Workshop 6: Heine-Borel Theorem, Topology in $\mathbb{R}^n$

Let's make one thing clear: compact sets are **not** intuitive. The open cover definition of a compact set is weird and does not, at least at a first glance, justify why we even call a compact set compact. In order to truly understand the nomenclature of a compact set and to gain an intuition behind them, we begin with a brief discussion behind why they are useful. In particular, they are useful for analyzing a new kind of function you are soon to learn: **uniformly continuous functions**.

Suppose that $E \subseteq \mathbb{R}$ and that $f : E \to \mathbb{R}$ is continuous. This means that for any $\epsilon > 0$ and $x \in E$, we can choose $\delta_x > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta_x$ for any $y \in E$. This definition is arguably complicated, but let's look at something: *the existence of* $\delta_x > 0$ *depends on the choice of* x. Can we find a $\delta > 0$ that works *independently* of the choice of x?

(a) Let $\delta = \inf\{\delta_x : x \in E\}$. If E is finite, is δ positive?

(b) If E is infinite and there exists $\epsilon > 0$ such that for every $\delta > 0$, there is $x, y \in E$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$, explain why $\inf\{\delta_x : x \in E\} = 0$. [*Hint*: Try a contradiction.]

(c) Now consider $(x - \delta_x, x + \delta_x) \cap E$, and let $y_1, y_2 \in (x - \delta_x, x + \delta_x) \cap E$. Show that $|f(y_1) - f(y_2)| < 2\epsilon$.

(d) Now suppose there is $x_1, \ldots, x_n$ such that $E \subseteq \bigcup_{i=1}^{n}(x_i - \delta_{x_i}, x_i + \delta_{x_i})$. Why would this potentially imply that f is both continuous and that the existence of δ is independent of x?

Notice that in part (d), the key idea is the finiteness in the number of intervals in the union. This leads to the definition of a compact set.

Definition 1. Let $E \subseteq \mathbb{R}$. We say that E is *compact* if for every ________________________ $\{G_\alpha\}_{\alpha \in A}$, there is a ________________________ of $\{G_\alpha\}_{\alpha \in A}$.

While not clear yet, a compact set essentially is a set within a particular region that has no "punctures". For example, a set like $[0, 1] \cup [2, 3]$ is compact since $[1, 3]$ and $[2, 3]$ is compact, but $[0, 1/2) \cup (1/2, 1]$ is not compact because $1/2$ is a "puncture". But do you notice something? The first set discussed is also closed, while the second example is not closed.

Question. Provide an example of a subset of $\mathbb{R}$ that is closed in $\mathbb{R}$ but is not compact.

If the example you provided is indeed a compact set, what property do you notice that it has? If you did your work correctly, it should obey the following major theorem.

Theorem 1. Let $E \subseteq \mathbb{R}$.

(i) If E is compact, then E is closed and bounded.

(ii) **(Heine-Borel Theorem)** If E is closed and bounded, then E is compact.

The proof of (i) is in your textbook. We will focus on just the proof of (ii). We follow the proof of the Heine-Borel Theorem in Gaughan's *Introduction to Analysis*.

(a) Let E be closed and bounded, and let $\{G_\alpha\}_{\alpha \in A}$ be an open cover of E. We will suppose for a contradiction that $\{G_\alpha\}_{\alpha \in A}$ has no finite subcover. Because E is bounded, there is some closed interval $[\alpha, \beta]$ such that ...

(b) Consider the midpoint γ_0 of $[\alpha, \beta]$, and consider $[\alpha, \gamma_0] \cap E$ and $[\gamma_0, \beta] \cap E$. Why can't both of these sets be covered by a finite subfamily of $\{G_\alpha\}_{\alpha \in A}$?

(c) Select the interval from above that cannot be covered by a finite subfamily and call it $[\alpha_1, \beta_1]$. What deduction can we make if we choose a midpoint γ_1 of this interval?

(d) Now you probably see the pattern that is emerging. A particular sequence $\{[\alpha_n, \beta_n]\}_{n=1}^{\infty}$ has been obtained. Based on the work you did above, answer the following questions:

   (i) Describe $\beta_n - \alpha_n$ in terms of $\beta - \alpha$.

   (ii) For any $n \in \mathbb{N}$, what is the "smallest" interval is $[\alpha_{n+1}, \beta_{n+1}]$ contained inside?

(iii) What property does $[\alpha_n, \beta_n] \cap E$ have with respect to $\{G_\alpha\}_{\alpha \in A}$? (*Hint*: Parts (b) and (c)).

(e) Using one of (i)-(iii), explain why $[\alpha_n, \beta_n] \cap E$ must be nonempty.

(f) By (e), for every $n \in \mathbb{N}$, there is $x_n \in [\alpha_n, \beta_n] \cap E$, so the set $P := \{x_n : n \in \mathbb{N}\}$ is nonempty. Suppose in one case that P is finite. What is an example of a case where P can indeed be finite?

(g) Continuing from (f), since P is finite, make a conclusion about $\bigcap_{n=1}^{\infty}([\alpha_n, \beta_n] \cap E)$. Why can't we use the Nested Interval Property or Compact Nested Interval Property?

(h) Since $\{G_\alpha\}_{\alpha \in A}$ is an open cover for E, then use (g) and the definition of an open set in $\mathbb{R}$ to draw a conclusion regarding a particular interval contained within an element of $\{G_\alpha\}_{\alpha \in A}$.

(i) Let $\epsilon > 0$. Given $(1/2^n)_{n=1}^{\infty}$ converges to 0, what conclusion can be drawn regarding $\beta_n - \alpha_n = \dfrac{1}{2^n}(\beta - \alpha)$?

(j) Using (g) and (i), what implication results from letting $x \in [\alpha_n, \beta_n] \cap E$?

(k) Using (h) and (j), draw a conclusion about a set from $\{G_\alpha\}_{\alpha \in A}$ that $[\alpha_n, \beta_n] \cap E$ is contained inside.

(l) What contradiction arises as a result of (k)?

Note that we still need to consider the case where P is infinite. This is left as an exercise for you in the extra practice section.

Topology on $\mathbb{R}^n$

Question. When considering the metric topology on $\mathbb{R}^n$, we must understand what a *metric space* is. Given a space X and a function $d : X \times X \to [0, \infty)$ satisfying the following:

(a) $d(x, x) = 0$ for all $x \in X$,

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$,

we say that (X, d) is a *metric space*, where d is the metric, or distance function, on the space X. More often than not, X is used in place of (X, d) when the context is clear.

1. Let $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be given by $d(x, y) = |x - y|$. Using the space below, show that (a), (b), and (c) hold, impyling that d is a metric on $\mathbb{R}$.

2. If we want a metric on $\mathbb{R}^n$, how should we define $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$?

3. Using your response from (2), write an appropriate definition for $V_\epsilon(\mathbf{x})$ for any $\epsilon > 0$ and $\mathbf{x} \in \mathbb{R}^n$.

4. Using (3), write a definition for a limit point for $\mathbb{R}^n$.

5. Write a definition for a convergent sequence in $\mathbb{R}^n$.

Exercise 1. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of nonempty compact subsets of $\mathbb{R}^n$. Assume that $F_{k+1} \subseteq F_k$ for all $k \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

(a) Reread the definition of a compact set in your textbook. Use this to deduce a reasonable definition for a compact subset of $\mathbb{R}^n$.

(b) Complete the proof of Exercise 1. In order to understand why no steps have been provided, consider the following question: does the proof of the Nested Compact Set Property in the textbook depend in any way on the inherent properties of the choice of the metric space itself?

Extra Practice

1. Finish the proof of the Heine-Borel Theorem by proving the infinite case for P. (Here you will need to use the assumption that E is closed!)

2. Let $d^* : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ be given by

$$d^*(\mathbf{p}, \mathbf{q}) = |p_1 - q_1| + |p_2 - q_2|$$

(note that $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$). Prove that d^* is a metric on $\mathbb{R}^2$. We refer to this as the ℓ_1-*metric* on $\mathbb{R}^2$. (The ℓ_p-metric, where $p \in [1, \infty)$, is given by $d_{\ell_p}(\mathbf{p}, \mathbf{q}) = (|p_1 - q_1|^p + |p_2 - q_2|^p)^{1/p}$. When $p = \infty$, $d_{l_\infty}(\mathbf{p}, \mathbf{q}) = \sup\{|p_1 - q_1|, |p_2 - q_2|\}$.)

3. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. We define a set $A \subseteq \mathbb{R}^n$ to be *bounded* if there exists $M \geq 0$ such that $||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \leq M$ for all $\mathbf{x} \in A$.

   (a) Reread the definition of the closure of a set in $\mathbb{R}$. Deduce a reasonable definition for the closure of a set in $\mathbb{R}^n$.

   (b) Prove that if $D \subseteq \mathbb{R}^n$ is bounded, then $\overline{D}$ is bounded.

   (c) Let $A \subseteq \mathbb{R}^n$. Recall the definition of a compact set that you deduced in Exercise 1(a). Prove that A is compact if and only if A is closed and bounded.