THE DISCONTINUITY OF HOMOLOGY: SIERPINSKI'S GASKET

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Abstract. This paper discusses the application of simplicial homology theory, Cech homology theory, and the use of topological inverse limits to compute the homology groups of Sierpinski Gasket approximations as well as the Sierpinski Gasket itself. Starting with a rigorous construction of the Sierpinski Gasket, we will compute the coordinatized chain modules and simplicial homology groups of the approximations of the Gasket, followed by finding the Čech homology groups of the Gasket. We conclude with the intuition-defying discussion of the relevance of the Jordan Curve Theorem with the Sierpinski Gasket and how its singular simplicial homology groups do not agree with its Čech homology groups.

1. Introduction

We introduce a formal construction of the Sierpinski Gasket by taking the approach of using a solid equilateral triangle as the initiator and the middle fourth triangle removed as the generator, which will be represented via an iterated function system of contracting similiarities. We use the IFS to obtain an infinite sequence of compact, nonempty subsets of $\mathbb{R}^2$ and use a result from [1], as well as its proof, to determine the resulting invariant set—the limit of the sequence under the Hausdorff metric—that is by definition the Sierpinski Gasket.

We will then use abstract simplicial complexes to represent each sequence element, and will compute the corresponding coordinatized simplicial chain modules and simplicial homology groups. Which will be followed by a discussion of Čech homology theory using and inverse topological limits using [4] and [5]. These will be used to compute both the homology groups of the sequence elements (verifying that Čech and simplicial homology agree) as well as the homology groups of the Gasket itself.

To finish the paper, we will discuss the relevance of the Jordan Curve Theorem and the mindboggling result that the Sierpinski Gasket separates $\mathbb{R}^2$ into an unbounded region and a bounded, connected region. From which it will follow that the singular simplicial homology group of the Sierpinski Gasket is one-dimensional, proving that the limit of the Betti numbers of simplicial homology groups does not converge to the Betti numbers of singular simplicial homology groups.

For convenience, all homology groups discussed that are not Čech homology groups are assumed to be $\mathbb{Z}$-modules. As a final note, this paper assumes familiarity with real analysis, metric spaces, point-set topology, module theory, and simplicial homology theory.

2. Basic Information

We will construct the Sierpinski Gasket $\mathbb{S}$ by taking a solid equilateral triangle S_0 with vertices $(-1, 0)$, $(1, 0)$, and $(0, \sqrt{3})$ to be the initiator. Among all possible initiators, S_0 is the most intuitive to work with, so S_0 will be the initiator set for all the major results of this paper. Because as we will soon see, we can start with *any* compact nonempty subset of $\mathbb{R}^2$, and through a particular iterated function system, will *always* obtain the Sierpinski Gasket. The case where the initiator is a set containing a single point results in something called the "chaos game".

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Before talking about IFSs, we first need to define what an IFS is and relevant terminology. Most of the literature in this section is derived from [1].

Definition 2.1. *Given metric space* $(\mathbb{X}, d_{\mathbb{X}})$*, a* similarity *of ratio* $r > 0$ *is a function* $\varrho : \mathbb{X} \to \mathbb{X}$ *satisfying*

$$d_{\mathbb{X}}(\varrho(x), \varrho(y)) = r d_{\mathbb{X}}(x, y)$$

for all $x, y \in \mathbb{X}$*. A* contraction mapping *or* contracting mapping *is a function* $\varphi : \mathbb{X} \to \mathbb{X}$ *such that for some* $0 \leq s < 1$,

$$d_{\mathbb{X}}(\varphi(x), \varphi(y)) \leq s d_{\mathbb{X}}(x, y).$$

Definition 2.2. *Given metric spaces* $(\mathbb{X}, d_{\mathbb{X}}), (\mathbb{Y}, d_{\mathbb{Y}})$,

(a) *An* iterated function system *is a list of functions* $\{f_i : \mathbb{X} \to \mathbb{Y}\}_{i \in I}$.
(b) *A* ratio list *is a finite list of positive numbers* $(r_1, r_2, \ldots, r_n)$*, and a* contracting ratio list *has the property* $r_i < 1$ *for each* i.
(c) *An* iterated function system $(f_i : \mathbb{X} \to \mathbb{Y})_{i=1}^n$ realizing a ratio list $(r_1, r_2, \ldots r_n)$ *is defined such that* f_i *is a similarity with ratio* r_i.
(d) *A nonempty compact set* $K \subseteq \mathbb{X}$ *is an* invariant set *or* attractor *of an IFS* $(f_1, f_2, \ldots, f_n)$ *if and only if* $K = \bigcup_{i=1}^n f_i[K]$.

We will proceed by providing an IFS with initiator S_0, for which we will prove that the IFS is a list of contracting similarities.

2.1. Construction of Sierpinski's Gasket.

Definition 2.3. *The IFS* $(\mathbf{F}_i : \mathbb{R}^2 \to \mathbb{R}^2)_{i=0}^2$ *is given by*

$$\mathbf{F}_0((x, y)) = 1/2(x, y) + (1 - 1/2)(-1, 0) = \left(\frac{1}{2}x - \frac{1}{2}, \frac{1}{2}y\right), \tag{1}$$

$$\mathbf{F}_1((x, y)) = 1/2(x, y) + (1 - 1/2)(1, 0) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right), \tag{2}$$

$$\mathbf{F}_2((x, y)) = 1/2(x, y) + (1 - 1/2)(0, \sqrt{3}) = \left(\frac{1}{2}x, \frac{1}{2}y + \frac{\sqrt{3}}{2}\right). \tag{3}$$

The reason the expressions in the middle are included are because $\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2$ are dilations of the plane with centers $(-1, 0)$, $(1, 0)$, and $(0, \sqrt{3})$, respectively, and have ratios $1/2$.

Example 2.4. Let T_0 be the solid equilateral triangle in $\mathbb{R}^2$ with vertices $(-1, 0)$, $(0, 0)$, and $(-1/2, \sqrt{3}/2)$, T_1 the solid equilateral triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$, and $(1/2, \sqrt{3}/2)$, and T_2 the solid equilateral triangle in $\mathbb{R}^2$ with the vertices $(-1/2, \sqrt{3}/2)$, $(1/2, \sqrt{3}/2)$, and $(0, \sqrt{3})$. Then

(a) $\mathbf{F}_0[S_0] = T_0$, (b) $\mathbf{F}_0[S_1] = T_1$, (c) $\mathbf{F}_0[S_2] = T_2$.

Since many of these proofs are computationally similar, we only prove the forward inclusion of (a). Let $(x, y) \in S_0$. We use a linear algebra approach based on the literature in Chapter 3 of [2]. The points $(-1, 0)$, $(1, 0)$, and $(0, \sqrt{3})$ are affinely independent, or that $(1, 0) - (-1, 0) = (2, 0)$ and $(0, \sqrt{3}) - (-1, 0) = (1, \sqrt{3})$ are linearly independent, so S_0 equals the convex hull of $\{(-1, 0), (1, 0), (0, \sqrt{3})\}$. Therefore

$$\begin{aligned}(x, y) &= \lambda_1(-1, 0) + \lambda_2(1, 0) + \lambda_3(0, \sqrt{3}) \\ &= (-\lambda_1 + \lambda_2, \lambda_3\sqrt{3}),\end{aligned}$$

for some $\lambda_1, \lambda_2, \lambda_3 \geq 0$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$. So,

$$\begin{aligned}\mathbf{F}_0(x,y) &= \left(\frac{1}{2}(-\lambda_1+\lambda_2) - \frac{1}{2}, \frac{1}{2}(\lambda_3\sqrt{3})\right) \\ &= \left(-\frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 - \frac{1}{2}\lambda_3 + \frac{1}{2}\right)(-1,0) + \left(\frac{3}{2}\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{2}\lambda_3 - \frac{1}{2}\right)(0,0) \\ &+ \lambda_3\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right),\end{aligned}$$

and we have that

$$-\frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 - \frac{1}{2}\lambda_3 + \frac{1}{2} + \frac{3}{2}\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{2}\lambda_3 - \frac{1}{2} + \lambda_3 = \lambda_1 + \lambda_2 + \lambda_3 = 1.$$

T_0 is the solid equilateral triangle formed by affinely independent points $(-1,0)$, $(0,0)$, and $(-1/2, \sqrt{3}/2)$, so T_0 is the convex hull of $\{(-1,0),(0,0),(-1/2,\sqrt{3}/2)\}$. Because $\mathbf{F}_0(x,y)$ is a convex combination of these three points, it follows that $\mathbf{F}_0(x,y) \in T_0$. Thus, $\mathbf{F}_0[S_0] \subseteq T_0$.

Figure 1 contains some examples of repeated iterations of $(\mathbf{F}_i : \mathbb{R}^2 \to \mathbb{R}^2)_{i=0}^2$ on S_0.

Proposition 2.5. *The IFS* $(\mathbf{F}_i : \mathbb{R}^2 \to \mathbb{R}^2)_{i=0}^2$ *realizes the ratio list* $(1/2, 1/2, 1/2)$.

Proof. Consider the case where $i = 1$. The other two cases are similar. Let $(x,y),(x_1,y_1) \in \mathbb{R}^2$. Using properties of the Euclidean norm $\|\cdot\|_2$, we have that

$$\begin{aligned}\|\mathbf{F}_1(x,y) - \mathbf{F}_1(x_1,y_1)\|_2 &= \left\|\left(\frac{1}{2}(x-x_1), \frac{1}{2}(y-y_1)\right)\right\|_2 \\ &= \frac{1}{2}\|(x,y)-(x_1,y_1)\|_2.\end{aligned}$$

Hence, $\mathbf{F}_1$ is a similarity of ratio $1/2$. □

Using the IFS $(\mathbf{F}_i : \mathbb{R}^2 \to \mathbb{R}^2)_{i=1}^2$, we will now proceed to rigorously define $\mathbb{S}$. The idea is to consider the sequence $(S_k)_{k=1}^\infty$ defined recursively where S_0 is given and

$$S_k = \mathbf{F}_0[S_{k-1}] \cup \mathbf{F}_1[S_{k-1}] \cup \mathbf{F}_2[S_{k-1}], k \in \mathbb{N}, \tag{4}$$

and to consider the "limiting" set. To do this, we need the Hausdorff metric, which makes the notion of convergence of sets precise.

Definition 2.6. *Let* $(\mathbb{X}, d_\mathbb{X})$ *be a metric space, and define* $\mathscr{H}(\mathbb{X})$ *to be the collection of all nonempty compact subsets of* $\mathbb{X}$.

(a) *If* $x \in \mathbb{X}$ *and* $r > 0$, *we define* $B_r(x) = \{y \in \mathbb{X} : d_\mathbb{X}(x,y) < r\}$.
(b) *If* $A \subseteq \mathbb{X}$ *and* $r > 0$, *the* open r-neighborhood *of* A *is*

$$N_r(A) = \{y \in \mathbb{X} : d_\mathbb{X}(x,y) < r \text{ for some } x \in A\} = \bigcup_{x \in \mathbb{X}} B_r(x).$$

(c) *We define the* Hausdorff metric *to be the mapping* $D : \mathscr{H}(\mathbb{X}) \times \mathscr{H}(\mathbb{X}) \to [0,\infty)$ *given by*

$$D(A,B) = \inf\{r > 0 : A \subseteq N_r(B) \text{ and } B \subseteq N_r(A)\}.$$

Theorem 2.7. *For any metric space* $\mathbb{X}$, $(\mathscr{H}(\mathbb{X}), D)$ *is a metric space.*

Proof. Can be found in [1]. □

Furthermore, to rigorously define $\mathbb{S}$, we will need the Contraction Mapping Theorem and its corollary, for which we refer the reader to [1] for both proofs.

Theorem 2.8. *Let $f : \mathbb{X} \to \mathbb{X}$ be a contraction mapping on a complete nonempty metric space $\mathbb{X}$. Then there exists unique $x' \in \mathbb{X}$ such that $f(x') = x'$.*

Corollary 2.9. *Let $\mathbb{X}$ be a complete metric space and $f : \mathbb{X} \to \mathbb{X}$ be a contraction mapping. If $x_0 \in \mathbb{X}$, and*

$$x_{n+1} = f(x_n) \text{ for } n \geq 0,$$

then $(x_n)_{n=1}^{\infty}$ converges to the unique $x' \in \mathbb{X}$ such that $f(x') = x'$ guaranteed by the Contraction Mapping Theorem.

By far the most important result needed to define $\mathbb{S}$ is the following theorem that describes the limiting behavior of sequences of sets generated by iterated function systems of contracting similarities. It essentially proves that fractals exist and are well-defined.

Theorem 2.10. *Let $\mathbb{X}$ be a nonempty complete metric space, let $(r_1, r_2, \ldots, r_n)$ be a contracting ratio list, and let $(f_1, f_2, \ldots, f_n)$ be an IFS of similarities in $\mathbb{X}$ that realize $(r_1, r_2, \ldots, r_n)$. Then there is a unique nonempty compact invariant set for the iterated function system.*

Proof. Suppose that $\mathbb{X}$ is complete, and let $F : \mathscr{H}(\mathbb{X}) \to \mathscr{H}(\mathbb{X})$ be given by $F(A) = \bigcup_{i=1}^{n} f_i[A]$. Because each f_i is continuous (a proof of this can be found in §2.2 of [1]) and that A is compact, then $f_i[A]$ is compact. Hence, $F[A]$ is compact.

We will show that F is a contraction mapping. Let $r = \max\{r_1, \ldots, r_n\}$. Because $(r_1, \ldots, r_n)$ is a contracting ratio list, $r < 1$. So, we will show that for every $A, B \in \mathscr{H}(\mathbb{X})$,

$$D(F(A), F(B)) \leq rD(A, B).$$

Let $q \in \mathbb{R}$ such that $q > D(A, B)$. By proving that $D(F(A), F(B)) < rq$, the arbitrary nature of q will imply $D(F(A), F(B)) \leq rD(A, B)$. So, by definition of D, we need to show that $F(A) \subseteq N_{rq}(F(B))$ and $F(B) \subseteq N_{rq}(F(A))$. We will only prove the first statement, as a similar argument proves the second statement.

Let $x \in F(A)$. Then $f_i(x') = x$ for some i and $x' \in A$. Because $D(A, B) < q$, it follows that $A \subseteq N_q(B)$. Hence, there exists $y' \in B$ such that $d_{\mathbb{X}}(x', y') < q$. Because f_i is a similarity of ratio r_i, $d_{\mathbb{X}}(f_i(x'), f_i(y')) \leq rd_{\mathbb{X}}(x', y') < rq$. Therefore, $d_{\mathbb{X}}(f_i(x'), f_i(y')) < rq$. Because $f_i'(y') \in F(B)$, $x = f_i(x') \in N_{rq}(F(B))$. Hence, $F(A) \subseteq N_{rq}(F(B))$. Since a similar proof shows $F(B) \subseteq N_{rq}(F(A))$, $D(F(A), F(B)) \leq rD(A, B)$. Therefore, F is a contraction mapping, so by Theorem 2.8, F has a unique fixed point. □

The next result follows directly from Theorem 2.10 and Corollary 2.9, and arguably can be called "The Fundamental Theorem of Fractal Geometry" and provides a constructive method to obtain a fractal. It says that no matter the initiator set, as long as its compact and nonempty, then the resulting sequence generated by an IFS of contracting similarities *always* converges and to the same set.

Corollary 2.11. *Let $\mathbb{X}$ be a nonempty complete metric space and A_0 be a compact nonempty subset of $\mathbb{X}$. Suppose $(f_1, f_2, \ldots, f_n)$ is defined like above. If*

$$A_{k+1} = \bigcup_{i=1}^{n} f_i[A_k]$$

for $k \geq 0$, then $(A_k)_{k=1}^{\infty}$ converges in the Hausdorff metric to the invariant set of the IFS.

We now have everything needed to formally define $\mathbb{S}$.

Definition/Theorem 2.12. ***(Sierpinski Gasket)*** *Let $(\mathbf{F}_i : \mathbb{R}^2 \to \mathbb{R}^2)_{i=0}^2$ be the iterated function system given by (1), (2), and (3) and $(S_k)_{k=1}^\infty$ to be the sequence of nonempty compact subsets of $\mathbb{R}^2$ given by (4). The invariant set*

$$\mathbb{S} = \mathbf{F}_0[\mathbb{S}] \cup \mathbf{F}_1[\mathbb{S}] \cup \mathbf{F}_2[\mathbb{S}] \tag{5}$$

guaranteed by Corollary 2.11 to be the limit under the Hausdorff metric of $(S_k)_{k=1}^\infty$ is defined to be the Sierpinski Gasket*.*

3. Main Results

3.1. **Simplicial Complex Construction of $\mathbb{S}$.** The idea of a simplicial complex representation of each S_k is, because each S_k naturally has a base-3 structure, to construct a function $a : \{0,1,2\}^\omega \to \mathbb{R}^2$ such that $\operatorname{Im}(a) = \mathbb{S}$ and that each $a(n_0, n_1, \ldots, n_k, 0, 0, 0, \ldots)$ represents every vertex of S_k. For simplicity, we will represent each $a(n_1, n_2, n_3, \ldots) = a_{n_1,n_2,n_3,\ldots}$ and $\mathbb{A} := \operatorname{Im}(\tau)$.

Definition 3.1. Suppose that $a_0 = (-1, 0)$, $a_1 = (1, 0)$, and $a_2 = (0, \sqrt{3})$. We define the set $\mathbb{A} = \{a_{n_0,n_1,n_2,n_3,\ldots} \in \mathbb{R}^2 : 0 \leq n_i \leq 2 \text{ for all } i \in \mathbb{N}\}$ to be the collection of points in $\mathbb{R}^2$ such that the following properties hold:

(i) If $n_0, n_1, \ldots, n_p \in \{0,1,2\}$ is a finite list of numbers, we define $a_{n_0,n_1,\ldots,n_p} = a_{n_0,n_1,\ldots,n_p,0,0,0,\ldots}$;
(ii) If $c \in \{0,1,2\}$ and $k = 0$, then $a_{n_0,n_1,\ldots,n_{k-1},c} = a_c$.
(iii) For any $n_0, n_1, \ldots, n_{k-2} \in \{0,1,2\}$,

$$\begin{aligned}
a_{n_0,n_1,\ldots,n_{k-2},0,0} &= a_{n_0,n_1,\ldots,n_{k-2},0}, & a_{n_0,n_1,\ldots,n_{k-2},1,0} &= \frac{a_{n_0,n_1,\ldots,n_{k-2},1} + a_{n_0,n_1,\ldots,n_{k-2},0}}{2},\\
a_{n_0,n_1,\ldots,n_{k-2},0,1} &= \frac{a_{n_0,n_1,\ldots,n_{k-2},0} + a_{n_0,n_1,\ldots,n_{k-2},1}}{2}, & a_{n_0,n_1,\ldots,n_{k-2},1,1} &= a_{n_0,n_1,\ldots,n_{k-2},1}\\
a_{n_0,n_1,\ldots,n_{k-2},0,2} &= \frac{a_{n_0,n_1,\ldots,n_{k-2},0} + a_{n_0,n_1,\ldots,n_{k-2},2}}{2}, & a_{n_0,n_1,\ldots,n_{k-2},1,2} &= \frac{a_{n_0,n_1,\ldots,n_{k-2},1} + a_{n_0,n_1,\ldots,n_{k-2},2}}{2},
\end{aligned}$$

$$\begin{aligned}
a_{n_0,n_1,\ldots,n_{k-2},2,0} &= \frac{a_{n_0,n_1,\ldots,n_{k-2},2} + a_{n_0,n_1,\ldots,n_{k-2},0}}{2},\\
a_{n_0,n_1,\ldots,n_{k-2},2,1} &= \frac{a_{n_0,n_1,\ldots,n_{k-2},2} + a_{n_0,n_1,\ldots,n_{k-2},1}}{2},\\
a_{n_0,n_1,\ldots,n_{k-2},2,2} &= a_{n_0,n_1,\ldots,n_{k-2},2}.
\end{aligned}$$

Note that the only new points generated in (ii) of the definition above are $a_{n_0,n_1,\ldots,n_{k-2},0,1}$, $a_{n_0,n_1,\ldots,n_{k-2},0,2}$, and $a_{n_0,n_1,\ldots,n_{k-2},1,2}$. But since we are using $a_{n_0,n_1,\ldots,n_{k-2},0}$, $a_{n_0,n_1,\ldots,n_{k-2},1}$, and $a_{n_0,n_1,\ldots,n_{k-2},2}$ to represent the generating set of any given 2-simplex T_{k-1} in S_{k-1}, the system above organizes the points of S_k such that the ith column of points generates $\mathbf{F}_i[T_{k-1}]$. By using this method, the task of defining $\operatorname{Abs}(S_k)$ for each integer $k \geq 0$ is made significantly easier than it would be otherwise.

However, in order for $\operatorname{Abs}(S_k)$ to yield information about each S_k, we first need to prove that the columns above do indeed generate each of the 2-simplicies of S_k.

Lemma 3.2. *Let $k \in \mathbb{Z}_{\geq 0}$ and $a_{n_0,n_1,\ldots,n_{k-1},n_k} \in \mathbb{A}$. Then $a_{n_0,n_1,\ldots,n_{k-1},0}$, $a_{n_0,n_1,\ldots,n_{k-1},1}$, and $a_{n_0,n_1,\ldots,n_{k-1},2}$ are affinely independent.*

Proof. We proceed by induction. For the case where $k = 0$, then by definition, $a_{n_0,n_1,\ldots,n_{k-1},0} = a_0$, $a_{n_0,n_1,\ldots,n_{k-1},1} = a_1$, and $a_{n_0,n_1,\ldots,n_{k-1},2} = a_2$. These points are clearly affinely independent.

Now suppose that for some $j \in \mathbb{N}$, we have that the statement in Lemma 3.2 holds. Let $n_0, n_1, \ldots, n_j \in \{0, 1, 2\}$. There are three separate cases to consider, and since they are all similar, we only consider the case where $n_j = 0$. Then

$$\begin{aligned} a_{n_0,n_1,\ldots,n_{j-1},n_j,0} &= a_{n_0,n_1,\ldots,n_{j-1},0}, \\ a_{n_0,n_1,\ldots,n_{j-1},n_j,1} &= \frac{a_{n_0,n_1,\ldots,n_{j-1},0} + a_{n_0,n_1,\ldots,n_{j-1},1}}{2}, \\ a_{n_0,n_1,\ldots,n_{j-1},n_j,2} &= \frac{a_{n_0,n_1,\ldots,n_{j-1},0} + a_{n_0,n_1,\ldots,n_{j-1},2}}{2}. \end{aligned} \tag{6}$$

We want to show that $a_{n_0,n_1,\ldots,n_{j-1},n_j,1} - a_{n_0,n_1,\ldots,n_{j-1},n_j,0}$ and $a_{n_0,n_1,\ldots,n_{j-1},n_j,2} - a_{n_0,n_1,\ldots,n_{j-1},n_j,0}$ are linearly independent. Let $r, s \in \mathbb{R}$ such that

$$r(a_{n_0,n_1,\ldots,n_{j-1},n_j,1} - a_{n_0,n_1,\ldots,n_{j-1},n_j,0}) + s(a_{n_0,n_1,\ldots,n_{j-1},n_j,2} - a_{n_0,n_1,\ldots,n_{j-1},n_j,0}) = (0,0).$$

Because

$$a_{n_0,n_1,\ldots,n_{j-1},n_j,1} - a_{n_0,n_1,\ldots,n_{j-1},n_j,0} = \frac{1}{2}(a_{n_0,n_1,\ldots,n_{j-1},1} - a_{n_0,n_1,\ldots,n_{j-1},0})$$

and

$$a_{n_0,n_1,\ldots,n_{j-1},n_j,2} - a_{n_0,n_1,\ldots,n_{j-1},n_j,0} = \frac{1}{2}(a_{n_0,n_1,\ldots,n_{j-1},2} - a_{n_0,n_1,\ldots,n_{j-1},0}),$$

then setting $\mathbf{u} = a_{n_0,n_1,\ldots,n_{j-1},1} - a_{n_0,n_1,\ldots,n_{j-1},0}$ and $\mathbf{v} = a_{n_0,n_1,\ldots,n_{j-1},2} - a_{n_0,n_1,\ldots,n_{j-1},0}$ yields

$$\begin{aligned} r(a_{n_0,n_1,\ldots,n_{j-1},n_j,1} - a_{n_0,n_1,\ldots,n_{j-1},n_j,0}) + s(a_{n_0,n_1,\ldots,n_{j-1},n_j,2} - a_{n_0,n_1,\ldots,n_{j-1},n_j,0}) &= \frac{1}{2}r\mathbf{u} + \frac{1}{2}s\mathbf{v} \\ &= (0,0). \end{aligned}$$

Because $\mathbf{u}$ and $\mathbf{v}$ are linearly independent by the induction hypothesis, it follows that $\frac{1}{2}r = \frac{1}{2}s = 0 \implies r = s = 0$. This proves the $n_j = 0$ case, and similar proofs yield the $n_j = 1$ and $n_j = 2$ cases, which completes the induction. □

Lemma 3.3. *Let $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d$ be affinely independent points (which implies that $\mathbf{x}_0$, $(\mathbf{x}_0 + \mathbf{x}_1)/2$, and $(\mathbf{x}_0 + \mathbf{x}_2)/2$ are affinely independent). Then $\mathbf{x} \in \operatorname{conv}\{\mathbf{x}_0, (\mathbf{x}_0 + \mathbf{x}_1)/2, (\mathbf{x}_0 + \mathbf{x}_2)/2\}$ if and only if $2\mathbf{x} - \mathbf{x}_i \in \operatorname{conv}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$ for $i = 0, 1,$ or 2.*

Proof. Let $\mathbf{x} \in \operatorname{conv}\{\mathbf{x}_0, (\mathbf{x}_0+\mathbf{x}_1)/2, (\mathbf{x}_0+\mathbf{x}_2)/2\}$. Then for unique $\lambda_0, \lambda_1, \lambda_2 \geq 0$ such that $\lambda_0+\lambda_1+\lambda_2 = 1$,

$$\begin{aligned} \mathbf{x} &= \lambda_0\mathbf{x}_0 + \lambda_1(\mathbf{x}_0 + \mathbf{x}_1)/2 + \lambda_2(\mathbf{x}_0 + \mathbf{x}_2)/2 \\ &= \frac{1}{2}(2\lambda_0 + \lambda_1 + \lambda_2)\mathbf{x}_0 + \frac{1}{2}\lambda_1\mathbf{x}_1 + \frac{1}{2}\lambda_2\mathbf{x}_2. \end{aligned}$$

We will consider the $i = 0$ case, since the other two cases are similar. By the previous line,

$$\begin{aligned} 2\mathbf{x} - \mathbf{x}_0 &= (2\lambda_0 + \lambda_1 + \lambda_2)\mathbf{x}_0 + \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 - \mathbf{x}_0 \\ &= (2\lambda_0 + \lambda_1 + \lambda_2 - 1)\mathbf{x}_0 + \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 \\ &= \lambda_0\mathbf{x}_0 + \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2. \end{aligned}$$

Because $\lambda_0 + \lambda_1 + \lambda_2 = 1$, it follows that $2\mathbf{x} - \mathbf{x}_0 \in \operatorname{conv}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$. Since all of these steps are equivalent, we obtain the desired logical equivalence in Lemma 3.3. □

Proposition 3.4. *Let $\mathbb{A}$ be defined as in Definition 3.1, and let $a_{n_0,n_1,\ldots,n_{k-1},n_k} \in \mathbb{A}$ for some $k \in \mathbb{Z}_{\geq 0}$ and $n_0, n_1, \ldots, n_k \in \{0,1,2\}$. Then*

$$\operatorname{conv}\{a_{n_0,n_1,\ldots,n_{k-1},0}, a_{n_0,n_1,\ldots,n_{k-1},1}, a_{n_0,n_1,\ldots,n_{k-1},2}\} = \mathbf{F}_{n_{k-1}} \ldots \mathbf{F}_{n_1}\mathbf{F}_{n_0}[S_0].$$

Proof. Again, we proceed using induction. Let $k \in \mathbb{Z}_{\geq 0}$. If $k = 0$, then $a_{n_0,n_1,\ldots,n_{k-1},0} = a_0$, $a_{n_0,n_1,\ldots,n_{k-1},1} = a_1$, and $a_{n_0,n_1,\ldots,n_{k-1},2} = a_2$. By definition, $\operatorname{conv}\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2\} = S_0$.

Now suppose that for some integer $j \geq 1$, the equality in Proposition 3.4 holds. Let $\mathbf{u}_0 = a_{n_0,n_1,\ldots,n_j,0}$, $\mathbf{u}_1 = a_{n_0,n_1,\ldots,n_j,1}$, $\mathbf{u}_2 = a_{n_0,n_1,\ldots,n_j,2} \in \mathbb{A}$ and $(x,y) \in \operatorname{conv}\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2\}$. We have three cases: $n_j = 0$, $n_j = 1$, or $n_j = 2$. We focus on only the $n_j = 0$ case. By Definition 3.2(iii),

$$\begin{aligned}
\mathbf{u}_0 &= a_{n_0,n_1,\ldots,n_{j-1},0}, \\
\mathbf{u}_1 &= \frac{a_{n_0,n_1,\ldots,n_{j-1},0} + a_{n_0,n_1,\ldots,n_{j-1},1}}{2}, \\
\mathbf{u}_2 &= \frac{\mathbf{v}_0 + a_{n_0,n_1,\ldots,n_{j-1},2}}{2}.
\end{aligned}$$

Since $\mathbf{v}_0 = a_{n_0,n_1,\ldots,n_{j-1},0}$, $\mathbf{v}_1 = a_{n_0,n_1,\ldots,n_{j-1},1}$, and $\mathbf{v}_2 = a_{n_0,n_1,\ldots,n_{j-1},2}$ are affinely independent by Lemma 3.2, then by Lemma 3.3, $(2x+1, 2y) \in \operatorname{conv}\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$. Therefore,

$$(x,y) = \mathbf{F}_{n_j}(2x+1, 2y) \in \mathbf{F}_{n_j}[\operatorname{conv}\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}] = \mathbf{F}_{n_j}\mathbf{F}_{n_{j-1}} \ldots \mathbf{F}_{n_1}\mathbf{F}_{n_0}[S_0]$$

by the induction hypothesis. Because all of these steps are equivalent,

$$\operatorname{conv}\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2\} = \mathbf{F}_{n_j}\mathbf{F}_{n_{j-1}} \ldots \mathbf{F}_{n_1}\mathbf{F}_{n_0}[S_0],$$

Proving the $n_j = 1$ and $n_j = 2$ cases will complete the induction. □

Proposition 3.4 tells us that at the kth step, the points $a_{n_0,n_1,\ldots,n_k} \in \mathbb{A}$ generate all the 2-simplicies of S_k. Therefore, it makes sense to use $\mathbb{A}$ to define an abstract simplicial complex to represent S_k because this abstract simplicial complex, by Proposition 3.4, will accurately represent S_k and will yield information about the simplicial homology groups of each S_k.

Definition 3.5. Let $k \in \mathbb{N}$. Define the sets $A_{k,0}$, $A_{k,1}$, $A_{k,2}$ as follows:

(a) $A_{k,0} = \{[a_{n_0,n_1,\ldots,n_k}]\}_{n_0,n_1,\ldots,n_k \in \{0,1,2\}}$,

(b) If $A_{k,i,j} = \{[a_{n_0,n_1,\ldots,n_{k-1},i}, a_{n_0,n_1,\ldots,n_{k-1},j}]\}_{n_0,n_1,\ldots,n_{k-1} \in \{0,1,2\}}$, where $i < j \in \{0,1,2\}$, then

$$A_{k,1} = A_{k,0,1} \cup A_{k,0,2} \cup A_{k,1,2},$$

(c) $A_{k,2} = \{[a_{n_0,n_1,\ldots,n_{k-1},0}, a_{n_0,n_1,\ldots,n_{k-1},1}, a_{n_0,n_1,\ldots,n_{k-1},2}]\}_{n_0,n_1,\ldots,n_{k-1} \in \{0,1,2\}}$.

Then we define the *abstract simplicial complex* $\mathcal{A}_k = Abs(S_k)$ *of* S_k to be

$$\operatorname{Abs}(S_k) = \{\emptyset\} \cup A_{k,0} \cup A_{k,1} \cup A_{k,2}. \tag{7}$$

The proof that $\operatorname{Abs}(S_k)$ is an abstract simplicial complex is somewhat trivial since it only involves looking at $A_{k,0}$, $A_{k,1}$, and $A_{k,2}$ individually and applying Definition 3.5 [2].

3.2. **Simplicial Homology Groups of $\mathbb{S}$.** We now proceed to define simplicial chain complexes over $\mathcal{A}_k$.

Definition 3.6. *Let $A_{k,0}, A_{k,1}, A_{k,2}$ be defined as in Definition 3.5, and let $A_{k,-1} = \{\emptyset\}$ and $A_{k,p} = \emptyset$ for every other $p \in \mathbb{Z}$. Define $C_p(\mathcal{A}_k)$ to be the free $\mathbb{Z}$-module over $A_{k,p}$, and specifically set $C_{-1}(\mathcal{A}_k) = \mathbf{0}$. We define $(C_p(\mathcal{A}_k), \partial)$ to be the* simplicial chain complex *over $\mathcal{A}_k$ given by the sequence of $\mathbb{Z}$-module* *homomorphisms* $\partial_{p,k} : C_p(\mathcal{A}_k) \to C_{p-1}(\mathcal{A}_k)$*, determined by the following actions on the generators of* $C_p(\mathcal{A}_k)$*:*

$$\partial_{1,k}([a_{n_0,n_1,\ldots,n_{k-1},i}, a_{n_0,n_1,\ldots,n_{k-1},j}]) = [a_{n_0,n_1,\ldots,n_{k-1},i}] - [a_{n_0,n_1,\ldots,n_{k-1},j}] \ (i < j \in \{0,1,2\}), \tag{8}$$

$$\partial_{2,k}([a_{n_0,n_1,\ldots,n_{k-1},0}, a_{n_0,n_1,\ldots,n_{k-1},1}, a_{n_0,n_1,\ldots,n_{k-1},2}]) = [a_{n_0,n_1,\ldots,n_{k-1},1}, a_{n_0,n_1,\ldots,n_{k-1},2}] - [a_{n_0,n_1,\ldots,n_{k-1},0}, a_{n_0,n_1,\ldots,n_{k-1},2}] + [a_{n_0,n_1,\ldots,n_{k-1},0}, a_{n_0,n_1,\ldots,n_{k-1},1}], \tag{9}$$

and $\partial_{p,k}$ *is the zero map for every other* $p \in \mathbb{Z}$*.*

Like with Definition 3.5, it is not difficult to see that $(C_p(\mathcal{A}_k), \partial)$ is a chain complex because of the fact that $\mathrm{Abs}(S_k)$ is an abstract simplicial complex and the definition of each $\partial_{p,k}$ [6].

We now discuss the first major theorem of this paper.

Theorem 3.7. *Let* $C_p(\mathcal{A}_k)$*, with* $k \in \mathbb{Z}_{\geq 0}$*, be defined as in Definition 3.6. Then* $C_0(\mathcal{A}_k) \cong \mathbb{Z}^{\frac{3}{2}(1+3^k)}$*,* $C_1(\mathcal{A}_k) \cong \mathbb{Z}^{3^{k+1}}$*,* $C_2(\mathcal{A}_k) \cong \mathbb{Z}^{3^k}$*, and* $C_p(\mathcal{A}_k) \cong \mathbf{0}$ *for every other* $p \in \mathbb{Z}$*.*

Visually, because these chain complexes are mappings among free $\mathbb{Z}$-modules, we can visualize the chain complex diagrams with respect to the coordinatized $\mathbb{Z}$-modules products isomorphic to the free $\mathbb{Z}$-modules over each A_i $(i = 0, 1, 2)$.

$$\begin{array}{lccccccccccccc}
(C(\mathcal{A}_0),\partial): & \cdots & \longleftarrow & \mathbf{0} & \longleftarrow & \mathbb{Z}^3 & \xleftarrow{\partial_{1,0}} & \mathbb{Z}^3 & \xleftarrow{\partial_{2,0}} & \mathbb{Z} & \longleftarrow & \mathbf{0} & \longleftarrow & \cdots \\
(C(\mathcal{A}_1),\partial): & \cdots & \longleftarrow & \mathbf{0} & \longleftarrow & \mathbb{Z}^6 & \xleftarrow{\partial_{1,1}} & \mathbb{Z}^9 & \xleftarrow{\partial_{2,1}} & \mathbb{Z}^3 & \longleftarrow & \mathbf{0} & \longleftarrow & \cdots \\
(C(\mathcal{A}_2),\partial): & \cdots & \longleftarrow & \mathbf{0} & \longleftarrow & \mathbb{Z}^{15} & \xleftarrow{\partial_{1,2}} & \mathbb{Z}^{27} & \xleftarrow{\partial_{2,2}} & \mathbb{Z}^9 & \longleftarrow & \mathbf{0} & \longleftarrow & \cdots \\
(C(\mathcal{A}_3),\partial): & \cdots & \longleftarrow & \mathbf{0} & \longleftarrow & \mathbb{Z}^{42} & \xleftarrow{\partial_{1,3}} & \mathbb{Z}^{81} & \xleftarrow{\partial_{2,3}} & \mathbb{Z}^{27} & \longleftarrow & \mathbf{0} & \longleftarrow & \cdots \\
\vdots & & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
(C(\mathcal{A}_k),\partial): & \cdots & \longleftarrow & \mathbf{0} & \longleftarrow & \mathbb{Z}^{\frac{3}{2}(1+3^k)} & \xleftarrow{\partial_{1,k}} & \mathbb{Z}^{3^{k+1}} & \xleftarrow{\partial_{2,k}} & \mathbb{Z}^{3^k} & \longleftarrow & \mathbf{0} & \longleftarrow & \cdots \\
\vdots & & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

For brevity, we define $\mathbf{v}_i = a_{n_0,n_1,\ldots,n_{j-1},i}$, where $i \in \{0,1,2\}$.

Proof. Let $k \in \mathbb{Z}_{\geq 0}$. If p is an integer such that $p \notin \{-1,0,1,2\}$, then $A_p = \emptyset$. Any $\mathbb{Z}$-module with an empty generating set equals the zero $\mathbb{Z}$-module, so $C_p(\mathcal{A}_k) \cong \mathbf{0}$. Any by Definition 3.6, $C_{-1}(\mathcal{A}_k) \cong \mathbf{0}$.

$\underline{p = 0}$: $C_0(\mathcal{A}_k)$ is the free $\mathbb{Z}$-module over $A_{k,0}$, so it suffices to calculate $|A_{k,0}|$, the cardinality of A_0. We will prove $|A_{k,0}| = \dfrac{3}{2}(1+3^k)$, and we do so via induction. If $k = 0$, then $A_{0,0} = \{[a_0], [a_1], [a_2]\}$. Clearly $|A_{0,0}| = 3 = \dfrac{3}{2}(1+3^0)$.

Now suppose that for some integer $j \geq 0$, $|A_{j,0}| = \dfrac{3}{2}(1+3^j)$, and consider

$$[\mathbf{v}_0], [\mathbf{v}_1], [\mathbf{v}_2] \in A_{j,0}.$$

By Definition 3.1, the 0-simplicies from $A_{j+1,0}$ are generated from all possible values of $n_0, n_1, \ldots, n_j$, and $[\mathbf{v}_0], [\mathbf{v}_1], [\mathbf{v}_2]$ themselves generate nine 0-simplicies in $A_{j+1,0}$. But only three of these 0-simplicies contain points that are not in $A_{j,0}$. So, since the Fundamental Principle of Counting implies there are 3^j possibilities for $\{[\mathbf{v}_0], [\mathbf{v}_1], [\mathbf{v}_2]\} \subseteq A_{j,0}$, and that $|A_{j,0}| = \dfrac{3}{2}(1+3^j)$ by the induction hypothesis, then by the FPoC again,

$$|A_{j+1,0}| = 3 \cdot 3^j + \frac{3}{2}(1+3^j) = \frac{2 \cdot 3^{j+1} + 3 + 3^{j+1}}{2} = \frac{3^{j+2}+3}{2} = \frac{3}{2}(1+3^{j+1}).$$

Therefore, $|A_{k,0}| = \dfrac{3}{2}(1+3^k)$, and since $A_{k,0}$ is the basis for $C_0(\mathcal{A}_k)$, a $\mathbb{Z}$-module, then $C_0(\mathcal{A}_k) \cong \mathbb{Z}^{\frac{3}{2}(1+3^k)}$.
$\underline{p=1}$: We proceed via induction. We have that $A_{1,0} = \{[a_0,a_1]\} \cup \{[a_0,a_2]\} \cup \{[a_1,a_2]\}$. Clearly $|A_{1,0}| = 3 = 3^{0+1}$. Now suppose that for some $j \in \mathbb{N}$, $|A_{1,j}| = 3^{j+1}$ and that $|A_{j,0,1}| = |A_{j,0,2}| = |A_{j,1,2}|$ as in Defintion 3.5. Then $|A_{j,m,n}| = 3^j$ for $m < n$. For each $[\mathbf{v}_0,\mathbf{v}_1],[\mathbf{v}_0,\mathbf{v}_2],[\mathbf{v}_1,\mathbf{v}_2] \in A_{j,1}$, we obtain by Definition 3.1 nine distinct 1-simplicies in $A_{j+1,1}$, three of each which are contained in $A_{j+1,m,n}$ with $m < n \in \{0,1,2\}$. Figure 2 indicates which set $A_{j+1,m,n}$ that each of the new simplicies belong to. Because $|A_{j,0,1}| = |A_{j,0,2}| = |A_{j,1,2}| = 3^j$, and that each $[\mathbf{v}_0,\mathbf{v}_1] \in A_{j,0,1}$, $[\mathbf{v}_0,\mathbf{v}_2] \in A_{j,0,2}$, $[\mathbf{v}_1,\mathbf{v}_2] \in A_{j,1,2}$ generate three new 1-simplicies each in $A_{j+1,0,1}, A_{j+1,0,2}, A_{j+1,1,2}$, then by the FPoC, $|A_{j+1,1}| = 3(|A_{j,0,1}| + |A_{j,0,2}| + |A_{j,1,2}|) = 3^{j+2}$. Therefore, $|A_{k+1,1}| = 3^{k+1}$, so like in the $p=0$ case, $C_1(\mathcal{A}_k) \cong \mathbb{Z}^{3^{k+1}}$.
$\underline{p=2}$: This is the easiest case. If $k=0$, then $A_{2,0} = \{[a_0,a_1,a_2]\}$, so clearly $|A_{2,0}| = 3^0$ Now suppose for some $j \in \mathbb{N}$, $|A_{2,j}| = 3^j$. Consider any $[\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2] \in A_{2,j}$. By Definitions 3.1 and 3.5, we obtain 2-simplicies $[\mathbf{v}_0,(\mathbf{v}_0+\mathbf{v}_1)/2,(\mathbf{v}_0+\mathbf{v}_2)/2],[(\mathbf{v}_0+\mathbf{v}_1)/2,\mathbf{v}_1,(\mathbf{v}_1+\mathbf{v}_2)/2],[(\mathbf{v}_0+\mathbf{v}_2)/2,(\mathbf{v}_1+\mathbf{v}_2)/2,\mathbf{v}_2] \in A_{j+1,2}$. Because there are 3^j 2-simplicies in $A_{j,2}$ and that for every 2-simplex in $A_{2,j}$, we obtain three new 2-simplicies in $A_{j+1,2}$, then by the FPoC, $|A_{j+1,2}| = 3 \cdot 3^j = 3^{j+1}$. Hence, $|A_{k,2}| = 3^k$, so we have that $C_2(\mathcal{A}_k) \cong \mathbb{Z}^{3^k}$. $\square$

With an understanding of the structure of the chain modules of each $\mathcal{A}_k$, we will now discuss the simplicial homology groups of $\mathbb{S}$. These will enable us to prove both the path-connected nature of each S_k as well as the number of holes in S_k and the behavior of the Betti numbers of each S_k as $k \to \infty$. Note that $H_p(\mathcal{A}_k) = \mathrm{Ker}(\partial_{p,k})/\mathrm{Im}(\partial_{p+1,k})$ over $\mathbb{Z}$.

Theorem 3.8. *Let $(C(\mathcal{A}_k),\partial)$ be defined as in Definition 3.6, and $H_p(\mathcal{A}_k)$ be the pth simplicial homology group of $\mathcal{A}_k$. Then for every $k \in \mathbb{Z}_{\geq 0}$,*

(a) $H_0(\mathcal{A}_k) \cong \mathbb{Z}$,

(b) $H_1(\mathcal{A}_k) \cong \begin{cases} \mathbf{0} & \text{if } k=0, \\ \mathbb{Z}^{(3^k-1)/2} & \text{if } k>0, \end{cases}$

and $H_p(\mathcal{A}_k) \cong \mathbf{0}$ for every other $p \in \mathbb{Z}$.

Partial Proof. If $p \in \mathbb{Z} - \{0,1\}$, then because $C_p(\mathcal{A}_k) \cong \mathbf{0}$ for $p<0$ and $p>2$, then $H_p(\mathcal{A}_k) \cong \mathbf{0}$. For $p=2$, since ∂_2 is defined to be the zero $\mathbb{Z}$-module homomorphism, we have that $H_2(\mathcal{A}_k) \cong \mathrm{Ker}(\partial_{2,k})/\mathbf{0} \cong \mathrm{Ker}(\partial_{2,k})$. Let x be a nonzero chain in $C_2(\mathcal{A}_k)$. Then $x = c_1 v_1 + \ldots c_j v_j$ for nonzero $c_1,\ldots,c_j \in \mathbb{Z}$ and distinct $v_1,\ldots,v_j \in A_{k,2}$. So,

$$\partial_2(x) = c_1\partial_2(v_1) + \ldots + c_j\partial_2(v_j).$$

In general, if a 1-simplex of any 2-dimensional abstract simplicial complex L is a subset of, or the boundary of, at least two 2-simplicies in L, then at least one pair of 2-simplicies in L have more than one point in common. Hence, because any pair v_i, v_j have at most one point in common (this can be proven using Definition 3.5) and are all distinct, then $\partial_2(v_1),\ldots,\partial_2(v_j)$ are all distinct. Therefore, $x \notin \mathrm{Ker}(\partial_{2,k})$, proving that $\mathrm{Ker}(\partial_{2,k}) = \mathbf{0}$ and thus $H_2(\mathcal{A}_k) \cong \mathbf{0}$. $\square$

As a note, an alternative way to prove that $H_2(\mathcal{A}_k) \cong \mathbf{0}$ is by finding the $(3^{k+1} \times 3^k)$ matrix representation of ∂_2, row reduce it via elementary row operations (both of which are not difficult to do) and then count the number of pivots to obtain $\mathrm{rank}(\partial_2) = 3^k$.

Proving parts (a) and (b) is not nearly as difficult as it seems as a consequence of an abundance of homology theorems, with Theorem 3.7, that we will discuss later in the paper. We do not even require matrix representations because of these theorems. Due to the similarities that $\mathcal{A}_k$ has with the nerves of open coverings of $\mathbb{S}$ that will be developed in the Čech homology section for $\mathbb{S}$, we defer the rest of the proof of Theorem 3.8 until then. We also will prove that particular subcomplexes of each $\mathcal{A}_k$ have equivalent simplicial homology groups to the nerves of the open covers, resulting in dimensions with powers of 3 like in (b) that differ by 1.

3.3. **Inverse Limit Spaces.** Much of the following discussion will be based from [4] and [5]. First, we need to discuss the notion of an *inverse limit* of topological spaces–the key to obtaining the homology groups of $\mathbb{S}$ itself. For convenience, in the following definition, $a \leq b$ will be taken to be equivalent to $b \geq a$.

Definition 3.9. (i) A directed set *is a set A endowed with a transitive, reflexive relation $\geq$ such that for any $a, b \in S$, there exists $c \in S$ such that $a \geq c \geq b$. We say that $\geq$* directs *S.* [5]

(ii) *A* cofinal subset *A' of a directed set A has the property that for any $a \in A$, there exists $a' \in A'$ such that $a' \geq a$.*

(iii) *Let $\{\mathbb{X}_\alpha\}_{\alpha \in A}$ be a collection of topological spaces indexed by the directed set $(A, \geq)$. An* inverse system *of $\{\mathbb{X}_\alpha\}_{\alpha \in A}$ is the collection of spaces from $\{\mathbb{X}_\alpha\}_{\alpha \in A}$ along with continuous mappings $\mathbb{X}_\alpha \xleftarrow{p_{\alpha\beta}} \mathbb{X}_\beta$ for each $\alpha \leq \beta$, and the following properties are satisfied:*
   (a) $(\mathbb{X}_\alpha \xleftarrow{p_{\alpha\alpha}} \mathbb{X}_\alpha) = 1_{\mathbb{X}_\alpha}$ *for all $\alpha \in A$,*
   (b) $(\mathbb{X}_\alpha \xleftarrow{p_{\alpha\beta}} \mathbb{X}_\beta) \circ (\mathbb{X}_\beta \xleftarrow{p_{\beta\gamma}} \mathbb{X}_\gamma) = (\mathbb{X}_\alpha \xleftarrow{p_{\alpha\gamma}} \mathbb{X}_\gamma)$ *for every $\alpha \leq \beta \leq \gamma$.*

   The mappings $p_{\alpha\beta}$ are called bonding morphisms*. We denote inverse systems by $\mathbf{X} = (\mathbb{X}_\alpha, p_{\alpha\beta}, A)$.*

(iv) *Let $\mathbf{X} = (\mathbb{X}_\alpha, p_{\alpha\beta}, A)$ be an inverse system. We define the* inverse limit*, $\varprojlim \mathbf{X}$, to be the subspace of $\prod\limits_{\alpha \in A} \mathbb{X}_\alpha$, under the product topology, given by*

$$\varprojlim \mathbf{X} := \left\{ (x_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} \mathbb{X}_\alpha : p_{\alpha\beta}(x_\beta) = x_\alpha \text{ for some } \beta \geq \alpha \right\}.$$

Example 3.10. The subset $\mathbb{N}$ of $\mathbb{R}$ is a cofinal subset of $\mathbb{R}$ because by the Archimedean property, for all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$.

Example 3.11. [4] Consider the set $C_0 = [0,1]$ and the IFS (g_1, g_2) given by $g_1(x) = \dfrac{1}{3}x$ and $g_1(x) = \dfrac{1}{3}x + \dfrac{2}{3}$ [1]. (g_1, g_2) realizes the ratio list $(1/2, 1/2)$, so by Corollary 2.11, $C_{k+1} = \bigcup_{i=1}^n f_i[C_k]$ for $k \geq 0$ converges under the Hausdorff metric to the invariant set of (g_1, g_2) guaranteed by Theorem 2.10. By definition, this set $\mathscr{C}$ is called the middle-thirds Cantor Set.

Define the inclusion maps $C_m \xleftarrow{p_{m,n}} C_n$, with $m < n \in \mathbb{Z}_{\geq 0}$, which are well-defined because $\{C_n\}$ is an descending sequence. This is continuous because inclusion maps of topological spaces are continuous. It is not difficult to see that these are bonding morphisms by Definition 3.9(iii). So, we have the sequence

$$C_0 \xleftarrow{p_{0,1}} C_1 \xleftarrow{p_{1,2}} C_2 \xleftarrow{p_{2,3}} \dots \xleftarrow{p_{n-1,n}} C_n \xleftarrow{p_{n,n+1}} C_{n+1} \xleftarrow{p_{n+1,n+2}} \dots,$$

Now $\varprojlim \mathbf{X}$ equals the set of points $(x_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}} C_k$ such that $x_k = p_{k,j}(x_j)$ for some $j > k$. Since $p_{k,j}(x_j) = x_j$, $x_k = x_j$. If there is $m \in \mathbb{N}$ such that $k < m < j$, then by definition of $p_{m,n}$

$$x_k = x_j = p_{k,j}(x_j) = p_{k,m}(p_{m,j}(x_j)) = p_{k,m}(x_m) = x_m.$$

Hence, for any $k \in \mathbb{Z}_{\geq 0}$, we have that $x_k = x_{k+1} = \ldots = x_j$ for some $j > k$. Hence, $\varprojlim \mathbf{X}$ equals the set of points $(x, x, \ldots, x, \ldots) \in \prod_{k \in \mathbb{N}} C_k$, with $x \in \mathscr{C}$. The function $x \mapsto (x, x, \ldots, x, \ldots)$ yields a homeomorphism betwen $\mathscr{C}$ and $\varprojlim \mathbf{X}$.

Before proceeding to the next section, we need to assure that the topological spaces used will guarantee that $\varprojlim \mathbf{X}$ is nonempty. The following theorem and proof can be found in [5].

Theorem 3.12. *Suppose that $\{\mathbb{X}_\alpha\}_{\alpha \in A}$ is a collection of compact, Hausdorff spaces. Then $\varprojlim \mathbf{X}$ is nonempty.*

And because sets of topological spaces can be extraordinarily large, it is quite helpful to consider the inverse systems of subsets of the sets of topological spaces. The following theorem from [4] will be crucial for the homology group results on $\mathbb{S}$.

Theorem 3.13. *If A' is a cofinal subset of A, then the inverse systems $(\mathbb{X}_\alpha, p_{\alpha\beta}, A)$ and $(\mathbb{X}_\alpha, p_{\alpha\beta}, A')$ have homeomorphic inverse limits.*

3.4. **Čech Homology Theory.** The usefulness of Čech homology groups and its dependence on inverse limit spaces lies in the fact that it addresses the gap in sets of convex geometric structures that simplicial homology groups does not address: infinite simplicial complexes. We now develop Čech homology theory, starting with Čech systems.

Definition/Theorem 3.14. *Let $\mathbb{X}$ be a compact Hausdorff space. We define a* Čech inverse system $\mathbf{X}$ *of $\mathbb{X}$ to have the following properties:*

1. *The set $\mathscr{P}(\mathbb{X})$ of finite open covers of $\mathbb{X}$ (nonempty by definition of $\mathbb{X}$), each endowed with the discrete topology, will make up the* topological spaces *of* $\mathbf{X}$.
2. *The* partial ordering $\geq$ *of* $\mathbf{X}$ *will be defined so that $\mathcal{V} \geq \mathcal{U}$ iff for any $V \in \mathcal{V}$, there is $U \in \mathcal{U}$ such that $V \subseteq U$ (we say that $\mathcal{V}$ refines $\mathcal{U}$)*
3. *Let $\mathcal{U} \xleftarrow{p_{\mathcal{U}\mathcal{V}}} \mathcal{V}$ be defined so that if $\mathcal{V}$ refines $\mathcal{U}$, then $p_{\mathcal{U}\mathcal{V}}$ assigns any $V \in \mathcal{V}$ to a fixed $U \in \mathcal{U}$ such that $V \subseteq U$. Because $p_{\mathcal{U}\mathcal{V}}$ is not unique, we define the* bonding morphisms *on* $\mathbf{X}$ *to be the set of homotopy equivalence classes $[p_{\mathcal{U}\mathcal{V}}]$ of projections from $\mathcal{V}$ to $\mathcal{U}$.*

We denote $\mathbf{X} = (\mathcal{U}, p_{\mathcal{U}\mathcal{V}}, \mathscr{P}(\mathbb{X}))$.

Proof. We will prove that the definitions for the bonding morphisms in (3) are indeed bonding morphisms but with respect to homotopy, which will become clearer near the end of the proof. First, we will prove that any projection mapping $\mathcal{U} \xleftarrow{p_{\mathcal{U}\mathcal{V}}} \mathcal{V}$ as defined above are homotopic. Since $\mathcal{U}$ and $\mathcal{V}$ are finite, we'll suppose $\mathcal{V} = \{V_1, \ldots, V_m\}$ and $\mathcal{U} = \{U_1, \ldots, U_n\}$. Let $p_{\mathcal{U}\mathcal{V}} : \mathcal{V} \to \mathcal{U}$ and $q_{\mathcal{U}\mathcal{V}} : \mathcal{V} \to \mathcal{U}$ be choices of projection mappings, and define $F : \mathcal{V} \times [0,1] \to \mathcal{U}$ by

$$F(V,t) = \begin{cases} p_{\mathcal{U}\mathcal{V}}(V) & \text{if } t = 0, \\ U_k & \text{if } 0 < t < 1, \\ q_{\mathcal{U}\mathcal{V}}(V) & \text{if } t = 1, \end{cases}$$

where $V \subseteq U_k$. If U is open and nonempty in $\mathcal{U}$, then since $\mathcal{V}$ is finite and has the discrete topology, $F^{-1}(U)$ must itself be open in $\mathcal{V}$, proving that $p_{\mathcal{U}\mathcal{V}}$ and $q_{\mathcal{U}\mathcal{V}}$ are homotopic. Using a similar variant of F given above also will prove that $[p_{\mathcal{V}\mathcal{V}}] = [1_{\mathcal{V}}]$ and $[p_{\mathcal{U}\mathcal{V}} p_{\mathcal{V}\mathcal{W}}] = [p_{\mathcal{U}\mathcal{W}}]$. $\square$

The point of homotopy equivalence classes of (3) is that, when obtaining the induced homomorphisms of the simplicial homology groups of the nerves of the open covers for Čech homology groups, the homomorphisms are unique to the homotopy equivalence classes.

We are almost ready to define Čech homology groups. As a note, we denote

$$\mathrm{Nerve}(\mathcal{U}) = \{\{\mathbb{X}_\gamma\}_{\gamma\in A'} \in \mathcal{P}(\mathcal{C}) : \bigcap_{\gamma\in A'} \mathbb{X}_\gamma \neq \emptyset \text{ for some } A' \subseteq A\},$$

for $\mathcal{U}$ in the power set $\mathcal{P}(\mathcal{C})$ of a collection $\mathcal{C}$, and that this is an abstract simplicial complex. In order to get simplicial maps from $\mathcal{U} \xleftarrow{p_{\mathcal{U}\mathcal{V}}} \mathcal{V}$, we define $\Gamma : \mathrm{Nerve}(\mathcal{V}) \to \mathrm{Nerve}(\mathcal{U})$ by

$$\Gamma([V_1, \ldots, V_j]) = [p_{\mathcal{U}\mathcal{V}}(V_1), \ldots, p_{\mathcal{U}\mathcal{V}}(V_j)]$$

To get a simplicial mapping out of this, we note that the geometric realization theorem implies the existence of a realization of $\mathrm{Nerve}(\mathcal{U})$ and $\mathrm{Nerve}(\mathcal{V})$ in $\mathbb{R}^{2\dim(\mathrm{Nerve}(\mathcal{U}))+1}$ and $\mathbb{R}^{2\dim(\mathrm{Nerve}(\mathcal{V}))+1}$, respectively. So, by associating each of the simplicies in $\mathrm{Nerve}(\mathcal{V})$ and $\mathrm{Nerve}(\mathcal{U})$ in their convex hulls of their respective underlying spaces, we obtain a simplicial map $\Gamma* : |\mathrm{Nerve}(\mathcal{V})| \to |\mathrm{Nerve}(\mathcal{U})|$.

Finally, we note that when the spaces that make up the inverse system are replaced with groups or R-modules, the bonding morphisms that make up the inverse system must be group or R-module homomorphisms. Both the topological and group definitions are necessary since the next definition depends on the definition of a Čech inverse system.

Definition/Theorem 3.15. *Let $\mathbb{X}$ be a compact Hausdorff space, $\mathscr{P}(\mathbb{X})$ the finite open covers of $\mathbb{X}$, R a ring with 1 and M an R-module, and let $H_p(\mathcal{U}, M)$ be the pth simplicial homology group of Nerve($\mathcal{U}$) over M. The inverse system $(H_p(\mathcal{U}, M), p_{\mathcal{U}\mathcal{V}*}, \mathscr{P}(\mathbb{X}))$ will be defined such that the following holds:*

1. ***Spaces**: For any $\mathcal{U} \in \mathscr{P}(\mathbb{X})$, the pth simplicial homology groups $H_p(\mathcal{U}, M)$ of* Nerve($\mathcal{U}$) *will be the spaces of $(H_p(\mathcal{U}, M), p_{\mathcal{U}\mathcal{V}*}, \mathscr{P}(\mathbb{X}))$.*
2. ***Bonding Morphisms**: From the projection maps $\mathcal{U} \xleftarrow{p_{\mathcal{U}\mathcal{V}}} \mathcal{V}$, we associate $\mathcal{U} \xleftarrow{p_{\mathcal{U}\mathcal{V}}} \mathcal{V}$ with the simplicial maps of Nerve($\mathcal{U}$) and Nerve($\mathcal{V}$). Consequently, we obtain induced homomorphisms $H_p(\mathcal{U}, M) \xleftarrow{p_{\mathcal{U}\mathcal{V}}*} H_p(\mathcal{V}, M)$, and by Theorem 8-3 in* [5]*, $H_p(\mathcal{U}, M) \xleftarrow{p_{\mathcal{U}\mathcal{V}}*} H_p(\mathcal{V}, M)$ is unique to $[p_{\mathcal{U}\mathcal{V}}]$. The homomorphisms $p_{\mathcal{U}\mathcal{V}}*$ will be the bonding morphisms of $(H_p(\mathcal{U}, M), p_{\mathcal{U}\mathcal{V}*}, \mathscr{P}(\mathbb{X}))$.*
3. ***Partial Ordering**: The same one given for $\mathscr{P}(\mathbb{X})$ in the definition of a Čech system.*

*We define the p*th Čech homology group *of $\mathbb{X}$ to be*

$$\check{H}_p(\mathbb{X}, M) := \varprojlim (H_p(\mathcal{U}, M), p_{\mathcal{U}\mathcal{V}*}, \mathscr{P}(\mathbb{X})), \tag{10}$$

with operations of componentwise addition and scalar multiplication.

A discussion in Chapter 8.1 of [5] reveals that the structure of $\check{H}_p(\mathbb{X}, M)$, namely the relationship between the elements of the coordinates, is given by the following proposition.

Proposition 3.16. *Let Γ^* be the simplicial mapping induced by $p_{\mathcal{U}\mathcal{V}*}$. If $(z_\mathcal{U}) \in \check{H}_p(\mathbb{X}, M)$, then if $(z_\mathcal{U})_\alpha = [z_{\mathcal{U},\alpha}] \in \check{H}_p(\mathcal{U}, M)$, $(z_\mathcal{V})_\beta = [z_{\mathcal{V},\beta}] \in \check{H}_p(\mathcal{V}, M)$, and if $\mathcal{V}$ refines $\mathcal{U}$, then $\Gamma^*(z_{\mathcal{V},\beta})$ is homologous to $z_{\mathcal{U},\alpha}$.*

3.5. **The Need for a Non-Integer Coefficient Ring.** Before proceeding to calculate the Čech homology groups of $\mathbb{S}$, a discussion about what coefficient ring to use is in order. The following example from [5], and something similar happens when working with the cycles in the nerves of open covers of each S_k, demonstrates that $\mathbb{Z}$ is insufficient in computing nontrivial 1-cycles.

Example 3.17. Let T_1 be the standard 2-torus in $\mathbb{R}^3$ centered at the origin. Now define T_2 to be a 2-torus contained inside the interior of T_1. Likewise, define T_3 to be a 2-torus contained inside the interior of T_2. Repeating this ad infinitum, we obtain by definition a solenoid $S := \bigcap_{n=1}^{\infty} T_n$. Figure 3 provides a picture of this process.

Define $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ to be a cofinal family of coverings of S by covering each T_n with finitely many open connected sets with diameter less than $1/(n-1/2)$, $\mathcal{U}_{n+1} > \mathcal{U}_n$ for all $n \in \mathbb{N}$, and that the geometric realization of each Nerve($\mathcal{U}_n$) is a polygonal simple closed curve containing 2^{n+1} 1-simplicies. Therefore, each projection $p_{\mathcal{U}_n\mathcal{U}_{n+1}} : \mathcal{U}_{n+1} \to \mathcal{U}_n$ have associated simplicial mappings such that every 1-simplex in Nerve($\mathcal{U}_n$) has exactly two 1-simplex preimages in Nerve($\mathcal{U}_{n+1}$).

Let $(z_1(\mathcal{U}_1), z_1(\mathcal{U}_2), z_1(\mathcal{U}_3), \ldots)$ be a sequence of 1-cycles in each $\mathcal{U}_k$ such that the property stated in Proposition 3.16 holds for each element $z_1(\mathcal{U}_k)$. We will assume that the open covers of $\mathcal{U}_n$ are situated so that all the 1-simplicies of $\mathcal{U}_n$ have the same coefficient r_n. For any projection $p_{\mathcal{U}_n,\mathcal{U}_{n+1}}$, its associated simplicial map Γ^* maps $z(\mathcal{U}_{n+1})$ to a 1-simplex that will have coefficient $2r_{n+1}$ since every 1-simplex of Nerve($\mathcal{U}_n$) has exactly two 1-simplex preimages. Since $\Gamma^*(z_1(\mathcal{U}_{n+1}))$ is homologous to $z_1(\mathcal{U}_n)$, and that $C_2(\text{Nerve}(\mathcal{U}_k)) \cong \mathbf{0}$ for every k, all the coefficients $r_n - 2r_{n+1}$ corresponding to the difference $\Gamma^*(z_1(\mathcal{U}_{n+1})) - z_1(\mathcal{U}_n)$ must equal zero due to the linear independence of the 1-simplicies, and since this cycle is composed of finitely many 1-simplicies, we have $r_n = 2r_{n+1}$. So, if we select $r_1 = r$, then we must have that $r_n = r/2^k$.

Obviously, not every ring satisfies this property of every r_n being nonzero and distinct, and $\mathbb{Z}$ is one of those rings because if each r_n is defined over $\mathbb{Z}$, then the only integer satisfying this recursive sequence is 0. Therefore, to obtain nontrivial 1-cycles, a field like $\mathbb{Q}$ suffices since it is infinite and not cyclic.

As a note, the use of the mapping operator Γ^* is an abuse of notation, and to actually map 1-cycles with their coefficients above would require the induced homomorphisms on the chain modules of Nerve($\mathcal{U}_n$) satisfying a commutativity property. But we avoided this in order to reduce pedantics and messy notation.

3.6. **The Čech Homology Groups of $\mathbb{S}$.** We will construct a cofinal family of finite open covers $\{\mathcal{U}_n\}$ of $\mathbb{S}$. The information for which many of these results are based are derived from [4], but many of the approaches we take are unique from [4].

Spaces for $\mathbb{S}$.

Definition/Theorem 3.18. *Let $\mathcal{U}_k$ be a finite open cover of $\mathbb{S}$ such that the following holds:*

(i) *We define $U_{-1} \in \mathcal{U}_{-1}$ to be the singleton 1-neighborhood open cover of $\mathbf{S}$ such that S_0 is covered.*

(ii) *For any $\mathbf{F}_{n_k} \ldots \mathbf{F}_{n_0}[S_0]$, the r-neighborhoods $U_{n_0,\ldots,n_k} = N_r(\mathbf{F}_{n_k} \ldots \mathbf{F}_{n_0}[S_0])$ make up the elements of $\mathcal{U}_n$ such that r is sufficiently small enough so that Nerve($\mathcal{U}_n$) contains only 1-simplicies given by $[U_{n_0,\ldots,n_{k-1},i}, U_{n_0,\ldots,n_{k-1},j}]$, where $i < j \in \{0,1,2\}$, along with*

$$[U_{n_0,\ldots,0,1,\ldots,1,1}, U_{n_0,\ldots,1,0,\ldots,0,0}], [U_{n_0,\ldots,0,2,\ldots,2,2}, U_{n_0,\ldots,2,0,\ldots,0,0}], [U_{n_0,\ldots,1,1,\ldots,1,2}, U_{n_0,\ldots,2,2,\ldots,2,1}],$$

and 0-simplicies given by the r-neighborhoods $[U_{n_0,\ldots,n_k}]$.

(iii) *For any $k \in \mathbb{Z}_{\geq 0}$ and $U_{n_0,\ldots,n_k} \in \mathcal{U}_k$, we define $U_{n_0,\ldots,n_k} = U_{n_0,\ldots,n_k,0,0,\ldots}$.*

Before defining the projection mappings, a comparison between Definitions 3.1, 3.5, and 3.18 is in order. By Definition 3.18(ii), the indices and the 1-simplicies match exactly what is given in Definition 3.5(ii), along with the 0-simplicies in Definition 3.5(i). However, the major difference between Definitions 3.5 and 3.18 is that for every Nerve($\mathcal{U}_k$) *there are no 2-simplicies* in Nerve($\mathcal{U}_k$). Moreover, in Definition 3.1(iii), where we only obtain 3 new points, *every* $U_{n_0,n_1,\ldots,n_k} \in \mathcal{U}_k$ is distinct. Namely, we have that

$U_{n_0,\ldots,0,0,\ldots0,1} \neq U_{n_0,\ldots,1,0,\ldots,0,0}$, $U_{n_0,\ldots,0,0,\ldots,0,2} \neq U_{n_0,\ldots,2,0,\ldots,0,0}$, and $U_{n_0,\ldots,1,1,\ldots,1,2} \neq U_{n_0,\ldots,2,1,\ldots,1,1}$. Meaning that if $A_{0,k+1}$ is the set of 0-simplicies of Nerve($\mathcal{U}_{k+1}$), then $|A_{0,k+1}| = 9|A_{0,k+1}|$ and not $3|A_{0,k+1}| - 3$ like in Theorem 3.7. Figure 4 makes this clearer to see.

Projection Mappings for $\mathbb{S}$.

Definition/Theorem 3.19. *Let $\mathcal{U}_k \xleftarrow{s_{\mathcal{U}_k,\mathcal{U}_{k+1}}} \mathcal{U}_{k+1}$ be the projection mapping defined such that*

$$s_{\mathcal{U}_{k+1},\mathcal{U}_k}(U_{n_0,n_1,\ldots,n_k,n_{k+1}}) = U_{n_0,n_1,\ldots,n_k}. \tag{11}$$

By Definition 3.19,

$$\begin{aligned} U_{n_0,n_1,\ldots,n_k,0} &\xmapsto{s_{\mathcal{U}_{k+1},\mathcal{U}_k}} U_{n_0,n_1,\ldots,n_k}, \\ U_{n_0,n_1,\ldots,n_k,1} &\xmapsto{s_{\mathcal{U}_{k+1},\mathcal{U}_k}} U_{n_0,n_1,\ldots,n_k}, \\ U_{n_0,n_1,\ldots,n_k,2} &\xmapsto{s_{\mathcal{U}_{k+1},\mathcal{U}_k}} U_{n_0,n_1,\ldots,n_k}. \end{aligned}$$

Also, because $\mathbf{F}_{n_{k+1}}\mathbf{F}_{n_k}\ldots\mathbf{F}_{n_0}\mathbf{F}_1[S_0] \subseteq \mathbf{F}_{n_k}\ldots\mathbf{F}_{n_0}[S_0]$, then

$$U_{n_0,n_1,\ldots,n_k,n_{k+1}} = N_{r_1}(\mathbf{F}_{n_{k+1}}\mathbf{F}_{n_k}\ldots\mathbf{F}_1\mathbf{F}_1[S_0]) \subseteq N_{r_2}(\mathbf{F}_{n_k}\ldots\mathbf{F}_1\mathbf{F}_1[S_0]) = U_{n_0,n_1,\ldots,n_k},$$

with $r_1 \leq r_2$. Therefore, $\mathcal{U}_{k+1}$ refines $\mathcal{U}_k$, so $s_{\mathcal{U}_k,\mathcal{U}_{k+1}}$ is a projection mapping.

Simplicial Homology Groups of Nerve($\mathcal{U}_k$). We are now in a position to begin the process of calculating the Čech homology groups of $\mathbb{S}$. First, we need the simplicial homology groups of each Nerve($\mathcal{U}_k$). Though we are working with Nerve($\mathcal{U}_k$), calculating its simplicial homology groups will yield the exact same dimensions of the simplicial homology groups of $\mathcal{A}_k$ when considering the subcomplex $\mathcal{A}_{k,0,1}$ of $\mathcal{A}_k$ containing only its 0 and 1-simplicies (so, we essentially can remove the 2-simplicies from $\mathcal{A}_k$ which increases the exponent of the base 3 in Theorem 3.8(b) by 1). This is because we are simply taking the midpoint 0-simplicies of the subcomplex and "stretching" them into 1-simplicies, as shown in Figure 5, and also removes the ambiguity of points from Definition 3.1. In other words, |Nerve($\mathcal{U}_k$)| and $|\mathcal{A}_{k,0,1}|$ are homotopy equivalent.

Proposition 3.20. *Let $k \in \mathbb{Z}_{\geq 0}$ and $\mathcal{U}_k$ be defined as in Definition 3.18. Then if |Nerve($\mathcal{U}_k$)| is a geometric realization of Nerve($\mathcal{U}_k$) in $\mathbb{R}^2$, and $\mathcal{A}_{k,0,1}$ is the subcomplex of $\mathcal{A}_k$ containing only its 0 and 1-simplicies with $|\mathcal{A}_k| = S_k$, then |Nerve($\mathcal{U}_k$)| is homotopy equivalent to $|\mathcal{A}_{k,0,1}|$. A picture to represent this homotopy equivalence is given in Figure 5.*

And because of the following theorem from [7], it follows that the simplicial homology groups $H_p(\mathcal{A}_{k,0,1}, \mathbb{Q}) \cong H_p(\mathcal{U}_k, \mathbb{Q})$.

Theorem 3.21. *Let K be a simplicial complex, $p \in \mathbb{Z}$, $H_p^{\Delta}(K)$ the simplicial homology group of K, and $H_p(|K|)$ the singular simplicial homology group of K. Then $H_p^{\Delta}(K) \cong H_p(|K|)$.*

The next two theorems from [7] and [2] makes the next theorem significantly easier to prove.

Theorem 3.22. *Let $\mathbb{X}$ be a topological space and K a simplicial complex. Let $H_p(\mathbb{X})$ be the pth singular simplicial homology group of $\mathbb{X}$ and $H_p^{\Delta}(K)$ the simplicial homology group of K.*

1. *(Path-Connectedness) If $\mathbb{X}$ is path-connected, then* $\dim(H_0(\mathbb{X})) = 1$.
2. *(Euler-Poincare Formula) If A_p is the set of all p-simplicies of K, and K is d-dimensional, then* $\sum_{i=0}^{d}(-1)^i|A_p| = \sum_{i=0}^{d}(-1)^i \dim(H_p^{\Delta}(K))$.

Theorem 3.23. *Let $(C(Nerve(\mathcal{U}_k)), \partial)$ be the simplicial chain complex of $Nerve(\mathcal{U}_k)$, and let $H_p(\mathcal{U}_k, \mathbb{Q})$, with $p \in \mathbb{Z}$, be the corresponding pth simplicial homology group as defined in Definition 3.15. Then*

(a) $H_0(\mathcal{U}_k, \mathbb{Q}) \cong \mathbb{Q}$,
(b) $H_1(\mathcal{U}_k, \mathbb{Q}) \cong \mathbb{Q}^{(3^{k+1}-1)/2}$,

and $H_p(\mathcal{U}_k, \mathbb{Q}) \cong \mathbf{0}$ for every other $p \in \mathbb{Z}$.

Proof. Because $C_p(\text{Nerve}(\mathcal{U}_k)) \cong \mathbf{0}$ for all $p \neq 0$ and $p \neq 1$, the fact that $\text{Nerve}(\mathcal{U}_k)$ has no 2-simplicies, and that $\partial_{0,k}$ is the zero homomorphism, then $H_p(\mathcal{U}_k) \cong \mathbf{0}$.

Denote $H_p^{\Delta}(\mathcal{U}_k, \mathbb{Q})$ to be the simplicial homology group of $\text{Nerve}(\mathcal{U}_k)$ and $H_p(\mathcal{U}_k, \mathbb{Q})$ the singular simplicial homology group of $|\text{Nerve}(\mathcal{U}_k)|$. Now let $[U_{n_0,n_1,\ldots,n_k}], [U_{m_0,m_1,\ldots,m_k}] \in \text{Nerve}(\mathcal{U}_k)$, and suppose without loss of generality that, in base 3, that $n_0 n_1 \ldots n_k < m_0 m_1 \ldots m_k$. Let m_j be the first nonzero digit such that for all $0 \leq \ell \leq j$, $m_\ell = 0$. Then $n_j \leq m_j$. It follows that for some minimal $j \leq q \leq k$, $n_q < m_q$. So, we have that $n_0 = m_0, n_1 = m_1, \ldots, n_{q-1} = m_{q-1}, n_q < m_q$.

Because there are an incredibly large number of cases, we will outline the general process for proving there is a simplicial path from $[U_{n_0,n_1,\ldots,n_k}]$ and $[U_{m_0,m_1,\ldots,m_k}]$. By Definition 3.18, there is 1-simplex $[U_{m_0,m_1,\ldots,m_k}, U_{m_0,m_1,\ldots,0}]$. From here, there are three cases: $m_{k-1} = 0, 1$, or 2. The 1 and 2 cases result in 1-simplicies $[U_{m_0,m_1,\ldots,0,2}, U_{m_0,m_1,\ldots,2,0}]$ or $[U_{m_0,m_1,\ldots,0,1}, U_{m_0,m_1,\ldots,1,0}]$. If $m_{k-1} = 0$, this individually would yield three cases corresponding to $m_{k-2} = 0, 1$ or 2, where 1 and 2 yields $[U_{m_0,m_1,\ldots,0,1,1}, U_{m_0,m_1,\ldots,1,0,0}]$ or $[U_{m_0,m_1,\ldots,0,2,2}, U_{m_0,m_1,\ldots,2,0,0}]$. In both cases, there are 1-simplicies $[U_{m_0,m_1,\ldots,0,1,0}, U_{m_0,m_1,\ldots,0,1,1}]$ or $[U_{m_0,m_1,\ldots,0,2,0}, U_{m_0,m_1,\ldots,0,2,2}]$, for which we can repeat the process of m_{k-1} for the 1 and 2 cases and eventually obtain 0 for m_{k-1}. But if $m_{k-2} = 0$, then like in the $m_{k-1} = 0$ case, we'll similarly obtain three more subcases $m_{k-3} = 0, 1$ or 2. Continuing in this manner, then for any $q + 1 \leq r \leq k - 1$, the cases $m_r = 0$ yield the 0-simplex $[U_{m_0,m_1,\ldots,m_{r-1},0,\ldots,0}]$, for which we have 1-simplex $[U_{m_0,m_1,\ldots,m_{r-2},0,2,\ldots,2}]$, $[U_{m_0,m_1,\ldots,m_{r-2},0,1,\ldots,1}, U_{m_0,m_1,\ldots,m_{r-1},0,\ldots,0}]$, or three more subcases with $m_{r-2} = 0, 1$ or 2. This is a finite process, so it must terminate with 1-simplex

$$[U_{m_0,m_1,\ldots,m_q,0,\ldots,0}, U_{m_0,m_1,\ldots,m_q,0,\ldots,1}] \text{ or } [U_{m_0,m_1,\ldots,m_q,0,\ldots,0}, U_{m_0,m_1,\ldots,m_q,0,\ldots,2}].$$

Therefore, this leads to the next three subcases: $m_q = 0, 1$, or 2. But because $0 \leq n_q < m_q$, we have $n_q = 0$ and $m_q = 1$ or $n_q = 0$ or 1 and $m_q = 2$. And due to the minimality of q, there are 1-simplicies $[U_{n_0,n_1,\ldots,n_{q-1},n_q,1,\ldots,1}, U_{n_0,n_1,\ldots,n_{q-1},1,0,\ldots,0}]$ or $[U_{n_0,n_1,\ldots,n_{q-1},n_q,2,\ldots,2}, U_{n_0,n_1,\ldots,n_{q-1},2,0,\ldots,0}]$. If $n_{q+1} = \ldots = n_k = 1$ or $n_{q+1} = \ldots = n_k = 2$, we are done. Otherwise, for some $q + 1 \leq t_1 \leq k$, we have that $n_{t_1} = 0$ or 2 in the former case or $n_{t_1} = 0$ or 1 in the latter case. Let's suppose t_1 is the minimal integer such that $n_{q+1} = \ldots = n_{t_1-1} = 1$ and $n_{t_1} \neq 1$ or $n_{q+1} = \ldots = n_{t_1-1} = 2$ and $n_{t_1} \neq 2$. Then $[U_{n_0,n_1,\ldots n_k}] = [U_{n_0,n_1,\ldots,n_q,1,1,\ldots,\neq 1,\ldots,n_k}]$ or $[U_{n_0,n_1,\ldots n_k}] = [U_{n_0,n_1,\ldots,n_q,2,2,\ldots,\neq 2,\ldots,n_k}]$.

Let's suppose in one case that $m_q = 1$, so that we work with the 1-simplex $[U_{n_0,n_1,\ldots,n_{q-1},n_q,1,\ldots,1}, U_{n_0,n_1,\ldots,n_{q-1},1,0,\ldots,0}]$. Since $[U_{n_0,n_1,\ldots n_k}] = [U_{n_0,n_1,\ldots,n_q,1,1,\ldots,\neq 1,\ldots,n_k}]$, we can follow a similar procedure as in the third paragraph above in order to eventually obtain the 1-simplex $[U_{n_0,n_1,\ldots,1,1,\ldots,\neq 1,n_{t_1+1},0,\ldots,0}, U_{n_0,n_1,\ldots,1,1,\ldots,\neq 1,n_{t_1+1},0,\ldots,1}]$ or $[U_{n_0,n_1,\ldots,1,1,\ldots,\neq 1,n_{t_1+1},0,\ldots,0}, U_{n_0,n_1,\ldots,1,1,\ldots,\neq 1,n_{t_1+1},0,\ldots,2}]$. Then we obtain cases $n_{t_1+1} = 0, 1$, or 2, and if $n_{t_1+1} = 0$, the process in Paragraph 3 suffices to obtain 1-simplex $[U_{n_0,n_1,\ldots,1,1,\ldots,\neq 1, n_{t_1+1},n_{t_1+2},0,\ldots,0}, U_{n_0,n_1,\ldots,1,1,\ldots,\neq 1,n_{t_1+1},n_{t_1+2},0,\ldots,1}]$ or $[U_{n_0,n_1,\ldots,1,1,\ldots,\neq 1,n_{t_1+1},n_{t_1+2},0,\ldots,0}, U_{n_0,n_1,\ldots,1,1,\ldots,\neq 1,n_{t_1+1}, n_{t_1+2},0,\ldots,2}]$ and similarly for the $n_{t_1+1} = 1$ and 2 cases.

Because the above process is finite, we must eventually obtain 1-simplex

$$[U_{n_0,n_1,\ldots,1,1,\ldots,\neq 1,n_{t_1+1},n_{t_1+2},\ldots,n_{k-1},0}, U_{n_0,n_1,\ldots,1,1,\ldots,\neq 1,n_{t_1+1},n_{t_1+2},\ldots,1}] \text{ or}$$
$$[U_{n_0,n_1,\ldots,1,1,\ldots,\neq 1,n_{t_1+1},n_{t_1+2},\ldots,n_{k-1},0}, U_{n_0,n_1,\ldots,1,1,\ldots,\neq 1,n_{t_1+1},n_{t_1+2},\ldots,n_{k-1},2}]$$

Since $n_k = 0, 1$ or 2, this completes the outline of the process of obtaining a simplicial path between $[U_{n_0,n_1,\ldots,n_k}]$ and $[U_{m_0,m_1,\ldots,m_k}]$. Hence, $|\mathrm{Nerve}(\mathcal{U}_k)|$ is path-connected.

Because $|\mathrm{Nerve}(\mathcal{U}_k)|$ is path-connected, then by Theorem 3.21, $H_0(\mathcal{U}_k, \mathbb{Q}) \cong \mathbb{Q}$. By Theorem 3.22(1), $H_0(\mathcal{U}_k, \mathbb{Q}) \cong H_0^\Delta(\mathcal{U}_k, \mathbb{Q})$, so $H_0^\Delta(\mathcal{U}_k, \mathbb{Q}) \cong \mathbb{Q}$. Therefore, by the Euler-Poincare Formula,

$$\dim(H_1^\Delta(\mathcal{U}_k, \mathbb{Q})) = |A_{k,0}| - |A_{k,1}| - 1,$$

where $A_{k,0}$ and $A_{k,1}$ are the sets of 0 and 1-simplicies in $\mathrm{Nerve}(\mathcal{U}_k)$, respectively. To compute $|A_{k,0}|$, we can follow a similar line of reasoning to the proof of Theorem 3.7, with the exception of obtaining nine 0-simplicies induced from $[U_{n_0,\ldots,0}], [U_{n_0,\ldots,1}], [U_{n_0,\ldots,2}] \in \mathrm{Nerve}(\mathcal{U}_{k-1})$ instead of only three. Since the base step $k = 0$ yields $3 = 3^1$ 0-simplicies, we can obtain $|A_{k,0}| = 3^{k+1}$. Likewise, computing $|A_{k,1}|$ can be performed via induction by recognizing that for any $\mathrm{Nerve}(\mathcal{U}_k)$, we obtain

$$|A_{k,1}| = \sum_{i=0}^{k} 3^i = \frac{1 - 3^{k+1}}{1 - 3} = \frac{1}{2}(3^{k+1} - 1).$$

Therefore,

$$\begin{aligned}\dim(H_1^\Delta(\mathcal{U}_k, \mathbb{Q})) &= 3^{k+1} - \frac{1}{2}(3^{k+1} - 1) - 1 \\ &= \frac{1}{2}(3^{k+1} - 1).\end{aligned}$$

Hence, $H_1^\Delta(\mathcal{U}_k, \mathbb{Q}) \cong \mathbb{Q}^{(3^{k+1}-1)/2}$. □

Now that the simplicial homology groups for the nerves of the open covers of $\mathbb{S}$ have been obtained, we are ready to compute the Čech homology groups of $\mathbb{S}$.

Theorem 3.24. *Let $\mathbb{S}$ be the Sierpinski Gasket. Then*

(a) $\dim(\check{H}_0(\mathbb{S})) = 1$, (b) $\dim(\check{H}_1(\mathbb{S})) = \infty$,

with $\dim(\check{H}_p(\mathbb{S})) = 0$ *for every other* $p \in \mathbb{Z}$.

Proof. The simplicial homology groups $H_p(\mathcal{U}_k, \mathbb{Q})$, the projection mappings $s_{\mathcal{U}_k,\mathcal{U}_{k+1}} : \mathcal{U}_{k+1} \to \mathcal{U}_k$ and the homomorphisms $H_p(\mathcal{U}_k, \mathbb{Q}) \xleftarrow{s_{\mathcal{U}_{k+1},\mathcal{U}_k}{}^*} H_p(\mathcal{U}_{k+1}, \mathbb{Q})$ induced by the simplicial mappings $\Gamma_{\mathcal{U}_k,\mathcal{U}_{k+1}} : |\mathrm{Nerve}(\mathcal{U}_{k+1})| \to |\mathrm{Nerve}(\mathcal{U}_k)|$ induced by $s_{\mathcal{U}_k,\mathcal{U}_{k+1}}$, and the ordering given by the refinement $\mathcal{U}_{k+1} > \mathcal{U}_k$ yields the inverse system $(H_p(\mathcal{U}_k, \mathbb{Q}), s_{\mathcal{U}_k,\mathcal{U}_{k+1}*}, \mathscr{P}_n(\mathbb{X}))$, with $\mathscr{P}_n(\mathbb{X}) = \{\mathcal{U}_k : k \geq -1\}$, by Definition 3.15.

For $p \neq 0$ and $p \neq 1$, because $H_p(\mathcal{U}_k, \mathbb{Q}) \cong \mathbf{0}$ by Theorem 3.23, it follows that $\check{H}_p(\mathcal{U}_k, \mathbb{Q}) = \{(0, 0, \ldots, 0, \ldots)\} = \mathbf{0}$.

For $p = 1$, we will first prove that each $H_p(\mathcal{U}_k, \mathbb{Q}) \xleftarrow{s_{\mathcal{U}_{k+1},\mathcal{U}_k}{}^*} H_p(\mathcal{U}_{k+1}, \mathbb{Q})$ is surjective. First we note that for any integer $k \geq 0$, it can be proven via induction that $\mathrm{Nerve}(\mathcal{U}_k)$ contains (all containing 1-cycles with coefficient 1)

3^k 1-cycles containing chain with 3 1-simplicies,

3^{k-1} 1-cycles containing chain with $3 \cdot 2$ 1-simplicies,

3^{k-2} 1-cycles containing chain with $3 \cdot 2^2$ 1-simplicies,

$\vdots$

1 1-cycle containing chain with $3 \cdot 2^k$ 1-simplicies,

and because these are all distinct and that $H_1(\mathcal{U}_k, \mathbb{Q}) \cong \mathrm{Ker}(\partial_1)$, the cosets containing exactly one of each of these cycles are linearly independent in $H_p(\mathcal{U}_k, \mathbb{Q})$. Also, $3^k + 3^{k-1} + \ldots + 1 = \dfrac{1}{2}(3^{k+1} - 1) = \dim(H_1(\mathcal{U}_k, \mathbb{Q}))$ by Theorem 3.23, and since $\mathbb{Q}$ is a field, it follows that this set of 1-cycles is a basis for $H_p(\mathcal{U}_k, \mathbb{Q})$. Likewise, by replacing k with $k+1$ in the list above, we obtain a basis of $\dfrac{1}{2}(3^{k+2} - 1)$ elements for $H_p(\mathcal{U}_{k+1}, \mathbb{Q})$.

Let $0 \le j \le k$ and x be a 1-cycle containing $3 \cdot 2^{k-j}$ 1-simplicies. If $j = k$, then x is simply any 1-cycle $[U_{n_0,\ldots,0}, U_{n_0,\ldots,1}] + [U_{n_0,\ldots,0}, U_{n_0,\ldots,2}] + [U_{n_0,\ldots,1}, U_{n_0,\ldots,2}]$. Then the 1-cycle, and only the 1-cycle,

$$[[U_{n_0,n_1,\ldots,n_{k-1},0,1}, U_{n_0,n_1,\ldots,n_{k-1},1,0}] + [U_{n_0,n_1,\ldots,n_{k-1},0,2}, U_{n_0,n_1,\ldots,n_{k-1},2,0}] + [U_{n_0,n_1,\ldots,n_{k-1},1,2}, U_{n_0,n_1,\ldots,n_{k-1},2,1}]]$$

$$\xmapsto{s_{\mathcal{U}_k,\mathcal{U}_{k+1}*}}$$

$$[[U_{n_0,\ldots,0}, U_{n_0,\ldots,1}] + [U_{n_0,\ldots,0}, U_{n_0,\ldots,2}] + [U_{n_0,\ldots,1}, U_{n_0,\ldots,2}]].$$

If $0 \le j < k$, then

$$\begin{aligned} x =& [U_{n_0,n_1,\ldots,0,1,\ldots,1}, U_{n_0,n_1,\ldots,1,0,\ldots,0}] + [U_{n_0,n_1,\ldots,0,2,\ldots,2}, U_{n_0,n_1,\ldots,2,0,\ldots,0}] + [U_{n_0,n_1,\ldots,1,1,\ldots,2}, U_{n_0,n_1,\ldots,2,1,\ldots,1}] \\ &+ [U_{n_0,n_1,\ldots,1,0,\ldots,0,0}, U_{n_0,n_1,\ldots,1,0,\ldots,0,2}] + \ldots + [U_{n_0,n_1,\ldots,1,2,\ldots,2,0}, U_{n_0,n_1,\ldots,1,2,\ldots,2,2}] \\ &+ [U_{n_0,n_1,\ldots,0,1,\ldots,1,1}, U_{n_0,n_1,\ldots,0,1,\ldots,1,2}] + \ldots + [U_{n_0,n_1,\ldots,0,2,\ldots,2,1}, U_{n_0,n_1,\ldots,0,2,\ldots,2,2}] \\ &+ [U_{n_0,n_1,\ldots,2,0,\ldots,0,0}, U_{n_0,n_1,\ldots,2,0,\ldots,0,1}] + \ldots + [U_{n_0,n_1,\ldots,2,1,\ldots,1,0}, U_{n_0,n_1,\ldots,2,1,\ldots,1,1}]. \end{aligned}$$

For every one of these 1-simplicies, we have that (using the induced homomorphism induced from the simplicial mapping of $s_{\mathcal{U}_k,\mathcal{U}_{k+1}}$)

$$\begin{aligned} &[U_{n_0,n_1,\ldots,0,1,\ldots,1,1}, U_{n_0,n_1,\ldots,1,0,\ldots,0,0}] \mapsto [U_{n_0,n_1,\ldots,0,1,\ldots,1}, U_{n_0,n_1,\ldots,1,0,\ldots,0}], \\ &[U_{n_0,n_1,\ldots,0,2,\ldots,2,2}, U_{n_0,n_1,\ldots,2,0,\ldots,0,0}] \mapsto [U_{n_0,n_1,\ldots,0,2,\ldots,2}, U_{n_0,n_1,\ldots,2,0,\ldots,0}], \\ &[U_{n_0,n_1,\ldots,1,2,\ldots,2,2}, U_{n_0,n_1,\ldots,2,1,\ldots,1,1}] \mapsto [U_{n_0,n_1,\ldots,1,2,\ldots,2}, U_{n_0,n_1,\ldots,2,1,\ldots,1}], \\ &[U_{n_0,n_1,\ldots,1,0,\ldots,0,0,2}, U_{n_0,n_1,\ldots,1,0,\ldots,0,2,0}] \mapsto [U_{n_0,n_1,\ldots,1,0,\ldots,0,0}, U_{n_0,n_1,\ldots,1,0,\ldots,0,2}], \\ &\vdots \\ &[U_{n_0,n_1,\ldots,1,2,\ldots,2,0,2}, U_{n_0,n_1,\ldots,1,2,\ldots,2,2,0}] \mapsto [U_{n_0,n_1,\ldots,1,2,\ldots,2,0}, U_{n_0,n_1,\ldots,1,2,\ldots,2,2}], \\ &[U_{n_0,n_1,\ldots,0,1,\ldots,1,1,2}, U_{n_0,n_1,\ldots,0,1,\ldots,1,2,1}] \mapsto [U_{n_0,n_1,\ldots,0,1,\ldots,1,1}, U_{n_0,n_1,\ldots,0,1,\ldots,1,2}], \\ &\vdots \\ &[U_{n_0,n_1,\ldots,0,2,\ldots,2,1,2}, U_{n_0,n_1,\ldots,0,2,\ldots,2,2,1}] \mapsto [U_{n_0,n_1,\ldots,0,2,\ldots,2,1}, U_{n_0,n_1,\ldots,0,2,\ldots,2,2}], \\ &[U_{n_0,n_1,\ldots,2,0,\ldots,0,0,1}, U_{n_0,n_1,\ldots,2,0,\ldots,0,1,0}] \mapsto [U_{n_0,n_1,\ldots,2,0,\ldots,0,0}, U_{n_0,n_1,\ldots,2,0,\ldots,0,1}], \end{aligned}$$

$$\vdots$$

$$[U_{n_0,n_1,\ldots,2,1,\ldots,1,0,1}, U_{n_0,n_1,\ldots,2,1,\ldots,1,1,0}] \mapsto [U_{n_0,n_1,\ldots,2,1,\ldots,1,0}, U_{n_0,n_1,\ldots,2,1,\ldots,1,1}].$$

Therefore, adding the 1-simplicies on the left and mapping them through $s_{\mathcal{U}_k,\mathcal{U}_{k+1}}$ yields x. And again, because 1-cycles in $H_1(\mathcal{U}_k, \mathbb{Q})$ have unique preimages, the sum of the 1-simplicies on the left is the *only* preimage of x. So, all $\frac{1}{2}(3^{k+1}-1)$ 1-cycles in $H_1(\mathcal{U}_k, \mathbb{Q})$ have unique preimages, and because $s_{\mathcal{U}_k,\mathcal{U}_{k+1}*}$ is a $\mathbb{Q}$-module homomorphism, it follows that $s_{\mathcal{U}_k,\mathcal{U}_{k+1}*}$ is surjective.

We will now prove that $\check{H}_1(\mathbb{S})$ is generated by infinitely many elements. Consider any integer $k \geq -1$ and any $([z_{-1}], [z_0], [z_1], \ldots, [z_k], \ldots) \in \check{H}_1(\mathbb{S})$, where $[z_k] = 0$ or $[z_k]$ is a basis element for $H_1(\mathcal{U}_k, \mathbb{Q})$. If $[z_k] = 0$, then because every nonzero element of $H_1(\mathcal{U}_k, \mathbb{Q})$ has a unique nonzero preimage, then there are exactly 3^{k+1} elements (the 1-cycles that map to a single 0-simplex) that map to $[z_k]$. Hence, this induces 3^{k+1} linearly independent elements in $\check{H}_1(\mathbb{S})$ since each of these 1-cycles that map to 0 are not linear combinations of one another. If $[z_k]$ is a basis element, then from the previous paragraph, all the possible basis elements by themselves yield $\frac{1}{2}(3^{k+1}-1)$ linearly independent elements in $\check{H}_1(\mathbb{S})$. Hence, $[z_k]$ yields $\frac{1}{2}(3^{k+2}-1)$ linearly independent elements in $\check{H}_1(\mathbb{S})$. Because distinct 1-cycles in each $H_1(\mathcal{U}_k, \mathbb{Q})$ are linearly independent, combining linearly independent sets obtained for every $k \geq -1$ yields linearly independent sets. Hence, since every $s_{\mathcal{U}_k,\mathcal{U}_{k+1}*}$ is surjective and that $\frac{1}{2}(3^{k+2}-1) \to \infty$ as $k \to \infty$ it follows that $\check{H}_1(\mathbb{S})$ is an infinitely generated $\mathbb{Q}$-module. Hence, $\dim(\check{H}_1(\mathbb{S})) = \infty$.

Proving (a) is not difficult. Just like in the proof for (b), the mappings $H_0(\mathcal{U}_k) \xleftarrow{s_{\mathcal{U}_k,\mathcal{U}_{k+1}}*} H_0(\mathcal{U}_{k+1})$ are surjective, but unlike in (b), every $H_0(\mathcal{U}_k, \mathbb{Q})$ is 1-dimensional. So, we can select any 0-simplex x_k in $\text{Nerve}(\mathcal{U}_k)$, and the coset $[x_k]$ will generate $H_0(\mathcal{U}_k, \mathbb{Q})$. Due to the surjectivity of each $s_{\mathcal{U}_k,\mathcal{U}_{k+1}}*$, we can obtain the sequence $([x_{-1}], [x_0], [x_1], \ldots, [x_k], \ldots) \in \check{H}_0(\mathbb{S})$. Due to the 1-dimensionality of each $H_0(\mathcal{U}_k, \mathbb{Q})$, the fact that each $s_{\mathcal{U}_k,\mathcal{U}_{k+1}}*$ is a $\mathbb{Q}$-module homomorphism, then $\text{span}\{([x_{-1}], [x_0], [x_1], \ldots, [x_k], \ldots)\} = \check{H}_0(\mathbb{S})$. Hence, $\dim(\check{H}_0(\mathbb{S})) = 1$. $\square$

3.7. **The Discontinuity of Singular Simplicial Homology.** The Cech homology groups of $\mathbb{S}$ and the Jordan curve theorem applied to $\mathbb{S}$ tell two very different stories that are extremely counterintuitive and justifies the title of this paper. The following will be based primarily on a combination of known results in algebraic topology combined with [8], where Sierpinski proves that the Sierpinski Gasket is a Jordan curve, or a simple closed curve (i.e. a subset of $\mathbb{R}^2$ homeomorphic to S^1)

By Theorem 3.23, if $\beta_1(\mathcal{U}_k)$ denotes the 1st Betti number of the singular simplicial homology group $H_1(\mathcal{U}_k, \mathbb{Q})$ of $|\text{Nerve}(\mathcal{U}_k)|$, which is isomorphic to the simplicial homology group $H_1^{\Delta}(\mathcal{U}_k, \mathbb{Q})$, then $\beta_1(\mathcal{U}_k) = \frac{1}{2}(3^{k+1}-1)$. Therefore,

$$\lim_{k\to\infty} \beta_1(\mathcal{U}_k) = \infty = \dim(\check{H}_1(\mathcal{U}_k, \mathbb{Q})). \tag{12}$$

Therefore, Čech homology appears to agree with the *limit* of the sequence of 1st Betti numbers. But what about $H_1(\mathbb{S})$ in singular simplicial homology? To understand how to proceed, we need the Jordan Curve Theorem.

Theorem 3.25. (Jordan Curve Theorem) *If $\mathbb{X} \subseteq \mathbb{R}^2$ is homeomorphic to S^1, then its complement $\mathbb{R}^2 \backslash \mathbb{X}$ has two connected components, one bounded, the other unbounded. Any neighborhood of any point on $\mathbb{X}$ meets both of these components.* [3]

Proof. This can be found in [3]. $\square$

The proof that the Sierpinski Gasket is a Jordan curve is omitted since the article is written in French, but essentially what the theorem tells us is that the unit circle S^1 can, surprisingly enough, be transformed homeomorphically into $\mathbb{S}$.

Theorem 3.26. The Sierpinski Gasket $\mathbb{S}$ is a Jordan curve.

So, by the Jordan Curve Theorem, the Sierpinski Gasket has a bounded, connect component. Which completely defies the intuition from the results of the computations of $\check{H}_p(\mathbb{S})$ which suggests that the opposite is true. In fact,

Theorem 3.27. *Let $H_1(\mathbb{S})$ be the singular simplicial homology group of $\mathbb{S}$. Because $H_1(S^1) \cong \mathbb{Z}$ by the Mayer-Vietoris theorem of singular simplicial homology, and that $\mathbb{S}$ is homeomorphic to S^1 by the Jordan Curve Theorem, then $H_1(\mathbb{S}) \cong H_1(S^1) \cong \mathbb{Z}$.*

Therefore, we can conclude that the 1st Cech homology groups and the 1st singular simplicial homology groups of compact Hausdorff spaces do *not* agree. Meaning, after more than 17 pages of discussion, we can finally declare the following theorem:

Theorem 3.28. (Discontinuity of Singular Simplicial Homology) *If $\beta_1 : \mathscr{X} \to \mathbb{R}$, where $\mathscr{X}$ is the set of all compact nonempty spaces under the Hausdorff metric, is the 1st Betti number of any $\mathbb{X} \in \mathscr{X}$, then β_1 is discontinuous.*

Proof. By Theorem 2.12, $\lim_{k\to\infty} S_k = \mathbb{S}$, but by Theorem 3.23, $\lim_{k\to\infty} \beta_0(S_k) = \infty$. And by Theorem 3.27, $\beta_0(\mathbb{S}) = 1$. ◻

When considering the Čech homogy, limits do not "pass through" the homology functor operator in the sense that

$$\text{(Čech Homology)} \qquad \varprojlim (H_1(\mathcal{U}), p_{\mathcal{U}\mathcal{V}*}, \mathscr{P}(\mathbb{X})) \ncong H_1\left(\lim_{n\to\infty} \mathbb{X}_n\right). \qquad \text{(Singular Simplicial Homology)}$$

So, not all homology groups paint the same picture for any given topological space.

(i) $H_1(\mathbb{S})$ describes the connected nature of the interior region of $\mathbb{S}$ and so is not a suitable candidate for describing the infinitely many holes in $\mathbb{S}$ and thus is not a suitable limit for the simplicial homology groups of each S_k.
(ii) $\check{H}_1(\mathbb{S})$ perfectly describes the infinitely many holes in $\mathbb{S}$ and is perfect as the limit of the simplicial homology groups of each S_k but fails to address the connected nature of the interior region of $\mathbb{S}$.

Figures

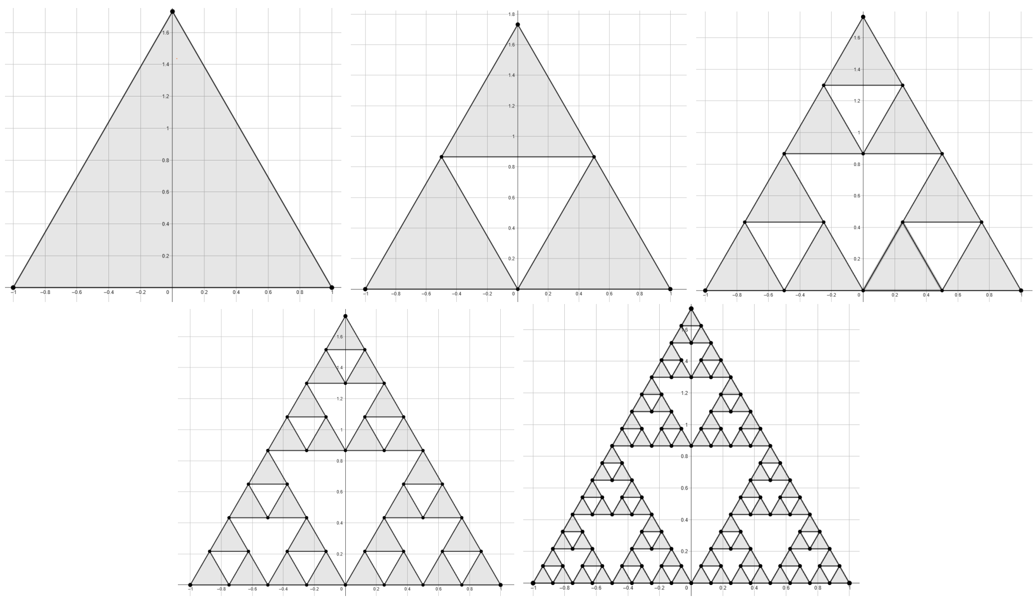

Figure 1. Each of these represent, from left to right, the sets S_0 and $S_k = \mathbf{F}_0[S_{k-1}] \cup \mathbf{F}_1[S_{k-1}] \cup \mathbf{F}_2[S_{k-1}]$ for $k = 1, 2, 3, 4$. Made using GeoGebra.

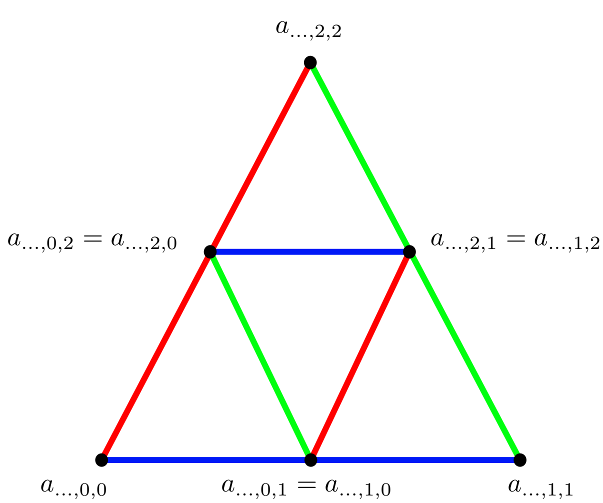

Figure 2. The blue 1-simplicies are the simplicies in $A_{j+1,0,1}$, the red 1-simplicies are the simplicies in $A_{j+1,0,2}$, and the green 1-simplicies are the simplicies in $A_{j+1,1,2}$.

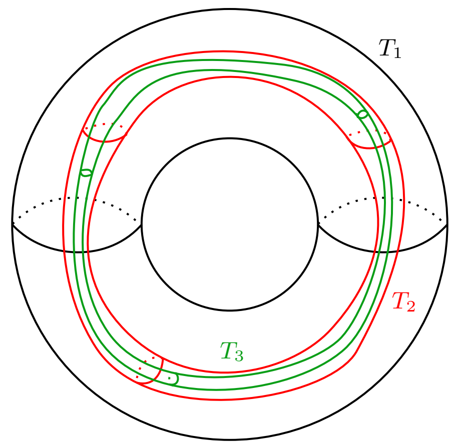

FIGURE 3. A picture of the construction of each T_i from the 2-torus, embedded in the interior of T_{i-1}.

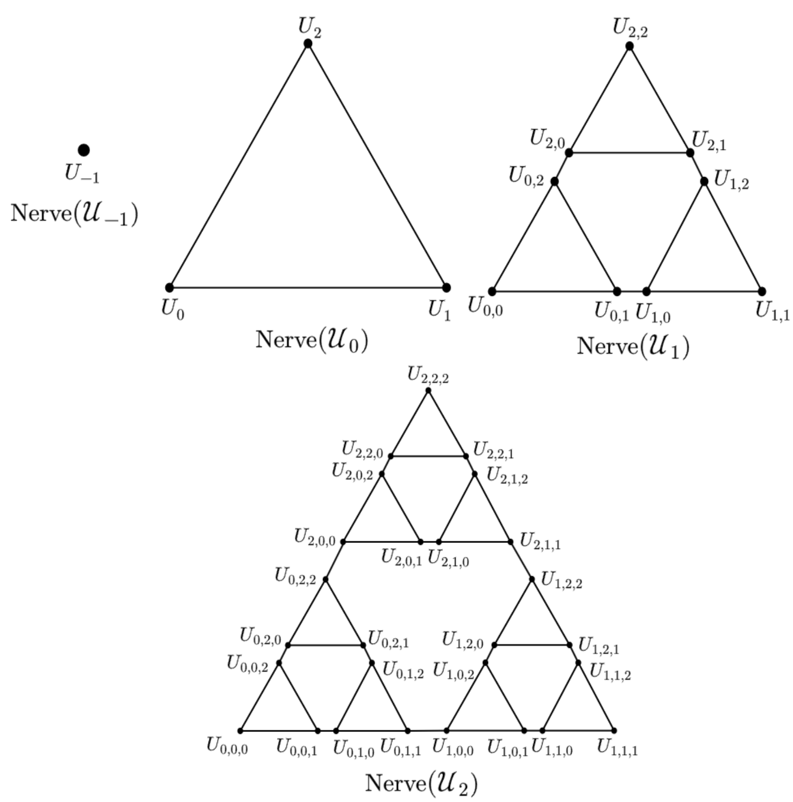

FIGURE 4. Each of these represent the nerves of $\mathcal{U}_{-1}, \mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2$ obtained when taking open covers of $\mathbb{S}$ according to Definition 3.18. Notice the $\mathbb{S}$-like structure of Nerve($\mathcal{U}_1$) and Nerve($\mathcal{U}_2$) with the exception that the midpoints of each triangle are replaced with two distinct points.

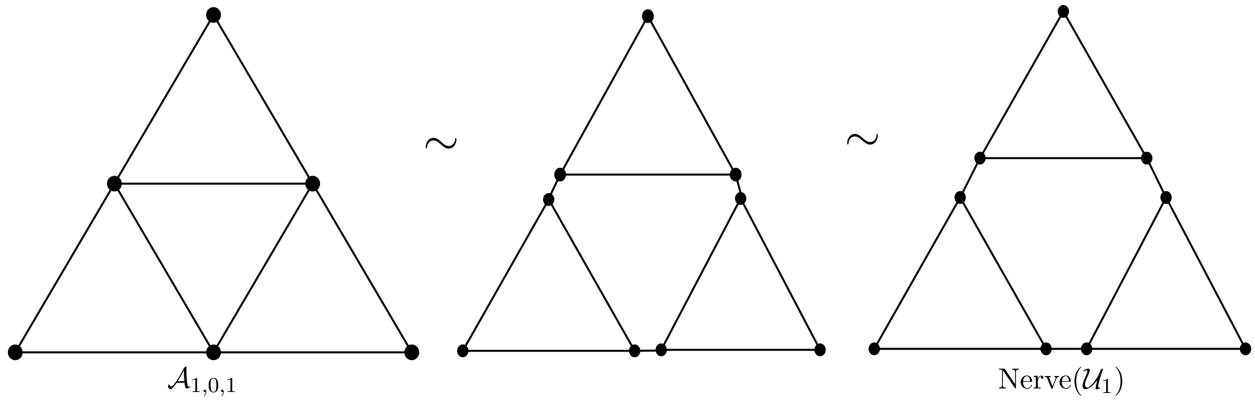

FIGURE 5. A pictorial representation of the homotopy equivalence between $|\mathcal{A}_{1,0,1}|$ and $|\text{Nerve}(\mathcal{U}_1)|$.

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