Homework: Propositional Logic
Due: Oct 18 7^{00} PM in class
(see syllabus for late policy)

Although you may do this using either *legible* writing or in print, I strongly encourage you to do the homework using LaTeX(required for grad students, though I won't enforce it for figures). Although there is a learning curve, it will benefit you greatly in the future, especially if you enter CS/Math research. All problems should be done using the approaches and notation as defined in this course, and with all steps justified. Problems are not weighted equally.

1. A job ad for Gaggle Inc. says:

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   It is necessary to major in either math, computer science, or
   philosophy, and be an A student. Exceptions will be made for logic
   minors, for whom there are no grade requirements.
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   You have also heard through the grapevine that it is sufficient to win a Fields Medal or Turing Award (regardless of what the ad may say).

   Express the above as one PC wff. You should of course state what your atoms represent. If you want partial credit, you should also provide wffs for individual parts, clearly writing out the corresponding phrases (I strongly recommend you do this to avoid getting a 0). Please do not do any form of reasoning to simplify the wff – that's not the point here, and may also introduce bugs.

2. Suppose we augment PC with an *exclusive or* binary operator, $\oplus$. Modify the model theoretic semantics to support $\oplus$.

   The main purpose of this problem is to get familiar with the notation, and it will thus be graded srictly on your use of proper notation.

3. Consider the wffs:

   $$\phi_1 \equiv p_1 \to (p_2 \to (p_3 \to p_4))$$
   $$\phi_2 \equiv (p_1 \wedge p_2 \wedge p_3) \to p_4$$

   (a) Technically speaking, neither ϕ_1 nor ϕ_2 is well-formed since neither is allowed by the formal syntax of propositional logic. Correct them. Note, however, that we will freely make such trivial 'errors' throughout this semester (as do most such courses).

   (b) Use truth tables (in the form defined in this course) to show that $\models \phi_1 \leftrightarrow \phi_2$.

   (c) After internalizing an intuitive understanding of this equality, propose an extension of it to n atoms.

   (d) State the number of rows in a truth table for proving the extension.

4. Consider

   $$\psi \equiv (\neg\phi_3 \vee (\phi_1 \wedge \phi_2)) \to p_{m+1}$$

   where the ϕ's are wffs and p's are atoms as usual. Suppose ϕ_1, ϕ_2, and ϕ_3 have n_1, n_2, and n_3 models respectively, where $n_1 > n_2 > n_3$. Also, $\mathcal{A} = \{p_1, p_2, p_3, \ldots, p_{m+1}\}$, and the ϕ's do not contain p_{m+1}. Determine the range for the number of models of ψ. For simplicity, you may assume that the n's are bounded by ranges for which you do not need to worry about floor and ceiling functions.

5. Prove the following sequent using our sequent calculus.

$$p_1, p_2, p_3 \wedge q \vdash_{SC} p_3$$

6. Prove using our sequent calculus that the following wff is valid:

$$(p \wedge \neg p) \rightarrow q$$

7. Prove the following sequent using our sequent calculus:

$$p \wedge \neg p, q \rightarrow r \vdash_{SC} r$$

8. **Grads only** The text proves one case of the [sequent calculus] completeness theorem. Prove it for case 2. Make sure to explicitly state the induction hypothesis, and make it very clear that you show knowledge of how your induction proceeds.

9. The tableau procedure we used assumed that wffs were preprocessed to remove the $\rightarrow$ operator (among others). Add tableau rules to support $\rightarrow$.

10. Show using tableaux that the following wff is valid:

$$(p \rightarrow q) \rightarrow (p \rightarrow (q \vee r))$$

    You may either eliminate implications or use your above rules for them. However, using other identities to simplify your wffs (other than for implications) does not qualify as a tableau proof.

11. Consider the wffs:

$$\phi_1 \equiv p_1 \vee \neg p_2 \vee p_3 \vee p_4$$
$$\phi_2 \equiv \neg p_2 \rightarrow p_3$$
$$\phi_3 \equiv p_1 \rightarrow p_3$$
$$\phi_4 \equiv p_3 \rightarrow (p_2 \vee p_4)$$
$$\phi_5 \equiv p_3 \rightarrow p_4$$
$$\phi_6 \equiv p_3 \vee \neg p_4$$
$$\phi_7 \equiv p_4$$

    Show using resolution that $(\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4 \wedge \phi_5 \wedge \phi_6) \rightarrow \phi_7$ is valid.

12. Prove property P2 of the resolution rule (see text). The proof itself is simple, and the point of this problem is to get practice writing rigorous proofs in standard CS/Math language.

13. Show using DPLL that the following wff (in clausal form) is ~~valid~~ unsatisfiable:

$$\{\{p_1, p_2, p_4\}, \{\neg p_1, p_2\}, \{\neg p_2, \neg p_3, p_4\}, \{\neg p_2, p_3\}, \{\neg p_4\}, \{\neg p_4, p_5\}, \{p_3, p_6\}, \{p_2, p_6\}\}$$

14. Consider the wff:

$$\phi \equiv (\neg p_5 \vee p_3 \vee \neg p_1 \vee \neg p_2) \wedge (\neg p_1 \vee p_6) \wedge p_2 \wedge (p_3 \vee p_1 \vee \neg p_4) \wedge (p_5 \vee \neg p_3 \vee p_1 \vee p_6) \wedge (\neg p_3 \vee p_1) \wedge (p_5 \vee \neg p_6 \vee \neg p_2)$$

    Use DPLL to show that ϕ is satisfiable, and identify the corresponding model. You only need to get one model.

Additional Notes

- For Problem 1 you are formalizing the statements, **not** conditions for getting a job (which in this case wouldn't make sense anyway since the sentences are mostly not conditions).
- You may use "..." in Problem 3 (and will need to!) as long as you make what you mean unambiguous.
- Obviously, you may not use a SAT solver or other tool, and need to show all steps.