# MATHEMATICAL MORPHOLOGY THEORY AND THE CONNECTEDNESS OF $\mathbb{S}$

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Let $\mathbf{x}_0 = (0,0)$, $\mathbf{x}_1 = (1,0)$, and $\mathbf{x}_2 = (0,1)$. Define $(\mathbf{F}_i : \mathbb{R}^2 \to \mathbb{R}^2)_{i=0}^2$ by $\mathbf{F}_i(x,y) = \mathbf{x}_i + \dfrac{1}{2}((x,y) - \mathbf{x}_i)$. Lastly, define $S_0 = [0,1] \times [0,1]$ and $S_{k+1} = \mathbf{F}_0[S_k] \cup \mathbf{F}_1[S_k] \cup \mathbf{F}_2[S_k]$ for all $k \geq 0$. The unique attractor $\mathbb{S}_{\text{right}} := \lim_{k\to\infty} S_k$ we will call the right-angled Sierpinski Gasket.

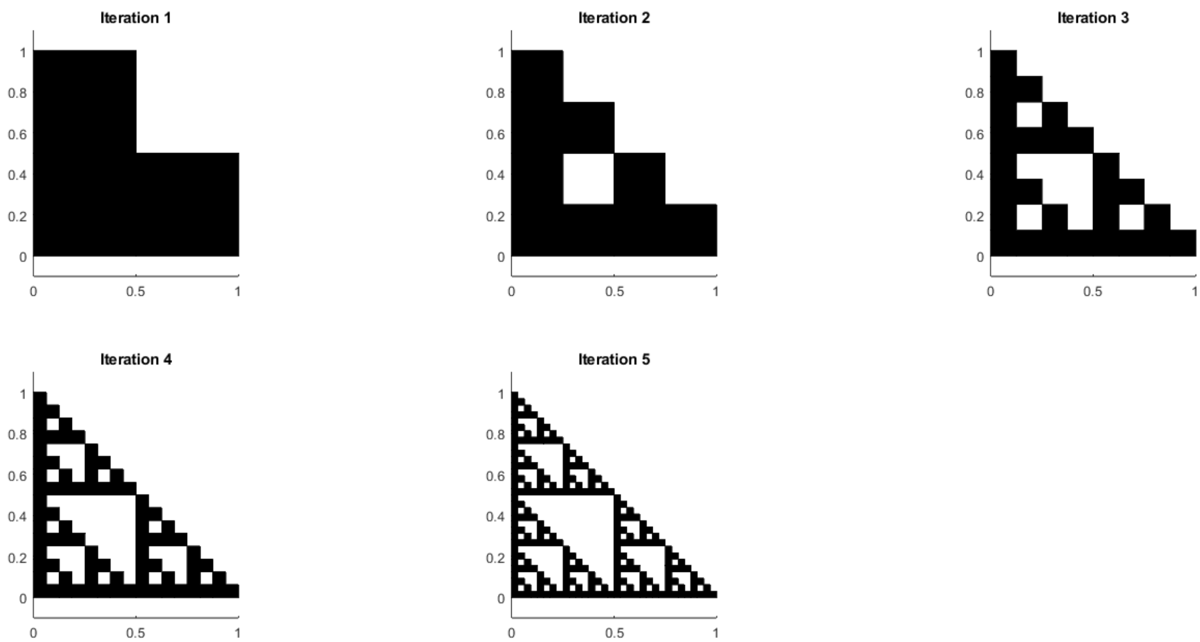


FIGURE 1. First five iterations of $(\mathbf{F}_i : \mathbb{R}^2 \to \mathbb{R}^2)_{i=0}^2$ on $[0,1]^2$. Generated using MATLAB.

It makes intuitive sense that Sierpinski Gasket variants are connected, yet actually proving this is not a trivial task. This paper focuses on using a discretization approach to prove that $\mathbb{S}_{\text{right}}$ is connected.

If $O = (0,0,\ldots,0) \in \mathbb{R}^n$, define $\Theta_k^{\{O\}} : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ by

$$\Theta_k^{\{O\}}(X) = \bigcup_{\substack{C_{k_1,\ldots,k_n} \in \mathcal{G}_{n,k}, \\ C_{k_1,\ldots,k_n} \cap X \neq \emptyset}} C_{k_1,k_2,\ldots,k_n} \tag{1}$$

where $\mathcal{G}_{n,k} = \{\prod_{i=1}^n [k_i/2^k, (k_i+1)/2^k] : k_i \in \mathbb{Z}\}$.

Before proving $\mathbb{S}_{\text{right}}$ is connected using the approach of [1], we need some background. First we need a lemma that states the descending nature of $(X_k)_{k=1}^\infty$.

**Lemma 0.1.** Define $Y_m = \Theta_m^{\{O\}}(X)$, where $X$ is a compact subset of $\mathbb{R}^n$. If $x \in X$ and $x \in C_{k_1,k_2,\ldots,k_n} \subseteq Y_m$, there exists $C_{m_1,m_2,\ldots,m_n} \subseteq Y_{m+1}$ such that $x \in C_{m_1,m_2,\ldots,m_n}$

*Proof.* Suppose $x \in X$ and $x \in C_{k_1,k_2,\ldots,k_n} \subseteq Y_m$. Then since $x = (x_1, x_2, \ldots, x_n)$ for some $x_1, x_2, \ldots, x_n \in \mathbb{R}$, it follows that $k_i/2^m \leq x_i \leq (k_i+1)/2^m$. Then $2k_i/2^{m+1} \leq x_i \leq (2k_i+2)/2^{m+1}$, so either $2k_i/2^{m+1} \leq$

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$x_i \leq (2k_i+1)/2^{m+1}$ or $(2k_i+1)/2^{m+1} < x_i \leq (2k_i+2)/2^{m+1}$. Therefore, there are a total of $2^n$ possible sets in $\mathcal{G}_{n,(m+1)}$ that $x$ is contained in, which proves the lemma. $\square$

**Lemma 0.2.** Define $Y_m = \Theta_m^{\{O\}}(X)$, where $X$ is a compact subset of $\mathbb{R}^n$. Then $(Y_m)_{m=1}^{\infty}$ is decreasing and $X = \lim_{m\to\infty} Y_m = \bigcap Y_m$.

*Proof.* Let $x \in \Theta_{m+1}^{\{O\}}(X)$. Then for some $C_{k_1,\ldots,k_n} \in \mathcal{G}_{n,m+1}$ with $C_{k_1,\ldots,k_n} \cap X \neq \emptyset$, $x \in C_{k_1,\ldots,k_n}$. Hence, $x \in \prod_{i=1}^n [k_i/2^{m+1}, (k_i+1)/2^{m+1}]$. For any $1 \leq i \leq n$, $k_i = 2m_i + r_i$ for some $m_i \in \mathbb{Z}$, with $r_i = 0$ or $1$. If $r_i = 0$, then $k_i/2^{m+1} = m_i/2^m$, so $[k_i/2^{m+1}, (k_i+1)/2^{m+1}] \subseteq [m_i/2^m, (k_i+2)/2^{m+1}] = [m_i/2^m, (m_i+1)/2^m]$. If $r_i = 1$, then $k_i = 2m_i+1$, so $(k_i-1)/2^{m+1} = m_i/2^m$. Since $(k_i-1)/2^{m+1} < k_i/2^{m+1}$, $[k_i/2^{m+1}, (k_i+1)/2^{m+1}] \subseteq [m_i/2^m, (2m_i+2)/2^{m+1}] = [m_i/2^m, (m_i+1)/2^m]$. Therefore, $x \in \prod_{i=1}^n [m_i/2^m, (m_i+1)/2^m] = C_{m_1,\ldots,m_n}$. Since $C_{m_1,\ldots,m_n} \subseteq \Theta_m^{\{O\}}(X) = Y_m$, we conclude that $Y_{m+1} \subseteq Y_m$.

Since $(Y_m)$ is decreasing, it follows that $Y_m \to \bigcap Y_m$ [2]. It suffices to show $X = \bigcap Y_m$. Let $y = (y_1, \ldots, y_n) \in X$. Because the dyadic rationals are unbounded, then for each $1 \leq i \leq n$, there is $k_{1,i} \in \mathbb{Z}$ such that $k_{1,i}/2 \leq y_i \leq (k_{1,i}+1)/2$. Therefore, $y \in C_{k_{1,1},k_{1,2},\ldots,k_{1,n}}$. By Lemma 1.3, there is a descending sequence $(C_{k_{m,1},k_{m,2},\ldots,k_{m,n}})_{m=1}^{\infty}$ such that $x \in C_{k_{m,1},k_{m,2},\ldots,k_{m,n}}$ for all $m \in \mathbb{N}$. Therefore, $y \in Y_m$ for all $m \in \mathbb{N}$, so $X \subseteq \bigcap Y_m$.

Conversely, suppose that $y \in \bigcap Y_m$. Then $y \in Y_m$ for every $m \in \mathbb{N}$, so for some $C_{k_1,\ldots,k_n} \subseteq Y_m$, $y \in C_{k_1,\ldots,k_n}$. Since $C_{k_1,\ldots,k_n} \cap X \neq \emptyset$, there exists $x_m \in X$ such that $x_m \in C_{k_1,\ldots,k_n}$. Hence, $\|x_m - y\|_2 \leq 2^{-m} \cdot \sqrt{n}$. Hence, $x_m \to y$, and because $X$ is compact, $y \in X$. Therefore, $\bigcap Y_m \subseteq X$. $\square$

**Lemma 0.3.** Let $(X_k)_{k=1}^{\infty}$ be a sequence such that $X_k$ is a **finite** union of boxes in $\mathcal{G}_{n,k}$. If $X_k$ is connected for sufficiently large $k \in \mathbb{N}$ and if $\lim_{k\to\infty} X_k$ exists, then $\lim_{k\to\infty} X_k$ is connected.

*Proof.* Suppose that $x, y \in \lim_{k\to\infty} X_k$. Since $\lim_{k\to\infty} X_k = \{x \in \mathbb{R}^n : x_k \to x \text{ for some } (x_k)_{k=1}^{\infty} \text{ with } x_k \in X_k\}$ [2], it follows that there are sequences $(x_k)_{k=1}^{\infty}$ and $(y_k)_{k=1}^{\infty}$ with $x_k, y_k \in X_k$ such that $x_k \to x$ and $y_k \to y$. We will use the following lemma from [1] for this proof:

**Lemma 0.4.** For $\epsilon > 0$, we define an *$\epsilon$-path* from $x$ to $y$ in $\mathbb{R}^n$ to be a finite sequence $(x_k)_{k=0}^{p}$ in $X$ such that $x_0 = x$, $x_p = y$, and $\|x_{k-1} - x_k\|_2 \leq \epsilon$ for all $1 \leq k \leq p$. So, a compact subset $X \subseteq \mathbb{R}^n$ is connected if and only if any pair of points in $X$ can be connected by an $\epsilon$-path in $X$ for any $\epsilon > 0$.

Let $\epsilon > 0$, and define $N \in \mathbb{N}$ such that $\dfrac{\sqrt{n}}{2^{N-2}} < \epsilon$, $\|x_m - x\|_2 < \epsilon$ and $\|y_m - y\|_2 < \epsilon$ for all $m \geq N$, and such that $H\left(X_m, \lim_{k\to\infty} X_k\right) < \dfrac{\epsilon}{4}$ for all $m \geq N$. Because $X_k$ is connected for sufficiently large $k \geq N$ and that $X_k$ is a finite union of boxes in $\mathcal{G}_{n,k}$, there is a chained finite sequence of boxes $(C_i)_{i=1}^{p}$ in $X_k$ such that $x_k \in C_1$ and $y_k \in C_p$. Now select $c_i \in C_i$ and $c_{i-1} \in C_{i-1}$, with $2 \leq i \leq p-1$. Because $H\left(X_k, \lim_{k\to\infty} X_k\right) < \dfrac{\epsilon}{4}$, then $C_i \subseteq X_k \subseteq N_{\epsilon/4}(X)$ and $C_{i-1} \subseteq X_k \subseteq N_{\epsilon/4}(X)$. Hence, there exists $a_i, a_{i-1} \in X$ such that $\|a_i - c_i\|_2 < \epsilon/4$ and $\|a_{i-1} - c_{i-1}\|_2 < \epsilon/4$. Because $C_i$ and $C_{i-1}$ intersect at their boundaries, then if $c_i^*$ is an corner point (a boundary point) where they intersect, it follows that

$$\|c_{i-1} - c_i^*\|_2 \leq \frac{\sqrt{n}}{2^k} < \frac{\epsilon}{4}$$

and

$$\|c_i^* - c_i\|_2 \leq \frac{\sqrt{n}}{2^k} < \frac{\epsilon}{4}.$$

Therefore,

$$\begin{aligned}\|a_{i-1}-a_i\|_2 &\leq \|a_{i-1}-c_{i-1}\|_2+\|c_{i-1}-c_i^*\|_2+\|c_i^*-c_i\|_2+\|a_i-c_i\|_2\\ &< \frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}\\ &= \epsilon.\end{aligned}$$

Because $x_k \in C_1$ and $y_k \in C_p$, then for any $c_1 \in C_1$ and $c_p \in C_p$, $\|x_k - c_1\| < \epsilon/4$ and $\|y_k - c_p\| < \epsilon/4$. Hence, by choosing $a_1 = x_k$ and $a_p = y_k$, we obtain an $\epsilon$-path $(a_1, a_2, \ldots, a_p)$. And because $\|x_k - x\|_2 < \epsilon$ and $\|y_k - y\|_2 < \epsilon$, then $(x, a_1, a_2, \ldots, a_p, y)$ is also an $\epsilon$-path.

Because $\mathbb{R}^n$ is complete, $\mathscr{H}(\mathbb{R}^n)$ is complete, so $\lim\limits_{k\to\infty} X_k$ is compact. Hence, by Lemma 1.6, $\lim\limits_{k\to\infty} X_k$ is connected. $\square$

**Theorem 0.5.** Let $X \subseteq \mathbb{R}^n$ be compact. Then $X$ is connected if and only if $Y_k$ is connected for all $k \in \mathbb{N}$.

*Proof.* Let $Y_k = \Theta_k^{\{O\}}(X)$ for all $k \in \mathbb{N}$. Suppose that every $Y_k$ is connected. By Lemmas 1.4 and 1.5, $X$ is connected.

Conversely, suppose that $X$ is connected and let $\epsilon > 0$. Define $N \in \mathbb{N}$ such that $\epsilon \leq 1/2^N$ and define $C_{k_1,\ldots,k_n}, C_{m_1,\ldots,m_n} \subseteq Y_N$. Choose (AoC) $x \in X \cap C_{k_1,\ldots,k_n}$ and $y \in X \cap C_{m_1,\ldots,m_n}$. Since $X$ is connected and compact, then there is an $\epsilon$-path $(a_1, a_2, \ldots, a_p)$ in $X$ from $x$ to $y$. Since $X = \bigcap Y_m$, each $a_i \in C_{r_{1,i},\ldots,r_{n,i}} \subseteq Y_k$ for some $r_1, \ldots, r_n \in \mathbb{Z}$. Now because $(a_1, \ldots, a_p)$ is an $\epsilon$-path, $\|a_{i-1} - a_i\|_2 \leq \epsilon \leq 1/2^N$.

Now suppose $C_{r_{1,i},\ldots,r_{n,i}} \neq C_{r_{1,i-1},\ldots,r_{n,i-1}}$. For two non-equal boxes $C_1, C_2 \in \mathcal{G}_{n,N}$ to be non-adjacent, we must have that for every $a \in C_1$ and $b \in C_2$, $\|a - b\|_2 > 1/2^N$. Because $\|a_{i-1} - a_i\|_2 \leq 1/2^N$, $C_{r_{1,i},\ldots,r_{n,i}} \neq C_{r_{1,i-1},\ldots,r_{n,i-1}}$ are adjacent. Therefore, the finite sequence $(C_{r_{1,i},\ldots,r_{n,i}})_{i=1}^p$ is a list of either, consecutively, equal or adjacent boxes. Therefore, $Y_k$ is path-connected, and thus connected. $\square$

Clearly more was proven in the proof of the converse of Theorem 1.7 than was needed. All that was needed was the $\epsilon$-path $(a_1, \ldots, a_p)$ and the fact that each $a_i$ was contained in $Y_k$. The rest of the proof implies mores: not only is $Y_k$ path-connected, but that for any two boxes in $Y_k$, a finite list of adjacent boxes can be found in $Y_k$ connecting the original two boxes. So long as $X$ is compact and connected, $\Theta_k^{\{O\}}(X)$ is a union of adjacent boxes for all $k \in \mathbb{N}$. Conversely, if $\Theta_k^{\{O\}}(X)$ is a union of adjacent boxes for all $k \in \mathbb{N}$, then $\Theta_k^{\{O\}}(X)$ is clearly connected, so by Theorem 1.7, $X$ is connected.

We have the following equivalences, assuming $X \subseteq \mathbb{R}^n$ is compact:

$$\begin{aligned} X \text{ is connected} &\iff \text{Every } \Theta_k^{\{O\}}(X) \text{ is connected} \iff \text{Every } \Theta_k^{\{O\}}(X) \text{ is a union of adjacent boxes}\\ &\iff \forall x, y \in X,\ \exists \epsilon\text{-path in } X \text{ for any } \epsilon > 0 \text{ from } x \text{ to } y.\end{aligned}$$

**Lemma 0.6.** $S_k = \Theta_k^{\{O\}}(\mathbb{S}_{\text{right}})$ for every $k \in \mathbb{N}$.

*Proof.* We proceed using induction, and consider the base case $k = 1$. Then

$$S_1 = \mathbf{F}_0[S_0] \cup \mathbf{F}_1[S_0] \cup \mathbf{F}_2[S_0] = ([0,1/2] \times [0,1/2]) \cup ([0,1/2] \times [1/2,1]) \cup ([1/2,1] \times [0,1/2]).$$

Since $\mathbf{F}_0[\mathbb{S}_{\text{right}}] \subseteq \mathbf{F}_0[S_0] = [0,1/2]^2$, $\mathbf{F}_1[\mathbb{S}_{\text{right}}] \subseteq \mathbf{F}_1[S_0] = [1/2,1] \times [0,1/2]$, and $\mathbf{F}_2[\mathbb{S}_{\text{right}}] \subseteq \mathbf{F}_2[S_0] = [0,1/2] \times [1/2,1]$, then considering $C_{0,0}, C_{1,0}, C_{0,1} \subseteq \mathcal{G}_{2,1}$, it follows that $C_{0,0} \cap \mathbb{S}_{\text{right}} \neq \emptyset$, $C_{0,1} \cap \mathbb{S}_{\text{right}} \neq \emptyset$, and $C_{1,0} \cap \mathbb{S}_{\text{right}} \neq \emptyset$ since $\mathbf{F}_0[\mathbb{S}_{\text{right}}], \mathbf{F}_1[\mathbb{S}_{\text{right}}], \mathbf{F}_2[\mathbb{S}_{\text{right}}] \subseteq \mathbb{S}_{\text{right}}$. Hence, if $(x, y) \in S_1$, then $(x, y) \in \Theta_1^{\{O\}}(\mathbb{S}_{\text{right}})$.

Conversely, suppose $(x, y) \in \Theta_1^{\{O\}}(\mathbb{S}_{\text{right}})$. $\square$

## Appendix: Sample Code

Code used in applied projects should be documented in the appendix (lines of code do not count towards page-count requirements). Here is an example showing how to typeset a code snippet.

Listing 1. AI-Assisted Generation of MATLab Function That Plots Sierpinski Gasket

```
function IFS_Fractal()
    % Define the three IFS functions as anonymous functions
    F{1} = @(x, y) deal(0.5*x, 0.5*y);                    % F_0
    F{2} = @(x, y) deal(0.5*x + 0.5, 0.5*y);              % F_1
    F{3} = @(x, y) deal(0.5*x, 0.5*y + 0.5);              % F_2

    % Initial square: [0,1] x [0,1]
    squares = {[0 1 1 0; 0 0 1 1]};  % Each square is a 2x4 matrix (x; y)

    % Number of iterations
    max_iter = 5;

    figure;
    for iter = 1:max_iter
        new_squares = {};
        for i = 1:length(squares)
            sq = squares{i};
            for f = 1:3
                [x_new, y_new] = F{f}(sq(1,:), sq(2,:));
                new_squares{end+1} = [x_new; y_new];
            end
        end
        squares = new_squares;

        % Plotting the current iteration
        subplot(2, 3, iter);
        hold on;
        axis equal on;
        title(['Iteration ' num2str(iter)]);
        for i = 1:length(squares)
            fill(squares{i}(1,:), squares{i}(2,:), 'k');
        end
    end
end
```

## References

[1] Michel Schmitt, "Digitization and Connectivity", *Mathematical Morphology and its Applications to Image and Signal Processing* (H. J. A. M. Heijmans and J. B. T. M. Roerdink, eds.), 91-98, Khrwer Academic Publishers, 1998.

[2] Gerald Edgar, *Measure, Topology, and Fractal Geometry*, Springer, 2nd ed., 2008.