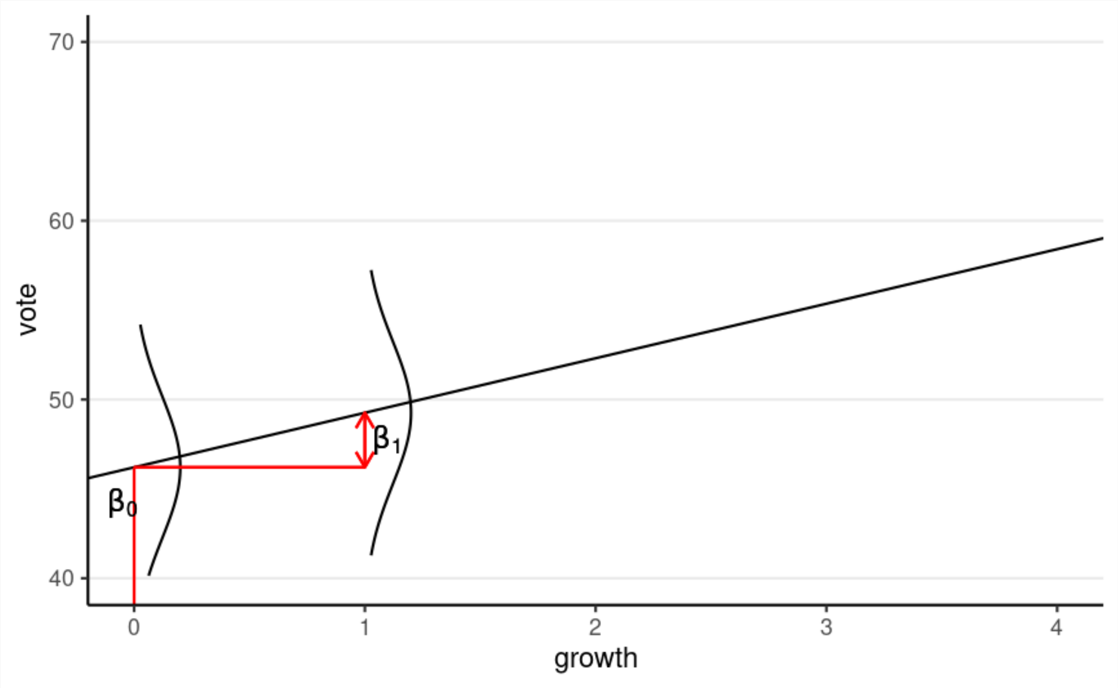

b_Intercept
b_growth
sigma
Chain
1
2
3
4
0
500
1000
1500
2000


b_Intercept
b_growth
sigma

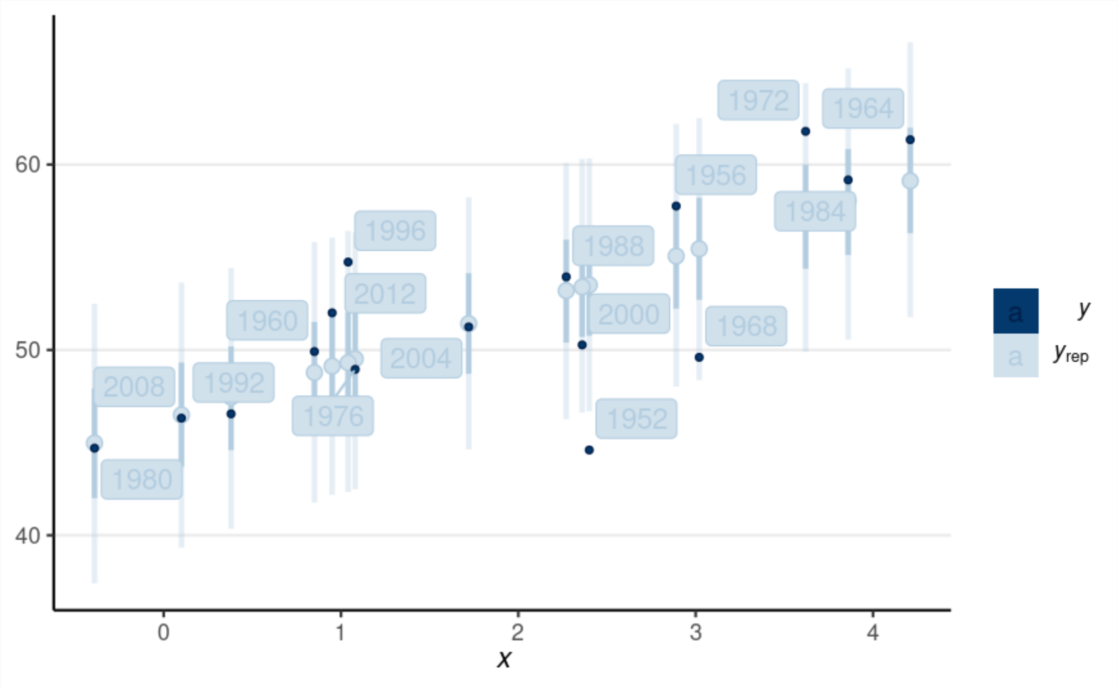
Meaning of Coefficients

When growth = 0, $\mathbf{vote} \sim N(\beta_0, \sigma)$

When growth = 1, $\mathbf{vote} \sim N(\beta_0 + \beta_1, \sigma)$



Posterior Predictive Check

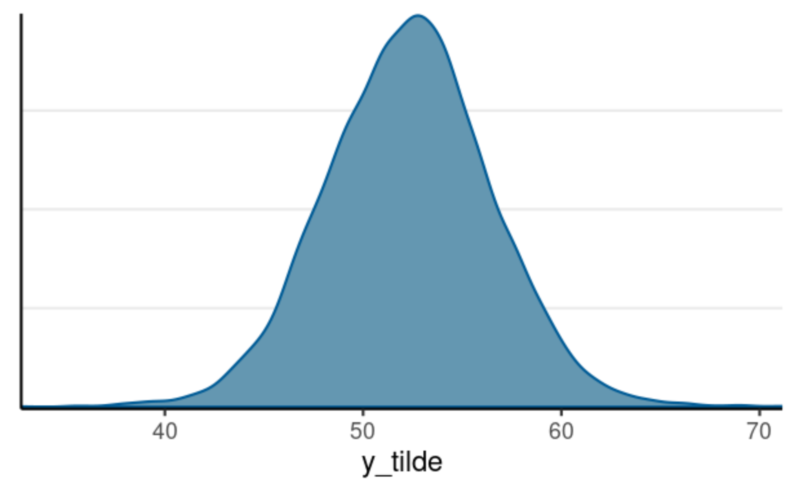


The model fits a majority of the data, but not everyone. The biggest discrepancy is 1952.

Prediction

Predicted vote share when growth = 2: $\tilde{y} \mid y \sim N(\beta_0 + \beta_1 \times 2, \sigma)$

```
pp_growth_eq_2 <- posterior_predict(m1_brm,
    newdata = list(growth = 2)
)
```



Probability of incumbent's vote share > 50% = 0.713

Table

```r
library(modelsummary)
msummary(m1_brm, estimate = "{estimate} [{conf.low}, {conf.high}]",
         statistic = NULL, fmt = 2)
```

	Model 1
b_Intercept	46.20 [42.76, 49.75]
b_growth	3.03 [1.56, 4.56]
sigma	3.88 [2.56, 5.51]
Num.Obs.	16
ELPD	-46.1
ELPD s.e.	3.6
LOOIC	92.3
LOOIC s.e.	7.2
WAIC	92.1
RMSE	24.97

Prediction vs. Explanation

Is personal income growth a reason a candidate/party got more vote share?

If so, what is the mechanism?

If not, what is responsible for the association?

Additional Notes

Outlier: use $Y_i \sim t_\nu(\mu_i, \sigma)$

Nonconstant σ

- One option is $\log(\sigma_i) = \beta_0^s + \beta_1^s X_i$

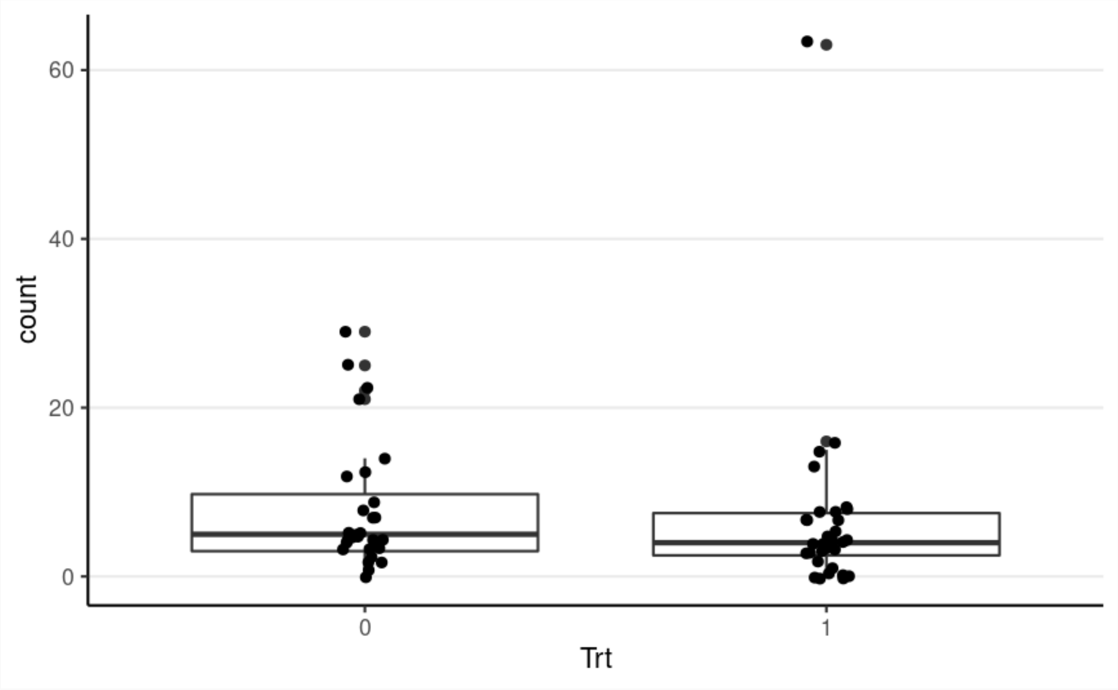
Check whether linearity holds

- Other options: splines, quadratic, log transform (i.e., lognormal model), etc

Poisson Regression

- `count`: The seizure count between two visits
- `Trt`: Either 0 or 1 indicating if the patient received anticonvulsant therapy

$$
\begin{aligned}
\text{count}_i &\sim \text{Pois}(\mu_i) \\
\log(\mu_i) &= \eta_i \\
\eta_i &= \beta_0 + \beta_1 \text{Trt}_i
\end{aligned}
$$



Poisson with log link

Predicted seizure rate = $\exp(\beta_0 + \beta_1) = \exp(\beta_0) \exp(\beta_1)$ for Trt = 1; $\exp(\beta_0)$ for Trt = 0

β_1 = mean difference in **log** rate of seizure; $\exp(\beta_1)$ = ratio in rate of seizure

```
m2 <- brm(count ~ Trt, data = epilepsy4,
          family = poisson(link = "log"))
```

Poisson with identity link

In this case, with one binary predictor, the link does not matter to the fit

$$
\begin{aligned}
\textbf{count}_i &\sim \textbf{Pois}(\mu_i) \\
\mu_i &= \eta_i \\
\eta_i &= \beta_0 + \beta_1 \textbf{Trt}_i
\end{aligned}
$$

β_1 = mean difference in the rate of seizure in two weeks

```r
m3 <- brm(count ~ Trt, data = epilepsy4,
          family = poisson(link = "identity"))
```

	log link	identity link
b_Intercept	2.07	7.97
	[1.95, 2.20]	[6.94, 8.96]
b_Trt1	−0.17	−1.25
	[−0.35, 0.02]	[−2.58, 0.16]
Num.Obs.	59	59
ELPD	−343.1	−345.1
ELPD s.e.	93.8	95.7
LOOIC	686.2	690.2
LOOIC s.e.	187.7	191.3
WAIC	688.5	687.8
RMSE	10.50	10.53