Chapter 3

Naked planet model with soil thermodynamics

3.1 Lunar regolith thermodynamic model

To simulate a transient transfer of energy into soil with enough thermal inertia to cause gradual nighttime cooling, we need to describe the thermodynamics of the lunar regolith. As a general rule, ”rocky, coherent surfaces and blocks with higher thermal inertia provide larger reservoirs of heat and remain warmer than the pulverized, fine-grained regolith” (Williams et al., 2017). Here we will initially limit ourselves to regolith to build our model.

Heat conduction into (and back out of) the lunar regolith is a nonlinear process that requires finite difference schemes to model through time. The thermodynamic definition of temperature is that it is a quantity proportional to the mean kinetic energy (or thermal energy) of the particles at some point. What we seek is a function $T(z,t)$ that describes the temperature T as it changes through time t and at various depths z in the soil column.

3.2 Derivation of 1-D heat conduction equation

Energy conservation for a small volume of lunar ground V can be described as

$$E_{soil} = \int_V \rho e \, dV \tag{3.1}$$

where E_{soil} is the total (extensive) energy of the volume of soil, ρ is the volumetric mass density, e is the energy per unit volume (intensive), and the integral is over the entire volume.

If there are no heating sources within the volume (radiogenic heating is only significant on a much larger scale, and assuming no phase changes within the regolith matrix), we can express the rate of change of the total energy of the volume (in J/s) as

$$\frac{dE_{soil}}{dt} = \int_V \rho \frac{\partial e}{\partial t} dV \tag{3.2}$$

The l.h.s. expresses the Lagrangian or total derivative for the system as a whole, and the r.h.s. includes the Eulerian or local derivative for each point within the system.

Since $\frac{\partial e}{\partial t} = c\frac{\partial T}{\partial t}$, where c is the specific heat,

$$\frac{dE_{soil}}{dt} = \int_V \rho c \frac{\partial T}{\partial t} dV \tag{3.3}$$

Letting $\dot{Q}$ be the rate at which energy enters the volume (rate of heat transfer), another expression of energy conservation would be

$$\frac{dE_{soil}}{dt} = \dot{Q} \tag{3.4}$$

Assuming no internal heat source, the rate of heat transfer in turn can be written in vector form as

$$\dot{Q} = -\int_A \mathbf{q}'' \cdot \mathbf{n}\, dA \tag{3.5}$$

where $\mathbf{q}''$ is the heat flux vector, $\mathbf{n}$ is the normal outward surface vector of the surface element dA, and the area integral is over the surface area of the system.

The divergence theorem allows us to transform the area integral into a volume integral:

$$\int_A \mathbf{q}'' \cdot \mathbf{n}\, dA = \int_V \nabla \cdot \mathbf{q}'' dV \tag{3.6}$$

Now we can write another expression for energy conservation in terms of the heat flux and temperature:

$$\int_V \rho c \frac{\partial T}{\partial t} dV = -\int_V \nabla \cdot \mathbf{q}'' dV \tag{3.7}$$

$$\int_V \rho c \frac{\partial T}{\partial t} dV + \int_V \nabla \cdot \mathbf{q}'' dV = 0 \tag{3.8}$$

$$\int_V \left(\rho c \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{q}'' \right) dV = 0 \tag{3.9}$$

$$\rho c \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{q}'' = 0 \tag{3.10}$$

Here we make use of Fourier's law of heat conduction (cf. Ingersoll and Zobel, 1913: "When different parts of a solid body are at different temperatures, heat flows from the hotter to the colder portions by a process of transference probably from molecule to molecule known as conduction.")

$$\mathbf{q}'' = -k\nabla T \tag{3.11}$$

where $\mathbf{q}''$ is the heat flux vector, k is mean thermal conductivity of the slab in $Wm^{-2}K^{-1}$, and ∇T is the gradient of the temperature field: $\nabla T = \frac{\partial T}{\partial x}\mathbf{i}, \frac{\partial T}{\partial y}\mathbf{j}, \frac{\partial T}{\partial z}\mathbf{k}$. Here z is zero at the surface and increases downward. The negative sign in Fourier's law indicates that the heating flows down the temperature gradient, from hotter to colder regions.

Neglecting horizontal heat flows, Fourier's law reduces to a vertical gradient:

$$\mathbf{q}'' = -k \frac{\partial T}{\partial z}\mathbf{i} \tag{3.12}$$

Now using this reduced form of Fourier's law, we can rewrite the energy conservation equation as

$$\rho c \frac{\partial T}{\partial t} - (\nabla \cdot k \frac{\partial T}{\partial z}\mathbf{i}) = 0 \tag{3.13}$$

$$\rho c \frac{\partial T}{\partial t} - (\nabla \cdot k \frac{\partial T}{\partial z} \mathbf{i}) = 0 \tag{3.14}$$

$$\rho c \frac{\partial T}{\partial t} - \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) = 0 \tag{3.15}$$

which gives us the one-dimensional soil heat conduction equation (HCE)

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) \tag{3.16}$$

In canonical form, the HCE is written as

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} + f \tag{3.17}$$

where α is the thermal diffusivity in units of m^2/s, defined as $\alpha = \frac{k}{\rho c}$, and f represents an internal source of heat. Heating from decay of radioactive elements is negligible for small volumes on the Moon (but significant on the global scale, as we have seen from our surface temperature calculations that require a geothermal flux q_g) so we assume $f \approx 0$. The parameter α is used as part of the stability criterion for numerical solutions of this class of equations.

Although this is a canonical Initial Value Boundary Problem (IVBP) with a parabolic partial differential equation (PDE), the quantities ρ, c, and k vary throughout the subsurface as a function of depth and temperature. The resulting nonlinearity excludes an analytical solution. Since the HCE derived above has a first order derivative in time, its solution, whether analytical or numerical, requires an initial condition, which in this case we can take to be the temperature field at time zero. The second order derivative in space makes two boundary conditions necessary, essentially at the top and at the bottom of the soil column. As we derive our first numerical solution, however, we will hold the thermal diffusivity α constant for simplicity.

3.3 Numerical solution (discretization and finite difference scheme)

There are many details involved in solving the heat conduction equation for lunar regolith, so it can be helpful to idealize the case to work out the essential details and add in complicating factors step by step so that our model provides an incrementally better and better simulation of reality.

There are a variety of numerical schemes that are available for solving a second-order PDE. The study of numerical methods is an academic field unto itself that lies at the intersection of applied mathematics, science, and engineering. Practice has shown that certain schemes have greater stability for the class of PDEs, in one state variable plus time, than others. In other words a stable solution minimizes error and converges realistic solutions. Here we will use the Crank-Nicholson finite differencing algorithm, a form of forward Euler time integration (also known as explicit timestepping). It is one of the most popular methods in practice.

We first need to discretize our variables by mapping the continuous state function (in our case $T(z,t)$) onto a grid or mesh of discrete z,t values over our domain of interest.

$$z_i = i\Delta z \quad i = 0, \ldots, N_z \tag{3.18}$$

$$t_n = n\Delta t \quad n = 0, \ldots, N_t \tag{3.19}$$

where N_z is the number of vertical layers, N_t is the number of timesteps, i is the layer index, n is the timestep index, and Δz, Δt represent the uniform discrete spacing between gridpoints in z,t space. As we will soon see, there are reasons to set grid spacing at each layer dynamically (for example, with geometrically increased grid spacing based on the penetration depth of the diurnal temperature wave in the soil column). For initial simplicity and understandability we will use a uniform grid with constant Δz. $T_i^{(n)}$ is then taken to approximate the value of the temperature function in grid space, $T(z_i,t_n)$.

We are now equipped to rewrite the HCE in grid space:

$$\frac{\partial}{\partial t} T(z_i,t_n) = \alpha \frac{\partial^2}{\partial z^2} T(z_i,t_n) \tag{3.20}$$

Later we will find that α also varies in time and space in lunar regolith, but for simplicity we will hold it constant as we first build our model. Using a forward difference in time and a central difference in space, we can write our HCE as

$$\frac{T_i^{(n+1)} - T_i^{(n)}}{\Delta t} = \alpha \frac{T_{i+1}^{(n)} - 2T_i^{(n)} + T_{i-1}^{(n)}}{(\Delta z)^2} \tag{3.21}$$

which is merely an algebraic equation which can be solved for our desired unknown. $T_i^{(n)}$ is the temperature at the current (spatial) gridpoint i and the current timestep n. The value of T at the same gridpoint but at the next timestep is $T_i^{(n+1)}$. Solving the discrete HCE for $T_i^{(n+1)}$ gives us

$$T_i^{(n+1)} = T_i^{(n)} + \alpha \frac{T_{i+1}^{(n)} - 2T_i^{(n)} + T_{i-1}^{(n)}}{(\Delta z)^2} \Delta t \tag{3.22}$$

One way to interpret each new temperature is that it is calculated by adding its old value to a weighted average of the temperature values surrounding it. This fits with the physical reality of a thermal diffusion model.

In choosing our grid spacing (and timestep duration), for the solution to be stable we need to make use of a stability criterion F based on the mesh's Fourier number. We define F as

$$F \equiv \alpha \frac{\Delta t}{(\Delta z)^2} \tag{3.23}$$

where a stable solution has the following stability criterion:

$$F \leq 0.5 \tag{3.24}$$

If it weren't for the fact that α, the thermal diffusivity, depends on a number of variables (temperature and density, in particular) in real soil, we'd be ready to simply implement (3.22) in some Python code and fire up a simulation of just another day on the Moon. That is, if things were easy and thermal diffusivity were a universal constant, we could just set α to some fixed value. If you've got some spare time you could try that out as an exercise, but what you might find is that unless α changes with temperature, your soil layers won't conduct heat with each other quickly enough when things start getting cold on the surface at sunset. This can make your code break in strange ways.

3.4 Nonlinear soil thermodynamic properties

In real environments, thermal diffusivity $\alpha = \frac{k}{\rho c}$, with units of $[m^2/s]$, depends upon both temperature and depth. Obviously with increasing overburden, ρ should increase with depth, and conductivity can be expected to change as well with changing density.

One recent standard textbook on partial differential equations (Olver, 2014) calls the heat conduction equation (more generally referred to as the diffusion equation) with variable diffusivity a "much thornier nonlinear diffusion problem" than the constant diffusivity case.

Before we tackle the top and bottom boundary conditions in the next sections, we can derive expressions for calculating soil properties (including temperature) at the inner mesh points.

Assuming no internal heating, the heat conduction equation can be written as

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} \tag{3.25}$$

or more specifically,

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c} \frac{\partial^2 T}{\partial z^2} \tag{3.26}$$

where once again k is the thermal conductivity with units of $Wm^{-2}K^{-1}$, ρ is the mass density with units of $kg\,m^{-3}$, and c is the specific heat capacity with units of $J\,kg^{-1}\,K^{-1}$.

If we make use of Fourier's law in one dimension: $q = -k\frac{\partial T}{\partial z}$,

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c} \frac{\partial q}{\partial z} \tag{3.27}$$

Discretizing the time step,

$$\Delta T = \frac{\Delta t}{\rho c} \frac{\partial q}{\partial z} \tag{3.28}$$

Using a forward difference formula to approximate the heat flux across each layer q_i,

$$q_i \approx k_i \frac{T_{i+1} + T_i}{\Delta z} \tag{3.29}$$

so that

$$T_i^{(n+1)} = T_i^{(n)} + \frac{\Delta t}{\rho_i c_i} \frac{\partial}{\partial z} q_i \tag{3.30}$$

since

$$\frac{\partial}{\partial z} q_i \approx \frac{q_i - q_{i-1}}{\Delta z} \tag{3.31}$$

$$\frac{\partial}{\partial z} q_i \approx k_i \frac{T_{i+1} - T_i}{\Delta z} - k_{i-1} \frac{T_i - T_{i-1}}{\Delta z} \tag{3.32}$$

$$T_i^{(n+1)} = T_i^{(n)} + \frac{\Delta t}{\rho_i c_i} \left\{ k_i \frac{T_{i+1} - T_i}{\Delta z} - k_{i-1} \frac{T_i - T_{i-1}}{\Delta z} \right\} \tag{3.33}$$

The soil parameters from (3.33) at each discrete level i, ρ_i, k_i, and c_i, all need to be calculated:

$$\rho_i = \rho_d - (\rho_d - \rho_s) e^{-z/H} \tag{3.34}$$

where H is a scale parameter $H \approx 0.06\,m$ (Hayne et al. 2017).

The contact conductivity is given by:

$$k_{C,i} = k_d - (k_d - k_s) \frac{\rho_d - \rho_i}{\rho_d - \rho_s} \tag{3.35}$$

which can in turn be used to calculate the overal thermal conductivity taking into account internal radiative fluxes:

$$k_i = k_{C,i}\left[1+\chi\left(\frac{T_i}{350\,K}\right)^3\right] \tag{3.36}$$

where χ is a radiative conductivity parameter, taken to be $\chi = 2.7$ by Hayne et al. 2017 and Vasavada et al. 2012.

$$c_i = c_0 + c_1 T_i + c_2 T_i^2 + c_3 T_i^3 + c_4 T_i^4 \tag{3.37}$$

where the coefficients $c_0 ... c_4$ are taken from Hayne et al. 2017:

c_0	$-3.6125\,J\,kg^{-1}\,K^{-1}$
c_1	$+2.7431\,J\,kg^{-1}\,K^{-1}$
c_2	$+2.3616\times10^{-3}\,J\,kg^{-1}\,K^{-1}$
c_3	$-1.2340\times10^{-5}\,J\,kg^{-1}\,K^{-1}$
c_4	$+8.9093\times10^{-9}\,J\,kg^{-1}\,K^{-1}$

3.5 Overview of boundary conditions

There are generally three options available for defining boundary conditions for PDEs. Here I will describe them in terms of the heat conduction equation.

Neumann boundary condition (isolated)

The system does not exchange energy with its surroundings. The derivative of the solution is specified.

Dirichlet boundary condition (equilibrium)

The system exchanges energy instantaneously with its surroundings, so that both sides of the boundary are at thermal equilibrium and thus have the same temperature. The value of the solution is specified.

Robin boundary condition (leaky / diffusive)

The system exchanges some energy with its surroundings according to some constant of proportionality. Information regarding both the value and the solution is specified.

Lunar regolith can be considered to have a Robin boundary condition at the surface which ”leaks” soil column energy to the surroundings by means of thermal radiation according to its temperature. Solar energy also passes downward through the boundary by means of conduction as well as some radiation that penetrates into the spaces between grains.

Below a certain depth, the regolith is no longer sensitive to diurnal temperature swings, is at equilibrium with its surroundings and thus has the same temperature as its most immediate surroundings. This is a Dirichlet condition.

3.5.1 Soil thermal model - lower boundary (Dirichlet)

The simplest way to express thermal equilibrium is to simply let the temperature of the bottom layer be equal to the temperature in the environment just below the bottom layer such that $T(z=D,t) = T^*_{bot}$ at the bottom boundary. This is a Dirichlet boundary condition, where T^* indicates a temperature just outside of the system.

3.5.2 Soil thermal model - upper boundary (Robin)

The energy balance at the surface is the familiar

$$q_{in} = q_{out} \tag{3.38}$$

where (just to be clear) q_{in} is the heat flux into the system (our point on the surface) and q_{out} is the heat flux out. The only inputs are the solar flux q_{solar}, the cosmic background heat flux q_c, and the ”geothermal” heat flux from the interior of the planet q_g. As we've already discussed, the maximum values of q_{solar} ($> 10^3\,W\,m^{-2}$) are orders of magnitude greater than the $10^{-3}\,W\,m^{-2}$ cosmic background flux q_c. The cosmic background radiation is sufficient to keep nighttime temperatures on an airless body above absolute zero, but only to the $2.7\,K$ ambient temperature of space. As we saw from our calculations of the energy balance in regions of permanent shadow on the Moon, the ”geothermal flux” or heat flux from the interior of the Moon q_g (essentially from radioactive decay) is sufficient to bring the Moon's minimum surface temperature up to a modeled $21\,K$ and an observed $26\,K$. In this particular model, since we have chosen to express the lower boundary condition as a Dirichlet condition with thermal equilibrium, the geothermal flux is accounted for by the constant bottom layer temperature.

However to approximate the observed equatorial surface minimum temperature of $100K$ we need to invoke a "soil" or regolith heat flux q_{soil} which represents the slow return of absorbed solar energy from the subsurface. Assuming a nighttime temperature of $100\,K$, we estimated a value of $q_{soil} \approx 5.5\,W\,m^{-2}$.

Now that we've got the picture, we can improve our previous approximations that involved a static, constant q_{soil} and use it in our dynamic soil model as a conduit between the subsurface and the surface. Something that is important to recognize is that during the daytime, q_{soil} could actually be negative, representing a loss of energy from the surface to the colder subsurface. Likewise when the surface gets cold, as it does very rapidly as the sun goes down, q_{soil} has a positive value in the sense that it contributes positive energy to the surface from the warmer layers below.

In terms of our input and output energy balance model $q_{in} = q_{out}$, it's very easy to express the input of energy to the surface "system":

$$q_{in} = q_{soil} + q_{solar} + q_{cosmic} \tag{3.39}$$

which we can simplify if we like to

$$q_{in} = q_{soil} + q_{surface} \tag{3.40}$$

where the $q_{surface}$ term simply represents all energy reaching the surface from above ($q_{surface} = q_{solar} + q_{cosmic}$).

Fourier's law in one dimension helps us to rewrite q_{soil} in terms of the vertical temperature gradient:

$$q_{soil} = -k \frac{\partial T}{\partial z}\bigg|_{z=0} \tag{3.41}$$

where the negative sign indicates that heat will only flow in the direction of positive z if the temperature decreases with increasing z (downward).

And so to be explicit, the input term to the surface energy balance is

$$q_{in} = -k \frac{\partial T}{\partial z}\bigg|_{z=0} + q_{surface} \tag{3.42}$$

Meanwhile the output term to the surface energy balance is radiative (according to the Stefan-Boltzmann relation):

$$q_{out} = \varepsilon \sigma T_s^4 \tag{3.43}$$

Putting this all together, we have the Robin radiative boundary condition

$$-k \frac{\partial T}{\partial z}\bigg|_{z=0} + q_{surface} = \varepsilon \sigma T_s^4 \tag{3.44}$$

which if we rewrite as

$$\varepsilon \sigma T_s^4 + k \frac{\partial T}{\partial z}\bigg|_{z=0} - q_{surface} = 0 \tag{3.45}$$

we can recognize as a fourth order polynomial equation.

3.5.3 Numerical solution of radiative boundary condition

Our goal is to model the temperature of the surface of the Moon in response to periodic solar forcing and the dynamic effects of soil thermal inertia. We have already derived expressions for the temperature $T_i^{(n+1)}$ in layer i at a new timestep $n+1$ at the bottom layer and at the internal points within the soil mesh. Now comes the trickiest part of our numerical model, solving for the surface temperature T_s.

Direct methods for finding the roots of fourth degree polynomials are difficult to use (Iyengar and Jain, 2009). The standard reference for numerical computation in the sciences (Press et al., 2007) states: ”There are no good, general methods for solving systems of more than one nonlinear equation [and] there never will be.”

Sounds like a meaningful challenge to me!

Our first step is to discretize the continuous Robin radiative boundary condition we derived in (3.44).

If we make use of a three-point numerical scheme to approximate the derivative in the diffusion term,

$$\frac{\partial T}{\partial z} \approx \frac{-3T_0 + 4T_1 - T_2}{2\Delta z} \tag{3.46}$$

where T_0 represents the temperature in layer 0 (the surface temperature), we can rewrite (3.44) as:

$$\varepsilon \sigma T_0^4 + k_0 \frac{-3T_0 + 4T_1 - T_2}{2\Delta z} - q_{surface} = 0 \tag{3.47}$$

Note that the value of the thermal conductivity k_i is specific to each particular layer. Therefore here we are referring to k_0, the current top layer (surface) thermal conductivity. Here we can make use of (3.36):

$$k_i = k_{C,i} \left[1 + \chi \left(\frac{T_i}{350\,K} \right)^3 \right] \tag{3.48}$$

which by substitution into (3.47) yields:

$$\varepsilon \sigma T_0^4 + k_{C,0} \left[1 + \chi \left(\frac{T_0}{350\,K} \right)^3 \right] \frac{-3T_0 + 4T_1 - T_2}{2\Delta z} - q_{surface} = 0 \tag{3.49}$$

Expanding the second term:

$$\left(k_{C,0} + k_{C,0} \chi \left(\frac{T_0}{350\,K} \right)^3 \right) \left(\frac{-3T_0 + 4T_1 - T_2}{2\Delta z} \right) \tag{3.50}$$

$$\left(k_{C,0} + k_{C,0} \chi \left(\frac{T_0}{350\,K} \right)^3 \right) \left(\frac{-3T_0}{2\Delta z} + \frac{4T_1 - T_2}{2\Delta z} \right) \tag{3.51}$$

*Is this actually correct? In case you can't quite remember exactly what works and what doesn't when decomposing fractions, you can test the decomposition we use here with a little Python script. The question is whether

$\frac{-3T_0+4T_1-T_2}{2\Delta z}$ is equivalent to $\frac{-3T_0}{2\Delta z}+\frac{4T_1-T_2}{2\Delta z}$. In the Python interpreter you can check this for plausible but random T values:

```
>>> import random
>>> T0 = random.random() * 400
>>> T1 = random.random() * 400
>>> T2 = random.random() * 400
>>> T0
10.397014739021326
>>> T1
68.84833555806256
>>> T2
186.66411573424267
>>> dz = 0.01
>>> left = (-3*T0 + 4*T1 -T2)/2/dz
>>> right = -3*T0/2/dz + (4*T1 - T2)/2/dz
>>> left
2876.9091140471787
>>> right
2876.909114047179
```

Now by the distributive property (FOIL/First Outside Inside Last... turns out middle school math is actually good for something: numerical methods for planetary climate modeling!), (3.51) is expanded to become something wildly complicated:

$$\left(k_{C,0}+k_{C,0}\chi\left(\frac{T_0}{350\,K}\right)^3\right)\left(\frac{-3T_0}{2\Delta z}+\frac{4T_1-T_2}{2\Delta z}\right) \tag{3.52}$$

$$first+outside+inside+last \tag{3.53}$$

$$k_{C,0}\frac{-3T_0}{2\Delta z}+k_{C,0}\frac{4T_1-T_2}{2\Delta z}+k_{C,0}\chi\left(\frac{T_0}{350\,K}\right)^3\frac{-3T_0}{2\Delta z}+k_{C,0}\chi\left(\frac{T_0}{350\,K}\right)^3\frac{4T_1-T_2}{2\Delta z} \tag{3.54}$$

Now to put the expanded second term back into (3.49):

$$\varepsilon\sigma T_0^4 + k_{C,0}\frac{-3T_0}{2\Delta z} + k_{C,0}\frac{4T_1 - T_2}{2\Delta z} + k_{C,0}\chi\left(\frac{T_0}{350\,K}\right)^3\frac{-3T_0}{2\Delta z} + k_{C,0}\chi\left(\frac{T_0}{350\,K}\right)^3\frac{4T_1 - T_2}{2\Delta z} - q_{surface} = 0 \tag{3.55}$$

and regrouping the terms into standard form (descending powers of T_0):

$$\varepsilon\sigma T_0^4 + k_{C,0}\chi\left(\frac{T_0}{350\,K}\right)^3\frac{-3T_0}{2\Delta z} + k_{C,0}\chi\left(\frac{T_0}{350\,K}\right)^3\frac{4T_1 - T_2}{2\Delta z} + k_{C,0}\frac{-3T_0}{2\Delta z} + k_{C,0}\frac{4T_1 - T_2}{2\Delta z} - q_{surface} = 0 \tag{3.56}$$

$$\varepsilon\sigma T_0^4 + \frac{-3k_{C,0}\chi}{2\Delta z(350\,K)^3}T_0^4 + \frac{k_{C,0}\chi}{(350\,K)^3}\left(\frac{4T_1 - T_2}{2\Delta z}\right)T_0^3 + \frac{-3k_{C,0}}{2\Delta z}T_0 + k_{C,0}\frac{4T_1 - T_2}{2\Delta z} - q_{surface} = 0 \tag{3.57}$$

$$\left(\varepsilon\sigma + \frac{-3k_{C,0}\chi}{2\Delta z(350\,K)^3}\right)T_0^4 + \frac{k_{C,0}\chi}{(350\,K)^3}\left(\frac{4T_1 - T_2}{2\Delta z}\right)T_0^3 + \frac{-3k_{C,0}}{2\Delta z}T_0 + k_{C,0}\frac{4T_1 - T_2}{2\Delta z} - q_{surface} = 0 \tag{3.58}$$

Written this way, we can recognize that (3.58) an algebraic (polynomial) equation of form

$$f(x) = P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0 \tag{3.59}$$

with coefficients

$$a_0 = \varepsilon\sigma + \frac{-3k_{C,0}\chi}{2\Delta z(350\,K)^3} \tag{3.60}$$

$$a_1 = \frac{k_{C,0}\chi}{(350\,K)^3}\left(\frac{4T_1 - T_2}{2\Delta z}\right) \tag{3.61}$$

$$a_2 = 0 \tag{3.62}$$

$$a_3 = \frac{-3k_{C,0}}{2\Delta z} \tag{3.63}$$

$$a_4 = k_{C,0}\frac{4T_1 - T_2}{2\Delta z} - q_{surface} \tag{3.64}$$

where a_4 has no direct dependence on the T_0.

Now that we have the Robin radiative boundary condition expressed as a discretized fourth-degree polynomial equation, when we implement this as Python code we can choose a root-finding algorithm to solve for the surface temperature T_0 at each timestep.

3.6 Initial condition

It's quite unrealistic, but we can set the initial temperature (at time $t_0 = 0$) throughout the soil column at $250\,K$. This will help us visualize the simulated diffusion of thermal energy through the soil column.

3.7 Python implementation of simple thermal model of lunar regolith

Quite a few components go into making this model work. If we write our code as a single procedural script, the rough organization might be:

```
# soilheat.py

# Import dependencies
# Set constants
# Set Moon parameters
# Set model parameters (modelruntime, dt, dz, Nt, Nz, etc.)
# Create numpy arrays
# Set initial conditions
# Time-stepping loop:
    # Calculate solar forcing
    # Set lower Dirichlet boundary condition
    # Solve upper Robin boundary condition
    # Update the new temperature array
```

```
    # Assign the old temp array to the new values for use in the
        next iteration
# Plot data
```

The core engine of our time-integrating model is a for control statement that loops through each timestep n up until the total number of timesteps Nt. Letting dt be the length of each timestep, the modelruntime is equal to Nt*dt. We could run our model for a split second, a few seconds, a few minutes, years, or even millions or billions of years, as some people do for astrophysical and paleoclimate models.

```
for n in range(0, Nt):
        (compute T at inner mesh points)
```

3.8 Hayne heat1d model

$$\frac{\partial^2 T}{\partial z^2} - \frac{1}{\alpha}\frac{\partial T}{\partial t} = 0 \tag{3.65}$$

$$\frac{\partial^2 T}{\partial z^2} - \frac{\rho c}{k}\frac{\partial T}{\partial t} = 0 \tag{3.66}$$

$$\frac{\partial}{\partial z} q_i \approx \frac{q_i - q_{i-1}}{\frac{1}{2}(\Delta z_i + \Delta z_{i-1})} \tag{3.67}$$

$$\approx \frac{2}{\Delta z_i + \Delta z_{i-1}} \left\{ k_i \frac{T_{i+1} - T_i}{\Delta z_i} - k_{i-1} \frac{T_i - T_{i-1}}{\Delta z_{i-1}} \right\} \tag{3.68}$$

so that

$$T_i^{(n+1)} = T_i^{(n)} + \frac{\Delta t}{\rho c} \left(\frac{2}{\Delta z_i + \Delta z_{i-1}} \right) \left(k_i \frac{T_{i+1} - T_i}{\Delta z_i} - k_{i-1} \frac{T_i - T_{i-1}}{\Delta z_{i-1}} \right) \tag{3.69}$$