Computing $H(S_n, S_{n+1})$

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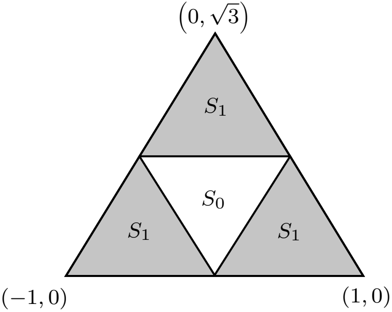
July 18, 2025

Summary

The Problem

Example 4.5

Let $(\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2 : \mathbb{R}^2 \to \mathbb{R}^2)_{i=0}^2$ be the iterated function system given by $\mathbf{F}_i(x) = x_i + \frac{1}{2}(x - x_i)$, where $x_0 = (-1, 0)$, $x_1 = (1, 0)$, and $x_2 = (0, \sqrt{3})$. Define S_0 to be the solid equilateral triangle with vertices x_0, x_1, and x_2 and $S_k = \mathbf{F}_0[S_{k-1}] \cup \mathbf{F}_1[S_{k-1}] \cup \mathbf{F}_2[S_{k-1}]$ for $k \geq 1$. Estimate, or find an upper bound, for the Hausdorff distance between S_0 and S_1.



Solution Pt. 1

Since $S_1 \subseteq S_0$, it follows that $H(S_0, S_1) = D(S_0, S_1) = \max_{\mathbf{x} \in S_0}(\min_{\mathbf{y} \in S_1} \|\mathbf{x} - \mathbf{y}\|_2)$. Because $S_1 \subseteq S_0$, then for any $\mathbf{x} \in S_1$, $\min_{\mathbf{y} \in S_1} \|\mathbf{x} - \mathbf{y}\|_2 = 0$, so $H(S_0, S_1) = \sup_{\mathbf{x} \in S_0 \setminus S_1}(\min_{\mathbf{y} \in S_1} \|\mathbf{x} - \mathbf{y}\|_2)$. The following lemma will be helpful in calculating $H(S_0, S_1)$.

Lemma 4.6

Let $K \in \mathscr{H}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n \setminus K$. If $r : K \to \mathbb{R}$ is given by $r(k) = \|k - x\|_2$, then $\min\limits_{k \in K} r(k) = r(k_0)$ for some $k_0 \in \partial K$, the boundary of K.

Solution Pt. 2

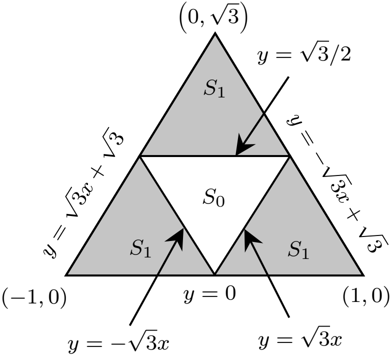
Proof.

Because K is compact, then by the Extreme Value Theorem, $\min_{k\in K} r(k)$ exists. So, there exists k_0 such that $r(k_0) = \min_{k\in K} r(k)$ Now suppose for a contradiction that $k_0 \notin \partial K$. Because compact subsets of metric spaces are closed, $K = \text{int}K \cup \partial K$. Thus, $k_0 \in \text{int}K$, so there exists $\gamma > 0$ such that $B_\gamma(k_0) \subseteq K$. Now consider the function $f : [0,1] \to \mathbb{R}^n$ given by $f(t) = tx + (1-t)\,k_0$, the function describing the line segment $f([0,1])$ between x and k_0. Because $B_\gamma(k_0) \subseteq K$, $r(k_0) \geq \gamma$. Also, since $k_0 \in f([0,1]) \cap B_\gamma(k_0)$ and that $f([0,1]) \cap \partial B_\gamma(k_0)$ are nonempty and that $B_\gamma(k_0)$ is convex, there exists $0 \leq t_0 \leq 1$ such that $\{f(t) : 0 \leq t < t_0\} \subseteq B_\gamma(k_0)$, where $\|k - f(t_0)\| = \gamma$. Therefore, there exists $0 \leq t' < t_0$ such that $r(f(t')) = \|f(t') - k_0\| = \epsilon/2 < \epsilon \leq r(k_0)$, contradicting the minimality of k_0. Therefore, $k_0 \in \partial K$.

$\square$

Solution Pt. 3

By Lemma 4.6, $H(S_0, S_1) = \sup_{\mathbf{x} \in S_0 \setminus S_1} (\min_{\mathbf{y} \in \partial S_1} \|\mathbf{x} - \mathbf{y}\|)$. Letting $(x, y) \in \partial S_1$, there are six possibilities for $\mathbf{y}$, which are given in the figure for S_0 and S_1 below.



Claim: $H(S_0, S_1) = \dfrac{\sqrt{3}}{6} < 2^0 = 1.$

Solution Pt. 4

We focus only on lines L_1, L_2, and L_3 given by $y = \sqrt{3}x$, $y = -\sqrt{3}x$, and $y = \sqrt{3}/2$, respectively, as it is clear from the figure that these lines are "closer" to S_0 than the other three lines. We have that

$$\sup\left\{\min_{\mathbf{y}\in\partial S_1}\|\mathbf{x}-\mathbf{y}\|_2 : \mathbf{x}\in S_0\right\} = \sup\{\min\bigcup_{\mathbf{y}\in L_1}\|\mathbf{x}-\mathbf{y}\|_2\cup$$

$$\bigcup_{\mathbf{y}\in L_2}\|\mathbf{x}-\mathbf{y}\|_2\cup\bigcup_{\mathbf{y}\in L_3}\|\mathbf{x}-\mathbf{y}\|_2 : \mathbf{x}\in S_0\backslash S_1\}$$

$$= \sup\{\min\{\min\bigcup_{L_1}\|\mathbf{x}-\mathbf{y}\|, \min\bigcup_{L_2}\|\mathbf{x}-\mathbf{y}\|, \min\bigcup_{L_3}\|\mathbf{x}-\mathbf{y}\|\}$$

$$: \mathbf{x}\in S_0\backslash S_1\}$$

If $A = \{(0,0),(0,\sqrt{3}/3),(1/2,\sqrt{3}/2)\}$, $B = \{(0,0),(0,\sqrt{3}/3),(-1/2,\sqrt{3}/2)\}$, and $C = \{(-1/2,\sqrt{3}/2),(0,\sqrt{3}/3),(1/2,\sqrt{3}/2)\}$, and also letting $A_0 = \{(0,0),(1/2,\sqrt{3}/2)\}$,

Solution Pt. 5

$B_0 = \{(0,0), (-1/2, \sqrt{3}/2)\}$, and $C_0 = \{(-1/2, \sqrt{3}/2), (1/2, \sqrt{3}/2)\}$, then $S_0 \backslash S_1 = \text{conv}A \backslash \text{conv}A_0 \cup \text{conv}B \backslash \text{conv}B_0 \cup \text{conv}C \backslash \text{conv}C_0$. We now state a lemma needed for the remainder of the solution.

Lemma 4.7

Let $\triangle ABC$ be an solid equilateral triangle with vertices A, B, and C in $\mathbb{R}^2$, and let M be the centroid of $\triangle ABC$. Then

(i) If $\mathbf{x} \in \triangle AMB$, then $\min\{d_{\mathbb{R}^2}(\mathbf{x}, \overline{AB}), d_{\mathbb{R}^2}(\mathbf{x}, \overline{BC}), d_{\mathbb{R}^2}(\mathbf{x}, \overline{AC})\} = d_{\mathbb{R}^2}(\mathbf{x}, \overline{AB})$;

(ii) If $\mathbf{x} \in \triangle AMC$, then $\min\{d_{\mathbb{R}^2}(\mathbf{x}, \overline{AB}), d_{\mathbb{R}^2}(\mathbf{x}, \overline{BC}), d_{\mathbb{R}^2}(\mathbf{x}, \overline{AC})\} = d_{\mathbb{R}^2}(\mathbf{x}, \overline{AC})$;

(iii) If $\mathbf{x} \in \triangle BMC$, then $\min\{d_{\mathbb{R}^2}(\mathbf{x}, \overline{AB}), d_{\mathbb{R}^2}(\mathbf{x}, \overline{BC}), d_{\mathbb{R}^2}(\mathbf{x}, \overline{AC})\} = d_{\mathbb{R}^2}(\mathbf{x}, \overline{BC})$.

Solution Pt. 6

Fix $(x_0, y_0) \in S_0 \backslash S_1$, Define $f, g, h : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} f(x) &= \|(x_0, y_0) - (x, \sqrt{3}x)\| \\ &= \sqrt{4x^2 - x(2\sqrt{3}y_0 + 2x_0) + (x_0^2 + y_0^2)}, \end{aligned}$$

$$\begin{aligned} g(x) &= \|(x_0, y_0) - (x, -\sqrt{3}x)\| \\ &= \sqrt{4x^2 + x(2\sqrt{3}y_0 - 2x_0) + (x_0^2 + y_0^2)}, \end{aligned}$$

and

$$\begin{aligned} h(x) &= \|(x_0, y_0) - (x, \sqrt{3}/2)\| \\ &= \sqrt{x^2 + x(-2x_0) + (x_0^2 + y_0^2 - \sqrt{3}y_0 + 3/4)}. \end{aligned}$$

Solution Pt. 7

Then

$$f'(x) = \frac{1}{2f(x)} \cdot (8x - 2\sqrt{3}y_0 - 2x_0),$$
$$g'(x) = \frac{1}{2g(x)} \cdot (8x + 2\sqrt{3}y_0 - 2x_0),$$
$$h'(x) = \frac{1}{2h(x)} \cdot (2x - 2x_0).$$

for which the values $x_{f,(x_0,y_0)} = \dfrac{x_0 + \sqrt{3}y_0}{4}$, $x_{g,(x_0,y_0)} = \dfrac{x_0 - \sqrt{3}y_0}{4}$, and $x_{h,(x_0,y_0)} = x_0$ satisfy $f'(x_{f,(x_0,y_0)}) = g'(x_{g,(x_0,y_0)}) = h'(x_{h,(x_0,y_0)}) = 0$. Therefore, if $\mathbf{x}_0 = (x_0, y_0)$, then $\min \bigcup_{L_1} \|\mathbf{x}_0 - \mathbf{y}\| = f(x_{f,\mathbf{x}_0})$, $\min \bigcup_{L_2} \|\mathbf{x}_0 - \mathbf{y}\| = g(x_{g,\mathbf{x}_0})$, and $\min \bigcup_{L_3} \|\mathbf{x}_0 - \mathbf{y}\| = h(x_{h,\mathbf{x}_0})$.

Solution Pt. 8

Now by Lemma 4.7,

$$D(S_0, S_1) = \sup \bigcup_{\mathbf{x}\in \text{conv}A\setminus\text{conv}A_0} \min \bigcup_{\mathbf{y}\in L_1} \|\mathbf{x}-\mathbf{y}\| \cup$$
$$\bigcup_{\mathbf{x}\in \text{conv}B\setminus\text{conv}B_0} \min \bigcup_{\mathbf{y}\in L_2} \|\mathbf{x}-\mathbf{y}\| \cup$$
$$\bigcup_{\mathbf{x}\in \text{conv}C\setminus\text{conv}C_0} \min \bigcup_{\mathbf{y}\in L_3} \|\mathbf{x}-\mathbf{y}\|,$$

and by the stability propery of sup and the previous differentiation results from Slide 11,

$$D(S_0, S_1) = \max\{\sup \bigcup_{\mathbf{x}\in \text{conv}A\setminus\text{conv}A_0} f(x_{f,\mathbf{x}}), \sup \bigcup_{\mathbf{x}\in \text{conv}B\setminus\text{conv}B_0} g(x_{g,\mathbf{x}}),$$
$$\sup \bigcup_{\mathbf{x}\in \text{conv}C\setminus\text{conv}C_0} h(x_{h,\mathbf{x}})\}.$$

Solution Pt. 9

To calculate the suprema, partial differentiation will be used alongside the contraposition of the fact that if a function has an extreme point, then all of its partial derivatives equal 0 at that point. Set $T := \{(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3 : \lambda_0 + \lambda_1 + \lambda_2 = 1, \lambda_0, \lambda_2 \geq 0, \lambda_1 > 0\}$, the barycentric coordinates of the points in $\mathrm{conv}A \backslash \mathrm{conv}A_0$, and define $r : T \to [0, \infty)$ by

$$\begin{aligned} r(\lambda_0, \lambda_1, \lambda_2) &= f(x_{f,(1/2\lambda_2, \sqrt{3}/3\lambda_1 + \sqrt{3}/2\lambda_2)}) \\ &= \frac{1}{2}[3((1/2)\lambda_2)^2 - 2\sqrt{3}(\sqrt{3}/3\lambda_1 + \sqrt{3}/2\lambda_2) \\ &+ (\sqrt{3}/3\lambda_1 + \sqrt{3}/2\lambda_2)^2]^{1/2} \\ &= \frac{\lambda_1}{2\sqrt{3}}. \end{aligned}$$

This function represents the minimal distance from $\mathrm{conv}A_0$ to a point in $\mathrm{conv}A \backslash \mathrm{conv}A_0$ using its barycentric coordinates with respect to $(0, 0)$, $(0, \sqrt{3}/3)$, and $(1/2, \sqrt{3}/2)$.

Solution Pt. 10

We see that $r_{\lambda_0}(\lambda_0, \lambda_1, \lambda_2) = 0 = r_{\lambda_2}(\lambda_0, \lambda_1, \lambda_2)$ and $r_{\lambda_1}(\lambda_0, \lambda_1, \lambda_2) = \dfrac{1}{2\sqrt{3}}$. Therefore, r has no extreme values, and by setting $f(t) := r(\lambda_0, t, \lambda_2)$ for $0 < t \leq 1$, we see $f(t) = t/2\sqrt{3}$. f is clearly increasing, so it follows that

$$\sup \bigcup_{\mathbf{x} \in \mathrm{conv} A \setminus \mathrm{conv} A_0} f(x_{f,\mathbf{x}}) = f(1) = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}.$$

The calculation for $\mathrm{conv} B \setminus \mathrm{conv} B_0$ is similar and also yields $\sqrt{3}/6$, so we only focus on $\mathrm{conv} C \setminus \mathrm{conv} C_0$. Define $s : T \to [0, \infty)$ by

$$\begin{aligned} s(\lambda_0, \lambda_1, \lambda_2) &= h(x_{h,(1/2(\lambda_2 - \lambda_0), \sqrt{3}/2(\lambda_0 + \lambda_2) + \sqrt{3}/3\lambda_1)}) \\ &= \left| \frac{\sqrt{3}}{2} - \left(\frac{\sqrt{3}}{2}(\lambda_0 + \lambda_2) + \frac{\sqrt{3}}{3}\lambda_1 \right) \right| \\ &= \frac{\sqrt{3}}{2} - \left(\frac{\sqrt{3}}{2}(\lambda_0 + \lambda_2) + \frac{\sqrt{3}}{3}\lambda_1 \right) \end{aligned}$$

since

$$\begin{aligned}\sqrt{3} \cdot 1/2 &= \sqrt{3} \cdot (1/2\lambda_0 + 1/2\lambda_1 + 1/2\lambda_2) \\ &> \sqrt{3} \cdot (1/2\lambda_0 + 1/3\lambda_1 + 1/2\lambda_2).\end{aligned}$$

We have that $s_{\lambda_0}(\lambda_0, \lambda_1, \lambda_2) = -\sqrt{3}/2 = s_{\lambda_2}(\lambda_0, \lambda_1, \lambda_2)$ and $s_{\lambda_1}(\lambda_0, \lambda_1, \lambda_2) = \sqrt{3}/3$, meaning there are no extreme values. Thus, we must check the boundary of conv$C$. We'll first look at the case where $\lambda_0 = 0$. Then setting

$$s_0(\lambda_1, \lambda_2) := s(0, \lambda_1, \lambda_2) = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\lambda_2 - \frac{\sqrt{3}}{3}\lambda_1$$

for which $s_{0_{\lambda_1}}(\lambda_1, \lambda_2) = -\sqrt{3}/3$ and $s_{0_{\lambda_2}}(\lambda_1, \lambda_2) = -\sqrt{3}/2$. Again, s_0 has no extreme values, meaning we must check the boundary of conv$\{(0, \sqrt{3}/3), (1/2, \sqrt{3}/2)\}$.

Solution Pt. 12

But these are simply the points $(0, \sqrt{3}/3)$ and $(1/2, \sqrt{3}/2)$, which correspond to $\lambda_1 = 1$ with $\lambda_0 = \lambda_2 = 0$ or $\lambda_2 = 1$ with $\lambda_0 = \lambda_1 = 0$. The latter case yields $s(0,0,1) = \sqrt{3}/2 - \sqrt{3}/2 = 0$, while the first yields $s(0,1,0) = \sqrt{3}/2 - \sqrt{3}/3 = \sqrt{3}/6$.
The other two cases where $\lambda_1 = 0$ or $\lambda_2 = 0$ to obtain the desired maximum value of s are similar and $\sqrt{3}/6$ is obtained in both cases.
Therefore,

$$H(S_0, S_1) = \max\{\sqrt{3}/6, \sqrt{3}/6, \sqrt{3}/6\} = \sqrt{3}/6.$$

Conjecture Results

Due to the work in Example 4.5 along with a result, to be proven, involving the Hutchinson operator, we can draw definitive conclusions of the following conjectures:

Conjecture 4.9

For any $n \in \mathbb{Z}_{\geq 0}$, $0 < H(S_n, S_{n+1}) \leq 2^{-n}$ and $0 < H(\mathbf{F}_i[S_n], \mathbf{F}_i[S_{n+1}]) \leq 2^{-(n+1)}$ for $i = 0, 1$, or 2. **TRUE!**

Conjecture 4.11

$H(S_n, \mathbb{S}) \leq 2^{-n+1}$ for every $n \in \mathbb{N}$. **TRUE!**

Conjecture 4.12

For any $n \in \mathbb{Z}_{\geq 0}$, $H(\mathbf{F}_i[S_n], \mathbf{F}_i[S_{n+1}]) = 2^{-(n+1)}$ and $H(S_n, S_{n+1}) = 2^{-n}$ for $i = 0, 1$, or 2. **FALSE!**

The Hutchinson Operator is a Contraction Mapping

Let $(\mathbb{X}, d)$ be a metric space and $(f_1, f_2, \ldots, f_n : \mathbb{X} \to \mathbb{X})$ be an iterated function system of contracting similarities on $\mathbb{X}$ with ratios $r_1, r_2, \ldots, r_n < 1$, respectively. Define $r := \max\{r_1, r_2, \ldots, r_n\}$. Then if $F : \mathscr{H}(\mathbb{X}) \to \mathscr{H}(\mathbb{X})$ is defined by

$$F(A) = \bigcup_{i=1}^{n} f_i[A],$$

then F is a contracting similarity on $\mathscr{H}(\mathbb{X})$ with ratio r.

The Hutchinson Operator Pt. 2

Proof.

The argument that follows is based from Gerald Edgar's *Measure, Topology, and Fractal Geometry*. Let $A, B \in \mathscr{H}(\mathbb{X})$ and suppose that $q \in \mathbb{R}$ such that $D_{\mathbb{X}}(A, B) < q$. Then $A \subseteq N_q(B)$ and $B \subseteq N_q(A)$. Now let $x \in F(A)$. We want to show that $x \in N_{rq}(F(B))$. We have that $f_i(x') = x$ for some $1 \leq i \leq n$ and $x' \in A$. Since $A \subseteq N_q(B)$, there exists $y' \in B$ such that $d(x', y') < q$. Hence, $d(f_i(x'), f_i(y')) = r_i d(x', y') < r_i q \leq rq$. Since $f_i(x') = x$ and $f_i(y') \in F(B)$, it follows that $F(A) \subseteq N_{rq}(F(B))$. A similar proof shows that $F(B) \subseteq N_{rq}(F(A))$. Therefore, $D_{\mathbb{X}}(F(A), F(B)) \leq rq$. And since q is any upper bound for $\{x \in \mathbb{R} : x \leq D_{\mathbb{X}}(A, B)\}$ and that $\sup\{x \in \mathbb{R} : x \leq D_{\mathbb{X}}(A, B)\} = D_{\mathbb{X}}(A, B)$, it follows that $D_{\mathbb{X}}(F(A), F(B)) \leq r D_{\mathbb{X}}(A, B)$. Hence, F is a contracting similarity on $\mathscr{H}(\mathbb{X})$ with ratio r. $\square$

The Conjectures (cont'd)

Proposition 4.9

For any $n \in \mathbb{Z}_{\geq 0}$, $0 < H(S_n, S_{n+1}) \leq 2^{-n} \cdot \dfrac{\sqrt{3}}{6}$ and $0 < H(\mathbf{F}_i[S_n], \mathbf{F}_i[S_{n+1}]) \leq 2^{-(n+1)} \cdot \dfrac{\sqrt{3}}{6}$ for $i = 0, 1$, or 2.

Proof.

We proceed via induction. For $n = 0$, we proved in Example 4.5 that $H(S_0, S_1) = \sqrt{3}/6$. Now suppose that for some integer $k \geq 0$, $H(S_k, S_{k+1}) \leq 2^{-k} \cdot \sqrt{3}/6$. Define $F : \mathscr{H}(\mathbb{R}^2) \to \mathscr{H}(\mathbb{R}^2)$ by $F(A) = \mathbf{F}_0[A] \cup \mathbf{F}_1[A] \cup \mathbf{F}_2[A]$, a union of contracting similarities on $\mathbb{R}^2$. Since the Hutchinson operator is a contraction mapping,

The Conjectures (cont'd)

Proof. (Pt. 2)

$$
\begin{aligned}
H(S_{k+1}, S_{k+2}) &= H(\mathbf{F}_0[S_k] \cup \mathbf{F}_1[S_k] \cup \mathbf{F}_2[S_k], \mathbf{F}_0[S_{k+1}] \cup \mathbf{F}_1[S_{k+1}] \\
&\cup \mathbf{F}_2[S_{k+1}]) \\
&= H(F(S_k), F(S_{k+1})) \\
&\leq \frac{1}{2} H(S_k, S_{k+1}) \\
&\leq \frac{1}{2} \cdot 2^{-k} \cdot \frac{\sqrt{3}}{6} = 2^{-(k+1)} \cdot \frac{\sqrt{3}}{6}.
\end{aligned}
$$

To prove the second inequality, define $G : \mathscr{H}(\mathbb{R}^2) \to \mathscr{H}(\mathbb{R}^2)$ by $G(A) = \mathbf{F}_i[A]$. Then by the previous inequality,

The Conjectures (cont'd)

Proof. (Pt. 3)

$$\begin{aligned} H(\mathbf{F}_i[S_n], \mathbf{F}_i[S_{n+1}]) &= H(G(S_n), G(S_{n+1})) \\ &\leq \frac{1}{2} H(S_n, S_{n+1}) \\ &\leq \frac{1}{2} \cdot 2^{-n} \cdot \sqrt{3}/6 \\ &= 2^{-(n+1)} \cdot \sqrt{3}/6. \end{aligned}$$

Hence, $H(\mathbf{F}_i[S_n], \mathbf{F}_i[S_{n+1}]) \leq 2^{-(n+1)} \cdot \sqrt{3}/6$. $\square$

The Conjectures (cont'd)

The Collage Theorem

Let $(\mathbb{X}, d)$ be a complete metric space. Let $L \in \mathscr{H}(\mathbb{X})$ be given, and let $\epsilon \geq 0$ be given. Consider an IFS $(f_0, f_1, \ldots f_n : \mathbb{X} \to \mathbb{X})$ of contracting similarities with ratio $0 \leq r < 1$. If

$$H\left(L, \bigcup_{i=0}^{n} f_i[L]\right) \leq \epsilon, \tag{1}$$

then

$$H(L, A) \leq \frac{\epsilon}{1-s},$$

where A is the unique attractor of $(f_0, f_1, \ldots f_n : \mathbb{X} \to \mathbb{X})$.

The Conjectures (cont'd)

Corollary 4.11

$H(S_n, \mathbb{S}) \leq 2^{-n+1} \cdot \sqrt{3}/6$ for every $n \in \mathbb{N}$.

Proof.

By Proposition 4.9, for every $n \in \mathbb{N}$, $H(S_n, S_{n+1}) \leq 2^{-n} \cdot \sqrt{3}/6$. Since $\mathbf{F}_0[S_n] \cup \mathbf{F}_1[S_n] \cup \mathbf{F}_2[S_n] = S_{n+1}$, then by the Collage Theorem,

$$H(S_n, \mathbb{S}) \leq \frac{2^{-n} \cdot \sqrt{3}/6}{1 - 1/2} = 2^{-n+1} \cdot \sqrt{3}/6. \qquad \square$$

The Conjectures (cont'd)

Corollary 4.11

For any $n \in \mathbb{Z}_{\geq 0}$, $H(\mathbf{F}_i[S_n], \mathbf{F}_i[S_{n+1}]) = 2^{-(n+1)}$ and $H(S_n, S_{n+1}) = 2^{-n}$ for $i = 0, 1$, or 2. **FALSE!**

Example 4.5 clearly indicated for $n = 0$ that $H(S_0, S_1) = \sqrt{3}/6 < 2^0 = 1$. Likewise, it follows by Proposition 4.9 that $H(\mathbf{F}_i[S_0], \mathbf{F}_i[S_1]) = \sqrt{3}/12 < 1/2$.

Possible Generalizations?

Possibly. There is a class of fractals called n-*flakes*, which are simply fractals constructed by applying iterated function systems of similitudes to regular n-gons. The Sierpinski Gasket is an example of a 3-flake since its initiator is an equilateral triangle, or regular 3-gon. Another example is the 5-*flake*, constructed by starting with a pentagon as the initiator. Using an IFS with 5 dilations centered at the extreme points of the pentagon, we obtain the following below.

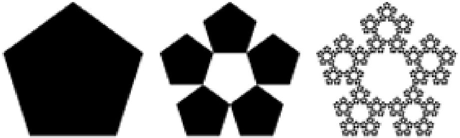


Figure 1: The 5-flake. [3]

Bibliography

- Gerald Edgar, *Measure, Topology, and Fractal Geometry*, Springer, 2nd ed., 2008.
- Michael F. Barnsley, *Fractals Everywhere: New Edition*, Dover, 2012.
- Steven Wilkinson and Blake Settle, "n-Flake Variations", *Bridges 2024 Conference Proceedings*, Northern Kentucky University, 2024.