Expressing a thermostat as a quadratic programing problem

1

I want to find a method to find the power consumption of a smart thermostat given a tariff profile and a model of the home the thermostat is responsible for.

The home is modelled as a simple thermal system. It has thermal mas cm insulation $\frac{1}{k}$ and a heater which when on outputs power P. If the external temperature is $T^e(t)$ then the temperature is governed by the first order differential equation

$$\frac{dT(t)}{dt} = -\frac{k}{cm}(T(t) - T^e(t)) + \frac{P\delta(t)}{cm} \tag{1}$$

where $\delta(t) \in (0, 1)$ determines the power setting of the heater. For a sufficiently small timestep Δt this can be approximated as an update rule

$$T_{n+1} = T_n - \frac{\Delta tk}{cm}(T_n - T_n^e) + \frac{\Delta tP\delta_n}{cm} \tag{2}$$

If the time period under consideration is discretized into N periods of Δt then T, T^e, δ can be represented as column vectors of size N and the update rule can be expressed as a matrix equation and simplified to give

$$\left(\mathbf{I}_{N+1\times N\times 1} - (1 - \frac{\Delta tk}{cm})\mathbf{U}\right)\underline{T} = \frac{\Delta tk}{cm}\mathbf{U}\underline{T}^e + \frac{P}{cm}\underline{\delta} \tag{3}$$

$$\underline{T} = \mathbf{\Phi}\underline{\delta} + \underline{\Psi} \tag{4}$$

where

$$\mathbf{\Phi} = \frac{P}{cm}\left(\mathbf{I}_{N+1\times N\times 1} - (1 - \frac{\Delta tk}{cm})\mathbf{U}\right)^{-1} \tag{5}$$

$$\underline{\Psi} = \left(\mathbf{I}_{N+1\times N\times 1} - (1 - \frac{\Delta tk}{cm})\mathbf{U}\right)^{-1}\left(\frac{\Delta tk}{cm}\mathbf{U}\underline{T}^{e}\right) \tag{6}$$

and

$$\mathbf{U} = \left[\begin{array}{c|c} \underline{0}_{1\times N} & 1 \\ \hline \mathbf{I}_N & \underline{0}_{N\times 1} \end{array}\right] \tag{7}$$

This formulation assumes a periodic nature such that the temperature at the end of the Nth teimestep is equal to the temperature at the start of the $0th$. Thus the given all the required environmental constants the temperature profile has been expressed as an affine function of the control input.

2

There are to costs associated with a given control input. The actuation cost is $\underline{\Lambda}^T\underline{\delta}$ where the Λ_n, is the unit price of electricity at $t = n\Delta t$. The utility cost is $(\underline{T}-\underline{T}^s)^T\mathbf{Q}(\underline{T}-\underline{T}^s)$ where T_n^s is the desired temperature at time $n\Delta t$ and $\mathbf{Q} = \text{diag}(\underline{q})$ where q_n is the coefficient of quadratic cost of deviation from the desired temperature at the same time.

The quantity to be minimised is therefore

$$f = (\underline{T} - \underline{T}^s)^T\mathbf{Q}(\underline{T} - \underline{T}^s) + \underline{\Lambda}^T P\underline{\delta} \tag{8}$$

Rather than having a very high dimensional $\underline{\delta}$ constrained to be either zero or one which is difficult to solve and very high dimensional we reduce the dimension of the input to M by defining

$$\underline{\delta} = \mathbf{D}\underline{u} \tag{9}$$

where

$$\mathbf{D} = \mathbf{I}_M \otimes \underline{1}_{J\times 1}, \qquad N = MJ \tag{10}$$

which gives

$$f = (\mathbf{\Phi}\mathbf{D}\underline{u} + \underline{\Psi} - \underline{T}^s)^T\mathbf{Q}(\mathbf{\Phi}\mathbf{D}\underline{u} + \underline{\Psi} - \underline{T}^s) + \underline{\Lambda}^T\mathbf{D}\underline{u} \tag{11}$$

where instead of the on/off state over a short time interval u now defines the mark/space ratio of a PWM system over a much longer period and so power consumption and temperatures are now true in expectation rather than in value. With some substitutions this gives a new function to be minimised

$$f' = \underline{u}^T\mathbf{R}\underline{u} + \underline{S}^T\underline{u} \tag{12}$$

where

$$\mathbf{R} = \mathbf{D}^T \mathbf{\Phi}^T \mathbf{Q} \mathbf{\Phi} \mathbf{D} \tag{13}$$

$$\underline{S}^T = 2(\underline{\Psi} - \underline{T}^s)^T \mathbf{Q} \mathbf{\Phi} \mathbf{D} + P \underline{\Lambda}^T \mathbf{D} \tag{14}$$

3

There are many possible constraint sets to give desired behavior. The minimum required constraint is that the control signal must be between zero and one at all times

$$0 \leq \delta_n \leq 1 \qquad \forall n \tag{15}$$

This is expressed in standard form for the quadratic problem as

$$\left[\begin{array}{c} \mathbf{I}_M \\ \hline -\mathbf{I}_M \end{array}\right] \underline{u} \leq \left[\begin{array}{c} \underline{1}_{M\times 1} \\ \hline \underline{0}_{M\times 1} \end{array}\right] \tag{16}$$

A constraint that is likely to be used is a maximum or minimum temperature requirement. $T_n \geq T^{min}$ or $T_n \leq T^{max}$. This is expressed in terms of $\underline{u}$ as

$$\underline{e}^T_{N+1,n} \mathbf{\Phi} \mathbf{D} \underline{u} \leq T^{max} - \underline{e}^T_n \underline{\Psi} \tag{17}$$

$$-\underline{e}^T_{N+1,n} \mathbf{\Phi} \mathbf{D} \underline{u} \leq -T^{min} + \underline{e}^T_n \underline{\Psi} \tag{18}$$

Where $\underline{e}_{M,i}$ is the column vector with M elements with 1 at position i and zero elsewhere. and the budget constraint

$$\Lambda^T \mathbf{D} \underline{u} \leq \beta \tag{19}$$

This leads to a full set of inequality constraints

$$\left[\begin{array}{c} \mathbf{I}_M \\ \hline -\mathbf{I}_M \\ \hline \underline{e}^T_{N+1,n} \mathbf{\Phi} \mathbf{D} \\ \vdots \\ \hline -\underline{e}^T_{N+1,n} \mathbf{\Phi} \mathbf{D} \\ \vdots \\ \hline \Lambda^T \mathbf{D} \end{array}\right] \underline{u} \leq \left[\begin{array}{c} \underline{1}_{M\times 1} \\ \hline \underline{0}_{M\times 1} \\ \hline T^{max}_n - \underline{e}^T_n \underline{\Psi} \\ \vdots \\ \hline -T^{min}_n + \underline{e}^T_n \underline{\Psi} \\ \vdots \\ \hline \beta \end{array}\right] \tag{20}$$

Another potentially useful constraint is to specify a mean temperature over a time period. If $\underline{\alpha}$ is a column vector of size $N+1$ with ones at the positions corresponding to times that are to be averaged over and zeros elsewhere, and T^{avg} is the desired mean over that period then the standard form constraint is

$$\underline{\alpha}^T \mathbf{\Phi} \mathbf{D} \underline{u} = \|\underline{\alpha}\|_0 T^{avg} - \underline{\alpha}^T \underline{\Psi} \tag{21}$$

4

The numbers need to come from somewhere:

[3] has produced a markov chain for the active occupancy of a building for a given number of occupants at ten minute resolution which is used to determine the times the temperature must be controlled.

[1] is a survey of various building statistics including floor area and occupancy number. The occupancy is drawn from a multinomial based on this with occupancy greater than six rounded down to six since this is the highest number a markov chain is available for. Floor area is drawn from an offset gamma fitted to the CABE data. They type of building is also listed so it is decided that a Binomial classification as 0.365 of being a flat and 0.635 as a house (2 floors) is appropriate.

[2] provides low, medium, high values for thermal mass per sqm floor area of $100, 250, 400 kJ/m^2K$. This is used as the mean and stddev of a Gaussian in log-spaces so that $cm = 10^3 e^x, \quad x : N(5.5, 0.7)$.

I decide that a floor is 3m high, $h = 3$. Looking out the window I decide that a house has one fifth of its external walls as windows while a flat has one half, but that a flat only has one half of its perimeter exposed while a house has the full perimeter and the ground and the roof with a slope of 30 degrees.

Insulation values are given in W/m^2K as 0.18 for external walls, 0.13 for floors and roofs, 1.4 for windows. Therefore the insulation value for a house related to internal area is

$$U = 4h\sqrt{\frac{A}{2}}(\frac{4}{5}0.18 + \frac{1}{5}1.4) + \frac{A}{2}0.13 + \frac{A}{2}\frac{2}{\sqrt{3}}0.13 = 3.6\sqrt{A} + 0.14A \tag{22}$$

while for a flat the relation is

$$U = 4h\sqrt{\frac{A}{2}} \times 0.5(\frac{1}{2}1.4 + \frac{1}{2}0.18) = 3.3\sqrt{A} \tag{23}$$

From the Google result for electric boilers I decide that power will be uniform distributed $(6, 15)kW$.

References

[1] Cabe dwelling size study. `http://data.gov.uk/dataset/floor-space-data-for-english-houses-and-flats-may-2010`, 2011. Accessed: 2014-07-02.

[2] The government's standard assessment procedure for energy rating of dwellings. `http://www.bre.co.uk/filelibrary/SAP/2012/SAP-2012_9-92.pdf`, 2012. Accessed: 2010-07-02.

[3] Ian Richardson, Murray Thomson, and David Infield. A high-resolution domestic building occupancy model for energy demand simulations. *Energy and Buildings*, 40(8):1560–1566, 2008.