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Sudhir R. Ghorpade
Balmohan V. Limaye

A Course in Calculus and Real Analysis

Second Edition

Springer

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Sudhir R. Ghorpade
Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai 400076, India

Balmohan V. Limaye
Department of Mathematics
Indian Institute of Technology Dharwad
Dharwad, Karnataka 580011, India

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Preface

It has been more than a decade since *A Course in Calculus and Real Analysis*, or in short, ACICARA, was first published. The response from students and teachers alike has been gratifying. Over the years, we have also received several comments and suggestions from readers. Thus we kept adding corrections that were pointed out by others or noticed by ourselves to the dynamic errata on the web page of the book. It was also felt that the inclusion of some additional topics would enhance the utility of the book. So when the publisher proposed that we bring out a new edition of ACICARA, we thought the time was ripe and accepted the suggestion. We then worked for about a year and a half to put together this second edition, which contains a substantial amount of new material. The goals of this book were enunciated in the preface to the original edition[1] and essentially remain the same. We briefly recall these below and also mention some salient features of the book.

Our primary goal is to give a self-contained and rigorous introduction to the calculus of functions of one variable, and in fact, a unified exposition of calculus and classical real analysis. At the same time, we have attempted to give due importance to computational techniques and applications of calculus. The topics covered are mostly standard, and the novelty, if any, lies in how we approach them. Throughout this text we have sought to make a distinction between the intrinsic definition of a geometric notion and its analytic characterization. Usually each important result is followed by two kinds of examples: one to illustrate the result and the other to show that a hypothesis cannot be dropped. When a concept is defined it appears in boldface. Definitions are not numbered but can be located using the index. Everything else (propositions, examples, remarks, etc.) is numbered serially in each chapter. The numbering of exercises now indicates the number of the chapter where they occur, or the symbol R, which corresponds to the Revision Exercises given at the end of Chapter 7. The end of a proof is marked by the symbol $\Box$, while the symbol $\diamond$

[1] Preface and the table of contents for the first edition are available on the webpage of the book: `http://www.math.iitb.ac.in/~srg/acicara/`.

marks the end of an example or a remark. Citations to other books and articles appear as a number in square brackets, and the bibliographic details can be found in the list of references. A list of symbols and abbreviations used in the text is given, in the order of their appearance, after the list of references. In the *Notes and Comments* at the end of each chapter, distinctive features of the exposition are mentioned and pointers to some relevant literature and further developments are provided. Exercises for each chapter are divided into two parts: Part A contains problems that are relatively routine, while Part B has problems that are of a theoretical nature or particularly challenging.

The major addition in this edition is a new chapter that discusses sequences and series of real-valued functions of a real variable, and their continuous counterpart, namely, improper integrals depending on a parameter. Another important addition consists of two appendices, of which the first outlines the construction of real numbers, while the second provides a self-contained proof of the fundamental theorem of algebra. Also, a section on cluster points of sequences is added to Chapter 2, and one on Riemann integrals over bounded sets is added to Chapter 6. Besides these, a number of minor revisions have been made in Chapters 1–9. Wherever appropriate, we have given references to *A Course in Multivariable Calculus and Analysis*, or in short, ACIMC, which is a sequel to ACICARA, published by Springer in 2010.

We thank IIT Bombay and IIT Dharwad for enabling us to work on this book. The figures in the book were drawn using PSTricks. We are grateful to Arunkumar Patil and Nirmala Limaye for creating the figures for the first and the second editions. Jonathan Lewin read large parts of preliminary versions of Chapters 9 and 10, and made many valuable comments and suggestions, for which we are grateful to him. We are also very thankful to Anjan Chakrabarty and Venkitesh Iyer, who read the entire manuscript and pointed out several corrections and provided useful suggestions. The editorial staff at Springer, New York, have always been helpful, and we thank all of them, especially Loretta Bartolini and Dimana Tzvetkova for their interest and kind cooperation. We are also grateful to David Kramer for his excellent copyediting. Last, but not least, we would like to thank our families for their support and understanding, without which this book would not have been possible.

We wish to express our collective gratitude to all those who took time to write to us with their comments, suggestions, and corrections in ACICARA. We would appreciate receiving comments on this edition as well. These can be sent by e-mail to either of us at sudhirghorpade@gmail.com and balmohan.limaye@gmail.com. Corrections, modifications, and relevant information will be posted at http://www.math.iitb.ac.in/∼srg/acicara, and we expect to maintain and periodically update this website.

Mumbai and Dharwad, India
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Sudhir Ghorpade
Balmohan Limaye

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1

Numbers and Functions

Let us begin at the beginning. When we learn the script of a language, such as the English language, we begin with the letters of the alphabet A, B, C, . . . ; when we learn the sounds of music, such as those of Western classical music, we begin with the notes Do, Re, Mi, ... Likewise, in mathematics, one begins with $1, 2, 3, \ldots$; these are the **positive integers**, or the **natural numbers**. We shall denote the set of positive integers by $\mathbb{N}$. Thus,

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

These numbers have been known since antiquity. Over the years, the number 0 was conceived[1] and subsequently, the negative integers. Together, these form the set $\mathbb{Z}$ of integers.[2] Thus,

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Quotients of integers are called **rational numbers**. We shall denote the set of all rational numbers by $\mathbb{Q}$. Thus,

$$\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0\right\}.$$

Geometrically, the integers can be represented by points on a line by fixing a base point (signifying the number 0) and a unit distance. Such a line is called the **number line**, and it may be drawn as in Figure 1.1. By suitably subdividing the segment between 0 and 1, we can also represent rational numbers such as $1/n$, where $n \in \mathbb{N}$, and this can, in turn, be used to represent any

[1] The invention of "zero", which also paves the way for the place-value system of enumeration, is widely credited to the Indians. Great psychological barriers had to be overcome when "zero" was being given the status of a legitimate number. For more on this, see the books of Kaplan [46] and Kline [48].

[2] The notation $\mathbb{Z}$ for the set of integers is inspired by the German word *Zahlen* for numbers.

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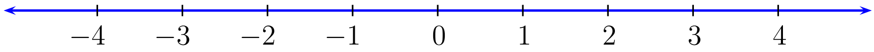

Fig. 1.1. The number line.

rational number by a unique point on the number line. It is seen that the rational numbers spread themselves rather densely on this line. Nevertheless, several gaps remain. For example, the "number" $\sqrt{2}$ can be represented by a unique point between 1 and 2 on the number line using simple geometric constructions, but as we shall see later, this is not a rational number. We are therefore forced to reckon with the so-called *irrational numbers*, which are precisely the "numbers" needed to fill the gaps left on the number line after marking all the rational numbers. The rational numbers and the irrational numbers together constitute the set $\mathbb{R}$, called the set of *real numbers*. The geometric representation of the real numbers as points on the number line naturally implies that there is an *order* among the real numbers. In particular, those real numbers that are greater than 0, that is, which correspond to points to the right of 0, are called *positive*.

1.1 Properties of Real Numbers

A formal definition of real numbers is given in Appendix A. For now, it will suffice to know the following.

There is a set $\mathbb{R}$ (whose elements are called real numbers), which contains the set $\mathbb{Q}$ of all rational numbers (and, in particular, the numbers 0 and 1) such that the following three types of properties are satisfied.

Algebraic Properties

The set $\mathbb{R}$ has the operations of addition (denoted by $+$) and multiplication (denoted by $\cdot$ or by juxtaposition), which extend the usual addition and multiplication of rational numbers and satisfy the following properties.

A1. $a + (b + c) = (a + b) + c$ *and* $a(bc) = (ab)c$ *for all* $a, b, c \in \mathbb{R}$.
A2. $a + b = b + a$ *and* $ab = ba$ *for all* $a, b \in \mathbb{R}$.
A3. *There are distinct elements* $0, 1 \in \mathbb{R}$ *satisfying the following:* $a + 0 = a$ *and* $a \cdot 1 = a$ *for all* $a \in \mathbb{R}$.
A4. *Given any* $a \in \mathbb{R}$, *there is* $a' \in \mathbb{R}$ *such that* $a + a' = 0$. *Further, if* $a \in \mathbb{R}$ *is such that* $a \neq 0$, *then there is* $a^* \in \mathbb{R}$ *such that* $aa^* = 1$.
A5. $a(b + c) = ab + ac$ *for all* $a, b, c \in \mathbb{R}$.

It is interesting to note that several simple properties of real numbers that one is tempted to take for granted can be derived as consequences of the above properties. For example, let us prove that $a \cdot 0 = 0$ for all $a \in \mathbb{R}$. First, by A3,

$0 + 0 = 0$. So, by A5, $a \cdot 0 = a(0 + 0) = a \cdot 0 + a \cdot 0$. Now, by A4, there is a $b' \in \mathbb{R}$ such that $a \cdot 0 + b' = 0$. Thus,

$$0 = a \cdot 0 + b' = (a \cdot 0 + a \cdot 0) + b' = a \cdot 0 + (a \cdot 0 + b') = a \cdot 0 + 0 = a \cdot 0,$$

where the third equality follows from A1 and the last equality follows from A3. This completes the proof! A number of similar properties are listed in the exercises, and we invite the reader to supply the proofs. These show, in particular, that given any $a \in \mathbb{R}$, an element $a' \in \mathbb{R}$ such that $a + a' = 0$ is unique; this element will be called the **negative** or the **additive inverse** of a and is denoted by $-a$. Likewise, if $a \in \mathbb{R}$ and $a \neq 0$, then an element $a^* \in \mathbb{R}$ such that $aa^* = 1$ is unique; this element is called the **reciprocal** or the **multiplicative inverse** of a and is denoted by a^{-1} or by $1/a$. Once all these formalities are understood, we will be free to replace expressions such as

$$a\,(1/b)\,,\;\; a + a,\;\; aa,\;\; (a + b) + c,\;\; (ab)c,\;\; a + (-b),$$

by the corresponding simpler expressions, namely,

$$a/b,\;\; 2a,\;\; a^2,\;\; a + b + c,\;\; abc,\;\; a - b.$$

Here, for instance, it is meaningful and unambiguous to write $a+b+c$, thanks to A1. More generally, given finitely many real numbers $a_1, \ldots, a_n$, the sum $a_1 + \cdots + a_n$ has an unambiguous meaning. To represent such sums, the "sigma notation" can be quite useful. Thus, $a_1 + \cdots + a_n$ is often denoted by $\sum_{i=1}^{n} a_i$ or sometimes simply by $\sum_i a_i$ or $\sum a_i$. Likewise, the product $a_1 \cdots a_n$ of the real numbers $a_1, \ldots, a_n$ has an unambiguous meaning, and it is often denoted by $\prod_{i=1}^{n} a_i$ or sometimes simply by $\prod_i a_i$ or $\prod a_i$. We remark that as a convention, the empty sum is defined to be zero, whereas an empty product is defined to be one. Thus, if $n = 0$, then $\sum_{i=1}^{n} a_i := 0$, whereas $\prod_{i=1}^{n} a_i := 1$.

Order Properties

The set $\mathbb{R}$ contains a subset $\mathbb{R}^+$, called the set of all positive real numbers, satisfying the following properties:

O1. *Given any $a \in \mathbb{R}$, exactly one of the following statements is true:*

$$a \in \mathbb{R}^+; \qquad a = 0; \qquad -a \in \mathbb{R}^+.$$

O2. *If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$ and $ab \in \mathbb{R}^+$.*

Given the existence of $\mathbb{R}^+$, we can define an *order relation* on $\mathbb{R}$ as follows. For $a, b \in \mathbb{R}$, define a to be **less than** b, and write $a < b$, if $b - a \in \mathbb{R}^+$. Sometimes, we write $b > a$ in place of $a < b$ and say that b is **greater than** a. With this notation, it follows from the algebraic properties that $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. Moreover, the following properties are easy consequences of A1–A5 and O1–O2:

(i) Given any $a, b \in \mathbb{R}$, exactly one of the following statements is true.

$$a < b; \qquad a = b; \qquad b < a.$$

(ii) If $a, b, c \in \mathbb{R}$ with $a < b$ and $b < c$, then $a < c$.
(iii) If $a, b, c \in \mathbb{R}$, with $a < b$, then $a+c < b+c$; further, if $c > 0$, then $ac < bc$, whereas if $c < 0$, then $ac > bc$.

Note that it is also a consequence of the properties above that $1 > 0$. Indeed, by (i), either $1 > 0$ or $1 < 0$. If we had $1 < 0$, then we obtain $-1 > 0$ and hence by (iii), $1 = (-1)(-1) > 0$, which is a contradiction. Therefore, $1 > 0$. A similar argument shows that $a^2 > 0$ for every $a \in \mathbb{R}$, $a \neq 0$.

The notation $a \leq b$ is often used to mean that either $a < b$ or $a = b$. Likewise, $a \geq b$ means that $a > b$ or $a = b$.

Let S be a subset of $\mathbb{R}$. We say that S is **bounded above** if there exists $\alpha \in \mathbb{R}$ such that $x \leq \alpha$ for all $x \in S$. Every such α is called an **upper bound** of S. We say that S is **bounded below** if there exists $\beta \in \mathbb{R}$ such that $x \geq \beta$ for all $x \in S$. Every such β is called a **lower bound** of S. The set S is said to be **bounded** if it is bounded above as well as bounded below; otherwise, S is said to be **unbounded**. Note that if $S = \emptyset$, that is, if S is the empty set, then every real number is an upper bound as well as a lower bound of S.

Examples 1.1. (i) The set $\mathbb{N}$ of positive integers is bounded below, and every real number $\beta \leq 1$ is a lower bound of $\mathbb{N}$. However, as we shall see later in Proposition 1.3, the set $\mathbb{N}$ is not bounded above.
(ii) The set S of reciprocals of positive integers, that is, $S := \{1, 1/2, 1/3, \dots\}$, is bounded. Every real number $\alpha \geq 1$ is an upper bound of S, whereas every real number $\beta \leq 0$ is a lower bound of S.
(iii) The set $S := \{x \in \mathbb{Q} : x^2 < 2\}$ is bounded. Here, for example, 2 is an upper bound, while -2 is a lower bound. $\diamond$

Let S be a subset of $\mathbb{R}$. An element $M \in \mathbb{R}$ is called a **supremum** or a **least upper bound** of the set S if

(i) M is an upper bound of S, that is, $x \leq M$ for all $x \in S$, and
(ii) $M \leq \alpha$ for every upper bound α of S.

It is easy to see from the definition that if S has a supremum, then it must be unique; we denote it by $\sup S$. Note that $\emptyset$ does not have a supremum. In practice, showing that a real number M is the supremum of a nonempty set S is equivalent to showing that M is an upper bound of S and that for every $\epsilon > 0$, there is some $a \in S$ such that $M - \epsilon < a \leq M$. (See Figure 1.2.)

An element $m \in \mathbb{R}$ is called an **infimum** or a **greatest lower bound** of the set S if

(i) m is a lower bound of S, that is, $m \leq x$ for all $x \in S$, and
(ii) $\beta \leq m$ for every lower bound β of S.

Fig. 1.2. Illustration of the supremum M and the infimum m of a subset S of $\mathbb{R}$.

Again, it is easy to see from the definition that if S has an infimum, then it must be unique; we denote it by $\inf S$. Note that $\emptyset$ does not have an infimum. In practice, showing that a real number m is the infimum of a nonempty set S is equivalent to showing that m is a lower bound of S and that for every $\epsilon > 0$, there is some $b \in S$ such that $m \le b < m + \epsilon$. (See Figure 1.2.)

For example, if $S := \{x \in \mathbb{R} : 0 < x \le 1\}$, then $\inf S = 0$ and $\sup S = 1$. In this example, $\inf S$ is not an element of S, but $\sup S$ is an element of S.

If the supremum of a set S is an element of S, then it is called the **maximum** of S, and denoted by $\max S$; likewise, if the infimum of S is an element of S, then it is called the **minimum** of S, and denoted by $\min S$.

The last and perhaps the most important property of $\mathbb{R}$ that we shall assume is the following.

Completeness Property

Every nonempty subset of $\mathbb{R}$ *that is bounded above has a supremum.*

The significance of the Completeness Property (which is also known as the Least Upper Bound Property) will become more and more apparent from the results proved in this as well as the subsequent chapters.

Proposition 1.2. *Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Then S has an infimum.*

Proof. Let $T = \{\beta \in \mathbb{R} : \beta \le a \text{ for all } a \in S\}$. Since S is bounded below, T is nonempty, and since S is nonempty, T is bounded above. Hence T has a supremum. Let $m := \sup T$. If $a \in S$, then a is an upper bound of T and hence $m \le a$. This shows that m is a lower bound of S. Further, if $\beta \in \mathbb{R}$ is a lower bound of S, then $\beta \in T$, and hence $\beta \le m$. Thus, $m = \inf S$. $\square$

Proposition 1.3 (Archimedean Property). *Given any $x \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $n > x$. Consequently, there is $\ell \in \mathbb{N}$ such that $-\ell < x$.*

Proof. Let $x \in \mathbb{R}$. Suppose there is no $n \in \mathbb{N}$ such that $n > x$. Then x is an upper bound of $\mathbb{N}$. Hence $\mathbb{N}$ has a supremum. Let $M = \sup \mathbb{N}$. Then $M - 1$ is not an upper bound of $\mathbb{N}$. Hence, there is $n_0 \in \mathbb{N}$ such that $M - 1 < n_0$. But then $n_0 + 1 \in \mathbb{N}$ and $M < n_0 + 1$, which is a contradiction, since M is an upper bound of $\mathbb{N}$. The second assertion about the existence of $\ell \in \mathbb{N}$ with $-\ell < x$ follows by applying the first assertion to $-x$. $\square$

By Proposition 1.3, we see that for every $x \in \mathbb{R}$, there are $\ell, n \in \mathbb{N}$ such that $-\ell < x < n$. The largest among the finitely many integers k satisfying $-\ell \le k \le n$ and also $k \le x$ is called the **integer part** of x and is denoted by $[x]$. Note that the integer part of x is characterized by the following properties:

$$[x] \in \mathbb{Z} \quad \text{and} \quad [x] \le x < [x] + 1.$$

Sometimes, the integer part of x is called the **floor** of x and is denoted by $\lfloor x \rfloor$. In the same vein, the smallest integer $\ge x$ is called the **ceiling** of x and is denoted by $\lceil x \rceil$. For example, $\lfloor \frac{3}{2} \rfloor = \lfloor 1 \rfloor = 1$, whereas $\lceil \frac{3}{2} \rceil = \lceil 2 \rceil = 2$.

Given any $a \in \mathbb{R}$ and $n \in \mathbb{N}$, we define the nth **power** a^n of a to be the product $a \cdots a$ of a with itself taken n times. Further, if $a \ne 0$, then we define $a^0 = 1$ and $a^{-n} = (1/a)^n$. In this way, integral powers of all nonzero real numbers are defined. The following elementary properties are immediate consequences of the algebraic properties and the order properties of $\mathbb{R}$:

(i) $(a_1 a_2)^n = a_1^n a_2^n$ for all $n \in \mathbb{Z}$ and $a_1, a_2 \in \mathbb{R}$ (with $a_1 a_2 \ne 0$ if $n \le 0$).
(ii) $(a^m)^n = a^{mn}$ and $a^{m+n} = a^m a^n$ for all $m, n \in \mathbb{Z}$ and $a \in \mathbb{R}$ (with $a \ne 0$ if $m \le 0$ or $n \le 0$).
(iii) If $n \in \mathbb{N}$ and $b_1, b_2 \in \mathbb{R}$ with $0 \le b_1 < b_2$, then $b_1^n < b_2^n$.

The first two properties above are sometimes called the **laws of exponents** or the **laws of indices** (for integral powers).

Proposition 1.4. *Given any $n \in \mathbb{N}$ and $a \in \mathbb{R}$ with $a \ge 0$, there exists a unique $b \in \mathbb{R}$ such that $b \ge 0$ and $b^n = a$.*

Proof. Uniqueness is clear, since $b_1, b_2 \in \mathbb{R}$ with $0 \le b_1 < b_2$ implies that $b_1^n < b_2^n$. To prove the existence of $b \in \mathbb{R}$ with $b \ge 0$ and $b^n = a$, note that the case $a = 0$ is trivial, and moreover, the case $0 < a < 1$ follows from the case $a > 1$ by taking reciprocals. Thus we will assume that $a \ge 1$. Let

$$S_a = \{x \in \mathbb{R} : x^n \le a\}.$$

Then S_a is a subset of $\mathbb{R}$, which is nonempty (since $1 \in S_a$) and bounded above (by a, for example). Define $b = \sup S_a$. Note that $b \ge 1 > 0$, since $1 \in S_a$. We will show that $b^n = a$ by showing that each of the inequalities $b^n < a$ and $b^n > a$ leads to a contradiction.

Note that by the Binomial Theorem, for every $\delta \in \mathbb{R}$,

$$(b+\delta)^n = b^n + \binom{n}{1} b^{n-1}\delta + \binom{n}{2} b^{n-2}\delta^2 + \cdots + \delta^n.$$

Now, suppose $b^n < a$. Let us define

$$\epsilon := a - b^n, \quad M := \max\left\{\binom{n}{k} b^{n-k} : k = 1, \ldots, n\right\}, \quad \text{and} \quad \delta := \min\left\{1, \frac{\epsilon}{nM}\right\}.$$

Then $M \geq 1$ and $0 < \delta^k \leq \delta$ for $k = 1, 2, \ldots, n$. Therefore,

$$(b+\delta)^n \leq b^n + M\delta + M\delta^2 + \cdots + M\delta^n \leq b^n + nM\delta \leq b^n + \epsilon = a.$$

Hence, $b + \delta \in S_a$. But this is a contradiction, since b is an upper bound of S_a.

Next, suppose $b^n > a$. This time, take $\epsilon = b^n - a$ and define M and δ as before. Similar arguments will show that

$$(b-\delta)^n \geq b^n - nM\delta \geq b^n - \epsilon = a.$$

But $b - \delta < b$, and hence $b - \delta$ cannot be an upper bound of S_a. This means that there is some $x \in S_a$ such that $b - \delta < x$. Therefore, $(b-\delta)^n < x^n \leq a$, which is a contradiction. Thus $b^n = a$. $\square$

Thanks to Proposition 1.4, we define, for $n \in \mathbb{N}$ and $a \in \mathbb{R}$ with $a \geq 0$, the nth **root** of a to be the unique real number b such that $b \geq 0$ and $b^n = a$; we denote this real number by $\sqrt[n]{a}$ or by $a^{1/n}$. In case $n = 2$, we simply write $\sqrt{a}$ instead of $\sqrt[2]{a}$. From the uniqueness of the nth root, the analogues of the properties (i), (ii), and (iii) stated just before Proposition 1.4 can be easily proved for nth roots instead of nth powers. More generally, given any $r \in \mathbb{Q}$, we write $r = m/n$, where $m, n \in \mathbb{Z}$ with $n > 0$, and define $a^r = (a^m)^{1/n}$ for $a \in \mathbb{R}$ with $a > 0$. Note that if also $r = p/q$ for some $p, q \in \mathbb{Z}$ with $q > 0$, then $(a^m)^{1/n} = (a^p)^{1/q}$ for $a \in \mathbb{R}$ with $a > 0$. This follows from the uniqueness of roots, because $m/n = p/q$ implies $mq = pn$ and so $(a^m)^q = (a^p)^n$. Thus, rational powers of positive real numbers are unambiguously defined. In general, for negative real numbers, nonintegral rational powers are not defined in $\mathbb{R}$. For example, $(-1)^{1/2}$ cannot equal any $b \in \mathbb{R}$, since $b^2 \geq 0$. However, in a special case, rational powers of negative real numbers can be defined. More precisely, if $n \in \mathbb{N}$ is odd and $a \in \mathbb{R}$ is positive, then we define $(-a)^{1/n} := -\left(a^{1/n}\right)$. It is easily seen that this is well-defined, and as a result, for $x \in \mathbb{R}$, $x \neq 0$, the rth **power** x^r is defined whenever $r \in \mathbb{Q}$ has an odd denominator, that is, when $r = m/n$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ with n odd. Finally, if r is any positive rational number, then we set $0^r = 0$. However, 0^0 is not defined.[3] For rational powers, wherever they are defined, analogues of the properties (i), (ii), and (iii) stated just before Proposition 1.4 are valid. These analogues can be easily proved using the uniqueness of roots and the corresponding properties of integral powers.

A real number that is not rational is called an **irrational number**. The possibility of taking nth roots provides a useful method to construct several examples of irrational numbers. For instance, we prove below a classical result that $\sqrt{2}$ is an irrational number. We recall first the familiar notion of divisibility in the set $\mathbb{Z}$ of integers. Given $m, n \in \mathbb{Z}$, we say that m **divides** n or that

[3] For an interesting historical discussion about 0^0, see the article by Knuth [50]. See also Exercise 7.15.

m is a **factor** of n (and write $m \mid n$) if $n = \ell m$ for some $\ell \in \mathbb{Z}$. Sometimes, we write $m \nmid n$ if m does not divide n. Two integers m and n are said to be **relatively prime** or **coprime** if the only integers that divide both m and n are 1 and -1. A basic property of rational numbers is that every $r \in \mathbb{Q}$ can be written as

$$r = \frac{p}{q}, \quad \text{where } p, q \in \mathbb{Z},\ q > 0,\ \text{and } p, q \text{ are relatively prime.}$$

The above representation of r is called the **reduced form** of r. The numerator (namely, p) and the denominator (namely, q) of a reduced form of r are uniquely determined by r. (See Exercise 1.41.)

Proposition 1.5. *No rational number has a square equal to* 2*. In other words,* $\sqrt{2}$ *is an irrational number.*

Proof. Suppose $\sqrt{2}$ is rational. Write $\sqrt{2}$ in the reduced form as p/q, where $p, q \in \mathbb{Z}$, $q > 0$, and p, q are relatively prime. Then $p^2 = 2q^2$. This implies that p^2 is even, and so p must be even (because if $p = 2k + 1$ for some $k \in \mathbb{Z}$, then $p^2 = 4k(k+1)+1$ is odd). Thus $p = 2m$ for some $m \in \mathbb{Z}$. But then $2m^2 = q^2$, which implies that q^2 is even, and so q must be even as well. This contradicts the assumption that p, q are relatively prime. Hence $\sqrt{2}$ is not rational. □

More generally, if $d \in \mathbb{N}$ is not the square of an integer, then it is not the square of a rational number, that is, $\sqrt{d}$ is irrational. A proof of this, and, in fact, of a more general fact, is sketched in Exercise 1.42.

The following result shows that the rational numbers as well as the irrational numbers spread themselves rather densely on the number line.

Proposition 1.6. *Given any* $a, b \in \mathbb{R}$ *with* $a < b$*, there exists a rational number as well as an irrational number between* a *and* b*.*

Proof. By Proposition 1.3, we can find $n \in \mathbb{N}$ such that $n > 1/(b-a)$. Let $m = [na] + 1$. Then $m - 1 \le na < m$, and hence

$$a < \frac{m}{n} \le \frac{na+1}{n} = a + \frac{1}{n} < a + (b-a) = b.$$

Thus we have found a rational number (namely, m/n) between a and b. See Figure 1.3. Now, $a + \sqrt{2} < b + \sqrt{2}$, and if r is a rational number between $a + \sqrt{2}$ and $b + \sqrt{2}$, then $r - \sqrt{2}$ is an irrational number between a and b. □

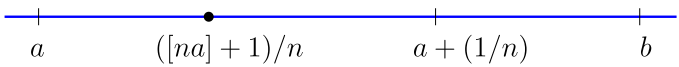

Fig. 1.3. Existence of a rational number between distinct real numbers a and b.

We shall now introduce some basic terminology that is useful in dealing with real numbers. We begin with the general notion of an interval, which can be viewed as a "connected" subset of $\mathbb{R}$ in the sense that it contains every point on the line segment joining any two of its points.

Let $I \subseteq \mathbb{R}$. We say that I is an **interval** if

$$a, b \in I \text{ and } a < b \Longrightarrow x \in I \text{ for all } x \in \mathbb{R} \text{ with } a \le x \le b.$$

A subset of an interval I that is also an interval is called a **subinterval** of I.

Here are some examples of intervals. Let $a, b \in \mathbb{R}$. Then the **open interval** from a to b is defined to be the set

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}.$$

The **closed interval** from a to b is defined to be the set

$$[a, b] := \{x \in \mathbb{R} : a \le x \le b\}.$$

A **half-open interval** or a **semiopen interval** from a to b is defined to be either of the following sets:

$$(a, b] := \{x \in \mathbb{R} : a < x \le b\} \quad \text{and} \quad [a, b) := \{x \in \mathbb{R} : a \le x < b\}.$$

It is clear that an open, closed, or semiopen interval from a to b is an interval in the general sense defined earlier, and also a bounded subset of $\mathbb{R}$. We will refer to these as **bounded intervals**. If I is an interval of the form $[a, b]$, (a, b), $[a, b)$ or $(a, b]$, where $a, b \in \mathbb{R}$ with $a < b$, then a is called the **left (hand) endpoint** of I, while b is called the **right (hand) endpoint** of I. Collectively, a and b are called the **endpoints** of I.

It is often useful to consider the symbols ∞ (called **infinity**) and $-\infty$ (called **minus infinity**), which may be thought as the fictional (right and left) endpoints of the number line. Thus

$$-\infty < a < \infty \quad \text{for all } a \in \mathbb{R}.$$

The set $\mathbb{R}$ together with the additional symbols ∞ and $-\infty$ is sometimes called the set of **extended real numbers**. We use the symbols ∞ and $-\infty$ to define, for $a \in \mathbb{R}$, the following **infinite intervals**:

$$\begin{array}{lll} (-\infty, a) := \{x \in \mathbb{R} : x < a\}, & (-\infty, a] := \{x \in \mathbb{R} : x \le a\}, & \\ (a, \infty) := \{x \in \mathbb{R} : x > a\}, & [a, \infty) := \{x \in \mathbb{R} : x \ge a\}, & (-\infty, \infty) := \mathbb{R}. \end{array}$$

Note that these infinite intervals are intervals in the general sense defined earlier. The infinite intervals are not bounded subsets of $\mathbb{R}$ and may be referred to as **unbounded intervals**. Intervals of the form (a, b), $(-\infty, a)$, (a, ∞), and $(-\infty, \infty)$, where $a, b \in \mathbb{R}$, are collectively referred to as **open intervals**. The next result says that the list of examples of intervals given above is exhaustive.

Proposition 1.7. *Let $I \subseteq \mathbb{R}$ be an interval. If I is bounded, then*

$$I = (a, b), \text{ or } [a, b], \text{ or } (a, b], \text{ or } [a, b) \quad \text{for some } a, b \in \mathbb{R} \text{ with } a \le b,$$

whereas if I is unbounded, then either $I = \mathbb{R}$ or

$$I = (a, \infty), \text{ or } [a, \infty), \text{ or } (-\infty, a), \text{ or } (-\infty, a] \quad \text{for some } a \in \mathbb{R}.$$

Proof. If $I = \emptyset$, then $I = (a, a)$ for every $a \in \mathbb{R}$. Suppose $I \neq \emptyset$. If I is bounded, then let $a := \inf I$ and $b := \sup I$. Note that a, b are well-defined real numbers, because of the Completeness Property and Proposition 1.2. Since I is an interval, it follows that (i) $I = (a, b)$, or (ii) $I = [a, b]$, or (iii) $I = [a, b)$, or (iv) $I = (a, b]$ according as (i) $a \notin I$ and $b \notin I$, or (ii) $a \in I$ and $b \in I$, or (iii) $a \in I$ and $b \notin I$, or (iv) $a \notin I$ and $b \in I$. In case I is neither bounded above nor bounded below, then $[\alpha, \beta] \subseteq I$ for all $\alpha, \beta \in \mathbb{R}$, and hence $I = \mathbb{R}$. Finally, if I is bounded above but not bounded below, then $I = (-\infty, a]$ or $(-\infty, a)$ according as $a \in I$ or $a \notin I$, where $a := \sup I$. Likewise, if I is bounded below but not bounded above, then $I = [a, \infty)$ or (a, ∞), where $a := \inf I$. □

Henceforth, when we write $[a, b]$, (a, b), $[a, b)$ or $(a, b]$, it will be tacitly assumed that a and b are real numbers and $a \leq b$.

Given any real number a, the **absolute value** or the **modulus** of a is denoted by $|a|$ and is defined by

$$|a| := \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

Note that $|a| \geq 0$, $|a| = |-a|$, and $|ab| = |a|\,|b|$ for all $a, b \in \mathbb{R}$. The notion of absolute value can sometimes be useful in describing certain intervals that are symmetric about a point. For example, if $a \in \mathbb{R}$ and ϵ is a positive real number, then $(a - \epsilon, a + \epsilon) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$.

1.2 Inequalities

In this section, we describe and prove some inequalities that will be useful to us in the sequel.

Proposition 1.8 (Basic Inequalities for Absolute Values). *Let $a, b \in \mathbb{R}$. Then*

(i) $|a + b| \leq |a| + |b|$,

(ii) $\big|\,|a| - |b|\,\big| \leq |a - b|$.

Proof. It is clear that $a \leq |a|$ and $b \leq |b|$. Thus, $a + b \leq |a| + |b|$. Likewise, $-(a + b) \leq |a| + |b|$. This implies (i). To prove (ii), note that by (i) we obtain $|a - b| \geq |(a - b) + b| - |b| = |a| - |b|$ and also $|a - b| = |b - a| \geq |b| - |a|$. □

The first inequality in the proposition above is sometimes referred to as the **Triangle Inequality**. An immediate consequence of this is that

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| \quad \text{for all } a_1, \ldots, a_n \in \mathbb{R}.$$

Proposition 1.9 (Basic Inequalities for Powers and Roots). *Let $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$. Then*

(i) $|a^n - b^n| \le nM^{n-1}|a-b|$, *where* $M := \max\{|a|, |b|\}$,
(ii) $|a^{1/n} - b^{1/n}| \le |a-b|^{1/n}$, *provided* $a \ge 0$ *and* $b \ge 0$.

Proof. (i) Consider the identity

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

Take the absolute value of both sides and use Proposition 1.8. The absolute value of the second factor on the right is bounded above by nM^{n-1}. This implies the inequality in (i).

(ii) We may assume, without loss of generality, that $a \ge b$. Let $c = a^{1/n}$ and $d = b^{1/n}$. Then $c - d \ge 0$ and by the Binomial Theorem,

$$c^n = [(c-d) + d]^n = (c-d)^n + \cdots + d^n \ge (c-d)^n + d^n.$$

Therefore,

$$a - b = c^n - d^n \ge (c-d)^n = [a^{1/n} - b^{1/n}]^n.$$

This implies the inequality in (ii). □

We remark that the basic inequality for powers in part (i) of Proposition 1.9 is valid, more generally, for rational powers. (See Exercise 1.48 (i).) As for part (ii), a slightly weaker inequality holds if instead of nth roots, we consider rational roots. (See Exercise 1.48 (ii).)

Proposition 1.10 (Binomial Inequalities). *Let* $n \in \mathbb{N}$. *Then*

$$(1+a)^n \ge 1 + na \quad \textit{for all } a \in \mathbb{R} \textit{ such that } 1 + a \ge 0.$$

More generally, if $a_1, \ldots, a_n \in \mathbb{R}$ *are such that* $1 + a_i \ge 0$ *for* $i = 1, \ldots, n$ *and* $a_1, \ldots, a_n$ *all have the same sign, then*

$$(1+a_1)(1+a_2)\cdots(1+a_n) \ge 1 + (a_1 + \cdots + a_n).$$

Proof. Clearly, the first inequality follows from the second by substituting $a_1 = \cdots = a_n = a$. To prove the second inequality, we use induction on n. The case $n = 1$ is obvious. If $n > 1$ and the result holds for $n - 1$, then

$$(1+a_1)(1+a_2)\cdots(1+a_n) \ge (1+b_n)(1+a_n),$$

where $b_n = a_1 + \cdots + a_{n-1}$. Now, b_n and a_n have the same sign, and hence

$$(1+b_n)(1+a_n) = 1 + b_n + a_n + b_n a_n \ge 1 + b_n + a_n.$$

This proves that $(1+a_1)(1+a_2)\cdots(1+a_n) \ge 1 + (a_1 + \cdots + a_n)$. □

Note that the first inequality in the proposition above is an immediate consequence of the Binomial Theorem when $a \ge 0$, although we have proved it in the more general case $a \ge -1$. We shall refer to the first inequality in Proposition 1.10 as the **Binomial Inequality**. On the other hand, we shall refer to the second inequality in Proposition 1.10 as the **Generalized Binomial Inequality**. We remark that the Binomial Inequality is valid, more generally, for rational powers. (See Exercise 1.48 (iii).)

Proposition 1.11 (A.M.-G.M. Inequality). *Let $n \in \mathbb{N}$ and let $a_1, \ldots, a_n$ be nonnegative real numbers. Then the arithmetic mean of $a_1, \ldots, a_n$ is greater than or equal to their geometric mean, that is,*

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

Moreover, equality holds if and only if $a_1 = \cdots = a_n$.

Proof. If some $a_i = 0$, then the result is obvious. Hence we shall assume that $a_i > 0$ for $i = 1, \ldots, n$. Let $g = (a_1 \cdots a_n)^{1/n}$ and $b_i = a_i/g$ for $i = 1, \ldots, n$. Then $b_1, \ldots, b_n$ are positive and $b_1 \cdots b_n = 1$. We shall now show, using induction on n, that $b_1 + \cdots + b_n \geq n$. This is clear if $n = 1$ or if each of $b_1, \ldots, b_n$ equals 1. Suppose $n > 1$ and not every b_i equals 1. Then $b_1 \cdots b_n = 1$ implies that among $b_1, \ldots, b_n$ there is a number < 1 as well as a number > 1. Relabeling $b_1, \ldots, b_n$ if necessary, we may assume that $b_1 < 1$ and $b_n > 1$. Let $c_1 = b_1 b_n$. Then $c_1 b_2 \cdots b_{n-1} = 1$, and hence by the induction hypothesis $c_1 + b_2 + \cdots + b_{n-1} \geq n - 1$. Now observe that

$$\begin{aligned} b_1 + \cdots + b_n &= (c_1 + b_2 + \cdots + b_{n-1}) + b_1 + b_n - c_1 \\ &\geq (n-1) + b_1 + b_n - b_1 b_n \\ &= n + (1 - b_1)(b_n - 1) \\ &> n, \end{aligned}$$

where the last inequality follows since $b_1 < 1$ and $b_n > 1$. This proves that $b_1 + \cdots + b_n \geq n$, and moreover, the inequality is strict unless $b_1 = \cdots = b_n = 1$. Substituting $b_i = a_i/g$, we obtain the desired result. □

Proposition 1.12 (Cauchy–Schwarz Inequality). *Let $n \in \mathbb{N}$ and let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be any real numbers. Then*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2} \left(\sum_{i=1}^{n} b_i^2 \right)^{1/2}.$$

Moreover, equality holds if and only if $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are proportional to each other, that is, if $a_i b_j = a_j b_i$ for all $i, j = 1, \ldots, n$.

Proof. Let $\alpha := \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2}$ and $\beta := \left(\sum_{i=1}^{n} b_i^2 \right)^{1/2}$. If $\alpha = 0$ or $\beta = 0$, then $a_1 = \cdots = a_n = 0$ or $b_1 = \cdots = b_n = 0$, and the desired inequality as well as the assertion about equality is clear. So assume that $\alpha \neq 0$ and $\beta \neq 0$. Now for all $a, b \in \mathbb{R}$. by considering $(a-b)^2$, we see that $ab \leq \frac{1}{2}(a^2 + b^2)$ and that equality holds if and only if $a = b$. Thus, for each $i = 1, \ldots, n$,

$$\frac{a_i}{\alpha} \cdot \frac{b_i}{\beta} \leq \frac{1}{2} \left(\frac{a_i^2}{\alpha^2} + \frac{b_i^2}{\beta^2} \right), \quad \text{and equality holds if and only if } \frac{a_i}{\alpha} = \frac{b_i}{\beta}.$$

By summing from $i = 1$ to n, we obtain

$$\sum_{i=1}^{n} a_i b_i \le \frac{\alpha\beta}{2}\left(\frac{\alpha^2}{\alpha^2} + \frac{\beta^2}{\beta^2}\right) = \alpha\beta,$$

and moreover, equality holds if and only if $a_i/\alpha = b_i/\beta$ for all $i = 1, \ldots, n$. This yields the desired result. □

Remark 1.13. It can be shown that the difference between the two sides of the Cauchy–Schwarz inequality is given by

$$\Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{j=1}^{n} b_j^2\Big) - \Big(\sum_{i=1}^{n} a_i b_i\Big)^2 = \sum_{1\le i<j\le n} (a_i b_j - a_j b_i)^2.$$

This is known as the **Lagrange Identity**, and it may be viewed as a one-line proof of Proposition 1.12. See also Exercise 1.15 for yet another proof. ◇

1.3 Functions and Their Geometric Properties

The concept of a function is of basic importance in calculus and real analysis. In this section, we begin with an informal description of this concept followed by a precise definition. Next, we outline some basic terminology associated with functions. Later, we give basic examples of functions, including polynomial functions, rational functions, and algebraic functions. Finally, we discuss a number of geometric properties of functions and state some results concerning them. These results are proved here without invoking any of the notions of calculus that are encountered in the subsequent chapters.

Typically, a function is described with the help of an expression in a single parameter (say x), which varies over a stipulated set; this set is called the *domain* of that function. For example, each of the expressions

$$\text{(i) } f(x) := 2x+1,\ x \in \mathbb{R}, \qquad \text{(ii) } f(x) := x^2,\ x \in \mathbb{R},$$
$$\text{(iii) } f(x) := 1/x,\ x \in \mathbb{R},\ x \neq 0, \qquad \text{(iv) } f(x) := x^3,\ x \in \mathbb{R},$$

defines a function f. In (i), (ii), and (iv), the domain is the set $\mathbb{R}$, whereas in (iii), the domain is the set $\mathbb{R} \setminus \{0\} := \{x \in \mathbb{R} : x \neq 0\}$. Note that each of the functions in (i)–(iv) takes its "values" in the set $\mathbb{R}$.

Given a function f having a subset D of $\mathbb{R}$ as its domain and taking values in $\mathbb{R}$, it is often useful to consider the **graph** of f, which is defined as the subset $\{(x, f(x)) : x \in D\}$ of the plane $\mathbb{R}^2$. In other words, this is the set of points on the *curve* given by $y = f(x),\ x \in D$, in the xy-plane. For example, the graphs of the functions in (i) and (ii) are shown in Figure 1.4, while the graphs of the functions in (iii) and (iv) above are shown in Figure 1.5.

In general, we can talk about a function from any set D to any set E, and this associates to each point of D a unique element of E. A formal definition of a function is given below. It may be seen that this, in essence, identifies a function with its graph!

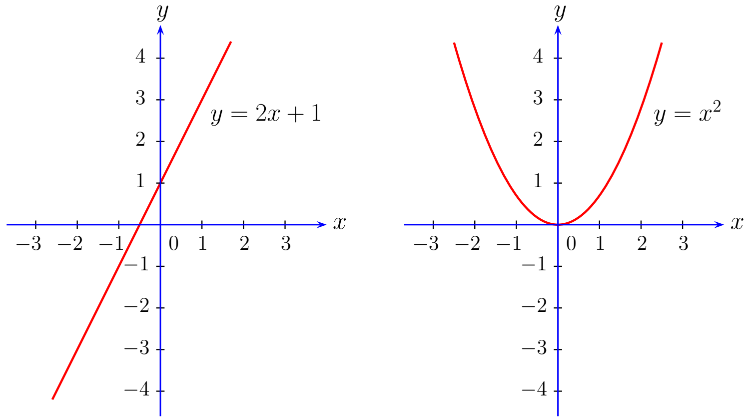

Fig. 1.4. Graphs of $f(x) = 2x + 1$ and $f(x) = x^2$.

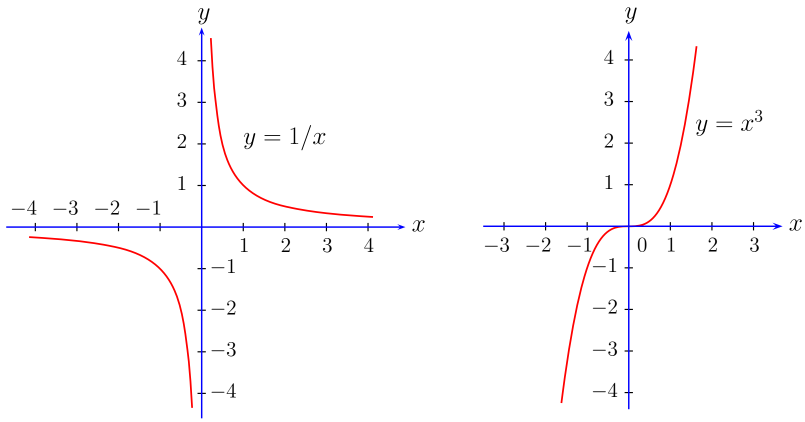

Fig. 1.5. Graphs of $f(x) = 1/x$ and $f(x) = x^3$.

Let D and E be any sets. We denote by $D \times E$ the set of all ordered pairs (x, y), where x varies over elements of D and y varies over elements of E. A **function** from D to E, or a **map** from D to E, is a subset f of $D \times E$ with the property that for each $x \in D$, there is a unique $y \in E$ such that $(x, y) \in f$. The set D is called the **domain** of f, and E is called the **codomain** of f.

Usually, we write $f : D \to E$ to indicate that f is a function from D to E. Also, instead of $(x, y) \in f$, we usually write $y = f(x)$, and call $f(x)$ the **value** of f at x. This may also be indicated by writing $x \mapsto f(x)$, and saying that f **maps** x to $f(x)$. Functions $f : D \to E$ and $g : D \to E$ are said to be **equal**, and we write $f = g$ if $f(x) = g(x)$ for all $x \in D$.

If $f : D \to E$ is a function, then the subset $f(D) := \{f(x) : x \in D\}$ of E is called the **range** of f. We say that f is **onto** or **surjective** if $f(D) = E$.

On the other hand, if f maps distinct points to distinct points, that is, if

$$x_1, x_2 \in D,\ f(x_1) = f(x_2) \Longrightarrow x_1 = x_2,$$

then f is said to be **one-one** or **injective**. If f is both one-one and onto, then it is said to be **bijective** or a **one-to-one correspondence**.

Finite, Countable, and Uncountable Sets

A set D is said to be **finite** if either it is the empty set or there exists $n \in \mathbb{N}$ such that there is a bijective map from $\{1, \ldots, n\}$ to D. A set that is not finite is said to be **infinite**. A set D is said to be **countable** if it is finite or if there is a bijective map from $\mathbb{N}$ to D. A set that is not countable is said to be **uncountable**. A set that is both countable and infinite is sometimes called **countably infinite** or **denumerable**. If D is a countable set and $f : E \to D$ is a bijective map, where either $E = \mathbb{N}$ or $E = \{1, \ldots, n\}$ for some nonnegative integer n, then $f(1), f(2), \ldots$ is called an **enumeration** of D.

For example, the set $\mathbb{Z}$ of all integers is countably infinite, since the map $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(2k-1) = k - 1$ and $f(2k) = -k$ for $k \in \mathbb{N}$ is easily seen to be bijective. Corollary 1.15 gives another important example of a countably infinite set. More examples can be easily generated if one notes that a subset of a countable set is countable. Examples of uncountable sets are given in Exercises 1.44 and 2.30. These show that the set $\mathbb{R}$ of real numbers is uncountable and imply that the set $\mathbb{R}\setminus\mathbb{Q}$ of irrational numbers is uncountable.

Proposition 1.14. *A countably infinite union of countable sets is countable, that is, if $\{A_n : n \in \mathbb{N}\}$ is a family of sets indexed by $\mathbb{N}$ such that A_n is countable for each $n \in \mathbb{N}$, then $\cup_{n\in\mathbb{N}} A_n$ is countable.*

Proof. Let us first assume that the sets $A_1, A_2, \ldots$ are disjoint. For $n \in \mathbb{N}$, let $a_{n,1}, a_{n,2}, \ldots$ be an enumeration of A_n . Writing $\{a_{i,j} : i, j \in \mathbb{N}\}$ as a two-dimensional array and moving diagonally, that is, considering

$$\underbrace{a_{1,1}}\,,\ \underbrace{a_{2,1}, a_{1,2}}\,,\ \underbrace{a_{3,1}, a_{2,2}, a_{1,3}}\,,\ \underbrace{a_{4,1}, a_{3,2}, a_{2,3}, a_{1,4}}\,,\ \ldots,$$

we obtain an enumeration of $\cup_{n\in\mathbb{N}} A_n$. Note that in this enumeration, for all $i, j \in \mathbb{N}$, the element $a_{i,j}$ is included only if it is present in A_i.

In case the sets $A_1, A_2, \ldots$ are not disjoint, let $B_n := A_n \setminus (A_1 \cup \cdots \cup A_{n-1})$ for $n \in \mathbb{N}$ and note that each B_n is countable (being a subset of the countable set A_n), the sets $B_1, B_2, \ldots$ are disjoint, and $\cup_{n\in\mathbb{N}} A_n = \cup_{n\in\mathbb{N}} B_n$. $\square$

Corollary 1.15. *The set $\mathbb{Q}$ of all rational numbers is countable.*

Proof. For $n \in \mathbb{N}$, let $A_n := \{m/n : m \in \mathbb{Z}\}$. For each $n \in \mathbb{N}$, the map from $\mathbb{Z}$ to A_n given by $m \mapsto m/n$ is bijective, and thus A_n is countable. Moreover, $\cup_{n\in\mathbb{N}} A_n = \mathbb{Q}$. Thus the desired result follows from Proposition 1.14. $\square$

Basic Notions Concerning Functions

Given any set D, the function $f : D \to D$ defined by $f(x) := x$ for all $x \in D$ is called the **identity function** on D. Given any sets D and E, a function $f : D \to E$ defined by $f(x) := c$ for all $x \in D$, where c is a fixed element of E, is called a **constant function**. Note that the identity function on D is bijective for every set D, whereas a constant function from a set D to a set E is neither one-one (unless D is a singleton set) nor onto (unless E is a singleton set). To look at more specific examples, note that $f : \mathbb{R} \to \mathbb{R}$ defined by (i) or by (iv) above is bijective, while $f : \mathbb{R} \to [0, \infty)$ defined by (ii) is onto but not one-one, and $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by (iii) is one-one but not onto.

If $f : D \to E$ and $g : D' \to E'$ are functions with $f(D) \subseteq D'$, then the function $h : D \to E'$ defined by $h(x) = g(f(x))$, $x \in D$, is called the **composite** of g with f, and is denoted by $g \circ f$ (read as g composed with f, or as f followed by g).

Note that any function $f : D \to E$ can be made an onto function by replacing the codomain E with its range $f(D)$; more formally, this may be done by looking at the function $\tilde{f} : D \to f(D)$ defined by $\tilde{f}(x) = f(x)$, $x \in D$. In particular, if $f : D \to E$ is one-one, then for every $y \in f(D)$, there exists a unique $x \in D$ such that $f(x) = y$. In this case, we write $x = f^{-1}(y)$. We thus obtain a function $f^{-1} : f(D) \to D$ such that $f^{-1} \circ f$ is the identity function on D and $f \circ f^{-1}$ is the identity function on $f(D)$. We call f^{-1} the **inverse function** of f.

For example, the inverse of $f : \mathbb{R} \to \mathbb{R}$ defined by (i) above is the function $f^{-1} : \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(y) = (y - 1)/2$ for $y \in \mathbb{R}$, whereas the inverse of $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by (iii) above is the function $f^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ given by $f^{-1}(y) = 1/y$ for $y \in \mathbb{R} \setminus \{0\}$.

In general, if a function $f : D \to E$ is not one-one, then we cannot talk about its inverse. However, sometimes it is possible to restrict the domain of a function to a smaller set, and then a "restriction" of f may become injective. For any subset C of D, the **restriction** of f to C is the function $f_{|C} : C \to E$, defined by $f_{|C}(x) = f(x)$ for $x \in C$. For example, if $f : \mathbb{R} \to \mathbb{R}$ is the function defined by (ii), then f is not one-one, but its restriction $f_{|[0,\infty)}$ is one-one and its inverse $g = \left(f_{|[0,\infty)}\right)^{-1}$ is given by $g(y) = \sqrt{y}$ for $y \in [0, \infty)$.

Suppose $D \subseteq \mathbb{R}$ is **symmetric** about the origin, that is, $-x \in D$ whenever $x \in D$. For example, D can be the whole real line $\mathbb{R}$ or an interval of the form $[-a, a]$ or the punctured real line $\mathbb{R} \setminus \{0\}$. A function $f : D \to \mathbb{R}$ is said to be an **even function** if $f(-x) = f(x)$ for all $x \in D$, whereas f is said to be an **odd function** if $f(-x) = -f(x)$ for all $x \in D$. For example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is an even function, whereas $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = 1/x$ and $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ are both odd functions. On the other hand, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ is neither even nor odd.

Geometrically speaking, given $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$, the fact that f is a function corresponds to the property that for every $x_0 \in D$, the vertical line $x = x_0$ in the xy-plane meets the graph of f in exactly one point. Further, the property that f is one-one corresponds to requiring, in addition, that for every $y_0 \in \mathbb{R}$, the horizontal line $y = y_0$ meet the graph of f in at most one point. On the other hand, the property that a point $y_0 \in \mathbb{R}$ is in the range $f(D)$ of f corresponds to requiring, in addition, that the horizontal line $y = y_0$ meet the graph of f in at least one point. In case the inverse function $f^{-1} : f(D) \to \mathbb{R}$ exists, then its graph is obtained from that of f by reflecting along the diagonal line $y = x$. Assuming that D is symmetric, to say that f is an even function corresponds to saying that the graph of f is symmetric with respect to the y-axis, whereas to say that f is an odd function corresponds to saying that the graph of f is symmetric with respect to the origin. Notice that if f is odd and one-one, then its range $f(D)$ is also symmetric, and $f^{-1} : f(D) \to \mathbb{R}$ is an odd function.

A function f is said to be **real-valued** if it takes values in $\mathbb{R}$, that is, if its codomain is $\mathbb{R}$. Henceforth, by a function we shall mean a real-valued function, unless mentioned otherwise. Given a set D and functions $f,\ g : D \to \mathbb{R}$, we can associate new functions $f+g : D \to \mathbb{R}$ and $fg : D \to \mathbb{R}$, called respectively the **sum** and the **product** of f and g defined componentwise, that is, by

$$(f+g)(x) := f(x) + g(x) \quad \text{and} \quad (fg)(x) := f(x)g(x) \quad \text{for } x \in D.$$

In case f is the constant function given by $f(x) = c$ for all $x \in D$, then fg is often denoted by cg and called the **multiple** of g (by c). We often write $f - g$ in place of $f + (-1)g$. In case $g(x) \neq 0$ for all $x \in D$, the **quotient** f/g is defined as a function from D to $\mathbb{R}$ given by $(f/g)(x) = f(x)/g(x)$ for $x \in D$.

Let D be a set. The function $f : D \to \mathbb{R}$ defined by $f(x) := 0$ for all $x \in D$ is called the **zero function** on D, and we may write $f = 0$ to indicate that f is the zero function (on D). Given any functions $f, g : D \to \mathbb{R}$, we write $f \leq g$ or $g \geq f$ to mean that $f(x) \leq g(x)$ for all $x \in D$. In particular, if $f \geq 0$, then we call f a **nonnegative function**. We define $\max(f,g), \min(f,g) : D \to \mathbb{R}$ by

$$\max(f,g)(x) := \max\{f(x), g(x)\} \quad \text{and} \quad \min(f,g)(x) := \min\{f(x), g(x)\}$$

for $x \in D$. Note that $\min(f,g) \leq f \leq \max(f,g)$ and $\min(f,g) \leq g \leq \max(f,g)$. We further define $|f| : D \to \mathbb{R}$ by $|f|(x) := |f(x)|$ for $x \in D$. Observe that

$$\max(f,g) = \frac{1}{2}\left(f + g + |f-g|\right) \quad \text{and} \quad \min(f,g) = \frac{1}{2}\left(f + g - |f-g|\right).$$

Basic Examples of Functions

Polynomials give rise to one of the most basic classes of functions. Let us first review some relevant algebraic facts about polynomials. A **polynomial** (in one variable x) with real coefficients is an expression[4] of the form

[4] For those who consider "expression" a vague term and wonder what x really is, a formal and pedantic definition of a polynomial (in one variable) can be given as

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0,$$

where n is a nonnegative integer and $c_0, c_1, \ldots, c_n$ are real numbers. We call $c_0, c_1, \ldots, c_n$ the **coefficients** of the above polynomial and more specifically, c_i is called the **coefficient** of x^i for $i = 0, 1, \ldots, n$. In case $c_n \neq 0$, the polynomial is said to have **degree** n, and c_n is said to be its **leading coefficient**. A polynomial (in x) whose leading coefficient is 1 is said to be **monic** (in x). Two polynomials are said to be equal if the corresponding coefficients are equal. In particular, $c_n x^n + \cdots + c_1 x + c_0$ is the **zero polynomial** if and only if $c_0 = c_1 = \cdots = c_n = 0$. The degree of the zero polynomial is not defined. If $p(x)$ is a nonzero polynomial, then its degree is denoted by $\deg p(x)$. Polynomials of degrees 1, 2, and 3 are often referred to as **linear**, **quadratic**, and **cubic** polynomials, respectively. Polynomials of degree zero as well as the zero polynomial are called **constant polynomials**. The set of all polynomials in x with real coefficients is denoted by $\mathbb{R}[x]$. Addition and multiplication of polynomials are defined in a natural manner. For example,

$$(x^2 + 2x + 3) + (x^3 + 2x^2 + 5) = x^3 + 3x^2 + 2x + 8$$

and

$$(x^2 + 2x + 3)(x^3 + 2x^2 + 5) = x^5 + 4x^4 + 7x^3 + 11x^2 + 10x + 15.$$

In general, for any $p(x), q(x) \in \mathbb{R}[x]$, the sum $p(x) + q(x)$ and the product $p(x)q(x)$ are polynomials in $\mathbb{R}[x]$. Moreover, if $p(x)$ and $q(x)$ are nonzero, then so is $p(x)q(x)$, and $\deg (p(x)q(x)) = \deg p(x) + \deg q(x)$, whereas $p(x) + q(x)$ is either the zero polynomial or $\deg (p(x) + q(x)) \leq \max\{\deg p(x), \deg q(x)\}$. We say that $q(x)$ **divides** $p(x)$ and write $q(x) \mid p(x)$ if $p(x) = q(x)r(x)$ for some $r(x) \in \mathbb{R}[x]$. We may write $q(x) \nmid p(x)$ if $q(x)$ does not divide $p(x)$.

If $p(x) = c_n x^n + \cdots + c_1 x + c_0 \in \mathbb{R}[x]$ and $\alpha \in \mathbb{R}$, then we denote by $p(\alpha)$ the real number $c_n \alpha^n + \cdots + c_1 \alpha + c_0$ and call it the **evaluation** of $p(x)$ at α. In case $p(\alpha) = 0$, we say that α is a (real) **root** of $p(x)$. There exist polynomials with no real roots. For example, the quadratic polynomial $x^2 + 1$ has no real root, since $\alpha^2 + 1 \geq 1 > 0$ for all $\alpha \in \mathbb{R}$. More generally, if $q(x) = ax^2 + bx + c$ is any quadratic polynomial (so that $a \neq 0$), then

$$4aq(x) = (2ax + b)^2 - (b^2 - 4ac).$$

---

follows. A polynomial with real coefficients is a function from the set $\{0, 1, 2, \ldots\}$ of nonnegative integers into $\mathbb{R}$ such that all except finitely many nonnegative integers are mapped to zero. Thus, the expression $c_n x^n + \cdots + c_1 x + c_0$ corresponds to the function that sends 0 to c_0, 1 to c_1, ..., n to c_n and m to 0 for all $m \in \mathbb{N}$ with $m > n$. In this setup, one can *define* x to be the unique function that maps 1 to 1, and all other nonnegative integers to 0. More generally, we may define x^n to be the function that maps n to 1, and all other integers to 0. We may also identify a real number a with the function that maps 0 to a and all the positive integers to 0. Now, with componentwise addition of functions, $c_n x^n + \cdots + c_1 x + c_0$ has a formal meaning, which is in accord with our intuition!

Consequently, $q(x)$ has a real root if and only if $b^2 - 4ac \geq 0$; indeed, if $b^2 - 4ac \geq 0$, then $\left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$ are the roots of $q(x)$. We call $b^2 - 4ac$ the **discriminant** of the quadratic polynomial $q(x) = ax^2 + bx + c$.

Quotients of polynomials, that is, expressions of the form $p(x)/q(x)$, where $p(x)$ is a polynomial and $q(x)$ is a nonzero polynomial, are called **rational functions**. Two rational functions $p_1(x)/q_1(x)$ and $p_2(x)/q_2(x)$ are regarded as equal if on cross-multiplying, the corresponding polynomials are equal, that is, if $p_1(x)q_2(x) = p_2(x)q_1(x)$. Sums and products of rational functions are defined in a natural manner. Basic facts about polynomials and rational functions are as follows:

(i) If a nonzero polynomial has degree n, then it has at most n roots. Consequently, if $p(x)$ is a polynomial with real coefficients such that $p(\alpha) = 0$ for all α in an infinite subset D of $\mathbb{R}$, then $p(x)$ is the zero polynomial.

(ii) (**Real Fundamental Theorem of Algebra**) Every nonzero polynomial with real coefficients can be factored as a finite product of linear polynomials and quadratic polynomials with negative discriminants.

(iii) (**Partial Fraction Decomposition**) Every rational function can be decomposed as the sum of a polynomial and finitely many rational functions of the form
$$\frac{A}{(x-\alpha)^i} \quad \text{or} \quad \frac{Bx + C}{(x^2 + \beta x + \gamma)^j},$$
where A, B, C and α, β, γ are real numbers and i, j are positive integers.

The factorization in (ii) is, in fact, unique up to a rearrangement of terms. In (iii), we can choose $(x-\alpha)^i$ and $(x^2 + \beta x + \gamma)^j$ to be among the factors of the denominator of the given rational function, and in that case, the partial fraction decomposition is also unique up to a rearrangement of terms. See Exercises 1.54 and 1.61 (and some of the preceding exercises) for a proof of (i) and (iii) above. A proof of (ii) is given in Appendix B. A simple and useful example of partial fraction decomposition is obtained by taking any distinct real numbers α, β and noting that

$$\frac{1}{(x-\alpha)(x-\beta)} = \frac{A_1}{x-\alpha} + \frac{A_2}{x-\beta}, \text{ where } A_1 = \frac{1}{\alpha - \beta} \text{ and } A_2 = \frac{1}{\beta - \alpha}.$$

More generally, if $p(x), q(x)$ are polynomials with $\deg p(x) < \deg q(x)$ and $q(x) = (x - \alpha_1) \cdots (x - \alpha_k)$, where $\alpha_1, \ldots, \alpha_k$ are distinct real numbers, then

$$\frac{p(x)}{q(x)} = \frac{A_1}{x - \alpha_1} + \cdots + \frac{A_r}{x - \alpha_k}, \qquad \text{where } A_i = \frac{p(\alpha_i)}{\prod\limits_{j \neq i} (\alpha_i - \alpha_j)} \text{ for } i = 1, \ldots, r.$$

This, then, is the partial fraction decomposition of $p(x)/q(x)$. In general, the partial fraction decomposition of a rational function can be more complicated. A typical example is the following:

$$\frac{x^5 - 4x^4 + 8x^3 - 13x^2 + 3x - 7}{x^4 - 3x^3 + x^2 + 4} = (x-1) + \frac{2}{(x-2)} - \frac{3}{(x-2)^2} + \frac{2x+1}{(x^2+x+1)}.$$

Now let us revert to functions. Evaluating polynomials at real numbers, we obtain functions known as polynomial functions. Thus, if $D \subseteq \mathbb{R}$, then a **polynomial function** on D is a function $f : D \to \mathbb{R}$ given by

$$f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 \quad \text{for } x \in D,$$

where n is a nonnegative integer and $c_0, c_1, \ldots, c_n$ are real numbers. Alternatively, we can view the polynomial functions on D as the class of functions obtained from the identity function on D and the constant functions from D to $\mathbb{R}$ by the construction of forming sums and products of functions. If D is an infinite set, then it follows from (i) above that a polynomial function on D and the corresponding polynomial determine each other uniquely. In this case, it is possible to identify them with each other, and permit polynomial functions to inherit some of the terminology applicable to polynomials. For example, a polynomial function is said to have **degree** n if the corresponding polynomial has degree n.

Rational functions give rise to real-valued functions on subsets D of $\mathbb{R}$, provided their denominators do not vanish at any point of D. Thus, a **rational function** on D is a function $f : D \to \mathbb{R}$ such that $f(x) = p(x)/q(x)$ for $x \in D$, where p and q are polynomial functions on D with $q(x) \neq 0$ for all $x \in D$.

Polynomial functions and rational functions (on $D \subseteq \mathbb{R}$) are special cases of *algebraic functions* (on D), which are defined as follows. A function $f : D \to \mathbb{R}$ is said to be an **algebraic function** if $y = f(x)$ satisfies a polynomial equation whose coefficients are polynomials in x, that is,

$$p_n(x)y^n + p_{n-1}(x)y^{n-1} + \cdots + p_1(x)y + p_0(x) = 0 \quad \text{for } x \in D,$$

where $n \in \mathbb{N}$ and $p_0(x), p_1(x), \ldots, p_n(x)$ are polynomials such that $p_n(x)$ is a nonzero polynomial. For example, the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) := \sqrt[n]{x}$ is an algebraic function since $y = f(x)$ satisfies the polynomial equation $y^n - x = 0$ for $x \in [0, \infty)$. It can be shown[5] that sums, products, and quotients of algebraic functions are algebraic. Here is a simple example that illustrates why such a property is true. Consider the sum $y = \sqrt{x} + \sqrt{x+1}$ of functions that are clearly algebraic. To show that this sum is algebraic, write $y - \sqrt{x} = \sqrt{x+1}$, square both sides, and simplify to get $y^2 - 1 = 2y\sqrt{x}$; now squaring once again, we obtain the equation $y^4 - 2(1+2x)y^2 + 1 = 0$, which is of the desired type. Algebraic functions also have the property that their radicals are algebraic. More precisely, if $f : D \to \mathbb{R}$ is algebraic and $f(x) \geq 0$ for all $x \in D$, then any root of f is algebraic, that is, for every $d \in \mathbb{N}$, the function $g : D \to \mathbb{R}$ defined by $g(x) := f(x)^{1/d}$ is algebraic. This follows simply by changing y to y^d in the algebraic equation satisfied

[5] A general proof of this requires some ideas from algebra. The interested reader is referred to [18] or [38].

by $y = f(x)$, and noting that the resulting equation is satisfied by $y = g(x)$. It is seen, therefore, that algebraic functions constitute a fairly large class of functions, which is *closed* under the basic operations of algebra. This class may be viewed as a basic stockpile of functions from which various examples can be drawn. A real-valued function that is not algebraic is called a **transcendental function**. The transcendental functions are also important in calculus, and we will discuss them in greater detail in Chapter 7.

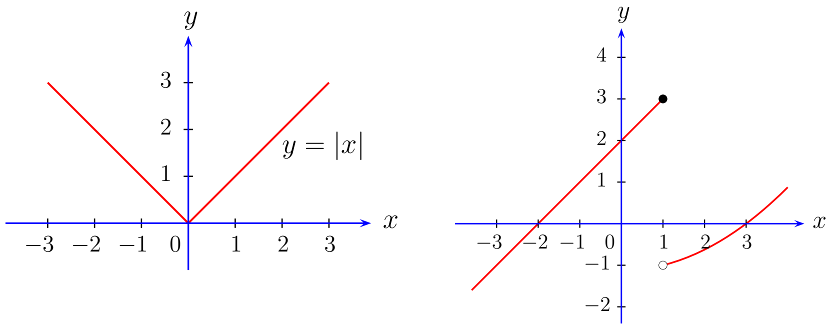

Fig. 1.6. Graphs of $f(x) := |x|$ and $f(x) := \begin{cases} x+2 & \text{if } x \le 1, \\ (x^2-9)/8 & \text{if } x > 1. \end{cases}$

Apart from algebra, a fruitful way to construct new functions is by piecing together known functions. For example, consider $f : \mathbb{R} \to \mathbb{R}$ defined by either of the following:

$$\text{(i) } f(x) := |x| = \begin{cases} x & \text{if } x \ge 0, \\ -x & \text{if } x < 0; \end{cases} \qquad \text{(ii) } f(x) := \begin{cases} x+2 & \text{if } x \le 1, \\ (x^2-9)/8 & \text{if } x > 1. \end{cases}$$

The graphs of these functions may be drawn as in Figure 1.6. Taking the integer part, or floor, of a real number gives rise to a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := [x]$, which we refer to as the **integer part function** or the **floor function**. Likewise, $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) := \lceil x \rceil$ is called the **ceiling function**. These two functions may also be viewed as examples of functions obtained by piecing together known functions, and their graphs are shown in Figure 1.7. As seen in Figures 1.6 and 1.7, it is often the case that the graphs of functions defined by piecing together different functions look broken or have beak-like edges. Also, in general, such functions are not algebraic. Nevertheless, such functions can be quite useful in constructing examples of unusual type.

Remark 1.16. Polynomials (in one variable) are analogous to integers. Likewise, rational functions are analogous to rational numbers. Algebraic functions

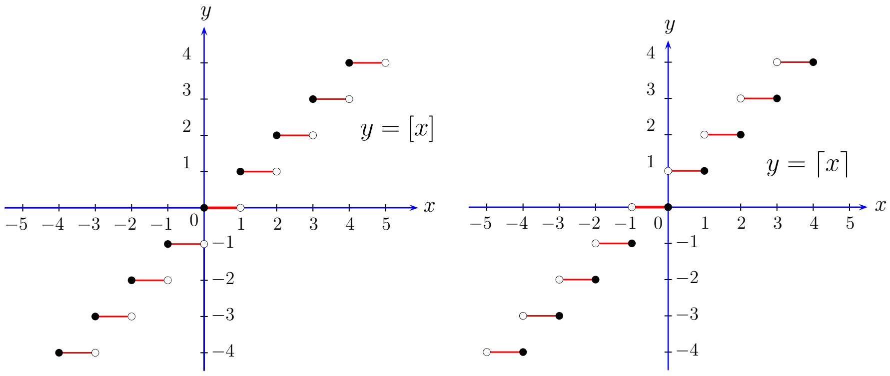

Fig. 1.7. Graphs of the integer part function and the ceiling function.

and transcendental functions also have analogues in arithmetic, which are defined as follows. A real number α is called an **algebraic number** if it satisfies a nonzero polynomial equation with integer coefficients. Numbers that are not algebraic are called **transcendental numbers**. For example, it can be easily seen that $\sqrt{2}$, $\sqrt{3}$, $\sqrt[5]{7}$, $\sqrt{2}+\sqrt{3}$ are algebraic numbers. Also, every rational number is an algebraic number. On the other hand, it is not easy to give concrete examples of transcendental numbers. Those interested are referred to the book of Baker [8] for the proof of transcendence of several well-known numbers. ◇

We shall now discuss a number of geometric properties of real-valued functions. We begin with the notion of bounded functions that are defined on arbitrary sets. Subsequently, we discuss notions that are applicable to real-valued functions defined on certain subsets of $\mathbb{R}$.

Bounded Functions

The notion of a bounded set has an analogue in the case of functions. In effect, we use for functions the terminology that is applicable to their range. More precisely, we make the following definitions.

Let D be a set and $f : D \to \mathbb{R}$ be a function. We say that

- f is **bounded above** on D if there is $\alpha \in \mathbb{R}$ such that $f(x) \le \alpha$ for all $x \in D$. Every such α is called an **upper bound** for f.
- f is **bounded below** on D if there is $\beta \in \mathbb{R}$ such that $f(x) \ge \beta$ for all $x \in D$. Every such β is called a **lower bound** for f.
- f is **bounded** on D if it is bounded above on D and also bounded below on D.

Notice that f is bounded on D if and only if there is $\gamma \in \mathbb{R}$ such that $|f(x)| \le \gamma$ for all $x \in D$. Every such γ is called a **bound** for the absolute value of f. Geometrically speaking, f is bounded above means that the graph of f lies below some horizontal line, while f is bounded below means that its graph lies above some horizontal line.

For example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := -x^2$ is bounded above on $\mathbb{R}$, while $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$ is bounded below on $\mathbb{R}$. However, neither of these functions is bounded on $\mathbb{R}$. On the other hand, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2/(x^2+1)$ gives an example of a function that is bounded on $\mathbb{R}$. For this function, we see readily that $0 \le f(x) < 1$ for all $x \in \mathbb{R}$. The bounds 0 and 1 are, in fact, optimal in the sense that

$$\inf\{f(x) : x \in \mathbb{R}\} = 0 \quad \text{and} \quad \sup\{f(x) : x \in \mathbb{R}\} = 1.$$

Of these, the first equality is obvious, since $f(x) \ge 0$ for all $x \in \mathbb{R}$ and $f(0) = 0$. To see the second equality, let α be an upper bound such that $\alpha < 1$. Then $1 - \alpha > 0$, and so we can find $n \in \mathbb{N}$ such that

$$\frac{1}{n} < 1 - \alpha \quad \text{and hence} \quad f(\sqrt{n-1}) = \frac{n-1}{n} = 1 - \frac{1}{n} > \alpha,$$

which is a contradiction. This shows that $\sup\{f(x) : x \in \mathbb{R}\} = 1$. Thus there is a qualitative difference between the infimum of (the range of) f, which is attained, and the supremum, which is not attained. This suggests the following general definition.

Let D be a set and $f : D \to \mathbb{R}$ a function. We say that

- f **attains its upper bound** on D if there is $c \in D$ such that
$$\sup\{f(x) : x \in D\} = f(c),$$
- f **attains its lower bound** on D if there is $d \in D$ such that
$$\inf\{f(x) : x \in D\} = f(d),$$
- f **attains its bounds** on D if it attains its upper bound on D and also attains its lower bound on D.

In case f attains its upper bound, we may write $\max\{f(x) : x \in D\}$ in place of $\sup\{f(x) : x \in D\}$. Likewise, if f attains its lower bound, then "inf" may be replaced by "min".

Monotonicity, Convexity, and Concavity

Monotonicity is a geometric property of a real-valued function defined on a subset of $\mathbb{R}$ that corresponds to its graph being increasing or decreasing. For example, consider Figure 1.8, where the graph on the left is increasing, while that on the right is decreasing.

A formal definition is as follows. Let $D \subseteq \mathbb{R}$ be such that D contains an interval I, and let $f : D \to \mathbb{R}$ be a function. We say that

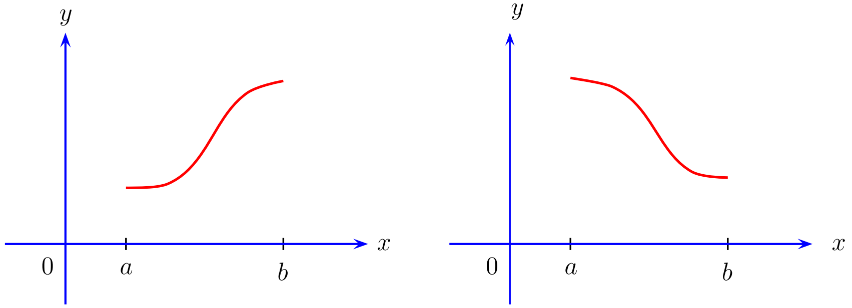

Fig. 1.8. Typical graphs of increasing and decreasing functions on $I = [a, b]$.

- f is (**monotonically**) **increasing** on I if

$$x_1, x_2 \in I,\ x_1 < x_2 \Longrightarrow f(x_1) \le f(x_2),$$

- f is (**monotonically**) **decreasing** on I if

$$x_1, x_2 \in I,\ x_1 < x_2 \Longrightarrow f(x_1) \ge f(x_2),$$

- f is **monotonic** on I if f is monotonically increasing on I or f is monotonically decreasing on I.

Next, we discuss more subtle properties of a function, known as convexity and concavity. Geometrically, these notions are easily described. A function is convex if the line segment joining any two points on its graph lies on or above the graph. A function is concave if any such line segment lies on or below the graph. An illustration is given in Figure 1.9. To formulate a more precise definition, one should first note that convexity or concavity can be defined relative to an interval I contained in the domain of a function f, and also that given any $x_1, x_2 \in I$ with $x_1 < x_2$, the equation of the line joining the corresponding points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph of f is given by

$$y - f(x_1) = m(x - x_1), \qquad \text{where} \qquad m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

So, once again let $D \subseteq \mathbb{R}$ be such that D contains an interval I, and let $f : D \to \mathbb{R}$ be a function. We say that

- f is **convex on** I or **concave upward on** I if

$$x_1, x_2, x \in I,\ x_1 < x < x_2 \Longrightarrow f(x) - f(x_1) \le \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1),$$

- f is **concave on** I or **concave downward on** I if

$$x_1, x_2, x \in I,\ x_1 < x < x_2 \Longrightarrow f(x) - f(x_1) \ge \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

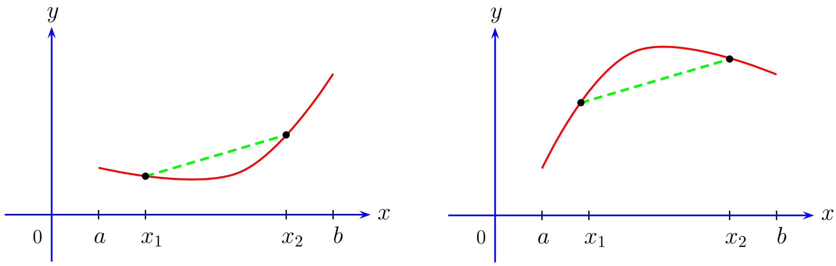

Fig. 1.9. Typical graphs of convex and concave functions on $I = [a, b]$.

An alternative way to formulate the definitions of convexity and concavity is as follows. First, note that for all $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, the points x between x_1 and x_2 are of the form $(1-t)x_1 + tx_2$ for some $t \in (0,1)$; in fact, t and x determine each other uniquely, since

$$x = (1-t)x_1 + tx_2 \iff t = \frac{x - x_1}{x_2 - x_1}.$$

Substituting this in the definition above, we see that f is convex on I if (and only if) $f((1-t)x_1 + tx_2) \le (1-t)f(x_1) + tf(x_2)$ for all $x_1, x_2 \in I$ with $x_1 < x_2$ and all $t \in (0,1)$. Of course, the roles of t and $1-t$ can be readily reversed, and with this in view, one need not assume that $x_1 < x_2$. Thus, f is convex on I if (and only if)

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2) \quad \text{for all } x_1, x_2 \in I \text{ and } t \in (0,1).$$

Similarly, f is concave on I if (and only if)

$$f(tx_1 + (1-t)x_2) \ge tf(x_1) + (1-t)f(x_2) \quad \text{for all } x_1, x_2 \in I \text{ and } t \in (0,1).$$

Examples 1.17. (i) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$ is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$. Indeed, if $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, then $(x_2^2 - x_1^2) = (x_2 - x_1)(x_2 + x_1)$ is positive if $x_1, x_2 \in [0, \infty)$ and negative if $x_1, x_2 \in (-\infty, 0]$. Further, f is convex on $\mathbb{R}$. To see this, note that if $x_1, x_2, x \in \mathbb{R}$ with $x_1 < x < x_2$, then $x - x_1 > 0$ and

$$x^2 - x_1^2 = (x + x_1)(x - x_1) < (x_2 + x_1)(x - x_1) = \frac{(x_2^2 - x_1^2)}{(x_2 - x_1)}(x - x_1).$$

(ii) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^3$ is increasing on $(-\infty, \infty)$. Indeed, if $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, that is, $x_2 - x_1 > 0$, then

$$x_2^3 - x_1^3 = (x_2 - x_1)(x_2^2 + x_2x_1 + x_1^2) = (x_2 - x_1)\left(\left(x_2 + \frac{x_1}{2}\right)^2 + \frac{3}{4}x_1^2\right) > 0.$$

Further, f is concave on $(-\infty, 0]$ and convex on $[0, \infty)$. To see this, first note that if $x_1 < x < x_2 \le 0$, then $x - x_1 > 0$, $x^2 > x_2^2$, and $x_1 x > x_1 x_2$, and so $x^3 - x_1^3 = (x^2 + x_1 x + x_1^2)(x - x_1)$ satisfies

$$x^3 - x_1^3 > (x_2^2 + x_1x_2 + x_1^2)(x - x_1) = \frac{(x_2^3 - x_1^3)}{(x_2 - x_1)}(x - x_1).$$

Also, if $0 \le x_1 < x < x_2$, then $x - x_1 > 0$, $x^2 < x_2^2$, and $x_1x \le x_1x_2$, and so in this case $x^3 - x_1^3 = (x^2 + x_1x + x_1^2)(x - x_1)$ satisfies

$$x^3 - x_1^3 < (x_2^2 + x_1x_2 + x_1^2)(x - x_1) = \frac{(x_2^3 - x_1^3)}{(x_2 - x_1)}(x - x_1).$$

(iii) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := |x|$ is decreasing on $(-\infty, 0]$, increasing on $[0, \infty)$, and convex on $\mathbb{R} = (-\infty, \infty)$. Indeed, the first two assertions about the monotonicity of f are obvious. The convexity of f is easily verified by noting that $f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2)$ for all $x_1, x_2 \in \mathbb{R}$ and $t \in (0, 1)$. ◇

Remark 1.18. In each of the examples above, we have in fact obtained a stronger conclusion than was needed to satisfy the definitions of increasing/decreasing and convex/concave functions. Namely, instead of the inequalities "$\le$" and "$\ge$", we obtained the corresponding strict inequalities "$<$" and "$>$". If one wants to emphasize this, the terminology of **strictly increasing**, **strictly decreasing**, **strictly convex**, or **strictly concave**, is employed. The definitions of these concepts are obtained by changing the inequality "$\le$" or "$\ge$" to the corresponding strict inequality "$<$" or "$>$" in the respective definitions. Also, we say that a function is **strictly monotonic** if it is strictly increasing or strictly decreasing. ◇

Local Extrema and Points of Inflection

Points where the graph of a function has peaks or dips, or where the convexity changes to concavity (or vice versa), are of great interest in calculus and its applications. We shall now formally introduce the terminology used in describing this type of behavior.

Let $D \subseteq \mathbb{R}$ and $c \in D$ be such that D contains $(c-r, c+r)$ for some $r > 0$. Given $f : D \to \mathbb{R}$, we say that

- f has a **local maximum** at c if there is $\delta > 0$ with $\delta \le r$ such that $f(x) \le f(c)$ for all $x \in (c - \delta, c + \delta)$,
- f has a **local minimum** at c if there is $\delta > 0$ with $\delta \le r$ such that $f(x) \ge f(c)$ for all $x \in (c - \delta, c + \delta)$.
- f has a **local extremum** at c if f has a local maximum at c or a local minimum at c,
- c is a **point of inflection** for f if there is $\delta > 0$ with $\delta \le r$ such that f is convex in $(c - \delta, c)$, while f is concave in $(c, c + \delta)$, or vice versa, that is, f is concave in $(c - \delta, c)$, while f is convex in $(c, c + \delta)$.

It may be noted that the terms **local maxima**, **local minima**, and **local extrema** are often used as plural forms of **local maximum**, **local minimum**, and **local extremum**, respectively.

Examples 1.19. (i) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := -x^2$ has a local maximum at 0.

(ii) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := |x|$ has a local minimum at 0. (See Figure 1.6.)

(iii) The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^3$ has a point of inflection at 0. (See Figure 1.5.) ◇

It is easy to see that if $D \subseteq \mathbb{R}$ contains an open interval of the form $(c - r, c + r)$ for some $r > 0$ and $f : D \to \mathbb{R}$ is a function such that f is decreasing on $(c - \delta, c]$ and increasing on $[c, c + \delta)$, for some $0 < \delta \le r$, then f must have a local minimum at c. But as the following example shows, the converse of this need not be true.

Example 1.20. Consider the function $f : [-1, 1] \to \mathbb{R}$ that is obtained by piecing together infinitely many zigzags as follows. On $[1/(n + 1), 1/n]$, we define f to be such that its graph is formed by the line segments PM and MQ, where P, Q are the points on the line $y = x$ whose x-coordinates are $1/n + 1$ and $1/n$, respectively, while M is the point on the line $y = 2x$ whose x-coordinate is the midpoint of the x-coordinates of P and Q. More precisely, for $n \in \mathbb{N}$, we define

$$f(x) := \begin{cases} 2(n+1)x - \dfrac{2n+1}{n+1} & \text{if } \dfrac{1}{n+1} \le x \le \dfrac{2n+1}{2n(n+1)}, \\ -2nx + \dfrac{2n+1}{n} & \text{if } \dfrac{2n+1}{2n(n+1)} \le x \le \dfrac{1}{n}. \end{cases}$$

Further, let $f(0) := 0$ and $f(x) := f(-x)$ for $x \in [-1, 0)$. The graph of this piecewise linear function can be drawn as in Figure 1.10. It is clear that f has a local minimum at 0. However, there is no $\delta > 0$ such that f is decreasing on $(-\delta, 0]$ and f is increasing on $[0, \delta)$.

A similar comment holds for the notion of local maximum. ◇

Remark 1.21. As before, in each of the examples above, the given function satisfies the property mentioned in a strong sense. For example, for $f : \mathbb{R} \to \mathbb{R}$ defined by (i), we not only have $f(x) \le f(0)$ in an interval around 0 but in fact, $f(x) < f(0)$ for each point x, except 0, in an interval around 0. To indicate this, the terminology **strict local maximum**, **strict local minimum**, **strict local extremum**, and **strict point of inflection** can be employed. The first two of these notions are defined by changing in 1 and 2 above the inequalities "$\le$" and "$\ge$" to the corresponding strict inequalities "$<$" and "$>$", and the condition "$x \in (c - \delta, c + \delta)$" by the condition "$x \in (c - \delta, c + \delta)$, $x \ne c$". To say that f has a strict local extremum at c just means that it has a strict local maximum or a strict local minimum at c. Finally, the notion of a strict point of inflection is defined by adding "strictly" before the words "convex" and "concave" in the above definition of a point of inflection. ◇

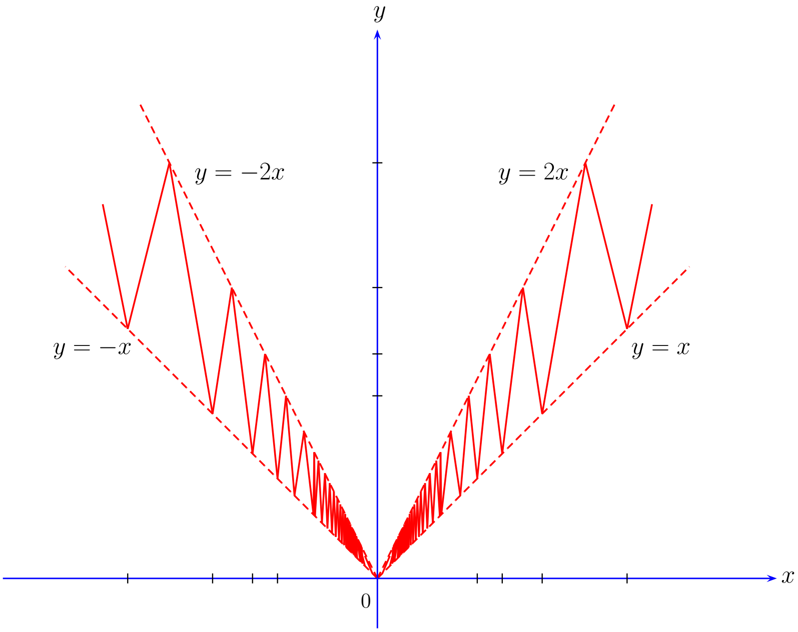

Fig. 1.10. Graph of the piecewise linear zigzag function in Example 1.20.

In Examples 1.17 and 1.19, which illustrate the geometric phenomena of increasing/decreasing functions, convexity/concavity, local maxima/minima, and points of inflection, the verification of the corresponding property has been fairly easy. In fact, we have looked at what are possibly the simplest functions that are prototypes of the above phenomena. But even here, the proofs of convexity or concavity in the case of functions given by x^2 and x^3 required some effort. As one considers functions that are more complicated, the verification of all these geometric properties can become increasingly difficult. Later in this book, we shall describe some results from calculus that can make such verification significantly simpler for a large class of functions. It is, nevertheless, useful to remember that the definition as well as the intuitive idea behind these properties is geometric, and as such, it is independent of the notions from calculus that we shall encounter in the subsequent chapters.

Intermediate Value Property

We now consider a geometric property of a function that corresponds, intuitively, to the idea that the graph of a function has no "breaks" or "disconnections". For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) := 2x + 1$ or by $f(x) := x^2$ or by $f(x) := |x|$, then the graph of f has apparently no "breaks". (See Figures 1.4 and 1.6.) But if $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) := \begin{cases} x+2 & \text{if } x \le 1, \\ (x^2-9)/8 & \text{if } x > 1, \end{cases}$$

then the graph of f does seem to have a "break". (See Figure 1.6.) This intuitive condition on the graph of a real-valued function f can be formulated by stating that every intermediate value of f is attained by f. More precisely, we make the following definition.

Let $D \subseteq \mathbb{R}$ and let I be an interval such that $I \subseteq D$. Also let $f : D \to \mathbb{R}$ be a function. We say that f has the **Intermediate Value Property**, or in short, f has the **IVP**, on I if for all $a, b \in I$ with $a < b$ and $r \in \mathbb{R}$,

$$r \text{ lies between } f(a) \text{ and } f(b) \Longrightarrow r = f(x) \text{ for some } x \in [a, b].$$

Note that if f has the IVP on I, and J is a subinterval of I, then f has the IVP on J.

Proposition 1.22. *Let I be an interval and let $f : I \to \mathbb{R}$ be a function. Then*

$$f \text{ has the IVP on } I \Longrightarrow f(I) \text{ is an interval.}$$

Proof. Let $c, d \in f(I)$ with $c < d$. Then $c = f(a)$ and $d = f(b)$ for some $a, b \in I$. If $r \in (c, d)$, then by the IVP for f on I, there is $x \in I$ between a and b such that $f(x) = r$. Hence $r \in f(I)$. It follows that $f(I)$ is an interval. $\square$

Remark 1.23. The converse of the above result is true for monotonic functions. To see this, suppose I is an interval and $f : I \to \mathbb{R}$ is a monotonic function such that $f(I)$ is an interval. Let $a, b \in I$ and $r \in \mathbb{R}$ be such that $a < b$ and r is between $f(a)$ and $f(b)$. If $r = f(a)$ or $r = f(b)$, then there is nothing to prove. Suppose $r \ne f(a)$ and $r \ne f(b)$. Since $f(I)$ is an interval, there is $x \in I$ such that $r = f(x)$. Now, if f is monotonically increasing on I, then $f(a) \le f(b)$; thus, $f(a) < f(x) < f(b)$, and consequently, $a \le x \le b$. Likewise, if f is monotonically decreasing on I, then $f(a) > f(x) > f(b)$, and consequently, $a \le x \le b$. This shows that f has the IVP on I.

However, in general, the converse of the result in Proposition 1.22 is not true. For example, if $I = [0, 4]$ and $f : I \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 2, \\ 6 - x & \text{if } 2 \le x \le 4, \end{cases}$$

then $f(I) = I$ is an interval, but f does not have the IVP on I. The latter follows, for example, since 3 lies between $0 = f(0)$ and $4 = f(2)$, but $3 \ne f(x)$ for any $x \in [0, 2]$. Note that in this example, f is one-one, but not monotonic on I. Also, $f([1, 3]) = [1, 2) \cup [3, 4]$ is not an interval $\diamond$

Proposition 1.24. *Let I be an interval and let $f : I \to \mathbb{R}$ be a function. Then*

$$f \text{ has the IVP on } I \Longleftrightarrow f(J) \text{ is an interval for every subinterval } J \text{ of } I.$$

Proof. The implication $\Longrightarrow$ follows from applying Proposition 1.22 to restrictions of f to subintervals of I. Conversely, suppose $f(J)$ is an interval for every subinterval J of I. Let $a, b \in I$ with $a < b$ and let $r \in \mathbb{R}$ lie between $f(a)$ and $f(b)$. Then $J := [a, b]$ is a subinterval of I and so $f(J)$ is an interval containing $f(a)$ and $f(b)$. Hence $r = f(x)$ for some $x \in J$. Thus, f has the IVP on I. $\square$

The relation between (strict) monotonicity and the IVP is made clearer by the following result.

Proposition 1.25. *Let I be an interval and let $f : I \to \mathbb{R}$ be a function. Then f is one-one and has the IVP on I if and only if f is strictly monotonic and $f(I)$ is an interval. In this case, $f^{-1} : f(I) \to \mathbb{R}$ is strictly monotonic and has the IVP on $f(I)$.*

Proof. Assume that f is one-one and has the IVP on I. By Proposition 1.22, $f(I)$ is an interval. Suppose f is not strictly monotonic on I. Then there are $x_1, x_2 \in I$ and $y_1, y_2 \in I$ such that

$$x_1 < x_2 \text{ but } f(x_1) \geq f(x_2) \quad \text{and} \quad y_1 < y_2 \text{ but } f(y_1) \leq f(y_2).$$

Let $a := \min\{x_1, y_1\}$ and $b := \max\{x_2, y_2\}$. Note that $a < b$. Now, suppose $f(a) \leq f(b)$. Then $f(x_1) \leq f(b)$, because otherwise, $f(x_1) > f(b) \geq f(a)$, and hence by the IVP of f on I, there is $z_1 \in [a, x_1]$ such that $f(z_1) = f(b)$. But since $z_1 \leq x_1 < x_2 \leq b$, this contradicts the assumption that f is one-one. Thus, $f(x_2) \leq f(x_1) \leq f(b)$. Again, by the IVP of f on I, there is $w_1 \in [x_2, b]$ such that $f(w_1) = f(x_1)$. But since $x_1 < x_2 \leq w_1$, this contradicts the assumption that f is one-one. Next, suppose $f(b) < f(a)$. Then $f(y_2) \leq f(a)$, because otherwise, $f(y_2) > f(a) > f(b)$, and hence by the IVP of f on I, there is $z_2 \in [y_2, b]$ such that $f(z_2) = f(a)$. But since $a \leq y_1 < y_2 \leq z_2$, this contradicts the assumption that f is one-one. Thus, $f(y_1) \leq f(y_2) \leq f(a)$. Again, by the IVP of f on I, there is $w_2 \in [a, y_1]$ such that $f(w_2) = f(y_2)$. But since $w_2 \leq y_1 < y_2$, this contradicts the assumption that f is one-one. It follows that f is strictly monotonic on I.

To prove the converse, assume that f is strictly monotonic on I and $f(I)$ is an interval. Then we have seen in Remark 1.23 above that f has the IVP on I. Also, strict monotonicity obviously implies that f is one-one.

Finally, suppose f is one-one and has the IVP on I. Then as seen above, f is strictly monotonic on I. This implies readily that f^{-1} is strictly monotonic on $f(I)$. Also, $f(I)$ is an interval and so is $I = f^{-1}(f(I))$. Hence by the equivalence proved above, f^{-1} has the IVP on $f(I)$. $\square$

Notes and Comments

It is often said, and believed, that mathematics is an exact science, and all the terms one uses in mathematics are always precisely defined. This is of course *true to a large extent. But one should realize that it is impossible to precisely define everything. Indeed, to define one term, we would have to use another, to define which we would have to use yet another, and so on. Since our vocabulary is finite (check!), we would soon land in a vicious circle! Mathematicians find a way out of this dilemma by agreeing to regard certain terms as undefined or primitive. Further, one also stipulates certain axioms or postulates that describe some "natural" properties that the primitive terms possess. Once this is done, every other term is defined using the primitive terms or the ones defined earlier. Also, a result is not accepted unless it is precisely proved using the axioms or the results proved before.*

The terms that are usually considered primitive or undefined in mathematics are "set" and "is an element of" (a set). One has a small number of axioms that postulate certain basic and seemingly obvious "facts" about sets. Taking these for granted, we can define just about everything else that one encounters in mathematics. The formal definition of a function given in this chapter is a good illustration of this phenomenon. Good references for the nitty-gritty about sets, or rather, the subject of axiomatic set theory, are the books by Enderton [27] and Halmos [37]. To define the real numbers, one begins with the set $\mathbb{Q}$ *of rational numbers and constructs a set that satisfies the properties we postulated for* $\mathbb{R}$*. For more on this, see Appendix A.*

The topic of inequalities, which we briefly discussed in Section 1.2, is now a subject in itself, and to get a glimpse of it, one can see the book of Hardy, Littlewood, and Pólya [41] or of Beckenbach and Bellman [10]. There are also specialized books like that of Bullen, Mitrinovic, and Vasić [17], which has, for example, more than 50 proofs of the A.M.-G.M. inequality! A more elementary and accessible introduction is the little booklet [53] of Korovkin.

The notion of a function is of basic importance not only in calculus and analysis, but in all of mathematics. Not surprisingly, it has evolved over the years, and the formal definition is a distilled form of various ideas one has about this notion. Classically, the notion of a function was supple enough to admit y *as an (implicit) function of* x *if the two are related by an equation* $F(x,y) = 0$*, even though for a given value of* x*, there could be multiple values of* y *satisfying* $F(x,y) = 0$*. But in modern parlance, there is no such thing as a multivalued function! Nonetheless, the so-called "multivalued functions" have played an important role in the development of calculus and other parts of mathematics. With this in view, we have included a discussion of algebraic functions, remaining within the confines of modern definitions and the subject of calculus. For a glimpse of the classical viewpoint, see the two-volume textbook of Chrystal [18], which is also an excellent reference for algebra in general. A relatively modern and accessible book on algebra, which includes a discussion of partial fraction decomposition, is the survey of Birkhoff and Mac Lane [13]. The so called Real Fundamental Theorem of Algebra, stated but not proved in this chapter, can be deduced from the Fundamental Theorem of Algebra, and a proof is given in Appendix B.*

For real-valued functions defined on intervals, we have discussed a number of geometric properties such as monotonicity, convexity, local extrema, and the Intermediate Value Property. Typically, these appear in calculus books in conjunction with the notions of differentiability and continuity. The reason to include these in the very first chapter is to stress the fact that these are geometric notions and should not be confused with various criteria involving differentiability or continuity that one develops to check them.

Exercises

Part A

1.1. Using only the algebraic properties A1–A5 on page 2, prove the following.
 (i) 0 is the unique real number such that $a+0=a$ for all $a \in \mathbb{R}$. In other words, if some $z \in \mathbb{R}$ is such that $a+z=a$ for all $a \in \mathbb{R}$, then $z=0$.
 (ii) 1 is the unique real number such that $a \cdot 1 = a$ for all $a \in \mathbb{R}$.
 (iii) Given any $a \in \mathbb{R}$, an element $a' \in \mathbb{R}$ such that $a + a' = 0$ is unique. (As noted before, this unique real number a' is denoted by $-a$.)
 (iv) Given any $a \in \mathbb{R}$ with $a \neq 0$, an element $a^* \in \mathbb{R}$ such that $a \cdot a^* = 1$ is unique. (This unique real number a^* is denoted by a^{-1} or by $1/a$.)
 (v) Let $a \in \mathbb{R}$. Then $-(-a) = a$. Further, if $a \neq 0$, then $\left(a^{-1}\right)^{-1} = a$.
 (vi) Let $a, b \in \mathbb{R}$. Then $a(-b) = -(ab)$ and $(-a)(-b) = ab$.

1.2. Given any $a \in \mathbb{R}$ and $k \in \mathbb{Z}$, the **binomial coefficient** associated with a and k is defined by

$$\binom{a}{k} = \begin{cases} \dfrac{a(a-1)\cdots(a-k+1)}{k!} & \text{if } k \geq 0, \\ 0 & \text{if } k < 0, \end{cases}$$

where for $k \geq 0$, we denote by $k!$ (read as k **factorial**) the product of the first k positive integers. Note that $0! = 1$ and $\binom{a}{0} = 1$ for every $a \in \mathbb{R}$.
 (i) Show that if $a, k \in \mathbb{Z}$ with $0 \leq k \leq a$, then

$$\binom{a}{k} = \frac{a!}{k!(a-k)!} = \binom{a}{a-k}.$$

 (ii) If $a \in \mathbb{R}$ and $k \in \mathbb{Z}$, then show that

$$\binom{a}{k} = \binom{a-1}{k} + \binom{a-1}{k-1}.$$

[Note: This identity is sometimes called the **Pascal triangle identity**. If we compute the values of the binomial coefficients $\binom{n}{k}$ for $n \in \mathbb{N}$ and $0 \leq k \leq n$, and write them in a triangular array such that the nth row consists of the numbers $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, then this array is called the **Pascal triangle**. It may be instructive to write the first few rows of the Pascal triangle and *see* what the identity means pictorially.]

(iii) Use the identity in (ii) and induction to prove the Binomial Theorem (for positive integral exponents). In other words, given $x, y \in \mathbb{R}$, prove that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ for all $n \in \mathbb{N}$.
[Note: Proving a statement defined for $n \in \mathbb{N}$ such as the above identity by **induction** means that we should prove it for the initial value $n = 1$, and further prove it for every $n \in \mathbb{N}$ with $n > 1$, by assuming either that it holds for $n-1$ or that it holds for all positive integers smaller than n. The technique of induction also works when $\mathbb{N}$ is replaced by any subset S of $\mathbb{Z}$ such that S is bounded below; the only difference would be that the initial value 1 would have to be changed to the least element of S.]

1.3. Use induction to prove the following statements for each $n \in \mathbb{N}$:
(i) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, (ii) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$,
(iii) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{i=1}^{n} i\right)^2$.

1.4. Use the algebraic properties and the order properties of $\mathbb{R}$ to prove that
(i) $a^2 > 0$ for every $a \in \mathbb{R}$ with $a \neq 0$.
(ii) $0 < (1/b) < (1/a)$ for all $a, b \in \mathbb{R}$ with $0 < a < b$.

1.5. Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then show that the set $U_S = \{\alpha \in \mathbb{R} : \alpha \text{ is an upper bound of } S\}$ is bounded below, $\min U_S$ exists, and $\sup S = \min U_S$. Likewise, if S is bounded below, then show that the set $L_S = \{\beta \in \mathbb{R} : \beta \text{ is a lower bound of } S\}$ is bounded above, $\max L_S$ exists, and $\inf S = \max L_S$.

1.6. Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above and $M \in \mathbb{R}$, then show that $M = \sup S$ if and only if M is an upper bound of S and for every $\epsilon > 0$, there exists $x \in S$ such that $M - \epsilon < x \leq M$. Likewise, if S is bounded below and $m \in \mathbb{R}$, then show that $m = \inf S$ if and only if m is a lower bound of S and for every $\epsilon > 0$, there exists $x \in S$ such that $m \leq x < m + \epsilon$.

1.7. Let S be a nonempty subset of $\mathbb{R}$ and $c \in \mathbb{R}$. Define the additive translate $c + S$ and the multiplicative translate cS of S as follows:

$$c + S = \{c + x : x \in S\} \quad \text{and} \quad cS = \{cx : x \in S\}.$$

If S is bounded, then show that $c + S$ and cS are bounded, and also

$$\sup(c+S) = c + \sup S \quad \text{and} \quad \inf(c+S) = c + \inf S,$$

whereas

$$\sup(cS) = \begin{cases} c \sup S & \text{if } c \geq 0, \\ c \inf S & \text{if } c \leq 0, \end{cases} \quad \text{and} \quad \inf(cS) = \begin{cases} c \inf S & \text{if } c \geq 0, \\ c \sup S & \text{if } c \leq 0. \end{cases}$$

1.8. Given any $x, y \in \mathbb{R}$ with $y > 0$, show that there is $n \in \mathbb{N}$ such that $ny > x$.
[Note: The above property is equivalent to the Archimedean Property, which was stated in Proposition 1.3.]

1.9. Given any $x, y \in \mathbb{R}$ with $x \neq y$, show that there exists $\delta > 0$ such that the intervals $(x-\delta, x+\delta)$ and $(y-\delta, y+\delta)$ have no point in common.
[Note: The above property is sometimes called the **Hausdorff property**.]

1.10. If $a, b \in \mathbb{R}$ with $a < b$, then show that there exist infinitely many rational numbers as well as infinitely many irrational numbers between a and b.

1.11. Show that $n! \leq 2^{-n}(n+1)^n$ for every $n \in \mathbb{N}$, and that equality holds if and only if $n = 1$.

1.12. Let $n \in \mathbb{N}$ and $a_1, \ldots, a_n$ be positive real numbers. Prove that

$$\sqrt[n]{a_1 \cdots a_n} \geq \frac{n}{r}, \quad \text{where} \quad r := \frac{1}{a_1} + \cdots + \frac{1}{a_n},$$

and that equality holds if and only if $a_1 = \cdots = a_n$.
[Note: The above result is sometimes called the **G.M.-H.M. Inequality** and n/r is called the **harmonic mean** of $a_1, \ldots, a_n$.]

1.13. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be real numbers such that $a_1 > \cdots > a_n > 0$.
(i) (**Abel Inequality**) Let $B_i := b_1 + \cdots + b_i$ for $i = 1, \ldots, n$. Also let $m := \min\{B_i : 1 \leq i \leq n\}$ and $M := \max\{B_i : 1 \leq i \leq n\}$. Show that $ma_1 \leq \sum_{i=1}^{n} a_i b_i \leq Ma_1$.
(ii) Show that the alternating sum $a_1 - a_2 + \cdots + (-1)^{n+1} a_n$ is always between 0 and a_1.

1.14. Let $n, m \in \mathbb{N}$ be such that $m \geq n$.
(i) Let $a_1, \ldots, a_n, \ldots, a_m$ be real numbers, and let A_n denote the arithmetic mean of $a_1, \ldots, a_n$, and let A_m denote the arithmetic mean of $a_1, \ldots, a_m$. Show that

$$A_n \leq A_m \text{ if } a_1 \leq \cdots \leq a_m \quad \text{and} \quad A_n \geq A_m \text{ if } a_1 \geq \cdots \geq a_m.$$

Further, show that if $m > n$, then equality holds if and only if $a_1 = \cdots = a_m$. (Hint: Induct on m.)
(ii) Let $x \in \mathbb{R}$ be such that $x \geq 0$. Use (i) to show that

$$\frac{x^m - 1}{m} \geq \frac{x^n - 1}{n} \text{ and if } m > n, \text{ then equality holds} \iff x = 1.$$

[Note: Exercise 1.46 gives an alternative proof of this inequality.]
(iii) Let $a \in \mathbb{R}$ and $r \in \mathbb{Q}$ be such that $1 + a \geq 0$ and $r \geq 1$. Use (ii) to show that $(1+a)^r \geq 1 + ra$. Further, show that if $r > 1$, then equality holds if and only if $a = 0$.
[Note: Exercise 1.48 (iii) gives an alternative proof of this inequality.]

1.15. Let $n \in \mathbb{N}$ and let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be any real numbers. Assume that not all $a_1, \ldots, a_n$ are zero. Consider the quadratic polynomial

$$q(x) = \sum_{i=1}^{n} (xa_i + b_i)^2.$$

Show that the discriminant Δ of $q(x)$ is nonpositive, and $\Delta = 0$ if and only if there is $c \in \mathbb{R}$ such that $b_i = ca_i$ for all $i = 1, \ldots, n$. Use this to give an alternative proof of Proposition 1.12.

1.16. Show that if $n \in \mathbb{N}$ and $a_1, \ldots, a_n$ are nonnegative real numbers, then $(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$. (Hint: Write $(a_1 + \cdots + a_n)^2$ as $t_1 + \cdots + t_n$, where $t_k := a_1 a_k + a_2 a_{k+1} + \cdots + a_{n-k+1} a_n + a_{n-k+2} a_1 + \cdots + a_n a_{k-1}$.) [Note: Exercise 1.33 gives an alternative approach to this inequality.]

1.17. Given a function $f : \mathbb{R} \to \mathbb{R}$, prove the following:

(i) If $f(xy) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, then $f(x) = 0$ for all $x \in \mathbb{R}$. [Note: It is, however, possible that there are nonzero functions defined on subsets of $\mathbb{R}$, such as $(0, \infty)$, that satisfy $f(xy) = f(x) + f(y)$ for all x, y in the domain of f. A prominent example of this is the logarithmic function, which will be discussed in Section 7.1.]

(ii) If $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$, then either $f(x) = 0$ for all $x \in \mathbb{R}$, or $f(x) = 1$ for all $x \in \mathbb{R}$, or $f(0) = 0$ and $f(1) = 1$. Further, f is either an even function or an odd function.

Give examples of even as well as odd functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

1.18. Prove that the absolute value function, that is, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$, is not a rational function.

1.19. Prove that the function $f : [0, \infty) \to \mathbb{R}$ defined by the following is an algebraic function:

$$\text{(i) } f(x) = 1 + \sqrt[3]{x}. \quad \text{(ii) } f(x) = \sqrt{x} + \sqrt{2x}, \quad \text{(iii) } f(x) = \sqrt{x} + \sqrt[3]{x}.$$

1.20. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by the following:

$$\text{(i) } f(x) = x^3, \quad \text{(ii) } f(x) = x^4, \quad \text{(iii) } f(x) = |x|, \quad \text{(iv) } f(x) = \sqrt{|x|}.$$

Sketch the graph of f and determine the points at which f has local extrema as well as the points of inflection of f, if any, in each case.

1.21. Let $f : (0, \infty) \to \mathbb{R}$ be the function defined by the following:

$$\text{(i) } f(x) = \sqrt{x}, \quad \text{(ii) } f(x) = x^{3/2}, \quad \text{(iii) } f(x) = \frac{1}{x}, \quad \text{(iv) } f(x) = \frac{1}{x^2}.$$

Sketch the graph of f and determine the points at which f has local extrema as well as the points of inflection of f, if any, in each case.

1.22. Given any $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$, define functions g_c, h_c, k_c, and ℓ_c from $\mathbb{R}$ to $\mathbb{R}$ as follows:

$$g_c(x) = f(x) + c, \quad h_c(x) = cf(x), \quad k_c(x) = f(x + c), \quad \ell_c(x) = f(cx).$$

If f is given by $f(x) = x^n$ for all $x \in \mathbb{R}$, then sketch the graph of g_c, h_c, k_c, and ℓ_c when $n = 1$, 2, or 3 and $c = 0$, 1, 2, -1, -2, $\frac{1}{2}$, or $-\frac{1}{2}$.

1.23. Consider $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ defined by the following. Determine whether f is bounded above/below on D. If so, find an upper/lower bound for f on D. Also, determine whether f attains its bounds.

(i) $D = (-1, 1)$ and $f(x) = x^2 - 1$, (ii) $D = (-1, 1)$ and $f(x) = x^3 - 1$,

(iii) $D = (-1, 1]$ and $f(x) = x^2 - 2x - 3$, (iv) $D = \mathbb{R}$ and $f(x) = \dfrac{1}{1 + x^2}$.

1.24. Let D be a bounded subset of $\mathbb{R}$ and let $f : D \to \mathbb{R}$ be a polynomial function. Prove that f is bounded on D.

1.25. Let I be an interval and let $f : I \to \mathbb{R}$ be monotonically increasing on I. Given any $r \in \mathbb{R}$, show that rf is a monotonically increasing function on I if $r \geq 0$ and a monotonically decreasing function on I if $r < 0$.

1.26. Let I be an interval and let $f : I \to \mathbb{R}$ be convex on I. Given any $r \in \mathbb{R}$, show that rf is a convex function on I if $r \geq 0$ and a concave function on I if $r < 0$.

1.27. Let I be an interval and let $f, g : I \to \mathbb{R}$ be convex on I. Show that the function $f + g$ is also convex on I.

1.28. Give an example of $f : (0, 1) \to \mathbb{R}$ such that f is
(i) strictly increasing and convex, (ii) strictly increasing and concave,
(iii) strictly decreasing and convex, (iv) strictly decreasing and concave.

1.29. Give an example of a nonconstant function $f : (-1, 1) \to \mathbb{R}$ such that f has a local extremum at 0, and 0 is a point of inflection for f.

1.30. Let I be an interval and let $f : I \to \mathbb{R}$ be a function.
  (i) If f is monotonically increasing as well as monotonically decreasing on I, then show that f is constant on I.
  (ii) If f is convex as well as concave on I, then show that f is given by a linear polynomial (that is, there are $a, b \in \mathbb{R}$ such that $f(x) = ax + b$ for all $x \in I$).

1.31. Let I be an interval and let $f : I \to \mathbb{R}$ be a function. Show that f is convex on I if and only if the slope of the chord joining $(x_1, f(x_1))$ and $(x, f(x))$ is less than or equal to the slope of the chord joining $(x, f(x))$ and $(x_2, f(x_2))$ for all $x_1 < x < x_2$ in I.

1.32. (**Jensen Inequality**) Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be convex on I. Show that $f(t_1x_1 + \cdots + t_nx_n) \leq t_1f(x_1) + \cdots + t_nf(x_n)$ for all $x_1, \ldots, x_n \in I$ and $t_1, \ldots, t_n \in [0, 1]$ with $t_1 + \cdots + t_n = 1$.

1.33. Show that if $n \in \mathbb{N}$ and $a_1, \ldots, a_n$ are nonnegative real numbers, then $(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$. (Hint: Exercise 1.32)

1.34. For $f, g : \mathbb{R} \to \mathbb{R}$, which of the following statements are true? Why?
  (i) If f and g have a local maximum at $x = c$, then so does $f + g$.
  (ii) If f and g have a local maximum at $x = c$, then so does fg. What if $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in \mathbb{R}$?
  (iii) If c is a point of inflection for f as well as for g, then it is a point of inflection for $f + g$.
  (iv) If c is a point of inflection for f as well as for g, then it is a point of inflection for fg.

Part B

1.35. Given any $\ell, m \in \mathbb{Z}$ with $\ell \neq 0$, prove that there are unique integers q and r such that $m = \ell q + r$ and $0 \leq r < |\ell|$. (Hint: Consider the least element of the subset $\{m - \ell n : n \in \mathbb{Z} \text{ with } m - \ell n \geq 0\}$ of $\mathbb{Z}$.)

1.36. Given any integers m and n, not both zero, a positive integer d satisfying

$$\text{(i) } d \mid m \text{ and } d \mid n \qquad \text{and} \qquad \text{(ii) } e \in \mathbb{Z},\ e \mid m \text{ and } e \mid n \Longrightarrow e \mid d$$

is called a **greatest common divisor**, or simply a **GCD**, of m and n. If $m = n = 0$, we set the GCD of m and n to be 0. Given any $m, n \in \mathbb{Z}$, show that a GCD of m and n exists and is unique; it is denoted by $\text{GCD}(m, n)$. Also show that $\text{GCD}(m, n) = um + vn$ for some $u, v \in \mathbb{Z}$. (Hint: Consider the least element of $\{um + vn : u, v \in \mathbb{Z} \text{ with } um + vn > 0\}$.)

1.37. Let $m, n \in \mathbb{Z}$. Show that m and n are relatively prime if and only if $\text{GCD}(m, n) = 1$. Also show that m and n are relatively prime if and only if $um + vn = 1$ for some $u, v \in \mathbb{Z}$. Is it true that if a positive integer d satisfies $um + vn = d$ for some $u, v \in \mathbb{Z}$, then $d = \text{GCD}(m, n)$?

1.38. Let $m, n \in \mathbb{Z}$ be relatively prime integers different from ± 1. Show by an example that the integers u, v satisfying $um + vn = 1$ need not be unique. Show, however, that there are unique $u, v \in \mathbb{Z}$ such that $um + vn = 1$ and $0 \le u < |n|$. In this case, show that $|v| < |m|$. (Hint: Exercise 1.35.)

1.39. Given any integers m and n, both nonzero, a positive integer ℓ satisfying

$$\text{(i) } m \mid \ell \text{ and } n \mid \ell \qquad \text{and} \qquad \text{(ii) } k \in \mathbb{N},\ m \mid k \text{ and } n \mid k \Longrightarrow \ell \mid k$$

is called a **least common multiple**, or simply an **LCM**, of m and n. If $m = 0$ or $n = 0$, we set the LCM of m and n to be 0. Given any $m, n \in \mathbb{Z}$, show that an LCM of m and n exists and is unique; it is denoted by $\text{LCM}(m, n)$. Also show that if m and n are nonnegative integers and we let $d = \text{GCD}(m, n)$ and $\ell = \text{LCM}(m, n)$, then $d\ell = mn$.

1.40. If $m, n, n' \in \mathbb{Z}$ are such that m and n are relatively prime and $m \mid nn'$, then show that $m \mid n'$. Deduce that if p is a **prime** (which means that p is an integer > 1 and the only positive integers that divide p are 1 and p) and if p divides a product of two integers, then it divides one of them. (Hint: Exercise 1.36.)

1.41. Prove that every rational number r can be written as

$$r = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z},\ q > 0 \text{ and } p, q \text{ are relatively prime,}$$

and moreover, the integers p and q are uniquely determined by r.

1.42. Show that if a rational number α satisfies a monic polynomial with integer coefficients, that is, if $\alpha \in \mathbb{Q}$ and $\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_1\alpha + c_0 = 0$ for some $n \in \mathbb{N}$ and $c_0, c_1, \ldots, c_{n-1} \in \mathbb{Z}$, then α must be an integer. Deduce that the following numbers are irrational:

$$\sqrt{2},\ \sqrt{3},\ \sqrt{6},\ \sqrt[3]{2},\ \sqrt[4]{11},\ \sqrt[5]{16},\ \sqrt{2} + \sqrt{3}.$$

1.43. If A and B are any countable sets, then show that the set $A \times B := \{(a, b) : a \in A \text{ and } b \in B\}$ is also countable. (Hint: Write $A \times B = \cup_{b \in B} A \times \{b\}$.)

1.44. Let $\{0,1\}^{\mathbb{N}}$ denote the set of all maps from $\mathbb{N}$ to the two-element set $\{0,1\}$. Prove that $\{0,1\}^{\mathbb{N}}$ is uncountable. (Hint: If $f : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ is any map and $h \in \{0,1\}^{\mathbb{N}}$ is the map that associates $n \in \mathbb{N}$ to 1 or to 0 according as $f(n)$ associates n to 0 or to 1, then h is not in the range of f.)

1.45. Let $I_n = [a_n, b_n]$, $n \in \mathbb{N}$, be closed and bounded intervals in $\mathbb{R}$ such that $I_n \supseteq I_{n+1}$ for each $n \in \mathbb{N}$. If $x = \sup\{a_n : n \in \mathbb{N}\}$ and $y = \inf\{b_n : n \in \mathbb{N}\}$, then show that $x \in I_n$ and $y \in I_n$ for every $n \in \mathbb{N}$.

1.46. Let $m, n \in \mathbb{N}$ and let $x \in \mathbb{R}$ be such that $x \geq 0$ and $x \neq 1$. Show that

$$\frac{x^m - 1}{m} > \frac{x^n - 1}{n} \quad \text{if } m > n \quad \text{and} \quad \frac{x^m - 1}{m} < \frac{x^n - 1}{n} \quad \text{if } m < n.$$

(Hint: It suffices to assume that $m > n$. Write $n(x^m - 1) - m(x^m - 1)$ as $(x-1)\big(n(x^n + x^{n+1} + \cdots + x^{m-1}) - (m-n)(1 + x + \cdots + x^{n-1})\big)$. Compare the $m-n$ elements $x^n, x^{n+1}, \ldots, x^{m-1}$ as well as the n elements $1, x, \ldots, x^{n-1}$ with x^n, when $x < 1$ and $x > 1$.)

1.47. Let $r \in \mathbb{Q}$ and $a, b \in \mathbb{R}$ be such that $a > 0$, $b > 0$, and $a \neq b$. Prove that

$$\begin{aligned} ra^{r-1}(a-b) < a^r - b^r < rb^{r-1}(a-b) &\quad \text{if } 0 < r < 1, \text{ and} \\ rb^{r-1}(a-b) < a^r - b^r < ra^{r-1}(a-b) &\quad \text{if } r > 1. \end{aligned}$$

(Hint: Write $r = m/n$, where $m, n \in \mathbb{N}$ and use Exercise 1.46 with $x = (a/b)^{1/n}$ and with $x = (b/a)^{1/n}$.)

1.48. Use Exercise 1.47 to deduce the following:
 (i) (**Basic Inequality for Rational Powers**) Let $a, b \in \mathbb{R}$ and $r \in \mathbb{Q}$ be such that $a \geq 0$, $b \geq 0$, and $r \geq 1$. Also, let $M := \max\{|a|, |b|\}$. Then $|a^r - b^r| \leq rM^{r-1}|a-b|$.
 (ii) (**Inequality for Rational Roots**) Let $a, b \in \mathbb{R}$ and $r \in \mathbb{Q}$ be such that $a \geq 0$, $b \geq 0$ and $0 < r < 1$. Then $|a^r - b^r| \leq 2|a-b|^r$. (Hint: We may suppose $0 < b < a$. If $b \leq a/2$, then $a^r - b^r \leq a^r \leq 2^r(a-b)^r$, whereas if $b > a/2$, then $a^r - b^r \leq rb^{r-1}(a-b) \leq r(a-b)^r$.)
 (iii) (**Binomial Inequality for Rational Powers**) Let $a \in \mathbb{R}$ and $r \in \mathbb{Q}$ be such that $1 + a \geq 0$ and $r \geq 1$. Then $(1+a)^r \geq 1 + ra$. Further, if $r > 1$, then $(1+a)^r > 1 + ra$.

1.49. Give an alternative proof of the A.M.-G.M. inequality as follows.
 (i) First prove the inequality for n numbers $a_1, \ldots, a_n$ when $n = 2^m$ using induction on m.
 (ii) In the general case, choose $m \in \mathbb{N}$ such that $2^m > n$, and apply (i) to the 2^m numbers $a_1, \ldots, a_n, g, \ldots, g$, with g repeated $2^m - n$ times, where $g := \sqrt[n]{a_1 a_2 \cdots a_n}$.

1.50. Use the Cauchy–Schwarz inequality to prove the **A.M.-H.M. inequality**, namely, if $n \in \mathbb{N}$ and $a_1, \ldots, a_n$ are positive real numbers, then prove that

$$\frac{a_1 + \cdots + a_n}{n} \geq \frac{n}{r}, \quad \text{where} \quad r := \frac{1}{a_1} + \cdots + \frac{1}{a_n}.$$

1.51. Given any $n \in \mathbb{N}$ and positive real numbers $a_1, \ldots, a_n$ let

$$M_p = \left(\frac{a_1^p + \cdots + a_n^p}{n} \right)^{1/p} \quad \text{for } p \in \mathbb{Q} \text{ with } p \neq 0.$$

Prove that if $p \in \mathbb{Q}$ is positive, then $M_p \leq M_{2p}$ and equality holds if and only if $a_1 = \cdots = a_n$. (Hint: Cauchy–Schwarz inequality.)
[Note: M_p is called the pth **power mean** of $a_1, \ldots, a_n$. In fact, M_1 is the arithmetic mean and M_{-1} is the harmonic mean, while M_2 is called the **root mean square** of $a_1, \ldots, a_n$. A general *power mean inequality*, which includes as a special case the above inequality $M_p \leq M_{2p}$, the A.M.-G.M. inequality, and the G.M.-H.M. inequality, is described in Revision Exercise R.27 given at the end of Chapter 7.]

1.52. Given any polynomials $\ell(x), p(x) \in \mathbb{R}[x]$ with $\ell(x) \neq 0$, prove that there exist unique polynomials $q(x)$ and $r(x)$ in $\mathbb{R}[x]$ such that

$$p(x) = q(x)\ell(x) + r(x) \text{ and either } r(x) = 0 \text{ or } \deg r(x) < \deg \ell(x).$$

(Hint: If $p(x) \neq 0$ and $\deg p(x) \geq \deg \ell(x)$, then use induction on $\deg p(x)$.)

1.53. Given any $p(x) \in \mathbb{R}[x]$ and $\alpha \in \mathbb{R}$, show that there is a unique polynomial $q(x) \in \mathbb{R}[x]$ such that $p(x) = (x - \alpha)q(x) + p(\alpha)$. Deduce that α is a root of $p(x)$ if and only if the polynomial $(x - \alpha)$ divides $p(x)$.

1.54. Show that a nonzero polynomial in $\mathbb{R}[x]$ of degree n has at most n roots in $\mathbb{R}$. (Hint: Exercise 1.53.)

1.55. Given any $p(x), q(x) \in \mathbb{R}[x]$, not both zero, a polynomial $d(x)$ in $\mathbb{R}[x]$ satisfying

(i) $d(x) \mid p(x)$ and $d(x) \mid q(x)$, and

(ii) $e(x) \in \mathbb{R}[x]$, $e(x) \mid p(x)$ and $e(x) \mid q(x) \Longrightarrow e(x) \mid d(x)$

is called a **greatest common divisor**, or simply a **GCD**, of $p(x)$ and $q(x)$. In case $p(x) = q(x) = 0$, we set the GCD of $p(x)$ and $q(x)$ to be 0. Prove that for all $p(x), q(x) \in \mathbb{R}[x]$, a GCD of $p(x)$ and $q(x)$ exists, and is unique up to multiplication by a nonzero constant, that is, if $d_1(x)$ as well as $d_2(x)$ is a GCD of $p(x)$ and $q(x)$, then $d_2(x) = cd_1(x)$ for some $c \in \mathbb{R}$ with $c \neq 0$. Further, show that every GCD of $p(x)$ and $q(x)$ can be expressed as $u(x)p(x) + v(x)q(x)$ for some $u(x), v(x) \in \mathbb{R}[x]$. (Hint: Consider a polynomial of least degree in the set $\{u(x)p(x) + v(x)q(x) : u(x), v(x) \in \mathbb{R}[x] \text{ with } u(x)p(x) + v(x)q(x) \neq 0\}$.)

1.56. Let $p(x), q(x) \in \mathbb{R}[x]$. Show that $p(x)$ and $q(x)$ are relatively prime if and only if $u(x)p(x) + v(x)q(x) = 1$ for some $u(x), v(x) \in \mathbb{R}[x]$. Is it true that if a nonzero polynomial $d(x) \in \mathbb{R}[x]$ satisfies $u(x)p(x) + v(x)q(x) = d(x)$ for some $u(x), v(x) \in \mathbb{R}[x]$, then $d(x)$ is a GCD of $p(x)$ and $q(x)$?

1.57. Let $p(x), q(x) \in \mathbb{R}[x]$ be relatively prime polynomials of positive degree. Show by an example that the polynomials $u(x), v(x) \in \mathbb{R}[x]$ such that $u(x)p(x) + v(x)q(x) = 1$ need not be unique. Show, however, that there are unique $u(x), v(x) \in \mathbb{R}[x]$ such that $u(x)p(x) + v(x)q(x) = 1$ and either $u(x) = 0$ or $\deg u(x) < \deg q(x)$. In this case show that either $v(x) = 0$ or $\deg v(x) < \deg p(x)$. (Hint: Exercise 1.52.)

1.58. Let $p(x)$, $q_1(x)$, and $q_2(x)$ be nonzero polynomials in $\mathbb{R}[x]$ of degrees m, d_1, and d_2, respectively. Assume that $q_1(x)$ and $q_2(x)$ are relatively prime and that d_1 and d_2 are positive. If $m < d_1+d_2$, then show that there are unique polynomials $u_1(x), u_2(x) \in \mathbb{R}[x]$ such that $p(x) = u_1(x)q_2(x) + u_2(x)q_1(x)$ and for $i = 1, 2$, either $u_i(x) = 0$ or $\deg u_i(x) < \deg q_i(x)$. Deduce that if $q(x) := q_1(x)q_2(x)$, then

$$\frac{p(x)}{q(x)} = \frac{u_1(x)}{q_1(x)} + \frac{u_2(x)}{q_2(x)}.$$

1.59. Let $q(x) \in \mathbb{R}[x]$ be a nonzero polynomial of degree n. Use the Real Fundamental Theorem of Algebra to write $q(x) = cq_1(x)^{e_1} \cdots q_k(x)^{e_k}$, where $c \in \mathbb{R}$ with $c \neq 0$, $e_1, \ldots, e_k$ are positive integers, and $q_1(x), \ldots, q_k(x)$ are distinct polynomials in $\mathbb{R}[x]$ that are either linear of the form $x - \alpha$ with $\alpha \in \mathbb{R}$ or quadratic of the form $x^2 + \beta x + \gamma$ with $\beta, \gamma \in \mathbb{R}$ such that $\beta^2 - 4\gamma < 0$. Show that the polynomials $q_i(x)^{e_i}$ and $q_j(x)^{e_j}$ are relatively prime for $i, j = 1, \ldots, k$ with $i \neq j$. Use Exercise 1.58 and induction to show that given any nonzero polynomial $p(x) \in \mathbb{R}[x]$ with $\deg p(x) < \deg q(x)$, there are unique polynomials $u_1(x), \ldots, u_k(x) \in \mathbb{R}[x]$ such that either $u_i(x) = 0$ or $\deg u_i(x) < e_i \deg q_i(x)$ for $i = 1, \ldots, k$ and

$$\frac{p(x)}{q(x)} = \frac{u_1(x)}{q_1(x)^{e_1}} + \cdots + \frac{u_k(x)}{q_k(x)^{e_k}}.$$

1.60. Let $p(x), q(x) \in \mathbb{R}[x]$ be nonzero polynomials of degrees m, n respectively. Let $e \in \mathbb{N}$ be such that $m < en$. Show that there are polynomials $A_1(x), \ldots, A_e(x)$ in $\mathbb{R}[x]$ such that $p(x) = A_1(x)q(x)^{e-1} + A_2(x)q(x)^{e-2} + \cdots + A_e(x)$, and for $i = 1, \ldots, e$, either $A_i(x) = 0$ or $\deg A_i(x) < n$. Deduce that

$$\frac{p(x)}{q(x)^e} = \frac{A_1(x)}{q(x)} + \frac{A_2(x)}{q(x)^2} + \cdots + \frac{A_e(x)}{q(x)^e}.$$

(Hint: Let $R_0(x) := p(x)$. Use Exercise 1.52 to successively find $A_i(x)$ and $R_i(x)$ such that $R_{i-1}(x) = A_i(x)q(x)^{e-i} + R_i(x)$ for $i = 1, \ldots, e$.)

1.61. Use Exercises 1.52, 1.58, 1.59, and 1.60 to show that every rational function has a partial fraction decomposition as stated in this chapter.

1.62. Let $f : [a, b] \to \mathbb{R}$ be a function and let c be a point of (a, b).
(i) If f is monotonically (resp. strictly) increasing on $[a, c]$ and on $[c, b]$, then show that f is monotonically (resp. strictly) increasing on $[a, b]$.
(ii) If f is convex (resp. strictly convex) on $[a, c]$ and on $[c, b]$, then is it true that f is convex (resp. strictly convex) on $[a, b]$?

1.63. Let I be an interval containing more than one point and let $f : I \to \mathbb{R}$ be a function. Define $\phi(x_1, x_2) := \big(f(x_1) - f(x_2)\big)/(x_1 - x_2)$ for $x_1, x_2 \in I$ with $x_1 \neq x_2$. Show that f is convex on I if and only if ϕ is a monotonically increasing function of x_1, that is, $\phi(x_1, x) \leq \phi(x_2, x)$ for all $x_1, x_2 \in I$ with $x_1 < x_2$ and $x \in I \setminus \{x_1, x_2\}$.

2

Sequences

The word *sequence* refers to a succession of certain objects. For us, these objects will be real numbers. A sequence of real numbers looks like an infinite succession such as

$$a_1, a_2, a_3, a_4, a_5, \ldots,$$

where the a_n's are real numbers. For example,

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

is a sequence of real numbers. If we see a sequence, it is natural to ask where it leads. For example, the above sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ seems to approach 0. We shall make this idea precise by defining the notion of convergence of sequences.

In Section 2.1 below, we begin with the formal definition of a sequence and go on to discuss a number of basic concepts and results. Along the way, we will look at numerous examples of sequences and see whether they approach a fixed number. Next, in Section 2.2, we consider the notion of a subsequence of a given sequence, and also a special class of sequences known as Cauchy sequences. We show that every sequence in $\mathbb{R}$ that satisfies a mild condition of being "bounded" has a convergent subsequence. Also, we show that a sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.

2.1 Convergence of Sequences

A **sequence** (in $\mathbb{R}$) is a real-valued function whose domain is the set $\mathbb{N}$ of all natural numbers. Usually, we shall denote sequences by (a_n), (b_n), and so on, or sometimes by (A_n), (B_n), and so on. The value of a sequence (a_n) at $n \in \mathbb{N}$ is given by a_n, and this is called the nth **term** of that sequence; the set $\{a_n : n \in \mathbb{N}\}$ is called the **set of terms** of the sequence (a_n). Note that although a sequence has infinitely many terms, the set of its terms need

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not be infinite. For instance, this is the case in the first two of following five examples of sequences (a_n) whose nth term is defined by

(i) $a_n := 1$, (ii) $a_n := (-1)^n$, (iii) $a_n := \dfrac{1}{n}$, (iv) $a_n := n$, (v) $a_n := (-1)^n n$.

If D is a countably infinite subset of $\mathbb{R}$, then an enumeration of D is a sequence having a special property that all of its terms are distinct, and the set of its terms is D.

We shall use the terminology "bounded", "unbounded", "bounded above", "bounded below" for a sequence just as we use them for any function. Thus the sequences defined in (i), (ii), and (iii) are bounded, but those defined in (iv) and (v) are not bounded. The sequence defined in (iv) is bounded below (by 1), but it is not bounded above, while the sequence defined in (v) is neither bounded above nor bounded below.

We say that a sequence (a_n) is **convergent** if there exists $a \in \mathbb{R}$ that satisfies the following condition: For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$|a_n - a| < \epsilon \quad \text{for all} \quad n \geq n_0.$$

In this case, we say that (a_n) **converges** to a or that a is a **limit** of (a_n), and write $a_n \to a$ (as $n \to \infty$). We write $a_n \not\to a$ if the sequence (a_n) does not converge to a. A sequence that is not convergent is said to be **divergent**.

Examples 2.1. (i) If $a_n := 1$ for $n \in \mathbb{N}$, then obviously $a_n \to 1$.

(ii) If $a_n := (-1)^n$ for $n \in \mathbb{N}$, then (a_n) is divergent. To see this, suppose (a_n) converges to $a \in \mathbb{R}$. Since a can be in at most one of the intervals $(-2, 0)$ and $(0, 2)$, we see that $|a - 1| \geq 1$ or $|a + 1| \geq 1$. Thus $|a - a_n| \geq 1$ for all even $n \in \mathbb{N}$ and $|a - a_n| \geq 1$ for all odd $n \in \mathbb{N}$. Hence if $\epsilon \in \mathbb{R}$ and $0 < \epsilon \leq 1$, then there is no $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq n_0$. Thus $a_n \not\to a$.

(iii) If $a_n := 1/n$ for $n \in \mathbb{N}$, then $a_n \to 0$. Indeed, by the Archimedean Property (Proposition 1.3), given any $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $n_0 > 1/\epsilon$. This implies that $|a_n - 0| < \epsilon$ for all $n \geq n_0$.

(iv) If $a_n := n$ for $n \in \mathbb{N}$, then (a_n) is divergent. Indeed, given $a \in \mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that $n_0 > a + 1$. This implies that $|a_n - a| \geq 1$ for all $n \geq n_0$. Thus $a_n \not\to a$. Similarly, if $a_n := (-1)^n n$ for $n \in \mathbb{N}$, then (a_n) is divergent.

Note that the convergence of a sequence (a_n) is not altered if a finite number of a_n's are replaced by some other b_n's. Thus if we replace $a_{n_1}, \ldots, a_{n_k}$ by $b_{n_1}, \ldots, b_{n_k}$ respectively, then the altered sequence converges if and only if the original sequence converges. With this in view, we may sometimes regard $(1/a_n)$ as a sequence if we know that all except finitely many a_n's are nonzero.

Proposition 2.2. (i) *A convergent sequence has a unique limit.*
(ii) *A convergent sequence is bounded.*

Proof. (i) Suppose $a_n \to a$ as well as $a_n \to b$. If $b \neq a$, let $\epsilon := |a - b|$. Since $a_n \to a$, there is $n_1 \in \mathbb{N}$ such that $|a_n - a| < \epsilon/2$ for all $n \geq n_1$, and since $a_n \to b$, there is $n_2 \in \mathbb{N}$ such that $|a_n - b| < \epsilon/2$ for all $n \geq n_2$. Let $n_0 := \max\{n_1, n_2\}$. Then

$$|a - b| \leq |a - a_{n_0}| + |a_{n_0} - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = |a - b|.$$

This contradiction shows that $b = a$.

(ii) Let $a_n \to a$. Then there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < 1$ for all $n > n_0$. If $\alpha := \max\{|a_1|, \ldots, |a_{n_0}|, |a| + 1\}$, then $|a_n| \leq \alpha$ for all $n \in \mathbb{N}$. Hence (a_n) is bounded. $\square$

Let $a_n \to a$. In view of part (i) of Proposition 2.2, we say that a is *the* limit of (a_n) and write

$$\lim_{n\to\infty} a_n = a.$$

Example 2.1 (ii) shows that the converse of part (ii) of the above proposition does not hold.

We now prove some results that are useful in proving convergence or divergence of a variety of sequences.

First we consider how the algebraic operations on $\mathbb{R}$ are related to the concept of the convergence of a sequence of real numbers. The following result is known as the **Limit Theorem for Sequences**.

Proposition 2.3. *Let $a_n \to a$ and $b_n \to b$. Then*

(i) $a_n + b_n \to a + b$,
(ii) $ra_n \to ra$ *for every* $r \in \mathbb{R}$,
(iii) $a_n b_n \to ab$,
(iv) *if* $a \neq 0$, *then there is* $m \in \mathbb{N}$ *such that* $a_n \neq 0$ *for all* $n \geq m$, *and* $1/a_n \to 1/a$.

Proof. Let $\epsilon > 0$ be given. There are $n_1, n_2 \in \mathbb{N}$ such that

$$|a_n - a| < \epsilon \text{ for all } n \geq n_1 \quad \text{ and } \quad |b_n - b| < \epsilon \text{ for all } n \geq n_2.$$

(i) Let $n_0 := \max\{n_1, n_2\}$. Then for all $n \geq n_0$,

$$|a_n + b_n - (a + b)| \leq |a_n - a| + |b_n - b| < \epsilon + \epsilon = 2\epsilon.$$

(ii) If $r = 0$, then the result is obvious. Let $r > 0$. Then for all $n \geq n_1$,

$$|ra_n - ra| = |r|\,|a_n - a| < |r|\epsilon.$$

(iii) By part (ii) of Proposition 2.2, there is $\alpha \in \mathbb{R}$ such that $|a_n| \leq \alpha$ for all $n \in \mathbb{N}$. Let $n_0 := \max\{n_1, n_2\}$. Then for all $n \geq n_0$,

$$\begin{aligned} |a_n b_n - ab| &= |a_n(b_n - b) + (a_n - a)b| \\ &\leq |a_n|\,|b_n - b| + |a_n - a|\,|b| \\ &\leq \alpha\epsilon + \epsilon|b| = (\alpha + |b|)\epsilon. \end{aligned}$$

(iv) Since $|a| > 0$, there is $m \in \mathbb{N}$ such that $|a_n - a| < |a|/2$ for all $n \geq m$. But then $|a_n| \geq |a| - |a - a_n| > |a|/2$ for all $n \geq m$. Let $n_0 := \max\{n_1, m\}$. Then $a_n \neq 0$ and

$$\left| \frac{1}{a_n} - \frac{1}{a} \right| = \frac{|a - a_n|}{|a_n|\,|a|} < \frac{2\epsilon}{|a|^2} \quad \text{for all } n \geq n_0.$$

Since $\epsilon > 0$ is arbitrary, the conclusions in (i), (ii), (iii), and (iv) follow. $\square$

With notation and hypotheses as in the above proposition, a combined application of parts (i) and (ii) of Proposition 2.3 shows that $a_n - b_n \to a - b$. Likewise, a combined application of parts (iii) and (iv) of Proposition 2.3 shows that if $b \neq 0$, then $a_n/b_n \to a/b$. Further, given any $m \in \mathbb{Z}$, successive applications of part (iii) or part (iv) of Proposition 2.3 show that $a_n^m \to a^m$, provided $a \neq 0$ in case $m < 0$.

Next, we show how the order relation on $\mathbb{R}$ and the operation of taking the kth root are preserved under convergence.

Proposition 2.4. *Let (a_n) and (b_n) be sequences and let a, b be real numbers such that $a_n \to a$ and $b_n \to b$. Then*

(i) $|a_n| \to |a|$.
(ii) $\max\{a_n, b_n\} \to \max\{a, b\}$ *and* $\min\{a_n, b_n\} \to \min\{a, b\}$.
(iii) *If there is $n_0 \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq n_0$, then $a \leq b$. Conversely, if $a < b$, then there is $m_0 \in \mathbb{N}$ such that $a_n < b_n$ for all $n \geq m_0$.*
(iv) *If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$ and $a_n^{1/k} \to a^{1/k}$ for every $k \in \mathbb{N}$.*

Proof. (i) Since $\big|\,|a_n| - |a|\,\big| \leq |a_n - a|$ for $n \in \mathbb{N}$, it follows that $|a_n| \to |a|$.

(ii) Let $\epsilon > 0$ be given. Then there are $n_1, n_2 \in \mathbb{N}$ such that

$$a - \epsilon < a_n < a + \epsilon \text{ for all } n \geq n_1 \quad \text{and} \quad b - \epsilon < b_n < b + \epsilon \text{ for all } n \geq n_2.$$

Let $n_0 := \max\{n_1, n_2\}$. Then we see that for each $n \geq n_0$,

$$\max\{a - \epsilon,\, b - \epsilon\} < \max\{a_n, b_n\} < \max\{a + \epsilon,\, b + \epsilon\}.$$

Since $\max\{a-\epsilon,\, b-\epsilon\} = \max\{a, b\} - \epsilon$ and $\max\{a+\epsilon,\, b+\epsilon\} = \max\{a, b\} + \epsilon$, it follows that $\max\{a_n, b_n\} \to \max\{a, b\}$. In a similar manner (or alternatively, by applying the result just proved to $(-a_n)$ and $(-b_n)$ and using part (ii) of Proposition 2.3), we obtain $\min\{a_n, b_n\} \to \min\{a, b\}$.

(iii) The first assertion in (iii) follows immediately from (ii). For the converse, suppose $a < b$. Let $\epsilon := (b - a)/2$. There is $m_1 \in \mathbb{N}$ such that $a_n < a + \epsilon$ for all $n \geq m_1$ and there is $m_2 \in \mathbb{N}$ such that $b_n > b - \epsilon$ for all $n \geq m_2$. Let $m_0 := \max\{m_1, m_2\}$. Then $a_n < a + \epsilon = (a + b)/2 = b - \epsilon < b_n$ for all $n \geq m_0$.

(iv) Part (ii) implies that $a \geq 0$. Let $k \in \mathbb{N}$ and $\epsilon > 0$ be given. Since $\epsilon^k > 0$, there is $n_2 \in \mathbb{N}$ such that $|a_n - a| < \epsilon^k$ for all $n \geq n_2$. Hence by the basic inequality for roots (part (ii) of Proposition 1.9), we obtain

$$|a_n^{1/k} - a^{1/k}| \le |a_n - a|^{1/k} < \epsilon \quad \text{for all } n \ge n_2.$$

It follows that $a_n^{1/k} \to a^{1/k}$. □

The converse of (i) above is not true. For example, if $a_n := (-1)^n$ for $n \in \mathbb{N}$, then $|a_n| \to 1$, but (a_n) is not convergent. Also, in (iii) above, it can happen that $a_n \to a$, $b_n \to b$, and $a_n < b_n$ for all $n \in \mathbb{N}$, but still $a = b$. For example, this happens when $a_n := 0$ and $b_n := 1/n$ for $n \in \mathbb{N}$ and $a = 0 = b$.

With notation and hypotheses as in the above proposition, a combined application of part (iii) of Proposition 2.3 and part (iv) of Proposition 2.4 shows that if $a_n \ge 0$ for all $n \in \mathbb{N}$, then $a_n^r \to a^r$, where r is any positive rational number, since we can write $r = m/k$, where $m, k \in \mathbb{N}$. This, together with part (iv) of Proposition 2.3, shows that if $a > 0$, then $a_n^r \to a^r$, where r is any negative rational number.

Proposition 2.5 (Sandwich Theorem). *Let (a_n), (b_n), (c_n) be sequences and let $c \in \mathbb{R}$ be such that $a_n \le c_n \le b_n$ for all $n \in \mathbb{N}$ and $a_n \to c$ as well as $b_n \to c$. Then $c_n \to c$.*

Proof. Let $\epsilon > 0$ be given. Since $a_n \to c$, there is $n_1 \in \mathbb{N}$ such that $a_n - c > -\epsilon$ for all $n \ge n_1$, and since $b_n \to c$, there is $n_2 \in \mathbb{N}$ such that $b_n - c < \epsilon$ for all $n \ge n_2$. Let $n_0 := \max\{n_1, n_2\}$. Then

$$-\epsilon < a_n - c \le c_n - c \le b_n - c < \epsilon \quad \text{for all } n \ge n_0.$$

It follows that $c_n \to c$. □

We now use the above result to show that the supremum and the infimum of a subset of $\mathbb{R}$ are limits of sequences in that subset.

Corollary 2.6. *Let E be a nonempty subset of $\mathbb{R}$.*

(i) *If E is bounded above and $a := \sup E$, then there is a sequence (a_n) such that $a_n \in E$ for all $n \in \mathbb{N}$ and $a_n \to a$.*
(ii) *If E is bounded below and $b := \inf E$, then there is a sequence (b_n) such that $b_n \in E$ for all $n \in \mathbb{N}$ and $b_n \to b$.*

Proof. To prove (i), suppose E is bounded above. Let $a := \sup E$. Then for every $n \in \mathbb{N}$, there is $a_n \in E$ such that $a_n > a - (1/n)$. Also since $a_n \le a$ for all $n \in \mathbb{N}$, by the Sandwich Theorem, we see that $a_n \to a$.

The proof of (ii) is similar. □

Examples 2.7. (i) Let $a \in \mathbb{R}$ with $|a| < 1$. Then

$$\lim_{n\to\infty} a^n = 0 \quad \text{and} \quad \lim_{n\to\infty} na^n = 0.$$

If $a = 0$, this is obvious. Suppose $a \ne 0$. Write $1/|a| = 1 + h$. Then $h > 0$, and by the Binomial Theorem,

$$\frac{1}{|a|^n} = (1+h)^n = 1 + nh + \frac{n(n-1)}{2}h^2 + \cdots + h^n \quad \text{for all } n \in \mathbb{N}.$$

Consequently, $1/|a|^n > nh$ for all $n \in \mathbb{N}$ and $1/|a|^n > n(n-1)h^2/2$ for all $n \in \mathbb{N}$ with $n > 1$. Hence $0 < |a|^n < 1/nh$ and $0 < n|a|^n < 2/(n-1)h^2$ for all $n \in \mathbb{N}$ with $n > 1$. By part (ii) of Proposition 2.3 and Example 2.1 (iii), $1/nh \to 0$ and $2/(n-1)h^2 \to 0$. Therefore, by the Sandwich Theorem, $|a|^n \to 0$ and $n|a|^n \to 0$, and thus $a^n \to 0$ and $na^n \to 0$. Using the former, we can find the limit of the sequence (A_n), where $A_n := 1 + a + \cdots + a^n$ for $n \in \mathbb{N}$. Since $A_n = (1 - a^{n+1})/(1-a)$ for $n \in \mathbb{N}$, it follows that

$$\lim_{n\to\infty}(1 + a + \cdots + a^n) = \frac{1}{1-a}.$$

(ii) Let $a \in \mathbb{R}$. Then

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0.$$

Choose $m \in \mathbb{N}$ such that $|a| < m$. Then for $n > m$,

$$0 \le \left|\frac{a^n}{n!}\right| = \frac{|a|^n}{m!}\left(\prod_{j=m+1}^{n}\frac{1}{j}\right) < \frac{|a|^n}{m!}\left(\frac{1}{m^{n-m}}\right) = \frac{m^m}{m!}\left(\frac{|a|}{m}\right)^n.$$

Since m is a constant and $(|a|/m)^n \to 0$ by (i) above, the Sandwich Theorem shows that $|a^n/n!| \to 0$, that is, $a^n/n! \to 0$.

(iii) Let $a \in \mathbb{R}$ and $a > 0$. Then

$$\lim_{n\to\infty} a^{1/n} = 1.$$

This is obvious if $a = 1$. Suppose $a > 1$. Let $h_n := a^{1/n} - 1$ for $n \in \mathbb{N}$. Then by the Binomial Inequality (Proposition 1.10),

$$a = (1 + h_n)^n \ge 1 + nh_n > nh_n,$$

and so $0 < h_n < a/n$ for all $n \in \mathbb{N}$. Since $a/n \to 0$, it follows from the Sandwich Theorem that $h_n \to 0$, that is, $a^{1/n} \to 1$. If $0 < a < 1$, let $b := 1/a$, so that $1/a^{1/n} = b^{1/n} \to 1$ because $b > 1$. Hence by part (iv) of Proposition 2.3, $a^{1/n} \to 1/1 = 1$.

(iv) Consider the sequence $(n^{1/n})$. As in (iii) above, let $h_n := n^{1/n} - 1$. Then $h_n \ge 0$ for all $n \in \mathbb{N}$, whereas for $n \ge 2$,

$$n = (1 + h_n)^n \ge 1 + nh_n + \frac{n(n-1)}{2}h_n^2 > \frac{n(n-1)}{2}h_n^2.$$

Thus $0 \le h_n < \sqrt{2/(n-1)}$ for $n \ge 2$. By part (iv) of Proposition 2.4 and Example 2.1 (iii), $\sqrt{2/(n-1)} \to 0$. Hence by the Sandwich Theorem, $h_n \to 0$, and so $n^{1/n} \to 1$.

(v) For $n \in \mathbb{N}$, let

$$C_n := \frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}.$$

Then $C_n \to 1$. To see this, for $n \in \mathbb{N}$, let

$$A_n := \sum_{k=1}^{n} \frac{n}{n^2+n} = \frac{n^2}{n^2+n} \quad \text{and} \quad B_n := \sum_{k=1}^{n} \frac{n}{n^2+1} = \frac{n^2}{n^2+1}.$$

Then $A_n \leq C_n \leq B_n$ for all $n \in \mathbb{N}$, and by Proposition 2.3,

$$A_n = \frac{1}{1+(1/n)} \to 1 \quad \text{and} \quad B_n = \frac{1}{1+(1/n^2)} \to 1.$$

Hence by the Sandwich Theorem, $C_n \to 1$. $\diamond$

We have seen in part (ii) of Proposition 2.2 that every convergent sequence is bounded. On the other hand, not every bounded sequence is convergent, as is shown by the example $a_n = (-1)^n$ for $n \in \mathbb{N}$. We shall now consider a class of sequences for which convergence is equivalent to boundedness.

A sequence (a_n) is called **(monotonically) increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. Likewise, (a_n) is called **(monotonically) decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is called **monotonic** if it is monotonically increasing or monotonically decreasing.

Proposition 2.8. (i) *A monotonically increasing sequence is convergent if and only if it is bounded above. Moreover, if a sequence (a_n) is monotonically increasing and bounded above, then*

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

(ii) *A monotonically decreasing sequence is convergent if and only if it is bounded below. Moreover, if a sequence (a_n) is monotonically decreasing and bounded below, then*

$$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}.$$

Proof. (i) Let (a_n) be a monotonically increasing sequence. Suppose it is bounded above. Then the set $\{a_n : n \in \mathbb{N}\}$ has a supremum. Let $a := \sup\{a_n : n \in \mathbb{N}\}$. Given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $a - \epsilon < a_{n_0}$. But since (a_n) is monotonically increasing, $a_{n_0} \leq a_n$ for all $n \geq n_0$. Hence

$$a - \epsilon < a_n \leq a < a + \epsilon \quad \text{for all } n \geq n_0.$$

Thus $a_n \to a$. Conversely, if (a_n) is convergent, then it is bounded above by part (ii) of Proposition 2.2.

(ii) A proof similar to the one above can be given. Alternatively, one may observe that if (a_n) is monotonically decreasing, and if we let $b_n := -a_n$, then (b_n) monotonically increasing. Also, (a_n) is bounded below if and only if (b_n) is bounded above, and in this case, $\inf\{a_n : n \in \mathbb{N}\} = -\sup\{b_n : n \in \mathbb{N}\}$. Also, by part (ii) of Proposition 2.3, $a_n \to a$ if and only if $-a_n \to -a$. Thus the desired results follow from (i) above. ☐

Corollary 2.9. *A monotonic sequence is convergent if and only if it is bounded.*

Proof. The result follows from parts (i) and (ii) of Proposition 2.8. ☐

Examples 2.10. (i) Consider the sequence (A_n) defined by

$$A_n := \sum_{k=0}^{n} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \quad \text{for } n \in \mathbb{N}.$$

Clearly, (A_n) is a monotonically increasing sequence. Also, for all $n \in \mathbb{N}$,

$$A_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 + 2\left(1 - \frac{1}{2^n}\right) < 3.$$

Hence (A_n) is bounded above. So by part (i) of Proposition 2.8, (A_n) is convergent.

(ii) Consider the sequence (B_n) defined by

$$B_n := \left(1 + \frac{1}{n}\right)^n \quad \text{for } n \in \mathbb{N}.$$

We show that (B_n) is convergent and its limit is equal to the limit of the sequence (A_n) considered in (i) above. By the Binomial Theorem,

$$B_n = \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k} = \sum_{k=0}^{n} 1\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)\frac{1}{k!}.$$

This implies that $B_n \leq A_n$ for all $n \in \mathbb{N}$.
To find a lower bound for B_n in terms of A_n, we use the generalized binomial inequality given in Proposition 1.10, and obtain for $k = 1, \ldots, n$,

$$(1-0)\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) \geq 1 - \left(0 + \frac{1}{n} + \cdots + \frac{k-1}{n}\right).$$

Now, since $0 + 1 + 2 + \cdots + (k-1) = (k-1)k/2$, we see that

$$1 - \frac{(k-1)k}{2n} \leq \frac{n(n-1)\cdots(n-k+1)}{n^k} \quad \text{for } k = 0, 1, \ldots, n.$$

Dividing by $k!$ and summing from $k = 0$ to $k = n$, we obtain

$$\sum_{k=0}^{n} \frac{1}{k!} - \frac{1}{2n} \sum_{k=2}^{n} \frac{1}{(k-2)!} \leq B_n.$$

Moreover, from (i) above,

$$\sum_{k=2}^{n} \frac{1}{(k-2)!} \leq A_n < 3 \quad \text{and hence} \quad A_n - \frac{3}{2n} < B_n \leq A_n \quad \text{for all } n \in \mathbb{N}.$$

Therefore, by the Sandwich Theorem, (B_n) is convergent, and its limit is equal to the limit of (A_n).
Alternatively, we may argue as follows. Let $A_n \to A$ and $\epsilon > 0$ be given. Then there is $n_0 \in \mathbb{N}$ such that $A-(\epsilon/2) < a_n$ for all $n \geq n_0$. In particular, $A - (\epsilon/2) < A_{n_0}$. For $n \in \mathbb{N}$, define

$$C_n := \sum_{k=0}^{n_0} 1 \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \frac{1}{k!}.$$

Then $C_n \leq B_n$ for all $n \geq n_0$. By parts (i) and (iii) of Proposition 2.3, $C_n \to \sum_{k=0}^{n_0}(1/k!) = A_{n_0}$ as $n \to \infty$. Hence there is $n_1 \in \mathbb{N}$ such that

$$A_{n_0} - \frac{\epsilon}{2} < C_n \quad \text{for all } n \geq n_1.$$

Now for all $n \geq \max\{n_0, n_1\}$,

$$A - \epsilon < A_{n_0} - \frac{\epsilon}{2} < C_n \leq B_n \leq A_n \leq A < A + \epsilon.$$

This proves that $B_n \to A$. We remark that yet another proof of the convergence of the sequence (B_n) is indicated in Exercise 2.8. The common limit of the sequences (A_n) and (B_n) is an important real number. This real number, denoted by e, will be introduced in Chapter 7. See Section 7.1 and, in particular, Corollary 7.7.

(iii) Consider the sequence (A_n) defined by

$$A_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad \text{for } n \in \mathbb{N}.$$

Clearly, (A_n) is a monotonically increasing sequence. Also, for $n \in \mathbb{N}$,

$$\begin{aligned} A_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right) \\ &\geq 1 + \frac{1}{2} + \frac{2}{4} + \cdots + \frac{2^{n-1}}{2^n} \\ &= 1 + \frac{n}{2}. \end{aligned}$$

Hence there is no $\alpha \in \mathbb{R}$ such that $A_n \leq \alpha$ for all $n \in \mathbb{N}$, that is, (A_n) is not bounded above. So (A_n) is not convergent. Let us modify the sequence (A_n) by changing the signs of alternate summands of its terms and define

$$B_n := 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n-1}\frac{1}{n} \quad \text{for } n \in \mathbb{N}.$$

We shall show that (B_n) is convergent. First note that $\frac{1}{2} \le B_n \le 1$. Let $C_n := B_{2n-1}$ and $D_n := B_{2n}$ for $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$,

$$C_{n+1} - C_n = \frac{1}{2n+1} - \frac{1}{2n} \le 0 \quad \text{and} \quad D_{n+1} - D_n = \frac{1}{2n+1} - \frac{1}{2n+2} \ge 0.$$

Hence (C_n) is a monotonically decreasing sequence that is bounded below by $\frac{1}{2}$, and (D_n) is a monotonically increasing sequence that is bounded above by 1. By Proposition 2.8, both (C_n) and (D_n) are convergent. Let $C_n \to C$ and $D_n \to D$. Now since $C_n - D_n = \frac{1}{2n} \to 0$, we obtain $C = D$. It follows that (B_n) is convergent and $B_n \to C$.

(iv) Consider the sequence (A_n) defined by

$$A_n := 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \quad \text{for } n \in \mathbb{N}.$$

Clearly, (A_n) is a monotonically increasing sequence. Also, for $n \in \mathbb{N}$,

$$\begin{aligned} A_n &\le 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 2 - \frac{1}{n} < 2. \end{aligned}$$

Hence (A_n) is bounded above. So by part (i) of Proposition 2.8, (A_n) is convergent.

(v) Let p be a rational number and consider the sequence (A_n) defined by

$$A_n := 1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p} \quad \text{for } n \in \mathbb{N}.$$

Clearly, (A_n) is a monotonically increasing sequence. We have seen in (iii) and (iv) above that if $p = 1$, then the sequence (A_n) is divergent, whereas if $p = 2$, then it is convergent. This implies that (A_n) is divergent if $p \le 1$, and it is convergent if $p \ge 2$, because for each $n \in \mathbb{N}$,

$$0 < \frac{1}{n} \le \frac{1}{n^p} \quad \text{if } p \le 1, \quad \text{while} \quad 0 < \frac{1}{n^p} \le \frac{1}{n^2} \quad \text{if } p \ge 2.$$

We now give an alternative argument that shows that (A_n) is convergent if $p > 1$. Suppose $p > 1$. For $n \in \mathbb{N}$,

$$\begin{aligned}
A_{2n+1} &= 1 + \left(\frac{1}{2^p} + \frac{1}{4^p} + \cdots + \frac{1}{(2n)^p}\right) + \left(\frac{1}{3^p} + \frac{1}{5^p} + \cdots + \frac{1}{(2n+1)^p}\right) \\
&< 1 + 2\left(\frac{1}{2^p} + \frac{1}{4^p} + \cdots + \frac{1}{(2n)^p}\right) \\
&= 1 + \frac{2}{2^p}\left(1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p}\right) \\
&= 1 + 2^{1-p} A_n \\
&< 1 + 2^{1-p} A_{2n+1}.
\end{aligned}$$

Since $2^{1-p} < 1$, we see that $A_{2n+1} < 1/(1-2^{p-1})$ for $n \in \mathbb{N}$. Also, since $A_{2n} < A_{2n+1}$ for all $n \in \mathbb{N}$, it follows that (A_n) is bounded above. So by part (i) of Proposition 2.8, we see that (A_n) is convergent.

(vi) Consider a sequence defined by a linear recurrence relation, that is, let $\alpha, \beta, \gamma \in \mathbb{R}$ and let (a_n) be defined by

$$a_1 := \alpha \quad \text{and} \quad a_{n+1} := \beta a_n + \gamma \quad \text{for } n \in \mathbb{N}.$$

Suppose α, β, γ are nonnegative and $\beta < 1$. Then $a_n \geq 0$ for all $n \in \mathbb{N}$. Now $a_2 - a_1 = \beta a_1 + \gamma - a_1 = \gamma - (1-\beta)\alpha$, whereas for $n \in \mathbb{N}$ with $n > 1$,

$$a_{n+1} - a_n = \beta a_n + \gamma - (\beta a_{n-1} + \gamma) = \beta(a_n - a_{n-1}).$$

This implies that (a_n) is monotonically increasing or decreasing according as $a_2 - a_1 \geq 0$ or $a_2 - a_1 \leq 0$. Suppose $a_2 - a_1 \geq 0$, that is, $\gamma \geq (1-\beta)\alpha$. Then

$$a_n \leq a_{n+1} = \beta a_n + \gamma \Longrightarrow (1-\beta)a_n \leq \gamma \Longrightarrow a_n \leq \frac{\gamma}{1-\beta}.$$

Thus (a_n) is monotonically increasing and bounded above. On the other hand, if $a_2 - a_1 < 0$, then (a_n) is (strictly) decreasing, and since each a_n is nonnegative, (a_n) is bounded below. So in any case, Proposition 2.8 shows that (a_n) is convergent. Let $a_n \to a$. Then $a_{n+1} \to a$, and since $a_{n+1} = \beta a_n + \gamma \to \beta a + \gamma$, we obtain $a = \beta a + \gamma$, that is, $a = \gamma/(1-\beta)$. $\diamond$

Remark 2.11. We introduce some notation for comparing the orders of magnitude of two sequences. This will also be useful in comparing the relative growth rates of sequences. Let (a_n) and (b_n) be any sequences in $\mathbb{R}$.

If there are $K > 0$ and $n_0 \in \mathbb{N}$ such that

$$|a_n| \leq K|b_n| \text{ for all } n \geq n_0, \quad \text{then we write} \quad a_n = O(b_n).$$

One reads the statement "$a_n = O(b_n)$" as "(a_n) is big-oh of (b_n)". If $b_n = 1$ for all large n, then we simply write $a_n = O(1)$, and this means that the sequence (a_n) is bounded. For example,

$$(-1)^n = O(1), \quad 10\,n + 100 = O(n), \quad \text{and} \quad (-1)^n \frac{10}{n} + \frac{100}{n\sqrt{n}} = O\left(\frac{1}{n}\right).$$

If for every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$|a_n| \le \epsilon |b_n| \text{ for all } n \ge n_0, \quad \text{then we write} \quad a_n = o(b_n).$$

One reads the statement "$a_n = o(b_n)$" as "(a_n) is little-oh of (b_n)". If $b_n \neq 0$ for all large n, then $a_n = o(b_n)$ means that $\lim_{n\to\infty}(a_n/b_n)$ exists and is zero. In particular, if $b_n = 1$ for all large n, then we simply write $a_n = o(1)$, and this means that $a_n \to 0$. For example,

$$\frac{(-1)^n}{n} = o(1), \quad 10\,n + 100 = o(n\sqrt{n}), \quad \text{and} \quad (-1)^n \frac{10}{n} + \frac{100}{n\sqrt{n}} = o\left(\frac{1}{\sqrt{n}}\right).$$

Now suppose $b_n \neq 0$ for all large n. We say that (a_n) is **asymptotically equivalent** to (b_n) and write $a_n \sim b_n$ if $\lim_{n\to\infty}(a_n/b_n) = 1$. For example,

$$n + \frac{1}{n} \sim n, \quad n^2 + 10\,n + 100 \sim n^2, \quad \text{and} \quad \frac{1}{n^2} + \frac{10}{n^3} + \frac{100}{n^4} \sim \frac{1}{n^2}.$$

Finally, suppose (b_n) is monotonic and $b_n > 0$ for all large n. We say that (a_n) and (b_n) have the same **growth rate** if $a_n \sim \ell\, b_n$ for some $\ell \in \mathbb{R}$ with $\ell \neq 0$. In case $a_n = o(b_n)$, then we say that the **growth rate** of (a_n) is less than the growth rate of (b_n). On the other hand, if $a_n = O(b_n)$, then we say that the **growth rate** of (a_n) is at most the growth rate of (b_n). $\diamond$

We shall now describe how in some cases ∞ or $-\infty$ can be regarded as a "limit" of a sequence (a_n). We say that (a_n) **tends** to ∞ or **diverges** to ∞ if for every $\alpha \in \mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that $a_n > \alpha$ for all $n \ge n_0$, and then we write $a_n \to \infty$. We write $a_n \not\to \infty$ if (a_n) does not tend to ∞. Similarly, we say that (a_n) **tends** to $-\infty$ or **diverges** to $-\infty$ if for every $\beta \in \mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that $a_n < \beta$ for all $n \ge n_0$, and then we write $a_n \to -\infty$. We write $a_n \not\to -\infty$ if (a_n) does not tend to $-\infty$.

Remark 2.12. Let (a_n) and (b_n) be any sequences of real numbers. Suppose $a_n \to \infty$. Then $a_n > 0$ for all large n, and $1/a_n \to 0$. Now suppose $b_n \to \ell$ for some $\ell \in \mathbb{R} \cup \{\infty\}$. Then it is easy to see that $a_n + b_n \to \infty$, and also that $a_n b_n \to \infty$ if $\ell > 0$, while $a_n b_n \to -\infty$ if $\ell < 0$. If $\ell = 0$, then nothing can be said about the convergence or divergence of $(a_n b_n)$, as the examples (i) $a_n := n$ and $b_n := 0$, (ii) $a_n := n$ and $b_n := 1/n$, (iii) $a_n := n$ and $b_n := 1/\sqrt{n}$, (iv) $a_n := n$ and $b_n := -1/\sqrt{n}$, (v) $a_n := n$ and $b_n := (-1)^n/n$ show. Similar conclusions hold if $a_n \to -\infty$ and $b_n \to \ell$ for some $\ell \in \mathbb{R} \cup \{-\infty\}$.

If $a_n \to \infty$ and $b_n \to -\infty$, then $a_n b_n \to -\infty$, but nothing can be said about the convergence of $a_n + b_n$, as the examples (i) $a_n := n$, $b_n := -n$, (ii) $a_n := n$, $b_n := -n + 1$, (iii) $a_n := n$, $b_n := -\sqrt{n}$, (iv) $a_n := n$, $b_n := -n^2$, (v) $a_n := n$, $b_n := -n + (-1)^n$ show. Some of these "indeterminate" cases can be tested using the method indicated in Remark 4.46. $\diamond$

The following result is a counterpart of Proposition 2.8 for sequences tending to ∞ or $-\infty$.

Proposition 2.13. *Let (a_n) be a sequence in $\mathbb{R}$.*

(i) *Suppse (a_n) is monotonically increasing. Then $a_n \to \infty$ if and only if (a_n) is not bounded above.*
(ii) *Suppse (a_n) is monotonically decreasing. Then $a_n \to -\infty$ if and only if (a_n) is not bounded below.*

Proof. (i) If $a_n \to \infty$, then it is clear that (a_n) is not bounded above. Conversely, if (a_n) is not bounded above, then for every $\alpha \in \mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that $a_{n_0} > \alpha$. Since (a_n) is monotonically increasing, we see that $a_n > \alpha$ for all $n \geq n_0$. Thus $a_n \to \infty$.

(ii) A proof similar to (i) can easily be given. □

Examples 2.14. (i) Let p be a positive rational number and $a_n := n^p$ for $n \in \mathbb{N}$. Then $a_n \to \infty$. Indeed, given any $\alpha \in \mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that $n_0 > |\alpha|^{1/p}$, and consequently, $a_n > \alpha$ for $n \geq n_0$.
(ii) Let $a_n := \sum_{k=1}^{n}(1/k)$. Then $a_n \to \infty$, since (a_n) is monotonically increasing, and as shown in Example 2.10 (iii), (a_n) is not bounded above.
(iii) Let $a \in \mathbb{R}$ be such that $|a| > 1$ and let $a_n := a^n$ for $n \in \mathbb{N}$. If $a > 1$, then $1/a_n = (1/a)^n \to 0$ (Example 2.7 (i)), and so $a_n \to \infty$. If $a < -1$, then let $b_n := a_{2n-1}$ and $c_n := a_{2n}$ for $n \in \mathbb{N}$. Clearly, $b_n = (a^2)^n/a \to -\infty$ and $c_n = (a^2)^n \to \infty$. Hence $a_n \not\to \infty$ and $a_n \not\to -\infty$. ◇

We end this section with a notion that is slightly weaker than convergence. Let (a_n) be a sequence in $\mathbb{R}$. Then the sequence (b_n) in $\mathbb{R}$ defined by

$$b_n := \frac{a_1 + \cdots + a_n}{n} \quad \text{for } n \in \mathbb{N}$$

is called the sequence of **arithmeric means** of (a_n). We say that (a_n) is **Cesàro convergent** if (b_n) is convergent.

Proposition 2.15. *Let (a_n) be a sequence in $\mathbb{R}$ and let (b_n) be the sequence of arithmetic means of (a_n). If $a \in \mathbb{R}$ and $a_n \to a$, then $b_n \to a$. Conversely, if (b_n) is convergent, then (a_n) is convergent if and only if $\frac{1}{n}\sum_{k=1}^{n}(a_n - a_k) \to 0$.*

Proof. Suppose $a_n \to a$ for some $a \in \mathbb{R}$. Let $\epsilon > 0$ be given. Then there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon/2$ for $n \geq n_0$. Now, for $n > n_0$,

$$|b_n - a| = \frac{1}{n}\left|\sum_{k=1}^{n}(a_k - a)\right| \leq \frac{1}{n}\sum_{k=1}^{n_0}|a_k - a| + \frac{1}{n}\sum_{k=n_0+1}^{n}|a_k - a|.$$

Thus if we choose $n_1 \in \mathbb{N}$ such that $n_1 > n_0$ and $\frac{1}{n_1}\sum_{k=1}^{n_0}|a_k - a| < \epsilon/2$, then

$$|b_n - a| < \frac{\epsilon}{2} + \frac{n - n_0}{n} \cdot \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for } n \geq n_1.$$

This proves that $b_n \to a$. Conversely, suppose (b_n) is convergent. Then clearly (a_n) is convergent if and only if $a_n - b_n \to 0$. But $a_n - b_n$ can be written as

$$a_n - \frac{a_1 + \cdots + a_n}{n} = \frac{(a_n - a_1) + \cdots + (a_n - a_n)}{n} = \frac{1}{n}\sum_{k=1}^{n}(a_n - a_k).$$

This yields the desired result. □

The above proposition shows that a convergent sequence is Cesàro convergent. However, a Cesàro convergent sequence may not be convergent. For example, if $a_n := (-1)^n$ and $b_n := (a_1 + \cdots + a_n)/n$ for $n \in \mathbb{N}$, then b_n is 0 or $-1/n$ according as n is even or odd. Hence (b_n) is convergent, but clearly, (a_n) is not convergent.

One can use Proposition 2.15 to quickly compute limits of some seemingly complicated sequences. For example, since $1/n \to 0$, we obtain

$$\frac{1}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \to 0, \quad \text{even though} \quad \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \to \infty.$$

More such examples are given in Exercise 2.13. (See also Exercise 2.24.)

2.2 Subsequences and Cauchy Sequences

Let (a_n) be a sequence. If $n_1, n_2, \ldots$ are positive integers such that $n_k < n_{k+1}$ for each $k \in \mathbb{N}$, then the sequence (a_{n_k}), whose terms are

$$a_{n_1}, a_{n_2}, \ldots,$$

is called a **subsequence** of (a_n). Note that $n_1 < n_2 < \cdots$ implies that $n_k \geq k$ for all $k \in \mathbb{N}$, and in particular, $n_k \to \infty$ as $k \to \infty$.

It is easy to see that a sequence (a_n) converges to a if and only if every subsequence of (a_n) converges to a. This follows by observing that (a_n) is itself a subsequence of (a_n) and on the other hand, if (a_{n_k}) is a subsequence of (a_n), then for every $n_0 \in \mathbb{N}$, there is $k_0 \in \mathbb{N}$ such that $n_k \geq n_0$ for all $k \geq k_0$.

Similarly, it can be seen that a sequence (a_n) tends to ∞ if and only if every subsequence of (a_n) tends to ∞, and that (a_n) tends to $-\infty$ if and only if every subsequence of (a_n) tends to $-\infty$.

We now prove a remarkable fact about monotonic subsequences.

Proposition 2.16. *Every sequence in $\mathbb{R}$ has a monotonic subsequence.*

Proof. Let (a_n) be a sequence in $\mathbb{R}$. Consider the "peaks" in (a_n), that is, those terms that are greater than all the succeeding terms. Let E be the set of all positive integers n for which a_n is a "peak", that is, let

$$E = \{n \in \mathbb{N} : a_n > a_m \text{ for all } m > n\}.$$

First assume that E is a finite set. Then there is $n_1 \in \mathbb{N}$ such that $n_1 > n$ for every $n \in E$. Since $n_1 \notin E$, there is $n_2 \in \mathbb{N}$ such that $n_2 > n_1$ and $a_{n_1} \leq a_{n_2}$.

Again, since $n_2 \notin E$, there is $n_3 \in \mathbb{N}$ such that $n_3 > n_2$ and $a_{n_2} \le a_{n_3}$. Having chosen n_k for $k \in \mathbb{N}$ in this manner, we note that $n_k \notin E$, and hence there is $n_{k+1} \in \mathbb{N}$ such that $n_{k+1} > n_k$ and $a_{n_k} \le a_{n_{k+1}}$. Thus we obtain a monotonically increasing subsequence (a_{n_k}) of (a_n).

Next, assume that E is an infinite set. If we enumerate E as $n_1, n_2, \ldots$, where $n_1 < n_2 < \cdots$, then since $n_k \in E$ for each $k \in \mathbb{N}$, we obtain $a_{n_k} > a_m$ for all $m > n_k$. In particular, taking $m = n_{k+1}$, we get $a_{n_k} > a_{n_{k+1}}$ for each $k \in \mathbb{N}$. Thus we obtain a monotonically decreasing subsequence (a_{n_k}) of (a_n).

In any case, we have proved that (a_n) has a monotonic subsequence. □

We shall use the above result to prove two important results in analysis, known as the Bolzano–Weierstrass Theorem and the Cauchy Criterion.

Proposition 2.17 (Bolzano–Weierstrass Theorem). *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof. Let (a_n) be a bounded sequence in $\mathbb{R}$. By Proposition 2.16, (a_n) has a monotonic subsequence (a_{n_k}). Since (a_n) is bounded, so is its subsequence (a_{n_k}). Hence by Corollary 2.9, (a_{n_k}) is convergent. □

For an alternative proof of the Bolzano–Weierstrass Theorem, see Exercise 2.38. The following result may be viewed as a stronger version of the Bolzano–Weierstrass Theorem.

Corollary 2.18. *Let (a_n) be a sequence in $\mathbb{R}$. If either (a_n) is bounded above and $a_n \not\to -\infty$, or if (a_n) is bounded below and $a_n \not\to \infty$, then (a_n) has a convergent subsequence.*

Proof. First assume that (a_n) is bounded above and $a_n \not\to -\infty$. The statement $a_n \not\to -\infty$ means there is $\beta \in \mathbb{R}$ such that for every $n_0 \in \mathbb{N}$, there is $n \in \mathbb{N}$ with $n > n_0$ and $a_n \ge \beta$. Hence there are positive integers $n_1 < n_2 < \cdots$ such that $a_{n_k} \ge \beta$ for each $k \in \mathbb{N}$. The subsequence (a_{n_k}) in $\mathbb{R}$ is thus bounded above as well as bounded below. So by the Bolzano–Weierstrass Theorem, (a_{n_k}) has a convergent subsequence. Finally, we note that a subsequence of (a_{n_k}) is a subsequence of (a_n) itself.

The case in which (a_n) is bounded below and $a_n \not\to \infty$ can be treated similarly. □

See Exercise 2.40 for an alternative proof of the above corollary, and in fact, a more elaborate version of it. A consequence of the Bolzano–Weierstrass Theorem is the following useful characterization of convergent sequences.

Proposition 2.19. *A sequence in $\mathbb{R}$ is convergent if and only if it is bounded and all of its convergent subsequences have the same limit.*

Proof. If a sequence (a_n) converges to a, then it is bounded by part (ii) of Proposition 2.2, and it is clear that every subsequence (and not just every convergent subsequence) of (a_n) converges to a.

Conversely, assume that (a_n) is a bounded sequence and there is $a \in \mathbb{R}$ such that every convergent subsequence of (a_n) converges to a. We claim that $a_n \to a$. For otherwise, there exist $\epsilon > 0$ and positive integers $n_1 < n_2 < \cdots$ such that $|a_{n_k} - a| \geq \epsilon$ for all $k \in \mathbb{N}$. By the Bolzano–Weierstrass Theorem, the bounded sequence (a_{n_k}) has a convergent subsequence, which cannot possibly converge to a. This is a contradiction. $\square$

In general, proving the convergence of a sequence (a_n) is difficult, since we must correctly guess the limit of (a_n) beforehand. There is a way of avoiding this guesswork, which we now describe.

A sequence (a_n) in $\mathbb{R}$ is called a **Cauchy sequence** if for every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon \quad \text{for all } n,\, m \geq n_0.$$

If (a_n) is a Cauchy sequence in $\mathbb{R}$, then clearly, $(a_{n+1} - a_n) \to 0$. The converse, however, does not hold. For example, if $a_n := \sqrt{n}$ for $n \in \mathbb{N}$, then

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0,$$

but (a_n) is not a Cauchy sequence, because given any $n_0 \in \mathbb{N}$,

$$a_{4n_0} - a_{n_0} = \sqrt{4n_0} - \sqrt{n_0} = \sqrt{n_0} \geq 1.$$

The following result gives a useful sufficient condition for a sequence to be Cauchy. A more general sufficient condition is given in Exercise 2.28.

Proposition 2.20. *Let (a_n) be a sequence of real numbers and let α be a real number such that $\alpha < 1$ and*

$$|a_{n+1} - a_n| \leq \alpha |a_n - a_{n-1}| \quad \textit{for all } n \in \mathbb{N} \textit{ with } n \geq 2.$$

Then (a_n) is a Cauchy sequence.

Proof. For $n \in \mathbb{N}$,

$$|a_{n+1} - a_n| \leq \alpha |a_n - a_{n-1}| \leq \alpha^2 |a_{n-1} - a_{n-2}| \leq \cdots \leq \alpha^{n-1} |a_2 - a_1|.$$

Hence for all $m,\, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned}
|a_m - a_n| &\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \\
&\leq |a_2 - a_1| \left(\alpha^{m-2} + \alpha^{m-3} + \cdots + \alpha^{n-1}\right) \\
&= |a_2 - a_1| \alpha^{n-1} \frac{(1 - \alpha^{m-n})}{(1 - \alpha)} \\
&\leq |a_2 - a_1| \alpha^{n-1} \frac{1}{1 - \alpha}.
\end{aligned}$$

If $a_2 = a_1$, then it is clear that $a_n = a_1$ for all $n \in \mathbb{N}$, and (a_n) is a Cauchy sequence. Suppose $a_2 \neq a_1$ and let $\epsilon > 0$ be given. Since $\alpha < 1$, we see that $\alpha^n \to 0$, as in Example 2.7 (i). Consequently, there is $n_0 \in \mathbb{N}$ such that

$$\alpha^{n-1} < \frac{\epsilon(1-\alpha)}{|a_2 - a_1|} \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq n_0.$$

It follows that $|a_m - a_n| < \epsilon$ for all m, $n \in \mathbb{N}$ with $m, n \geq n_0$. Thus (a_n) is a Cauchy sequence. $\square$

It may be noted that the condition

$$|a_{n+1} - a_n| \leq \alpha |a_n - a_{n-1}| \quad \text{for some } \alpha < 1$$

in the above proposition cannot be weakened to $|a_{n+1} - a_n| < |a_n - a_{n-1}|$. For example, if $a_n := \sqrt{n}$ for $n \in \mathbb{N}$, then

$$|a_{n+1} - a_n| = \sqrt{n+1} - \sqrt{n} < \sqrt{n} - \sqrt{n-1} = |a_n - a_{n-1}|,$$

but as we have just seen, the sequence (a_n) is not Cauchy.

It is easy to see that a convergent sequence is Cauchy. We shall prove this, and also the converse, in Proposition 2.22. For now, we give a useful necessary condition for a sequence to be Cauchy, and a useful sufficient condition for a Cauchy sequence to be convergent.

Proposition 2.21. *Let (a_n) be a Cauchy sequence in $\mathbb{R}$. Then*

(i) *(a_n) is bounded.*
(ii) *If (a_n) has a convergent subsequence, then (a_n) is convergent.*

Proof. (i) Since (a_n) is Cauchy, there is $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq n_0$. Hence

$$|a_n| \leq |a_n - a_{n_0}| + |a_{n_0}| < 1 + |a_{n_0}| \quad \text{for all } n \geq n_0.$$

If we let $\alpha := \max\{|a_1|, \ldots, |a_{n_0 - 1}|, 1 + |a_{n_0}|\}$, then $|a_n| \leq \alpha$ for all $n \in \mathbb{N}$. Hence (a_n) is bounded.

(ii) Suppose (a_n) has a convergent subsequence (a_{n_k}). Let $a_{n_k} \to a$. We show that in fact $a_n \to a$. Let $\epsilon > 0$ be given. Since (a_n) is Cauchy, there is $n_0 \in \mathbb{N}$ such that

$$|a_n - a_m| < \frac{\epsilon}{2} \quad \text{for all } n, m \geq n_0.$$

Further, since $a_{n_k} \to a$, there is $k_0 \in \mathbb{N}$ such that

$$|a_{n_k} - a| < \frac{\epsilon}{2} \quad \text{for all } k \geq k_0.$$

Also, since $n_1 < n_2 < \cdots$, there is $j \in \mathbb{N}$ such that $j \geq k_0$ and $n_j \geq n_0$. Now

$$|a_n - a| \leq |a_n - a_{n_j}| + |a_{n_j} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } n \geq n_0.$$

Hence (a_n) is convergent. $\square$

We are now ready to prove the Cauchy Criterion, which says that in $\mathbb{R}$, the notions of convergent sequences and Cauchy sequences are equivalent.

Proposition 2.22 (Cauchy Criterion). *A sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.*

Proof. Let (a_n) be a convergent sequence, and let $a_n \to a$. Given any $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon/2$ for all $n \geq n_0$. Consequently,

$$|a_n - a_m| \leq |a_n - a| + |a - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } n, m \geq n_0.$$

Hence (a_n) is a Cauchy sequence.

Conversely, let (a_n) be a Cauchy sequence. By part (i) of Proposition 2.21, (a_n) is bounded. Hence by the Bolzano–Weierstrass Theorem, (a_n) has a convergent subsequence. Thus, by part (ii) of Proposition 2.21, we conclude that (a_n) is convergent. □

The following example shows how the Cauchy Criterion can be used to prove the convergence of a sequence.

Example 2.23. Consider the sequence (a_n) defined by

$$a_1 := 1 \quad \text{and} \quad a_{n+1} := 1 + \frac{1}{a_n} \quad \text{for } n \in \mathbb{N}.$$

First note that $a_n \geq 1$ for all $n \in \mathbb{N}$ and hence

$$a_n a_{n-1} = \left(1 + \frac{1}{a_{n-1}}\right) a_{n-1} = a_{n-1} + 1 \geq 2 \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq 2.$$

Since

$$a_{n+1} - a_n = \left(1 + \frac{1}{a_n}\right) - \left(1 + \frac{1}{a_{n-1}}\right) = \frac{1}{a_n} - \frac{1}{a_{n-1}} = \frac{a_{n-1} - a_n}{a_n a_{n-1}},$$

we see that

$$|a_{n+1} - a_n| \leq \frac{1}{2}|a_n - a_{n-1}| \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq 2.$$

Hence by Proposition 2.20, (a_n) is a Cauchy sequence, and by Proposition 2.22, it is convergent. Let $a_n \to a$. Then $a_{n+1} \to a$, and since $a_{n+1} = 1 + (1/a_n)$, we see that $a = 1 + (1/a)$. Also, $a_n \geq 1$ for all $n \in \mathbb{N}$ implies that $a \geq 1$. Hence $a = (1 + \sqrt{5})/2$.

It may be noted that (a_n) is not a monotonic sequence. In fact, for every $n \in \mathbb{N}$ with $n \geq 2$, $a_n \leq a_{n+1}$ if and only if $a_{n-1} \geq a_n$. So we cannot appeal to Proposition 2.22 to deduce the convergence of (a_n). ◇

We remark that the Completeness Property of $\mathbb{R}$ is crucially used (via Corollary 2.9 and the Bolzano–Weierstrass Theorem) in the proof that every Cauchy sequence in $\mathbb{R}$ is convergent. Conversely, assuming that every Cauchy sequence in $\mathbb{R}$ is convergent, it is possible to establish the Completeness Property of $\mathbb{R}$. (See Exercise 2.42.) In view of this, the result in Proposition 2.22 is sometimes referred to as the **Cauchy completeness** of $\mathbb{R}$. For more on Cauchy completeness and related matters, see Remark A.16 in Appendix A.

2.3 Cluster Points of Sequences

A sequence of real numbers may not converge, but it may have convergent subsequences. This leads to a useful notion of a cluster point, and the related notions of limsup and liminf. These will be discussed in this section.

Let (a_n) be a sequence in $\mathbb{R}$. A real number a is called a **cluster point** of (a_n) if there is a subsequence (a_{n_k}) of (a_n) such that $a_{n_k} \to a$.

Note that Proposition 2.19 can be paraphrased as follows: A sequence in $\mathbb{R}$ is convergent if and only if it is bounded and has exactly one cluster point. The unique cluster point of a convergent sequence is evidently its limit.

Examples 2.24. (i) Let $a_n := (-1)^n$ for $n \in \mathbb{N}$. Then (a_n) is bounded and it has exactly two cluster points, namely 1 and -1.

(ii) Suppose $a_n := n$ if $n \in \mathbb{N}$ is odd and $a_n := 1/n$ if $n \in \mathbb{N}$ is even. Then (a_n) is not bounded, and (a_n) has only one cluster point, namely 0.

(iii) Let $a_n := n$ for $n \in \mathbb{N}$. Then (a_n) has no cluster point.

(iv) Let $r_1, r_2, \ldots$ be an enumeration of $\mathbb{Q}$. Then every $a \in \mathbb{R}$ is a cluster point of the sequence (r_n). To see this, note that for each $k \in \mathbb{N}$, the interval $(a, a + \frac{1}{k})$ contains infinitely many rational numbers. ◇

Proposition 2.25. *Let (a_n) be a sequence in $\mathbb{R}$.*

(i) *Suppose (a_n) is bounded above. For $n \in \mathbb{N}$, define $M_n := \sup\{a_n, a_{n+1}, \ldots\}$. Then (M_n) is a monotonically decreasing sequence. If (M_n) is bounded below, then $\inf\{M_n : n \in \mathbb{N}\}$ is the largest cluster point of (a_n).*

(ii) *Suppose (a_n) is bounded below. For $n \in \mathbb{N}$, define $m_n := \inf\{a_n, a_{n+1}, \ldots\}$. Then (m_n) is a monotonically increasing sequence. If (m_n) is bounded above, then $\sup\{m_n : n \in \mathbb{N}\}$ is the smallest cluster point of (a_n).*

Proof. (i) It is clear that $M_n \le M_{n+1}$ for all $n \in \mathbb{N}$. Thus (M_n) is monotonically decreasing. Suppose (M_n) is bounded below and $M := \inf\{M_n : n \in \mathbb{N}\}$. If $a \in \mathbb{R}$ is any cluster point of (a_n), then there is a subsequence (a_{n_k}) of (a_n) such that $a_{n_k} \to a$. Now let $n \in \mathbb{N}$ be given. Then there is $k_0 \in \mathbb{N}$ such that $n_k \ge n$ for all $k \ge k_0$. Consequently, $a_{n_k} \le \sup\{a_n, a_{n+1}, \ldots\} = M_n$ for all $k \ge k_0$. By letting $k \to \infty$, we obtain $a \le M_n$. Since $n \in \mathbb{N}$ was arbitrary, we see that $a \le M$. Thus to show that M is the largest cluster point of (a_n), it suffices to show that M is a cluster point of (a_n). To this end, first note that there is $\ell_1 \in \mathbb{N}$ such that $M \le M_{\ell_1} < M + 1$. Now $M - 1 < M_{\ell_1} = \sup\{a_n : n \ge \ell_1\}$. So there is $n_1 \ge \ell_1$ such that

$$M - 1 < a_{n_1} \le M_{\ell_1} < M + 1.$$

Next, there is $\ell_2 \in \mathbb{N}$ such that $M \le M_\ell \le M_{\ell_2} < M + \frac{1}{2}$ for all $\ell \ge \ell_2$. Replacing ℓ_2 by a greater integer if necessary, we may assume that $\ell_2 > n_1$. Now $M - \frac{1}{2} < M \le M_{\ell_2} = \sup\{a_n : n \ge \ell_2\}$. So there is $n_2 \ge \ell_2$ such that

$$n_1 < n_2 \quad \text{and} \quad M - \frac{1}{2} < a_{n_2} \le M_{\ell_2} < M + \frac{1}{2}.$$

Continuing in this way, we find positive integers $n_1, n_2, \ldots$ such that

$$n_k < n_{k+1} \quad \text{and} \quad M - \frac{1}{k} < a_{n_k} < M + \frac{1}{k} \quad \text{for all } k \in \mathbb{N}.$$

It follows that (a_{n_k}) is a subsequence of (a_n) and $a_{n_k} \to M$. This proves (i).

(ii) A proof similar to (i) can easily be given. □

Remark 2.26. Let (a_n) and (M_n) be as in part (i) of Proposition 2.25. If (M_n) is bounded below, then $\inf\{M_n : n \in \mathbb{N}\} = \lim_{n\to\infty} M_n$. On the other hand,

$$(M_n) \text{ is not bounded below} \iff M_n \to -\infty \iff a_n \to -\infty.$$

The first equivalence follows from part (ii) of Proposition 2.13. For the second equivalence, note that if $M_n \to -\infty$, then clearly $a_n \to -\infty$, since $a_n \leq M_n$ for all $n \in \mathbb{N}$. For the converse, suppose $a_n \to -\infty$. Then for every $\beta \in \mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that $a_n < \beta$ for all $n \geq n_0$. Hence $M_{n_0} = \sup\{a_n : n \geq n_0\} \leq \beta$, and so $M_n \leq M_{n_0} \leq \beta$ for all $n \geq n_0$. Thus $M_n \to -\infty$.

Next, suppose (a_n) and (m_n) are as in part (ii) of Proposition 2.25. In case (m_n) is bounded above, $\sup\{m_n : n \in \mathbb{N}\} = \lim_{n\to\infty} m_n$. On the other hand,

$$(m_n) \text{ is not bounded above} \iff m_n \to \infty \iff a_n \to \infty.$$

The proof of these equivalences is similar to that in the last paragraph. ◇

In view of Proposition 2.25 and Remark 2.26, we associate to each sequence of real numbers certain extended real numbers as follows.

Let (a_n) be a sequence in $\mathbb{R}$. Define the **limit superior** of (a_n) by

$$\limsup_{n\to\infty} a_n := \begin{cases} \infty & \text{if } (a_n) \text{ is not bounded above,} \\ \lim\limits_{n\to\infty} M_n & \text{if } (a_n) \text{ is bounded above and } a_n \not\to -\infty, \\ -\infty & \text{if } a_n \to -\infty, \end{cases}$$

where $M_n := \sup\{a_n, a_{n+1}, \ldots\}$ for $n \in \mathbb{N}$. Note that (M_n) is well-defined if (a_n) is bounded above, and if in addition, $a_n \not\to -\infty$, then $\lim_{n\to\infty} M_n$ exists. Also, note that if $a_n \to -\infty$, then (a_n) is bounded above. Thus the three cases in the above definition are mutually exclusive, and they exhaust all possibilities. Similarly, define the **limit inferior** of (a_n) by

$$\liminf_{n\to\infty} a_n := \begin{cases} -\infty & \text{if } (a_n) \text{ is not bounded below,} \\ \lim\limits_{n\to\infty} m_n & \text{if } (a_n) \text{ is bounded below and } a_n \not\to \infty, \\ \infty & \text{if } a_n \to \infty, \end{cases}$$

where $m_n := \inf\{a_n, a_{n+1}, \ldots\}$ for $n \in \mathbb{N}$.

Examples 2.27. (i) Let $a_n := (-1)^n$ for $n \in \mathbb{N}$. Then $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$.

(ii) Suppose $a_n := n$ if $n \in \mathbb{N}$ is odd and $a_n := 1/n$ if $n \in \mathbb{N}$ is even. Then $\limsup_{n\to\infty} a_n = \infty$ and $\liminf_{n\to\infty} a_n = 0$.
(iii) Let $a_n := n$ for $n \in \mathbb{N}$. Then $\limsup_{n\to\infty} a_n = \infty = \liminf_{n\to\infty} a_n$.
(iv) Let $r_1, r_2, \ldots$ be an enumeration of $\mathbb{Q}$. Then $\limsup_{n\to\infty} r_n = \infty$ and $\liminf_{n\to\infty} a_n = -\infty$.
(v) Let $a_n := (-1)^n n$ for $n \in \mathbb{N}$. Then (a_n) has no cluster point, i contrast to (iv) above, but $\limsup_{n\to\infty} a_n = \infty$ and $\liminf_{n\to\infty} a_n = -\infty$. $\diamond$

Remarks 2.28. Let (a_n) be a sequence in $\mathbb{R}$.

(i) It is clear from the definition of limit superior and limit inferior that (a_n) is bounded if and only if both $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are real numbers. Moreover, in this case, by Proposition 2.25, we easily see that the set C of all cluster points of (a_n) is nonempty, and

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \ldots\} = \max C$$

and

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, \ldots\} = \min C.$$

(ii) $\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$. Indeed, if (a_n) is bounded, then the inequality follows from (i), since $\min C \le \max C$. On the other hand, $\limsup_{n\to\infty} a_n = \infty$ or $\liminf_{n\to\infty} a_n = -\infty$ according as (a_n) is not bounded above or not bounded below, and so the inequality holds readily.

(iii) By Proposition 2.19, a sequence in $\mathbb{R}$ is convergent if and only if it is bounded and has exactly one cluster point. Hence (i) also shows that

$$(a_n) \text{ is convergent} \iff \limsup_{n\to\infty} a_n \text{ and } \liminf_{n\to\infty} a_n \text{ are in } \mathbb{R} \text{ and are equal.}$$

In this case, $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$. Also, note that

$$a_n \to \infty \iff \liminf_{n\to\infty} a_n = \infty = \limsup_{n\to\infty} a_n,$$
$$a_n \to -\infty \iff \liminf_{n\to\infty} a_n = -\infty = \limsup_{n\to\infty} a_n.$$

These characterizations are immediate consequences of the definitions. $\diamond$

Notes and Comments

The concept of the convergence of a sequence is extremely crucial in calculus and analysis. It is the point of departure from "discrete mathematics" to "continuous mathematics". It makes precise the idea of serially numbered real numbers coming arbitrarily close to a fixed real number. In the next chapter, *this concept will be further extended to state what is meant by the "limit" of a function defined on a subset of* $\mathbb{R}$*.*

The arguments used to prove the convergence of sequences in some of our examples are not so standard. The convergence of the sequence (a_n)*, where* $a_n = 1+(1/2^p)+\cdots+(1/n^p)$ *for* $n \in \mathbb{N}$ *and* $p > 1$*, is proved following an article of Cohen and Knight [20]. Also, the proof (using the generalized binomial inequality) of the convergence of the sequence* (b_n)*, where* $b_n = (1 + (1/n))^n$ *for* $n \in \mathbb{N}$*, is based on the article of Lyon and Ward [63]. Several examples of sequences that we have discussed in this chapter are, in fact, examples of "infinite series" in disguise. We shall study them systematically in Chapter 9.*

The Bolzano–Weierstrass Theorem given in Section 2.2 is the cornerstone of much of mathematical analysis. Many of the properties of a nice class of functions (the "continuous" functions, which we shall introduce in the next chapter) are based on this result. Classically, the Bolzano–Weierstrass Theorem is proved by considering a bounded interval that contains infinitely many terms of the sequence, dividing it in equal halves, and picking a half that contains infinitely many terms of the sequence. This process can be continued, and it leads to a nested sequence of intervals in which a limit of a subsequence is trapped. (See Exercises 2.37 and 2.38.) Another standard approach to proving the Bolzano–Weierstrass Theorem as well as the Cauchy Criterion is to use the notion of a cluster point introduced in Section 2.3. (See Exercises 2.40 and 2.41.)We have bypassed both these methods and instead used a neat result that every sequence in $\mathbb{R}$ *has a monotonic subsequence. This result is easy to prove and can be of interest in itself. It appears, for example, in the books of Spivak [76] (Lemma on page 451) and Ross [69] (Theorem 11.4).*

Exercises

Part A

2.1. Which of the following sequences are bounded? Which of them are convergent? In case of convergence, find the limit.
(i) $a_n := 1/n^2$, (ii) $a_n := \sqrt{n}$, (iii) $a_n := n/(2n+1)$,
(iv) $a_n := \sqrt{n}(\sqrt{n+1} - \sqrt{n})$, (v) $a_n := n^{3/2}(\sqrt{n^3+1} - \sqrt{n^3})$.

2.2. Let (a_n) and (b_n) be sequences in $\mathbb{R}$. Under which of the following conditions is the sequence $(a_n b_n)$ convergent? Justify.
(i) (a_n) is convergent.
(ii) (a_n) is convergent and (b_n) is bounded.
(iii) (a_n) converges to 0 and (b_n) is bounded.
(iv) (a_n) and (b_n) are convergent.

2.3. Let $a, b \in \mathbb{R}$ and let (a_n) be a sequence in $\mathbb{R}$ such that $a_n \to a$ and $a_n \geq b$ for all $n \in \mathbb{N}$. Show that $a \geq b$. Give an example in which $a_n > a$ for all $n \in \mathbb{N}$, but $a_n \to a$.

2.4. Let a and x be real numbers. If (b_n) and (c_n) are sequences in $\mathbb{R}$ such that

$$\lim_{n\to\infty} b_n = 0 = \lim_{n\to\infty} c_n \quad \text{and} \quad a - b_n \le x \le a + c_n \quad \text{for } n \in \mathbb{N},$$

then show that $x = a$.

2.5. If (a_n) is a sequence in $\mathbb{R}$ such that $a_n \neq 0$ for all $n \in \mathbb{N}$, and if $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ exists and is less than 1, then show that $a_n \to 0$.

2.6. If $k \in \mathbb{N}$ and $x \in \mathbb{R}$ with $|x| < 1$, then show that $\lim\limits_{n\to\infty} n^k x^n = 0$.

2.7. For $n \in \mathbb{N}$, let $a_n := n^{1/n}$. Show that $a_1 < a_2 < a_3$ and $a_n > a_{n+1}$ for all $n \ge 3$. Use this to show that the sequence (a_n) is convergent.

2.8. Let $B_n := \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N}$. Show that the sequence (B_n) is monotonically increasing. Deduce that (B_n) is convergent. (Hint: Given $n \in \mathbb{N}$, use the A.M.-G.M. inequality for $a_1 = \cdots = a_n := 1 + (1/n)$ and $a_{n+1} := 1$. Also, note that $B_n \le 3$ for all $n \in \mathbb{N}$.)

2.9. Show that the sequence (a_n) is convergent and find its limit if (a_n) is given by the following.

(i) $a_1 := 1$ and $a_{n+1} := (3a_n + 2)/6$ for $n \in \mathbb{N}$.
(ii) $a_1 := 1$ and $a_{n+1} := a_n/(2a_n + 1)$ for $n \in \mathbb{N}$.
(iii) $a_1 := 1$ and $a_{n+1} := 2a_n/(4a_n + 1)$ for $n \in \mathbb{N}$.
(iv) $a_1 := 2$ and $a_{n+1} := \sqrt{1 + a_n}$ for $n \in \mathbb{N}$.
(v) $a_1 := 1$ and $a_{n+1} := \sqrt{2 + a_n}$ for $n \in \mathbb{N}$.
(vi) $a_1 := 2$ and $a_{n+1} := (1/2) + \sqrt{a_n}$ for $n \in \mathbb{N}$.
(vii) $a_1 := 1$ and $a_{n+1} := (1/2) + \sqrt{a_n}$ for $n \in \mathbb{N}$.

2.10. Let $a_n := 1/2 + 1/4 + \cdots + 1/(2n)$ and $b_n := 1 + 1/3 + \cdots + 1/(2n - 1)$ for $n \in \mathbb{N}$. Show that $a_n \to \infty$ and $b_n \to \infty$. (Hint: Example 2.10 (iii).)

2.11. Show that $(n!)^{1/n} \to \infty$.

2.12. Suppose α and β are real numbers such that $0 \le \beta \le \alpha$. Let

$$a_1 := \alpha, \;\; b_1 := \beta \quad \text{and} \quad a_{n+1} := \frac{a_n + b_n}{2}, \;\; b_{n+1} := \sqrt{a_n b_n} \quad \text{for } n \in \mathbb{N}.$$

Show that (a_n) is a monotonically decreasing sequence that is bounded below by β, and (b_n) is a monotonically increasing sequence that is bounded above by α. Further, show that $0 \le a_n - b_n \le (\alpha - \beta)/2^{n-1}$ for $n \in \mathbb{N}$. Deduce that (a_n) and (b_n) are convergent and have the same limit.
[Note: The common limit of the sequences (a_n) and (b_n) is called the **arithmetic–geometric mean** of the nonnegative real numbers α and β. It was introduced and studied by Gauss. For further details, see [22].]

2.13. Show that the following sequences are convergent and find their limits.

(i) $b_n := \dfrac{1 + a + a^2 + \cdots + a^n}{n+1}$, where $a \in (-1, 1)$, (ii) $b_n := \dfrac{2^{n+1} - 1}{(n+1)2^n}$,

(iii) $b_n := \dfrac{1}{n}\left(\dfrac{2}{5} + \dfrac{5}{11} + \cdots + \dfrac{n^2+1}{2n^2+3}\right)$, (iv) $b_n := \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \frac{(k+1)^k}{k^k}$.

2.14. Let (a_n) be a sequence in $\mathbb{R}$. Show that (a_n) is not bounded above if and only if (a_n) has a subsequence (a_{n_k}) such that $a_{n_k} \to \infty$. Also, show that (a_n) is not bounded below if and only if (a_n) has a subsequence (a_{n_k}) such that $a_{n_k} \to -\infty$.

2.15. Suppose $a \in \mathbb{R}$ or $a = \infty$ or $a = -\infty$. Show that if a monotonic sequence (a_n) in $\mathbb{R}$ has a subsequence (a_{n_k}) such that $a_{n_k} \to a$, then $a_n \to a$.

2.16. Prove that a sequence (a_n) in $\mathbb{R}$ has no convergent subsequence if and only if $|a_n| \to \infty$.

2.17. Let $A_n := 1 + (1/2) + \cdots + (1/n)$ for $n \in \mathbb{N}$. Show that $(A_{n+1} - A_n) \to 0$ as $n \to \infty$, but (A_n) is not a Cauchy sequence.

2.18. Let $A_n := 1 + (1/2^2) + \cdots + (1/n^2)$ for $n \in \mathbb{N}$. Show that there is no real number $\alpha < 1$ such that $|A_{n+1} - A_n| \le \alpha |A_n - A_{n-1}|$ for all $n \in \mathbb{N}$ with $n \ge 2$, but (A_n) is a Cauchy sequence.

2.19. Let (a_n) be a sequence of real numbers and let $a \in \mathbb{R}$. Show that $a_n \to a$ if and only if a is a cluster point of every subsequence of (a_n). (Hint: Proposition 2.19.)

2.20. Let (a_n) be a sequence in $\mathbb{R}$.
(i) If (a_n) is convergent, then show that (a_n) has a unique cluster point. Also, show that the converse is not true.
(ii) If $a_n \to \infty$ or if $a_n \to -\infty$, then show that (a_n) has no cluster point. Also, show that the converse is not true.

2.21. Determine $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ if (a_n) is as defined below.
(i) $a_n := (-1)^n \left(1 + \frac{1}{n}\right)$ for $n \in \mathbb{N}$,
(ii) $a_1 := 0$ and for $k \in \mathbb{N}$, $a_{2k} := a_{2k-1}/2$ and $a_{2k+1} := (1/2) + a_{2k}$. (Hint: $a_{2k-1} = 1 - 2^{1-k}$ for all $k \in \mathbb{N}$.)

Part B

2.22. Let $x \in \mathbb{R}$ and $x > 0$. Define

$$A_n := 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \quad \text{and} \quad B_n := \left(1 + \frac{x}{n}\right)^n \quad \text{for } n \in \mathbb{N}.$$

Show that (A_n) and (B_n) are convergent and have the same limit.

2.23. Show that the number $e := \lim_{n\to\infty} \sum_{k=0}^{n} (1/k!)$ is irrational. (Hint: For every $n \in \mathbb{N}$, $0 < e - \sum_{k=0}^{n}(1/k!) < (1/n!n)$. Multiply by $n!$.)

2.24. Let (a_n) be a sequence in $\mathbb{R}$ and let (b_n) be the sequence of arithmetic means of (a_n). Show that if $a_n \to \infty$, then $b_n \to \infty$, and also that if $a_n \to -\infty$, then $b_n \to -\infty$. Further, show that the converse is not true.

2.25. Suppose α, β, and γ are positive real numbers. Let $a_1 := \alpha$ and $a_{n+1} := a_n/\beta a_n + \gamma$ for $n \in \mathbb{N}$. Show that (a_n) is convergent. Further, if $a := \lim_{n\to\infty} a_n$, then show that $a = 0$ if $\gamma \ge 1$ and $a = (1-\gamma)/\beta$ otherwise. (Hint: Consider the cases $\alpha\beta + \gamma \ge 1$ and $\alpha\beta + \gamma < 1$.)

2.26. Suppose α and β are nonnegative real numbers. Let $a_1 := \alpha$ and $a_{n+1} := \sqrt{\beta + a_n}$ for $n \in \mathbb{N}$. Show that (a_n) is convergent. Further, if $a := \lim_{n\to\infty} a_n$, then show that $a = 0$ if $\alpha = 0 = \beta$, and $a = (1 + \sqrt{1 + 4\beta})/2$ otherwise. (Hint: Consider the cases $\sqrt{\alpha + \beta} \le \alpha$ and $\sqrt{\alpha + \beta} > \alpha$.)

2.27. Suppose $\alpha, \beta \in \mathbb{R}$ are nonnegative. Let $a_1 := \alpha$ and $a_{n+1} := \beta + \sqrt{a_n}$ for $n \in \mathbb{N}$. Show that (a_n) is convergent. Further, if $a := \lim_{n\to\infty} a_n$, then show that $a = 0$ if $\alpha = 0 = \beta$, and $a = (1 + 2\beta + \sqrt{1+4\beta})/2$ otherwise. (Hint: Consider the cases $\sqrt{\alpha} + \beta \le \alpha$ and $\sqrt{\alpha} + \beta > \alpha$.)

2.28. Let (a_n) and (b_n) be sequences in $\mathbb{R}$ such that $|a_{n+1} - a_n| \le b_n$ for all $n \in \mathbb{N}$. Also let $B_n := \sum_{k=1}^{n} b_k$ for $n \in \mathbb{N}$. Show that if (B_n) is convergent, then (a_n) is convergent. (Hint: Cauchy criterion)

2.29. Let y be any real number with $0 \le y < 1$. Define sequences (b_n) and (y_n) iteratively as follows. Let $y_1 := 10y$ and $b_1 := [y_1]$, and for each $n \in \mathbb{N}$, let $y_{n+1} := 10(y_n - b_n)$ and $b_{n+1} := [y_{n+1}]$. Show that b_n are integers with $0 \le b_n \le 9$ and y_n are real numbers with $0 \le y_n < 10$ for each $n \in \mathbb{N}$ and moreover,

$$y = \frac{b_1}{10} + \frac{b_2}{10^2} + \cdots + \frac{b_n}{10^n} + \frac{y_{n+1}}{10^{n+1}}.$$

Deduce that $0 \le y_{n+1}/10^{n+1} < 1/10^n$ for each $n \in \mathbb{N}$, and consequently,

$$y = \lim_{n\to\infty} \left(\frac{b_1}{10} + \frac{b_2}{10^2} + \cdots + \frac{b_n}{10^n} \right).$$

[Note: It is customary to call $b_1, b_2, \ldots,$ the **digits** of y and write the above expression as $y = 0.b_1b_2\ldots$ and call it the **decimal expansion** of y.]

2.30. Show that the set $[0, 1)$ is uncountable. Deduce that $\mathbb{R}$ is uncountable. (Hint: If $f : \mathbb{N} \to [0, 1)$ is bijective, then consider $y \in [0, 1)$ whose nth digit is 1 or 0 according as the nth digit of $f(n)$ is zero or nonzero.)

2.31. Given any $m \in \mathbb{N}$, show that there is a unique nonnegative integer k such that $10^k \le m < 10^{k+1}$. Use Exercise 1.35 repeatedly to show that there are unique integers $a_0, a_1, \ldots, a_k$ between 0 and 9 such that

$$m = a_0 + a_1(10) + a_2(10^2) + \cdots + a_k\left(10^k\right).$$

2.32. Given any $x \in \mathbb{R}$, show that there is a nonnegative integer k and integers $a_k, a_{k-1}, \ldots, a_1, a_0, b_1, b_2, \ldots$ between 0 and 9 such that

$$x = \pm \lim_{n\to\infty} \left(a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_0 + \frac{b_1}{10} + \frac{b_2}{10^2} + \cdots + \frac{b_n}{10^n} \right).$$

(Hint: If $|x| < 1$, set $k = 0 = a_0$ and apply Exercise 2.29 to $y := |x|$. If $|x| \ge 1$, apply Exercise 2.31 to $m := [|x|]$ and Exercise 2.29 to $y := |x| - m$.) [Note: It is customary to call $a_k, a_{k-1}, \ldots, a_0, b_1, b_2, \ldots$ the **digits** of x and write the above expression for x as $x = \pm a_k a_{k-1} \ldots a_0.b_1b_2\ldots$ and call it the **decimal expansion** of x.]

2.33. Given any $y \in [0, 1)$, let (y_n) and (b_n) be the sequences associated to y as in Exercise 2.29. We say that the decimal expansion of y is **finite** if $y_n = 0$ for some $n \in \mathbb{N}$ and **recurring** if it not finite but $y_i = y_j$ for some $i, j \in \mathbb{N}$ with $i < j$. Show that if $y \in [0, 1)$ is a rational number, then its decimal expansion is either finite or recurring. (Hint: Write y in reduced form as $y = p/q$. Let $r_0 := p$. Use Exercise 1.35 successively to find integers $q_1, r_1, q_2, r_2, \ldots$ such that $10r_{i-1} = qq_i + r_i$ and $0 \le r_i < q$ for $i \ge 1$. Now $y_i = 10r_{i-1}/q$ and the r_i's take only finitely many values.) [Note: The converse also holds. See Remark 9.2.]

2.34. Show that the results of Exercises 2.29, 2.31, 2.32, and 2.33 are valid with the number 10 replaced by any integer $d > 1$ and the number 9 by $d - 1$. [Note: The corresponding limiting expression of a real number x is called the **d-ary expansion** of x. When $d = 2$, it is called the **binary expansion**, and when $d = 3$, it is called the **ternary expansion**.]

2.35. Define $a_1 := 1$ and $a_{n+1} := (1 + (-1)^n 2^{-n})\, a_n$ for $n \in \mathbb{N}$.
(i) For every $n \in \mathbb{N}$, show that

$$|a_{n+1}| \le \left(1 + \frac{1}{2^n}\right)\left(1 + \frac{1}{2^{n-1}}\right)\cdots\left(1 + \frac{1}{2}\right) \le \left(\frac{n+1}{n}\right)^n < 3.$$

(Hint: Use the A.M.-G.M. inequality.)
(ii) Use (i) above to show that $|a_{n+1} - a_n| < 3/2^n$ for all $n \in \mathbb{N}$. Deduce, using Exercise 2.28, that (a_n) is a Cauchy sequence.
(iii) Conclude that (a_n) is convergent. Is (a_n) monotonic?

2.36. Assuming only the algebraic and the order properties of $\mathbb{R}$, and assuming that every monotonically decreasing sequence that is bounded below is convergent in $\mathbb{R}$, establish the Completeness Property of $\mathbb{R}$. (Hint: Consider $S \subseteq \mathbb{R}$, $a_0 \in S$, and an upper bound α_0 of S. If $(a_0 + \alpha_0)/2$ is an upper bound of S, let $a_1 := a_0$ and $\alpha_1 := (a_0 + \alpha_0)/2$; otherwise, there is $a_1 \in S$ such that $(a_0 + \alpha_0)/2 < a_1$, and in this case, let $\alpha_1 := \alpha_0$. Continuing in this manner, obtain a monotonically decreasing sequence (α_n) that is bounded below.) (Compare part (ii) of Proposition 2.8.)

2.37. **(Nested Interval Theorem)** For $n \in \mathbb{N}$, let $I_n := [a_n, b_n]$ be closed and bounded intervals in $\mathbb{R}$. If $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$, then show that there is $x \in \mathbb{R}$ such that $x \in I_n$ for all $n \in \mathbb{N}$. If, in addition, $|b_n - a_n| \to 0$, then show that such $x \in \mathbb{R}$ is unique. (Hint: Exercise 1.45)

2.38. Use the Nested Interval Theorem in Exercise 2.37 to prove the Bolzano–Weierstrass Theorem.

2.39. Use the the Bolzano–Weierstrass Theorem to prove that every sequence in $\mathbb{R}$ has a monotonic subsequence.

2.40. Let (a_n) be a sequence in $\mathbb{R}$. Prove Corollary 2.18 by showing that if (a_n) is bounded above and $a_n \not\to -\infty$, then (a_n) has a subsequence that converges to $\limsup_{n\to\infty} a_n$, while if (a_n) is bounded below and $a_n \not\to \infty$, then (a_n) has a subsequence that converges to $\liminf_{n\to\infty} a_n$.

2.41. Let (a_n) be a Cauchy sequence in $\mathbb{R}$. Prove that (a_n) is convergent by showing that it is bounded and $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$.

2.42. Assuming only the algebraic and the order properties of $\mathbb{R}$, and assuming that every Cauchy sequence in $\mathbb{R}$ is convergent, establish the Completeness Property of $\mathbb{R}$. (Hint: Consider $S \subseteq \mathbb{R}$ and a_n, α_n as in the hint for Exercise 2.36. Then $\alpha_n - a_n \le (\alpha_0 - a_0)/2^n$ for all $n \in \mathbb{N}$.)

3

Continuity and Limits

In the previous chapter, we studied real sequences, that is, real-valued functions defined on the subset $\mathbb{N}$ of $\mathbb{R}$. In this chapter we shall consider real-valued functions whose domains are arbitrary subsets of $\mathbb{R}$. The basic question we address is the following: Let $D \subseteq \mathbb{R}$, $c \in \mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function. Must there exist a real number ℓ such that whenever x in D is near c, $f(x)$ is near ℓ? In order to answer this and related questions, we develop the concepts of the continuity of a function and of the limit of a function.

In Section 3.1 below, we introduce the notion of continuity and derive a number of elementary results. Next, in Section 3.2, we examine this notion in relation to various properties of functions introduced in Section 1.3. In particular, we establish some important properties of real-valued continuous functions defined on a closed and bounded subset of $\mathbb{R}$ or on an interval in $\mathbb{R}$. These will turn out to be of basic importance in our subsequent development of calculus and analysis. The fundamental notion of limit of a function is discussed in Section 3.3. Our treatment will be based on the notion of convergence of sequences and the results proved in Chapter 2.

3.1 Continuity of Functions

Let $D \subseteq \mathbb{R}$. Consider a function $f : D \to \mathbb{R}$ and a point $c \in D$. We say that f is **continuous** at c if

$$(x_n) \text{ any sequence in } D \text{ such that } x_n \to c \Longrightarrow f(x_n) \to f(c).$$

If f is not continuous at c, we say that f is **discontinuous** at c. In case f is continuous at every $c \in D$, we say that f is continuous on D.

Examples 3.1. (i) Let a and b be real numbers and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := ax + b$ for $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$. To see this, let $c \in \mathbb{R}$ and let (x_n) be any sequence in $\mathbb{R}$ such that $x_n \to c$. By parts (i)

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and (ii) of Proposition 2.3, $ax_n + b \to ac + b$, that is, $f(x_n) \to f(c)$. Thus f is continuous on $\mathbb{R}$.

(ii) Let $f(x) := |x|$ for $x \in \mathbb{R}$. (See Figure 1.6.) Then f is continuous on $\mathbb{R}$. This follows by noting that $|x_n| \to |c|$ whenever $c \in \mathbb{R}$ and $x_n \to c$, because $|\,|x_n| - |c|\,| \le |x_n - c|$ for all $n \in \mathbb{N}$ by part (ii) of Proposition 1.8.

(iii) Let $f(x) := [x]$ for $x \in \mathbb{R}$. (See Figure 1.7.) If $c \in \mathbb{Z}$, then f is not continuous at c, since $c-(1/n) \to c$ and $f(c-(1/n)) = c-1$ for all $n \in \mathbb{N}$, but $f(c) = c$, and so $f(c-(1/n)) \not\to f(c)$. On the other hand, if $c \in \mathbb{R}\setminus\mathbb{Z}$, then f is continuous at c. To see this, let $\epsilon := \min\{c-[c],\ [c]+1-c\}$. Then $\epsilon > 0$, and if $x_n \to c$, then there is $n_0 \in \mathbb{N}$ such that $|x_n - c| < \epsilon$, that is, $c - \epsilon < x_n < c + \epsilon$, and so $[c] < x_n < [c]+1$ for all $n \ge n_0$. Thus $f(x_n) = [x_n] = [c]$ for all $n \ge n_0$. Therefore $f(x_n) \to f(c)$.

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be the **Dirichlet function** defined by

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let $c \in \mathbb{R}$. Then for each $n \in \mathbb{N}$, by Proposition 1.6, there is a rational number x_n and an irrational number y_n between c and $c + (1/n)$. Now $x_n \to c$ and $y_n \to c$, but $f(x_n) = 1$ and $f(y_n) = 0$ for all $n \in \mathbb{N}$. This shows that f is discontinuous at every $c \in \mathbb{R}$.

(v) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x & \text{if } x \text{ is rational,} \\ -x & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is continuous only at 0. Indeed, if (x_n) is any sequence in $\mathbb{R}$ such that $x_n \to 0$, then $-x_n \to 0$ as well, and hence $f(x_n) \to 0$. On the other hand, if $c \in \mathbb{R}$ with $c \ne 0$, then as in (iv) above, we can find sequences (x_n) and (y_n) such that $x_n \to c$ and $y_n \to c$, but $f(x_n) \to c$ and $f(y_n) \to -c$. Thus f is discontinuous at c. $\diamond$

We show that continuity is preserved by algebraic operations on functions.

Proposition 3.2. *Let $D \subseteq \mathbb{R}$, $c \in D$, and let $f, g : D \to \mathbb{R}$ be functions that are continuous at c. Then*

(i) *$f + g$ is continuous at c,*
(ii) *rf is continuous at c for every $r \in \mathbb{R}$,*
(iii) *fg is continuous at c,*
(iv) *if $f(c) \ne 0$, then there is $\delta > 0$ such that $f(x) \ne 0$ for all $x \in D$ satisfying $|x - c| < \delta$; also, $1/f : D \cap (c-\delta, c+\delta) \to \mathbb{R}$ is continuous at c,*
(v) *if there is $\delta > 0$ such that $f(x) \ge 0$ for $x \in D$ with $0 < |x - c| < \delta$, then $f(c) \ge 0$ and for every $k \in \mathbb{N}$, the function $f^{1/k} : D \cap (c-\delta, c+\delta) \to \mathbb{R}$ is continuous at c.*

Proof. Let (x_n) be any sequence in D such that $x_n \to c$. Then $f(x_n) \to f(c)$ and $g(x_n) \to g(c)$.

By parts (i), (ii), and (iii) of Proposition 2.3, it follows that

$$\begin{aligned}(f+g)(x_n) = f(x_n) + g(x_n) &\to f(c) + g(c) = (f+g)(c),\\ (rf)(x_n) = rf(x_n) &\to rf(c) = (rf)(c) \text{ for every } r \in \mathbb{R},\\ (fg)(x_n) = f(x_n)g(x_n) &\to f(c)g(c) = (fg)(c).\end{aligned}$$

This proves (i), (ii), and (iii).

Next, assume that $f(c) \neq 0$. If there were no $\delta > 0$ such that $f(x) \neq 0$ for all $x \in D$ with $|x - c| < \delta$, then we could obtain a sequence (c_n) in D with $|c_n - c| < 1/n$ and $f(c_n) = 0$ for all $n \in \mathbb{N}$. But then $c_n \to c$ and $f(c_n) \not\to f(c)$, which is a contradiction. Thus there is $\delta > 0$ such that $f(x) \neq 0$ whenever $x \in D$ and $|x - c| < \delta$. Now since $x_n \to c$, there is $n_0 \in \mathbb{N}$ such that $|x_n - c| < \delta$ for all $n \geq n_0$, and so $f(x_n) \neq 0$. By part (iv) of Proposition 2.3, we see that

$$\left(\frac{1}{f}\right)(x_n) = \frac{1}{f(x_n)} \to \frac{1}{f(c)} = \left(\frac{1}{f}\right)(c).$$

This proves (iv).

Finally, suppose $f(x) \geq 0$ whenever $x \in D$ and $0 < |x - c| < \delta$. Applying part (iv) of Proposition 2.4 to the sequence $(f(c_n))$, where $c_n := c + (\delta/2n)$ for $n \in \mathbb{N}$, we see that $f(c) \geq 0$. Moreover, for every $k \in \mathbb{N}$, by applying part (iv) of Proposition 2.4 to the sequence $(f(x_n))$, we obtain

$$\left(f^{1/k}\right)(x_n) = (f(x_n))^{1/k} \to (f(c))^{1/k} = f^{1/k}(c).$$

This proves (v). □

With notation and hypotheses as in the proposition above, a combined application of its parts (i) and (ii) shows that the difference $f - g$ is continuous at c. Likewise, a combined application of parts (iii) and (iv) shows that if $g(c) \neq 0$, then the quotient f/g is continuous at c. Further, since every positive rational number r is equal to n/k, where $n, k \in \mathbb{N}$, a combined application of parts (v) and (iii) shows that if there is $\delta > 0$ such that $f(x) \geq 0$ whenever $x \in D$ and $|x - c| < \delta$, then the function f^r is continuous at c for every positive rational number r. Similarly, a combined application of parts (v), (iv), and (iii) shows that if $f(c) > 0$, then the function f^r is continuous at c for every negative rational number r.

We show next that continuity is preserved by forming the absolute value of a function as well as by forming the maximum and the minimum of functions.

Proposition 3.3. *Let $D \subseteq \mathbb{R}$, $c \in D$, and let $f, g : D \to \mathbb{R}$ be functions that are continuous at c. Then*

(i) $|f|$ *is continuous at* c,
(ii) $\max\{f, g\}$ *and* $\min\{f, g\}$ *are continuous at* c.

Proof. Let (x_n) be a sequence in D such that $x_n \to c$. Then $f(x_n) \to f(c)$, and so by part (i) of Proposition 2.4, $|f(x_n)| \to |f(c)|$. This proves (i). Further, $g(x_n) \to g(c)$ as well, and so by part (ii) of Proposition 2.4, we see that $\max\{f(x_n), g(x_n)\} \to \max\{f(c), g(c)\}$ and $\min\{f(x_n), g(x_n)\} \to \min\{f(c), g(c)\}$. This yields (ii). $\square$

The converse of part (i) of Proposition 3.2 is not true. For example, if $f : \mathbb{R} \to \mathbb{R}$ is the variant of the Dirichlet function given by $f(x) := 1$ if $x \in \mathbb{Q}$ and $f(x) := -1$ if $x \in \mathbb{R} \setminus \mathbb{Q}$, then $|f|$ is continuous on $\mathbb{R}$, but f is not continuous at every $c \in \mathbb{R}$.

Corollary 3.4. *Let $D \subseteq \mathbb{R}$, $c \in D$, and let $f_1, \ldots, f_n : D \to \mathbb{R}$ be continuous at c. Then the functions $\max(f_1, \ldots, f_n), \min(f_1, \ldots, f_n) : D \to \mathbb{R}$ defined by $\max(f_1, \ldots, f_n)(x) := \max\{f_1(x), \ldots, f_n(x)\}$ and $\min(f_1, \ldots, f_n)(x) := \min\{f_1(x), \ldots, f_n(x)\}$ for $x \in D$, are continuous at c.*

Proof. The result follows from part (ii) of Proposition 3.3 using induction on n by noting that if $n > 1$, then $\max(f_1, \ldots, f_n) = \max\left(\max(f_1, \ldots, f_{n-1}),\, f_n\right)$ and $\min(f_1, \ldots, f_n) = \min\left(\min(f_1, \ldots, f_{n-1}),\, f_n\right)$. $\square$

We now show that the composition of continuous functions is continuous.

Proposition 3.5. *Let $D, E \subseteq \mathbb{R}$, and let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions such that $f(D) \subseteq E$. Let $c \in D$ be given. Assume that f is continuous at c and g is continuous at $f(c)$. Then $g \circ f : D \to \mathbb{R}$ is continuous at c.*

Proof. Let (x_n) be any sequence in D such that $x_n \to c$. Then $f(x_n) \to f(c)$, since f is continuous at c. Now $(f(x_n))$ is a sequence in E and g is continuous at $f(c)$. Hence

$$(g \circ f)(x_n) = g(f(x_n)) \to g(f(c)) = (g \circ f)(c).$$

It follows that $g \circ f$ is continuous at c. $\square$

One can piece together continuous functions to construct a continuous function, as the following result shows.

Proposition 3.6. *Let $D \subseteq \mathbb{R}$ and $c \in D$. Suppose*

$$D_1 := \{x \in D : x \le c\} \quad \text{and} \quad D_2 := \{x \in D : c \le x\}.$$

Let $f_1 : D_1 \to \mathbb{R}$ and $f_2 : D_2 \to \mathbb{R}$ be functions such that $f_1(c) = f_2(c)$. Then the function $f : D \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in D_1, \\ f_2(x) & \text{if } x \in D_2, \end{cases}$$

is continuous on D if f_1 is continuous on D_1 and f_2 is continuous on D_2.

Proof. If $c_1 \in D$ and $c_1 < c$, then the continuity of f_1 at c_1 implies the continuity of f at c_1. Similarly, if $c_2 \in D$ and $c < c_2$, then the continuity of f_2 at c_2 implies the continuity of f at c_2. Hence we only need to show that f is continuous at c. Let (x_n) be any sequence in D such that $x_n \to c$. If there is $n_1 \in \mathbb{N}$ such that $x_n \le c$ for all $n \ge n_1$, then $f(x_n) = f_1(x_n)$ for all $n \ge n_1$, and the continuity of f_1 at c implies that $f(x_n) \to f_1(c) = f(c)$. Similarly, if there is $n_2 \in \mathbb{N}$ such that $c \le x_n$ for all $n \ge n_2$, then $f(x_n) = f_2(x_n)$ for all $n \ge n_2$, and the continuity of f_2 at c implies that $f(x_n) \to f_2(c) = f(c)$. In the remaining case, there are positive integers $\ell_1 < \ell_2 < \cdots$ and $m_1 < m_2 < \cdots$ such that $x_{\ell_k} \le c < x_{m_k}$ for all $k \in \mathbb{N}$, and $\mathbb{N} = \{\ell_k : k = 1, 2, \ldots\} \cup \{m_k : k = 1, 2, \ldots\}$. Clearly $x_{\ell_k} \to c$ and $x_{m_k} \to c$ as $k \to \infty$. Moreover, $f(x_{\ell_k}) = f_1(x_{\ell_k})$ and $f(x_{m_k}) = f_2(x_{m_k})$ for all $k \in \mathbb{N}$. By the continuity of f_1 at c, we obtain $f(x_{\ell_k}) \to f_1(c) = f(c)$, and by the continuity of f_2 at c, we obtain $f(x_{m_k}) \to f_2(c) = f(c)$. It follows that $f(x_n) \to f(c)$. Thus f is continuous at c. We therefore conclude that f is continuous on D. $\square$

Examples 3.7. (i) Consider a polynomial function $p : \mathbb{R} \to \mathbb{R}$ given by

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \qquad \text{for} \quad x \in \mathbb{R}.$$

Applying Proposition 3.2 repeatedly, we find that p is continuous on $\mathbb{R}$. Again, if $q : \mathbb{R} \to \mathbb{R}$ is another polynomial function, then the rational function p/q is continuous at a point $c \in \mathbb{R}$ if $q(c) \neq 0$. For example, if $D = \mathbb{R} \setminus \{1\}$ and

$$f(x) := \frac{x^4 + 3x + 2}{x - 1} \quad \text{for} \quad x \in D,$$

then f is continuous on D.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := |x^4 - 3x^3 + 2x - 1|^{1/4} \qquad \text{for} \quad x \in \mathbb{R}.$$

By Propositions 3.2 and 3.5, we see that f is continuous on $\mathbb{R}$.

(iii) Consider a rational number r. Let $D := [0, \infty)$ if $r \ge 0$ and $D := (0, \infty)$ if $r < 0$. For $x \in D$, define $g(x) := x^r$. Since the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ is continuous on $\mathbb{R}$ and $g(x) = f^r(x)$ for $x \in D$, it follows from Proposition 3.2 and the remark following its proof that g is continuous on D.

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x^2 & \text{if } x \le 0, \\ x & \text{if } x > 0. \end{cases}$$

Then by Proposition 3.6, f is continuous on $\mathbb{R}$.

(v) Let $f : [-1,1] \to \mathbb{R}$ denote the zigzag function given in Example 1.20. If $c \in [-1,1]$ and $c \neq 0$, then Proposition 3.6 implies that f is continuous at c. To show that f is continuous at 0 as well, let (x_n) be a sequence in $[-1,1]$ such that $x_n \to 0$. It can easily be seen that $0 \leq f(x_n) \leq 2|x_n|$ for all $n \in \mathbb{N}$, and so $f(x_n) \to 0$. Thus f is continuous at 0. $\diamond$

We conclude this section by giving a criterion for the continuity of a function at a point that does not involve convergence of sequences.

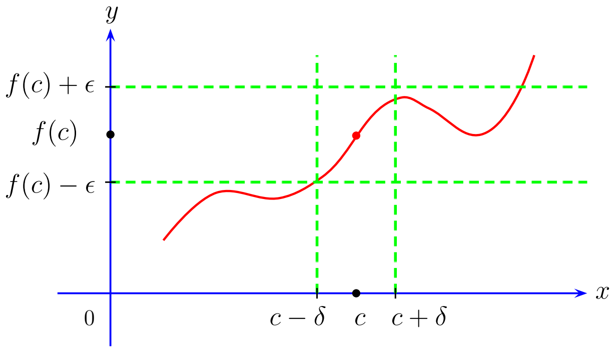

Fig. 3.1. Illustration of the ϵ-δ condition for continuity.

Proposition 3.8. *Let $D \subseteq \mathbb{R}$, $c \in D$, and let $f : D \to \mathbb{R}$ be a function. Then f is continuous at c if and only if f satisfies the following ϵ-δ condition: For every $\epsilon > 0$, there is $\delta > 0$ such that*

$$x \in D \text{ and } |x-c| < \delta \Longrightarrow |f(x) - f(c)| < \epsilon.$$

Proof. Let f be continuous at c. Suppose for a moment that the ϵ-δ condition does not hold. This means that there is $\epsilon > 0$ such that for every $\delta > 0$, there is $x \in D$ satisfying

$$|x-c| < \delta, \quad \text{but} \quad |f(x) - f(c)| \geq \epsilon.$$

Then there is a sequence (x_n) in D such that $|x_n - c| < 1/n$, but $|f(x_n) - f(c)| \geq \epsilon$ for all $n \in \mathbb{N}$. But then $x_n \to c$ and $f(x_n) \not\to f(c)$. This contradicts the continuity of f at c.

Conversely, assume the ϵ-δ condition. Let (x_n) be any sequence in D such that $x_n \to c$. Let $\epsilon > 0$ be given. Then there is $\delta > 0$ such that

$$x \in D \text{ and } |x-c| < \delta \Longrightarrow |f(x) - f(c)| < \epsilon.$$

Since $x_n \to c$, there is $n_0 \in \mathbb{N}$ such that $|x_n - c| < \delta$ for all $n \geq n_0$. Hence $|f(x_n) - f(c)| < \epsilon$ for all $n \geq n_0$. Thus $f(x_n) \to f(c)$. This shows that f is continuous at c. $\square$

The ϵ-δ condition in the above result is illustrated in Figure 3.1. As an application of the ϵ-δ condition, we show that the sign of a function is preserved in a neighborhood of a point of continuity.

Corollary 3.9. *Let $D \subseteq \mathbb{R}$, $c \in D$, and let $f : D \to \mathbb{R}$ be a function that is continuous at c. If $f(c) > 0$, then there is $\delta > 0$ such that $f(x) > 0$ whenever $x \in D$ and $|x - c| < \delta$. Likewise, if $f(c) < 0$, then there is $\delta > 0$ such that $f(x) < 0$ whenever $x \in D$ and $|x - c| < \delta$.*

Proof. If $f(c) > 0$ or $f(c) < 0$, then $\epsilon := |f(c)| > 0$. Hence by Proposition 3.8, there is $\delta > 0$ such that for $x \in D$ with $|x - c| < \delta$,

$$|f(x) - f(c)| < \epsilon, \text{ that is, } -|f(c)| < f(x) - f(c) < |f(c)|,$$

and hence $f(x) > 0$ or $f(x) < 0$ according as $f(c) > 0$ or $f(c) < 0$. □

3.2 Basic Properties of Continuous Functions

In this section we examine relationships between the continuity of a function and various geometric properties of a function considered earlier in Section 1.3. Also, we shall introduce the notion of a uniformly continuous function and discuss its relationship to continuity.

Continuity and Boundedness

A bounded function need not be continuous. For instance, we cite the Dirichlet function given in Example 3.1 (iv). Also, a continuous function need not be bounded. For example, let $D_1 := [0, \infty)$ and $f_1(x) := x$ for $x \in D_1$, or $D_2 := (0, 1]$ and $f_2(x) := 1/x$ for $x \in D_2$. An obvious reason why the continuous function f_1 is unbounded is that its domain D_1 is unbounded. To identify the reason why the continuous function f_2 is unbounded on its domain D_2, we introduce the following concept.

Let $D \subseteq \mathbb{R}$. We say that D is a **closed set** if

$$(x_n) \text{ any sequence in } D \text{ and } x_n \to x \Longrightarrow x \in D.$$

Notice that the interval $(0, 1]$ is not a closed set, since $(1/n) \in (0, 1]$ for each $n \in \mathbb{N}$ and $(1/n) \to 0$, but $0 \notin (0, 1]$. Similarly, it can be seen that the following intervals are not closed sets:

$$(a, b], \quad [a, b), \quad (a, b), \quad (a, \infty), \quad (-\infty, b), \qquad \text{where } a, b \in \mathbb{R}.$$

On the other hand, the following intervals are closed sets:

$$[a, b], \quad [a, \infty), \quad (-\infty, b], \quad (-\infty, \infty), \qquad \text{where} \quad a, b \in \mathbb{R}.$$

To show that the interval $[a, b]$ is a closed set, consider any sequence (x_n) in $[a, b]$ such that $x_n \to x$. Since $a \le x_n \le b$ and $x_n \to x$, part (iii) of Proposition 2.4 shows that $a \le x \le b$, that is, $x \in [a, b]$. Similar proofs can be given to show that $[a, \infty)$ and $(-\infty, b]$ are closed sets. It is obvious that $\mathbb{R} = (-\infty, \infty)$ is a closed set.

We now show that if a function defined on a closed and bounded set is continuous, then it is necessarily bounded. In fact, we shall show that such a function attains its bounds on its domain.

Proposition 3.10. *Let D be a nonempty, closed, and bounded subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$ be a continuous function. Then*

(i) *f is a bounded function, and*
(ii) *f attains its bounds on D, that is, there are r and s in D such that*

$$f(r) = \inf\{f(x) : x \in D\} \quad \textit{and} \quad f(s) = \sup\{f(x) : x \in D\}.$$

Proof. (i) Suppose f is not bounded on D. Then for every $n \in \mathbb{N}$, there is $x_n \in D$ such that $|f(x_n)| > n$. Since D is a bounded set, the sequence (x_n) is bounded. By the Bolzano–Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_k}). If $x_{n_k} \to x$, then $x \in D$, since D is a closed set. Also, since f is continuous at x, we see that $f(x_{n_k}) \to f(x)$. Being convergent, the sequence $(f(x_{n_k}))$ is bounded by part (ii) of Proposition 2.2. But $|f(x_{n_k})| > n_k$ for every $k \in \mathbb{N}$ and $n_k \to \infty$ as $k \to \infty$. This contradiction shows that f is a bounded function on D.

(ii) Since the function f is bounded on D, there are m, $M \in \mathbb{R}$ such that

$$m := \inf\{f(x) : x \in D\} \quad \text{and} \quad M := \sup\{f(x) : x \in D\}.$$

By Corollary 2.6, there are sequences (r_n) and (s_n) in D such that $f(r_n) \to m$ and $f(s_n) \to M$. Since D is a bounded set, the sequences (r_n) and (s_n) are bounded. By the Bolzano–Weierstrass Theorem, (r_n) has a convergent subsequence, say (r_{n_k}), and (s_n) has a convergent subsequence, say (s_{m_j}). If $r_{n_k} \to r$ and $s_{m_j} \to s$, then r and s belong to D, since D is a closed set. Also, $f(r_{n_k}) \to f(r)$ and $f(s_{m_j}) \to f(s)$, since f is continuous at r and s. Hence

$$f(r) = \lim_{k\to\infty} f(r_{n_k}) = \lim_{n\to\infty} f(r_n) = m$$

and

$$f(s) = \lim_{j\to\infty} f(s_{m_j}) = \lim_{n\to\infty} f(s_n) = M.$$

Thus f attains its bounds on D. □

Examples 3.11. (i) Let a and b be real numbers such that $a < b$. Since the interval $[a, b]$ is a closed and bounded subset of $\mathbb{R}$, it follows from the preceding result that every continuous function defined on $[a, b]$ is bounded and attains its bounds on $[a, b]$. For example, if $f(x) := x$ for $x \in [-1, 2]$, then f attains its lower bound at -1, and it attains its upper bound at 2. Also, if $f(x) := x^2$ for $x \in [-1, 2]$, then f attains its lower bound at 0 and its upper bound at 2. In general, it is not easy to determine the lower and the upper bounds of a continuous function on $[a, b]$ and to locate the points in $[a, b]$ at which they are attained. We shall return to this question when we consider applications of "differentiation" in Section 5.1.

(ii) If a subset D of $\mathbb{R}$ is not closed, then a continuous function on D may not be bounded on D, and even if it is bounded, it may not attain its bounds on D. For example, let $D := (a, b]$. If $f(x) := 1/(x - a)$ for $x \in D$, then f is continuous on D, but it is not bounded on D. Also, if $f(x) := x$ for $x \in D$, then f is continuous and bounded on D, but it does not attain its lower bound on D, since $\inf\{f(x) : x \in D\} = a$ and $a \notin D$.

(iii) If a subset D of $\mathbb{R}$ is not bounded, then a continuous function on D may not be bounded on D, and even if it is bounded on D, it may not attain its lower or upper bound on D. For example, let $D = [a, \infty)$. If $f(x) := x$ for $x \in D$, then f is continuous on D, but it is not bounded on D. Also, if $f(x) := (x - a)/(x - a + 1)$ for $x \in D$, then f is continuous and bounded on D, but it does not attain its upper bound on D, because $\sup\{f(x) : x \in D\} = 1$ and $f(x) \neq 1$ for all $x \in D$. ◇

Continuity and Monotonicity

It is easy to see that a function that is monotonic on an interval need not be continuous. For example, if $f(x) := [x]$ for $x \in \mathbb{R}$, then f is monotonic on $\mathbb{R}$, but it is discontinuous at every $c \in \mathbb{Z}$. (See Example 3.1 (iii).) Similarly, a continuous function defined on an interval need not be monotonic there, as the example $f(x) := |x|$ for $x \in \mathbb{R}$ shows. However, we now prove a peculiar result that says that if a function is strictly monotonic on an interval, then its inverse function (defined on the range of the given function) is continuous, irrespective of the continuity of the given function. Note also that the range of the given function, that is, the domain of the inverse function, need not be an interval. This phenomenon is illustrated by Figure 3.2.

Proposition 3.12. *Let I be an interval and let $f : I \to \mathbb{R}$ be a function that is strictly monotonic on I. Then $f^{-1} : f(I) \to \mathbb{R}$ is continuous.*

Proof. Since f is strictly monotonic on I, we see that f is one-one and its inverse $f^{-1} : f(I) \to \mathbb{R}$ is well-defined. Consider $d \in f(I)$. Then there is unique $c \in I$ such that $f(c) = d$.

Assume first that f is strictly increasing on I. Let $\epsilon > 0$ be given. Suppose that c is neither the left endpoint nor the right endpoint of the interval I. Then there are $c_1, c_2 \in I$ such that

$$c - \epsilon < c_1 < c < c_2 < c + \epsilon.$$

Let $d_1 := f(c_1)$ and $d_2 := f(c_2)$. Since f is strictly increasing on I, we see that $d_1 < d < d_2$, and since f^{-1} is also strictly increasing on $f(I)$, we obtain

$$y \in f(I), \;\; d_1 < y < d_2 \Longrightarrow c_1 = f^{-1}(d_1) < f^{-1}(y) < f^{-1}(d_2) = c_2,$$

so that $f^{-1}(d) - \epsilon < f^{-1}(y) < f^{-1}(d) + \epsilon$. Thus if we let $\delta := \min\{d - d_1, d_2 - d\}$, we see that $\delta > 0$ and

$$y \in f(I), \;\; |y - d| < \delta \Longrightarrow |f^{-1}(y) - f^{-1}(d)| < \epsilon.$$

Hence f^{-1} is continuous at d. (See Figure 3.2.) If $c = f^{-1}(d)$ is the left endpoint of the interval I, then since f and f^{-1} are strictly increasing on I and $f(I)$ respectively, it follows that

$$y \in f(I) \Longrightarrow d \leq y \quad \text{and} \quad f^{-1}(d) \leq f^{-1}(y),$$

and so the earlier argument works if we let $\delta := d_2 - d$. If $c = f^{-1}(d)$ is the right endpoint of the interval I, then similarly,

$$y \in f(I) \Longrightarrow y \leq d \quad \text{and} \quad f^{-1}(y) \leq f^{-1}(d),$$

and so the earlier argument works if we let $\delta := d - d_1$.

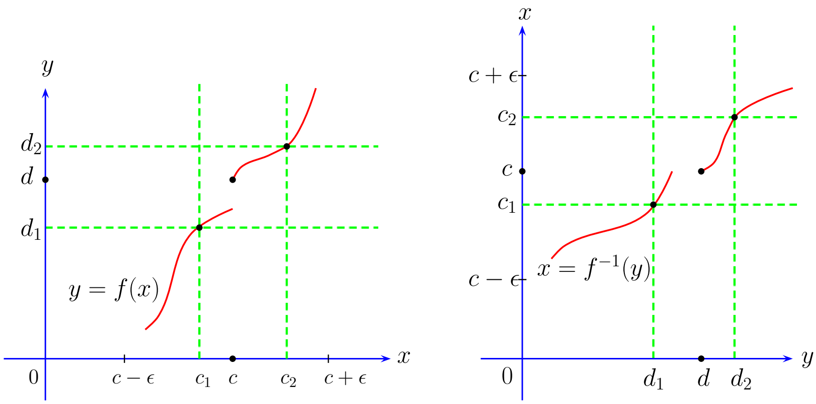

Fig. 3.2. A discontinuous strictly increasing function with continuous inverse.

Thus in all the cases, Proposition 3.8 shows that f^{-1} is continuous at d. Since d is an arbitrary point of $f(I)$, we see that $f^{-1} : f(I) \to \mathbb{R}$ is a continuous function. It may be noted that $f(I)$ need not be an interval, as Figure 3.2 shows.

If f is strictly decreasing on I, then $-f$ is strictly increasing on I, and by what was proved above, we see that $(-f)^{-1} : (-f)(I) \to \mathbb{R}$ is continuous. Since $f^{-1}(y) = (-f)^{-1}(-y)$ for every $y \in f(I)$, it follows from Proposition 3.5 that f^{-1} is a continuous function. $\square$

Example 3.13. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := x + [x]$. If $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$, then

$$f(x_1) = x_1 + [x_1] < x_2 + [x_1] \leq x_2 + [x_2] = f(x_2).$$

Hence f is strictly increasing on $\mathbb{R}$. If $m \in \mathbb{Z}$, then

$$f(x) = x + m \quad \text{for} \quad x \in [m, m+1).$$

Thus $f(\mathbb{R})$ is the union of the semiopen intervals $[2m, 2m+1)$, $m \in \mathbb{Z}$, that is, $f(\mathbb{R}) = \{y \in \mathbb{R} : [y] \text{ is even}\}$, and if $m \in \mathbb{Z}$, then

$$f^{-1}(y) = y - m \quad \text{for} \quad y \in [2m, 2m+1).$$

In other words,

$$f^{-1}(y) = y - \frac{[y]}{2} \quad \text{for} \quad y \in f(\mathbb{R}).$$

Observe that f^{-1} is continuous at each point of $f(\mathbb{R})$, even though f is not continuous at every $m \in \mathbb{Z}$. (See Figure 3.3.) $\diamond$

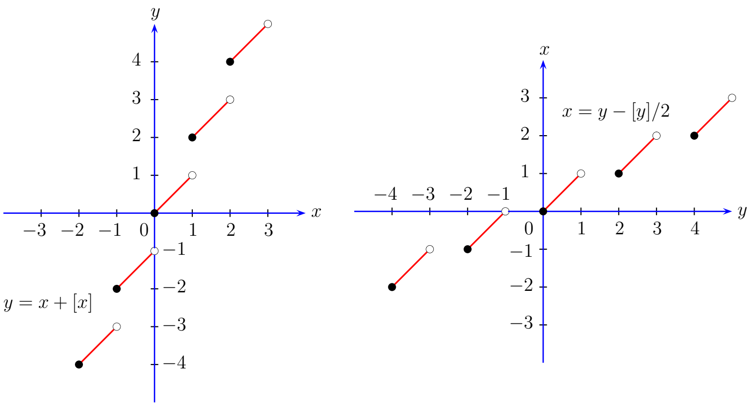

Fig. 3.3. Graphs of $f(x) = x + [x]$ and its inverse $f^{-1}(y) = y - [y]/2$.

Corollary 3.14. *Let I be an interval and let $f : I \to \mathbb{R}$ be a strictly monotonic function such that $f(I)$ is an interval. Then f is one-one and continuous.*

Proof. Since f is strictly monotonic on I, we see that f is one-one and its inverse $f^{-1} : f(I) \to \mathbb{R}$ is well-defined. Let $J := f(I)$ and $g := f^{-1}$. Then g is strictly monotonic on the interval J, and hence by Proposition 3.12, $g^{-1} : g(J) \to \mathbb{R}$ is continuous, that is, $f : I \to \mathbb{R}$ is continuous. $\square$

We shall prove in Proposition 3.17 that the converse of the above corollary holds. We shall later prove a stronger version of the above corollary (where strict monotonicity is replaced by monotonicity) in Proposition 3.45.

Continuity and Convexity

It is easy to see that a continuous function defined on an interval need not be either convex on that interval or concave on that interval. For example, if $f(x) := x^3$ for $x \in \mathbb{R}$, then f is continuous on $\mathbb{R}$, but it is neither convex nor concave on $\mathbb{R}$. On the other hand, we shall show below that a convex or a concave function on an open interval is necessarily continuous.

Proposition 3.15. *Let I be an open interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be convex on I or concave on I. Then f is continuous on I.*

Proof. First, suppose f is convex. Let $c \in I$. Then there is $r > 0$ such that $[c-r, c+r] \subseteq I$. Let $M := \max\{f(c-r), f(c+r)\}$. Then for each $x \in [c-r, c+r]$, there is $t \in [0, 1]$ such that $x = (1-t)(c-r) + t(c+r)$, and hence

$$f(x) \le (1-t)f(c-r) + tf(c+r) \le (1-t)M + tM = M.$$

Given any $\epsilon > 0$ with $\epsilon \le 1$, and $x \in \mathbb{R}$, we claim that

$$|x - c| \le r\epsilon \Longrightarrow x \in I \text{ and } |f(x) - f(c)| \le \epsilon\big(M - f(c)\big).$$

Suppose $|x - c| \le r\epsilon$. Then $x \in [c-r, c+r]$, since $\epsilon \le 1$, and so $x \in I$. Define

$$y := c + \frac{x-c}{\epsilon} \quad \text{and} \quad z := c - \frac{x-c}{\epsilon}.$$

Then $|y - c| = |z - c| = |x - c|/\epsilon \le r$, and so $y, z \in [c-r, c+r]$. Moreover,

$$x = (1-\epsilon)c + \epsilon y \quad \text{and} \quad c = \frac{1}{1+\epsilon}x + \frac{\epsilon}{1+\epsilon}z.$$

Since f is convex and $0 < \epsilon \le 1$, we see that

$$f(x) \le (1-\epsilon)f(c) + \epsilon f(y), \quad \text{that is,} \quad f(x) - f(c) \le \epsilon\big(f(y) - f(c)\big).$$

The last inequality implies that $f(x) - f(c) \le \epsilon\big(M - f(c)\big)$. Also, we see that

$$f(c) \le \frac{1}{1+\epsilon}f(x) + \frac{\epsilon}{1+\epsilon}f(z), \quad \text{that is,} \quad (1+\epsilon)f(c) \le f(x) + \epsilon f(z).$$

The last inequality implies that $f(c) - f(x) \le \epsilon\big(f(z) - f(c)\big) \le \epsilon\big(M - f(c)\big)$. It follows that $|f(x) - f(c)| \le \epsilon\big(M - f(c)\big)$, and thus the claim is established. The claim together with Proposition 3.8 implies the continuity of f at c.

If f is concave, it suffices to apply the result just proved to $-f$. □

An alternative proof of continuity of a convex function on an open interval is sketched in Exercise 3.44. We remark that at an endpoint of an interval, a convex (or a concave) function may be discontinuous. For example, consider $I := [-1, 1]$ and $f : I \to \mathbb{R}$ given by

$$f(x) := \begin{cases} |x| & \text{if } |x| < 1, \\ 2 & \text{if } x = -1 \text{ or } 1. \end{cases}$$

Clearly, f is convex on I, but f is discontinuous at 1 as well as at -1.

Continuity and Intermediate Value Property

The following important result shows that a continuous function on an interval always has the Intermediate Value Property (IVP).

Proposition 3.16 (Intermediate Value Theorem). *Let I be an interval and let $f : I \to \mathbb{R}$ be a continuous function. Then f has the IVP on I. In particular, $f(I)$ is an interval.*

Proof. Let a, b in I with $a < b$. Then $[a, b] \subseteq I$. Let r be an intermediate value between $f(a)$ and $f(b)$, that is, $r \in (f(a), f(b))$ or $r \in (f(b), f(a))$.

Assume first that $f(a) < f(b)$, so that $r \in (f(a), f(b))$. Define

$$S := \{x \in [a, b] : f(x) < r\}.$$

Now $a \in S$, since $f(a) < r$. Hence $S \neq \emptyset$. Also, the set S is bounded above by b. If $c := \sup S$, then by part (i) of Corollary 2.6, there is a sequence (c_n) in S such that $c_n \to c$. Since $c \in [a, b] \subseteq I$, f is continuous at c. Hence $f(c_n) \to f(c)$. Also, $f(c_n) < r$, since $c_n \in S$ for $n \in \mathbb{N}$. So by part (iii) of Proposition 2.4, $f(c) \leq r$. We note that $c \neq b$, because $r < f(b)$. Let

$$b_n := c + \frac{b - c}{n} \in [a, b] \quad \text{for each } n \in \mathbb{N}.$$

Clearly, $b_n \to c$. The continuity of f at c implies that $f(b_n) \to f(c)$. But since $b_n > c$ and $c = \sup S$, we see that $b_n \notin S$, that is, $f(b_n) \geq r$ for all $n \in \mathbb{N}$. As before, part (iii) of Proposition 2.4 shows that $f(c) \geq r$. In particular, $c \neq a$. Thus $c \in (a, b)$ and $f(c) = r$.

The case $r \in (f(b), f(a))$ can be proved similarly. □

The above result (together with Propositions 1.25 and 3.12) has the following striking consequence.

Proposition 3.17 (Continuous Inverse Theorem). *Let I be an interval and let $f : I \to \mathbb{R}$ be a one-one continuous function. Then the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is continuous. In fact, f is strictly monotonic on I, and $f(I)$ is an interval.*

Proof. By the Intermediate Value Theorem (Proposition 3.16), the one-one function f has the IVP. Hence by Proposition 1.25, f is strictly monotonic, and $f(I)$ is an interval. Thus by Proposition 3.12, f^{-1} is continuous. □

The above result shows that the converse of Corollary 3.14 holds. However, the converse of the Intermediate Value Theorem does not hold in general, that is, a discontinuous function may have the IVP on an interval I. We illustrate this by the following example, which is modeled after the function $f : [0, 1] \to \mathbb{R}$ given by $f(0) := 0$ and $f(x) = \sin(1/x)$ for $x \in (0, 1]$, which will be discussed in Section 7.3. (Compare Figure 7.12 and Exercise 7.43.)

Example 3.18. Let $D := [0,1]$ and consider a "criss-cross" function defined on D whose graph is obtained by the line segments joining $(1,1)$ to $(\frac{1}{2},-1)$, $(\frac{1}{2},-1)$ to $(\frac{1}{3},1)$, $(\frac{1}{3},1)$ to $(\frac{1}{4},-1)$, and so on. (See Figure 3.4.) More precisely, let $f : D \to \mathbb{R}$ be defined as follows: $f(0) := 0$ and for $x \in (0,1]$,

$$f(x) := \begin{cases} 2k(k+1)x - 2k - 1 & \text{if } \dfrac{1}{k+1} \le x \le \dfrac{1}{k},\ k \in \mathbb{N},\ k \text{ odd}, \\ -2k(k+1)x + 2k + 1 & \text{if } \dfrac{1}{k+1} \le x \le \dfrac{1}{k},\ k \in \mathbb{N},\ k \text{ even}. \end{cases}$$

We note that $|f(x)| \le 1$ for all $x \in [0,1]$. Also, for every $k \in \mathbb{N}$, the function f assumes every value between -1 and 1 on the interval $[1/(k+1), 1/k]$. Let $I_k := [1/(k+1), 1/k]$ and for $x \in I_k$ for $k = 1, 2, \ldots$, define $f_k(x) := f(x)$. Then f is continuous on I_k and $f_k(1/(k+1)) = f_{k+1}(1/(k+1))$ for each $k \in \mathbb{N}$. Since $(0,1] = \bigcup_{k=1}^{\infty} I_k$, by Proposition 3.6 we see that f is continuous on $(0,1]$.

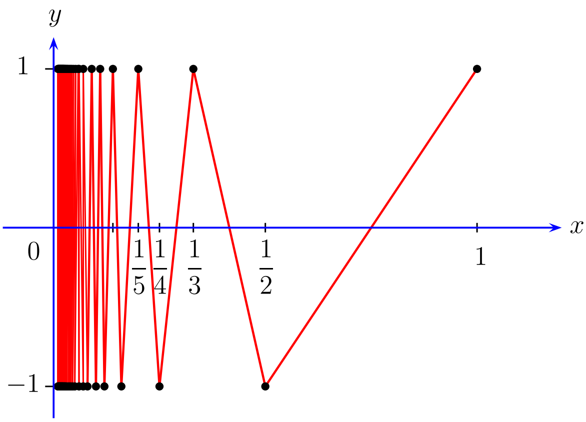

Fig. 3.4. Graph of the criss-cross function in Example 3.18.

On the other hand, f is not continuous at 0. To see this, let $x_n := 1/(2n-1)$ and $y_n := 1/(2n)$ for $n \in \mathbb{N}$. Then $x_n \to 0$ and $y_n \to 0$, but $f(x_n) = 1$, while $f(y_n) = -1$ for all $n \in \mathbb{N}$, so that $f(x_n) \not\to f(0)$ and $f(y_n) \not\to f(0)$. This argument in fact shows that f cannot be made continuous at 0 by redefining its value at 0.

Next, we show that f has the IVP on the interval $[0,1]$. Let $[a,b]$ be any subinterval of $[0,1]$. If $a > 0$, then f is continuous on $[a,b]$, and hence f assumes every value between $f(a)$ and $f(b)$ by the Intermediate Value Theorem (Proposition 3.16). Also, if $a = 0$ and $b > 0$, then there is a positive integer k such that $(1/k) < b$. Now $a < 1/(k+1) < 1/k < b$, and f assumes every value between -1 and 1 on the interval $[1/(k+1), 1/k]$. This shows that f has the IVP on $[0,1]$ although it is not continuous on $[0,1]$. $\diamond$

Remark 3.19. The following partial converse of the Intermediate Value Theorem holds. If a one-one function defined on an interval has the IVP, then it is continuous. This can be seen by noting that such a function is strictly monotonic and its range is an interval (Proposition 1.25), and then appealing to Corollary 3.14. Further, Proposition 3.45 will show that every monotonic function having the IVP on an interval is continuous. $\diamond$

Uniform Continuity

We introduce a concept that in general is stronger than the concept of continuity of a function. It will be useful in Chapter 6 when we consider "integrable" functions.

Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function. We say that f is **uniformly continuous** on D if

$$(x_n), (y_n) \text{ any sequences in } D \text{ and } x_n - y_n \to 0 \Longrightarrow f(x_n) - f(y_n) \to 0.$$

The following result establishes a relationship between the continuity and the uniform continuity of a function.

Proposition 3.20. *Let $D \subseteq \mathbb{R}$. Every uniformly continuous function on D is continuous on D. Moreover, if D is a closed and bounded set, then every continuous function on D is uniformly continuous on D.*

Proof. Let $f : D \to \mathbb{R}$ be given. First assume that f is uniformly continuous on D. If $c \in D$ and (x_n) is any sequence in D such that $x_n \to c$, then let $y_n := c$ for all $n \in \mathbb{N}$. Since $x_n - y_n \to 0$, we obtain $f(x_n) - f(c) = f(x_n) - f(y_n) \to 0$, that is, $f(x_n) \to f(c)$. Thus f is continuous at c. Since this holds for every $c \in D$, f is continuous on D.

Now assume that D is a closed and bounded set and f is continuous on D. Suppose f is not uniformly continuous on D. Then there are sequences (x_n) and (y_n) in D such that $x_n - y_n \to 0$, but $f(x_n) - f(y_n) \not\to 0$. Consequently, there exist $\epsilon > 0$ and positive integers $n_1 < n_2 < \cdots$ such that $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$ for all $k \in \mathbb{N}$. Since D is a bounded set, the sequence (x_{n_k}) is bounded. By the Bolzano–Weierstrass Theorem, it has a convergent subsequence, say $(x_{n_{k_j}})$. Let us denote the sequences $(x_{n_{k_j}})$ and $(y_{n_{k_j}})$ by $(\tilde{x}_j)$ and $(\tilde{y}_j)$ for simplicity. Let $\tilde{x}_j \to c$. Then $c \in D$, since D is a closed set. Because $x_n - y_n \to 0$, we see that $\tilde{x}_j - \tilde{y}_j \to 0$ and hence $\tilde{y}_j \to c$ as well. Since f is continuous at c, we obtain $f(\tilde{x}_j) \to f(c)$ and $f(\tilde{y}_j) \to f(c)$. Thus

$$f(\tilde{x}_j) - f(\tilde{y}_j) \to f(c) - f(c) = 0.$$

But this is a contradiction, since $|f(\tilde{x}_j) - f(\tilde{y}_j)| \geq \epsilon$ for all $j \in \mathbb{N}$. Hence f is uniformly continuous on D. $\square$

We remark that the continuity of a function on a set D is a local concept, that is, a function is defined to be continuous on D if it is continuous at every $c \in D$. On the other hand, the uniform continuity of a function on a set D takes into account the behavior of the function on the entire set D. In this sense, uniform continuity is a global concept.

Examples 3.21. (i) Since the interval $[a, b]$ is a closed and bounded set, it follows from the preceding proposition that every continuous function on $[a, b]$ is uniformly continuous on $[a, b]$. This result will be of crucial importance in our discussion of Riemann integration in Chapter 6.

(ii) If a subset D of $\mathbb{R}$ is not closed, then a continuous function on D may not be uniformly continuous on D. For example, consider $D := (a, b]$ and $f : D \to \mathbb{R}$ defined by $f(x) := 1/(x - a)$. Clearly f is continuous on D. But f is not uniformly continuous on D. To see this, let

$$x_n := a + \frac{b-a}{n} \quad \text{and} \quad y_n := a + \frac{b-a}{n+1}, \quad \text{for } n \in \mathbb{N}.$$

Then $x_n - y_n = (b - a)/n(n + 1) \to 0$, but

$$f(x_n) - f(y_n) = \frac{n - (n+1)}{b-a} = \frac{1}{a-b} \quad \text{for all } n \in \mathbb{N},$$

and hence $f(x_n) - f(y_n) \not\to 0$. An alternative way to show that f is not uniformly continuous is to note that D is a bounded set, but f is not a bounded function, and use Exercise 3.35 (i). For an example of a bounded continuous function on a bounded set that is not uniformly continuous, consider the criss-cross function of Example 3.18 on $D := (0, 1]$.

(iii) If a subset D of $\mathbb{R}$ is not bounded, then a continuous function on D may not be uniformly continuous on D. For example, let $D = [a, \infty)$ and $f(x) = x^2$ for $x \in D$. Then f is continuous on D. But f is not uniformly continuous on D. To see this, let

$$x_n := a + n \quad \text{and} \quad y_n := a + n - \frac{1}{n}, \quad n \in \mathbb{N}.$$

Then $x_n - y_n = 1/n \to 0$, but

$$f(x_n) - f(y_n) = (a + n)^2 - \left(a + n - \frac{1}{n}\right)^2 = 2 + \frac{2a}{n} - \frac{1}{n^2}$$

for all $n \in \mathbb{N}$, and so $f(x_n) - f(y_n) \not\to 0$. $\diamond$

Finally, we give a criterion for the uniform continuity of a function that does not involve convergence of sequences. The following result may be compared with the ϵ-δ condition for continuity given in Proposition 3.8.

Proposition 3.22. *Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function. Then f is uniformly continuous on D if and only if f satisfies the following uniform ϵ-δ condition: For every $\epsilon > 0$, there is $\delta > 0$ such that*

$$x, y \in D \text{ and } |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \epsilon.$$

Proof. Let f be uniformly continuous on D. Suppose there is $\epsilon > 0$ such that for every $\delta > 0$, there are x and y in D such that $|x - y| < \delta$, but $|f(x) - f(y)| \geq \epsilon$. Considering $\delta := 1/n$ for $n \in \mathbb{N}$, we obtain sequences (x_n) and (y_n) in D such that $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \epsilon$ for all $n \in \mathbb{N}$. Then $x_n - y_n \to 0$, but $f(x_n) - f(y_n) \not\to 0$. This contradicts the assumption that f is uniformly continuous on D.

Conversely, assume that the uniform ϵ-δ condition holds. Let (x_n) and (y_n) be any sequences in D such that $x_n - y_n \to 0$. Let $\epsilon > 0$ be given. Then there is $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$, whenever $x, y \in D$ and $|x - y| < \delta$. Since $x_n - y_n \to 0$, we can find $n_0 \in \mathbb{N}$ such that $|x_n - y_n| < \delta$ for all $n \geq n_0$. But then $|f(x_n) - f(y_n)| < \epsilon$ for all $n \geq n_0$. Thus $f(x_n) - f(y_n) \to 0$. Hence f is uniformly continuous on D. $\square$

Note that the difference between the uniform ϵ-δ condition in the above result and the ϵ-δ condition in Proposition 3.8 is that in the former, δ depends only on ϵ, whereas in the latter, δ depends not only on ϵ, but also possibly on the choice of the point c.

3.3 Limits of Functions of a Real Variable

In Chapter 2 we have seen what is meant by the limit of a sequence. As we know, a sequence is a function whose domain is the set $\mathbb{N}$ of all natural numbers. We shall now define the concept of a limit of a function at a point in $\mathbb{R}$ around which there are sufficiently many points of the domain. For defining this concept, we shall utilize the notion of sequences. For proving properties of limits of functions, we shall relate the notion of limit to that of continuity, and then use the results of the previous section. We begin with the notion of a limit point of a set, which can be viewed as a legitimate point at which limits of functions defined on that set can be considered.

Let $D \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. Then c is called a **limit point** (or an **accumulation point**) of D if there is a sequence (x_n) in $D \setminus \{c\}$ such that $x_n \to c$.

Examples 3.23. (i) If $c \in \mathbb{R}$ and if D contains either $(c - r, c)$ or $(c, c + r)$ for some $r > 0$, then c is a limit point of D. To see this, it suffices to consider the sequence (x_n) in D defined by $x_n := c - r/2n$ for $n \in \mathbb{N}$ or by $x_n := c + r/2n$ for $n \in \mathbb{N}$. In particular, if $D := (c - r, c + r)$ or if $D := (c - r, c) \cup (c, c + r)$ for some $r > 0$, then c is a limit point of D, and in fact, every point of $[c - r, c + r]$ is a limit point of D. Likewise, if $D := \mathbb{R}$ or if $D := \mathbb{R} \setminus \{c\}$, then every point of $\mathbb{R}$ is a limit point of D.

(ii) If $D := \{1/n : n \in \mathbb{N}\}$, then 0 is the only limit point of D.
(iii) If D is a finite set or if $D = \mathbb{Z}$, then D has no limit point.

An equivalent definition of limit points can be obtained from the following.

Proposition 3.24. *Let $D \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. Then c is a limit point of D if and only if for every $r > 0$, there is $x \in D$ such that $0 < |x - c| < r$.*

Proof. Suppose c is a limit point of D. Then there is a sequence (x_n) in $D \setminus \{c\}$ such that $x_n \to c$. Let $r > 0$ be given. Since $x_n \to c$, there is $n_0 \in \mathbb{N}$ such that $|x_n - c| < r$ for all $n \geq n_0$. In particular, $x_{n_0} \in D$ satisfies $0 < |x_{n_0} - c| < r$.

Conversely, if for every $r > 0$, there is $x \in D$ such that $0 < |x - c| < r$, then taking $r = 1/n$ for $n \in \mathbb{N}$, we obtain a sequence (x_n) in $D \setminus \{c\}$ such that $x_n \to c$. $\square$

Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. Also, let $f : D \to \mathbb{R}$ be a function. We say that a **limit** of f as x tends to c exists if there is a real number ℓ such that

$$(x_n) \text{ any sequence in } D \setminus \{c\} \text{ and } x_n \to c \Longrightarrow f(x_n) \to \ell.$$

We then write

$$f(x) \to \ell \ \text{ as } \ x \to c \qquad \text{or} \qquad \lim_{x \to c} f(x) = \ell.$$

Note that since c is a limit point of D, there does exist a sequence in $D \setminus \{c\}$ that converges to c. In particular, it follows from part (i) of Proposition 2.2 that $\lim_{x \to c} f(x)$ is unique whenever it exists.

Examples 3.25. (i) Consider the function whose graph is as in Figure 3.5. More precisely, let $D := \mathbb{R}$, $c := 0$, and let $f : D \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 1 & \text{if } x < 0, \\ 2 & \text{if } x = 0, \\ x + 1 & \text{if } x > 0. \end{cases}$$

Then 0 is clearly a limit point of D. Moreover, $\lim_{x \to 0} f(x) = 1$. To see this, let (x_n) be a sequence in $D \setminus \{0\}$ such that $x_n \to 0$. If $x_n < 0$ for some $n \in \mathbb{N}$, then $f(x_n) - 1 = 1 - 1 = 0$, and if $x_n > 0$ for some $n \in \mathbb{N}$, then $f(x_n) - 1 = (x_n + 1) - 1 = x_n$. It follows that $f(x_n) \to 1$. Thus $\ell = 1$ is the limit of f as x tends to 0. Note that $f(0) = 2$.

(ii) Let $D := \mathbb{R} \setminus \{0\}$, $c := 0$, and let $f : D \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

If $x_n := (-1)^n/n$ for $n \in \mathbb{N}$, then (x_n) is a sequence in $D \setminus \{0\}$ and $x_n \to 0$, but since $f(x_n) = (-1)^n$ for $n \in \mathbb{N}$, the sequence $(f(x_n))$ is

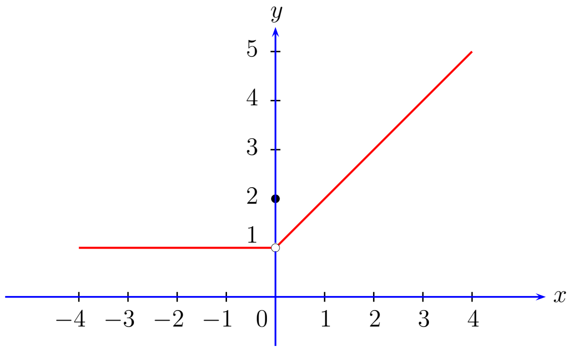

Fig. 3.5. Graph of the function in Example 3.25 (i).

divergent. Hence a limit of f as x tends to 0 does not exist. This can also be seen by considering $y_n := 1/n$ and $z_n := -1/n$ for $n \in \mathbb{N}$ and observing that $y_n \to 0$ and $z_n \to 0$, whereas $f(y_n) \to 1$ and $f(z_n) \to -1$.

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let $c \in \mathbb{R}$. Then a limit of $f(x)$ as x tends to c does not exist. To see this, choose a rational number x_n and an irrational number y_n in $(c, c + 1/n)$ for each $n \in \mathbb{N}$. Then (x_n) and (y_n) are sequences in $\mathbb{R} \setminus \{c\}$ that converge to c, but $f(x_n) \to 1$ and $f(y_n) \to 0$. Thus there can be no $\ell \in \mathbb{R}$ such that $f(x) \to \ell$ as $x \to c$. ◇

We now relate the concepts of continuity and limit.

Proposition 3.26. *Let $D \subseteq \mathbb{R}$, and let $c \in D$ be a limit point of D. Also, let $f : D \to \mathbb{R}$ be a function. Then f is continuous at c if and only if $\lim_{x\to c} f(x)$ exists and is equal to $f(c)$.*

Proof. Assume that f is continuous at c. Let (x_n) be any sequence in $D \setminus \{c\}$ such that $x_n \to c$. By the continuity of f at c, we see that $f(x_n) \to f(c)$. Thus $\lim_{x\to c} f(x)$ exists and equals $f(c)$.

Conversely, assume that $\lim_{x\to c} f(x)$ exists and is equal to $f(c)$. Let (x_n) be any sequence in D such that $x_n \to c$. If there is $n_0 \in \mathbb{N}$ such that $x_n = c$ for all $n \geq n_0$, then clearly $f(x_n) \to f(c)$. Otherwise, there are positive integers $n_1, n_2, \ldots$ such that $n_1 < n_2 < \cdots$ and $\{n \in \mathbb{N} : x_n \neq c\} = \{n_k : k \in \mathbb{N}\}$. Now, (x_{n_k}) is a sequence in $D\setminus\{c\}$ that converges to c, and therefore, $f(x_{n_k}) \to f(c)$. Since $f(x_n) = f(c)$ for all $n \in \mathbb{N} \setminus \{n_k : k \in \mathbb{N}\}$, it follows that $f(x_n) \to f(c)$. Hence f is continuous at c. □

Example 3.27. The function $f : \mathbb{R} \to \mathbb{R}$ given in Example 3.25(i) is not continuous at 0, since $\lim_{x\to 0} f(x) = 1$ and $f(0) = 2$. On the other hand, if we define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) := \begin{cases} 1 & \text{if } x \le 0, \\ x+1 & \text{if } x > 0, \end{cases}$$

then $\lim_{x\to 0} g(x) = 1$ and $g(0) = 1$. Hence g is continuous at 0. $\diamond$

Corollary 3.28. *Let $D \subseteq \mathbb{R}$, and let $c \in \mathbb{R}$ be a limit point of D. Given a function $f : D \to \mathbb{R}$ and $\ell \in \mathbb{R}$, let $F : D \cup \{c\} \to \mathbb{R}$ be the function defined by*

$$F(x) := \begin{cases} f(x) & \textit{if } x \in D \setminus \{c\}, \\ \ell & \textit{if } x = c. \end{cases}$$

Then $\lim_{x\to c} f(x)$ exists and is equal to ℓ if and only if F is continuous at c.

Proof. Since $F(x) = f(x)$ for all $x \in D\setminus\{c\}$, it is clear that $\lim_{x\to c} f(x)$ exists if and only if $\lim_{x\to c} F(x)$ exists, and in this case the two limits are equal. Now since c is a limit point of $D\cup\{c\}$ and $F(c) = \ell$, the desired result follows by applying Proposition 3.26 to the function F defined on $D \cup \{c\}$. $\square$

We now prove some results that are useful in calculating the limits of several functions. First we consider how the algebraic operations on $\mathbb{R}$ are related to limits of functions of a real variable. The following result is known as the **Limit Theorem for Functions**.

Proposition 3.29. *Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. Also, let $\ell, m \in \mathbb{R}$ and let $f, g : D \to \mathbb{R}$ be functions such that*

$$\lim_{x\to c} f(x) = \ell \quad \textit{and} \quad \lim_{x\to c} g(x) = m.$$

Then

(i) $\lim\limits_{x\to c} (f+g)(x) = \ell + m,$

(ii) $\lim\limits_{x\to c} (rf)(x) = r\ell$ *for every* $r \in \mathbb{R}$,

(iii) $\lim\limits_{x\to c} (fg)(x) = \ell m,$

(iv) *if $\ell \neq 0$, then there is $\delta > 0$ such that $f(x) \neq 0$ for all $x \in D$ satisfying $0 < |x - c| < \delta$; also, c is a limit point of $\{x \in D : 0 < |x - c| < \delta\}$ and*

$$\lim_{x\to c} \left(\frac{1}{f}\right)(x) = \frac{1}{\ell},$$

(v) *if there is $\delta > 0$ such that $f(x) \ge 0$ for all $x \in D$ with $0 < |x - c| < \delta$, then $\ell \ge 0$, and for every $k \in \mathbb{N}$,*

$$\lim_{x\to c} f^{1/k}(x) = \ell^{1/k}.$$

Proof. Let $F, G : D \cup \{c\} \to \mathbb{R}$ be the functions defined by

$$F(x) := \begin{cases} f(x) & \text{if } x \in D \setminus \{c\}, \\ \ell & \text{if } x = c, \end{cases} \quad \text{and} \quad G(x) := \begin{cases} g(x) & \text{if } x \in D \setminus \{c\}, \\ m & \text{if } x = c. \end{cases}$$

Then by Corollary 3.28, both F and G are continuous at c. Also, it is clear from Corollary 3.9 that if $\ell \neq 0$, then there is $\delta > 0$ such that $f(x) \neq 0$ for all $x \in D$ satisfying $0 < |x - c| < \delta$; moreover, Proposition 3.24 readily shows that c is a limit point of $\{x \in D : 0 < |x - c| < \delta\}$. Thus the assertions in (i), (ii), (iii), (iv), and (v) follow from parts (i), (ii), (iii), (iv), and (v) of Proposition 3.2 applied to the functions F and G defined on $D \cup \{c\}$. □

With notation and hypotheses as in the proposition above, a combined application of its parts (i) and (ii) shows that $\lim_{x\to c}(f - g)(x) = \ell - m$. Likewise, a combined application of parts (iii) and (iv) shows that if $m \neq 0$, then $\lim_{x\to c}(f/g)(x) = \ell/m$. Likewise, a combined application of parts (iii) and (v) shows that if r is any positive rational number and $f(x) \geq 0$ for all $x \in D$, then $\ell \geq 0$ and $\lim_{x\to c} f^r(x) = \ell^r$, since $r = m/k$, where m, $k \in \mathbb{N}$. This, together with part (iv), shows that if $\ell > 0$, then $\lim_{x\to c} f^r(x) = \ell^r$ for every negative rational number r.

Next, we show how the order relation on $\mathbb{R}$ is preserved under limits.

Proposition 3.30. *Let D, c, f, g, ℓ, and m be as in Proposition 3.29. If there is $\delta > 0$ such that*

$$f(x) \leq g(x) \text{ for all } x \in D \text{ satisfying } 0 < |x - c| < \delta,$$

then $\ell \leq m$. Conversely, if $\ell < m$, then there is $\delta > 0$ such that

$$f(x) < g(x) \text{ for all } x \in D \text{ satisfying } 0 < |x - c| < \delta.$$

In particular, if there is $\delta > 0$ such that $g(x) \geq 0$ for all $x \in D$ satisfying $0 < |x - c| < \delta$, then $\lim_{x\to c} g(x) \geq 0$, and conversely, if $\lim_{x\to c} g(x) > 0$, then there is $\delta > 0$ such that $g(x) > 0$ for all $x \in D$ satisfying $0 < |x - c| < \delta$.

Proof. Let F, G be as in the proof of Proposition 3.29. Define $H : D \cup \{c\} \to \mathbb{R}$ by $H := G - F$. Then the first assertion follows from part (v) of Proposition 3.2 applied to H, while the converse follows from Corollary 3.9 applied to H. The particular case follows by taking f to be the zero function. □

The following result is analogous to the Sandwich Theorem for sequences.

Proposition 3.31 (Sandwich Theorem). *Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. Also, let $\ell \in \mathbb{R}$ and let $f, g, h : D \to \mathbb{R}$ be such that*

$$f(x) \leq h(x) \leq g(x) \text{ for all } x \in D \quad \text{and} \quad \lim_{x\to c} f(x) = \ell = \lim_{x\to c} g(x).$$

Then

$$\lim_{x\to c} h(x) = \ell.$$

Proof. Let (x_n) be any sequence in $D\setminus\{c\}$ such that $x_n \to c$. Then $f(x_n) \to \ell$, $g(x_n) \to \ell$, and $f(x_n) \le h(x_n) \le g(x_n)$ for all $n \in \mathbb{N}$. Hence $h(x_n) \to \ell$ by the Sandwich Theorem for sequences. This proves that $\lim_{x\to c} h(x) = \ell$. $\square$

The following illustration of the Sandwich Theorem for limits of functions is analogous to the evaluation of $\lim_{x\to 0} x\sin(1/x)$ given in Example 7.20.

Example 3.32. Consider the criss-cross function $f : [0,1] \to \mathbb{R}$ given in Example 3.18. Define $f_1 : [-1,1] \to \mathbb{R}$ by

$$f_1(x) = \begin{cases} xf(x) & \text{if } x \in [0,1], \\ -xf(-x) & \text{if } x \in [-1,0). \end{cases}$$

Since $-1 \le f(x) \le 1$ for all $x \in [0,1]$, we obtain $-|x| \le f_1(x) \le |x|$ for all $x \in [-1,1]$. Hence by the Sandwich Theorem, $\lim_{x\to 0} f_1(x) = 0$. Moreover, since $f_1(0) = 0$, we see that f_1 is continuous at 0. $\diamond$

The following result is an analogue of the ϵ-δ condition for continuity.

Proposition 3.33. *Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. Also, let $f : D \to \mathbb{R}$ be a function. Then $\lim_{x\to c} f(x)$ exists if and only if there is $\ell \in \mathbb{R}$ satisfying the following ϵ-δ condition: For every $\epsilon > 0$, there is $\delta > 0$ such that*

$$x \in D \text{ and } 0 < |x - c| < \delta \Longrightarrow |f(x) - \ell| < \epsilon.$$

Proof. Assume that $\lim_{x\to c} f(x)$ exists and is equal to ℓ. Let $F : D \cup \{c\} \to \mathbb{R}$ be as in Corollary 3.28. Then F is continuous at c, and by Proposition 3.8, F satisfies the ϵ-δ condition for continuity. Consequently, the ϵ-δ condition in the statement of the result is satisfied.

Conversely, let $\ell \in \mathbb{R}$ be such that the ϵ-δ condition in the statement of the result is satisfied. Then the function $F : D \cup \{c\} \to \mathbb{R}$ defined as in Corollary 3.28 satisfies the ϵ-δ condition for continuity. Hence by Proposition 3.8 and Corollary 3.28, we conclude that $\lim_{x\to c} f(x)$ exists. $\square$

Example 3.34. Let $f : [0,1] \to \mathbb{R}$ be the **Thomae function** defined by

$$f(x) := \begin{cases} 1/q & \text{if } x = p/q, \text{ where } p, q \in \mathbb{Z},\ q > 0, \text{ and } p, q \text{ are coprime}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$$

We will use the ϵ-δ condition to show that $\lim_{x\to c} f(x) = 0$ for every $c \in [0,1]$. Let $c \in [0,1]$ and let $\epsilon > 0$ be given. Then there is $n_0 \in \mathbb{N}$ such that $n_0 > 1/\epsilon$. Note that the set of rational numbers p/q in $[0,1]$ such that $p, q \in \mathbb{Z}$ with $0 < q < n_0$ and p, q are coprime is finite. Let us choose $\delta > 0$ such that $(c-\delta, c+\delta) \setminus \{c\}$ contains none of these finitely many rational numbers. Thus for $x \in [0,1]$ with $0 < |x - c| < \delta$, either $f(x) = 0$ (when x is irrational) or $f(x) = 1/q$ for some $q \in \mathbb{N}$ with $q \ge n_0$ (when x is rational). It follows that

$$x \in [0,1] \text{ and } 0 < |x - c| < \delta \Longrightarrow |f(x)| \leq \frac{1}{n_0} < \epsilon.$$

Thus $\lim_{x\to c} f(x) = 0$. By Proposition 3.26, f is continuous at each irrational number in $[0, 1]$ and discontinuous at each rational number in $[0, 1]$. ◇

Next, we consider an analogue of the Cauchy Criterion for the convergence of a sequence (Proposition 2.22) for limits of functions of a real variable.

Proposition 3.35 (Cauchy Criterion). *Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. Also, let $f : D \to \mathbb{R}$ be a function. Then $\lim_{x\to c} f(x)$ exists if and only if for every $\epsilon > 0$, there is $\delta > 0$ such that*

$$x, y \in D,\ 0 < |x - c| < \delta \text{ and } 0 < |y - c| < \delta \Longrightarrow |f(x) - f(y)| < \epsilon.$$

Proof. Assume that $\lim_{x\to c} f(x)$ exists and is equal to a real number ℓ. Let $\epsilon > 0$ be given. By Proposition 3.33, there is $\delta > 0$ such that

$$x \in D \text{ and } 0 < |x - c| < \delta \Longrightarrow |f(x) - \ell| < \frac{\epsilon}{2}.$$

Hence for $x, y \in D$ satisfying $0 < |x - c| < \delta$ and $0 < |y - c| < \delta$,

$$|f(x) - f(y)| \leq |f(x) - \ell| + |\ell - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conversely, assume that the condition given in the statement of the result holds. Let $\epsilon > 0$ be given. Then there is $\delta > 0$ such that

$$x, y \in D, 0 < |x - c| < \delta \text{ and } 0 < |y - c| < \delta \Longrightarrow |f(x) - f(y)| < \frac{\epsilon}{2}.$$

Since c is a limit point of D, there is a sequence (x_n) in $D \setminus \{c\}$ such that $x_n \to c$. Then there is $n_0 \in \mathbb{N}$ such that $|x_n - c| < \delta$ for all $n \geq n_0$. Hence

$$|f(x_n) - f(x_m)| < \frac{\epsilon}{2} \quad \text{for all } n, m \geq n_0.$$

Thus $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$. By the Cauchy Criterion for sequences, there is $\ell \in \mathbb{R}$ such that $f(x_n) \to \ell$. Hence there is $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$ and $|f(x_{n_1}) - \ell| < \epsilon/2$. Since $0 < |x_{n_1} - c| < \delta$, it follows that

$$x \in D \text{ and } 0 < |x - c| < \delta \Longrightarrow |f(x) - \ell| \leq |f(x) - f(x_{n_1})| + |f(x_{n_1}) - \ell| < \epsilon.$$

Consequently, by Proposition 3.33, $\lim_{x\to c} f(x)$ exists and is equal to ℓ. □

We now consider one-sided limits. Let D be a subset of $\mathbb{R}$ and let $c \in \mathbb{R}$ be such that c is a limit point of $D \cap (-\infty, c)$, that is, there is a sequence (x_n) in D with $x_n < c$ for all $n \in \mathbb{N}$ and $x_n \to c$. We say that a **left (hand) limit** of f as x tends to c (from the left) exists if there is a real number ℓ such that

$$(x_n) \text{ any sequence in } D,\ x_n < c, \text{ and } x_n \to c \Longrightarrow f(x_n) \to \ell.$$

We then write

$$f(x) \to \ell \text{ as } x \to c^- \quad \text{ or } \quad \lim_{x\to c^-} f(x) = \ell.$$

Since c is a limit point of $D\cap(-\infty, c)$, it follows from part (i) of Proposition 2.2 that if $\lim_{x\to c^-} f(x)$ exists, then it is unique.

Similarly, if $c \in \mathbb{R}$ is such that c is a limit point of $D\cap(c,\infty)$, then we define the **right (hand) limit** of $f(x)$ as x tends c (from the right) by replacing the requirement $x_n < c$ by the requirement $x_n > c$ in the definition above. We then write

$$f(x) \to \ell \text{ as } x \to c^+ \quad \text{ or } \quad \lim_{x\to c^+} f(x) = \ell.$$

Remark 3.36. Results similar to Propositions 3.29, 3.30, 3.31, 3.33, and 3.35 hold for left limits and for right limits. The proofs are similar. $\diamond$

The notions of left limits and right limits, when both are defined, are related to the notion of limit by the following result.

Proposition 3.37. *Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be such that c is a limit point of $D \cap (-\infty, c)$ as well as of $D \cap (c, \infty)$. Also, let $f : D \to \mathbb{R}$ be a function. Then*

$$\lim_{x\to c} f(x) \textit{ exists } \iff \textit{both } \lim_{x\to c^-} f(x) \textit{ and } \lim_{x\to c^+} f(x) \textit{ exist and are equal.}$$

In this case, $\lim_{x\to c} f(x) = \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$. If in addition $c \in D$, then

$$f \textit{ is continuous at } c \iff \lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c^+} f(x).$$

Proof. If $\lim_{x\to c} f(x)$ exists, then clearly both $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist and are equal to $\lim_{x\to c} f(x)$.

Conversely, suppose $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist and are equal to some $\ell \in \mathbb{R}$. Let $D_1 := \big(D \cap (-\infty, c)\big) \cup \{c\}$ and $D_2 := \big(D \cap (c,\infty)\big) \cup \{c\}$. Note that $D_1 \cup D_2 = D \cup \{c\}$. Consider $F_i : D_i \to \mathbb{R}$ for $i = 1, 2$, and $F : D_1 \cup D_2 \to \mathbb{R}$ defined by

$$F_i(x) := \begin{cases} f(x) & \text{if } x \in D_i \setminus \{c\}, \\ \ell & \text{if } x = c, \end{cases} \quad \text{and} \quad F(x) := \begin{cases} f(x) & \text{if } x \in D \setminus \{c\}, \\ \ell & \text{if } x = c. \end{cases}$$

It is clear that

$$\lim_{x\to c^-} f(x) = \lim_{x\to c} F_1(x) \quad \text{and} \quad \lim_{x\to c^+} f(x) = \lim_{x\to c} F_2(x).$$

Hence by Corollary 3.28, we see that both F_1 and F_2 are continuous at c, and consequently, by Proposition 3.6, F is continuous at c. Thus it follows from Corollary 3.28 that $\lim_{x\to c} f(x)$ exists and is equal to ℓ.

In case $c \in D$, the last assertion follows from Proposition 3.26. $\square$

Limits at Infinity and Infinite "Limits"

Suppose $D \subseteq \mathbb{R}$ is such that D is not bounded above. Then there is a sequence in D that tends to ∞. Consider a function $f : D \to \mathbb{R}$. We say that a **limit** of f as x tends to infinity exists if there is a real number ℓ such that

$$(x_n) \text{ any sequence in } D \text{ and } x_n \to \infty \Longrightarrow f(x_n) \to \ell.$$

We then write

$$f(x) \to \ell \text{ as } x \to \infty \quad \text{ or } \quad \lim_{x\to\infty} f(x) = \ell.$$

Since there exists a sequence (x_n) in D such that $x_n \to \infty$, it follows from part (i) of Proposition 2.2 that if $\lim_{x\to\infty} f(x)$ exists, then it is unique.

It is easy to see that results similar to Propositions 3.29, 3.30, and 3.31 hold for limits as $x \to \infty$. We now give an analogue of the ϵ-δ condition (Proposition 3.33) and the Cauchy criterion (Proposition 3.35) for such limits.

Proposition 3.38. *Suppose $D \subseteq \mathbb{R}$ is not bounded above and $f : D \to \mathbb{R}$ is a function. Then $\lim_{x\to\infty} f(x)$ exists if and only if there is $\ell \in \mathbb{R}$ satisfying the following ϵ-α condition: For every $\epsilon > 0$, there is $\alpha \in \mathbb{R}$ such that*

$$x \in D \textit{ and } x \geq \alpha \Longrightarrow |f(x) - \ell| < \epsilon.$$

Proof. Assume that $\lim_{x\to\infty} f(x)$ exists and is equal to a real number ℓ. Suppose for a moment that the ϵ-α condition does not hold. This means that there is $\epsilon > 0$ such that for every $\alpha \in \mathbb{R}$, there is $x \in D$ satisfying $x \geq \alpha$, but $|f(x) - \ell| \geq \epsilon$. By choosing $\alpha = n$ for each $n \in \mathbb{N}$, we may obtain a sequence (x_n) in D such that $x_n \geq n$, but $|f(x_n) - \ell| \geq \epsilon$ for all $n \in \mathbb{N}$. Now $x_n \to \infty$ and $f(x_n) \not\to \ell$. This contradicts the assumption that $\lim_{x\to\infty} f(x) = \ell$.

Conversely, assume the ϵ-α condition. Let (x_n) be any sequence in D such that $x_n \to \infty$. Let $\epsilon > 0$ be given. Then there is $\alpha \in \mathbb{R}$ such that

$$x \in D \text{ and } x \geq \alpha \Longrightarrow |f(x) - \ell| < \epsilon.$$

Since $x_n \to \infty$, there is $n_0 \in \mathbb{N}$ such that $x_n \geq \alpha$, and hence $|f(x_n) - \ell| < \epsilon$, for all $n \geq n_0$. Thus $f(x_n) \to \ell$. So $\lim_{x\to\infty} f(x)$ exists and equals ℓ. □

Proposition 3.39 (Cauchy Criterion). *Suppose $D \subseteq \mathbb{R}$ is not bounded above and $f : D \to \mathbb{R}$ is a function. Then $\lim\limits_{x\to\infty} f(x)$ exists if and only if the following ϵ-α condition holds: For every $\epsilon > 0$, there is $\alpha \in \mathbb{R}$ such that*

$$x, y \in D,\ x \geq \alpha,\ \textit{and } y \geq \alpha \Longrightarrow |f(x) - f(y)| < \epsilon.$$

Proof. Assume that $\lim_{x\to\infty} f(x)$ exists and is equal to a real number ℓ. Let $\epsilon > 0$ be given. Then there is $\alpha \in \mathbb{R}$ such that

$$x \in D \text{ and } x \geq \alpha \Longrightarrow |f(x) - \ell| < \frac{\epsilon}{2}.$$

Hence for $x, y \in D$ satisfying $x \geq \alpha$ and $y \geq \alpha$, we obtain

$$|f(x) - f(y)| \leq |f(x) - \ell| + |\ell - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conversely, assume that the ϵ-α condition holds. Let $\epsilon > 0$ be given. Then there is $\alpha \in \mathbb{R}$ such that

$$x, y \in D, x \geq \alpha \text{ and } y \geq \alpha \Longrightarrow |f(x) - f(y)| < \frac{\epsilon}{2}.$$

By our hypothesis, there is a sequence (x_n) in D such that $x_n \to \infty$. Hence there is $n_0 \in \mathbb{N}$ such that $x_n \geq \alpha$ for all $n \geq n_0$. Consequently,

$$|f(x_n) - f(x_m)| < \frac{\epsilon}{2} \quad \text{for all } n, m \geq n_0.$$

Thus $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$. By the Cauchy Criterion for sequences, there is $\ell \in \mathbb{R}$ such that $f(x_n) \to \ell$. Hence there is $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$ and $|f(x_{n_1}) - \ell| < \epsilon/2$. Since $x_{n_1} \geq \alpha$, it follows that

$$x \in D \text{ and } x \geq \alpha \Longrightarrow |f(x) - \ell| \leq |f(x) - f(x_{n_1})| + |f(x_{n_1}) - \ell| < \epsilon.$$

Consequently, by Proposition 3.38, $\lim_{x\to\infty} f(x)$ exists and is equal to ℓ. □

Remark 3.40. We shall now explain how one can compare the orders of magnitude of two functions just as we compared the orders of magnitude of sequences in Remark 2.11. Let $D \subseteq \mathbb{R}$ be such that (a, ∞) is contained in D for some $a \in \mathbb{R}$. Also, let f, $g : D \to \mathbb{R}$ be any functions.

If there are $K > 0$ and $\alpha \in (a, \infty)$ such that

$$|f(x)| \leq K|g(x)| \text{ for all } x \geq \alpha, \text{ then we write } f(x) = O(g(x)) \text{ as } x \to \infty.$$

One reads the statement "$f(x) = O(g(x))$ as $x \to \infty$" as "$f(x)$ is big-oh of $g(x)$ as x tends to infinity". If $g(x) = 1$ for all large x, then we simply write $f(x) = O(1)$ as $x \to \infty$, and this means that f is bounded. For example,

$$\frac{[x]}{x} = O(1), \quad 10\,[x] + 100 = O(x), \quad \text{and} \quad \frac{10}{[x]} + \frac{100}{x\sqrt{x}} = O\left(\frac{1}{x}\right) \text{ as } x \to \infty.$$

If for every $\epsilon > 0$, there is $\alpha \in (a, \infty)$ such that

$$|f(x)| \leq \epsilon|g(x)| \text{ for all } x \geq \alpha, \quad \text{then we write } f(x) = o(g(x)) \text{ as } x \to \infty.$$

One reads the statement "$f(x) = o(g(x))$" as "$f(x)$ is little-oh of $g(x)$ as x tends to infinity". If $g(x) \neq 0$ for all large $x \in \mathbb{R}$, then $f(x) = o(g(x))$ as $x \to \infty$ means that $\lim_{x\to\infty}(f(x)/g(x)) = 0$. In particular, if $g(x) = 1$ for all large x, then we simply write $f(x) = o(1)$ as $x \to \infty$, and this means that $f(x) \to 0$ as $x \to \infty$. For example,

$$\frac{1}{[x]} = o(1), \quad 10\,x + 100 = o(x\sqrt{x}), \quad \text{and} \quad \frac{10}{[x]} + \frac{100}{x\sqrt{x}} = o\left(\frac{1}{\sqrt{x}}\right) \text{ as } x \to \infty.$$

Now suppose $g(x) \neq 0$ for all large x. We say that f is **asymptotically equivalent** to g and write $f \sim g$ if $\lim_{x\to\infty}(f(x)/g(x)) = 1$. For example,

$$x + \frac{1}{x} \sim x, \quad x^2 + 10\,x + 100 \sim x^2, \quad \text{and} \quad \frac{1}{x^2} + \frac{10}{x^3} + \frac{100}{x^4} \sim \frac{1}{x^2}.$$

Finally, suppose g is monotonically increasing on (a, ∞) and $g(x) \neq 0$ for all large x. We say that f and g have the same **growth rate** as $x \to \infty$ if $f \sim \ell g$ for some $\ell \in \mathbb{R}$ with $\ell \neq 0$. In case $f(x) = o(g(x))$ as $x \to \infty$, then we say that the **growth rate** of f is less than the growth rate of g as $x \to \infty$. On the other hand, if $f(x) = O(g(x))$ as $x \to \infty$, then we say that the **growth rate** of f is at most the growth rate of g as $x \to \infty$.

In a similar manner, we can talk about orders of magnitude and growth rates of real-valued functions on $D \subseteq \mathbb{R}$ as $x \to c^-$, provided $(c - r, c) \subseteq D$ for some $r > 0$, or as $x \to c^+$, provided $(c, c + r) \subseteq D$ for some $r > 0$, or as $x \to -\infty$, provided $(-\infty, a) \subseteq D$ for some $a \in \mathbb{R}$. $\diamond$

As in the case of sequences, we now describe how in some cases ∞ or $-\infty$ can be regarded as a "limit" of a function of a real variable. Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. We say that $f(x)$ tends to ∞ as x tends to c if

$$(x_n) \text{ any sequence in } D \setminus \{c\} \text{ and } x_n \to c \Longrightarrow f(x_n) \to \infty.$$

We then write

$$f(x) \to \infty \text{ as } x \to c.$$

We give an analogue of Proposition 3.33 for functions tending to infinity.

Proposition 3.41. *Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. Also, let $f : D \to \mathbb{R}$ be a function. Then $f(x) \to \infty$ as $x \to c$ if and only if the following α-δ condition holds: For every $\alpha \in \mathbb{R}$, there is $\delta > 0$ such that*

$$x \in D \text{ and } 0 < |x - c| < \delta \Longrightarrow f(x) > \alpha.$$

Proof. Assume that $f(x)$ tend to ∞ as x tends to c. Suppose for a moment that the α-δ condition does not hold. This means that there is $\alpha \in \mathbb{R}$ such that for every $\delta > 0$, there is $x \in D$ satisfying

$$0 < |x - c| < \delta, \text{ but } f(x) \leq \alpha.$$

By choosing $\delta = 1/n$ for each $n \in \mathbb{N}$, we obtain a sequence (x_n) in $D \setminus \{c\}$ such that $|x_n - c| < 1/n$, but $f(x_n) \leq \alpha$ for all $n \in \mathbb{N}$. Now $x_n \to c$ and $f(x_n) \not\to \infty$. This contradicts the assumption that $f(x) \to \infty$ as $x \to c$.

Conversely, assume that the α-δ condition holds. Let (x_n) be a sequence in $D \setminus \{c\}$ such that $x_n \to c$. Let $\alpha \in \mathbb{R}$ be given. Then there is $\delta > 0$ such that

$$x \in D \text{ and } 0 < |x - c| < \delta \Longrightarrow f(x) > \alpha.$$

Since $x_n \to c$, there is $n_0 \in \mathbb{N}$ such that $|x_n - c| < \delta$ for all $n \geq n_0$. Hence $f(x_n) > \alpha$ for all $n \geq n_0$. Thus $f(x_n) \to \infty$. So $f(x) \to \infty$ as $x \to c$. $\square$

In a similar manner, we may define "$f(x) \to -\infty$ as $x \to c$", and formulate an equivalent "β-δ condition". Also, one-sided limits (as $x \to c^-$ or as $x \to c^+$) can be similarly treated. Further, we may define

$$f(x) \to \infty \text{ as } x \to \infty \quad \text{and} \quad f(x) \to -\infty \text{ as } x \to \infty$$

as well as

$$f(x) \to \infty \text{ as } x \to -\infty \quad \text{and} \quad f(x) \to -\infty \text{ as } x \to -\infty$$

analogously.

If $f(x) \to \infty$ and $g(x) \to \ell$, where $\ell \in \mathbb{R}$ or $\ell = \infty$ or $\ell = -\infty$, then results regarding the existence of the "limits" of $f(x) + g(x)$ and $f(x)g(x)$ can be stated on the lines of the results stated in Remark 2.12.

Examples 3.42. (i) $\displaystyle\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to-\infty} \frac{1}{x}$, whereas

$$\frac{1}{x} \to \infty \text{ as } x \to 0^+ \quad \text{and} \quad \frac{1}{x} \to -\infty \text{ as } x \to 0^-.$$

(ii) $\displaystyle\lim_{x\to\infty} \frac{x^2 + 2x + 3}{4x^2 + 5x + 6} = \frac{1}{4}$, since

$$\frac{x^2 + 2x + 3}{4x^2 + 5x + 6} = \frac{1 + \dfrac{2}{x} + \dfrac{3}{x^2}}{4 + \dfrac{5}{x} + \dfrac{6}{x^2}} \quad \text{for all } x \in \mathbb{R},\ x \neq 0.$$

(iii) $x^3 \to \infty$ as $x \to \infty$ and $x^3 \to -\infty$ as $x \to -\infty$.

The concept of a limit involving ∞ or $-\infty$ is useful in considering "asymptotes of curves". Roughly speaking, a straight line is considered to be an **asymptote** of a curve if it comes arbitrarily close to that curve. A classification of the asymptotes, depending on their slopes, is given below.

Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function. We tacitly assume that D satisfies an appropriate condition needed for defining the relevant limit.

- A straight line given by $y = b$, where $b \in \mathbb{R}$, is called a **horizontal asymptote** of the curve $y = f(x)$ if

$$\lim_{x\to\infty} (f(x) - b) = 0 \quad \text{or} \quad \lim_{x\to-\infty} (f(x) - b) = 0.$$

- A straight line given by $y = ax + b$, where $a, b \in \mathbb{R}$ and $a \neq 0$, is called an **oblique asymptote** of the curve $y = f(x)$ if

$$\lim_{x\to\infty} (f(x) - ax - b) = 0 \quad \text{or} \quad \lim_{x\to-\infty} (f(x) - ax - b) = 0.$$

- A straight line given by $x = c$, where $c \in \mathbb{R}$, is called a **vertical asymptote** of the curve $y = f(x)$ if one or more of the following holds:

$$\begin{array}{ll} f(x) \to \infty \text{ as } x \to c^-, & f(x) \to -\infty \text{ as } x \to c^-, \\ f(x) \to \infty \text{ as } x \to c^+, & f(x) \to -\infty \text{ as } x \to c^+. \end{array}$$

Examples 3.43. (i) Consider $D := (-\infty, 0) \cup (1, \infty)$ and $f : D \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{2x-1}{x-1} & \text{if } x > 1, \\[2ex] \dfrac{3x^2+4x+1}{x} & \text{if } x < 0. \end{cases}$$

Then $f(x) = 2+[1/(x-1)]$ for $x > 1$, and so $\lim_{x\to\infty}(f(x)-2) = 0$. Hence the straight line given by $y = 2$ is a horizontal asymptote of the curve $y = f(x)$. Also, $f(x) \to \infty$ as $x \to 1^+$. Hence the straight line given by $x = 1$ is a vertical asymptote of the curve $y = f(x)$.

Also, $f(x) = 3x+4+(1/x)$ for $x < 0$, and so $\lim_{x\to-\infty}(f(x)-3x-4) = 0$. Hence the straight line given by $y = 3x + 4$ is an oblique asymptote of the curve $y = f(x)$. Also, $f(x) \to -\infty$ as $x \to 0^-$. Hence the straight line given by $x = 0$ is a vertical asymptote of the curve $y = f(x)$.

We can use this information to draw the graph of f as in Figure 3.6.

(ii) Let P and Q be nonzero polynomial functions that do not have a common real root. Consider $D := \mathbb{R} \setminus E$, where E is the set of all real roots of the polynomial function Q. Define $f(x) = P(x)/Q(x)$ for $x \in D$.

If the degree of P is equal to the degree of Q, then

$$f(x) = b + \frac{R(x)}{Q(x)} \quad \text{for } x \in D,$$

where $b \in \mathbb{R}$ and R is a polynomial function whose degree is less than the degree of Q. Since $R(x)/Q(x) \to 0$ as $x \to \infty$ and also as $x \to -\infty$, we see that the straight line given by $y = b$ is a horizontal asymptote of the curve $y = f(x)$.

If the degree of P is greater than the degree of Q by 1, then

$$f(x) = ax + b + \frac{R(x)}{Q(x)} \quad \text{for } x \in D,$$

where $a, b \in \mathbb{R}$, $a \neq 0$, and R is a polynomial function whose degree is less than the degree of Q. Again, since $R(x)/Q(x) \to 0$ as $x \to \infty$ and also as

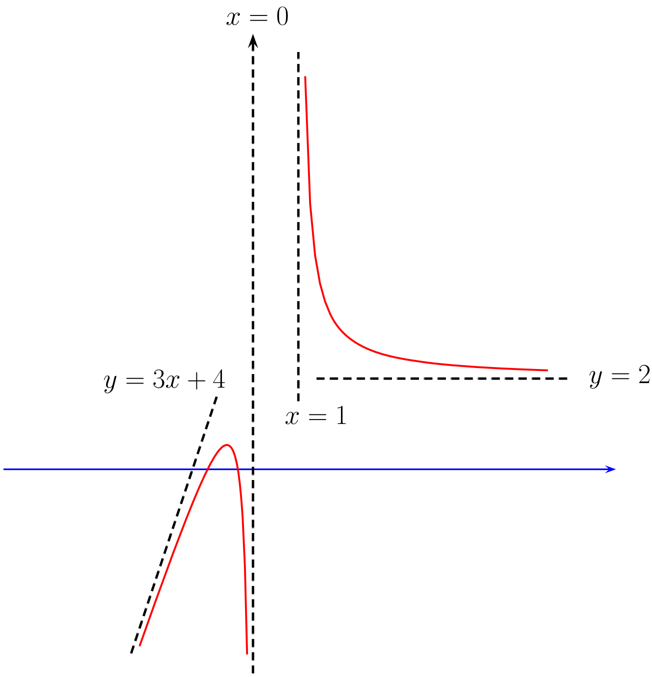

Fig. 3.6. Graph of $f(x) = \begin{cases} (2x-1)/(x-1) & \text{if } x > 1, \\ (3x^2+4x+1)/x & \text{if } x < 0, \end{cases}$ with its horizontal, oblique, and vertical asymptotes.

$x \to -\infty$, we see that the straight line given by $y = ax + b$ is an oblique asymptote of the curve $y = f(x)$.

Let now $c \in E$, that is, $c \in \mathbb{R}$ and $Q(c) = 0$. Then it is easy to see that $f(x) \to \infty$ or $f(x) \to -\infty$ (depending on the signs of the leading coefficients of the polynomial functions P and Q) as $x \to c^-$ and also as $x \to c^+$. Hence the straight line given by $x = c$ is a vertical asymptote of the curve $y = f(x)$.

We now consider limits of monotonic functions. The results given below may be compared with the corresponding results for limits of monotonic sequences (Propositions 2.8 and 2.13).

Proposition 3.44. *Suppose $a \in \mathbb{R}$ or $a = -\infty$, and $b \in \mathbb{R}$ or $b = \infty$. Let $f : (a, b) \to \mathbb{R}$ be a monotonically increasing function. Then*

(i) $\lim_{x \to b^-} f(x)$ *exists if and only if f is bounded above; in this case,*

$$\lim_{x \to b^-} f(x) = \sup\{f(x) : x \in (a, b)\}.$$

Also, $f(x) \to \infty$ as $x \to b^-$ if and only if f is not bounded above.

(ii) $\lim_{x\to a^+} f(x)$ *exists if and only if* f *is bounded below; in this case,*

$$\lim_{x\to a^+} f(x) = \inf\{f(x) : x \in (a,b)\}.$$

Also, $f(x) \to \infty$ *as* $x \to a^+$ *if and only if* f *is not bounded below.*

Proof. (i) Suppose f is bounded above. Let $M := \sup\{f(x) : x \in (a,b)\}$. Given any $\epsilon > 0$, there is $c \in (a,b)$ such that $M - \epsilon < f(c)$. Now since f is monotonically increasing, $M - \epsilon < f(x)$ for all $x \in (c,b)$. To show that $f(x) \to M$ as $x \to b^-$, let (b_n) be a sequence in (a,b) such that $b_n \to b$. Then since $c < b$, there is $n_0 \in \mathbb{N}$ such that $c < b_n$ for all $n \geq n_0$. Hence $M - \epsilon < f(b_n)$ for all $n \geq n_0$. On the other hand, $f(b_n) \leq M$ for all $n \in \mathbb{N}$. This shows that $f(b_n) \to M$. Thus $f(x) \to M$ as $x \to b^-$.

Assume now that f is not bounded above. Let $\alpha \in \mathbb{R}$. Then there is $c \in (a,b)$ such that $f(c) > \alpha$. Since f is monotonically increasing, $f(x) > \alpha$ for all $x \in (c,b)$. Again, let (b_n) be a sequence in (a,b) such that $b_n \to b$. Since $c < b$, there is $n_0 \in \mathbb{N}$ such that $c < b_n$ for all $n \geq n_0$. Hence $\alpha < f(b_n)$ for all $n \geq n_0$. This shows that $f(b_n) \to \infty$. Thus $f(x) \to \infty$ as $x \to b^-$.

(ii) The proof of this part is similar to the proof of part (i) above. □

A result similar to the one above holds for a monotonically decreasing function. (See Exercise 3.31.)

We shall now use Proposition 3.44 to prove the converse of the Intermediate Value Theorem (Proposition 3.16) for monotonic functions.

Proposition 3.45. *Let* I *be an interval and let* $f : I \to \mathbb{R}$ *be a function that is monotonic on* I*. If* f *has the IVP on* I*, then* f *is continuous on* I*.*

Proof. Assume first that f is monotonically increasing on I and that f has the IVP on I. Consider $c \in I$.

Suppose that c is neither the left endpoint nor the right endpoint of the interval I. Then there are $c_1, c_2 \in I$ such that $c_1 < c < c_2$. Now the function f is bounded above on the interval (c_1, c) and it is bounded below on the interval (c, c_2) by $f(c)$. Hence by Proposition 3.44, $f(x) \to \ell_1$ as $x \to c^-$ and $f(x) \to \ell_2$ as $x \to c^+$, where

$$\ell_1 := \sup\{f(x) : c_1 < x < c\} \quad \text{and} \quad \ell_2 := \inf\{f(x) : c < x < c_2\}.$$

Clearly, $\ell_1 \leq f(c) \leq \ell_2$. In fact, since f has the IVP on I, we see that $\ell_1 = f(c) = \ell_2$. Thus by Proposition 3.37, f is continuous at c.

If c is the left endpoint of the interval I, then the above argument shows that $f(x) \to \ell_2$ as $x \to c^+$ and $\ell_2 = f(c)$, while if c is the right endpoint of the interval I, then $f(x) \to \ell_1$ as $x \to c^-$ and $\ell_1 = f(c)$.

Thus in all cases, f is continuous at c. Since c is an arbitrary point of I, we see that f is continuous on I.

If f is monotonically decreasing on I and f has the IVP on I, then $-f$ is monotonically increasing on I and $-f$ has the IVP on I. Hence by what has been proved above, $-f$ is continuous on I, that is, f is continuous on I. □

Notes and Comments

Most books on calculus and analysis treat limits of functions of a real variable first and then discuss the continuity of such a function. We follow, however, the reverse order. The definition of continuity of a function given here relies on the concept of a limit of a sequence, which is introduced in Chapter 2; the domain of such a function can be an arbitrary subset of $\mathbb{R}$*. Such an approach is unusual but not new. See, for example, the book by Goffman [34]. After establishing basic properties of continuous functions, we relate the continuity of a function to various geometric properties of the function such as boundedness, monotonicity, convexity/concavity, and the intermediate value property. This includes a remarkable result, which states that the inverse of a strictly monotonic function defined on an interval is always continuous. The notion of uniform continuity is also introduced using sequences, and it is shown that continuous real-valued functions defined on closed and bounded intervals in* $\mathbb{R}$ *are uniformly continuous.*

Our definition of the limit of a function at a point of $\mathbb{R}$ *that is a limit point of the domain of the function also uses the concept of a limit of a sequence. We also relate the existence of a limit of a function on a subset* D *of* $\mathbb{R}$ *at a point* c *to the continuity of an associated function on* $D \cup \{c\}$*. Most of the basic properties of limits of functions are then deduced easily from the corresponding properties of continuous functions.*

The ϵ-δ *definitions of limits and continuity are often seen as a nemesis for a beginner in calculus. The approach taken here of utilizing the limits of sequences to introduce continuity and limits of functions of a real variable seems to be simple-minded and easier to understand. We have shown the equivalence of the sequential and the* ϵ-δ *definitions toward the end of our discussion of continuity and of limits.*

Exercises

Part A

3.1. State whether there is a function $f : [0, 3] \to \mathbb{R}$ that is continuous at 2 and satisfies
 (i) $f(x) := (x^3 - 3x - 2)/(x - 2)$ for $x \neq 2$,
 (ii) $f(x) := \begin{cases} x & \text{if } x \in [1, 2), \\ x/2 & \text{if } x \in (2, 3]. \end{cases}$

3.2. State whether there is a continuous function $f : [0, 3] \to \mathbb{R}$ such that for every $n \in \mathbb{N}$,
 (i) $f((2n - 1)/n) = (-1)^n$, (ii) $f((2n + 1)/n) = 2^{1/n}$.

3.3. Let k be an odd positive integer and $f(x) := \sqrt[k]{x}$ for $x \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$.

3.4. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, then show that (i) f is continuous at every $c \in \mathbb{R}$ and (ii) $f(sx) = sf(x)$ for all $s, x \in \mathbb{R}$. Deduce that there exists $r \in \mathbb{R}$ such that $f(x) = rx$ for all $x \in \mathbb{R}$.

3.5. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, then show that f is continuous at every $c \in \mathbb{R}$.
[Note: An important example of such a function, known as the exponential function, will be given in Section 7.1.]

3.6. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 1, then show that f is continuous at every $c \in \mathbb{R}$, except possibly at $c = 0$. Give an example of such a function that is continuous at 1 as well as at 0. Also, give an example of such a function that is continuous at 1, but not at 0. (Compare Exercise 1.17.)

3.7. Let $f : (0, \infty) \to \mathbb{R}$ satisfy $f(xy) = f(x) + f(y)$ for all $x, y \in (0, \infty)$. If f is continuous at 1, then show that f is continuous at every $c \in (0, \infty)$.
[Note: An important example of such a function, known as the logarithmic function, will be given in Section 7.1.]

3.8. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} ax & \text{if } x \le 0, \\ \sqrt{x} & \text{if } x > 0, \end{cases}$$

where $a \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$.

3.9. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} x & \text{if } x \text{ is rational}, \\ 1 - x & \text{if } x \text{ is irrational}. \end{cases}$$

Show that f is continuous only at $1/2$.

3.10. Let $D := \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and let $f : D \to \mathbb{R}$ be a function. Show that f is continuous at $1/n$ for every $n \in \mathbb{N}$, and f is continuous at 0 if and only if $f(1/n) \to f(0)$.

3.11. Let $f : [0, 2] \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} x & \text{if } 0 \le x < 1, \\ 3 - x & \text{if } 1 \le x \le 2. \end{cases}$$

Show that f assumes every value between 0 and 2 exactly once on $[0, 2]$, but f is not continuous on $[0, 2]$.

3.12. Let $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} 3x/2 & \text{if } 0 \le x < \frac{1}{2}, \\ (3x-1)/2 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Show that $f([0, 1]) = [0, 1]$. Is f continuous on $[0, 1]$? Does f have the IVP on $[0, 1]$?

3.13. Show that the cubic $x^3 - 6x + 3$ has exactly three real roots. (Hint: Find $f(-3)$, $f(0)$, $f(1)$, and $f(2)$, and use the IVP.)

3.14. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Use the IVP to show that given $c_1, \ldots, c_n \in [a, b]$, there is $c \in [a, b]$ such that

$$f(c) = \frac{f(c_1) + \cdots + f(c_n)}{n}.$$

3.15. If a function f satisfies the given conditions, then can it be continuous?

(i) $f : [1, 10] \to \mathbb{R}$, $f(1) = 0$, $f(10) = 11$, range of $f \subseteq [-1, 0] \cup [1, 11]$.

(ii) $f : [0, 1] \to \mathbb{R}$ and range of $f = (-1, 1)$.

(iii) $f : [-1, 1] \to \mathbb{R}$ and range of $f = [0, \infty)$.

3.16. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} x/(1+x) & \text{if } x \geq 0, \\ x/(1-x) & \text{if } x < 0. \end{cases}$$

Show that f is continuous and bounded on $\mathbb{R}$. Also, prove that

$$\inf\{f(x) : x \in \mathbb{R}\} = -1 \quad \text{and} \quad \sup\{f(x) : x \in \mathbb{R}\} = 1,$$

but there do not exist r, s in $\mathbb{R}$ such that $f(r) = -1$ and $f(s) = 1$.

3.17. Let D and E be subsets of $\mathbb{R}$ such that D is closed and bounded. If $f : D \to E$ is bijective and continuous, then show that $f^{-1} : E \to D$ is continuous. (Hint: Proposition 2.19.) In particular, this result holds if $D = [a, b]$. (Compare Proposition 3.17.)

3.18. Analyze the following functions for uniform continuity:

(i) $f(x) := x$, $x \in \mathbb{R}$, (ii) $f(x) := 1/x$, $x \in (0, 1]$,

(iii) $f(x) := x^2$, $x \in (0, 1)$, (iv) $f(x) := \sqrt{1 - x^2}$, $x \in [-1, 1]$.

3.19. Let D and E be subsets of $\mathbb{R}$, and let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions such that the range of f is contained in E. If f is uniformly continuous on D and g is uniformly continuous on E, then show that $g \circ f$ is uniformly continuous on D.

3.20. Determine the limit points of the subset D of $\mathbb{R}$ if

(i) $D := \{n + (1/n) : n \in \mathbb{N}\}$, (ii) $D := \{2^n : n \in \mathbb{Z}\}$, (iii) $D := \mathbb{Q}$.

3.21. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} x & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then $\lim_{x \to 0} f(x) = 0$, but there is a sequence (x_n) such that $x_n \to 0$ and $f(x_n) \not\to 0$. Explain.

3.22. Show that $\lim_{x \to 1} f(x)$ does not exist if $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ is given by

(i) $f(x) := [x] - x$, (ii) $f(x) := \dfrac{|x-1|}{x-1}$, (iii) $f(x) := \dfrac{[x-1]}{x-1}$.

3.23. Consider $f, g : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Under which of the following conditions does $\lim_{x \to c} f(x)g(x)$ exist? Justify.

(i) $\lim_{x\to c} f(x)$ exists.
(ii) $\lim_{x\to c} f(x)$ exists and g is bounded on $\{x \in \mathbb{R} : 0 < |x - c| < \delta\}$ for some $\delta > 0$.
(iii) $\lim_{x\to c} f(x) = 0$ and g is bounded on $\{x \in \mathbb{R} : 0 < |x - c| < \delta\}$ for some $\delta > 0$.
(iv) $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist.

3.24. Prove that the following limits exist.

(i) $\displaystyle\lim_{x\to 0} x[x]$, (ii) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$, (iii) $\displaystyle\lim_{x\to\infty} \frac{7x-1}{x^2}$, (iv) $\displaystyle\lim_{x\to\infty} \frac{x^4+x}{x^4+1}$,
(v) $\displaystyle\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{7+\sqrt{x+5}}}$, (vi) $\displaystyle\lim_{x\to 1} \frac{|x-1|+1}{x+|x+1|}$, (vii) $\displaystyle\lim_{x\to 3}([x]-[2x-1])$.

3.25. Show that $f(x) \to \infty$ as $x \to \infty$ if $f : [0, \infty) \to \mathbb{R}$ is given by
(i) $f(x) := \dfrac{3x^2+1}{2x+1}$, (ii) $f(x) := [x]$.

3.26. Let f and g be polynomial functions given by

$$f(x) := a_n x^n + \cdots + a_1 x + a_0 \quad \text{and} \quad g(x) := b_m x^m + \cdots + b_1 x + b_0,$$

where $a_n, \ldots, a_0$, $b_m, \ldots, b_0$ are in $\mathbb{R}$, $a_n \neq 0$, and $b_m \neq 0$. Show that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \begin{cases} 0 & \text{if } m > n, \\ a_m/b_m & \text{if } m = n. \end{cases}$$

In case $m < n$, show that

$$\frac{f(x)}{g(x)} \to \infty \text{ as } x \to \infty \text{ if } \frac{a_n}{b_m} > 0\,, \text{ and } \frac{f(x)}{g(x)} \to -\infty \text{ as } x \to \infty \text{ if } \frac{a_n}{b_m} < 0.$$

3.27. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. If $\lim_{x\to c} f(x)$ exists, then show that

$$\lim_{h\to 0^+} \big(f(c+h) - f(c-h)\big) = 0.$$

Is the converse true? Justify your answer.

3.28. **(Limit of Composition)** Let $D \subseteq \mathbb{R}$ and let $s_0 \in \mathbb{R}$ be a limit point of D. Also, let $u : D \to \mathbb{R}$ be a function such that $\lim_{s\to s_0} u(s)$ exists. Let $t_0 := \lim_{s\to s_0} u(s)$. Suppose $E \subseteq \mathbb{R}$ is such that $u(D \setminus \{s_0\}) \subseteq E$ and consider a function $v : E \to \mathbb{R}$. Prove the following.
(i) If t_0 is a limit point of E and $\lim_{t\to t_0} v(t)$ exists, and if in addition, $u(s) \neq t_0$ for every $s \in D \setminus \{s_0\}$, then $\lim_{s\to s_0} v \circ u(s) = \lim_{t\to t_0} v(t)$.
(ii) If $t_0 \in E$ and v is continuous at t_0, then $\lim_{s\to s_0} v \circ u(s) = v(t_0)$.
Show also that the hypothesis in (i) that $u(s) \neq t_0$ for every $s \in D \setminus \{s_0\}$ or the hypothesis in (ii) about the continuity of v at t_0 cannot be dropped.

3.29. Given $\epsilon > 0$, find $\delta > 0$ such that $|f(x) - \ell| < \epsilon$ whenever $0 < |x - c| < \delta$ if
(i) $f(x) := x^2 + 1,\ c = 1,\ \ell = 2$, (ii) $f(x) := \dfrac{1}{x},\ c \neq 0,\ \ell = \dfrac{1}{c}$,
(iii) $f(x) := \dfrac{3x^2+7x+2}{2x+4},\ c = -2,\ \ell = -5/2$.

3.30. Find the asymptotes of the following curves:
(i) $y = \dfrac{x}{x+1}$, $x \neq -1$, (ii) $y = \dfrac{x}{x-1}$, $x \neq 1$, (iii) $y = \dfrac{x^2}{x^2-1}$, $x \neq \pm 1$,
(vi) $y = \dfrac{x^2}{x^2+1}$, (v) $y = \dfrac{x^2+1}{x}$, $x \neq 0$, (vi) $y = \dfrac{x^2+x-2}{x-2}$, $x \neq 2$.

3.31. Let $f : (a,b) \to \mathbb{R}$ be a monotonically decreasing function. Prove the following results. (Compare Proposition 3.44.)
(i) $\lim_{x \to b^-} f(x)$ exists if and only if f is bounded below; in this case,
$$\lim_{x \to b^-} f(x) = \inf\{f(x) : x \in (a,b)\}.$$
Also, $f(x) \to -\infty$ as $x \to b^-$ if and only if f is not bounded below.
(ii) $\lim_{x \to a^+} f(x)$ exists if and only if f is bounded above; in this case,
$$\lim_{x \to a^+} f(x) = \sup\{f(x) : x \in (a,b)\}.$$
Also, $f(x) \to \infty$ as $x \to a^+$ if and only if f is not bounded above.

Part B

3.32. Let $k \in \mathbb{N}$ and $f(x) := x^{1/k}$ for $x \in [0,\infty)$. If $\epsilon \in \mathbb{R}$ is such that $0 < \epsilon \leq 1$, define $\delta := \min\{(1+\epsilon)^n - 1, 1 - (1-\epsilon)^n\}$. Show that $\delta > 0$ and
$$x \in [0,\infty) \text{ and } |x-1| < \delta \Longrightarrow |f(x) - 1| < \epsilon.$$
Also, show that δ is the greatest real number for which this holds.

3.33. Let $f : [a,b] \to \mathbb{R}$ be a continuous function satisfying $f(a) = f(b)$. Show that there are c, $d \in [a,b]$ such that $d - c = (b-a)/2$ and $f(c) = f(d)$. Deduce that for every $\epsilon > 0$, there are x, $y \in [a,b]$ such that $0 < y - x < \epsilon$ and $f(x) = f(y)$.

3.34. Prove that a function $f : (a,b) \to \mathbb{R}$ is convex if and only if it is continuous on (a,b) and satisfies
$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{f(x_1)+f(x_2)}{2} \quad \text{for all } x_1, x_2 \in (a,b).$$
(Hint: To prove convexity, first show that
$$f\left(\frac{x_1+\cdots+x_n}{n}\right) \leq \frac{f(x_1)+\cdots+f(x_n)}{n} \quad \text{for all } x_1, \ldots, x_n \in (a,b)$$
by observing that it holds if $n = 2^k$, where $k \in \mathbb{N}$, and it holds for every $n \in \mathbb{N}$ whenever it holds for $n+1$; then show that $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for all $x, y \in (a,b)$ and $\lambda \in \mathbb{Q}$ with $0 < \lambda < 1$, and finally use the continuity of f to complete the argument.)

3.35. Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function. Prove the following.

(i) If D is bounded and f is uniformly continuous on D, then f is bounded on D. Is this true if f is merely continuous on D?

(ii) Let (x_n) be a Cauchy sequence in D. If f is uniformly continuous on D, then $(f(x_n))$ is also a Cauchy sequence. Is this true if f is merely continuous on D?

3.36. Let $f : (a, b) \to \mathbb{R}$ be a continuous function. Show that f can be extended to a continuous function on $[a, b]$ if and only if f is uniformly continuous on (a, b). (Hint: Exercise 3.35 (ii) and Proposition 3.20.)

3.37. Suppose $f : D \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le \alpha |x - y|^r$ for all x, y in D and some constants $\alpha \in \mathbb{R}$, $r \in \mathbb{Q}$, $r > 0$. Show that f is uniformly continuous on D.

3.38. Let $r \in \mathbb{Q}$ and $r \ge 0$. If $f : [0, \infty) \to \mathbb{R}$ is defined by $f(x) = x^r$, show that f is uniformly continuous if and only if $r \le 1$. (Hint: If $r \le 1$, then $|x^r - y^r| \le 2|x - y|^r$ for all $x, y \in [0, \infty)$ by Exercise 1.48 (ii). If $r > 1$, then consider $x_n := n$, $y_n := n + (1/n^{r-1})$ for $n \in \mathbb{N}$.)

3.39. Let f, $g : D \to \mathbb{R}$ be uniformly continuous on D. Are the functions $f + g$, fg, $1/f$ (provided $f(x) \ne 0$ for all $x \in D$) uniformly continuous on D? What if D is a bounded subset of $\mathbb{R}$? What if D is a closed subset of $\mathbb{R}$? What if $D = [a, b]$? Justify your answers.

3.40. If $D \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ is a limit point of D, then show that for every $r > 0$, the set $\{x \in D : 0 < |x - c| < r\}$ is infinite.

3.41. Let (a_n) be a sequence in $\mathbb{R}$ and let $D := \{a_n : n \in \mathbb{N}\}$ be the set of its terms. If $c \in \mathbb{R}$ is limit point of D, then show that c is a cluster point of (a_n). Give an example to show that a cluster point of a sequence need not be a limit point of the set of its terms.

3.42. Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. Given any $f : D \to \mathbb{R}$, show that $\lim_{x \to c} f(x)$ exists if and only if the following conditions hold:

(i) For every sequence (x_n) in $D \setminus \{c\}$ such that $x_n \to c$, the sequence $(f(x_n))$ is bounded.

(ii) If (x_n) and (y_n) are any sequences in $D \setminus \{c\}$ such that $x_n \to c$ and $y_n \to c$, and moreover, both $(f(x_n))$ and $(f(y_n))$ are convergent, then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$.

(Hint: Proposition 2.19.)

3.43. Let $f : (a, b) \to \mathbb{R}$ be a monotonically increasing function. Show that for every $c \in (a, b)$, both $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ exist, and

$$\lim_{x \to c^-} f(x) = \sup_{a<x<c} f(x) \le f(c) \le \inf_{c<x<b} f(x) = \lim_{x \to c^+} f(x).$$

Also, if $d \in (a, b)$ and $c < d$, then show that

$$\lim_{x \to c^+} f(x) \le \lim_{x \to d^-} f(x).$$

Further, show that similar results hold for a monotonically decreasing function.

3.44. Let I be an open interval and let $f : I \to \mathbb{R}$ be convex on I or concave on I. Use Proposition 3.37 to show that f is continuous. (Hint: Given $c \in I$, choose $c_1, c_2 \in I$ such that $c_1 < c < c_2$. If f is convex on I, then

$$f(c_1) + \frac{f(c) - f(c_1)}{c - c_1}(x - c_1) \leq f(x) \leq f(c) + \frac{f(c_2) - f(c)}{c_2 - c}(x - c)$$

for all $x \in [c, c_2]$ and

$$f(c) + \frac{f(c_2) - f(c)}{c_2 - c}(x - c) \leq f(x) \leq f(c_1) + \frac{f(c) - f(c_1)}{c - c_1}(x - c_1)$$

for all $x \in [c_1, c]$. Compare Proposition 3.15.)

4

Differentiation

Differentiation is a process that associates to a real-valued function f another function f', called the derivative of f. This process is *local* in the sense that the value of f' at a point c depends only on the values of f in a small interval around c. The concept of differentiation originated from two classical problems:

- The geometric problem of determining a tangent at a point to a curve in the plane.
- The physical problem of determining the speed or the velocity of an object, such as a particle or a vehicle or a planet.

The notion of a derivative, which we shall study in this chapter, and the fact that it can often be computed effectively, turns out to be a key to solving the above two problems. Furthermore, the notion of a derivative has an enormous number of applications, some of which will be considered in Chapter 5.

In the first section below, we begin by describing in greater detail the second problem above. This leads to the definition of differentiability. The concept of differentiability of a function is intimately related to the continuity of an associated function, and this connection is made explicit by a lemma of Carathéodory. We first prove the Carathéodory Lemma and then use it to the fullest extent possible to derive a number of basic properties of differentiation. Next, in Section 4.2, we present results known as the Mean Value Theorem and the Taylor Theorem, which are extremely useful in calculus and analysis. In Section 4.3, we show that for differentiable functions, geometric properties of functions such as monotonicity, convexity, and concavity can be effectively determined by looking at their derivatives. Finally, in Section 4.4, we describe L'Hôpital's Rules, which show how differentiation can be used to compute certain limits.

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Undergraduate Texts in Mathematics, https://doi.org/10.1007/978-3-030-01400-1_4

4.1 Derivative and Its Basic Properties

Suppose we are traveling in a car from one place to another. If at an instant t_1 we have covered a distance $s_1 = s(t_1)$ from the starting point and at another instant t_2 we have covered a distance $s_2 = s(t_2)$, then it is clear that the average speed for the journey between these two instants is

$$\frac{\text{distance}}{\text{time}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

Now, what if we want to know the precise speed at a particular instant t_0? If the procedure above is followed blindly, then the answer would come out as the undefined quotient $0/0$, and that does not make sense. So, a natural thing to do is to consider the average speed

$$\frac{s(t_0 + h) - s(t_0)}{(t_0 + h) - t_0} = \frac{s(t_0 + h) - s(t_0)}{h},$$

where h is rather small (but can be positive or negative), so that $t_0 + h$ varies over points close to t_0. It is conceivable that as h approaches 0, the quotient above approaches what the speed at t_0 should be. Also, it is clear that the notion of limit, which was discussed in the previous chapter, would readily make the last statement precise. Thus, we simply set[1]

$$\text{the } \textbf{instantaneous speed} \text{ at } t_0 = \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h}.$$

The idea here can easily be extended from a "distance function" s to an arbitrary real-valued function f. It is, however, desirable that to form quotients such as those above, near a point $x = c$, the function should at least be defined at points around c. We thus make the following definition.

Let D be a subset of $\mathbb{R}$. An element $c \in D$ is said to be an **interior point** of D if there is $r > 0$ such that $(c - r, c + r) \subseteq D$. A function $f : D \to \mathbb{R}$ is said to be **differentiable** at an interior point c of D if the limit

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}, \quad \text{that is,} \quad \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

exists. In this case, the value of the limit is denoted by $f'(c)$ and is called the **derivative** of f at c.

If $D \subseteq \mathbb{R}$ is such that every point of D is an interior point of D, then a function $f : D \to \mathbb{R}$ is said to be **differentiable on** D if f is differentiable at

[1] In this example, the distance function s is evidently increasing, and thus the limit here would be nonnegative. For an arbitrary linear motion, the function s may not be increasing, and thus the limit could also be negative. It is then customary to call it the **instantaneous velocity** rather than the instantaneous speed. In general, speed is given by the absolute value of the velocity.

every point of D. In case f is differentiable on D, we obtain a new function from D to $\mathbb{R}$ whose value at $c \in D$ is $f'(c)$. Quite naturally, this function is denoted by f' and is called the **derivative (function)** of f.

Some alternative notations for the derivative f' are

$$\frac{df}{dx}, \text{ or also } \frac{dy}{dx} \text{ when one writes } y = f(x).$$

Likewise, $f'(c)$ is sometimes denoted by

$$\left.\frac{df}{dx}\right|_{x=c}, \text{ or } \left.\frac{dy}{dx}\right|_{x=c}.$$

At times, physicists use the notation $\dot{f}$ instead of f'.

Examples 4.1. (i) If $f : \mathbb{R} \to \mathbb{R}$ is a constant function, then clearly $f'(c) = 0$ for each $c \in \mathbb{R}$. So the derivative of a constant function is the zero function.

(ii) If $f : \mathbb{R} \to \mathbb{R}$ is the identity function given by $f(x) = x$, then

$$\frac{f(c+h) - f(c)}{h} = \frac{(c+h) - c}{h} = 1 \quad \text{for all } c, h \in \mathbb{R} \text{ with } h \neq 0.$$

It follows that f is differentiable on $\mathbb{R}$ and $f'(x) = 1$ for all $x \in \mathbb{R}$.

(iii) If $f : \mathbb{R} \to \mathbb{R}$ is the absolute value function given by $f(x) = |x|$, then

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|}{h} \quad \text{for all } h \in \mathbb{R} \text{ with } h \neq 0,$$

and from part (ii) of Example 3.25, we see that the limit of this quotient as $h \to 0$ does not exist. So f is not differentiable at $c = 0$. However, f is differentiable at each $c \in \mathbb{R}$, $c \neq 0$, and $f'(c)$ is 1 if $c > 0$ and -1 if $c < 0$.

(iv) If $f : (-1, 1) \to \mathbb{R}$ is defined by $f(x) = \sqrt{x^2 + x^3} = |x|\sqrt{x+1}$, then

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|\sqrt{h+1}}{h} \quad \text{for all } h \in \mathbb{R} \text{ with } h \neq 0,$$

and thus, as in the previous example, the limit of this quotient as $h \to 0$ does not exist. So f is not differentiable at $c = 0$. See Figure 4.1.

(v) If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \sqrt[3]{x^2} = x^{2/3}$, then

$$\frac{f(0+h) - f(0)}{h} = \frac{1}{\sqrt[3]{h}} \quad \text{for all } h \in \mathbb{R} \text{ with } h \neq 0,$$

and the limit of this quotient as $h \to 0$ clearly does not exist. So f is not differentiable at $c = 0$. See Figure 4.1. $\diamond$

Now let us turn to a geometric interpretation of the notion of derivative and in particular, a "solution" to the first problem stated at the beginning of this chapter. So let $D \subseteq \mathbb{R}$ be such that every point of D is an interior point of

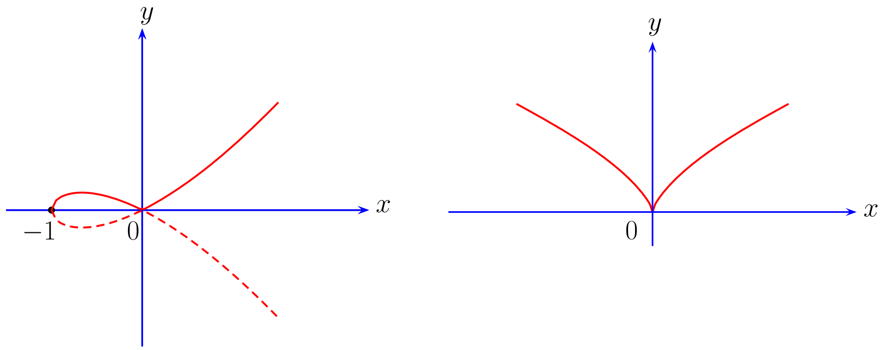

Fig. 4.1. Graphs of (iv) $y = \sqrt{x^2 + x^3}$ and (v) $y = \sqrt[3]{x^2}$.

D and let $f : D \to \mathbb{R}$ be a function. Given any $c \in D$ and $h \neq 0$ such that $c + h \in D$, the quotient

$$\frac{f(c+h) - f(c)}{h}$$

gives the slope of the chord joining the points $(c, f(c))$ and $(c + h, f(c + h))$ on the curve $y = f(x)$, $x \in D$. As $h \to 0$, these chords seem to approach a "tangent" to the curve $y = f(x)$ at the point $(c, f(c))$. It is, therefore, reasonable to *define* the tangent to the curve $y = f(x)$ at the point $(c, f(c))$ to be the line given by the equation

$$y - f(c) = m(x - c), \quad \text{where} \quad m = f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

provided the limit above exists, that is, provided f is differentiable at c. Notice that the form of the equation for the tangent is such that a vertical line (such as the one given by $x =$ constant) can never be a tangent to a curve of the form $y = f(x)$. Further, the (geometric) condition that there is a unique nonvertical tangent to the curve $y = f(x)$ at a point $(c, f(c))$ is equivalent to the (analytic) condition that f is differentiable at c. Thus, intuitively speaking, to say that f is differentiable at c means that the graph of f is "smooth" at $(c, f(c))$, that is to say, the graph has a unique nonvertical tangent at c. In this case, the graph of f has no breaks or sharp edges or cusps at c. This is similar to the intuitive meaning of the continuity of f at c, namely, that the graph of f is unbroken at c. For an illustration of these remarks, take a look at Examples 4.1 (iii), (iv), and (v) as well as Figure 1.6 on page 21 and Figure 4.1 above.

We shall now describe a number of basic properties of derivatives, and to prove these, the following characterization of differentiability will be useful.

Proposition 4.2 (Carathéodory Lemma). *Let $D \subseteq \mathbb{R}$ and let c be an interior point of D. Then a function $f : D \to \mathbb{R}$ is differentiable at c if and only if there exists a function $f_1 : D \to \mathbb{R}$ such that $f(x) - f(c) = (x - c)f_1(x)$*

for all $x \in D$*, and* f_1 *is continuous at* c*. Moreover, if these conditions hold, then* $f'(c) = f_1(c)$*.*

Proof. Given any $\ell \in \mathbb{R}$, consider $f_1 : D \to \mathbb{R}$ defined by

$$f_1(x) := \begin{cases} \dfrac{f(x) - f(c)}{x - c} & \text{if } x \in D,\ x \neq c, \\ \ell & \text{if } x = c. \end{cases}$$

Then clearly, $f(x) - f(c) = (x - c)f_1(x)$ for all $x \in D$. Also, by Corollary 3.28, f is differentiable at c, and $f'(c) = \ell$ if and only if f_1 is continuous at c. This implies the desired result. □

Let $D \subseteq \mathbb{R}$ and let c be an interior point of D. Given a function $f : D \to \mathbb{R}$, a function $f_1 : D \to \mathbb{R}$ satisfying

(i) $f(x) - f(c) = (x - c)f_1(x)$ for all $x \in D$, and (ii) f_1 is continuous at c

is called an **increment function** associated with f and c. It is clear that such an increment function, if it exists, is uniquely determined by f and c. The Carathéodory Lemma can be paraphrased by saying that differentiability of a function f at a point c is equivalent to the existence of an increment function associated with f and c, and in this case the derivative of f at c is the value of the increment function at c.

An immediate corollary of the Carathéodory Lemma is the following.

Proposition 4.3. *Let* $D \subseteq \mathbb{R}$ *and let* c *be an interior point of* D*. If a function* $f : D \to \mathbb{R}$ *is differentiable at* c*, then* f *is continuous at* c*.*

Proof. Let f_1 be the increment function associated with f and c. Continuity of f_1 at c implies the continuity of f at c, since $f(x) = f(c) + (x - c)f_1(x)$ for all $x \in D$. □

Note that the converse of the above proposition is not true. In other words, continuity need not imply differentiability. For example, the absolute value function is continuous at 0, but not differentiable at 0. An alternative way to state the above proposition is that if a function is not continuous at a point, then it is not differentiable at that point. This is used in the following.

Example 4.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x^2 & \text{if } x \text{ is rational}, \\ -x^2 & \text{if } x \text{ is irrational}. \end{cases}$$

As in Example 3.1 (v), we see that f is continuous only at 0. Hence f is not differentiable at each $c \in \mathbb{R}$ with $c \neq 0$. On the other hand, if we let f_1 be the function in Example 3.1 (v), then f_1 is continuous at 0, and clearly $f(x) - f(0) = xf_1(x)$ for all $x \in \mathbb{R}$. Hence f is differentiable only at 0. ◇

Remark 4.5. Since differentiability implies continuity, all the properties of continuous functions such as those discussed in Section 3.2 are inherited by differentiable functions. On the other hand, it may be worthwhile to examine whether the conditions that imply continuity also imply differentiability. Often, this is not the case. For example, we have shown in Corollary 3.14 that if I is an interval and $f : I \to \mathbb{R}$ is a strictly monotonic function such that $f(I)$ is an interval, then f is continuous. However, such a function need not be differentiable. To see this, consider $I := [0, 2]$ and $f : I \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1, \\ 3x - 2 & \text{if } 1 < x \le 2. \end{cases}$$

Then f is strictly increasing and $f(I) = [0, 4]$ is an interval, but f is not differentiable at 1, since

$$\lim_{x \to c^-} \frac{f(x) - f(1)}{x - 1} = 1 \ne 3 = \lim_{x \to c^+} \frac{f(x) - f(1)}{x - 1}.$$

The same example shows that the hypothesis of Proposition 3.45, namely that I is an interval and $f : I \to \mathbb{R}$ is monotonic and has the IVP on I, does not imply the differentiability of f on I. As another example, we have indicated in Exercise 3.44 that if I is an interval and $f : I \to \mathbb{R}$ is convex on I, or concave on I, then f is continuous at every interior point of I. However, convexity or concavity does not imply differentiability. To see this, consider the absolute value function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = |x|$. As seen earlier, f is convex on $[-1, 1]$, but not differentiable at an interior point of $[-1, 1]$, namely at 0. Similarly, $-f$ is concave on $[-1, 1]$, but not differentiable at 0. ◇

We shall now see when and how derivatives of sums, scalar multiples, products, reciprocals, and roots of functions can be determined.

Proposition 4.6. *Let $D \subseteq \mathbb{R}$, let c be an interior point of D. Also, let $f, g :\ D \to \mathbb{R}$ be functions that are differentiable at c. Then*

(i) *$f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$,*
(ii) *rf is differentiable at c and $(rf)'(c) = rf'(c)$ for every $r \in \mathbb{R}$,*
(iii) *fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$,*
(iv) *if $f(c) \ne 0$, then there is $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq D$ and $f(x) \ne 0$ for all $x \in (c - \delta, c + \delta)$; moreover, the function $1/f : (c - \delta, c + \delta) \to \mathbb{R}$ is differentiable at c, and*

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{f(c)^2},$$

(v) if $f(c) > 0$, then there is $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq D$ and $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$; moreover, for every $k \in \mathbb{N}$, the function $f^{1/k}$: $(c - \delta, c + \delta) \to \mathbb{R}$ is differentiable at c and

$$\left(f^{1/k}\right)'(c) = \frac{1}{k} f(c)^{(1/k)-1} f'(c).$$

Proof. Let f_1 and g_1 denote, respectively, the increment functions associated with f and g and the point c. Using part (i) of Proposition 3.2, we easily see that $f_1 + g_1$ is the increment function associated with $f + g$ and c. Likewise, using part (ii) of Proposition 3.2, we see that rf_1 is the increment function associated with rf and c for every $r \in \mathbb{R}$. This proves (i) and (ii).

Next, for all $x \in D$, the difference $f(x)g(x) - f(c)g(c)$ can be written as

$$\big(f(x) - f(c)\big)g(x) + f(c)\big(g(x) - g(c)\big) = (x - c)\big([f_1(x)g(x) + f(c)g_1(x)\big).$$

Moreover, by Proposition 4.3, g is continuous at c, and thus by parts (i), (ii), and (iii) of Proposition 3.2, the function $f_1 g + f(c)g_1$ is continuous at c. This implies (iii).

Since c is an interior point of D, by Proposition 4.3 and part (iv) of Proposition 3.2, we see that if $f(c) \neq 0$, then there is $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq D$ and $f(x) \neq 0$ for all $x \in (c - \delta, c + \delta)$, and moreover, the function $1/f : (c-\delta, c+\delta) \to \mathbb{R}$ is continuous at c. Thus, if $f(c) \neq 0$, then

$$\frac{1}{f(x)} - \frac{1}{f(c)} = \frac{-\big(f(x) - f(c)\big)}{f(x)f(c)} = (x - c)\left(\frac{-f_1(x)}{f(x)f(c)}\right) \quad \text{for } x \in (c-\delta, c+\delta).$$

This yields (iv), since the function $-f_1/f(c)f$ is continuous at c.

Finally, suppose $k \in \mathbb{N}$ and $f(c) > 0$. Since c is an interior point of D, it follows from Corollary 3.9 that there is $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq D$ and $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$. For simplicity, let us write $F(x) := f(x)^{1/k}$ for $x \in (c - \delta, c + \delta)$. Then part (v) of Proposition 3.2 shows that $F : (c - \delta, c + \delta) \to \mathbb{R}$ is continuous at c, and for all $x \in (c - \delta, c + \delta)$,

$$f(x) - f(c) = \big(F(x) - F(c)\big)\big(F(x)^{k-1} + F(c)F(x)^{k-2} + \cdots + F(c)^{k-1}\big).$$

Now since $F(x) > 0$ for all $x \in (c - \delta, c + \delta)$, we obtain

$$F(x) - F(c) = (x - c)\left(\frac{f_1(x)}{F(x)^{k-1} + F(c)F(x)^{k-2} + \cdots + F(c)^{k-1}}\right).$$

This implies (v), since the function $f_1/\left(F^{k-1} + F(c)F^{k-2} + \cdots + F(c)^{k-1}\right)$ is continuous at c. $\square$

Remark 4.7. With notation and hypotheses as in the above proposition, a combined application of its parts (i) and (ii) shows that the difference $f - g$ is differentiable at c and $(f - g)'(c) = f'(c) - g'(c)$. Likewise, a combined application of parts (iii) and (iv) shows that if $g(c) \neq 0$, then f/g is differentiable at c and its derivative is given by the following **quotient rule**:

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

Further, given any $n \in \mathbb{N}$, repeated application of part (iii) of the above proposition (or, if you prefer, induction on n) shows that the nth power f^n is differentiable at c and $(f^n)'(c) = nf(c)^{n-1}f'(c)$. Moreover, if $f(c) \neq 0$, then using the previous formula and part (iv), we see that

$$\left(\frac{1}{f^n}\right)'(c) = -\frac{nf(c)^{n-1}f'(c)}{f(c)^{2n}} = -nf(c)^{-n-1}f'(c).$$

Since the derivative of a constant function is zero, it follows for every $m \in \mathbb{Z}$, f^m is differentiable at c and

$$(f^m)'(c) = mf(c)^{m-1}f'(c), \quad \text{provided } f(c) \neq 0 \text{ in case } m < 0.$$

Furthermore, given any $r \in \mathbb{Q}$, we can write $r = m/k$, where $m \in \mathbb{Z}$ and $k \in \mathbb{N}$, and then the last formula together with part (v) of the above proposition shows that if $f(c) > 0$, then the rth power $f^r = (f^m)^{1/k}$ is differentiable at c and

$$\left((f^m)^{\frac{1}{k}}\right)'(c) = \frac{1}{k} f^{m\left(\frac{1}{k}-1\right)}(c)\,(f^m)'(c) = \frac{m}{k} f(c)^{m\left(\frac{1}{k}-1\right)+m-1} f'(c).$$

In other words, for every $r \in \mathbb{Q}$,

$$(f^r)'(c) = rf(c)^{r-1}f'(c),$$

provided $f(c) > 0$ if $r \notin \mathbb{Z}$ and $f(c) \neq 0$ if $r \in \mathbb{Z}$ with $r < 0$. ◇

Example 4.8. As a particular case of the results in Remark 4.7 and Examples 4.1 (i), (ii), we see that the nth power function is differentiable on $\mathbb{R}$ for each nonnegative integer n, and

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Moreover, the above result is valid for negative integral powers, provided $x \neq 0$, and for rational nonintegral powers, provided $x > 0$. In particular,

$$\frac{d}{dx}(x^r) = rx^{r-1} \quad \text{for every } r \in \mathbb{Q} \text{ and } x \in (0, \infty).$$

◇

Example 4.9. Using Proposition 4.6 and Example 4.8, it follows that every polynomial function is differentiable on $\mathbb{R}$ and every rational function is differentiable at each point of $\mathbb{R}$ where it is defined. Moreover, the derivatives of such functions can also be readily computed. For instance, if $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ is given by $f(x) = (x^4 + 3x + 2)/(x - 1)$, then

$$f'(x) = \frac{(4x^3 + 3)(x - 1) - (x^4 + 3x + 2)}{(x - 1)^2} = \frac{3x^4 - 4x^3 - 5}{(x - 1)^2}$$

for $x \neq 1$. ◇

To compute the derivative of a composite function $u = g(y)$, where y, in turn, is a function of x, say $y = f(x)$, the Chain Rule described below is quite useful. Roughly speaking, the Chain Rule can be stated as follows:

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}.$$

It may be tempting to prove this by just canceling out dy. But that wouldn't be correct, because we haven't defined the quantities dy and dx by themselves, even though we have defined $\frac{dy}{dx}$. We give below a precise statement of the Chain Rule and a proof using the Carathéodory Lemma.

Proposition 4.10 (Chain Rule). *Let $D, E \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$, $g : E \to \mathbb{R}$ be functions such that $f(D) \subseteq E$. Suppose c is an interior point of D such that $f(c)$ is an interior point of E. If f is differentiable at c and g differentiable at $f(c)$, then $g \circ f$ is differentiable at c and*

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Proof. Let $f_1 : D \to \mathbb{R}$ be the increment functions associated with f and c. Then

$$f(x) - f(c) = (x - c)f_1(x) \quad \text{for all } x \in D.$$

Also, let $g_1 : E \to \mathbb{R}$ be the increment functions associated with g and $f(c)$. Then

$$g(y) - g(f(c)) = (y - f(c))g_1(y) \quad \text{for all } y \in E.$$

Since $f(D) \subseteq E$, we can use the above two equations to obtain

$$g(f(x)) - g(f(c)) = \big(f(x) - f(c)\big)g_1(f(x)) = (x - c)g_1(f(x))f_1(x) \quad \text{for } x \in D.$$

Hence by Propositions 4.3 and 3.5, we see that the function $(g_1 \circ f) \cdot f_1 : D \to \mathbb{R}$ is continuous at c. Hence by the Carathéodory Lemma, $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g_1(f(c))f_1(c) = g'(f(c))f'(c)$. $\square$

Example 4.11. Consider $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = (4x^3 + 3)^7 + 2$. We can of course compute $F'(x)$ by expanding the seventh power and using Proposition 4.6 together with the formula for the derivative of the nth power function. But it is simpler to view F as the composite $g \circ f$, where $u = g(y) = y^7 + 2$ and $y = f(x) = 4x^3 + 3$, and apply the Chain Rule. This gives

$$F'(x) = g'(f(x))f'(x) = \left(7(4x^3 + 3)^6\right)\left(12x^2\right) = 84x^2\left(4x^3 + 3\right)^6$$

for $x \in \mathbb{R}$. $\diamond$

We shall now prove a general result about the derivatives of inverse functions. Roughly speaking, this result can be stated as follows:

$$\frac{dx}{dy} \text{ is the reciprocal of } \frac{dy}{dx} \quad \text{when } \frac{dy}{dx} \text{ is nonzero.}$$

A precise statement and a proof using the Carathéodory Lemma appears below.

Proposition 4.12 (Differentiable Inverse Theorem). *Let I be an interval and let c be an interior point of I. Suppose $f : I \to \mathbb{R}$ is a one-one and continuous function. Let $f^{-1} : f(I) \to I$ be the inverse function. Then $f(c)$ is an interior point of $f(I)$. Moreover, if f is differentiable at c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$ and*

$$\left(f^{-1}\right)'(f(c)) = \frac{1}{f'(c)}.$$

Proof. Let $J := f(I)$. By Proposition 3.17, f is strictly monotonic and J is an interval. Hence $f(c)$ is an interior point of J. Suppose f is differentiable at c and $f'(c) \neq 0$. Let $f_1 : I \to \mathbb{R}$ be the increment function associated with f and c. Then f_1 is continuous at c with $f_1(c) = f'(c)$ and

$$f(x) - f(c) = (x - c)f_1(x) \quad \text{for all } x \in I.$$

Since f is one-one, $f(x) \neq f(c)$ for any $x \in I$ with $x \neq c$, and therefore, $f_1(x) \neq 0$. Also, $f_1(c) = f'(c) \neq 0$. Thus f_1 is never zero on I, and so $1/f_1$ is defined on I. Further, since f_1 is continuous at c, so is $1/f_1$. Hence the equation displayed above can be written as $(x - c) = \big(f(x) - f(c)\big)/f_1(x)$ for all $x \in I$. This implies that

$$f^{-1}(y) - f^{-1}(f(c)) = (y - f(c))\frac{1}{f_1\left(f^{-1}(y)\right)} \quad \text{for all } y \in J.$$

Moreover, by Propositions 3.5 and 3.17, we see that $f_1 \circ f^{-1}$ is continuous at $f(c)$. Hence by part (iv) of Proposition 3.2, the reciprocal of $f_1 \circ f^{-1}$ is also continuous at $f(c)$. Thus, by the Carathéodory Lemma, it follows that f^{-1} is differentiable at $f(c)$ and $\left(f^{-1}\right)'(f(c)) = 1\big/f_1\left(f^{-1}(f(c))\right) = 1/f'(c)$. $\square$

Differentiation applied successively leads to the notion of higher derivatives. More formally, suppose $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ is a function that is differentiable at every point of an interval $(c-\delta, c+\delta) \subseteq D$. Then the derivative function f' is defined on $(c - \delta, c + \delta)$. In case f' is differentiable at c, then we say that f is **twice differentiable** at c and denote the derivative of f' at c by $f''(c)$. The quantity $f''(c)$ is called the **second derivative** (or the **second-order derivative**) of f at c. Further, if f' is differentiable at every point of an interval about c, then the second derivative function f'' is defined on this interval. In case f'' is also differentiable at c, then we say that f is **thrice differentiable** at c and denote the derivative of f'' at c by $f'''(c)$. Similarly, one defines n-times differentiability of f and the nth derivative $f^{(n)}(c)$ for every $n \in \mathbb{N}$. The notations

$$\left.\frac{d^2 f}{dx^2}\right|_{x=c}, \quad \left.\frac{d^3 f}{dx^3}\right|_{x=c}, \quad \text{and} \quad \left.\frac{d^n f}{dx^n}\right|_{x=c}$$

are sometimes used instead of $f''(c)$, $f'''(c)$, and $f^{(n)}(c)$, respectively. In case f is n-times differentiable at c for every $n \in \mathbb{N}$, then f is said to be **infinitely differentiable** at c.

To compute higher derivatives, the following formula, which is known as the **Leibniz Rule for Derivatives**, can be quite useful:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} = f^{(n)}g + nf^{(n-1)}g' + \cdots + nf'g^{(n-1)} + fg^{(n)}.$$

Here f, g are real-valued functions, both of which are assumed to be n-times differentiable at c, and $\binom{n}{k}$ denotes the binomial coefficient as defined in Exercise 1.2. Also, $f^{(0)}$ and $g^{(0)}$ denote, by convention, f and g respectively. The Leibniz Rule for derivatives can be easily proved by induction on n, using Proposition 4.6 and the Pascal triangle identity (Exercise 1.2 (ii)):

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

Examples 4.13. (i) If r is any rational number and $f : (0, \infty) \to (0, \infty)$ is given by $f(x) = x^r$, then f is infinitely differentiable at every $c \in (0, \infty)$, and $f^{(n)}(c) = r(r-1)\cdots(r-n+1)c^{r-n}$.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x|x|$. It is easily seen that f is differentiable on $\mathbb{R}$ and $f'(x) = 2|x|$ for $x \in \mathbb{R}$. Note that f' is not differentiable at 0. For a more general example of this kind, see Exercise 4.13.

Let us now consider the case of a real-valued function defined on a closed interval. While we can talk about the differentiability of such a function at any interior point, the definition we have given of differentiability does not apply to the endpoints. To take care of this omission, we introduce the notions of left derivative and right derivative as follows.

Let $D \subseteq \mathbb{R}$ and $c \in D$ be such that $(c - r, c] \subseteq D$ for some $r > 0$. If the left limit

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c},$$

exists, then it is called the **left (hand) derivative** of f at c and is denoted by $f'_-(c)$. In the case $[c, c+r) \subseteq D$ for some $r > 0$, the **right (hand) derivative** of f at c is defined similarly and is denoted by $f'_+(c)$.

If c is an interior point of D, then it follows from Proposition 3.37 that f is differentiable at c if and only if both $f'_-(c)$ and $f'_+(c)$ exist and are equal.

For example, if $f : \mathbb{R} \to \mathbb{R}$ is the absolute value function, then $f'_-(0) = -1$, whereas $f'_+(0) = 1$. On the other hand, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^{2/3}$, then neither $f'_-(0)$ nor $f'_+(0)$ exists. In each of these examples, we find that the function f is not differentiable at 0.

We say that a real-valued function f defined on a closed interval $[a, b]$ is **differentiable** if f is differentiable at every point of (a, b), and moreover if $f'_+(a)$ and $f'_-(b)$ exist. In this case, the function $f' : [a, b] \to \mathbb{R}$ defined by

$$f'(x) = \begin{cases} f'_+(a) & \text{if } x = a, \\ f'(c) & \text{if } x = c \in (a, b), \\ f'_-(b) & \text{if } x = b, \end{cases}$$

is called the **derivative** of f on $[a, b]$. If f' is differentiable on $[a, b]$, then f is said to be **twice differentiable** on $[a, b]$, and we denote the derivative of f' on $[a, b]$ by f''. More generally, the nth derivative $f^{(n)}$ of f on $[a, b]$ is defined for every $n \in \mathbb{N}$ in a similar way.

Remark 4.14. It should be noted that the Carathéodory Lemma continues to be valid for derivatives at the endpoints of a function $f : [a, b] \to \mathbb{R}$. The proof is identical to that of Proposition 4.2, provided we take limits as $h \to 0^+$ in case $c = a$ and as $h \to 0^-$ in case $c = b$. As a consequence, results similar to Propositions 4.3 and 4.6 as well as those in Remark 4.7 are valid for functions $f : D \to \mathbb{R}$ when $D = [a, b]$ and $c = a$ or $c = b$. Moreover, the Chain Rule (Proposition 4.10) is also valid if D is an interval and c is an endpoint of D such that $f(c)$ is an endpoint or an interior point of an interval contained in E. Likewise, the Differentiable Inverse Theorem (Proposition 4.12) is valid if c is an endpoint of I, provided in the conclusion we write "$f(c)$ is an endpoint of J" instead of "$f(c)$ is an interior point of J". The proof is essentially the same as before. $\diamond$

Tangents and Normals to Curves

We discussed earlier the notion of tangent to plane curves of the form $y = f(x)$. We shall now see how it can be extended to plane curves of more general type. Also, we will discuss the related notion of normal in the context of more general curves.

Plane curves of the form $y = f(x)$ admit generalizations to two distinct, yet overlapping, classes of plane curves. These are as follows.

- **Parametrically Defined Curves**: These are the plane curves C given by $(x(t), y(t))$, where x, y are real-valued functions[2] defined on some subset D of $\mathbb{R}$, and the parameter t varies over the points of D. Usually, we express this by simply saying that C is the (parametrically defined) curve $(x(t), y(t))$, $t \in D$. For example, the rectangular hyperbola is the curve $(t, 1/t)$, $t \in \mathbb{R} \setminus \{0\}$.
- **Implicitly Defined Curves**: These are the plane curves C given by an equation of the form $F(x, y) = 0$, where F is a real-valued function defined on some subset E of the plane $\mathbb{R}^2$, and (x, y) varies over the points of E. Usually, we express this by simply saying that C is the (implicitly defined) curve $F(x, y) = 0$, $(x, y) \in E$. The reference to the domain E of F is skipped if $E = \mathbb{R}^2$. For example, the circle centered at the origin with unit radius is the curve $x^2 + y^2 - 1 = 0$.

Notice that if $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ is any function, then the curve $y = f(x)$ can be viewed as a parametrically defined curve $(x(t), y(t))$, $t \in D$, where

[2] Generally, one requires that the set D be an interval and the two functions $x, y : D \to \mathbb{R}$ be continuous. In most applications, this will be so, but we do not make it a part of the definition.

$x(t) := t$ and $y(t) := f(t)$. Also, it can be viewed as an implicitly defined curve $F(x, y) = 0$, $(x, y) \in E$, where $E = D \times \mathbb{R}$ and $F : E \to \mathbb{R}$ is given by $F(x, y) := y - f(x)$.

If C is a parametrically defined curve $(x(t), y(t))$, $t \in D$, and t_0 is an interior point of D such that both x and y are differentiable at t_0 and $x'(t_0), y'(t_0)$ are not both zero, then we define the **tangent** to C at the point $(x(t_0), y(t_0))$ to be the line

$$\big(y - y(t_0)\big)x'(t_0) - \big(x - x(t_0)\big)y'(t_0) = 0.$$

The line passing through $(x(t_0), y(t_0))$ and perpendicular to the tangent at this point, namely, the line given by

$$\big(x - x(t_0)\big)x'(t_0) + \big(y - y(t_0)\big)y'(t_0) = 0,$$

is called the **normal** to the curve C at the point $(x(t_0), y(t_0))$. In case $x'(t_0)$ or $y'(t_0)$ does not exist or $(x'(t_0), y'(t_0)) = (0, 0)$, we say that the tangent to C (as well as the normal to C) at $(x(t_0), y(t_0))$ is *not defined*. It may be noted that the definition of a tangent to a parametrically defined curve is consistent with the previous definition of a tangent to a curve of the form $y = f(x)$. More generally, if $x'(t_0) \neq 0$ and y can be considered a function of x in an open interval about $x_0 := x(t_0)$, then by the Chain Rule,

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \left(\left.\frac{dy}{dt}\right|_{t=t_0}\right)\left(\left.\frac{dx}{dt}\right|_{t=t_0}\right)^{-1} = \frac{y'(t_0)}{x'(t_0)}.$$

The Chain Rule also helps us to formulate the notion of tangents to implicitly defined curves. For example, if $F(x, y) = x^2 + y^2 - 25$, then $F(x, y) = 0$ defines the circle of radius 5 centered at the origin. To find the tangent at a point, say $(3, 4)$, we differentiate $F(x, y)$ with respect to x, treating y as a function of x. Using the Chain Rule, we thus obtain

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{and hence} \quad \left.\frac{dy}{dx}\right|_{(3,4)} = -\left.\frac{x}{y}\right|_{(3,4)} = -\frac{3}{4}.$$

This suggests that the tangent to this circle at the point $(3, 4)$ is given by the line $y - 4 = -\frac{3}{4}(x - 3)$, that is, $3x + 4y - 25 = 0$. In general, given any equation $F(x, y) = 0$, we can try to differentiate with respect to x, treating y as a function of x. This process is known as **implicit differentiation**, and it leads to an equation of the type

$$P(x, y) + Q(x, y)\frac{dy}{dx} = 0.$$

At a point $(x_0, y_0) \in \mathbb{R}^2$ on the curve $F(x, y) = 0$ (that is, $(x_0, y_0) \in \mathbb{R}^2$ satisfying $F(x_0, y_0) = 0$) with the additional property that $P(x_0, y_0)$ and $Q(x_0, y_0)$ are defined and $Q(x_0, y_0) \neq 0$, we *define* the tangent to $F(x, y) = 0$ at (x_0, y_0) to be the line given by

$$y - y_0 = m(x - x_0), \quad \text{where} \quad m = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} = -\frac{P(x_0,y_0)}{Q(x_0,y_0)}.$$

It may be checked that this definition is consistent with our previous definition in the case $F(x,y) = y - f(x)$ for some function f that is differentiable at x_0; note that in this case,

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = \left.\frac{dy}{dx}\right|_{x=x_0}.$$

Sometimes, when dealing with curves defined by $F(x,y) = 0$, it is useful to reverse the roles of x and y. Thus, we may also differentiate with respect to y, treating x as a function of y. This leads to an equation of the type

$$R(x,y) + S(x,y)\frac{dx}{dy} = 0.$$

Now suppose $(x_0,y_0) \in \mathbb{R}^2$ is a point on the curve $F(x,y) = 0$ such that $\frac{dy}{dx}$ is not defined at (x_0,y_0). (Roughly speaking, this corresponds to the case $Q(x_0,y_0) = 0$.) If, however, $R(x_0,y_0)$ and $S(x_0,y_0)$ are defined and if $S(x_0,y_0) \neq 0$, then we *define* the tangent to $F(x,y) = 0$ at (x_0,y_0) to be the line given by

$$x - x_0 = \widetilde{m}(y - y_0), \quad \text{where} \quad \widetilde{m} = \left.\frac{dx}{dy}\right|_{(x_0,y_0)} = -\frac{R(x_0,y_0)}{S(x_0,y_0)}.$$

It may be noted that if both $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are defined and are nonzero at a point (x_0,y_0) on the curve $F(x,y) = 0$, then it follows from the Differentiable Inverse Theorem (Proposition 4.12) that $\widetilde{m} = 1/m$, and hence the lines obtained by either of the two approaches are identical. If, however, both $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are not defined, or if both of them are zero at (x_0,y_0), then we say that the tangent to the curve $F(x,y) = 0$ at the point (x_0,y_0) is *not defined*.[3]

As before, at a point (x_0,y_0) on a plane curve $F(x,y) = 0$ where the tangent is defined, we define the **normal** to be the unique line passing through (x_0,y_0) and perpendicular to the tangent to this curve at (x_0,y_0).

For example, for the circle $x^2 + y^2 - 25 = 0$, the tangent at the point $(5,0)$ can be determined by the latter method of differentiating with respect to y, treating x as a function of y. Indeed, we see that the tangent is given by the vertical line $x - 5 = 0$. On the other hand, for the curve $y^2 - x^2 - x^3 = 0$, neither $\frac{dy}{dx}$ nor $\frac{dx}{dy}$ is defined at the origin, and hence the tangent at the origin is not defined.

[3] For algebraic plane curves $F(x,y) = 0$, where $F(x,y)$ is a polynomial in two variables, there is an alternative, purely algebraic, method to determine tangents at a point. In fact, this algebraic approach could be used to *define* tangents at every point on the curve including those points at which the tangent is not defined as far as calculus is concerned. We stick to the method of calculus in this book but briefly outline the algebraic approach in Exercise 4.45, and refer to the book of Abhyankar [1] for more details.

4.2 Mean Value Theorem and Taylor Theorem

The Mean Value Theorem, or for short, the MVT, is a result that in geometric terms may be described as follows. For every nice curve of the form $y = f(x)$, $x \in [a, b]$, there exists a point $(c, f(c))$ on the curve, where $c \in (a, b)$, at which the tangent is parallel to the line joining the endpoints $(a, f(a))$ and $(b, f(b))$ of the curve. (See Figure 4.2.) Here, by "nice" we mean that the curve is unbroken, that is, f is continuous, and that tangents can be drawn everywhere except perhaps at the endpoints, that is, f is differentiable on (a, b).

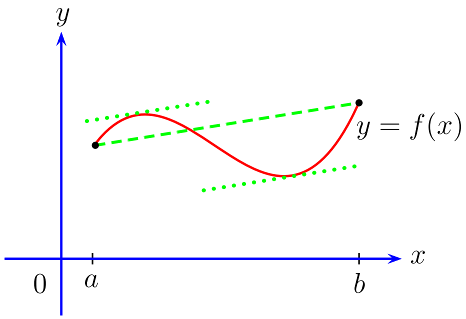

Fig. 4.2. Illustration of the MVT: Tangents at some intermediate points are parallel to the line joining the endpoints $(a, f(a))$ and $(b, f(b))$.

As we shall see in the sequel, the MVT is a very useful result in calculus. In particular, an extension of the MVT, known as the Taylor Theorem, will allow us to approximate a large class of functions by polynomial functions.

A special case of the MVT is that in which the endpoints of the curve $y = f(x)$, $x \in [a, b]$, lie on a horizontal line, that is, $f(a) = f(b)$. In this case, the MVT amounts to asserting the existence of a point $(c, f(c))$ on the curve, where $c \in (a, b)$, at which the tangent is parallel to the x-axis, that is, $f'(c) = 0$. We will, in fact, prove this special case first, and deduce the MVT as a consequence. This special case, known as the Rolle Theorem, will, in turn, be deduced from the following simple but useful fact about a point of local extremum, that is, a point of local maximum or local minimum, of a function.

Lemma 4.15. *Let $D \subseteq \mathbb{R}$ and let c be an interior point of D. If $f : D \to \mathbb{R}$ is differentiable at c and has a local extremum at c, then $f'(c) = 0$.*

Proof. Suppose f is differentiable at c. By the Carathéodory Lemma, there is $f_1 : D \to \mathbb{R}$ such that f_1 is continuous at c and

$$f(x) - f(c) = (x - c)f_1(x) \quad \text{for all } x \in D.$$

Now, if f has a local maximum at c, then there is $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq D$ and $f(x) - f(c) \leq 0$ for all $x \in (c-\delta, c+\delta)$. So, in this case,

$$f_1(x) \geq 0 \quad \text{for all } x \in (c-\delta, c) \quad \text{and} \quad f_1(x) \leq 0 \quad \text{for all } x \in (c, c+\delta).$$

Similarly, if f has a local minimum at c, then there is $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq D$ and $f(x) - f(c) \geq 0$ for all $x \in (c-\delta, c+\delta)$. So, in this case,

$$f_1(x) \leq 0 \quad \text{for all } x \in (c-\delta, c) \quad \text{and} \quad f_1(x) \geq 0 \quad \text{for all } x \in (c, c+\delta).$$

In any case, by the continuity of f_1 at c, it follows that $f_1(c) = 0$, that is, $f'(c) = 0$. □

We pause to give an interesting application of the above lemma before moving on to the Rolle Theorem and the MVT. This is a result that is sometimes ascribed to Darboux and called the **IVP for derivatives**. To put this result in perspective, let us first note that if I is an interval and a function $f : I \to \mathbb{R}$ is **continuously differentiable**, that is, if f' exists and is continuous, then by Proposition 3.16, f' has the IVP on I. But in general, f' need not be continuous.[4] Yet, as the following result shows, f' has the IVP.

Proposition 4.16. *Let I be an interval and let $f : I \to \mathbb{R}$ be a differentiable function. Then the derivative function f' has the IVP on I.*

Proof. Let $a, b \in I$ with $a < b$, and let $r \in \mathbb{R}$ lie between $f'(a)$ and $f'(b)$. First, suppose $f'(a) < r < f'(b)$. Consider $g : [a, b] \to \mathbb{R}$ defined by

$$g(x) = f(x) - rx \quad \text{for } x \in [a, b].$$

Then g is differentiable and hence continuous. Consequently, by Proposition 3.10, g attains its minimum at some $c \in [a, b]$. We shall show that $c \in (a, b)$. To this end, first note that

$$\lim_{x \to a^+} \frac{g(x) - g(a)}{x - a} = g'(a) = f'(a) - r < 0.$$

Therefore, by an analogue of part (i) of Proposition 3.30 for right limits, there is $\delta > 0$ such that for all $x \in (a, a+\delta)$,

$$\frac{g(x) - g(a)}{x - a} < 0 \quad \text{and hence} \quad g(x) < g(a).$$

It follows that g cannot attain its minimum at a. In a similar way, since $g'(b) = f'(b) - r > 0$, we see that there is $\delta > 0$ such that $g(x) < g(b)$ for all $x \in (b-\delta, b)$. So g cannot attain its minimum at b as well. This shows that $c \in (a, b)$. Hence by Lemma 4.15, we conclude that $g'(c) = 0$, that is, $f'(c) = r$. In case $f'(b) > r > f'(a)$, we obtain the same conclusion by considering $g(x) := rx - f(x)$. Thus f' has the IVP on I. □

[4] Simple examples to show that the derivative of a differentiable function may not be continuous can be constructed using trigonometric functions. See, for instance, Example 7.21. In fact, the derivative of a differentiable function on a closed and bounded interval need not be bounded. See Exercise 7.49.

The above result is sometimes useful for showing that a given function cannot be the derivative of any other function. For example, if $f : [-1, 1] \to \mathbb{R}$ is the integer part function given by $f(x) := [x]$, then it is clear that f attains the values -1, 0, and 1 but none in between. Hence, $f \neq F'$ for any differentiable function $F : [-1, 1] \to \mathbb{R}$.

Proposition 4.17 (Rolle Theorem). *Let $a, b \in \mathbb{R}$ with $a < b$. Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and suppose $f(a) = f(b)$. Then there is $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Since f is a continuous function on the closed and bounded set $[a, b]$, it follows from Proposition 3.10 that f is bounded and attains its bounds on $[a, b]$. Thus, there are $c_1, c_2 \in [a, b]$ such that

$$f(c_1) = \max\{f(x) : x \in [a, b]\} \quad \text{and} \quad f(c_2) = \min\{f(x) : x \in [a, b]\}.$$

Now if c_1 or c_2 is an interior point of $[a, b]$, then by Lemma 4.15, $f'(c_1) = 0$ or $f'(c_2) = 0$, and the result is proved. Otherwise, both c_1 and c_2 are endpoints of $[a, b]$, and since $f(a) = f(b)$, we see that $f(c_1) = f(c_2)$. Thus, the maximum and the minimum values of f on $[a, b]$ coincide. Hence f is constant on $[a, b]$, and therefore, $f'(c) = 0$ for every $c \in (a, b)$. □

The Rolle Theorem can be used together with the IVP of continuous functions to check the uniqueness and the existence of roots in certain intervals, especially for polynomials with real coefficients. This is illustrated by the following examples.

Examples 4.18. (i) If $f(x) := x^3 + px + q$ for $x \in \mathbb{R}$, where $p, q \in \mathbb{R}$ and $p > 0$, then f has a unique real root. To see this, note that if f had more than one real root, then there would exist $a, b \in \mathbb{R}$ with $a < b$ and $f(a) = f(b) = 0$. Hence by the Rolle Theorem, there would exist $c \in (a, b)$ such that $f'(c) = 0$. But $f'(x) = 3x^2 + p$ is not zero for every $x \in \mathbb{R}$, since $p > 0$. On the other hand, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$, and thus f takes negative as well as positive values. Hence, $f(c) = 0$ for some $c \in \mathbb{R}$, since f has the IVP on $\mathbb{R}$. Thus f has a unique real root.

(ii) If $f(x) := x^4 + 2x^3 - 2$ for $x \in \mathbb{R}$, then f has a unique root in $[0, \infty)$. Indeed, $f'(x) = 4x^3 + 6x^2 > 0$ for all $x \in (0, \infty)$. So by the Rolle Theorem, f has at most one root in $[0, \infty)$, On the other hand, since $f(0) = -2 < 0$ and $f(1) = 1 > 0$, the IVP implies that f has at least one root in $[0, 1]$. ◇

Note that in the above examples, the functions were clearly continuous and differentiable. The following examples show that the conclusion of the Rolle Theorem may not be true if any one of the three conditions on f is dropped.

Examples 4.19. (i) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) := x$ for $x \in [0, 1)$ and $f(1) := 0$. Then f is differentiable on $(0, 1)$ and $f(0) = f(1) = 0$, but $f'(c) = 1 \neq 0$ for every $c \in (0, 1)$. The Rolle Theorem does not apply here, since f is not continuous on $[0, 1]$. [In fact, continuity fails only at $x = 1$.]

(ii) Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) := |x|$ for $x \in [-1, 1]$. Then f is continuous on $[-1, 1]$ and $f(-1) = f(1) = 1$. But $f'(c) = 1$ or -1 for $c \neq 0$. the Rolle Theorem does not apply here, since f is not differentiable on $(-1, 1)$. (In fact, differentiability fails only at $x = 0$.)

(iii) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) := x$ for $x \in [0, 1]$. Then f is continuous on $[0, 1]$ and differentiable on $(0, 1)$ but $f'(c) = 1 \neq 0$ for every $c \in (0, 1)$. The Rolle Theorem does not apply here, since $f(0) \neq f(1)$. ◇

We are now ready to state and prove the Mean Value Theorem. It may be remarked that there are, in fact, several versions of the Mean Value Theorem. The one we state below is the most commonly used. It is usually ascribed to Lagrange and sometimes referred to as the Lagrange Mean Value Theorem.

Proposition 4.20 (Mean Value Theorem). *Let $a, b \in \mathbb{R}$ with $a < b$. Also, let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there is $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Consider $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = f(x) - f(a) - s(x - a), \quad \text{where} \quad s := \frac{f(b) - f(a)}{b - a}.$$

Then $F(a) = 0$ and our choice of the constant s is such that $F(b) = 0$. So the Rolle Theorem applies to F, and as a result, there is $c \in (a, b)$ such that $F'(c) = 0$. This implies that $f'(c) = s$, as desired. □

Remark 4.21. If we write $b = a + h$, then the conclusion of the MVT may be stated as follows:

$$f(a + h) = f(a) + hf'(a + \theta h) \quad \text{for some } \theta \in (0, 1).$$

The equivalence with the MVT is easily verified. ◇

Corollary 4.22 (Mean Value Inequality). *If a function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), and if $m_1, M_1 \in \mathbb{R}$ are such that $m_1 \le f'(x) \le M_1$ for all $x \in (a, b)$, then*

$$m_1(b - a) \le f(b) - f(a) \le M_1(b - a).$$

Proof. The desired inequality is an immediate consequence of the MVT. □

The corollary below is perhaps the most important consequence of the MVT in calculus.

Corollary 4.23. *Let I be an interval containing more than one point, and let $f : I \to \mathbb{R}$ be any function. Then f is a constant function on I if and only if f' exists and is identically zero on I.*

Proof. If f is a constant function on I, then it is obvious that f' exists on I and $f'(x) = 0$ for all $x \in I$. Conversely, suppose f' exists and vanishes identically on I. Let $x_1, x_2 \in I$ with $x_1 < x_2$. Then $[x_1, x_2] \subseteq I$, and by applying the MVT to the restriction of f to $[x_1, x_2]$, we obtain

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \quad \text{for some } c \in (x_1, x_2).$$

Since $f'(c) = 0$, we obtain $f(x_1) = f(x_2)$. Since x_1, x_2 are arbitrary elements of I with $x_1 < x_2$, it follows that f is a constant function on I. □

Remark 4.24. If $D \subseteq \mathbb{R}$ is not an interval, then there can be nonconstant differentiable functions on D whose derivative is identically zero. For example, if $D = (0,1) \cup (1,2)$ is a disjoint union of two open intervals and $f : D \to \mathbb{R}$ is defined by $f(x) = 1$ if $x \in (0,1)$ and $f(x) = 2$ if $x \in (1,2)$, then f is differentiable and f' is identically zero on D but f is not a constant function. Thus, the hypothesis that I is an interval is essential in Corollary 4.23. ◇

The MVT or the mean value inequality may also be used to approximate a differentiable function around a point. For example, if $m \in \mathbb{N}$ and $f(x) = \sqrt{x}$ for $x \in [m, m+1]$, then

$$\sqrt{m+1} - \sqrt{m} = f(m+1) - f(m) = f'(c) = \frac{1}{2\sqrt{c}}$$

for some $c \in \mathbb{R}$ such that $m < c < m+1$. Hence

$$\frac{1}{2\sqrt{m+1}} < \sqrt{m+1} - \sqrt{m} < \frac{1}{2\sqrt{m}}.$$

For example, by putting $m = 1$, we obtain

$$1 + \frac{1}{2\sqrt{2}} < \sqrt{2} < 1 + \frac{1}{2} \quad \text{and hence} \quad \frac{4}{3} < \sqrt{2} < \frac{3}{2}.$$

Similarly, putting $m = 2$, 3, and 4, we can obtain estimates for $\sqrt{3}$ and $\sqrt{5}$. (See Exercise 4.29 (i).)

Suppose we want to approximate a function f of a real variable x by a polynomial P around a point a at which f is defined. Naturally, we require $f(a) = P(a)$. Then the simplest approximation is the constant polynomial given by $P(x) = f(a)$, and the MVT can be used to estimate the error $f(x) - P(x) = f(x) - f(a)$. Next, if we require $f(a) = P(a)$ and further, $f'(a) = P'(a)$, then we may consider a linear polynomial instead of the constant polynomial $f(a)$. To be able to evaluate easily a linear polynomial P at a, let us write $P(x) = b_0 + b_1(x-a)$. Then the conditions $f(a) = P(a)$ and $f'(a) = P'(a)$ are equivalent to $b_0 = f(a)$ and $b_1 = f'(a)$. In general, if f has derivatives up to the nth order, then an nth-degree polynomial given by

$$P(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n$$

will satisfy the conditions

$$f(a) = P(a),\ f'(a) = P'(a),\ f''(a) = P''(a),\ \ldots,\ f^{(n)}(a) = P^{(n)}(a)$$

if we take

$$b_0 = f(a),\ b_1 = f'(a),\ b_2 = \frac{f''(a)}{2!},\ \ldots,\ b_n = \frac{f^{(n)}(a)}{n!}.$$

Note that these values are obtained simply by successively differentiating $P(x)$, substituting $x = a$, and then comparing with the corresponding derivative of f at a. This time, the error in the nth-degree polynomial approximation $P(x)$ can be estimated by the following generalization of the MVT.

Proposition 4.25 (Taylor Theorem). *Let $a, b \in \mathbb{R}$ with $a < b$. Also, let n be a nonnegative integer and let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f', f'', \ldots, f^{(n)}$ exist on $[a, b]$ and further, $f^{(n)}$ is differentiable on (a, b). Then there is $c \in (a, b)$ such that*

$$f(b) = f(a) + f'(a)(b - a) + \cdots + \frac{f^{(n)}(a)}{n!}(b - a)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}(b - a)^{n+1}.$$

Proof. For $x \in [a, b]$, let

$$P(x) := f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Consider $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) := f(x) - P(x) - s(x - a)^{n+1}, \quad \text{where} \quad s := \frac{f(b) - P(b)}{(b - a)^{n+1}}.$$

Then $F(a) = 0$, and our choice of s is such that $F(b) = 0$. So the Rolle Theorem applies to F, and as a result, there is $c_1 \in (a, b)$ such that $F'(c_1) = 0$. Next, if $n \geq 1$, then $f'(a) = P'(a)$, and so $F'(a) = 0$ as well. Now the Rolle Theorem applies to the restriction of F' to $[a, c_1]$, and so there is $c_2 \in (a, c_1)$ such that $F''(c_2) = 0$. Further, if $n \geq 2$, then $F''(a) = 0$, and hence there is $c_3 \in (a, c_2)$ such that $F'''(c_3) = 0$. Continuing in this way, we see that there is $c := c_{n+1} \in (a, c_n)$ such that $F^{(n+1)}(c) = 0$. Now $P^{(n+1)}$ is identically zero, since P is a polynomial of degree n. In particular, $P^{(n+1)}(c) = 0$. Hence $f^{(n+1)}(c) = s(n + 1)!$, which, in turn, yields the desired result. $\square$

Remarks 4.26. (i) Note that the MVT corresponds to the case $n = 0$ of the Taylor Theorem. The case $n = 1$ is sometimes called the **Extended Mean Value Theorem**. It says that if f is as in the Taylor Theorem with $n = 1$, then

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(c)}{2}(b - a)^2 \quad \text{for some } c \in (a, b).$$

If there are $m_2, M_2 \in \mathbb{R}$ such that $m_2 \le f''(x) \le M_2$ for all $x \in (a,b)$, then this implies the **Extended Mean Value Inequality**, which says that

$$m_2 \frac{(b-a)^2}{2} \le f(b) - f(a) - f'(a)(b-a) \le M_2 \frac{(b-a)^2}{2}.$$

(ii) In our statement of the Taylor Theorem, the point a was the left endpoint of the interval on which the function f was defined. There is an analogous version for the right endpoint. Namely, if $f : [a,b] \to \mathbb{R}$ is as in the statement of the Taylor Theorem, then there is $c \in (a,b)$ such that

$$f(a) = f(b) + f'(b)(a-b) + \cdots + \frac{f^{(n)}(b)}{n!}(a-b)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(a-b)^{n+1}.$$

This can be proved in a similar manner or alternatively deduced from our version of the Taylor Theorem by applying it to the function $g : [a,b] \to \mathbb{R}$ defined by $g(x) := f(a+b-x)$ for $x \in [a,b]$, and noting that $g^{(k)}(a) = (-1)^k f^{(k)}(b)$ for $k = 0, 1, \ldots, n$ and $g^{(n+1)}(x) = (-1)^{n+1} f^{(n+1)}(a+b-x)$ for $x \in (a,b)$. It follows that if I is any interval, a is any point of I, and $f : I \to \mathbb{R}$ is such that $f', f'', \ldots, f^{(n)}$ exist on I and $f^{(n+1)}$ exists at every interior point of I, then for every $x \in I$, $x \ne a$, there is c between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

The last expression is sometimes referred to as the **Taylor Formula** for f around a. The polynomial given by

$$P_n(x) := f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the nth **Taylor polynomial** of f around a. The difference $R_n = f - P_n$ is called the **remainder** of order n. Note that the Taylor Formula for f around a shows that the remainder R_n is given by

$$R_n(x) := \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x.$$

The above expression for $R_n(x)$ is sometimes called the **Lagrange form of the remainder** in the Taylor Formula. This is to distinguish it from some alternative expressions for the remainder in the Taylor Formula that appear in Exercises 4.51 and 6.42. ◇

The following corollary of the Taylor Formula generalizes Corollary 4.23 and gives a characterization of polynomial functions on intervals.

Corollary 4.27. *Let I be an interval containing more than one point, and let $f : I \to \mathbb{R}$ be any function. Let n be a nonnegative integer. Then f is a polynomial function of degree $\le n$ on I if and only if $f^{(n+1)}$ exists and is identically zero on I.*

Proof. If f is a polynomial function on I of degree $\leq n$, that is, if there are $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that

$$f(x) = a_n x^n + \cdots + a_1 x + a_0 \quad \text{for all } x \in I,$$

then it is obvious that $f^{(n+1)}(x) = 0$ for all $x \in I$. To prove the converse, it suffices to fix some $a \in I$ and use the Taylor formula for f around a. □

Example 4.28. Let $r \in \mathbb{Q}$ and let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) := (1+x)^r$. Given any nonnegative integer k and $c \in (-1, 1)$, we clearly have

$$f^{(k)}(c) = r(r-1)\cdots(r-k+1)(1+x)^{r-k}.$$

Hence, the nth Taylor polynomial of f around 0 is given by

$$P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{r(r-1)\cdots(r-k+1)}{k!} x^k = \sum_{k=0}^{n} \binom{r}{k} x^k.$$

Notice that if r equals a nonnegative integer n, then $f^{(n+1)}$ is identically zero, and so the remainder of order n is zero. Thus in this case, by the Taylor Theorem, we recover the binomial expansion for $(1+x)^n$. ◇

Usually, the nth Taylor polynomial of f around a provides a progressively better approximation to f around a as n increases. We shall study this aspect in greater detail in Section 5.3. For the moment, let us revisit the estimates for $\sqrt{2}$ that were obtained from the MVT and see what happens when we use the Taylor Theorem. So let $m \in \mathbb{N}$ and let $f : [m, m+1] \to \mathbb{R}$ be given by $f(x) := \sqrt{x}$. The Taylor Formula for f around m, with $n = 1$, gives

$$f(x) = f(m) + f'(m)(x-m) + \frac{f''(c)}{2!}(x-m)^2 \quad \text{for some } c \text{ between } m \text{ and } x.$$

In particular, for $x = m+1$, we obtain

$$\sqrt{m+1} = \sqrt{m} + \frac{1}{2\sqrt{m}} - \frac{1}{8c\sqrt{c}} \quad \text{for some } c \in (m, m+1).$$

For example, by putting $m = 1$, we see that

$$1 + \frac{1}{2} - \frac{1}{8} < \sqrt{2} < 1 + \frac{1}{2} - \frac{1}{16\sqrt{2}} \text{ and hence } \frac{11}{8} < \sqrt{2} < 1 + \frac{1}{2} - \frac{1}{16(3/2)} = \frac{35}{24},$$

where in the last inequality we have used the estimate $\sqrt{2} < \frac{3}{2}$, which is obvious from

$$\sqrt{2} < 1 + \frac{1}{2} - \frac{1}{16\sqrt{2}}.$$

The resulting bounds $\frac{11}{8} = 1.375$ and $\frac{35}{24} \approx 1.4584$ are, in fact, better than the bounds $\frac{4}{3} \approx 1.33$ and $\frac{3}{2} = 1.5$ obtained using the MVT. Needless to say, the higher-order Taylor polynomials would give even better bounds. In this way, if you are stranded on an island without your calculator and a demon demands to know a reasonably correct value of $\sqrt{2}$, then the Taylor Theorem can save the day for you!

4.3 Monotonicity, Convexity, and Concavity

Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a function. Recall from Chapter 1 that f is said to be **(monotonically) increasing** on I if $x_1, x_2 \in I$, $x_1 < x_2$ implies $f(x_1) \le f(x_2)$. Also, f is said to be **(monotonically) decreasing** on I if $x_1, x_2 \in I$, $x_1 < x_2$ implies $f(x_1) \ge f(x_2)$. One says that f is **monotonic** on I if it is monotonically increasing on I or monotonically decreasing on I. The function f is said to be **strictly increasing** on I if $x_1, x_2 \in I$, $x_1 < x_2$, implies $f(x_1) < f(x_2)$. Also, f is said to be **strictly decreasing** on I if $x_1, x_2 \in I$, $x_1 < x_2$, implies $f(x_1) > f(x_2)$. One says that f is **strictly monotonic** on I if it is strictly increasing on I or strictly decreasing on I.

As has been pointed out and illustrated in Chapter 1, the notions of monotonicity and strict monotonicity are purely geometric, and a priori they have no relationship to derivatives. However, in the case of differentiable functions, there is an intimate relationship between derivatives and the notions of monotonicity and strict monotonicity. The key idea can be easily grasped by looking at the graph of a function. The tangents to the graph of an increasing function have positive slopes, whereas the tangents to the graph of a decreasing function have negative slopes. (See, for example, the graph of $y = x^2$ in Figure 1.4 on page 14.) A more precise analytic formulation of this is given in the proposition below. In practice, this greatly simplifies checking whether a differentiable function is increasing or decreasing.

Proposition 4.29. *Let I be an interval containing more than one point, and let $f : I \to \mathbb{R}$ be a differentiable function. Then*

(i) *f' is nonnegative throughout $I \iff f$ is monotonically increasing on I.*
(ii) *f' is nonpositive throughout $I \iff f$ is monotonically decreasing on I.*
(iii) *f' is positive throughout $I \Longrightarrow f$ is strictly increasing on I.*
(iv) *f' is negative throughout $I \Longrightarrow f$ is strictly decreasing on I.*

Proof. Suppose $x_1, x_2 \in I$ with $x_1 < x_2$. Then $[x_1, x_2] \subseteq I$, and we can apply the MVT to the restriction of f to $[x_1, x_2]$ to obtain

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \quad \text{for some } c \in (x_1, x_2).$$

Thus, if f' is nonnegative throughout I, then $f(x_1) \le f(x_2)$, whereas if f' is nonpositive throughout I, then $f(x_1) \ge f(x_2)$. This proves the implication "$\Longrightarrow$" in (i) and (ii). Moreover, we also see from the MVT that if f' is positive throughout I, then $f(x_1) < f(x_2)$, whereas if f' is negative throughout I, then $f(x_1) > f(x_2)$. This proves (iii) and (iv).

Now, given any $x_0 \in I$ and $0 \ne h \in \mathbb{R}$ such that $x_0 + h \in I$, the quotient

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

is always nonnegative if f is monotonically increasing and always nonpositive if f is monotonically decreasing. Therefore, $f'(x_0) \ge 0$ if f is monotonically increasing on I, whereas $f'(x_0) \le 0$ if f is monotonically decreasing on I. This proves the implication "$\Longleftarrow$" in (i) and (ii). $\square$

The following corollary is essentially obtained by combining the first two and the last two parts of the above proposition.

Corollary 4.30. *Let I be an interval containing more than one point, and let $f : I \to \mathbb{R}$ be a differentiable function. Then*

(i) *f' does not change sign throughout $I \iff f$ is monotonic on I.*
(ii) *f' is nonzero throughout $I \Longrightarrow f$ is strictly monotonic on I.*

Proof. Using parts (i) and (ii) of Proposition 4.29, we obtain (i), while using the IVP for f' (Proposition 4.16) together with parts (iii) and (iv) of Proposition 4.29, we obtain (ii). $\square$

Examples 4.31. (i) Consider the polynomial function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 7.$$

Then f is differentiable, and one can easily check that

$$f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x-1)(x-2)(x-3).$$

Therefore, $f'(x) \ge 0$ if $x \ge 3$ or $1 \le x \le 2$, whereas $f'(x) \le 0$ if $x \le 1$ or $2 \le x \le 3$. Thus, f is monotonically increasing on $[1, 2]$ and on $[3, \infty)$, whereas f is monotonically decreasing on $[2, 3]$ and on $(-\infty, 1]$. In fact, since f' vanishes only at $x = 1$, 2, and 3, we see that f is strictly increasing on $(1, 2)$ and on $(3, \infty)$, whereas f is strictly decreasing on $(2, 3)$ and on $(-\infty, 1)$. Notice that in an example such as this, it would be extremely difficult to arrive at the above conclusions directly from the definition.

(ii) Let $n \in \mathbb{N}$ and consider the nth power function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^n$. Then $f'(x) = nx^{n-1}$ for $x \in \mathbb{R}$. First, assume that n is odd. Then $f'(x) \ge 0$ for all $x \in \mathbb{R}$. Thus, f is monotonically increasing on $\mathbb{R}$. In fact, since f' vanishes only at $x = 0$, we see that f is strictly increasing on $(-\infty, 0)$ as well as on $(0, \infty)$. Next, assume that n is even. Then $f'(x) \ge 0$ for $x \ge 0$ and $f'(x) \le 0$ for $x \le 0$. Thus, f is monotonically increasing on $[0, \infty)$ and monotonically decreasing on $(-\infty, 0]$. In fact, since f' vanishes only at $x = 0$, we see that f is strictly increasing on $(0, \infty)$ and strictly decreasing on $(-\infty, 0)$. Notice that in this example, we can reach these conclusions directly from the definition. In fact, we can do a little better. Namely, we can easily see that if n is odd, then f is strictly increasing on $\mathbb{R}$, whereas if n is even, then f is strictly increasing on $[0, \infty)$ and strictly decreasing on $(-\infty, 0]$. $\diamond$

As the last example shows, the converse of the implication in part (iii) of Proposition 4.29 is not true. In fact, it suffices to note that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is strictly increasing but $f'(0) = 0$. Similarly, the function

$g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = -x^3$ is strictly decreasing but $g'(0) = 0$, which shows that the converse of the implication in part (iv) of Proposition 4.29 is not true as well. As a consequence, the converse of part (ii) of Corollary 4.30 is not true. In other words, these parts give only sufficient conditions for a differentiable function to be strictly increasing or strictly decreasing or strictly monotonic. However, with a little more effort, it is possible to give a necessary and sufficient condition, as shown by the proposition below.

Proposition 4.32. *Let I be an interval containing more than one point, and let $f : I \to \mathbb{R}$ be a differentiable function. Then*

(i) *f is strictly increasing on I if and only if f' is nonnegative throughout I and f' does not vanish identically on any subinterval of I containing more than one point.*
(ii) *f is strictly decreasing on I if and only if f' is nonpositive throughout I and f' does not vanish identically on any subinterval of I containing more than one point.*

Proof. Let f be strictly increasing on I. Then by part (i) of Proposition 4.29, f' is nonnegative throughout I. Moreover, if f' were to vanish identically on any subinterval J of I containing more than one point, then by Corollary 4.23, f would be constant on J, and this is a contradiction, because f is strictly increasing. For the converse, first note that by part (i) of Proposition 4.29, f is monotonically increasing on I. Further, if $f(x_1) = f(x_2)$ for some $x_1, x_2 \in I$ with $x_1 < x_2$, then f is constant throughout $[x_1, x_2]$ and hence f' vanishes identically on $[x_1, x_2]$, which is a contradiction. This proves (i).

The assertion (ii) is proved similarly. □

We now turn to the notions of convexity and concavity. Let us recall that if I is an interval in $\mathbb{R}$, then $f : I \to \mathbb{R}$ is said to be convex on I if the graph of f lies below the line joining any two points on it, whereas $f : I \to \mathbb{R}$ is said to be concave on I if the graph of f lies above the line joining any two points on it. In other words, f is **convex** on I if $f(x) \le L(x)$ for all $x_1, x_2, x \in I$ with $x_1 < x < x_2$, whereas f is **concave** on I if $f(x) \ge L(x)$ for all $x_1, x_2, x \in I$ with $x_1 < x < x_2$, where

$$L(x) := f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) \quad \text{for } x \in I.$$

Recall also that the function f is **strictly convex** (resp. **strictly concave**) on I if $f(x) < L(x)$ (resp. $f(x) > L(x)$) for all $x_1, x_2, x \in I$ with $x_1 < x < x_2$.

As noted before, convexity and concavity are purely geometric notions, and a priori they have no relationship to derivatives. However, in the case of differentiable functions, there is an intimate relationship between derivatives and the notions of convexity and concavity. The key idea can once again be gleaned by looking at the graphs. Namely, if we draw tangents at each point, then as we move from left to right, the slopes increase if the function is convex, whereas the slopes decrease if the function is concave. (See, for example, the graph of $y = x^3$ in Figure 1.5.) A more precise analytic formulation of this is given in the proposition below. In practice, this greatly simplifies checking whether a differentiable function is convex or concave.

Proposition 4.33. *Let I be an interval containing more than one point, and let $f : I \to \mathbb{R}$ be a differentiable function. Then*

(i) *f' is monotonically increasing on $I \iff f$ is convex on I.*
(ii) *f' is monotonically decreasing on $I \iff f$ is concave on I.*
(iii) *f' is strictly increasing on $I \iff f$ is strictly convex on I.*
(iv) *f' is strictly decreasing on $I \iff f$ is strictly concave on I.*

Proof. First, assume that f' is monotonically increasing on I. Let $x_1, x_2, x \in I$ be such that $x_1 < x < x_2$. By the MVT, there are $c_1 \in (x_1, x)$ and $c_2 \in (x, x_2)$ satisfying

$$f(x) - f(x_1) = f'(c_1)(x - x_1) \quad \text{and} \quad f(x_2) - f(x) = f'(c_2)(x_2 - x).$$

Now $c_1 < c_2$ and f' is monotonically increasing on I, and so

$$\frac{f(x) - f(x_1)}{x - x_1} = f'(c_1) \le f'(c_2) = \frac{f(x_2) - f(x)}{x_2 - x}.$$

Collecting only the terms involving $f(x)$ on the left side, we obtain

$$f(x)\left(\frac{1}{x - x_1} + \frac{1}{x_2 - x}\right) \le \frac{f(x_1)}{x - x_1} + \frac{f(x_2)}{x_2 - x}.$$

Multiplying throughout by $(x - x_1)(x_2 - x)/(x_2 - x_1)$, we see that

$$f(x) \le \frac{f(x_1)(x_2 - x) + f(x_2)(x - x_1)}{x_2 - x_1} = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1),$$

where the last equality follows by writing $x_2 - x = (x_2 - x_1) - (x - x_1)$. Thus f is convex on I.

Conversely, assume that f is convex on I. Let $x_1, x_2, x \in I$ be such that $x_1 < x < x_2$. Then

$$f(x) \le f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) = f(x_2) - \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_2 - x),$$

where the last equality follows by writing $x - x_1 = (x_2 - x_1) - (x_2 - x)$. As a consequence, the slopes of chords are increasing, that is,

$$\frac{f(x) - f(x_1)}{x - x_1} \le \frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_2) - f(x)}{x_2 - x}.$$

Taking limits as $x \to x_1^+$ and $x \to x_2^-$, we obtain

$$f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2).$$

Thus, f' is monotonically increasing on I. This proves (i). Moreover, the arguments in the preceding paragraph, with $\leq$ replaced by $<$, also prove that if f' is strictly increasing on I, then f is strictly convex on I. Conversely, assume that f is strictly convex on I. Then by part (i) above, f' is monotonically increasing on I. Further, if $f'(x_1) = f'(x_2)$ for some $x_1, x_2 \in I$ with $x_1 < x_2$, then f' is constant throughout $[x_1, x_2]$, and so f'' is identically zero on $[x_1, x_2]$. Hence by Corollary 4.27 with $n = 1$, there are constants $a_0, a_1 \in \mathbb{R}$ such that $f(x) = a_1 x + a_0$ for all $x \in [x_1, x_2]$. But this contradicts the strict convexity of f. Thus, (iii) is proved.

The corresponding results (ii) and (iv) about concave and strictly concave functions are proved similarly. Alternatively, (ii) and (iv) follow from applying (i) and (iii) to $-f$. □

For twice differentiable functions, testing convexity or concavity can sometimes be simpler using the following result.

Proposition 4.34. *Let I be an interval containing more than one point, and let $f : I \to \mathbb{R}$ be a twice differentiable function. Then*

(i) *f'' is nonnegative throughout $I \iff f$ is convex on I.*
(ii) *f'' is nonpositive throughout $I \iff f$ is concave on I.*
(iii) *f'' is positive throughout $I \implies f$ is strictly convex on I.*
(iv) *f'' is negative throughout $I \implies f$ is strictly concave on I.*

Proof. Apply Proposition 4.33 to f and Proposition 4.29 to f'. □

The following corollary is obtained by combining the first two and the last two parts of the above proposition.

Corollary 4.35. *Let I be an interval containing more than one point, and let $f : I \to \mathbb{R}$ be a twice differentiable function. Then*

(i) *f'' does not change sign throughout $I \iff f$ is convex on I or f is concave on I.*
(ii) *f'' is nonzero throughout $I \implies f$ is strictly convex on I or f is strictly concave on I.*

Proof. Apply Proposition 4.33 to f and Corollary 4.30 to f'. □

Examples 4.36. (i) Consider the polynomial function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := x^4 + 2x^3 - 36x^2 + 62x + 5.$$

Then f is twice differentiable with $f'(x) = 4x^3 + 6x^2 - 72x + 62$ and

$$f''(x) = 12x^2 + 12x - 72 = 12(x + 3)(x - 2).$$

Therefore, $f''(x) \geq 0$ if $x \geq 2$ or $x \leq -3$, whereas $f''(x) \leq 0$ if $-3 \leq x \leq 2$. Thus f is convex on $[2, \infty)$ and on $(-\infty, -3]$, whereas f is concave on $[-3, 2]$. In fact, since f'' vanishes only at $x = -3$ and 2, we see that f is strictly convex on $(2, \infty)$ and on $(-\infty, -3)$, whereas f is strictly concave on $(-3, 2)$. Notice that in an example such as this, it would be extremely difficult to arrive at the above conclusions directly from the definition.

(ii) Let $n \in \mathbb{N}$ and consider the nth power function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^n$. Then $f''(x) = n(n-1)x^{n-2}$ for $x \in \mathbb{R}$. Thus if n is even, then f is convex on $\mathbb{R}$, whereas if n is odd and $n > 1$, then f is convex on $[0, \infty)$ and concave on $(-\infty, 0]$. In case $n = 1$, f is convex as well as concave on $\mathbb{R}$. In case $n > 1$, f'' vanishes only at $x = 0$, and hence if n is even, then f is strictly convex on $(0, \infty)$ as well as on $(-\infty, 0)$, whereas if n is odd and $n > 1$, then f is strictly convex on $(0, \infty)$ and strictly concave on $(-\infty, 0)$. Notice that in this example as well, it is not very easy to arrive at the above conclusions directly from the definition when n is large. (Compare Examples 1.17 (i), (ii).) However, we can directly appeal to Proposition 4.33 instead of Proposition 4.34 to get a stronger conclusion. Namely, if n is even, then $f'(x) = nx^{n-1}$ is strictly increasing on $\mathbb{R}$ and hence f is strictly convex on $\mathbb{R}$, whereas if n is odd and $n > 1$, then $f'(x) = nx^{n-1}$ is strictly increasing on $[0, \infty)$ and strictly decreasing on $(-\infty, 0]$, and hence f is strictly convex on $[0, \infty)$ and strictly concave on $(-\infty, 0]$. $\diamond$

The converse of the implication in part (ii) of Corollary 4.35 is not true. For example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^4$ is strictly convex on $\mathbb{R}$, since f' is strictly increasing on $\mathbb{R}$, but $f''(0) = 0$. Similarly, $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := -x^4$ is strictly concave on $\mathbb{R}$, but $g''(0) = 0$. Thus, part (ii) of Corollary 4.35 gives only a sufficient condition for a twice differentiable function to be strictly convex or strictly concave. However, with a little more effort, it is possible to give a necessary and sufficient condition, as shown by the proposition below.

Proposition 4.37. *Let I be an interval containing more than one point, and let $f : I \to \mathbb{R}$ be a twice differentiable function. Then*

(i) *f is strictly convex on I if and only if f'' is nonnegative throughout I and f'' does not vanish identically on any subinterval of I containing more than one point.*

(ii) *f is strictly concave on I if and only if f'' is nonpositive throughout I and f'' does not vanish identically on any subinterval of I containing more than one point.*

Proof. Applying Proposition 4.32 to f' and using parts (iii) and (iv) of Proposition 4.33, we get the desired results. $\square$

4.4 L'Hôpital's Rule

In this section, we shall describe a useful method for finding limits that is known as L'Hôpital's Rule.[5] Actually, there are several versions of L'Hôpital's Rule, and the formal statements of these will appear in the form of propositions or as a part of some remarks.

In its simplest form, L'Hôpital's Rule says the following. Suppose f, g are real-valued differentiable functions in an interval $(c-r, c+r)$ about a point c and suppose $f(c) = g(c) = 0$. If it so happens that the quotient $f'(x)/g'(x)$ of the derivatives is defined in an open interval around c (so that $g'(x) \neq 0$ in this interval), and moreover,

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = \frac{f'(c)}{g'(c)},$$

then the quotient $f(x)/g(x)$ has a limit as $x \to c$, and it is, in fact, the same limit as that of the quotient of the derivatives, that is,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

Since $g(c) = 0$, it follows from the MVT that the quotient $f(x)/g(x)$ is defined (that is, $g(x) \neq 0$) for all $x \neq c$ in the interval about c where $g'(x) \neq 0$.

For example, let us take $c = 0$ and $f, g : (-1, 1) \to \mathbb{R}$ given by

$$f(x) := \sqrt{1+x^2} - \sqrt{1-x^2} \quad \text{and} \quad g(x) := x \quad \text{for } x \in (-1, 1).$$

Then f, g are differentiable on $(-1, 1)$ and $f(0) = g(0) = 0$, while $g'(x) = 1$ is nonzero for every $x \in (-1, 1)$. Moreover,

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{x\left((1+x^2)^{-1/2} + (1-x^2)^{-1/2}\right)}{1} = \frac{0}{1} = \frac{f'(0)}{g'(0)}.$$

Hence from L'Hôpital's Rule, we can conclude that

$$\lim_{x \to 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x} = 0.$$

In this example, we could have avoided L'Hôpital's Rule and instead rationalized the quotient $f(x)/g(x)$ (that is, multiplied the numerator and denominator by $\sqrt{1+x^2} + \sqrt{1-x^2}$) to compute the limit. However, algebraic tricks such as rationalization become increasingly unwieldy if instead of $\sqrt{1+x^2} - \sqrt{1-x^2}$, $f(x)$ were given by $(1+x^2)^{3/2} - (1-x^2)^{3/2}$ or $(1+x^2)^{5/2} - (1-x^2)^{7/2}$. But L'Hôpital's Rule can still be applied to compute the limit just as easily.

[5] L'Hôpital, sometimes written L'Hospital, is pronounced *Lowpeetal*.

The reason why L'Hôpital's Rule works is quite simple. One just has to observe that

$$\lim_{x\to c}\frac{f'(x)}{g'(x)}=\frac{f'(c)}{g'(c)}=\frac{\displaystyle\lim_{x\to c}\frac{f(x)-f(c)}{x-c}}{\displaystyle\lim_{x\to c}\frac{g(x)-g(c)}{x-c}}=\lim_{x\to c}\frac{f(x)-f(c)}{g(x)-g(c)}=\lim_{x\to c}\frac{f(x)}{g(x)},$$

where the last step follows since $f(c)=g(c)=0$.

It turns out that we can get rid of some of the assumptions in the simple formulation of L'Hôpital's Rule given above. Indeed, in the true spirit of dealing with limits as $x\to c$, we need not require that the concerned functions be defined at the point c. Thus the condition $f(c)=g(c)=0$ may be replaced by the conditions

$$\lim_{x\to c} f(x)=0 \quad\text{and}\quad \lim_{x\to c} g(x)=0,$$

while the condition about the quotient of derivatives may be replaced by the condition

$$\frac{f'(x)}{g'(x)}\to\ell \quad \text{as } x\to c,$$

assuming, of course, that the above quotient is defined in an open interval about c except possibly at c. Now for the proof we will have to contend with some problems. First there is a minor problem that $f(c)$ and $g(c)$ are no longer defined. This is easily handled by simply *defining* $f(c)=g(c)=0$. A more serious problem is that $f'(c)$ and $g'(c)$ no longer make sense. To handle this, one has to deal directly with quotients such as $\big(f(x)-f(c)\big)/\big(g(x)-g(c)\big)$. What we need, in fact, is the following generalization of the MVT.

Proposition 4.38 (Cauchy Mean Value Theorem). *If $f,g:[a,b]\to\mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b), then there is $c\in(a,b)$ such that*

$$g'(c)\big(f(b)-f(a)\big)=f'(c)\big(g(b)-g(a)\big).$$

Proof. If $g(b)=g(a)$, the result follows by applying the Rolle Theorem to g. Otherwise, we consider $F:[a,b]\to\mathbb{R}$ defined by

$$F(x)=f(x)-f(a)-s\big(g(x)-g(a)\big), \quad \text{where} \quad s=\frac{f(b)-f(a)}{g(b)-g(a)}.$$

Then $F(a)=0$ and our choice of the constant s is such that $F(b)=0$. So the Rolle Theorem applies to F, and as a result, there is $c\in(a,b)$ such that $F'(c)=0$. This implies that $sg'(c)=f'(c)$, as desired. □

We are now ready to prove the first version of L'Hôpital's Rule. This one is called L'Hôpital's Rule for $\frac{0}{0}$ indeterminate forms, since it applies to limits of quotients in which both the numerator and the denominator tend to 0. For convenience, we state and prove below the version for left limits and remark later how other versions of L'Hôpital's Rule for $\frac{0}{0}$ indeterminate forms can be derived.

Proposition 4.39 (L'Hôpital's Rule for $\frac{0}{0}$ Indeterminate Forms). *Let $c \in \mathbb{R}$, $r > 0$, and let $f, g : (c - r, c) \to \mathbb{R}$ be differentiable functions such that*

$$\lim_{x \to c^-} f(x) = 0 \quad \text{and} \quad \lim_{x \to c^-} g(x) = 0.$$

Suppose $g'(x) \neq 0$ for all $x \in (c - r, c)$, and

$$\frac{f'(x)}{g'(x)} \to \ell \text{ as } x \to c^-.$$

Then $g(x) \neq 0$ for all $x \in (c - r, c)$ and

$$\frac{f(x)}{g(x)} \to \ell \text{ as } x \to c^-.$$

Here ℓ can be a real number or ∞ or $-\infty$.

Proof. Extend f, g to $(c - r, c]$ by putting $f(c) = g(c) = 0$. Then f and g are continuous on $(c - r, c]$ and differentiable on $(c - r, c)$. Hence if $g(x) = 0$ for some $x \in (c - r, c)$, then by the Rolle Theorem, $g'(y) = 0$ for some $y \in (x, c)$, which is a contradiction. This shows that $g(x) \neq 0$ for all $x \in (c - r, c)$. Next, let (x_n) be any sequence in $(c - r, c)$ such that $x_n \to c$. Then by the Cauchy Mean Value Theorem, we see that for each $n \in \mathbb{N}$,

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(c)}{g(x_n) - g(c)} = \frac{f'(c_n)}{g'(c_n)} \quad \text{for some } c_n \in (x_n, c).$$

Now $x_n \to c$ implies that $c_n \to c$, and so $f'(c_n)/g'(c_n) \to \ell$. Consequently, $f(x_n)/g(x_n) \to \ell$. This shows that $f(x)/g(x) \to \ell$ as $x \to c^-$. $\square$

Remarks 4.40. (i) L'Hôpital's Rule for $\frac{0}{0}$ indeterminate forms is also valid for right limits. The statement and proof are identical to the above, except we replace the interval $(c - r, c)$ by $(c, c + r)$ and the symbols $x \to c^-$ by $x \to c^+$. Combining the versions for left limits and right limits, we obtain L'Hôpital's Rule for (two-sided) limits of $\frac{0}{0}$ indeterminate forms, which may be stated as follows.

Let $c \in \mathbb{R}$ and $D = (c - r, c) \cup (c, c + r)$ for some $r > 0$. Let $f, g : D \to \mathbb{R}$ be differentiable functions such that $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$. Suppose $g'(x) \neq 0$ for all $x \in D$, and $f'(x)/g'(x) \to \ell$ as $x \to c$. Then $f(x)/g(x) \to \ell$ as $x \to c$. Here ℓ can be a real number or ∞ or $-\infty$.

(ii) Analogues of L'Hôpital's Rule for $\frac{0}{0}$ indeterminate forms are also valid if instead of considering limits as $x \to c$, where c is a real number, we consider limits as $x \to \infty$ or as $x \to -\infty$. For example, a statement for limits as $x \to \infty$ would be as follows.

Let $a \in \mathbb{R}$ and let $f, g : (a, \infty) \to \mathbb{R}$ be differentiable functions such that $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$. Suppose $g'(x) \neq 0$ for all $x \in (a, \infty)$, and

$f'(x)/g'(x) \to \ell$ as $x \to \infty$. Then $f(x)/g(x) \to \ell$ as $x \to \infty$. Here ℓ can be a real number or ∞ or $-\infty$.

As for the proof, it suffices to assume that $a > 0$ and apply L'Hôpital's Rule for right limits as $x \to 0^+$ to the functions $F, G : (0, 1/a) \to \mathbb{R}$ defined by $F(x) := f(1/x)$ and $G(x) := g(1/x)$. $\diamond$

The following examples illustrate how limits of certain functions that are in $\frac{0}{0}$ indeterminate form, or that can be converted to $\frac{0}{0}$ indeterminate form, can be computed by making one or more applications of L'Hôpital's Rule. The verification that L'Hôpital's Rule is indeed applicable (that is, the hypotheses of Proposition 4.39 are satisfied) in each case is left as an exercise.

Examples 4.41. (i) $\displaystyle\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-4} = \lim_{x\to 2} \frac{x/\sqrt{x^2+5}}{2x} = \frac{2/3}{4} = \frac{1}{6}$.

(ii) $\displaystyle\lim_{x\to 1} \frac{x^3-3x^2+3x-1}{x^3+x^2-5x+3} = \lim_{x\to 1} \frac{3x^2-6x+3}{3x^2+2x-5} = \lim_{x\to 1} \frac{6x-6}{6x+2} = 0$.

(iii) $\displaystyle\lim_{x\to\infty} \left(x^3+4x^2+13x+1\right)^{1/3} - x = \lim_{y\to 0^+} \frac{\left(1+4y+13y^2+y^3\right)^{1/3}-1}{y}$.

Using L'Hôpital's Rule, we see that the above limit exists and is equal to

$$\lim_{y\to 0^+} \frac{1}{3}\left(1+4y+13y^2+y^3\right)^{-2/3}\left(4+26y+3y^2\right) = \frac{4}{3}. \qquad \diamond$$

Now we describe another version of L'Hôpital's Rule, which is useful in computing limits of $\frac{\infty}{\infty}$ indeterminate forms, that is, of quotients of functions such that both the numerator and the denominator tend to infinity. It turns out here that the formulation as well as the proof of this rule is valid even when the numerator does not tend to infinity. But we may still refer to it as L'Hôpital's Rule for $\frac{\infty}{\infty}$ indeterminate forms. As before, we state and prove below the version for left limits. This time the statement and proof are such that they are applicable to left limits as x approaches a real number and also to limits as $x \to \infty$.

Proposition 4.42 (L'Hôpital's Rule for $\frac{\infty}{\infty}$ Indeterminate Forms). *Let I be the interval $[a, c)$, where $a \in \mathbb{R}$, and either $c \in \mathbb{R}$ with $a < c$ or $c = \infty$. Let $f, g : I \to \mathbb{R}$ be differentiable functions such that $|g(x)| \to \infty$ as $x \to c^-$. Suppose $g'(x) \neq 0$ for all $x \in I$ and*

$$\frac{f'(x)}{g'(x)} \to \ell \text{ as } x \to c^-.$$

Then

$$\frac{f(x)}{g(x)} \to \ell \text{ as } x \to c^-.$$

Here ℓ can be a real number or ∞ or $-\infty$.

Proof. Since $g'(x) \neq 0$ for all $x \in I$, by part (ii) of Corollary 4.29, either g is strictly increasing on I or g is strictly decreasing on I. Replacing g and f by $-g$ and $-f$ if necessary, we assume that g is strictly increasing on I. Since $g(a) \leq g(x)$ for all $x \in I$ and $|g(x)| \to \infty$ as $x \to c^-$, part (i) of Proposition 3.44 shows that in fact, $g(x) \to \infty$ as $x \to c^-$. In particular, there is $a_1 \in I$ such that $g(x) > 0$ for all $x \in (a_1, c)$.

To begin with, let us suppose ℓ is a real number. Let $\epsilon > 0$ be given. Since $f'(x)/g'(x) \to \ell$ as $x \to c^-$, there is $a_2 \in (a_1, c)$ such that

$$\ell - \epsilon < \frac{f'(x)}{g'(x)} < \ell + \epsilon \quad \text{for all } x \in (a_2, c).$$

Let $h_\epsilon, h_{2\epsilon} : I \to \mathbb{R}$ be defined by

$$h_\epsilon := f - (\ell - \epsilon)g \quad \text{and} \quad h_{2\epsilon} := f - (\ell - 2\epsilon)g.$$

Then $h'_{2\epsilon}(x) > h'_\epsilon(x) > 0$ for all $x \in (a_2, c)$. Therefore, by part (iii) of Proposition 4.29, the functions h_ϵ and $h_{2\epsilon}$ are strictly increasing on (a_2, c). We claim that there is some $a_3 \in (a_2, c)$ such that $h_{2\epsilon}(x) > 0$ for all $x \in (a_3, c)$. To see this, assume the contrary. Then we can find an increasing sequence (x_n) in (a_2, c) such that $x_n \to c$ and $h_{2\epsilon}(x_n) \leq 0$ for all $n \in \mathbb{N}$. Now, since $g(x) \to \infty$ as $x \to c^-$, we see that $g(x_n) \to \infty$. On the other hand, since h_ϵ is (strictly) increasing on (a_2, c), we obtain $h_\epsilon(x_1) \leq h_\epsilon(x_n)$ for all $n \in \mathbb{N}$, and hence

$$\epsilon g(x_n) = h_{2\epsilon}(x_n) - h_\epsilon(x_n) \leq 0 - h_\epsilon(x_1), \text{ that is, } g(x_n) \leq \frac{-h_\epsilon(x_1)}{\epsilon}$$

for all $n \in \mathbb{N}$. This contradicts the condition that $g(x_n) \to \infty$. So, our claim is proved. Thus, there is $a_3 \in (a_2, c)$ such that

$$h_{2\epsilon}(x) = f(x) - (\ell - 2\epsilon)g(x) > 0, \text{ that is, } \ell - 2\epsilon < \frac{f(x)}{g(x)} \quad \text{for all } x \in (a_3, c).$$

In a similar way, we see that there is $a_4 \in (a_2, c)$ such that

$$\frac{f(x)}{g(x)} < \ell + 2\epsilon \quad \text{for all } x \in (a_4, c).$$

Thus, if we let $a_5 := \max\{a_3, a_4\}$, then

$$\ell - 2\epsilon < \frac{f(x)}{g(x)} < \ell + 2\epsilon \quad \text{for all } x \in (a_5, c).$$

Since $\epsilon > 0$ is arbitrary, this proves that $f(x)/g(x) \to \ell$ as $x \to c^-$.

Next, suppose $\ell = \infty$. In this case, we can proceed as above, and the arguments are, in fact, simpler. Let $\alpha \in \mathbb{R}$ be given. Then there is $a_2 \in (a_1, c)$ such that $f'(x)/g'(x) > \alpha$ for all $x \in (a_2, c)$. Let $h_\alpha, h_{\alpha-1} : I \to \mathbb{R}$ be defined by $h_\alpha := f - \alpha g$ and $h_{\alpha-1} := f - (\alpha - 1)g$. Then $h'_{\alpha-1}(x) > h'_\alpha(x) > 0$ for all

$x \in (a_2, c)$. Therefore, by part (iii) of Proposition 4.29, the functions h_α and $h_{\alpha-1}$ are strictly increasing on (a_2, c). We claim that there is some $a_3 \in (a_2, c)$ such that $h_{\alpha-1}(x) > 0$ for all $x \in (a_3, c)$. To see this, assume the contrary. Then we can find an increasing sequence (x_n) in (a_2, c) such that $x_n \to c$ and $h_{\alpha-1}(x_n) \le 0$ for all $n \in \mathbb{N}$. Now, since $g(x) \to \infty$ as $x \to c^-$, we see that $g(x_n) \to \infty$. On the other hand, since h_α is (strictly) increasing on (a_2, c), we obtain $h_\alpha(x_1) \le h_\alpha(x_n)$ for all $n \in \mathbb{N}$, and hence

$$g(x_n) = h_{\alpha-1}(x_n) - h_\alpha(x_n) \le 0 - h_\alpha(x_1) \quad \text{for all } n \in \mathbb{N}.$$

This contradicts the condition that $g(x_n) \to \infty$. So our claim is proved. Thus, there is $a_3 \in (a_2, c)$ such that

$$h_{\alpha-1}(x) = f(x) - (\alpha - 1)g(x) > 0, \text{ that is, } \frac{f(x)}{g(x)} > \alpha - 1 \quad \text{for all } x \in (a_3, c).$$

Since $\alpha \in \mathbb{R}$ is arbitrary, this proves that $f(x)/g(x) \to \infty$ as $x \to c^-$.

The case $\ell = -\infty$ is proved similarly. □

Remark 4.43. L'Hôpital's Rule for $\frac{\infty}{\infty}$ indeterminate forms is valid for right limits as x approaches a real number, and for limits as $x \to -\infty$. The statement is analogous to Proposition 4.42, and is also proved similarly. Combining the versions for left limits and right limits, we obtain L'Hôpital's Rule for (two-sided) limits of $\frac{\infty}{\infty}$ indeterminate forms, which may be stated as follows:

Let $c \in \mathbb{R}$ and $D = (c - r, c) \cup (c, c + r)$ for some $r > 0$. Let $f, g : D \to \mathbb{R}$ be differentiable functions such that $|g(x)| \to \infty$ as $x \to c$. Suppose $g'(x) \ne 0$ for all $x \in D$, and $f'(x)/g'(x) \to \ell$ as $x \to c$. Then $f(x)/g(x) \to \ell$ as $x \to c$. Here ℓ can be a real number or ∞ or $-\infty$. ◇

Examples 4.44. (i) By successive applications of L'Hôpital's Rule for $\frac{\infty}{\infty}$ indeterminate forms, we see that

$$\lim_{x\to\infty} \frac{x^2 + 2x + 3}{3x^2 + 2x + 1} = \lim_{x\to\infty} \frac{2x + 2}{6x + 2} = \lim_{x\to\infty} \frac{2}{6} = \frac{1}{3}.$$

(ii) Simplifying and then applying L'Hôpital's Rule for $\frac{\infty}{\infty}$ indeterminate forms (twice), we obtain

$$\lim_{x\to\infty} \frac{x^3}{x^2 - 1} - \frac{x^3}{x^2 + 1} = \lim_{x\to\infty} \frac{2x^3}{x^4 - 1} = \lim_{x\to\infty} \frac{6x^2}{4x^3} = \lim_{x\to\infty} \frac{3}{2x} = 0. \quad \diamond$$

While L'Hôpital's Rule is extremely useful in computing limits, it is not a panacea! There are situations in which it is not applicable. These are illustrated by the following examples.

Examples 4.45. (i) If $f, g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) := x$ and $g(x) := \sqrt{1 + x^2}$, then $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$. If we try to apply L'Hôpital's Rule, then we get a loop:

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{\sqrt{1+x^2}} &= \lim_{x\to\infty}\frac{1}{2x/2\sqrt{1+x^2}}\\ &= \lim_{x\to\infty}\frac{\sqrt{1+x^2}}{x}\\ &= \lim_{x\to\infty}\frac{2x/2\sqrt{1+x^2}}{1}\\ &= \lim_{x\to\infty}\frac{x}{\sqrt{1+x^2}}.\end{aligned}$$

However, the desired limit exists and can be found directly as follows:

$$\lim_{x\to\infty}\frac{x}{\sqrt{1+x^2}} = \lim_{x\to\infty}\frac{1}{\sqrt{(1/x^2)+1}} = \frac{1}{\sqrt{0+1}} = 1.$$

(ii) If we were to apply L'Hôpital's Rule indiscriminately to calculate limits of quotients such as $(x+1)/x$ as $x \to 0$, we would obtain

$$\lim_{x\to 0}\frac{x+1}{x} = \lim_{x\to 0}\frac{1}{1} = 1.$$

But in fact, the limit does not exist; indeed,

$$\frac{x+1}{x} = 1 + \frac{1}{x} \to \infty \text{ as } x \to 0^+ \quad \text{and} \quad \frac{x+1}{x} \to -\infty \text{ as } x \to 0^-.$$

In this case, L'Hôpital's Rule was not applicable, since the given quotient is neither in $\frac{0}{0}$ form nor in $\frac{\infty}{\infty}$ form as $x \to 0$. ◇

Remarks 4.46. (i) Evaluating limits of seemingly different indeterminate forms such as $0 \cdot \infty$ and $\infty - \infty$ is also possible using L'Hôpital's Rule, since such forms can be reduced to $\frac{0}{0}$ indeterminate forms. More precisely, if $c \in \mathbb{R}$ and $f, g : (c-r, c) \to \mathbb{R}$ are differentiable functions such that

$$f(x) \to 0 \quad \text{and} \quad g(x) \to \infty \text{ or } -\infty \quad \text{as } x \to c^-,$$

then there is $\delta > 0$ such that $\delta < r$ and $g(x) \neq 0$ for all $x \in (c-\delta, c)$. Now, $1/g(x) \to 0$ as $x \to c^-$ and

$$f(x)g(x) = \frac{f(x)}{1/g(x)} \quad \text{for } x \in (c-\delta, c),$$

and thus a $0 \cdot \infty$ indeterminate form is converted to a $\frac{0}{0}$ indeterminate form to which L'Hôpital's Rule can be applied. Likewise, if $f(x) \to \infty$ and $g(x) \to \infty$, then there is $\delta > 0$ such that $\delta < r$ and $f(x) > 0$ as well as $g(x) > 0$ for all $x \in (c-\delta, c)$. Now we can write

$$f(x) - g(x) = \frac{\big(1/g(x)\big) - \big(1/f(x)\big)}{\big(1/f(x)g(x)\big)} \quad \text{for } x \in (c-\delta, c),$$

and thus an $\infty - \infty$ indeterminate form is converted to a $\frac{0}{0}$ indeterminate form to which L'Hôpital's Rule can be applied.

(ii) The power of L'Hôpital's Rule will be especially evident when we add to our repertoire of functions the logarithmic, exponential, and trigonometric functions and try to compute limits involving these functions. This will also enable us to deal with other indeterminate forms such as 0^0, ∞^0, and 1^∞. These variants of L'Hôpital's Rule are explained in Remark 7.13. Examples of limits involving the logarithmic, exponential, and trigonometric functions appear in Example 7.5 (ii), and in Revision Exercises R.18 and R.19 given at the end of Chapter 7, and in all these, L'Hôpital's Rule is particularly useful. In Examples 7.20 and 7.21, it will be shown that the converse of L'Hôpital's Rule, for $\frac{\infty}{\infty}$ and for $\frac{0}{0}$ indeterminate forms, does not hold in general. $\diamond$

Notes and Comments

In this chapter, we have derived all the basic properties of differentiation by appealing to a characterization of differentiability in terms of continuity and the relevant properties of continuous functions. With this approach, the proofs seem to become simpler. Another advantage is that we obtain formulas for the sum, product, quotient, composite, and the inverse of functions in the course of proving their differentiability, and it is not necessary to know them beforehand. The said characterization of differentiability appears, for example, in the book of Bartle and Sherbert [9], where it is ascribed to Carathéodory, and used to derive the Chain Rule and the Differentiable Inverse Theorem. Here, we have used it more extensively.

The Mean Value Theorem (MVT) and, more generally, the Taylor Theorem are among the most useful results in calculus. The importance of the MVT in calculus mainly stems from the fact that it is crucial in characterizing constant functions, monotonic functions, and convex/concave functions. Such characterizations can be proved using only the mean value inequality, which is obtained here as a corollary of the MVT. On the other hand, it is possible to give an alternative proof of the mean value inequality using properties of Riemann integration and without recourse to the MVT. This has prompted several articles with rather colorful titles. See, for example, the papers by Bers [12], Boas [14], Cohen [19], and Smith [75].

The study of convex functions, which was initiated in Chapter 1 and further continued in this chapter, is now a subject in itself. A quick and elegant introduction can be found in the first chapter of the little classic on gamma functions by Artin [6]. For more on the subject of convex analysis, see the introductory text of van Tiel [83].

Most books on calculus discuss L'Hôpital's Rule for $\frac{0}{0}$ *and* $\frac{\infty}{\infty}$ *indeterminate forms but prove only the former. On the other hand, some relatively advanced books such as Rudin [71] give a sleek proof applicable to both versions at once.*

A unified proof such as that in Rudin [71] appears to have been inspired by the article of Taylor [81], where it is given as an improved version of a proof by Wazewski. For pedagogical reasons, we have chosen to avoid the sleek unified proof and given instead separate proofs for the two versions. The proof in the $\frac{0}{0}$ case is quite standard and follows quickly from the Cauchy Mwan Value Theorem, thanks to our sequential approach to limits. The proof in the $\frac{\infty}{\infty}$ case uses the Intermediate Value Property of derivatives and is essentially based on an argument of Lettenmeyer, which is outlined in the article of Taylor [81].

Exercises

Part A

4.1. Use the definition of a derivative to find $f'(x)$ if
(i) $f(x) = x^2$, $x \in \mathbb{R}$, (ii) $f(x) = 1/x$, $0 \neq x \in \mathbb{R}$,
(iii) $f(x) = \sqrt{x^2+1}$, $x \in \mathbb{R}$, (iv) $f(x) = 1/\sqrt{2x+3}$, $x \in (-3/2, \infty)$.

4.2. Let $D \subseteq \mathbb{R}$ and let c be an interior point of D. If $f, g : D \to \mathbb{R}$ are differentiable at c and $g(c) \neq 0$, then determine an increment function associated to f/g and c.

4.3. Let $f : (a, b) \to \mathbb{R}$ be a function such that

$$|f(x+h) - f(x)| \leq C|h|^r \quad \text{for all } x, x+h \in (a, b),$$

where C is a constant and $r \in \mathbb{Q}$ with $r > 1$. Show that f is differentiable on (a, b) and compute $f'(x)$ for $x \in (a, b)$.

4.4. If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then show that

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c-h)}{2h}$$

exists and equals $f'(c)$. Is the converse true?

4.5. Let $f : (0, \infty) \to \mathbb{R}$ satisfy $f(xy) = f(x) + f(y)$ for all $x, y \in (0, \infty)$. If f is differentiable at 1, show that f is differentiable at every $c \in (0, \infty)$ and $f'(c) = f'(1)/c$. In fact, show that f is infinitely differentiable. If $f'(1) = 2$, find $f^{(n)}(3)$.

4.6. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f is differentiable at 0, then show that f is differentiable at every $c \in \mathbb{R}$ and $f'(c) = f'(0)f(c)$. In fact, show that f is infinitely differentiable. If $f'(0) = 2$, find $f^{(n)}(1)$ for $n \in \mathbb{N}$, in terms of $f(1)$.

4.7. Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x)g(y) + g(x)f(y)$ and $g(x+y) = g(x)g(y) - f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f and g are differentiable at 0, then show that f and g are differentiable at every $c \in \mathbb{R}$, and moreover, $f'(c) = g'(0)f(c) + f'(0)g(c)$ and $g'(c) = g'(0)g(c) - f'(0)f(c)$. In fact, show that f and g are infinitely differentiable.

4.8. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f(x-y) = f(x)g(y) - g(x)f(y)$ and $g(x-y) = g(x)g(y) + f(x)f(y)$ for all $x, y \in \mathbb{R}$. If $f'_+(0)$ exists, then show that f and g are differentiable at every $c \in \mathbb{R}$, and $f'(c) = f'(0)g(c)$ and $g'(c) = -f'(0)f(c)$. In fact, show that f and g are infinitely differentiable. If $f'_+(0) = 2$, find $f^{(n)}(1)$ and $g^{(n)}(1)$ in terms of $f(1)$ and $g(1)$. (Hint: Prove that f is an odd function, g is an even function, f and g are differentiable at 0, and $g'(0) = 0$. Use Exercise 4.7.)

4.9. Find the points on the curve $x^2 + xy + y^2 = 7$ at which (i) the tangent is parallel to the x-axis, (ii) the tangent is parallel to the y-axis.

4.10. Find the equation of the tangent at $(\frac{1}{4}, 4)$ to the parametrically defined curve $x(t) = t^{-2}$, $y(t) = \sqrt{t^2 + 12}$ for $t \in (0.1, 1)$.

4.11. Find values of the constants a, b, and c for which the graphs of the two functions $f(x) = x^2 + ax + b$ and $g(x) = x^3 - c$, $x \in \mathbb{R}$, intersect at the point (1, 2) and have the same tangent there.

4.12. Find the tangents to the implicitly defined curve $x^2y + xy^2 = 6$ at points for which $x = 1$. Also, compute $\dfrac{d^2y}{dx^2}$ at these points.

4.13. Given $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) := x^n$ if $x \geq 0$ and $f_n(x) := -x^n$ if $x < 0$. Show that f_n is $(n-1)$-times differentiable on $\mathbb{R}$, $f_n^{(n-1)}$ is continuous on $\mathbb{R}$, but $f_n^{(n)}(0)$ does not exist.

4.14. Let $D \subseteq \mathbb{R}$ be symmetric about the origin, that is, $-x \in D$ whenever $x \in D$. If $c \in D$ and $f : D \to \mathbb{R}$ is either an even or an odd function, then show that the left derivative $f'_-(c)$ at c exists if and only if the right derivative $f'_+(-c)$ at $-c$ exists. Further, if either (and hence both) of these derivatives exists, then show that $f'_-(c) = -f'_+(-c)$ if f is even, and $f'_-(c) = f'_+(-c)$ if f is odd. Deduce that if f is differentiable, then f' is an odd (resp. even) function according as f is an even (resp. odd) function.

4.15. Let I be an interval, $c \in I$, and let $f : I \to \mathbb{R}$ be any function. Let, as usual, $|f| : I \to \mathbb{R}$ be the function defined by $|f|(x) = |f(x)|$ for $x \in I$.
  (i) Suppose $(c, c+r) \subseteq I$ for some $r > 0$ and $f'_+(c)$ exists. Then show that $|f|'_+(c)$ exists.
  (ii) Suppose $(c-r, c) \subseteq I$ for some $r > 0$ and $f'_-(c)$ exists. Then show that $|f|'_-(c)$ exists.
  (iii) Suppose $(c-r, c+r) \subseteq I$ for some $r > 0$ and $f'(c)$ exists. Then show that $|f|'(c)$ exists if and only if either there is $\delta > 0$ such that $\delta \leq r$ and $f(x)$ has the same sign for all $x \in (c-\delta, c+\delta)$, or $f(c) = f'(c) = 0$.

4.16. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points on the curve $y = ax^2 + bx + c$. If $P_3 = (x_3, y_3)$ lies on the arc P_1P_2 and the tangent to the curve at P_3 is parallel to the chord P_1P_2, show that $x_3 = (x_1 + x_2)/2$.

4.17. Show that the x-axis is a normal to the curve $y^2 = x$ at $(0, 0)$. If three normals can be drawn to this curve from a point $(a, 0)$, show that a must be greater than $\frac{1}{2}$. Find the value of a such that the two normals, other than the x-axis, are perpendicular to each other.

4.18. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f(a)$ and $f(b)$ are of different signs and $f'(x) \neq 0$ for all $x \in (a,b)$, then show that there is a unique $x_0 \in (a,b)$ such that $f(x_0) = 0$.

4.19. Show that the cubic $2x^3 + 3x^2 + 6x + 10$ has exactly one real root.

4.20. Let $n \in \mathbb{N}$ and $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists in (a,b). If f vanishes at $n+1$ distinct points in $[a,b]$, then show that $f^{(n)}$ vanishes at least once in (a,b).

4.21. Let $f : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \sqrt{2x - x^2} & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \sqrt{-2x - x^2} & \text{if } -\frac{1}{2} \leq x \leq 0. \end{cases}$$

Show that $f(\frac{1}{2}) = f(-\frac{1}{2})$ but $f'(x) \neq 0$ for all x with $0 < |x| < \frac{1}{2}$. Does this contradict the Rolle Theorem? Justify your answer.

4.22. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) < f(b)$, then show that $f'(c) > 0$ for some $c \in (a,b)$.

4.23. Let $a > 0$ and $f : [-a,a] \to \mathbb{R}$ be continuous. Suppose $f'(x)$ exists and $f'(x) \leq 1$ for all $x \in (-a,a)$. If $f(a) = a$ and $f(-a) = -a$, then show that $f(x) = x$ for every $x \in (-a,a)$.

4.24. In each of the following cases, find a function f that satisfies all the given conditions, or else show that no such function exists.
(i) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 1$,
(ii) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 2$,
(iii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 100$ for all $x > 0$,
(iv) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 1$ for all $x < 0$.

4.25. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose $f(a) = a$ and $f(b) = b$. Show that there is $c \in (a,b)$ such that $f'(c) = 1$. Further, show that there are distinct $c_1, c_2 \in (a,b)$ such that $f'(c_1) + f'(c_2) = 2$. More generally, show that for every $n \in \mathbb{N}$, there are n distinct points $c_1, \ldots, c_n \in (a,b)$ such that $f'(c_1) + \cdots + f'(c_n) = n$.

4.26. Let a function $f : [a,b] \to \mathbb{R}$ be continuous and its second derivative f'' exist everywhere on the open interval (a,b). Suppose the line segment joining $(a, f(a))$ and $(b, f(b))$ intersects the graph of f at a third point $(c, f(c))$, where $a < c < b$. Prove that $f''(t) = 0$ for some $t \in (a,b)$.

4.27. Use the MVT to prove that $na^{n-1}(b-a) \leq b^n - a^n \leq nb^{n-1}(b-a)$ for all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$ such that $0 < a \leq b$.

4.28. Use the MVT to prove that

$$\frac{1}{3(m+1)^{2/3}} < (m+1)^{1/3} - m^{1/3} < \frac{1}{3m^{2/3}} \quad \text{for all } m \in \mathbb{N}.$$

4.29. Use the MVT to prove the following inequalities.
(i) $\dfrac{13}{8} < \sqrt{3} < \dfrac{7}{4}$ and $\dfrac{20}{9} < \sqrt{5} < \dfrac{9}{4}$.
(ii) $\dfrac{19}{16} < 2^{1/3} < \dfrac{4}{3}$, $\dfrac{17}{9} < 7^{1/3} < \dfrac{23}{12}$, and $\dfrac{1298}{625} < 9^{1/3} < \dfrac{25}{12}$.

4.30. Use the MVT to show that $10.049 < \sqrt{101} < 10.05$ and $10.24 < \sqrt{105} < 10.25$. Also, find better estimates using the Taylor Theorem with $n = 1$, that is, using the Extended MVT.

4.31. Let $f : (a, b) \to \mathbb{R}$ and let $c \in (a, b)$ be such that f is continuous at c and $f'(x)$ exist for every $x \in (a, c) \cup (c, b)$. If $\lim_{x \to c} f'(x)$ exists, then show that $f'(c)$ exists and is equal to this limit.

4.32. (i) Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) \le g(a)$ and $f'(x) \le g'(x)$ for all $x \in (a, b)$, then show that $f(b) \le g(b)$.

(ii) Use (i) to show that $15x^2 \le 8x^3 + 12 \le 18x^2$ for all $x \in [1.25, 1.5]$. Deduce that the range of the function $h : [1.25, 1.5] \to \mathbb{R}$ given by $h(x) = (2x^3 + 3)/3x^2$ is contained in $[1.25, 1.5]$.

4.33. Find the nth Taylor polynomial of f around a, that is,

$$P_n(x) = f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n \quad \text{for } x \in \mathbb{R},$$

when $a = 0$ and $f(x)$ equals: (i) $\dfrac{1}{1 - x}$, (ii) $\dfrac{1}{1 + x}$, (iii) $\dfrac{x}{1 + x^2}$.

4.34. Let I be an interval containing more than one point and let $f : I \to \mathbb{R}$ be a function.

(i) Assume that f is differentiable. If f' is nonnegative on I and f' vanishes at only a finite number of points on any bounded subinterval of I, then show that f is strictly increasing on I.

(ii) Assume that f is twice differentiable. If f'' is nonnegative on I and f'' vanishes at only a finite number of points on any bounded subinterval of I, then show that f is strictly convex on I.

(iii) Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x - 2n)^3 + 2n$, where $n \in \mathbb{Z}$ is such that $x \in [2n - 1, 2n + 1)$. Show that f is differentiable on $\mathbb{R}$ and f'' exists on $(2n - 1, 2n + 1)$, but $f''_+(2n + 1) = 6$, whereas $f''_-(2n+1) = -6$ for each $n \in \mathbb{N}$. Also show that f is strictly increasing on $\mathbb{R}$ although $f'(2n) = 0$ for each $n \in \mathbb{N}$. (Compare (i) above and Revision Exercise R.12 given at the end of Chapter 7.)

(iv) Consider $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = (x - 2n)^4 + 8nx$, where $n \in \mathbb{Z}$ is such that $x \in [2n - 1, 2n + 1)$. Show that g is twice differentiable on $\mathbb{R}$ and g''' exists on $(2n - 1, 2n + 1)$, but $g'''_+(2n + 1) = 24$, whereas $g'''_-(2n + 1) = -24$ for each $n \in \mathbb{N}$. Also show that g is strictly convex on $\mathbb{R}$ although $g''(2n) = 0$ for each $n \in \mathbb{N}$. (Compare (ii) above and Revision Exercise R.13 given at the end of Chapter 7.)

4.35. Let I be an interval in $\mathbb{R}$ and let $c \in I$ be an interior point. If $f : I \to \mathbb{R}$ is monotonically increasing and if the left and right derivatives of f at c, namely $f'_-(c)$ and $f'_+(c)$, exist, then show that $f'_-(c) \ge 0$ and $f'_+(c) \ge 0$. Further, give examples of monotonically increasing functions $f : I \to \mathbb{R}$ for which $f'_-(c) < f'_+(c)$ or for which $f'_-(c) > f'_+(c)$.

4.36. Let $f : [a, b] \to \mathbb{R}$ be such that f' is continuous on $[a, b]$ and f'' exists on (a, b). Show that $f''(c)\big(f(b) - f(a)\big) = f'(c)\big(f'(b) - f'(a)\big)$ for some $c \in (a, b)$.

4.37. Let $f, g, h : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that there is $c \in (a, b)$ such that the 3×3 determinant

$$\begin{vmatrix} f(a) & f(b) & f'(c) \\ g(a) & g(b) & g'(c) \\ h(a) & h(b) & h'(c) \end{vmatrix}$$

is zero, that is, $f(a)\big(g(b)h'(c) - h(b)g'(c)\big) - f(b)\big(g(a)h'(c) - h(a)g'(c)\big) + f'(c)\big(g(a)h(b) - h(a)g(b)\big) = 0$. Deduce that if $h(x) = 1$ for all $x \in [a, b]$, we obtain the conclusion of the Cauchy Mean Value Theorem (Proposition 4.38). What does the result say if $g(x) = x$ and $h(x) = 1$ for all $x \in [a, b]$?

4.38. Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If there is $\alpha \in \mathbb{R}$ such that $|f'(x)| \le \alpha |g'(x)|$ for all $x \in (a, b)$ and if $g'(x) \neq 0$ for all $x \in (a, b)$, then show that

$$|f(b) - f(a)| \le \alpha |g(b) - g(a)|.$$

Is this true if the condition $g'(x) \neq 0$ for all $x \in (a, b)$ is omitted?

4.39. Evaluate the following limits:

(i) $\displaystyle\lim_{x\to 1} \frac{(2x - x^4)^{1/2} - x^{1/3}}{1 - x^{3/4}}$, (ii) $\displaystyle\lim_{x\to\infty} \frac{5x^2 - 3x}{7x^2 + 1}$,

(iii) $\displaystyle\lim_{x\to\infty} \left(x - \sqrt{x + x^2}\right)$, (iv) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x+2}}{\sqrt{x+1}}$.

4.40. Show that if $f, g : \mathbb{R} \to \mathbb{R}$ are functions defined by

$$f(x) = \begin{cases} x + 2 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x + 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

then

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 1 \quad \text{but} \quad \lim_{x\to 0} \frac{f(x)}{g(x)} = 2.$$

Does this contradict L'Hôpital's Rule?

4.41. Consider the following application of L'Hôpital's Rule:

$$\lim_{x\to 1} \frac{3x^2 - 2x - 1}{x^2 - x} = \lim_{x\to 1} \frac{6x - 2}{2x - 1} = \lim_{x\to 1} \frac{6}{2} = 3.$$

Is it correct? Justify.

4.42. Define $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by $f(x) := 1/(x - 1)$ for $x \in \mathbb{R} \setminus \{1\}$ and $g(x) := x$ for $x \in \mathbb{R}$. Show that

$$\frac{f'(x)}{g'(x)} \to -\infty \text{ as } x \to 1^+, \quad \text{but} \quad \frac{f(x)}{g(x)} \to \infty \text{ as } x \to 1^+.$$

Does this contradict L'Hôpital's Rule? Justify.

Part B

4.43. Show that the Thomae function f defined in Example 3.34 is not differentiable at any $y \in [0,1]$. (Hint: If $y \notin \mathbb{Q}$, consider (b_n) and (y_n) as in Exercise 2.29, and let $h_n := y - 0.b_1b_2\ldots b_n$ for $n \in \mathbb{N}$. Then (h_n) is a sequence in $\mathbb{R} \setminus \mathbb{Q}$ with $h_n \to 0$ and $f(y - h_n)/h_n \geq 1$ for all $n \in \mathbb{N}$.)

4.44. Let $f : (a,b) \to \mathbb{R}$ and $c \in (a,b)$. Show that the following are equivalent:
 (i) f is differentiable at c.
 (ii) There exist $\alpha \in \mathbb{R}$, $\delta > 0$, and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that

$$f(c+h) = f(c) + \alpha h + h\epsilon_1(h) \text{ for all } h \in (-\delta,\delta) \text{ and } \lim_{h\to 0} \epsilon_1(h) = 0.$$

 (iii) There exists $\alpha \in \mathbb{R}$ such that $\displaystyle\lim_{h\to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = 0.$

If the above conditions hold, then show that $f'(c) = \alpha$.

4.45. Let C be an algebraic plane curve, that is, let C be implicitly defined by $F(x,y) = 0$, where $F(x,y)$ is a nonzero polynomial in two variables x and y with coefficients in $\mathbb{R}$. Let the (total) degree of $F(x,y)$ be n. Let $P = (x_0, y_0)$ be a point on C, so that $F(x_0, y_0) = 0$.
 (i) If we let $X := x - c$ and $Y := y - d$ and define $g(X,Y) := f(x,y)$, then show that $g(X,Y)$ is a polynomial in X and Y with $g(0,0) = 0$. Deduce that there is a unique $m \in \mathbb{N}$ such that $m \leq n$ and

$$g(X,Y) = g_m(X,Y) + g_{m+1}(X,Y) + \cdots + g_n(X,Y),$$

 where $g_i(X,Y)$ is either the zero polynomial or a nonzero homogeneous polynomial of degree i, for $m \leq i \leq n$, and $g_m(X,Y) \neq 0$. We denote the integer m by $\mathrm{mult}_P(C)$, and call it the **multiplicity** of C at the point P.
 (ii) Show that a tangent to the curve C at the point P is defined (as far as calculus is concerned) if and only if $\mathrm{mult}_P(C) = 1$. Moreover, if $\mathrm{mult}_P(C) = 1$, then there are $\alpha_1, \beta_1 \in \mathbb{R}$ such that $g_1(X,Y) = \alpha_1 X + \beta_1 Y$, and then the line $\alpha_1(x-c) + \beta_1(y-d) = 0$ is the tangent to C at P.
 (iii) If $F(x,y) := y - f(x)$ for some polynomial $f(x)$ in one variable x, and if C denotes the corresponding curve given by $F(x,y) = 0$, then show that $\mathrm{mult}_P(C) = 1$ for every P on C.
 (iv) Determine the integer $m = \mathrm{mult}_P(C)$ and a factorization of $g_m(X,Y)$ when $P = (0,0)$ and C is the curve implicitly defined by $F(x,y) := y^2 - x^2 - x^3 = 0$, or by $F(x,y) := y^2 - x^3 = 0$.

[Note: In the algebraic approach to tangents, the tangent lines to the curve C at the point P are the lines given by $\alpha(x-c) + \beta(y-d) = 0$, where α, β are such that $\alpha X + \beta Y$ is a factor of $g_m(X,Y)$.]

4.46. Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I and differentiable at every interior point of I. If there is $\alpha \in \mathbb{R}$ such that $|f'(x)| \leq \alpha$ for all interior points x of I, then show that f is uniformly continuous on I. Is the converse true? In other words, is it true that if $f : I \to \mathbb{R}$ is uniformly continuous on I and differentiable at every interior point of I, then there is a constant α such that $|f'(x)| \le \alpha$ for all interior points x of I?

4.47. Let $f : [a, b] \to \mathbb{R}$ be such that f' is continuous on $[a, b]$ and f'' exists on (a, b). Given any $\xi \in [a, b]$, show that there is $c \in (a, b)$ such that

$$f(\xi) - f(a) = \frac{f(b) - f(a)}{b - a}(\xi - a) + \frac{f''(c)}{2}(\xi - a)(\xi - b).$$

4.48. Let $f(x)$ be a polynomial. Then $c \in \mathbb{R}$ is called a **root** of $f(x)$ of **multiplicity** m if $f(x) = (x - c)^m g(x)$ for some polynomial $g(x)$ with $g(c) \neq 0$.

(i) Let $f(x)$ have r roots (counting multiplicities) in an open interval (a, b). Show that the polynomial $f'(x)$ has at least $r-1$ roots in (a, b). Also, give an example in which $f'(x)$ has more than $r - 1$ roots in (a, b). More generally, for $k \in \mathbb{N}$, show that the polynomial $f^{(k)}(x)$ has at least $r - k$ roots in (a, b).

(ii) If $f^{(k)}(x)$ has s roots in (a, b), what can you conclude about the number of roots of $f(x)$ in (a, b)?

4.49. Let $f(x)$ be a polynomial of degree n. Given any $a \in \mathbb{R}$, show that

$$f(x) = f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n, \quad \text{for } x \in \mathbb{R}.$$

Deduce that a is a root of $f(x)$ of multiplicity m if and only if $f(a) = f'(a) = \cdots = f^{(m-1)}(a) = 0$ and $f^{(m)}(a) \neq 0$. Further, show that if a is a root of f of multiplicity m, then

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h^m} = \frac{f^{(m)}(a)}{m!}.$$

4.50. Give an alternative proof of the Taylor Theorem with a single application of the Rolle Theorem by proceeding as follows. Let the notation and hypotheses be as in the statement of the Taylor Theorem (Proposition 4.25). Also, as in the proof of the Taylor Theorem, for $x \in [a, b]$, let

$$P(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = f(x) + f'(x)(b - x) + \frac{f''(x)}{2!}(b - x)^2 + \cdots + \frac{f^{(n)}(x)}{n!}(b - x)^n + s(b - x)^{n+1},$$

where $s = \big(f(b) - P(b)\big)/(b - a)^{n+1}$. Show that $g(a) = g(b) = f(b)$. Apply the Rolle Theorem to g to deduce the Taylor Theorem.

4.51. Let the notation and hypotheses be as in the Taylor Theorem (Proposition 4.25). If $p \in \mathbb{N}$ with $p \le n+1$, then show that there is $c \in (a,b)$ such that

$$f(b) = f(a)+f'(a)(b-a)+\cdots+\frac{f^{(n)}(a)}{n!}(b-a)^n+\frac{f^{(n+1)}(c)}{n!p}(b-a)^p(b-c)^{n-p+1}.$$

(Hint: Proceed as in the previous exercise, except change the $(n+1)$th power to the pth power in the definitions of $g(x)$ and s.) Show that the Taylor Theorem is a special case of this result with $p = n+1$. Further, show that if I is an interval containing more than one point, $a \in I$, and $f : I \to \mathbb{R}$ is such that $f', f'', \ldots, f^{(n)}$ exist on I and $f^{(n+1)}$ exists at each interior point of I, then given $x \in I$, there is c between a and x such that

$$f(x) = P_n(x)+R_{n,p}(x), \quad \text{where} \quad R_{n,p}(x) = \frac{f^{(n+1)}(c)}{n!p}(x-a)^p(x-c)^{n-p+1},$$

and where $P_n(x)$ is the nth Taylor polynomial of f around a.
[Note: The remainder term in the above result, namely $R_{n,p}(x)$, is called the **Schlömilch form of the remainder**. It reduces to the Lagrange form of remainder when $p = n+1$, whereas it is called the **Cauchy form of the remainder** when $p = 1$.]

4.52. Let I be an interval containing more than one point and let $f : I \to \mathbb{R}$ be a convex function.
   (i) Show that for every interior point c of I, both $f'_-(c)$ and $f'_+(c)$ exist and $f'_-(c) \le f'_+(c)$. (Hint: Exercise 1.63)
   (ii) Show that $f'_+(x_1) \le f'_-(x_2)$ for all $x_1, x_2 \in I$ with $x_1 < x_2$.

4.53. Let $m \in \mathbb{N}$ and $f, g : [a,b] \to \mathbb{R}$ be such that $f, f', \ldots, f^{(m-1)}$ as well as $g, g', \ldots, g^{(m-1)}$ are continuous on $[a,b]$ and $f^{(m)}$, $g^{(m)}$ exist on (a,b). Suppose $f'(a) = f''(a) = \cdots = f^{(m-1)}(a) = 0$ and $g'(a) = g''(a) = \cdots = g^{(m-1)}(a) = 0$, but $g^{(m)}(x) \ne 0$ for all $x \in (a,b)$. Prove that there exist $c_1, \ldots, c_m \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c_1)}{g'(c_1)} = \frac{f''(c_2)}{g''(c_2)} = \cdots = \frac{f^{(m)}(c_m)}{g^{(m)}(c_m)}.$$

[Note: This generalizes the Cauchy Mean Value Theorem.]

4.54. Let $c \in \mathbb{R}$, $r > 0$, and $f : (c-r, c+r) \to \mathbb{R}$ be such that $f''(c)$ exists. Show that

$$\lim_{h \to 0^+} \frac{f(c+h)+f(c-h)-2f(c)}{h^2}$$

exists and is equal to $f''(c)$. Give an example of a differentiable function f on $(c-r, c+r)$ such that the above limit exists, but $f''(c)$ does not exist. (Hint: L'Hôpital's Rule.)

4.55. Let $c \in \mathbb{R}$, $r > 0$, $f : (c-r, c+r) \to \mathbb{R}$, and $n \in \mathbb{N}$ be such that $f^{(n)}(c)$ exists. Use the L'Hôpital's Rule to show that

$$\lim_{h \to 0} \frac{f(c+h)-f(c)-hf'(c)-\cdots-h^{n-1}\left(f^{(n-1)}(c)/(n-1)!\right)}{h^n} = \frac{f^{(n)}(c)}{n!}.$$

5

Applications of Differentiation

The notion of differentiation is remarkably effective in studying the geometric properties of functions. We have seen already how derivatives are useful in determining monotonicity, convexity, or concavity for differentiable functions defined on an interval. We shall study similar applications in this chapter.

First, in Section 5.1 we will see how one can determine the absolute (or global) minimum or maximum of a large class of functions. Next, in Section 5.2 we shall describe a number of useful tests to determine the local minima or maxima of a function and to detect the points of inflection. In this way, we shall be able to locate the ups and downs, the peaks and dips, and the twists and turns in the graph of a real-valued function. This information is extremely useful in curve sketching, that is, in drawing graphs of real-valued functions and identifying their key features. In Section 5.3, we revisit the idea of approximating functions by simpler functions, which we discussed in Chapter 4 in connection with the MVT and the Taylor Theorem. We shall discuss here in greater detail the most widely used methods of approximation, namely linear and quadratic approximations. Finally, in Section 5.4, we discuss a method of Picard for finding fixed points of functions, and a method of Newton for finding zeros of functions.

5.1 Absolute Minimum and Maximum

We have seen in Proposition 3.10 that a continuous real-valued function defined on a closed and bounded subset of $\mathbb{R}$ is bounded and attains its bounds. In other words, if $D \subseteq \mathbb{R}$ is closed and bounded, and $f : D \to \mathbb{R}$ is continuous, then the **absolute minimum** and the **absolute maximum** of f on D, namely

$$m := \min\{f(x) : x \in D\} \quad \text{and} \quad M := \max\{f(x) : x \in D\},$$

exist, and moreover, there are $r, s \in D$ such that $m = f(r)$ and $M = f(s)$. A question that arises naturally is the following: Knowing the function f, how

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does one find the **absolute extrema** m and M, and the points r and s where they are attained? It turns out that we can considerably narrow down the search for the points where the absolute extrema are attained if we look at the derivative of f. To make this precise, let us first formulate a couple of definitions.

Given $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$, a point $c \in D$ is called a **critical point** of f if c is an interior point of D such that either f is not differentiable at c, or f is differentiable at c and $f'(c) = 0$.

Given $D \subseteq \mathbb{R}$, by a **boundary point** of D we shall mean a real number c for which there is a sequence (x_n) of points of D and a sequence (y_n) of real numbers not in D such that $x_n \to c$ and $y_n \to c$. It is easy to see that $c \in \mathbb{R}$ is a boundary point of D if and only if for every $r > 0$, the interval $(c-r, c+r)$ contains a point of D as well as a point not belonging to D. For example, if $D = [a, b]$ or $D = (a, b)$, then the endpoints a and b are the boundary points of D, whereas the points of (a, b) are the interior points of D.

Proposition 5.1. *Let D be a closed and bounded subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$ be continuous. Then the absolute minimum as well as the absolute maximum of f is attained either at a critical point of f or at a boundary point of D.*

Proof. By Proposition 3.10, f attains its absolute minimum as well as its absolute maximum on D. Let $c \in D$ be a point at which the absolute minimum of f is attained. Suppose c is an interior point of D. Then clearly, f has a local minimum at c. Hence if f is differentiable at c, then by Lemma 4.15, $f'(c) = 0$. It follows that c must be a critical point of D. Thus, c is either a critical point of f or a boundary point of D.

A similar argument applies to a point at which the absolute maximum of f is attained. □

In practice, the critical points of a function and the boundary points of its domain are few in number. Thus, in view of the above proposition, we have a simple recipe to determine the absolute extrema and the points where they are attained. Namely, determine the critical points of a function and the boundary points of its domain; then calculate the values at these points, and compare these values. The greatest value among them is the absolute maximum, while the least value is the absolute minimum. This recipe is illustrated by the following examples.

Examples 5.2. (i) Consider $f : [-1, 2] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} -x & \text{if } -1 \le x \le 0, \\ 2x^3 - 4x^2 + 2x & \text{if } 0 < x \le 2. \end{cases}$$

Let us try to find the absolute extrema of f. First note that by Proposition 3.6, f is continuous on $[0, 2]$. Next, f is not differentiable at 0, since $f'_-(0) = -1$ and $f'_+(0) = 2$. On the other hand,

$$f'(x) = \begin{cases} -1 & \text{if } -1 \le x < 0, \\ 6x^2 - 8x + 2 = 2(3x-1)(x-1) & \text{if } 0 < x < 2. \end{cases}$$

So, $f'(x) = 0$ only at $x = \frac{1}{3}$ and $x = 1$. It follows that $x = 0$, $x = \frac{1}{3}$, and $x = 1$ are the only critical points of f. The boundary points of our domain $[-1, 2]$ are -1 and 2. Thus we make the following table:

x	-1	0	$\frac{1}{3}$	1	2
$f(x)$	1	0	$\frac{8}{27}$	0	4

From this we conclude that the absolute minimum of f is 0, which is attained at $x = 0$ as well as at $x = 1$, whereas the absolute maximum of f is 4, which is attained at $x = 2$. Note that although f has a local maximum at $x = \frac{1}{3}$, it is not the absolute maximum of f.

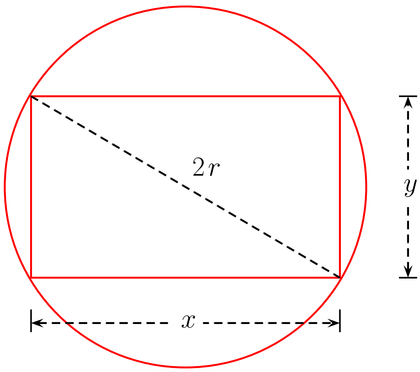

Fig. 5.1. Rectangle inscribed in a circle of radius r.

(ii) Let us show that among all rectangles that can be inscribed in a given circle, the square has the greatest area. Let r be the radius of the given circle. If x is the length and y the breadth of an inscribed rectangle (see Figure 5.1), then $0 \le x, y \le 2r$ and $x^2 + y^2 = (2r)^2 = 4r^2$. Now, the area of the rectangle could be viewed as a function $A : [0, 2r] \to \mathbb{R}$ given by $A(x) := xy = x\sqrt{4r^2 - x^2}$. To compute $A'(x)$, we may use implicit differentiation. For example, the equation $x^2 + y^2 = 4r^2$ implies that

$$2x + 2y\frac{dy}{dx} = 0,$$

and hence at points where $y \ne 0$, that is, $x \ne 2r$, we obtain

$$\frac{dA}{dx} = y + x\frac{dy}{dx} = y - \frac{x^2}{y} = \frac{y^2 - x^2}{y} = \frac{4r^2 - 2x^2}{y}.$$

Hence, for $0 \le x < 2r$,

$$\frac{dA}{dx} = 0 \iff x = \sqrt{2}\,r.$$

Thus $x = \sqrt{2}\,r$ is the only critical point of A, and so we make the following table:

x	0	$\sqrt{2}\,r$	$2r$
$A(x)$	0	$2r^2$	0

From this we conclude that the area $A(x)$ of the rectangle is maximal when $x = \sqrt{2}\,r = y$, that is, when the rectangle is, in fact, a square. $\diamond$

5.2 Local Extrema and Points of Inflection

Heuristically speaking, a local maximum of a function corresponds to a peak or a pinnacle in its graph, while a local minimum is something like a dip or a depression. Let us also recall the formal definition from Chapter 1. If $D \subseteq \mathbb{R}$ and c is an interior point in D, then $f : D \to \mathbb{R}$ is said to have a **local minimum** at c if there is $\delta > 0$ such that

$$(c - \delta, c + \delta) \subseteq D \quad \text{and} \quad f(x) \ge f(c) \text{ for all } x \in (c - \delta, c + \delta).$$

On the other hand, f is said to have a **local maximum** at c if there is $\delta > 0$ such that

$$(c - \delta, c + \delta) \subseteq D \quad \text{and} \quad f(x) \le f(c) \text{ for all } x \in (c - \delta, c + \delta).$$

Also, recall that f is said to have a **strict local minimum** [resp. **strict local maximum**] at c if there is $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq D$ and $f(x) > f(c)$ [resp. $f(x) < f(c)$] for all $x \in (c - \delta, c + \delta)$, $x \ne c$.

Further, f has a **local extremum** at c if f has a local maximum at c or a local minimum at c.

For example, consider a function whose graph looks as in Figure 5.2. In fact, this is the graph of the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 8 & \text{if } x \le -2, \\ x^4 - 2x^2 & \text{if } x \in (-2, 2), \\ 10 - x & \text{if } x \ge 2. \end{cases}$$

We see that at $x = -1$ and $x = 1$, the function has a strict local minimum, whereas at $x = 0$ and $x = 2$, it has a strict local maximum. At $x = -2$, it

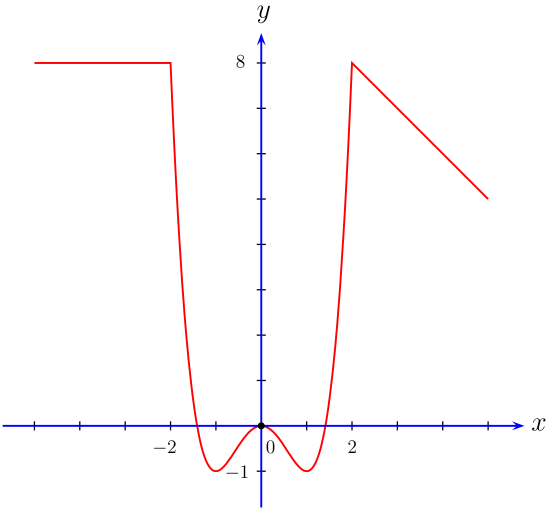

Fig. 5.2. Illustration of local extrema as peaks and dips.

has a local maximum that is not strict. In fact, there is a (nonstrict) local minimum as well as a local maximum at every point in $(-\infty, -2)$ where the function is constant, that is, its graph has a plateau.

To get an idea of the relation between derivatives and the notions of local minimum/maximum, we may look at the behavior of the above graph around its peaks or dips (or even a plateau). We see that as we approach a dip (local minimum) from the left, the graph is decreasing and the tangents have negative slopes, and further, as we continue to the right, the graph is increasing and the tangents have positive slopes. Similarly, as we approach a peak (local maximum) from the left, the graph is increasing and the tangents have positive slopes, and further, as we continue to the right, the graph is decreasing and the tangents have negative slopes. In case the tangent is defined at a peak or a dip, then it is necessarily horizontal, that is, it has slope zero. We have already seen the analytic formulation of the latter property in the form of Lemma 4.15. This lemma says that if $f : D \to \mathbb{R}$ is differentiable at an interior point c of $D \subseteq \mathbb{R}$, then the vanishing of $f'(c)$ is a **necessary condition** for f to have a local extremum at c. We have seen examples (such as $f(x) := x^3$ for $x \in \mathbb{R}$ and $c := 0$) that show that this condition is not sufficient to guarantee a local extremum. However, the above observations about the behavior of the graph lead to some **sufficient conditions for a local extremum**. We first state the result for a local minimum.

Proposition 5.3. *Let $D \subseteq \mathbb{R}$, c an interior point of D, and $f : D \to \mathbb{R}$.*

(i) **(First Derivative Test for a Local Minimum)** *If f is continuous at c, and also,*

(a) *f is differentiable on $(c-r, c) \cup (c, c+r)$ for some $r > 0$, and*
(b) *there is $\delta > 0$ with $\delta \le r$ such that $f'(x) \le 0$ for all $x \in (c-\delta, c)$, and $f'(x) \ge 0$ for all $x \in (c, c+\delta)$,*

then f has a local minimum at c.

(ii) **(Second Derivative Test for a Local Minimum)** *If f is twice differentiable at c and satisfies $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*

Proof. (i) If the conditions in (i) are satisfied and $\delta > 0$ is as in subpart (b) of (i), then by parts (i) and (ii) of Proposition 4.29, we see that f is decreasing on $(c-\delta, c)$ and increasing on $(c, c+\delta)$. So for $x, y \in (c-\delta, c)$ and $z, w \in (c, c+\delta)$,

$$f(x) \ge f(y) \text{ whenever } x < y \quad \text{and} \quad f(z) \le f(w) \text{ whenever } z < w.$$

Since f is continuous at c, by taking limits as $y \to c^-$ and as $z \to c^+$, and using Proposition 3.37, we see that $f(x) \ge f(c)$ for all $x \in (c-\delta, c+\delta)$. Thus, f has a local minimum at c.

(ii) If $f''(c)$ exists, then it is tacitly assumed that f' exists on $(c-r, c+r)$ for some $r > 0$ with $(c-r, c+r) \subseteq D$. Now, if $f'(c) = 0$ and $f''(c) > 0$, then

$$\lim_{x \to c} \frac{f'(x)}{x-c} = \lim_{x \to c} \frac{f'(x) - f'(c)}{x-c} = f''(c) > 0.$$

Thus, by part (i) of Proposition 3.30, there is $\delta > 0$ such that $\delta < r$ and

$$\frac{f'(x)}{x-c} > 0 \quad \text{for all } x \in (c-\delta, c) \cup (c, c+\delta).$$

In view of this, we see that f satisfies the conditions in (i) above. Hence f has a local minimum at c. □

The corresponding result for a local maximum is similar.

Proposition 5.4. *Let $D \subseteq \mathbb{R}$, c an interior point of D, and $f : D \to \mathbb{R}$.*

(i) **(First Derivative Test for a Local Maximum)** *If f is continuous at c, and also,*

(a) *f is differentiable on $(c-r, c) \cup (c, c+r)$ for some $r > 0$, and*
(b) *there is $\delta > 0$ with $\delta \le r$ such that $f'(x) \ge 0$ for all $x \in (c-\delta, c)$, and $f'(x) \le 0$ for all $x \in (c, c+\delta)$,*

then f has a local maximum at c.

(ii) **(Second Derivative Test for a Local Maximum)** *If f is twice differentiable at c and satisfies $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

Proof. Similar to the proof of Proposition 5.3. □

Remarks 5.5. (i) An informal, but easy, way to remember the First Derivative Test for a local extremum is as follows:

$$f' \text{ changes from } - \text{ to } + \text{ at } c \Longrightarrow f \text{ has a local minimum at } c;$$
$$f' \text{ changes from } + \text{ to } - \text{ at } c \Longrightarrow f \text{ has a local maximum at } c.$$

The First Derivative Test does not require f to be differentiable at c. For example, we can use it to ascertain that the absolute value function has a local minimum at 0. However, the assumption about the continuity of f at c cannot be dropped. For example, consider $f : (-1, 1) \to \mathbb{R}$ defined by $f(0) := 0$ and $f(x) := 1 - x$ if $x \in (-1, 0)$, while $f(x) := x - 1$ if $x \in (0, 1)$. Then f is decreasing on $(-1, 0)$ and increasing on $(0, 1)$, but f does not have a local minimum at 0. Also $-f$ is is increasing on $(-1, 0)$ and decreasing on $(0, 1)$, but it does not have a local maximum at 0.

(ii) The Second Derivative Test for a local extremum is valid under a restrictive hypothesis, namely twice differentiability, and usually needs more checking (values of both the derivatives). But it has the advantage of being short and easy to remember.

(iii) While the First Derivative Test and the Second Derivative Test provide sufficient conditions for a local extremum, neither of them is necessary, that is, a function can have a local extremum at a point but may not satisfy the hypotheses of either of these tests. We have, in fact, seen in Chapter 1 an example of a function, namely the piecewise linear zigzag function in Example 1.20, that has a local minimum at 0 but there is no $\delta > 0$ such that the function is decreasing on $(-\delta, 0]$ and increasing on $[0, \delta)$. It can easily be seen that this function does not satisfy the hypotheses of the First Derivative Test as well as of the Second Derivative Test, even though it has a local minimum at 0. Another such example is given in Exercise 7.44. Some simpler examples appear in Examples 5.6 (ii) and (iii) below. Notice that the negatives of these functions provide examples of functions that have a local maximum but do not satisfy the hypotheses of any of the tests.

(iv) If in subpart (b) of the First Derivative Test for a Local Minimum, we change the inequalities $f'(x) \le 0$ and $f'(x) \ge 0$ to the corresponding strict inequalities $f'(x) < 0$ and $f'(x) > 0$, then we can conclude that f has a strict local minimum at the corresponding point. A similar comment holds in the case of a local maximum. More generally, we can reach the conclusion about f having a strict local extremum if in addition to the hypotheses of the First Derivative Test, we require that f' does not vanish identically on any subinterval of $(c - \delta, c)$ or of $(c, c + \delta)$ containing more than one point. Thus f' is allowed to vanish at a few stray points but not on a continuous segment with c as one of its endpoints. On the other hand, if the hypotheses of the Second Derivative Test is satisfied, then the function has, in fact, a strict local extremum. These assertions are easily proved by gleaning through the proofs of Propositions 5.3 and 5.4 and using Propositions 4.29 and 4.32. $\diamond$

Examples 5.6. (i) Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \frac{1}{x^4 - 2x^2 + 7}.$$

Note that since $x^4 - 2x^2 + 7 = (x^2 - 1)^2 + 6 > 0$ for all $x \in \mathbb{R}$, the function f is well-defined and differentiable on $\mathbb{R}$. Moreover,

$$f'(x) = \frac{-(4x^3 - 4x)}{(x^4 - 2x^2 + 7)^2} = \frac{-4x(x-1)(x+1)}{(x^4 - 2x^2 + 7)^2} \quad \text{for } x \in \mathbb{R}.$$

Thus, f' vanishes only at $x = -1, 0, 1$. Now we can make a table as follows:

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Sign of f'	$+$	$-$	$+$	$-$

In view of this, from the First Derivative Test, we can conclude that f has a local minimum at $x = 0$ and local maxima at $x = -1$ and $x = 1$. Notice that in this example, it would be quite complicated to compute f'' and use the Second Derivative Test.

(ii) Consider $f : (-1, 1) \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x^2 & \text{if } 0 < |x| < 1, \\ -1 & \text{if } x = 0. \end{cases}$$

Then it is clear that $f(0) < f(x)$ for all nonzero $x \in (-1, 1)$, and thus f has a strict local minimum at $x = 0$. However, the conditions of the First Derivative Test are not satisfied. Indeed, f is differentiable on $(-1, 0)$ as well as on $(0, 1)$ and f' changes sign from $-$ to $+$ at $x = 0$, but f is not continuous at 0.

(iii) Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^4$. Then $f(0) = 0 < f(x)$ for all nonzero $x \in \mathbb{R}$, and thus f has a strict local minimum at $x = 0$. However, the conditions of the Second Derivative Test are not satisfied. Indeed, f is twice differentiable and $f'(0) = 0$, but $f''(0)$ is not positive. ◇

Points of Inflection

We shall now move on to a more subtle attribute of (the graph of) a real-valued function, namely, the geometric notion of a point of inflection, which was defined in Chapter 1. Briefly, this is a point at which convexity changes to concavity or vice versa. More precisely, given any $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$, an interior point $c \in D$ is said to be a **point of inflection** for f if there is $\delta > 0$ with $(c - \delta, c + \delta) \subseteq D$ such that f is convex on $(c - \delta, c)$ and concave on $(c, c + \delta)$, or vice versa. Also, recall that c is said to be a **strict point of inflection** of f if there is $\delta > 0$ with $(c - \delta, c + \delta) \subseteq \mathbb{R}$ such that f is strictly convex on $(c - \delta, c)$ and strictly concave on $(c, c + \delta)$, or vice versa.

A typical example is $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := x^3$ (see Figure 1.5 (iv)), for which 0 is a point of inflection; in fact, 0 is a strict point of inflection for this function.

Characterizations of convexity and concavity in terms of derivatives discussed in Chapter 4 lead to the following result about points of inflection.

Proposition 5.7 (Necessary and Sufficient Conditions for a Point of Inflection). *Let $D \subseteq \mathbb{R}$, c an interior point of D, and $f : D \to \mathbb{R}$.*

(i) *Suppose f is differentiable on $(c-r, c) \cup (c, c+r)$ for some $r > 0$. Then c is a point of inflection for f if and only if there is $\delta > 0$ with $\delta \le r$ such that f' is monotonically increasing on $(c - \delta, c)$, whereas f' is monotonically decreasing on $(c, c + \delta)$, or vice versa.*

(ii) *Suppose f is twice differentiable on $(c - r, c) \cup (c, c + r)$ for some $r > 0$. Then c is a point of inflection for f if and only if there is $\delta > 0$ with $\delta \le r$ such that f'' is nonnegative throughout $(c-\delta, c)$, whereas f'' is nonpositive throughout $(c, c + \delta)$, or vice versa.*

Proof. Part (i) follows from parts (i) and (ii) of Proposition 4.33, while part (ii) follows from parts (i) and (ii) of Proposition 4.34. □

The above results can be used to obtain weaker but concise conditions that are necessary or sufficient for an interior point in the domain of a function to be a point of inflection.

Proposition 5.8. *Let $D \subseteq \mathbb{R}$, c an interior point of D, and $f : D \to \mathbb{R}$.*

(i) **(Necessary Condition for a Point of Inflection)** *Let f be twice differentiable at c. If c is a point of inflection for f, then $f''(c) = 0$.*

(ii) **(Sufficient Conditions for a Point of Inflection)** *Let f be thrice differentiable at c. If $f''(c) = 0$ and $f'''(c) \neq 0$, then c is a point of inflection for f.*

Proof. (i) If $f''(c)$ exists, then it is tacitly assumed that f' exists on $(c-r, c+r)$ for some $r > 0$ with $(c - r, c + r) \subseteq D$. If c is a point of inflection for f, then by part (i) of Proposition 5.7, there is $\delta > 0$ with $\delta \le r$ such that

$$f' \text{ is increasing on } (c - \delta, c) \text{ and decreasing on } (c, c + \delta), \text{ or vice versa.}$$

Now f', being differentiable at c, is continuous at c, and therefore,

$$f'(x) \le f'(c) \text{ for all } x \in (c - \delta, c) \quad \text{and} \quad f'(c) \ge f'(x) \text{ for all } x \in (c, c + \delta),$$

or vice versa (that is, the inequalities $\le$ and $\ge$ above are interchanged). In particular, f' has a local extremum at c. Hence by Lemma 4.15, $f''(c) = 0$.

(ii) If $f'''(c)$ exists, then it is tacitly assumed that f'' exists on $(c-r, c+r)$ for some $r > 0$ with $(c - r, c + r) \subseteq D$. Suppose now that $f''(c) = 0$ and $f'''(c) \neq 0$. We may first assume that $f'''(c) < 0$. Then

$$\lim_{x \to c} \frac{f''(x)}{x - c} = \lim_{x \to c} \frac{f''(x) - f''(c)}{x - c} = f'''(c) < 0.$$

Hence by part (i) of Proposition 3.30, there is $\delta > 0$ with $\delta \leq r$ such that $f''(x)/(x - c) < 0$ for all $x \in (c - \delta, c) \cup (c, c + \delta)$. Consequently,

$$f''(x) > 0 \text{ for all } x \in (c - \delta, c) \quad \text{and} \quad f''(x) < 0 \text{ for all } x \in (c, c + \delta).$$

Thus, by part (ii) of Proposition 5.7, c is a point of inflection for f. A similar argument holds if $f''(c) = 0$ and $f'''(c) > 0$. □

Remarks 5.9. (i) The condition in part (i) of the above proposition is not sufficient. Consider, for example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^4$. Then 0 is not a point of inflection for f, but $f''(0) = 0$.

(ii) The condition in part (ii) is not necessary. Consider, for example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^5$. Then 0 is a point of inflection for f, but $f'''(0) = 0$.

(iii) If in part (i) of Proposition 5.7 we change the words "monotonically increasing" and "monotonically decreasing" to "strictly increasing" and "strictly decreasing", respectively, then we obtain a necessary and sufficient condition that c is a strict point of inflection for f. Likewise, in part (ii) of Proposition 5.7, if in addition to the condition about the sign of f'', we require that f'' not vanish identically on any subinterval of $(c - \delta, c)$ or of $(c, c + \delta)$ containing more than one point, then we obtain a necessary and sufficient condition for c to be a strict point of inflection for f. On the other hand, if the sufficient condition in part (ii) of Proposition 5.8 is satisfied, then c is, in fact, a strict point of inflection for f. These assertions are easily proved by gleaning through the proofs of Propositions 5.7 and 5.8 and appealing to parts (iii) and (iv) of Proposition 4.33 as well as parts (i) and (ii) of Proposition 4.37. ◇

As an illustration of the various tests obtained in this section and also in Section 4.3, let us work out an example in which we first use these tests to identify several features of the function or its graph. We shall also see how, equipped with this knowledge, one can make a rough sketch of the graph.

Example 5.10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^3 - 6x^2 + 9x + 1$. Then

$$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3) \quad \text{and} \quad f''(x) = 6x - 12 = 6(x - 2).$$

Thus, $f'(x) = 0$ only at $x = 1$ and 3, while $f''(x) = 0$ only at $x = 2$. Moreover, we can make tables as follows:

Interval	$(-\infty, 1)$	$(1, 3)$	$(3, \infty)$
Sign of f'	$+$	$-$	$+$

Interval	$(-\infty, 2)$	$(2, \infty)$
Sign of f''	$-$	$+$

In view of this (together with Propositions 4.29, 5.3, 5.4, 4.33, 5.7, and 5.8), we obtain the following:

- f is (strictly) increasing on $(-\infty, 1)$ as well as on $(3, \infty)$, and f is (strictly) decreasing on $(1, 3)$.
- f has a (strict) local maximum at $x = 1$ and a (strict) local minimum at $x = 3$.
- f is (strictly) concave on $(-\infty, 2)$ and (strictly) convex on $(2, \infty)$.
- 2 is a (strict) point of inflection for f.

Now we can make a rough sketch of the curve $y = f(x)$ by plotting a few points (for example, $f(0) = 1$, $f(1) = 5$, $f(2) = 3$, $f(3) = 1$, and $f(4) = 5$) and using the above facts. It will look like the graph in Figure 5.3. $\diamond$

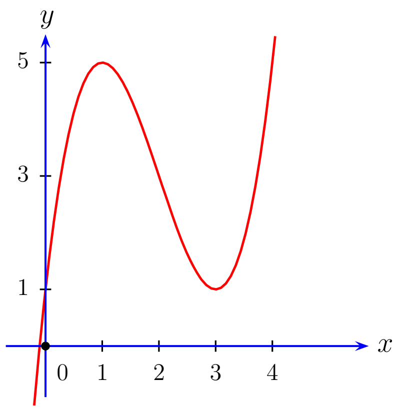

Fig. 5.3. Graph of $y = x^3 - 6x^2 + 9x + 1$

5.3 Linear and Quadratic Approximations.

By way of motivating the Taylor Theorem, we discussed in Chapter 4 how the MVT and its generalizations are helpful in evaluating functions approximately. In this section, we shall formalize these aspects and give some basic features of the simplest of such approximations that are used in practice, namely, the linear and quadratic approximations.

Let $D \subseteq \mathbb{R}$ and let c be an interior point of D. If $f : D \to \mathbb{R}$ is differentiable at c, then the function $L : \mathbb{R} \to \mathbb{R}$ defined by

$$L(x) := f(c) + f'(c)(x - c) \quad \text{for } x \in \mathbb{R}$$

is called the **linear approximation** to f around c. Note that $L(x)$ is the first Taylor polynomial of f around c. Geometrically speaking, $y = L(x)$ represents a line, which is precisely the tangent to the curve $y = f(x)$ at the point

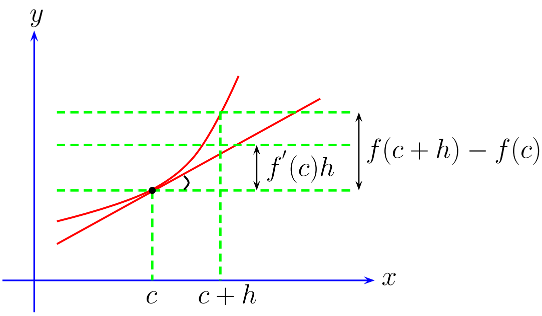

Fig. 5.4. Linear approximation or tangent line approximation around a point.

$(c, f(c))$. For this reason, L is also called the **tangent line approximation** to f around c.
The difference

$$e_1(x) := f(x) - L(x) \quad \text{for } x \in D$$

is called the **error** at x in the linear approximation to f around c.

Proposition 5.11. *Let $D \subseteq \mathbb{R}$ and let c be an interior point of D. If a function $f : D \to \mathbb{R}$ is differentiable at c, then the linear approximation L to f around c is indeed an approximation to f around c, that is,*

$$\lim_{x \to c} L(x) = f(c), \quad \textit{or equivalently,} \quad \lim_{x \to c} e_1(x) = 0.$$

In fact, $e_1(x)$ rapidly approaches zero as $x \to c$ in the sense that

$$\lim_{x \to c} \frac{e_1(x)}{x - c} = 0.$$

Moreover, given any $b \in D$ with $b \neq c$, if I_b denotes the open interval with c and b as its endpoints and if f' exists and is continuous on I_b as well as its endpoints c and b, f'' exists on I_b, and $|f''(x)| \leq M_2(b)$ for all $x \in I_b$, then a bound for the error is given by

$$|e_1(b)| \leq \frac{M_2(b)}{2} |b - c|^2.$$

Proof. It is obvious from the definition of L that

$$\lim_{x \to c} L(x) = f(c), \quad \text{or equivalently,} \quad \lim_{x \to c} e_1(x) = 0.$$

The assertion about rapid vanishing of $e_1(x)$ follows by noting that

$$\lim_{x \to c} \frac{e_1(x)}{x - c} = \lim_{x \to c} \frac{f(x) - f(c) - f'(c)(x - c)}{x - c} = f'(c) - f'(c) = 0.$$

Finally, if $b \in D$ and $b \neq c$, then applying the Taylor Formula (Remark 4.26), with $I = I_b$ and $n = 1$, we see that there is ξ between c and b such that

$$e_1(b) = f(b) - f(c) - f'(c)(b-c) = \frac{f''(\xi)}{2}(b-c)^2.$$

This implies the desired error bound for $|e_1(b)|$. □

Example 5.12. Let $D := \{x \in \mathbb{R} : x \neq 1\}$ and consider $f : D \to \mathbb{R}$ defined by $f(x) := 1/(1-x)$. Then $f'(x) = 1/(1-x)^2$ for $x \in D$, and in particular, $f'(0) = 1$. Thus, the linear approximation to f around 0 is given by

$$L(x) = f(0) + f'(0)(x-0) = 1 + x \quad \text{for } x \in \mathbb{R}.$$

See Figure 5.5. The error bound e_1 for $f - L$ in this case can be worked out as follows. Given $b \in (-1, 1)$, $b \neq 0$, let us consider two cases. First, suppose $b > 0$ and $I_b = (0, b)$. Then

$$|f''(x)| = \frac{2}{(1-x)^3} \leq \frac{2}{(1-b)^3} \quad \text{for } x \in I_b.$$

Thus, in this case we may take $M_2(b) = 2/(1-b)^3$ and by Proposition 5.11 conclude that $|e_1(b)| \leq b^2/(1-b)^3$. For instance, when $0 < b < 0.1$, we obtain $|e_1(b)| \leq (0.1)^2/(0.9)^2 < 0.014$. Next, suppose $b < 0$ and $I_b = (b, 0)$. Then

$$|f''(x)| = \frac{2}{(1-x)^3} \leq 2 \quad \text{for } x \in I_b.$$

So in this case we may take $M_2(b) = 2$ and by Proposition 5.11 conclude that $|e_1(b)| \leq b^2$. For instance, if $-0.1 < b < 0$, then $|e_1(b)| \leq (0.1)^2 = 0.01$. ◇

As the above example shows, linear approximation gives a reasonable approximation to the values of a function around a point where it is differentiable. However, if one wants to do better, then one may take recourse to quadratic approximation, which is available provided the relevant second derivative exists.

Let, as before, $D \subseteq \mathbb{R}$ and let c be an interior point of D. If $f : D \to \mathbb{R}$ is twice differentiable at c, then the function $Q : \mathbb{R} \to \mathbb{R}$ defined by

$$Q(x) := f(c) + (x-c)f'(c) + \frac{(x-c)^2}{2} f''(c) \quad \text{for } x \in \mathbb{R}$$

is called the **quadratic approximation** to f around c. Note that $Q(x)$ is the second Taylor polynomial of f around c. Geometrically speaking, $y = Q(x)$ represents a parabola passing through the point $(c, f(c))$ such that this parabola and the curve $y = f(x)$ have a common tangent at $(c, f(c))$. The difference

$$e_2(x) := f(x) - Q(x) \quad \text{for } x \in D$$

is called the **error** at x in the quadratic approximation to f around c.

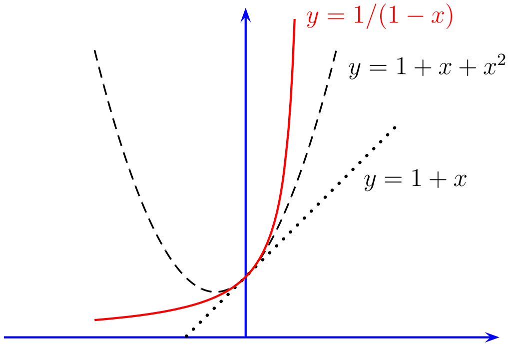

Fig. 5.5. Linear and quadratic approximations to $f(x) := 1/(1-x)$ around 0.

Proposition 5.13. *Let $D \subseteq \mathbb{R}$ and c be an interior point of D. If $f : D \to \mathbb{R}$ is twice differentiable at c, then the quadratic approximation Q to f around c is indeed an approximation to f around c, that is,*

$$\lim_{x\to c} Q(x) = f(c), \quad \textit{or equivalently,} \quad \lim_{x\to c} e_2(x) = 0.$$

In fact, $e_2(x)$ approaches zero as $x \to c$ doubly rapidly in the sense that

$$\lim_{x\to c} \frac{e_2(x)}{(x-c)^2} = 0.$$

Moreover, given any $b \in D$ with $b \neq c$, if I_b denotes the open interval with c and b as its endpoints and if f'' exists and is continuous on I_b as well as at its endpoints c and b, f''' exists on I_b, and $|f'''(x)| \leq M_3(b)$ for all $x \in I_b$, then we obtain the following error bound:

$$|e_2(b)| \leq \frac{M_3(b)}{3!} |b-c|^3.$$

Proof. It is obvious from the definition of Q that

$$\lim_{x\to c} Q(x) = f(c), \quad \text{or equivalently,} \quad \lim_{x\to c} e_2(x) = 0.$$

The assertion about doubly rapid vanishing of $e_2(x)$ follows by noting that by L'Hôpital's Rule for $\frac{0}{0}$ indeterminate forms,

$$\lim_{x\to c} \frac{e_2(x)}{(x-c)^2} = \lim_{x\to c} \frac{f'(x) - f'(c) - f''(c)(x-c)}{2(x-c)} = \frac{1}{2}\big(f''(c) - f''(c)\big) = 0.$$

Finally, if $b \in D$ and $b \neq c$, then applying the Taylor Formula (Remark 4.26), with $I = I_b$ and $n = 2$, we see that there is $\eta \in I_b$ such that

$$e_2(b) = f(b) - f(c) - f'(c)(b-c) - \frac{f''(c)}{2}(b-c)^2 = \frac{f'''(\eta)}{3!}(b-c)^3.$$

This implies the desired inequality for $|e_2(b)|$. $\square$

Now let us revisit Example 5.12 and see what the quadratic approximation and the corresponding error bound look like.

Example 5.14. Consider $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) := 1/(1-x)$ and $c := 0$. Then $f'(x) = 1/(1-x)^2$ and $f''(x) = 2/(1-x)^3$ for $x \in (-1, 1)$. In particular, $f'(0) = 1$ and $f''(0) = 2$. Thus, the quadratic approximation to f around 0 is given by

$$Q(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 = 1 + x + x^2.$$

See Figure 5.5. The error bound e_2 for $f - Q$ in this case can be worked out as follows. Given $b \in (-1, 1)$, $b \neq 0$, let us consider two cases. First, suppose $b > 0$ and $I_b = (0, b)$. Then

$$|f'''(x)| = \frac{6}{(1-x)^4} \leq \frac{6}{(1-b)^4} \quad \text{for } x \in I_b.$$

Thus, in this case we may take $M_3(b) = 6/(1-b)^4$ and by Proposition 5.13 conclude that $|e_2(b)| \leq |b|^3/(1-b)^4$. For instance, when $0 < b < 0.1$, we obtain $|e_2(b)| \leq (0.1)^3/(0.9)^4 < 0.0016$. Next, suppose $b < 0$ and $I_b = (b, 0)$. Then

$$|f''(x)| = \frac{6}{(1-x)^4} \leq 6 \quad \text{for } x \in I_b.$$

So in this case we may take $M_3(b) = 6$ and by Proposition 5.13 conclude that $|e_2(b)| \leq |b|^3$. For instance, if $-0.1 < b < 0$, then $|e_2(b)| \leq (0.1)^3 = 0.001$. $\diamond$

It may be noted that in the above example, the estimates have become sharper than those in Example 5.12. In a similar way, if we were to consider cubic approximations, quartic approximations, and so on, the estimates would become more and more sharp. These higher-degree approximations and the corresponding error bounds can be obtained in an analogous manner. See Exercise 5.21.

5.4 Picard and Newton Methods

The title of this section refers to methods that can be used to obtain approximate solutions to the following two interrelated problems:

- The problem of finding a **fixed point** of a function, namely, if $D \subseteq \mathbb{R}$ and $f : D \to D$, then the problem is to find $x \in D$ such that $f(x) = x$.
- The problem of finding a solution of an equation, namely, if $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$, then the problem is to find $x \in D$ such that $f(x) = 0$.

To see that these problems are interrelated, it suffices to note that if $f : D \to D$ and if we set $F(x) = f(x) - x$, then finding a fixed point of f is equivalent to finding a solution to $F(x) = 0$. On the other hand, if $f : D \to \mathbb{R}$ and if we set $F(x) = x + h(x)f(x)$, where $h : D \to \mathbb{R}$ is so chosen that $h(x) \neq 0$ and $x + h(x)f(x) \in D$ for all $x \in D$, then finding a solution of $f(x) = 0$ is equivalent to finding a fixed point of $F : D \to D$.

Finding a Fixed Point

Let us first take up the problem of finding fixed points. For simplicity, we shall restrict ourselves to the case of functions from a closed and bounded interval of $\mathbb{R}$ into itself. In this case, the existence of a fixed point is guaranteed if the function satisfies a mild condition such as continuity.

Proposition 5.15. *If $f : [a, b] \to [a, b]$ is continuous, then f has a fixed point.*

Proof. Let $F : [a, b] \to \mathbb{R}$ be defined by $F(x) = f(x) - x$. Since $a \le f(x) \le b$ for all $x \in [a, b]$, we see that

$$F(a) = f(a) - a \ge 0 \quad \text{and} \quad F(b) = f(b) - b \le 0.$$

Also, since f is continuous, so is F. Hence by Proposition 3.16, F has the IVP on $[a, b]$. So, there is $c \in [a, b]$ such that $F(c) = 0$, that is, $f(c) = c$. □

Examples 5.16. (i) While a fixed point in $[a, b]$ exists for a continuous function $f : [a, b] \to [a, b]$, it need not be unique. Consider, for example, $f : [0, 1] \to [0, 1]$ defined by $f(x) := x$, where every point of $[0, 1]$ is a fixed point of f.

(ii) The condition that f be defined on a closed subset of $\mathbb{R}$ is essential for the existence of a fixed point. For example, if $f : [0, 1) \to \mathbb{R}$ is defined by $f(x) := (1 + x)/2$, then f maps $[0, 1)$ into itself, and f is continuous. But f has no fixed point in $[0, 1)$. Indeed, $(1 + x)/2 = x$ only when $x = 1$.

(iii) The condition that f be defined on a bounded subset of $\mathbb{R}$ is essential for the existence of a fixed point. For example, if $f : [1, \infty) \to \mathbb{R}$ is defined by $f(x) := x + (1/x)$, then f maps $[1, \infty)$ into itself, and f is continuous. But clearly, f has no fixed point in $[1, \infty)$.

(iv) The condition that f be defined on an interval in $\mathbb{R}$ is essential for the existence of a fixed point. For example, if $D = [-2, -1] \cup [1, 2]$ and $f : D \to \mathbb{R}$ is defined by $f(x) := -x$, then f maps D into itself, and f is continuous. But f has no fixed point in D. ◇

Suppose we know that a function $f : [a, b] \to [a, b]$ has a fixed point. Then a natural question is whether we can find it. It is not easy, in general, to find it exactly. A simple and effective method given by Picard seeks to achieve what may be the next best alternative to finding a fixed point exactly, namely, to find it approximately. In geometric terms, the basic idea of the Picard method can be described as follows.

First, pick any point $P_0 = (x_0, f(x_0))$ on the curve $y = f(x)$. Project P_0 horizontally to a point Q_0 on the diagonal line $y = x$, and then project the point Q_0 vertically onto the curve $y = f(x)$ to obtain a point $P_1 = (x_1, f(x_1))$. Again, project P_1 horizontally to Q_1 on $y = x$ and then project Q_1 vertically onto $y = f(x)$ to obtain $P_2 = (x_2, f(x_2))$. This process can be be repeated a number of times. Often, it will weave a cobweb in which the fixed point of f, that is, the point of intersection of the curve $y = f(x)$ and the diagonal line $y = x$, gets trapped. See Figure 5.6 (i). In fact, we shall see that such trapping occurs if the slopes of the tangents to the curve $y = f(x)$ are smaller (in absolute value) than the slope of the diagonal line $y = x$. When the slope condition is not met, then the points $P_0, P_1, P_2, \ldots$ may move away from a fixed point. See Figure 5.6 (ii).

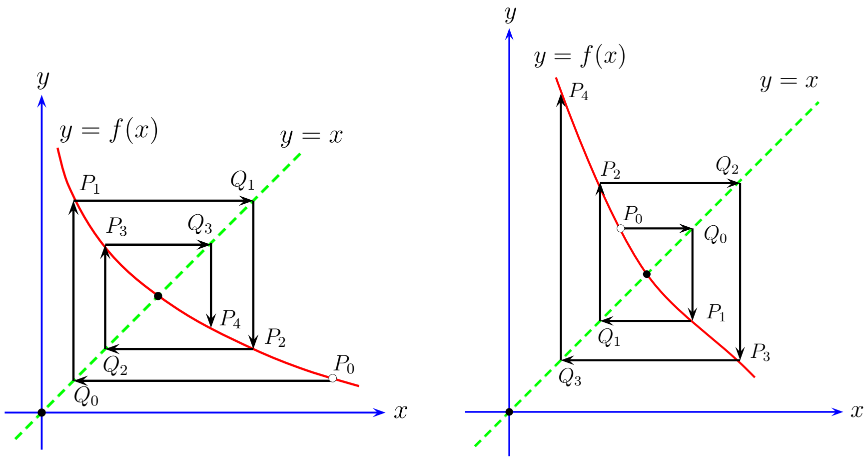

Fig. 5.6. Picard sequence that is (i) converging to a fixed point, and (ii) diverging away from a fixed point.

In analytic terms, the **Picard method** can be described as follows. Given any $x_0 \in [a, b]$, we recursively define a sequence (x_n) by

$$x_n = f(x_{n-1}) \quad \text{for } n \in \mathbb{N}.$$

Such a sequence (x_n) is called a **Picard sequence** for the function f (with its initial point x_0). It is clear that if a Picard sequence (x_n) for f is convergent and f is continuous, then the limit x of (x_n) is a fixed point of f. Indeed,

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = f\left(\lim_{n\to\infty} x_{n-1}\right) = \lim_{n\to\infty} f(x_{n-1}) = \lim_{n\to\infty} x_n = x.$$

A sufficient condition for the convergence of a Picard sequence, which is a formal analogue of the geometric condition on slopes mentioned above, is given by the following result. It is to be noted here that the same condition guarantees the uniqueness of a fixed point.

Proposition 5.17 (Picard Convergence Theorem). *If $f : [a, b] \to [a, b]$ is continuous on $[a, b]$ and differentiable on (a, b) with $|f'(x)| < 1$ for all $x \in (a, b)$, then f has a unique fixed point. Furthermore, every Picard sequence for f is convergent and converges to the unique fixed point of f.*

Proof. By Proposition 5.15, f has a fixed point in $[a, b]$. If there are fixed points $c_1, c_2 \in [a, b]$ with $c_1 \neq c_2$, then by the MVT, there is $c \in (a, b)$ such that

$$|c_1 - c_2| = |f(c_1) - f(c_2)| = |f'(c)||c_1 - c_2| < |c_1 - c_2|,$$

which is a contradiction. Thus, f has a unique fixed point.

Let c^* denote the unique fixed point of f. Consider any $x_0 \in [a, b]$, and let (x_n) be the Picard sequence for f with its initial point x_0. Now, given any $n \in \mathbb{N}$, by the MVT, there is c_{n-1} between x_{n-1} and c^* such that

$$x_n - c^* = f(x_{n-1}) - f(c^*) = f'(c_{n-1})\,(x_{n-1} - c^*)\,.$$

As a consequence, $|x_n - c^*| \leq |x_{n-1} - c^*|$ for all $n \in \mathbb{N}$. We shall now show that $x_n \to c^*$. First, note that since $x_n \in [a, b]$ for $n \geq 0$, the sequence (x_n) is bounded. Thus, by Proposition 2.19, it suffices to show that every convergent subsequence of (x_n) has c^* as its limit. Let $x \in \mathbb{R}$ and let (x_{n_k}) be a subsequence of (x_n) such that $x_{n_k} \to x$. Then

$$\left|x_{n_{k+1}} - c^*\right| \leq |x_{n_k+1} - c^*| \leq |x_{n_k} - c^*|\,.$$

But both the sequences $(|x_{n_k} - c^*|)$ and $\left(\left|x_{n_{k+1}} - c^*\right|\right)$ converge to $|x - c^*|$. So by the Sandwich Theorem, $|x_{n_k+1} - c^*| \to |x - c^*|$ as $k \to \infty$. On the other hand,

$$\lim_{k\to\infty} |x_{n_k+1} - c^*| = \lim_{k\to\infty} |f(x_{n_k}) - f(c^*)| = |f(x) - f(c^*)|.$$

It follows that $|f(x) - f(c^*)| = |x - c^*|$. Now, if $x \neq c^*$, then by the MVT, there is $c \in (a, b)$ such that

$$|x - c^*| = |f(x) - f(c^*)| = |f'(c)||x - c^*| < |x - c^*|,$$

which is a contradiction. This proves that $x_n \to c^*$. □

Remark 5.18. Our proof of the Picard Convergence Theorem becomes simpler if instead of assuming $|f'(x)| < 1$ for all $x \in (a, b)$, we make the stronger assumption that there is $\alpha < 1$ such that $|f'(x)| < \alpha$ for all $x \in (a, b)$. In this case, we can also obtain the "rate of convergence" for the Picard sequence. (See Exercise 5.32.) An alternative set of conditions for the convergence of the Picard sequence is given in Exercise 5.28. The Picard Convergence Theorem itself admits several generalizations and extensions. Some of these are outlined in Exercise 5.29. ◇

Examples 5.19. (i) Consider $f : [0, 2] \to \mathbb{R}$ defined by $f(x) := (1 + x)^{1/5}$. Then $0 \leq f(x) \leq 3^{1/5} < 2$ for all $x \in [0, 2]$. Thus, f maps the interval $[0, 2]$ into itself. Also, f is continuous on $[0, 2]$, differentiable on $(0, 2)$, and

$$|f'(x)| = \frac{1}{5(1 + x)^{4/5}} \leq \frac{1}{5} < 1 \quad \text{for } x \in [0, 2].$$

Thus, by the Picard convergence theorem, f has a unique fixed point, and successively better approximations to this fixed point are given by the successive terms of a Picard sequence for f. For example, if we take $x_0 = 0$, then the first few terms of the corresponding Picard sequence for f are roughly given by $x_1 = 1$, $x_2 = 1.148698$, $x_3 = 1.1652928$, and $x_4 = 1.1670872$. It may be noted that finding a fixed point of f is equivalent to finding the root of the quintic polynomial $x^5 - x - 1$ in the interval $[0, 2]$.

(ii) Consider $f : [0,1] \to \mathbb{R}$ defined by $f(x) := x^2/2$. Then f maps $[0,1]$ into itself. Moreover, f is continuous on $[0,1]$, differentiable on $(0,1)$, and $|f'(x)| = |x| < 1$ for all $x \in (0,1)$. Thus, by the Picard convergence theorem, f has a unique fixed point, and every Picard sequence for f will converge to this fixed point. Indeed, it is easily verified that 0 is the only fixed point of f in $[0,1]$. Note that in this case there is no $\alpha < 1$ such that $|f'(x)| < \alpha$ for all $x \in (0,1)$.

(iii) The condition $|f'(x)| < 1$ for all $x \in (a,b)$ is essential for the uniqueness of a fixed point. For example, if $f : [a,b] \to \mathbb{R}$ is defined by $f(x) := x$, then f maps $[a,b]$ into itself, and every point of $[a,b]$ is a fixed point of f. Here, $f'(x) = 1$ for all $x \in (a,b)$.

(iv) When the condition $|f'(x)| < 1$ for all $x \in (a,b)$ is not satisfied, a function can still have a unique fixed point c^*, but the Picard sequence (x_n) with its initial point $x_0 \neq c$ may not converge to c^*. For example, if $f : [-1,1] \to \mathbb{R}$ is defined by $f(x) := -x$, then f maps $[-1,1]$ into itself, f is differentiable, and $|f'(x)| = 1$ for all $x \in [-1,1]$. Clearly, $c^* = 0$ is the unique fixed point of f, but if $x_0 \neq 0$, then the corresponding Picard sequence looks like $-x_0, x_0, -x_0, x_0, \ldots$; in other words, it oscillates between x_0 and $-x_0$ and never reaches the fixed point. In geometric terms, the cobweb that we hope to weave just traces out a square over and over again. $\diamond$

Remark 5.20. When the hypothesis of the Picard Convergence Theorem is satisfied, a Picard sequence for $f : [a,b] \to [a,b]$ with arbitrary $x_0 \in [a,b]$ as its initial point will converge to a fixed point. It is natural to expect that if x_0 is closer to the fixed point, then the convergence will be rapid. But since we do not know the fixed point to begin with, it may not be clear how one picks a "good" initial point x_0. To this end, observe that a fixed point of f is necessarily in its range. The range is usually smaller than $[a,b]$. Thus, it is better that x_0 be picked from $f([a,b])$. For example, if $f : [0,1] \to [0,1]$ is given by $f(x) := (x+1)/4$, then the range of f equals $\left[\frac{1}{4}, \frac{1}{2}\right]$, and so we should choose x_0 to be in this smaller subinterval. In fact, this simple observation can be extended further. A fixed point of f lies not only in the range of f but also in the ranges of the composites $f \circ f$, $f \circ f \circ f$, and so on. Thus, if R_n is the range of the n-fold composite $f \circ \cdots \circ f$, then a fixed point is in each R_n as n varies over $\mathbb{N}$. If only a single point belongs to each R_n $(n \in \mathbb{N})$, then we have found our fixed point! In fact, the Picard method amounts to starting with any $x_0 \in [a,b]$ and considering the image of x_0 under the n-fold composite $f \circ \cdots \circ f$ of f. For example, if, as before, $f : [0,1] \to [0,1]$ is given by $f(x) = (x+1)/4$, then it is easy to see that the n-fold composite of f, say f_n, and its range are given by

$$f_n(x) = \frac{x + (4^n - 1)/3}{4^n} \text{ and } R_n := f_n([0,1]) = \left[\frac{1}{3}\left(1 - \frac{1}{4^n}\right), \frac{1}{3}\left(1 + \frac{2}{4^n}\right)\right].$$

It is clear, therefore, that $\frac{1}{3}$ is the only point in each R_n ($n \in \mathbb{N}$), and this is the unique fixed point of f (as can also be verified directly from the definition of f). Of course, in general, it is not practical to determine the ranges of the n-fold composites of f for all $n \in \mathbb{N}$. So it is simpler to use the Picard method. But the Picard method will be more effective if the above observations are used to some extent in choosing the initial point. ◇

Finding a Solution of an Equation

We now turn to the second problem mentioned earlier, namely, the problem of finding a solution of $f(x) = 0$, where f is a real-valued function defined on a subset of $\mathbb{R}$. For simplicity, we shall restrict ourselves to the case in which f is defined on a closed and bounded interval of $\mathbb{R}$. In this case, the existence of a solution is guaranteed if the IVP is available.

Proposition 5.21. *If $f : [a,b] \to \mathbb{R}$ is continuous and if $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has a solution in $[a,b]$.*

Proof. The result is an immediate consequence of the fact that a continuous function on $[a,b]$ has the IVP. □

Suppose we know that $f : [a,b] \to \mathbb{R}$ is such that the equation $f(x) = 0$ has a solution. Then a natural question is whether we can find it. It is not easy, in general, to find an exact solution.[1] A method given by Newton seeks to achieve what may be the next best alternative to finding an exact solution, namely, to find an approximate solution. In geometric terms, the basic idea of the Newton method can be described as follows.

First, pick any point $P_0 = (x_0, f(x_0))$ on the curve $y = f(x)$. Draw a tangent to this curve at P_0, and if it intersects the x-axis at $(x_1, 0)$, then

[1] In fact, this is a very difficult problem even for the nicest of functions, namely polynomial functions. In the special case of linear and quadratic equations, there are simple and well-known formulas for their solutions. For the solutions of cubic and quartic equations, there are more intricate formulas, ascribed to Cardan and Ferrari, which express the solutions in terms of the coefficients of the polynomial using the basic operations of algebra, namely, addition, subtraction, multiplication, division, and extraction of roots. After several unsuccessful attempts to find a similar formula for a general polynomial equation of degree 5 or more, it was proved by Abel that no such formula exists. In other words, a general equation of degree 5 or more is not *solvable by radicals*. An elegant proof of Abel's result was given by Galois, who also gave a criterion for an equation to be solvable by radicals. For more on these topics, we refer to the book of Tignol [84].

consider the corresponding point $P_1 = (x_1, f(x_1))$. Again, draw the tangent to $y = f(x)$ at P_1 and if it intersects the x-axis at $(x_2, 0)$, then consider the corresponding point $P_2 = (x_2, f(x_2))$. This process can be repeated a number of times. Often, it will rapidly bring us near to the point of intersection of the curve $y = f(x)$ and the x-axis, that is, to the solution of $f(x) = 0$. (See Figure 5.7.) It is clear, however, that the procedure will fail if at some point, the tangent is parallel to the x-axis.

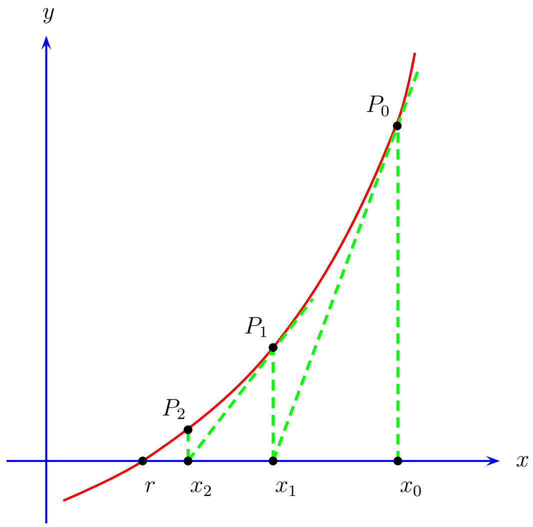

Fig. 5.7. Newton sequence approaching a solution of an equation.

In analytic terms, the **Newton method** (sometimes also called the **Newton–Raphson method**) can be described as follows. Choose any $x_0 \in [a, b]$ such that $f'(x_0)$ exists and $f'(x_0) \neq 0$. Given any $n \in \mathbb{N}$ and $x_{n-1} \in [a, b]$ such that $f'(x_{n-1}) \neq 0$, we let x_n be the root of the linear approximation

$$L(x) = f(x_{n-1}) + f'(x_{n-1})(x - x_{n-1})$$

to f around x_{n-1}. In other words,

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

Such a sequence (x_n) is called a **Newton sequence** for the function f (with its initial point x_0). It is clear that if a Newton sequence (x_n) for f is convergent and f' is bounded, then the limit x of (x_n) satisfies $f(x) = 0$. Indeed,

$$f(x) = f\left(\lim_{n\to\infty} x_{n-1}\right) = \lim_{n\to\infty} f(x_{n-1}) = \lim_{n\to\infty} f'(x_{n-1})(x_{n-1} - x_n) = 0.$$

A sufficient condition for the convergence of a Newton sequence can be derived from the Picard Convergence Theorem as follows.

Proposition 5.22 (Convergence of a Newton Sequence). *Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ with $f'(x) \neq 0$ for all $x \in [a, b]$, and*

$$x - \frac{f(x)}{f'(x)} \in [a, b] \quad \textit{for all } x \in [a, b].$$

Assume that f' is continuous on $[a, b]$, differentiable on (a, b), and

$$\left| \frac{f(x)f''(x)}{[f'(x)]^2} \right| < 1 \quad \textit{for all } x \in (a, b).$$

Then there is a unique $x^* \in [a, b]$ such that $f(x^*) = 0$. Furthermore, the Newton sequence for f with any initial point $x_0 \in [a, b]$ converges to x^*.

Proof. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) := x - \frac{f(x)}{f'(x)} \quad \text{for } x \in [a, b].$$

Then F is continuous on $[a, b]$, differentiable on (a, b), and F maps the interval $[a, b]$ into itself. Notice that $x \in [a, b]$ is a fixed point of F if and only if $f(x) = 0$. Moreover, for all $x \in [a, b]$,

$$|F'(x)| = \left| 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} \right| = \left| \frac{f(x)f''(x)}{[f'(x)]^2} \right| < 1.$$

Therefore, by the Picard Convergence Theorem, F has a unique fixed point x^* in $[a, b]$, which is then the unique root of f in $[a, b]$. Furthermore, if $x_0 \in [a, b]$ is any initial point, then the Newton sequence for f is, in fact, the Picard sequence for F, and hence it converges to x^*. ☐

Examples 5.23. (i) Consider $f : [5/4, 3/2] \to \mathbb{R}$ defined by $f(x) := x^3 - 3$. Then f is continuous on $[5/4, 3/2]$ and

$$f\left(\frac{5}{4}\right) = \frac{125}{64} - 3 < 0, \quad \text{while} \quad f\left(\frac{3}{2}\right) = \frac{27}{8} - 3 > 0.$$

Hence, by Proposition 5.21, f has a root in $[5/4, 3/2]$. In this case, the iterative formula for the Newton sequence is given by

$$x_n = x_{n-1} - \frac{x_{n-1}^3 - 3}{3x_{n-1}^2} = \frac{2}{3}x_{n-1} + \frac{1}{x_{n-1}^2} \quad \text{provided } x_{n-1} \neq 0.$$

Thus, if we take $x_0 = \frac{5}{4}$, then we obtain

$$x_1 = 1.473333\ldots, \; x_2 = 1.442900\ldots, \; x_3 = 1.442249\ldots \; .$$

On the other hand, if we take $x_0 = \frac{3}{2}$, then we obtain

$$x_1 = 1.444444\ldots, \ x_2 = 1.442252\ldots, \ x_3 = 1.442249\ldots\ .$$

This indicates that both Newton sequences converge to the same limit, which is approximately $1.442249\ldots$. In fact, this is quite in accordance with the theory, because

$$x - \frac{f(x)}{f'(x)} = \frac{2x^3+3}{3x^2} \in [5/4,\, 3/2] \quad \text{for all } x \in [5/4,\, 3/2],$$

since $15x^2 \le 8x^3+12$ and $4x^3+6 \le 9x^2$. (See Exercise 4.32 (ii).) Moreover,

$$\left|\frac{f(x)f''(x)}{f'(x)^2}\right| = \left|\frac{(x^3-3)6x}{9x^4}\right| = \frac{2}{3}\frac{|x^3-3|}{x^3} \le \frac{2}{3} < 1 \quad \text{for } x \in (5/4,\, 3/2).$$

Thus the hypothesis of Proposition 5.22 is satisfied. Hence the equation $f(x) = 0$ has a unique solution in $[5/4,\, 3/2]$, and every Newton sequence converges to it.

(ii) To illustrate how the Newton sequence behaves when there is more than one solution, consider $f : [-2, 4] \to \mathbb{R}$ defined by $f(x) := x^2 - 2x - 3 = (x+1)(x-3)$. See Figure 5.8. Clearly $f(x) = 0$ has two solutions $x = -1$ and $x = 3$. Now, $f'(x) = 2(x-1)$, and thus the Newton sequence for f with any initial point $x_0 \neq 1$ is given by

$$x_n = x_{n-1} - \frac{x_{n-1}^2 - 2x_{n-1} - 3}{2(x_{n-1}-1)} = \frac{x_{n-1}^2+3}{2(x_{n-1}-1)}, \quad \text{provided } x_{n-1} \neq 1.$$

It is not difficult to show that if $x_0 < 1$, then (x_n) converges to the root -1 of f, whereas if $x_0 > 1$, then (x_n) converges to the root 3 of f. (See Exercise 5.23.)

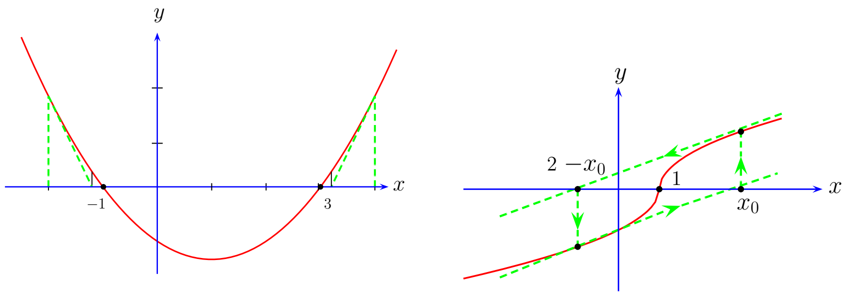

Fig. 5.8. Newton method for $f(x) := x^2 - 2x - 3$ and $f(x) := \begin{cases} \sqrt{x-1} & \text{if } x \ge 1, \\ -\sqrt{1-x} & \text{if } x < 1. \end{cases}$

(iii) Consider $f : [-10, 10] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} \sqrt{x-1} & \text{if } x \geq 1, \\ -\sqrt{1-x} & \text{if } x < 1. \end{cases}$$

In this case,

$$f'(x) = \begin{cases} 1/(2\sqrt{x-1}) & \text{if } x > 1, \\ 1/(2\sqrt{1-x}) & \text{if } x < 1. \end{cases}$$

The Newton sequence for f with any initial point $x_0 \neq 1$ is given by

$$x_n = x_{n-1} - 2(x_{n-1} - 1) = -x_{n-1} + 2.$$

Since $x_n - 1 = -(x_{n-1} - 1)$, we see that $x_n = 1 + (-1)^n(x_0 - 1)$. Thus the Newton sequence oscillates between x_0 and $2 - x_0$. See Figure 5.8. ◇

As the above examples show, a Newton sequence may not always converge to the desired root, but when it does converge, the rate of convergence is quite rapid, and just a few iterations give us values that are fairly close to the desired root. The conditions for convergence given in Proposition 5.22, which is a consequence of the Picard Convergence Theorem, are rather unwieldy and difficult to check in practice. However, there are alternative sets of sufficient conditions such as those given by the following result.

Proposition 5.24. *Let $f : [a, b] \to \mathbb{R}$ be such that $f(r) = 0$ for some $r \in [a, b]$. If f' is nonzero throughout $[a, b]$ and f' is monotonic on $[a, b]$, then r is the unique solution of $f(x) = 0$ in $[a, b]$, and the Newton sequence for f with any initial point $x_0 \in [a, b]$ converges to r.*

Proof. Since $f(r) = 0$ and f' is nonzero throughout $[a, b]$, it follows from the Rolle Theorem that r is the unique solution of $f(x) = 0$ in $[a, b]$. Moreover, since f' has the IVP on $[a, b]$ (Proposition 4.16), we see that f' is either positive throughout $[a, b]$ or negative throughout $[a, b]$. Also, since f' is monotonic on $[a, b]$, there are four possible cases according as f' is positive or f' is negative, and f' is monotonically increasing or f' is monotonically decreasing.

To begin with, suppose f' is positive and monotonically increasing on $[a, b]$. Choose an arbitrary initial point $x_0 \in [a, b]$. Let (x_n) denote the Newton sequence for f with its initial point x_0. If $x_0 = r$, then clearly, $x_n = r$ for all $n \in \mathbb{N}$, and so $x_n \to r$. Now assume that $x_0 > r$. Then by the MVT, there is $c \in (r, x_0)$ such that

$$\frac{f(x_0)}{x_0 - r} = \frac{f(x_0) - f(r)}{x_0 - r} = f'(c).$$

Moreover, since f' is positive and monotonically increasing on $[a, b]$, we see that $0 < f'(c) \leq f'(x_0)$. Hence $f(x_0) > 0$ and $(f(x_0)/f'(x_0)) \leq x_0 - r$. Thus

$$r = x_0 - (x_0 - r) \leq x_0 - \frac{f(x_0)}{f'(x_0)} < x_0.$$

Since $x_1 := x_0 - (f(x_0)/f'(x_0))$, we see that $r \leq x_1 < x_0$. (See Figure 5.9 (i).) Now, if $x_1 \neq r$, then $x_1 > r$, and we can proceed as before to obtain $r \leq x_2 < x_1$. Continuing in this way, we see that the Newton sequence (x_n) has the property that either $x_n = r$ for some $n \in \mathbb{N}$ (in which case $x_m = r$ for all $m > n$) or (x_n) is strictly decreasing and bounded below by r. In the latter event, by part (ii) of Proposition 2.8, $x_n \to s$ for some $s \in [a, b]$ with $r \leq s$. But then, $f(s) = 0$ and hence $s = r$. Thus, in any event, $x_n \to r$.

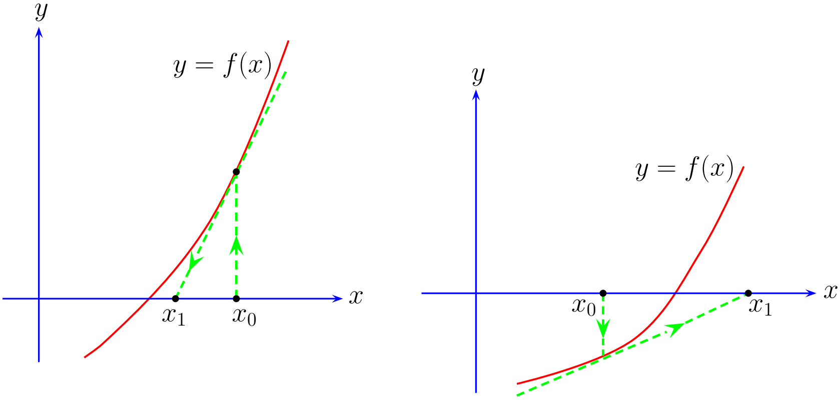

Fig. 5.9. Newton iterates for f when f' is positive and monotonically increasing.

Next, assume that $x_0 < r$. (See Figure 5.9 (ii).) Using the MVT as before, we see this time that there is $d \in (x_0, r)$ such that

$$\frac{f(x_0)}{x_0 - r} = \frac{f(x_0) - f(r)}{x_0 - r} = f'(d).$$

Again, since f' is positive and monotonically increasing on $[a, b]$, we see that $0 < f'(x_0) \leq f'(d)$. Hence $(f(x_0)/f'(x_0)) \leq x_0 - r$. Thus

$$x_1 := x_0 - \frac{f(x_0)}{f'(x_0)} \geq r.$$

This means that we are in one of the previous cases, in which the initial value is $\geq r$. Consequently, $x_n \to r$. This proves the proposition when f' is positive and monotonically increasing on $[a, b]$.

If f' is negative and monotonically decreasing on $[a, b]$, then it suffices to consider $-f$ and note that the Newton sequences for $-f$ and f are identical, provided both have the same initial point.

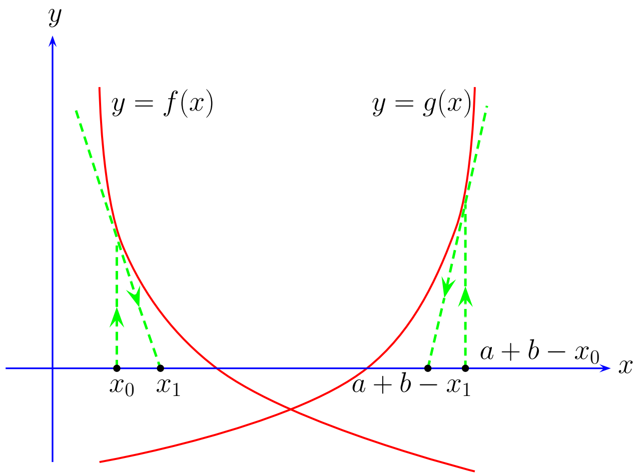

Fig. 5.10. Newton iterates for $f : [a,b] \to \mathbb{R}$ when f' is negative and monotonically increasing, and for its "reflection" $g : [a,b] \to \mathbb{R}$ defined by $g(x) := f(a+b-x)$.

If f' is negative and monotonically increasing, then it suffices to consider its "reflection" along the vertical line $x = (b-a)/2$, that is, the function $g : [a,b] \to \mathbb{R}$ defined by $g(x) := f(a+b-x)$ for $x \in [a,b]$, and note the following. First, g is differentiable and $s := a+b-r$ is a solution of $g(x) = 0$ in $[a,b]$. Next, g' is positive and monotonically increasing on $[a,b]$. Further, if (x_n) is the Newton sequence for f with its initial point x_0, then $(a+b-x_n)$ is the Newton sequence for g with its initial point $a+b-x_0$. Finally, if $a+b-x_n \to s$, then $x_n \to r$. See Figure 5.10.

If f' is positive and monotonically decreasing, then it suffices to consider $-f$ and use the result of the previous paragraph. □

Corollary 5.25. *Suppose $f : [a,b] \to \mathbb{R}$ is twice differentiable and $f(r) = 0$ for some $r \in [a,b]$. If f' is nonzero throughout $[a,b]$ and f'' does not change sign throughout $[a,b]$, then r is the unique solution of $f(x) = 0$ in $[a,b]$ and the Newton sequence for f with any initial point $x_0 \in [a,b]$ converges to r.*

Proof. Applying part (i) of Corollary 4.30 to f', we see that f' is monotonic on $[a,b]$. Now use Proposition 5.24. □

To end this section, we remark that if $f(x)$ is a polynomial of degree ≥ 2, then $f'(x)$ and $f''(x)$ are nonzero polynomials of smaller degree, and, in particular, they have finitely many roots. Thus, the real line can be divided into finitely many intervals in each of which f' and f'' are nonzero and do not change signs. In particular, for any root r of f, we can find $a, b \in \mathbb{R}$ such that the restriction of f to $[a,b]$ satisfies the hypotheses of Corollary 5.25. In this way, we may say that the Newton method is always applicable to polynomials, provided we keep away from points at which the derivative vanishes.

Notes and Comments

The applications of differentiation to various tests for local extrema and points of inflection are bread-and-butter topics in calculus courses, so much so that many students think of these tests as definitions of the concepts such as local minimum, local maximum, and point of inflection. However, these concepts are basically of a geometric nature. In fact, this was the reason why we introduced these concepts in Chapter 1 before discussing the notion of derivative. In a similar way, many students try to use the Second Derivative Test when asked to find the absolute extrema of a real-valued function (on, say, a closed and bounded interval). The fact of the matter is that for finding absolute extrema, this test is neither necessary nor sufficient! To emphasize this point, we have arranged the discussion of absolute minima and maxima before the discussion of the Second Derivative Test, which is useful in finding local maxima and minima.

The method of Picard that we have discussed in the last section of this chapter is perhaps a starting point of an area of mathematics, known as fixed point theory, that has grown considerably over the years. Fixed point theorems such as the Picard Convergence Theorem and its generalizations are extremely useful in proving the existence and uniqueness of solutions of certain differential equations with prescribed initial conditions. For an introduction, we refer to the delightful book of Simmons [74]. The method of Newton for finding approximate solutions can be found toward the beginning of any book on numerical analysis. The fact that it converges very rapidly is almost folklore. But precise results about conditions that ensure the convergence of Newton sequences seem a bit difficult to locate. Results similar to the last proposition in this chapter can be found, for example, in the little booklet of Vilenkin [85] and the substantive book on calculus by Klambauer [47].

Exercises

Part A

5.1. Find the greatest and the least values of $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$ and f are given by the following:
 (i) $D := [0, 2]$ and $f(x) := 4x^3 - 8x^2 + 5x$,
 (ii) $D := \mathbb{R}$ and $f(x) := (x + 2)^2/(x^2 + x + 1)$,
 (iii) $D := [-2, 5]$ and $f(x) := 1 + 12|x| - 3x^2$.

5.2. Given any constants $a, b \in \mathbb{R}$ with $a > b$, find the value of x at which the difference $\left(x/\sqrt{x^2 + a^2}\right) - \left(x/\sqrt{x^2 + b^2}\right)$ has the maximum value.

5.3. If $n \in \mathbb{N}$ is odd and the polynomial $1 + x + (1/2!)x^2 + \cdots + (1/n!)x^n$ has only one real root $x = c$, then show that

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} \geq \frac{c^{n+1}}{(n+1)!} \quad \text{for all } x \in \mathbb{R}.$$

5.4. A window is to be made in the form of a rectangle surmounted by a semicircular portion with diameter equal to the base of the rectangle. The rectangular portion is to be of clear glass and the semicircular portion is to be of colored glass admitting only half as much light per square foot as the clear glass. If the total perimeter of the window frame is to be p feet, find the dimensions of the window that will admit the maximum amount of light.

5.5. The stiffness of a rectangular beam is proportional to the product of its breadth and the cube of its thickness but is not related to its length. Find the proportions of the stiffest beam that can be cut from a cylindrical log of diameter d inches.

5.6. A post office will accept a box for shipment only if the sum of its length and its girth (that is, the distance around) does not exceed 84 inches. Find the dimensions of the largest acceptable box with a square end.

5.7. A wire of length ℓ inches is cut into two pieces, one being bent to form a square and the other to form an equilateral triangle. How should the wire be cut (i) if the sum of the two areas is minimal? (ii) if the sum of the two areas is maximal?

5.8. Let $D \subseteq \mathbb{R}$ and let c be an interior point of D. If $f : D \to \mathbb{R}$ is twice differentiable at c and $f''(c) \neq 0$, then prove that f has a local extremum at c if and only if $f'(c) = 0$.

5.9. Let $D \subseteq \mathbb{R}$, let c be an interior point of D, and let $f : D \to \mathbb{R}$ be differentiable at c. If c is a point of inflection for f, then is it necessarily true that $f'(c) = 0$? On the other hand, if $f'(c) = 0$, then is it necessarily true that either f has a local extremum at c or c is a point of inflection for f? (Compare Example 7.21.)

5.10. Find the local maxima and the local minima of $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = x^m(1 - x)^n$ for $x \in [0, 1]$, where m and n are positive integers.

5.11. For which constants $a, b, c, d \in \mathbb{R}$ does the function $f(x) = ax^3 + bx^2 + cx + d,\ x \in \mathbb{R}$, have (i) a local maximum at -1, (ii) 1 as its point of inflection, and (iii) $f(-1) = 10$ and $f(1) = -6$?

5.12. Sketch the following curves after locating intervals of increase/decrease, intervals of convexity/concavity, points of local maxima/minima, and points of inflection. How many times and approximately where does the curve cross the x-axis?
(i) $y = 2x^3 + 2x^2 - 2x - 1$ (ii) $y = x^3 - 6x^2 + 9x + 1$,
(iii) $y = x^2/(x^2 + 1)$, (iv) $y = 1/(1 + x^2)$,
(v) $y = x/(x - 1)$, $x \neq 1$ (vi) $y = x/(x + 1)$, $x \neq -1$,
(vii) $y = x^2/(x^2 - 1)$, $x \neq \pm 1$, (viii) $y = (x^2 + 1)/x$, $x \neq 0$,
(ix) $y = 1 + 12|x| - 3x^2$, $x \in [-2, 5]$, (x) $y = (x^2 + x - 2)/(x - 2)$, $x \neq 2$.

5.13. Sketch a continuous curve $y = f(x)$ having the following properties:
$f(-2) = 8,\ f(0) = 4,\ f(2) = 0;\ f'(2) = f'(-2) = 0;$
$f'(x) > 0$ for $|x| > 2$, $f'(x) < 0$ for $|x| < 2$;
$f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$.

5.14. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x - 2n)^3 + 2n$, where $n \in \mathbb{Z}$ is such that $x \in [2n - 1, 2n + 1)$. Show that $2n$ is a point of inflection for f, for each $n \in \mathbb{N}$. (Compare Exercise 4.34.)

5.15. Use linear approximation to find an approximate value of
(i) $(8.01)^{4/3} + (8.01)^2 - (8.01)^{-1/3}$, (ii) $(9.1)^{3/2} + (9.1)^{-1/2}$.

5.16. (i) Find an approximate value of $\sqrt{3}$ using the linear approximation to $f(x) = \sqrt{x}$ for x around 4.
(ii) Let $f(x) = \sqrt{x} + \sqrt{x+1} - 4$. Show that there is a unique $x_0 \in (3, 4)$ such that $f(x_0) = 0$. Using the linear approximation to f around 3, find an approximation x_1 of x_0. Find x_0 exactly and determine the error $|x_1 - x_0|$.

5.17. For the functions $f : [-1, \infty) \to \mathbb{R}$ and $g : (-\infty, 1) \to \mathbb{R}$ defined by $f(x) := \sqrt{1+x}$ and $g(x) := 1/\sqrt{1-x}$, find the following.
(i) The linear approximation $L(x)$ around 0.
(ii) An estimate for the error $e_1(x)$ when $x > 0$ and when $x < 0$. Also, find an upper bound for $|e_1(x)|$ that is valid for all $x \in (0, 0.1)$, and an upper bound for $|e_1(x)|$ that is valid for all $x \in (-0.1, 0)$.
(iii) The quadratic approximation $Q(x)$ around 0.
(iv) An estimate for the error $e_2(x)$ when $x > 0$ and when $x < 0$. Also, an upper bound for $|e_2(x)|$ that is valid for all $x \in (0, 0.1)$, and an upper bound for $|e_2(x)|$ that is valid for all $x \in (-0.1, 0)$.

5.18. Let $D \subseteq \mathbb{R}$ and let c be an interior point of D. Suppose $F : D \to \mathbb{R}$ is the polynomial function defined by $F(x) = a_0 + a_1(x - c) + a_2(x - c)^2$ for $x \in D$. If a function $f : D \to \mathbb{R}$ is differentiable at c, then show that F is the linear approximation to f around c if and only if

$$f(c) = F(c) = a_0, \quad f'(c) = F'(c) = a_1, \quad \text{and} \quad a_2 = 0,$$

whereas if f is twice differentiable at c, then show that F is the quadratic approximation to f around c if and only if

$$f(c) = F(c) = a_0, \quad f'(c) = F'(c) = a_1, \quad \text{and} \quad f''(c) = F''(c) = a_2.$$

5.19. Let $f(x) := \sqrt{x} + \left(1/\sqrt{x}\right)$ for $x > 0$. Write down the linear and the quadratic approximations $L(x)$ and $Q(x)$ to $f(x)$ around 4. Find the errors $f(4.41) - L(4.41)$ and $f(4.41) - Q(4.41)$.

5.20. Let $D \subseteq \mathbb{R}$ and let c be an interior point of D. If $f : D \to \mathbb{R}$ is continuous at c and we let $e_0(x) = f(x) - f(c)$ for $x \in D$, then show that $e_0(x) \to 0$ as $x \to c$. Moreover, given any $b \in D$ with $b \neq c$, if I_b denotes the open interval with c and b as its endpoints, and if f' exists on I_b and $|f'(x)| \leq M_1(b)$ for all $x \in I_b$, then show that $|e_0(b)| \leq M_1(b)|b - c|$.

5.21. Let $D \subseteq \mathbb{R}$ and let c be an interior point of D. If $f : D \to \mathbb{R}$ is n-times differentiable at c and $P_n(x)$ denotes the nth Taylor polynomial of f around c and if $e_n(x) := f(x) - P_n(x)$ for $x \in D$, then show that the limit of $e_n(x)/x - c)^n$ as $x \to c$ is zero. Moreover, given any $b \in D$ with $b \neq c$, if I_b denotes the open interval with c and b as its endpoints,

$f', \ldots, f^{(n)}$ are continuous on the closed interval between c and b, $f^{(n+1)}$ exists on I_b, and if $|f^{(n+1)}(x)| \le M_{n+1}(b)$ for all $x \in I_b$, then show that

$$|e_n(b)| \le \frac{M_{n+1}(b)}{(n+1)!}|b-c|^{n+1}.$$

(Hint: L'Hôpital's Rule for $\frac{0}{0}$ indeterminate forms and Taylor Formula.)

5.22. Consider $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 1/(1+x^2)$. Use the Picard Convergence Theorem to show that f has a unique fixed point in $[0,1]$ and every Picard sequence with its initial point $x_0 \in [0,1]$ will converge to this fixed point. Compute the first few values of the Picard sequence for f when $x_0 = 0$.

5.23. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 2x - 3$. If $x_0 \neq 1$, then show that the Newton sequence with its initial point x_0 converges to -1 if $x_0 < 1$, and to 3 if $x_0 > 1$,

5.24. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4 - x^3 - 75$. Show that there is a unique $r_1 \in [3,4]$ such that $f(r_1) = 0$ and there is a unique $r_2 \in [-3,-2]$ such that $f(r_2) = 0$. Use the Newton method with initial point

(i) $x_0 = 3$, (ii) $x_0 = -3$,

to find approximate values of the solutions r_1 and r_2 of $f(x) = 0$.

5.25. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x-1)^{1/3}$. Show that the Newton sequence for f with its initial point $x_0 \neq 1$ is unbounded.

Part B

5.26. Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$f'(x) = 6(x-1)(x-2)^2(x-3)^3(x-4)^4 \quad \text{for all } x \in \mathbb{R}.$$

Find all the points (in $\mathbb{R}$) at which f has a local extremum. Also, find all the points of inflection for f.

5.27. Let I be an interval in $\mathbb{R}$, $f : I \to \mathbb{R}$, and let c be an interior point of I.

(i) Suppose there is $n \in \mathbb{N}$ such that $f^{(2n)}$ exists at c, and $f'(c) = f''(c) = \cdots = f^{(2n-1)}(c) = 0$. If $f^{(2n)}(c) < 0$, then show that f has a strict local maximum at c, whereas if $f^{(2n)}(c) > 0$, then show that f has a strict local minimum at c. (Hint: Taylor Formula.)

(ii) Suppose there is $n \in \mathbb{N}$ such that $f^{(2n+1)}$ exists at c, and $f''(c) = f'''(c) = \cdots = f^{(2n)}(c) = 0$. If $f^{(2n+1)}(c) \neq 0$, then show that c is a strict point of inflection for f. (Hint: Taylor Formula.)

(iii) Suppose that f is infinitely differentiable at c and $f'(c) = 0$, but $f^{(k)}(c) \neq 0$ for some $k \in \mathbb{N}$. Show that either f has a strict local extremum at c, or c is a strict point of inflection for f.

5.28. Let $f : [a,b] \to [a,b]$ be continuous and monotonic. Then show that for every $x_0 \in [a,b]$, the Picard sequence for f with its initial point x_0 converges to a fixed point of f. (Hint: Show that the Picard sequence (x_n) is monotonic by considering separately the cases $x_0 \le x_1$ and $x_0 \ge x_1$.)

5.29. Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be such that $f(D) \subseteq D$. Prove the following generalizations and extensions of the Picard Convergence Theorem.

(i) If D is closed and f is a **contraction** (that is, there is $\alpha \in \mathbb{R}$ with $0 \le \alpha < 1$ such that $|f(x) - f(y)| \le \alpha|x - y|$ for all $x, y \in D$), then f has a unique fixed point, and every Picard sequence converges to this fixed point. Give an example to show that if f is a contraction but D is not closed, then f need not have a fixed point. (Hint: See Example 5.16 (ii).)

(ii) If D is closed and bounded, and f is **contractive** (that is, $|f(x) - f(y)| < |x - y|$ for all $x, y \in D$, $x \neq y$), then f has a unique fixed point, and every Picard sequence converges to this fixed point. Give an example to show that if f is contractive but D is not closed and bounded, then f need not have a fixed point. (Hint: See the proof of Proposition 5.17 and Example 5.16 (iii).)

(iii) If D is a closed and bounded interval, and f is **nonexpansive** (that is, $|f(x) - f(y)| \le |x - y|$ for all $x, y \in D$), then f has a fixed point in D but it may not be unique. Give an example to show that if f is nonexpansive but D is not a closed and bounded interval, then f need not have a fixed point. (Hint: See the proof of Proposition 5.15 and Example 5.16 (iv).)

5.30. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function such that f' is bounded on (a, b), and f has a root $r \in (a, b)$. For $x \in (a, b)$, $x \neq r$, let J_x denote the open interval between r and x. Assume that if $f(x) > 0$, then f is convex on J_x, while if $f(x) < 0$, then f is concave on J_x. Show that the Newton sequence with any initial point $x_0 \in (a, b)$ converges to a root of f in (a, b).

5.31. Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : (a, b) \to \mathbb{R}$ be any function.

(i) Suppose f is differentiable and there is $c \in (a, b)$ such that $f(c) = c$. If f' is continuous at c and $|f'(c)| < 1$, then show that there is a closed subinterval I of (a, b) with $c \in I$ such that f maps I into itself, c is the only fixed point of f in I, and the Picard sequence with any initial point $x_0 \in I$ converges to c.

(ii) Suppose f is twice differentiable, $f'(x) \neq 0$ for all $x \in (a, b)$, and there is $r \in (a, b)$ such that $f(r) = 0$. If f'' is continuous at c, then show that there is a closed subinterval I of (a, b) with $r \in I$ such that r is the only solution of $f(x) = 0$ in I, and the Newton sequence with any initial point $x_0 \in I$ converges to r.

5.32. **(Linear convergence of the Picard method)** Let $f : [a, b] \to [a, b]$ be a continuous function such that f' exists and is bounded on (a, b). If f has a fixed point $c^* \in [a, b]$, then show that there is a constant $\alpha \in \mathbb{R}$ such that for any Picard sequence (x_n) for f with its initial point x_0 in $[a, b]$,

$$|x_n - c^*| \le \alpha |x_{n-1} - c^*| \quad \text{for all } n \in \mathbb{N}.$$

Deduce that $|x_n - c^*| \le \alpha^n |x_0 - c^*|$ for all $n \in \mathbb{N}$.

[Note: In case $\alpha < 1$, the former inequality shows that the terms of the Picard sequence give a successively better approximation of the fixed point c^* of f. This inequality can also be interpreted to say that the Picard sequence **converges linearly**.]

5.33. **(Quadratic convergence of the Newton method)** Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is continuous on $[a, b]$, $f'(x) \neq 0$ for all $x \in [a, b]$, f'' exists and is bounded on (a, b). If $f(r) = 0$ for some $r \in [a, b]$, then show that r is the unique solution of $f(x) = 0$ in $[a, b]$ and there is a constant $\alpha \in \mathbb{R}$ such that for any Newton sequence (x_n) for f with its initial point x_0 in $[a, b]$,

$$|x_n - r| \leq \alpha\, |x_{n-1} - r|^2, \quad \text{provided } x_{n-1}, x_n \in [a, b].$$

Deduce that $|x_n - r| \leq \alpha^{2^n - 1}|x_0 - r|^{2^n}$, provided $x_1, \ldots, x_n \in [a, b]$.
[Note: In case $\alpha < 1$, the former inequality shows that the terms of the Newton sequence give a successively better approximation of the solution r of $f(x) = 0$. This inequality can also be interpreted to say that the Newton sequence **converges quadratically**.]

5.34. Let (x_n) be a sequence in $\mathbb{R}$ and let $c \in \mathbb{R}$ be such that $x_n \to c$. Assume that there is $n_0 \in \mathbb{N}$ such that $x_n \neq c$ for all $n \geq n_0$. If there is a real number p such that

$$\alpha := \lim_{n\to\infty} \frac{|x_n - c|}{|x_{n-1} - c|^p}$$

exists and is nonzero, then p is called the **order of convergence** of (x_n) to c, and α is called the corresponding **asymptotic error constant**.

(i) Let $f : (a, b) \to (a, b)$ and $x_0 \in (a, b)$ be such that the Picard sequence (x_n) for f with its initial point x_0 converges to some $x^* \in (a, b)$. If f is continuous at x^*, then show that x^* is a fixed point of f. Further, if f is p times differentiable at x^* and $f'(x^*) = \cdots = f^{(p-1)}(x^*) = 0$, but $f^{(p)}(x^*) \neq 0$, then show that the order of convergence of (x_n) to x^* is p, and the corresponding asymptotic error constant is $|f^{(p)}(x^*)|/p!$. (Hint: For $n \in \mathbb{N}$, write $x_n - x^*$ as

$$f(x_{n-1}) - f(x^*) - f'(x^*)(x_{n-1} - x^*) - \cdots - \frac{f^{(p-1)}(x^*)}{(p-1)!}(x_{n-1} - x^*)^{p-1},$$

and use L'Hôpital's Rule.)

(ii) Let $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$ be such that f is differentiable and $f'(x) \neq 0$ for all $x \in (a, b)$. Assume that the Newton sequence (x_n) for f with its initial point x_0 converges to some $r \in (a, b)$. If f' is bounded on (a, b), then show that r is a solution of $f(x) = 0$. Further, if f is twice differentiable at r and $f''(r) \neq 0$, then show that the order of convergence of (x_n) to r is 2, and the corresponding asymptotic error constant is $|f''(r)|/2|f'(r)|$. (Hint: Note that $f(x_{n-1})(x_n - r) = [f'(x_{n-1}) - (f(x_{n-1}) - f(r))/(x_{n-1} - r)](x_{n-1} - r)$ for $n \in \mathbb{N}$.)

6

Integration

In this chapter, we embark upon a project that is of a very different kind as compared to our development of calculus and analysis so far, namely the project of finding the "area" of a planar region. This leads us to the theory of integration propounded by Riemann. Although this theory may seem unrelated to continuity and differentiability of functions, it has deep underlying connections. These connections manifest themselves mainly in the form of a central result known as the Fundamental Theorem of Calculus. In Section 6.1 below, we motivate and formulate a definition of the Riemann integral. Later in this section we prove a useful characterization of the integrability of functions, and also a key property of the Riemann integral known as domain additivity. Next, in Section 6.2, we establish a number of basic properties of integrable functions. The Fundamental Theorem of Calculus and several of its consequences are proved in Section 6.3. In Section 6.4, we show that the Riemann integral of a function can be approximated by certain sums involving its values at more or less randomly chosen points. This approach yields an alternative definition of the Riemann integral via a result of Darboux.

6.1 Riemann Integral

Consider a nonnegative bounded function defined on an interval $[a, b]$. Let us investigate whether we can assign a meaning to what can be called the "area" of the region that lies under the graph of such a function, between the lines $x = a$, $x = b$ and above the x-axis. To conform to the usual expectation, the only *assumption* we make is that the area of a rectangle $[x_1, x_2] \times [y_1, y_2]$ is equal to $(x_2 - x_1)(y_2 - y_1)$. Following Archimedes, our problem can be approached by subdividing the interval $[a, b]$ into a finite number of subintervals and then finding the sum of the areas of rectangles inscribed within the region and the sum of the areas of rectangles that circumscribe the region. See Figure 6.1.

While the sum of the areas of the inscribed rectangles ought to be at most the expected "area" of the region, the sum of the areas of the circumscribing

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Undergraduate Texts in Mathematics, https://doi.org/10.1007/978-3-030-01400-1_6

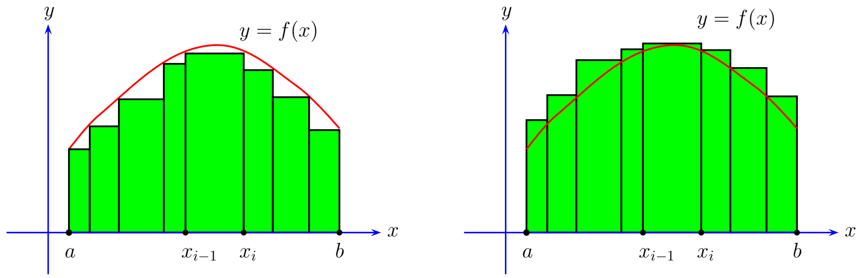

Fig. 6.1. Approximating the "area" under a curve by means of inscribed or circumscribing rectangles.

rectangles ought to be at least the expected "area". Further, if the given function is well-behave in some sense, both these sums should come close to the expected "area" of the region if the subdivision of the interval is made finer.

To proceed formally, we introduce the following concept. By a **partition** of an interval $[a, b]$ we mean a finite ordered set $\{x_0, x_1, \ldots, x_n\}$ of points in $[a, b]$ such that n is a nonnegative integer and

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Note that in accordance with our conventions, when we write $[a, b]$, it is tacitly assumed that a, b are real numbers and $a \le b$. In the trivial case $a = b$, the only possible partition of $[a, b]$ is $\{x_0\}$, where $x_0 = a = b$. If $a < b$, then several partitions are possible; for example, $\mathsf{P}_n := \{x_0, x_1, \ldots, x_n\}$, where

$$x_i = a + \frac{i(b-a)}{n} \quad \text{for } i = 0, 1, \ldots, n,$$

is a partition of $[a, b]$, which subdivides the interval $[a, b]$ into n subintervals, each of length $(b - a)/n$. We may refer to P_n as the **partition of** $[a, b]$ **into** n **equal parts**. It is clear that as n becomes larger, the subdivision of $[a, b]$ corresponding to P_n becomes uniformly finer.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let us define

$$m(f) := \inf\{f(x) : x \in [a, b]\} \quad \text{and} \quad M(f) := \sup\{f(x) : x \in [a, b]\}.$$

Given a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and $i = 1, \ldots, n$, we call $[x_{i-1}, x_i]$ the ith **subinterval of** $[a, b]$ **induced by** P, and we define

$$m_i(f) := \inf\{f(x) : x \in [x_{i-1}, x_i]\} \;\text{ and }\; M_i(f) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.$$

Clearly,

$$m(f) \le m_i(f) \le M_i(f) \le M(f) \quad \text{for all } i = 1, \ldots, n.$$

We define the **lower sum** and the **upper sum** for the function f with respect to the partition P as follows:

$$L(P,f) := \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) \quad \text{and} \quad U(P,f) := \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}).$$

We note that if f is nonnegative, then the lower sum is the sum of the areas of the inscribed rectangles and the upper sum is the sum of the areas of the circumscribing rectangles mentioned earlier. In the trivial case $a = b$, there is only one partition of $[a,b]$, namely $\mathsf{P}_0 := \{a\}$, and for this, $L(\mathsf{P}_0, f) = 0$ and $U(\mathsf{P}_0, f) = 0$ according to the usual convention that an empty sum is zero.

Proposition 6.1. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then for every partition P of $[a,b]$,*

$$m(f)(b-a) \le L(P,f) \le U(P,f) \le M(f)(b-a).$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Since $m(f) \le m_i(f) \le M_i(f) \le M(f)$ for each $i = 1, \ldots, n$ and $\sum_{i=1}^{n}(x_i - x_{i-1}) = b - a$, the desired inequalities follow. $\square$

Our goal is to look for partitions of $[a,b]$ with respect to which the lower sums are as large as possible and the upper sums are as small as possible, so that the expected “area” will get tightly caught between the lower sums and the upper sums. With this in mind, we define

$$L(f) := \sup\{L(P,f) : P \text{ is a partition of } [a,b]\}$$

and

$$U(f) := \inf\{U(P,f) : P \text{ is a partition of } [a,b]\}.$$

It is natural to expect that $L(f) \le U(f)$. To prove this, we need the following concepts. Given a partition P of $[a,b]$, we say that a partition P^* of $[a,b]$ is a **refinement** of P if every point of P is also a point of P^*. Given partitions P_1 and P_2 of $[a,b]$, the partition P^* consisting entirely of the points of P_1 and the points of P_2 is called the **common refinement** of P_1 and P_2.

Lemma 6.2. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function.*

(i) *If P is a partition of $[a,b]$, and P^* is a refinement of P, then*

$$L(P,f) \le L(P^*,f) \quad \textit{and} \quad U(P^*,f) \le U(P,f),$$

and consequently,

$$U(P^*,f) - L(P^*,f) \le U(P,f) - L(P,f).$$

(ii) *If P_1 and P_2 are partitions of $[a,b]$, then $L(P_1,f) \le U(P_2,f)$.*

(iii) $L(f) \le U(f)$.

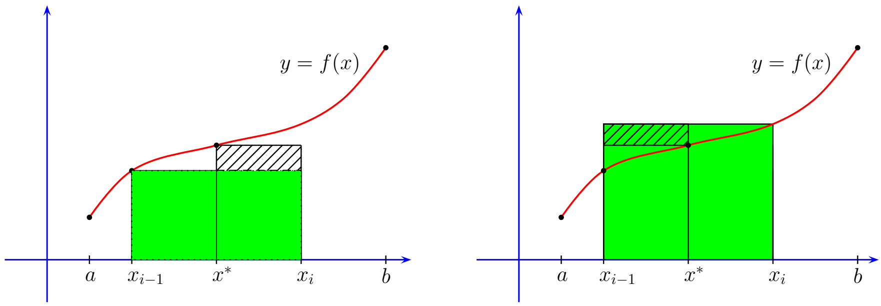

Fig. 6.2. By addition of a point x^* to the ith subinterval of a partition of $[a, b]$, the shaded area is added to the lower sum, but removed from the upper sum.

Proof. (i) Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. First let us assume that a refinement P^* of P contains just one additional point $x^* \in (a, b)$. Then $x_{i-1} < x^* < x_i$ for some $i \in \{1, \ldots, n\}$. (See Figure 6.2.) Using the abbreviations ℓ and r for "left" and "right" respectively, define

$$M_\ell^* := \sup\{f(x) : x \in [x_{i-1}, x^*]\} \quad \text{and} \quad M_r^* := \sup\{f(x) : x \in [x^*, x_i]\}.$$

Clearly, M_ℓ^* and M_r^* are both less than or equal to $M_i(f)$. Also, we can write $x_i - x_{i-1} = (x_i - x^*) + (x^* - x_{i-1})$ and therefore,

$$\begin{aligned} U(P, f) - U(P^*, f) &= M_i(f)(x_i - x_{i-1}) - M_\ell^*(x^* - x_{i-1}) - M_r^*(x_i - x^*) \\ &= (M_i(f) - M_\ell^*)(x^* - x_{i-1}) + (M_i(f) - M_r^*)(x_i - x^*) \\ &\geq 0 + 0 = 0. \end{aligned}$$

If a refinement P^* of P contains several additional points, we repeat the above argument several times and again obtain $U(P^*, f) \leq U(P, f)$. The proof of $L(P, f) \leq L(P^*, f)$ is similar. Subtracting, we obtain $U(P^*, f) - L(P^*, f) \leq U(P, f) - L(P, f)$.

(ii) Let P^* denote the common refinement of partitions P_1 and P_2. Then in view of (i) above,

$$L(P_1, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P_2, f).$$

(iii) Let us fix a partition P_0 of $[a, b]$. By (ii) above, $L(P_0, f) \leq U(P, f)$ for every partition P of $[a, b]$. Hence $L(P_0, f)$ is a lower bound of the set $\{U(P, f) : P \text{ is a partition of } [a, b]\}$. Since $U(f)$ is the greatest lower bound of this set, $L(P_0, f) \leq U(f)$. Now since P_0 is an arbitrary partition of $[a, b]$, we see that $U(f)$ is an upper bound of the set $\{L(P_0, f) : P_0 \text{ is a partition of } [a, b]\}$. Since $L(f)$ is the least upper bound of this set, we obtain $L(f) \leq U(f)$. $\square$

If a bounded function $f : [a, b] \to \mathbb{R}$ is nonnegative, and if we wish to define the "area" of the region lying under the graph of f, between the lines $x = a$,

$x = b$, and above the x-axis with the help of inscribed and circumscribing rectangles, then the "area" must be at least $L(f)$ and it can be at most $U(f)$. Thus, if $L(f) = U(f)$, then it would be appropriate to define the area to be this common value. This motivates the following definition.

Let $f : [a, b] \to \mathbb{R}$ be a function. Then f is said to be **integrable** (on $[a, b]$) if f is bounded and $L(f) = U(f)$. In this case, the common value $L(f) = U(f)$ is called the **Riemann integral** of f (on $[a, b]$) and it is denoted by

$$\int_a^b f(x)dx \quad \text{or simply by} \quad \int_a^b f.$$

The notation $\int_a^b f(x)dx$ emphasizes that f is "integrated" as a function of the "variable" x. While the Riemann integral of f does not depend on the name of the variable, this notation is useful when f depends on several parameters. The number $L(f)$ is called the **lower Riemann integral** of f and the number $U(f)$ the **upper Riemann integral** of f. Thus a bounded function on $[a, b]$ is integrable if its lower Riemann integral is equal to its upper Riemann integral. In the trivial case $a = b$, it is clear from the definitions that f is bounded and $L(f) = 0 = U(f)$. Thus f is integrable on $[a, a]$ and

$$\int_a^a f(x)dx = 0.$$

If a function $f : [a, b] \to \mathbb{R}$ is integrable and nonnegative, then the **area** of the region R_f under the curve given by $y = f(x)$, $x \in [a, b]$, is defined to be

$$\text{Area}(R_f) := \int_a^b f(x)dx, \text{ where } R_f := \{(x, y) \in \mathbb{R}^2 : a \le x \le b,\, 0 \le y \le f(x)\}.$$

We shall show later that in general, if $f : [a, b] \to \mathbb{R}$ is any integrable function, then its Riemann integral is equal to $\text{Area}(R_{f^+}) - \text{Area}(R_{f^-})$, where f^+ and f^- are the positive and the negative parts of f. (See Remark 6.21.) In this sense, the Riemann integral of f represents, in general, the "signed area" delineated by the curve $y = f(x)$, $x \in [a, b]$.

Admittedly, the definition of a Riemann integral of a bounded function $f : [a, b] \to \mathbb{R}$ is rather involved. This is because we need to consider lower sums for the function f with respect to all possible partitions of $[a, b]$ and calculate their supremum on the one hand, and also consider the corresponding upper sums and calculate their infimum on the other. We shall presently give several examples to illustrate what it takes to decide whether a bounded function on $[a, b]$ is integrable. When the definition of a Riemann integral has been well understood, it will be relatively easy to deduce its interesting properties and use them to obtain important results.

Now we give an elementary but useful estimate for the absolute value of a Riemann integral.

Proposition 6.3 (Basic Inequality for Riemann Integrals). *Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Let $\alpha, \beta \in \mathbb{R}$ be such that $\beta \le f \le \alpha$. Then $\beta(b-a) \le L(f) \le U(f) \le \alpha(b-a)$. Hence if f is integrable, then*

$$\beta(b-a) \le \int_a^b f(x)dx \le \alpha(b-a).$$

In particular, if $|f| \le \alpha$, then

$$\left| \int_a^b f(x)dx \right| \le \alpha(b-a).$$

Proof. Since $\beta \le f(x) \le \alpha$ for all $x \in \mathbb{R}$, we see that $\beta \le m(f)$ and $M(f) \le \alpha$. Let P be any partition of $[a, b]$. Then by Proposition 6.1,

$$\beta(b-a) \le m(f)(b-a) \le L(P, f) \quad \text{and} \quad U(P, f) \le M(f)(b-a) \le \alpha(b-a).$$

Consequently, $\beta(b-a) \le L(f) \le U(f) \le \alpha(b-a)$. Hence if f is integrable, then $L(f) = U(f) = \int_a^b f(x)dx$, and so

$$\beta(b-a) \le \int_a^b f(x)dx \le \alpha(b-a).$$

If $|f| \le \alpha$, then letting $\beta := -\alpha$, we obtain the desired conclusion. □

We work out a few examples to show how the integrability of a function can be investigated from first principles.

Examples 6.4. (i) Let $r \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be the constant function defined by $f(x) := r$. Clearly, f is bounded. Moreover, $m_i(f) = r = M_i(f)$ for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, and $i = 1, \ldots, n$. Hence

$$L(P, f) = U(P, f) = \sum_{i=1}^{n} r(x_i - x_{i-1}) = r(b-a),$$

and so $L(f) = r(b-a) = U(f)$. Thus f is integrable, and its Riemann integral is equal to $r(b-a)$.

(ii) Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be the **Dirichlet function** defined by

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is a rational number,} \\ 0 & \text{if } x \text{ is an irrational number.} \end{cases}$$

Clearly, f is bounded. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Since for each $i = 1, \ldots, n$, the interval $[x_{i-1}, x_i]$ contains a rational number and an irrational number, we see that $m_i(f) = 0$ and $M_i(f) = 1$. Hence

$$L(P, f) = \sum_{i=1}^{n} 0 \cdot (x_i - x_{i-1}) = 0, \quad \text{but} \quad U(P, f) = \sum_{i=1}^{n} 1 \cdot (x_i - x_{i-1}) = b-a.$$

Thus $L(f) = 0 \ne (b-a) = U(f)$. It follows that f is not integrable.

(iii) Let $f : [a, b] \to \mathbb{R}$ be the identity function defined by $f(x) := x$. Clearly, f is bounded. If $a = b$, then obviously f is integrable and $\int_a^b f(x)dx = 0$. Suppose $a < b$. Let $n \in \mathbb{N}$ and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Since $M_i(f) = x_i$ and $m_i(f) = x_{i-1}$ for $i = 1, \ldots, n$, we see that

$$U(P, f) = \sum_{i=1}^{n} x_i(x_i - x_{i-1}) \quad \text{and} \quad L(P, f) = \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1}).$$

Hence

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (x_i - x_{i-1})^2$$

and

$$U(P, f) + L(P, f) = \sum_{i=1}^{n} (x_i^2 - x_{i-1}^2) = b^2 - a^2.$$

It follows that

$$U(P, f) = \frac{b^2 - a^2}{2} + \frac{1}{2} \sum_{i=1}^{n} (x_i - x_{i-1})^2$$

and

$$L(P, f) = \frac{b^2 - a^2}{2} - \frac{1}{2} \sum_{i=1}^{n} (x_i - x_{i-1})^2.$$

In case $P := \mathsf{P}_n$ is the partition of $[a, b]$ into n equal parts, then

$$\sum_{i=1}^{n} (x_i - x_{i-1})^2 = \sum_{i=1}^{n} \frac{(b-a)^2}{n^2} = \frac{(b-a)^2}{n}.$$

Since $U(f) \le U(P, f)$ and $L(f) \ge L(P, f)$, we see that for every $n \in \mathbb{N}$,

$$U(f) \le \frac{b^2 - a^2}{2} + \frac{(b-a)^2}{2n} \quad \text{and} \quad L(f) \ge \frac{b^2 - a^2}{2} - \frac{(b-a)^2}{2n}.$$

By letting $n \to \infty$, we obtain $U(f) \le (b^2 - a^2)/2$ and $L(f) \ge (b^2 - a^2)/2$. On the other hand, by part (iii) of Lemma 6.2, $L(f) \le U(f)$. It follows that $L(f) = (b^2 - a^2)/2 = U(f)$, that is, f is integrable and

$$\int_a^b f(x)dx = \frac{b^2 - a^2}{2}.$$

This result will be generalized in Example 6.27 (i). ◇

The foregoing examples indicate that it is not easy to determine whether a bounded function on $[a, b]$ is integrable, and when the function is in fact integrable, it may be even more difficult to evaluate its Riemann integral. We shall first take up the question of determining whether a bounded function on $[a, b]$ is integrable. We give a simple criterion for this purpose and use it extensively in Section 6.2, not only to find large classes of integrable functions, but also to obtain many important properties of the Riemann integral. The more involved question concerning the evaluation of the Riemann integral will be discussed in Sections 6.3, 6.4, and later in Section 8.6.

Proposition 6.5 (Riemann Condition). *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\epsilon > 0$, there is a partition P_ϵ of $[a, b]$ such that*

$$U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon.$$

Proof. Suppose that the stated condition is satisfied. Then for every $\epsilon > 0$,

$$0 \leq U(f) - L(f) \leq U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon.$$

Hence $L(f) = U(f)$, that is, f is integrable.

Conversely, assume that f is integrable. Let $\epsilon > 0$ be given. By the definitions of $U(f)$ and $L(f)$, there are partitions Q_ϵ and $\widetilde{Q}_\epsilon$ of $[a, b]$ such that

$$U(Q_\epsilon, f) < U(f) + \frac{\epsilon}{2} \quad \text{and} \quad L(\widetilde{Q}_\epsilon, f) > L(f) - \frac{\epsilon}{2}.$$

Let P_ϵ be the common refinement of Q_ϵ and $\widetilde{Q}_\epsilon$. By part (i) of Lemma 6.2,

$$L(f) - \frac{\epsilon}{2} < L(\widetilde{Q}_\epsilon, f) \leq L(P_\epsilon, f) \leq U(P_\epsilon, f) \leq U(Q_\epsilon, f) < U(f) + \frac{\epsilon}{2}.$$

Since $L(f) = U(f)$, we obtain $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$, as desired. □

An immediate consequence of the above characterization is the following. Here and hereinafter, by a sequence (P_n) of partitions of $[a, b]$ we mean a map that associates to each $n \in \mathbb{N}$ a partition P_n of $[a, b]$.

Corollary 6.6. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if there is a sequence (P_n) of partitions of $[a, b]$ such that*

$$U(P_n, f) - L(P_n, f) \to 0.$$

In this case,

$$L(P_n, f) \to L(f) = \int_a^b f(x)dx \quad \textit{and} \quad U(P_n, f) \to U(f) = \int_a^b f(x)dx.$$

Proof. If f is integrable, then we can use the Riemann condition with $\epsilon = 1/n$ for each $n \in \mathbb{N}$ to obtain a desired sequence of partitions. Conversely, if there is a sequence (P_n) of partitions of $[a, b]$ such that $U(P_n, f) - L(P_n, f) \to 0$, then for every $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that $U(P_n, f) - L(P_n, f) < \epsilon$

for all $n \geq n_0$, and so the Riemann condition is satisfied with $P_\epsilon = P_{n_0}$, and hence by Proposition 6.5, f is integrable.

If (P_n) is a sequence of partitions such that $U(P_n, f) - L(P_n, f) \to 0$, then

$$0 \leq L(f) - L(P_n, f) \leq U(f) - L(P_n, f) \leq U(P_n, f) - L(P_n, f)$$

for every $n \in \mathbb{N}$, and hence $L(P_n, f) \to L(f)$. Similarly,

$$0 \leq U(P_n, f) - U(f) \leq U(P_n, f) - L(f) \leq U(P_n, f) - L(P_n, f)$$

for every $n \in \mathbb{N}$, and hence $U(P_n, f) \to U(f)$. $\square$

We give below a nontrivial example to illustrate the relative ease gained by the use of Corollary 6.6, as compared to the difficulty we faced earlier in showing the integrability of the identity function (Example 6.4 (iii)).

Example 6.7. Let $a, b \in \mathbb{R}$ with $0 \leq a < b$, $m \in \mathbb{N}$, and let $f(x) := x^m$ for $x \in [a, b]$. Clearly, f is bounded. For $n \in \mathbb{N}$, let $\mathsf{P}_n = \{x_0, x_1, \ldots, x_n\}$ be the partition of $[a, b]$ into n equal parts. Then $x_i - x_{i-1} = (b-a)/n$, $M_i(f) = x_i^m$, and $m_i(f) = x_{i-1}^m$ for $i = 1, \ldots, n$, and so

$$U(\mathsf{P}_n, f) - L(\mathsf{P}_n, f) = \frac{b-a}{n}\sum_{i=1}^{n}(x_i^m - x_{i-1}^m) = \frac{b-a}{n}(b^m - a^m) \to 0 \text{ as } n \to \infty.$$

Hence by Corollary 6.6, f is integrable, and moreover, $U(\mathsf{P}_n, f) \to \int_a^b f(x)dx$. In particular, if $m = 2$, then

$$\begin{aligned} U(\mathsf{P}_n, f) &= \frac{b-a}{n}\sum_{i=1}^{n}\left(a + i\frac{b-a}{n}\right)^2 \\ &= a^2(b-a) + 2\frac{a(b-a)^2}{n^2}\frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3}\frac{n(n+1)(2n+1)}{6} \end{aligned}$$

and therefore

$$\int_a^b f(x)dx = \lim_{n\to\infty} U(\mathsf{P}_n, f) = a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3} = \frac{b^3 - a^3}{3}.$$

The evaluation in the case $m = 1$ is simpler and is left to the reader. For an arbitrary m, the integral will be evaluated in Example 6.27 (i) using methods developed later in this chapter. For a direct approach, see Exercise 6.36. $\diamond$

Next, we prove an important and useful result that says that if $[a, b]$ is divided into two subintervals, then the Riemann integral of $f : [a, b] \to \mathbb{R}$ is the sum of the Riemann integrals of the restrictions of f to the two subintervals. This corresponds to the geometric notion that if a region R splits into two nonoverlapping regions R_1 and R_2, then the area of R is equal to the sum of the areas of R_1 and R_2.

Proposition 6.8 (Domain Additivity of Riemann Integrals). *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $c \in [a, b]$. Then f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and on $[c, b]$. In this case,*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Proof. If $c = a$ or $c = b$ (and in particular, if $a = b$), then the desired result is a trivial consequence of the definitions. Thus we assume that $a < b$ and $c \in (a, b)$. Now suppose f is integrable on $[a, b]$. Let $\epsilon > 0$ be given. Then by the Riemann condition, there is a partition P_ϵ of $[a, b]$ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$. Adjoining c to the points of P_ϵ, if c is not already a point of P_ϵ, we obtain a refinement $P_\epsilon^* = \{x_0, x_1, \dots, x_k, \dots, x_n\}$ of P_ϵ, where $c = x_k$ for some $k \in \{1, \dots, n-1\}$. Part (i) of Lemma 6.2 shows that

$$0 \le U(P_\epsilon^*, f) - L(P_\epsilon^*, f) \le U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon.$$

Now $Q_\epsilon^* := \{x_0, x_1, \dots, x_k\}$ is a partition of $[a, c]$, and if g denotes the restriction of f to $[a, c]$, then $U(Q_\epsilon^*, g) - L(Q_\epsilon^*, g)$ is equal to

$$\sum_{i=1}^k \big(M_i(g) - m_i(g)\big)(x_i - x_{i-1}) = \sum_{i=1}^k \big(M_i(f) - m_i(f)\big)(x_i - x_{i-1}).$$

Since $1 \le k < n$, we see that

$$\begin{aligned} U(Q_\epsilon^*, g) - L(Q_\epsilon^*, g) &\le \sum_{i=1}^n \big(M_i(f) - m_i(f)\big)(x_i - x_{i-1}) \\ &= U(P_\epsilon^*, f) - L(P_\epsilon^*, f) \\ &< \epsilon. \end{aligned}$$

Hence the Riemann condition shows that g is integrable, that is, f is integrable on $[a, c]$. Similarly, it can be seen that f is integrable on $[c, b]$.

Conversely, assume that f is integrable on $[a, c]$ and on $[c, b]$. Let g and h denote the restrictions of f to $[a, c]$ and to $[c, b]$ respectively. Let $\epsilon > 0$ be given. By the Riemann condition, there are partitions Q_ϵ of $[a, c]$ and R_ϵ of $[c, b]$ such that

$$U(Q_\epsilon, g) - L(Q_\epsilon, g) < \frac{\epsilon}{2} \quad \text{and} \quad U(R_\epsilon, h) - L(R_\epsilon, h) < \frac{\epsilon}{2}.$$

Let P_ϵ denote the partition of $[a, b]$ obtained from the points of Q_ϵ followed by the points of R_ϵ. Then P_ϵ contains the point c. So we obtain the decomposition

$$U(P_\epsilon, f) = U(Q_\epsilon, g) + U(R_\epsilon, h) \quad \text{and} \quad L(P_\epsilon, f) = L(Q_\epsilon, g) + L(R_\epsilon, h).$$

Consequently, $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon/2 + \epsilon/2 = \epsilon$. Thus, by the Riemann condition, f is integrable on $[a, b]$. The above decomposition also implies that

$$L(P_\epsilon, f) \le \int_a^c f(x)dx + \int_c^b f(x)dx \le U(P_\epsilon, f).$$

Since $L(P_\epsilon, f) \le \int_a^b f(x)dx \le U(P_\epsilon, f)$ and $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$, we obtain

$$\left| \int_a^c f(x)dx + \int_c^b f(x)dx - \int_a^b f(x)dx \right| < \epsilon.$$

Finally, since $\epsilon > 0$ is arbitrary, we see that

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

as desired. □

We shall see in the next two sections that domain additivity plays a crucial role in several proofs.

Remark 6.9. In accordance with our convention stated in Chapter 1, we have assumed that $a \le b$ while considering $[a, b]$ and defining the Riemann integral of $f : [a, b] \to \mathbb{R}$. We emphasize that the Riemann integral of f is defined over the *subset* $\{x \in \mathbb{R} : a \le x \le b\}$ of $\mathbb{R}$ and that we have not associated any direction or orientation with this subset. With this in mind, and in order to obtain uniformity of presentation and simplicity of notation, we define

$$\int_b^a f(x)dx := -\int_a^b f(x)dx.$$

The above equality is merely a *convention*; it is not a *result* that follows from our definitions, except when $b = a$. In view of this convention, domain additivity (Proposition 6.8) implies that

$$\int_c^d f(x)dx = \int_a^d f(x)dx - \int_a^c f(x)dx$$

for all points c and d in $[a, b]$. ◇

6.2 Integrable Functions

In this section we shall use the Riemann condition to prove the integrability of a wide variety of functions. We shall also consider algebraic and order properties of the Riemann integral. Readers who wish to directly look at the fundamental connection between differentiation and Riemann integration may pass on to the next section, assuming only that every continuous function on $[a, b]$ is integrable. This is proved in part (ii) of the following proposition.

Proposition 6.10. *Let $f : [a, b] \to \mathbb{R}$ be a function.*

(i) *If f is monotonic, then it is integrable.*
(ii) *If f is continuous, then it is integrable.*

Proof. Both (i) and (ii) hold trivially if $a = b$. Thus we assume that $a < b$.

(i) Suppose $f : [a, b] \to \mathbb{R}$ is monotonically increasing. Then f is bounded, since $f(a) \le f(x) \le f(b)$ for all $x \in [a, b]$. For $n \in \mathbb{N}$, let $\mathsf{P}_n = \{x_0, x_1, \ldots, x_n\}$ be the partition of $[a, b]$ into n equal parts. Then $f(x_{i-1}) \le f(x) \le f(x_i)$ for all $x \in [x_{i-1}, x_i]$, and hence $M_i(f) = f(x_i)$ and $m_i(f) = f(x_{i-1})$ for each $i = 1, \ldots, n$. Consequently,

$$U(\mathsf{P}_n, f) - L(\mathsf{P}_n, f) = \frac{b-a}{n} \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1}) \big) = \frac{b-a}{n} (f(b) - f(a)) \to 0.$$

Thus by Corollary 6.6, it follows that f is integrable. A similar proof holds if f is monotonically decreasing.

(ii) Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then f is bounded by part (i) of Proposition 3.10. Also, by Proposition 3.20, f is uniformly continuous. Hence by Proposition 3.22, for each $n \in \mathbb{N}$, there is $\delta_n > 0$ such that

$$x, y \in [a, b] \text{ and } |x - y| < \delta_n \Longrightarrow |f(x) - f(y)| < \frac{1}{n}.$$

For each $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ with $k_n > (b-a)/\delta_n$, and consider the partition $\mathsf{Q}_n = \{x_0, x_1, \ldots, x_{k_n}\}$ of $[a, b]$ into k_n equal parts. Then for $i = 1, \ldots, k_n$,

$$x, y \in [x_{i-1}, x_i] \Longrightarrow |x - y| \le \frac{b-a}{k_n} < \delta_n \Longrightarrow |f(x) - f(y)| < \frac{1}{n}.$$

Moreover, by part (ii) of Proposition 3.10, there are $x_i^*, y_i^* \in [x_{i-1}, x_i]$ such that $f(x_i^*) = M_i(f)$ and $f(y_i^*) = m_i(f)$ for each $i = 1, \ldots, k_n$. Consequently,

$$U(\mathsf{Q}_n, f) - L(\mathsf{Q}_n, f) = \frac{b-a}{k_n} \sum_{i=1}^{k_n} \big(f(x_i^*) - f(y_i^*) \big) < \frac{b-a}{k_n} \cdot \frac{k_n}{n} = \frac{b-a}{n} \to 0.$$

Thus by Corollary 6.6, it follows that f is integrable. □

The above proposition shows that monotonicity or continuity of a function is a sufficient condition for its integrability. Since a monotonic function need not be continuous (for example, $f(x) := 0$ if $a \le x \le (a+b)/2$ and $f(x) := 1$ if $(a+b)/2 < x \le b$) and a continuous function need not be monotonic (for example, $f(x) := |x-(a+b)/2|$ if $x \in [a, b]$), it follows that neither monotonicity nor continuity is a necessary condition for integrability. We now show that a function obtained by piecing together integrable functions is integrable.

Proposition 6.11. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $c \in [a, b]$. If f is integrable on $[a, c-\delta]$ for every $\delta > 0$ with $a \le c-\delta$, and f is integrable on $[c+\delta, b]$ for every $\delta > 0$ with $c + \delta \le b$, then f is integrable on $[a, b]$.*

Proof. Suppose f satisfies the conditions stated in the proposition. In case f is a constant function on $[a, b]$, then there is nothing to prove. Assume that f is not constant on $[a, b]$. Then $\alpha := M(f) - m(f) > 0$. Let $\epsilon > 0$ be given. We shall show that there is a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$.

First, suppose $c \in (a, b)$. Choose $\delta > 0$ such that $a \leq c - \delta < c + \delta \leq b$ and $\delta < \epsilon/6\alpha$. Let g, h denote the restrictions of the function f to $[a, c - \delta]$ and $[c + \delta, b]$ respectively. Since g and h are integrable, by the Riemann condition, there are partitions Q and R of $[a, c - \delta]$ and $[c + \delta, b]$ such that

$$U(Q, g) - L(Q, g) < \frac{\epsilon}{3} \quad \text{and} \quad U(R, h) - L(R, h) < \frac{\epsilon}{3}.$$

Let P be the partition of $[a, b]$ obtained from the points of Q followed by the points of R. Then

$$U(P, f) = U(Q, g) + 2\delta M_\delta + U(Q, h) \text{ and } L(P, f) = L(Q, g) + 2\delta m_\delta + L(Q, h),$$

where $M_\delta := \sup\{f(x) : x \in [c-\delta, c+\delta]\}$ and $m_\delta := \inf\{f(x) : x \in [c-\delta, c+\delta]\}$. Since $M_\delta - m_\delta \leq M(f) - m(f) = \alpha$, we see that

$$U(P, f) - L(P, f) < \frac{\epsilon}{3} + 2\delta\alpha + \frac{\epsilon}{3} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

So by the Riemann condition, f is integrable on $[a, b]$. In case $c = a$ or $c = b$, a similar argument using Q or R, as appropriate, and using $\epsilon/2\alpha$ and $\epsilon/2$ in place of $\epsilon/6\alpha$ and $\epsilon/3$, respectively, shows that f is integrable on $[a, b]$. □

Corollary 6.12. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $c \in [a, b]$. If f is monotonic or continuous on $[a, c)$ as well as $(c, b]$, then f is integrable on $[a, b]$. Consequently, if f has only a finite number of discontinuities in $[a, b]$, then f is integrable on $[a, b]$.*

Proof. The first assertion follows from Propositions 6.10 and 6.11. For the second assertion, suppose the only discontinuities of f are at $c_1, \ldots, c_n \in [a, b]$ with $c_1 < c_2 < \cdots < c_n$. Choose $b_1, \ldots, b_{n-1} \in (a, b)$ such that $c_i < b_i < c_{i+1}$ for $i = 1, \ldots, n - 1$. Let $b_0 := a$ and $b_n := b$. Then by the first assertion, f is integrable on $[b_{i-1}, b_i]$ for each $i = 1, \ldots, n$. Hence by repeated applications of Proposition 6.8, we see that f is integrable on $[a, b]$. □

We shall now use Proposition 6.11 to prove an interesting property of the Riemann integral. Roughly speaking, it says that if the values of an integrable function are arbitrarily changed at a finite number of points, then the modified function is also integrable and its Riemann integral is equal to the Riemann integral of the given function.

Proposition 6.13. *Let $g : [a, b] \to \mathbb{R}$ be integrable, and let $f : [a, b] \to \mathbb{R}$ be such that $\{x \in [a, b] : f(x) \neq g(x)\}$ is finite. Then f is integrable and*

$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

Proof. First, suppose $\{x \in [a,b] : f(x) \neq g(x)\} = \{c\}$ for some $c \in [a,b]$. Then by Proposition 6.11, we see that f is integrable. We shall now show that $\int_a^c f(x)dx = \int_a^c g(x)dx$. If $c = a$, this is clear from the definition. Suppose $c > a$. Let $\alpha := \max\{M(g), f(c)\}$. Then $|f(x)| \leq \alpha$ and $|g(x)| \leq \alpha$ for all $x \in [a,b]$. For any $\delta > 0$ with $a \leq c - \delta$, since $f = g$ on $[a, c-\delta]$, we see that

$$\left| \int_a^c f(x)dx - \int_a^c g(x)dx \right| = \left| \int_{c-\delta}^c f(x)dx \right| + \left| \int_{c-\delta}^c g(x)dx \right| \leq 2\alpha\delta.$$

Since $\delta > 0$ can be arbitrarily small, $\int_a^c f(x)dx = \int_a^c g(x)dx$. In a similar manner, we see that $\int_c^b f(x)dx = \int_c^b g(x)dx$. Hence by domain additivity for f and g, it follows that $\int_a^b f(x)dx = \int_a^b g(x)dx$.

If f differs from g at $c_1 < \cdots < c_n$ for some $c_1, \ldots, c_n \in [a,b]$, the integrability of f is obtained by repeated applications of the result proved above by arguing as in the proof of the second assertion in Corollary 6.12. □

Examples 6.14. (i) Let $f(x) := [x]$, the integer part of x, for all $x \in [a,b]$. Since f is monotonically increasing, it is integrable.

(ii) Let $f(x) := |x|$, the absolute value of x, for all $x \in [a,b]$. Since f is continuous, it is integrable.

(iii) Let $f : [a,b] \to \mathbb{R}$ be a polynomial function, or more generally, a rational function whose denominator does not vanish at any point in $[a,b]$. Then f is continuous on $[a,b]$, and hence it is integrable.

(iv) Let $f : [0,1] \to \mathbb{R}$ be the "infinite-steps function" given by

$$f(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1/n & \text{if } 1/(n+1) < x \leq 1/n \text{ for some } n \in \mathbb{N}. \end{cases}$$

(See Figure 6.3.) Since f is monotonically increasing on $[0,1]$, it is integrable. Note that f is discontinuous at infinitely many points in $[0,1]$.

(v) Let $f : [-1,1] \to \mathbb{R}$ be the zigzag function defined in Example 1.19 (iv). Since f is continuous (as shown in Example 3.7 (v)), it is integrable. Note that f is alternately increasing and decreasing on infinitely many subintervals of $[-1,1]$.

(vi) Let $f : [0,1] \to \mathbb{R}$ be the "broken-line function" given by

$$f(x) := \begin{cases} 0 & \text{if } x = 0 \text{ or } x = 1/n \text{ for some } n \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

(See Figure 6.3.) Here Corollary 6.12 is not directly applicable to f. Nevertheless, we can use it to show that f is integrable as follows. Let $\epsilon > 0$ be given, and choose $n_0 \in \mathbb{N}$ such that $1/n_0 < \epsilon/2$. Let $a_1 := 1/n_0$, and let g denote the restriction of f to $[a_1, 1]$. Since g is bounded and it is discontinuous only at $1/j$ for $1 \leq j \leq n_0$, Corollary 6.12 shows that g is integrable. By the Riemann condition, there is a partition Q_ϵ of $[a_1, 1]$

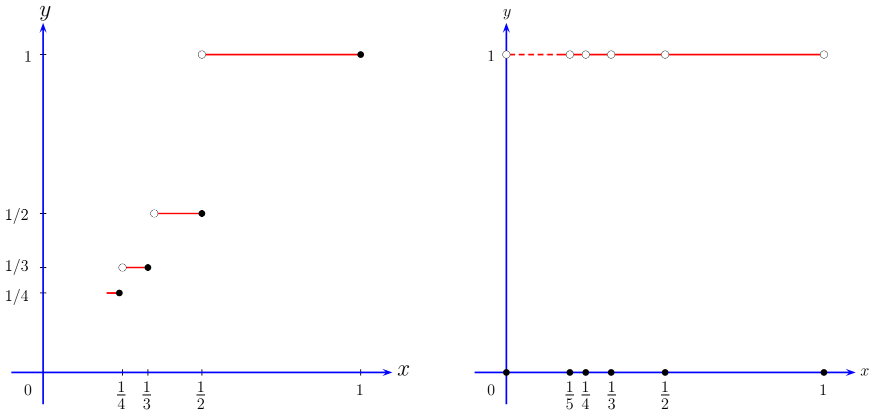

Fig. 6.3. Graphs of an "infinite-steps function" and a "broken-line function"

such that $U(Q_\epsilon, g) - L(Q_\epsilon, g) < \epsilon/2$. Let P_ϵ denote the partition of $[0,1]$ obtained by adding the point 0 to the partition Q_ϵ of $[a_1, 1]$. Since

$$\sup\{f(x) : 0 \le x \le a_1\} = 1 \quad \text{and} \quad \inf\{f(x) : 0 \le x \le a_1\} = 0,$$

we see that

$$U(P_\epsilon, f) = 1\cdot(a_1 - 0) + U(Q_\epsilon, g) \quad \text{and} \quad L(P_\epsilon, f) = 0\cdot(a_1 - 0) + L(Q_\epsilon, g).$$

Thus

$$U(P_\epsilon, f) - L(P_\epsilon, f) = (a_1 - 0) + U(Q_\epsilon, g) - L(Q_\epsilon, g) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence by the Riemann condition, f is integrable.

(vii) Consider the **Thomae function** $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1/q & \text{if } x = p/q, \text{ where } p, q \in \mathbb{Z},\ q > 0, \text{ and } p, q \text{ are coprime,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

We show that f is integrable. Since there is an irrational number between any two real numbers, we see that $L(P, f) = 0$ for every partition P of $[a, b]$. Let $\epsilon > 0$ be given and let $n_0 \in \mathbb{N}$ be such that $1/n_0 < \epsilon/2$. Since there are only finitely many rational numbers in $[0,1]$ having denominators less than n_0, it follows that the set $\{x \in [0,1] : f(x) \ge \epsilon/2\}$ is finite, say $\{c_1, \ldots, c_\ell\}$. Let $P_\epsilon = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[0,1]$ such that $(x_i - x_{i-1}) < \epsilon/4\ell$ for $i = 1, \ldots, n$. Now $c_1, \ldots, c_\ell$ belong to at most 2ℓ subintervals of P_ϵ, and if $x \in [0,1]$ belongs to any of the remaining subintervals, then $f(x) < \epsilon/2$. Also, $f(x) \le 1$ for all $x \in [0,1]$. Hence

$$U(P_\epsilon, f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) < 1\left(\frac{\epsilon}{4\ell}\right)2\ell + \frac{\epsilon}{2}\sum_{i=1}^{n}(x_i - x_{i-1}) = \epsilon.$$

Thus $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon - 0 = \epsilon$. The Riemann condition implies that f is integrable. Moreover,

$$\int_0^1 f(x)dx = \sup\{L(P, f) : P \text{ is a partition of } [0, 1]\} = 0.$$

We remark that f is discontinuous at each rational number in $[0, 1]$ and f is not monotonic on any subinterval $[c, d]$ with $0 \le c < d \le 1$. ◇

Remark 6.15. The integrability of a bounded function $f : [a, b] \to \mathbb{R}$ is intimately related to the nature of the set of points in $[a, b]$ at which f is discontinuous. This connection is further explored in Section 6.5. Some related results of a more advanced nature and a few relevant references are mentioned in the Notes and Comments at the end of this chapter. ◇

Algebraic and Order Properties

First we consider how Riemann integration behaves with respect to the algebraic operations on functions.

Proposition 6.16. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions. Then*

(i) *$f + g$ is integrable and $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$,*
(ii) *rf is integrable for every $r \in \mathbb{R}$ and $\int_a^b (rf)(x)dx = r \int_a^b f(x)dx$,*
(iii) *fg is integrable,*
(iv) *if there exists $\delta > 0$ such that $|f(x)| \ge \delta$ and all $x \in [a, b]$, then $1/f$ is integrable,*
(v) *if $f(x) \ge 0$ for all $x \in [a, b]$, then for every $k \in \mathbb{N}$, the function $f^{1/k}$ is integrable.*

Proof. Let $\epsilon > 0$ be given. By the Riemann condition, there are partitions Q and R of $[a, b]$ such that

$$U(Q, f) - L(Q, f) < \epsilon \quad \text{and} \quad U(R, g) - L(R, g) < \epsilon.$$

Let P_ϵ be the common refinement of Q and R. Then by part (i) of Lemma 6.2,

$$U(P_\epsilon, f) - L(P_\epsilon, f) \le U(Q, f) - L(Q, f) < \epsilon$$

and

$$U(P_\epsilon, g) - L(P_\epsilon, g) \le U(R, g) - L(R, g) < \epsilon.$$

(i) Let $P_\epsilon = \{x_0, x_1, \dots, x_n\}$. For all $i = 1, \dots, n$,

$$M_i(f + g) \le M_i(f) + M_i(g) \quad \text{and} \quad m_i(f + g) \ge m_i(f) + m_i(g).$$

Multiplying both sides of these inequalities by $x_i - x_{i-1}$ and summing from $i = 1$ to $i = n$, we obtain

$$U(P_\epsilon, f+g) \le U(P_\epsilon, f) + U(P_\epsilon, g) \quad \text{and} \quad L(P_\epsilon, f+g) \ge L(P_\epsilon, f) + L(P_\epsilon, g).$$

Hence

$$U(P_\epsilon, f+g) - L(P_\epsilon, f+g) \le U(P_\epsilon, f) - L(P_\epsilon, f) + U(P_\epsilon, g) - L(P_\epsilon, g) < \epsilon + \epsilon = 2\epsilon.$$

Hence by the Riemann condition, the function $f+g$ is integrable. Further, if we let $\alpha := U(P_\epsilon, f) + U(P_\epsilon, g)$ and $\beta := L(P_\epsilon, f) + L(P_\epsilon, g)$, then

$$\beta \le L(P_\epsilon, f+g) \le L(f+g) = \int_a^b (f+g)(x)dx = U(f+g) \le U(P_\epsilon, f+g) \le \alpha.$$

Also,

$$\beta \le L(f) + L(g) = \int_a^b f(x)dx + \int_a^b g(x)dx = U(f) + U(g) \le \alpha.$$

Thus we see that

$$\left| \int_a^b f(x)dx + \int_a^b g(x)dx - \int_a^b (f+g)(x)dx \right| \le \alpha - \beta < 2\epsilon.$$

Since this is true for every $\epsilon > 0$, we obtain

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

(ii) Let $r \in \mathbb{R}$. If $r = 0$, then $rf(x) = 0$ for all $x \in [a,b]$, and (ii) follows easily. Now assume that $r > 0$. Then for every partition P of $[a,b]$,

$$L(P, rf) = rL(P, f) \quad \text{and} \quad U(P, rf) = rU(P, f).$$

Hence

$$L(rf) = rL(f) = rU(f) = U(rf).$$

On the other hand, if $r < 0$, then for every partition P of $[a,b]$,

$$L(P, rf) = rU(P, f) \quad \text{and} \quad U(P, rf) = rL(P, f),$$

and so

$$L(rf) = rU(f) = rL(f) = U(rf).$$

In both cases, we see that rf is integrable and

$$\int_a^b (rf)(x)dx = r\int_a^b f(x)dx.$$

(iii) For all $i = 1, \ldots, n$, and $x, y \in [x_{i-1}, x_i]$,

$$\begin{aligned}(fg)(x)-(fg)(y)&=f(x)\big(g(x)-g(y)\big)+\big(f(x)-f(y)\big)g(y)\\&\le|f(x)|\,|g(x)-g(y)|+|g(y)|\,|f(x)-f(y)|\\&\le M(|f|)\big(M_i(g)-m_i(g)\big)+M(|g|)\big(M_i(f)-m_i(f)\big).\end{aligned}$$

Taking the supremum for x in $[x_{i-1},x_i]$ and the infimum for y in $[x_{i-1},x_i]$, we obtain

$$M_i(fg)-m_i(fg)\le M(|f|)\big(M_i(g)-m_i(g)\big)+M(|g|)\big(M_i(f)-m_i(f)\big).$$

Multiplying both sides of this inequality by x_i-x_{i-1} and summing from $i=1$ to $i=n$, we obtain

$$\begin{aligned}&U(P_\epsilon,fg)-L(P_\epsilon,fg)\\&\le M(|f|)\big(U(P_\epsilon,g)-L(P_\epsilon,g)\big)+M(|g|)\big(U(P_\epsilon,f)-L(P_\epsilon,f)\big)\\&<\big(M(|f|)+M(|g|)\big)\epsilon.\end{aligned}$$

Since $\epsilon>0$ arbitrary, the Riemann condition shows that fg is integrable.

(iv) Let $\delta>0$ be such that $|f(x)|\ge\delta$ for all $x\in[a,b]$. Then for all $i=1,\dots,n$ and $x,y\in[x_{i-1},x_i]$,

$$\frac{1}{f(x)}-\frac{1}{f(y)}=\frac{f(y)-f(x)}{f(x)f(y)}\le\frac{|f(x)-f(y)|}{|f(x)|\,|f(y)|}\le\frac{1}{\delta^2}\big(M_i(f)-m_i(f)\big).$$

Taking the supremum for x in $[x_{i-1},x_i]$ and the infimum for y in $[x_{i-1},x_i]$, we obtain

$$M_i\left(\frac{1}{f}\right)-m_i\left(\frac{1}{f}\right)\le\frac{1}{\delta^2}\big(M_i(f)-m_i(f)\big)$$

and consequently

$$U\left(P_\epsilon,\frac{1}{f}\right)-L\left(P_\epsilon,\frac{1}{f}\right)\le\frac{1}{\delta^2}\big(U(P_\epsilon,f)-L(P_\epsilon,f)\big)<\frac{\epsilon}{\delta^2}.$$

Again, since $\epsilon>0$ is arbitrary while $\delta>0$ is fixed, the Riemann condition shows that the function $1/f$ is integrable.

(v) Let $k\in\mathbb{N}$ and for simplicity, write $F=f^{1/k}$. First we assume that there exists $\delta>0$ such that $f(x)\ge\delta$ for all $x\in[a,b]$. For all x,y in $[a,b]$, we see that $f(x)-f(y)=F(x)^k-F(y)^k$ can be written as

$$\big(F(x)-F(y)\big)\big(F(x)^{k-1}+F(x)^{k-2}F(y)+\cdots+F(x)F(y)^{k-2}+F(y)^{k-1}\big).$$

Now

$$F(x)^{k-j}F(y)^{j-1}\ge\delta^{(k-j)/k}\delta^{(j-1)/k}=\delta^{(k-1)/k}>0\quad\text{for } j=1,\dots,k,$$

and so

$$F(x) - F(y) = \frac{f(x) - f(y)}{F(x)^{k-1} + F(x)^{k-2}F(y) + \cdots + F(x)F(y)^{k-2} + F(y)^{k-1}}$$
$$\leq \frac{f(x) - f(y)}{k\delta^{(k-1)/k}}.$$

If $P = \{x_0, x_1, \ldots, x_n\}$ is any partition of $[a, b]$ and $x, y \in [x_{i-1}, x_i]$ for some $i = 1, \ldots, n$, then

$$F(x) - F(y) \leq \frac{|f(x) - f(y)|}{k\delta^{(k-1)/k}} \leq \frac{M_i(f) - m_i(f)}{k\delta^{(k-1)/k}}.$$

Taking the supremum for x in $[x_{i-1}, x_i]$ and the infimum for y in $[x_{i-1}, x_i]$, we obtain

$$M_i(F) - m_i(F) \leq \frac{M_i(f) - m_i(f)}{k\delta^{(k-1)/k}} \quad \text{for } i = 1, \ldots, n.$$

Multiplying both sides of this inequality by $x_i - x_{i-1}$ and summing from $i = 1$ to $i = n$, we obtain

$$U(P, F) - L(P, F) \leq \frac{1}{k\delta^{(k-1)/k}} \big(U(P, f) - L(P, f)\big).$$

Since f is integrable, the Riemann condition shows that F is also integrable.

Next, we consider the general case of an arbitrary nonnegative integrable function f on $[a, b]$. Let $\delta > 0$ be given and define $g : [a, b] \to \mathbb{R}$ by $g(x) := f(x)+\delta$. For simplicity, write $G = g^{1/k}$. Then g is integrable by part (i) above, and $g(x) \geq \delta$ for all $x \in [a, b]$. It follows from what we have proved above that G is integrable. Moreover, since f is nonnegative,

$$G - \delta^{1/k} = (f + \delta)^{1/k} - \delta^{1/k} \leq f^{1/k} = F \leq (f + \delta)^{1/k} = G,$$

and therefore,

$$L(G - \delta^{1/k}) \leq L(F) \leq U(F) \leq U(G).$$

But

$$L(G-\delta^{1/k}) = L(G)-\delta^{1/k}(b-a) = \int_a^b G(x)dx-\delta^{1/k}(b-a) = U(G)-\delta^{1/k}(b-a).$$

This shows that

$$0 \leq U(F) - L(F) \leq U(G) - L(G - \delta^{1/k}) = \delta^{1/k}(b - a).$$

Since $\delta^{1/k} \to 0$ as $\delta \to 0$, we see that $F = f^{1/k}$ is integrable. □

We remark that there is no simple way to express the Riemann integral of fg in terms of the Riemann integrals of f and g. In Proposition 6.28, we shall give a method of evaluating the Riemann integral of fg under additional assumptions.

With notation and hypotheses as in Proposition 6.16, a combined application of its parts (i) and (ii) shows that the difference $f-g$ is integrable and

$$\int_a^b (f-g)(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

Further, given any $n \in \mathbb{N}$, successive applications of part (iii) of the above proposition show that the nth power f^n is integrable. Likewise, a combined application of parts (iii) and (iv) shows that if there exists $\delta > 0$ such that $|g(x)| \geq \delta$ for all $x \in [a,b]$, then the quotient f/g is integrable. Also, a combined application of parts (iii) and (v) shows that if $f(x) \geq 0$ for all $x \in [a,b]$, then given any positive $r \in \mathbb{Q}$, the rth power f^r is integrable, since $r = n/k$, where $n, k \in \mathbb{N}$.

The results obtained in Proposition 6.16 are in line with analogous results for continuity and differentiability of functions (Propositions 3.2 and 4.6). On the other hand, a composition of integrable functions need not be integrable. This is in contrast to the fact that continuity and differentiability are preserved by compositions of functions (Propositions 3.5 and 4.10).

Example 6.17. Let $f : [0,1] \to \mathbb{R}$ be the Thomae function and define

$$g(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } 0 < x \leq 1. \end{cases}$$

We have seen in Example 6.14 that f is integrable. Also, since $g : [0,1] \to \mathbb{R}$ is continuous on $(0,1]$, it follows from Corollary 6.12 that g is integrable. Now the composite functions $g \circ f$ and $f \circ g$ are both defined on $[0,1]$, and

$$f \circ g(x) = 1 \quad \text{for all } x \in [0,1], \quad \text{whereas} \quad g \circ f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Thus from Examples 6.4 (i) and (ii), we see that $f \circ g$ is integrable, whereas $g \circ f$ is not integrable. ◇

Remark 6.18. In Example 6.17, the function f is not continuous. It is also possible to construct a *continuous* function $f : [0,1] \to [0,1]$ and an integrable function $g : [0,1] \to \mathbb{R}$ such that $g \circ f$ is not integrable. (See Problem 28 of [70, Chap. 3] or the article [61].) On the other hand, if $f : [a,b] \to \mathbb{R}$ is integrable and $g : [m(f), M(f)] \to \mathbb{R}$ is continuous, then $g \circ f$ is integrable. (See Exercise 6.38.) ◇

Next, we consider how Riemann integration behaves with respect to the order relation on functions.

Proposition 6.19. *Let $f, g : [a,b] \to \mathbb{R}$ be integrable.*

(i) *If $f \leq g$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.*

(ii) *The function* $|f|$ *is integrable and* $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f|(x)dx$.

Proof. (i) Assume that $f(x) \leq g(x)$ for all $x \in [a, b]$. Then $U(P, f) \leq U(P, g)$ for every partition P of $[a, b]$, and so $U(f) \leq U(g)$. Since f and g are integrable,

$$\int_a^b f(x)dx = U(f) \leq U(g) = \int_a^b g(x)dx.$$

(ii) Let $\epsilon > 0$ be given. By the Riemann condition, there is a partition P_ϵ of $[a, b]$ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$. Let $P_\epsilon = \{x_0, x_1, \ldots, x_n\}$. For all $i = 1, \ldots, n$ and $x, y \in [x_{i-1}, x_i]$,

$$|f|(x) - |f|(y) \leq |f(x) - f(y)| \leq M_i(f) - m_i(f).$$

Taking the supremum for x in $[x_{i-1}, x_i]$ and the infimum for y in $[x_{i-1}, x_i]$, we obtain

$$M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f) \quad \text{ for } i = 1, \ldots, n.$$

Multiplying both sides of this inequality by $x_i - x_{i-1}$ and summing from $i = 1$ to $i = n$, we obtain

$$U(P_\epsilon, |f|) - L(P_\epsilon, |f|) \leq U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon.$$

Now the Riemann condition shows that $|f|$ is integrable. Further, since $-|f|(x) \leq f(x) \leq |f|(x)$ for all $x \in [a, b]$, by part (i) above we see that

$$\int_a^b -|f|(x)dx \leq \int_a^b f(x)dx \leq \int_a^b |f|(x)dx.$$

But $\int_a^b -|f|(x)dx = -\int_a^b |f|(x)dx$ by part (ii) of Proposition 6.16. Hence

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f|(x)dx,$$

as desired. $\square$

Corollary 6.20. *Let* $n \in \mathbb{N}$ *and let* $f_1, \ldots, f_n : [a, b] \to \mathbb{R}$ *be integrable. Then the functions* $\max(f_1, \ldots, f_n), \min(f_1, \ldots, f_n) : [a, b] \to \mathbb{R}$ *are integrable.*

Proof. Since for every $n > 1$, $\max(f_1, \ldots, f_n) = \max\left(\max(f_1, \ldots, f_{n-1}),\, f_n\right)$ and $\min(f_1, \ldots, f_n) = \min\left(\min(f_1, \ldots, f_{n-1}),\, f_n\right)$, the desired result will follow using induction on n if we show that whenever $f, g : [a, b] \to \mathbb{R}$ are integrable, then so are $\max(f, g)$ and $\min(f, g)$. But the latter follows from parts (i) and (ii) of Proposition 6.16 and part (ii) of Proposition 6.19 using $\max(f, g) = \frac{1}{2}\left(f + g + |f - g|\right)$ and $\min(f, g) = \frac{1}{2}\left(f + g - |f - g|\right)$. $\square$

Remark 6.21. Given $f : [a, b] \to \mathbb{R}$, define $f^+, f^- : [a, b] \to \mathbb{R}$ by

$$f^+(x) = \max\{f(x), 0\} \quad \text{and} \quad f^-(x) = -\min\{f(x), 0\} \quad \text{for } x \in [a, b].$$

Note that $f = f^+ - f^-$. Moreover, f^+ and f^- are nonnegative functions. The functions f^+ and f^- are known as the **positive part** and the **negative part** of f, respectively. By Corollary 6.20 together with parts (i) and (ii) of Proposition 6.16, we see that f is integrable if and only if both f^+ and f^- are integrable, and then

$$\int_a^b f(x)dx = \int_a^b f^+(x)dx - \int_a^b f^-(x)dx = \text{Area}(R_{f^+}) - \text{Area}(R_{f^-}),$$

where $R_{f^+} := \{(x, y) \in \mathbb{R}^2 : a \le x \le b \text{ and } 0 \le y \le f^+(x)\}$ and $R_{f^-} := \{(x, y) \in \mathbb{R}^2 : a \le x \le b \text{ and } 0 \le y \le f^-(x)\}$.

As remarked earlier, this suggests that the Riemann integral of f on $[a, b]$ can be interpreted as the "signed area" of the planar region delineated by the curve $y = f(x)$, $x \in [a, b]$. ◇

6.3 Fundamental Theorem of Calculus

Differentiation and integration are the two most important processes in calculus and analysis. As we have remarked in the introduction of Chapter 4, differentiation is a local process, that is, the value of the derivative at a point depends only on the values of the function in a small interval about that point. On the other hand, integration is a global process in the sense that the integral of a function depends on the values of the function on the entire interval. Further, these processes are defined in entirely different manners without any apparent connection between them. Indeed, from a geometric point of view, differentiation corresponds to finding (slopes of) tangents to curves, while integration corresponds to finding areas under curves. At first glance, there seems to be no reason for these two geometric processes to be intimately related.

In this section, we shall obtain a wonderful result, known as the Fundamental Theorem of Calculus, or for short, the FTC, which says that the processes of differentiating a function and integrating it are inverse to each other. Roughly speaking, if one first integrates a function and then differentiates it, one gets back the original function. Also, if one differentiates a function on an interval and then integrates it, again one gets back the original function. We remark that the proof of the FTC uses just the Riemann condition and domain additivity, and also the MVT. In particular, it does not depend on any of the results about integrable functions proved in Section 6.2.

Let us recall that if $f : [a, b] \to \mathbb{R}$ is differentiable, then we obtain a new function $f' : [a, b] \to \mathbb{R}$, called the derivative of f. Likewise, if $f : [a, b] \to \mathbb{R}$ is integrable, then we obtain a new function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)dt \quad \text{for } x \in [a, b].$$

Indeed, by Proposition 6.8, f is integrable on $[a, x]$ for every $x \in [a, b]$. Hence the function F is well-defined on $[a, b]$. Moreover, $F(a) = 0$. We begin by noting an important property of the new function F associated with f.

Proposition 6.22. *Let $f : [a, b] \to \mathbb{R}$ be integrable and let $F : [a, b] \to \mathbb{R}$ be defined by*

$$F(x) := \int_a^x f(t)dt.$$

Then F is continuous on $[a, b]$. In fact, F satisfies a **Lipschitz condition** *on $[a, b]$, namely there is $\alpha > 0$ such that*

$$|F(x) - F(y)| \le \alpha|x - y| \quad \textit{for all } x, y \in [a, b].$$

Proof. Since f is integrable on $[a, b]$, it is bounded on $[a, b]$, that is, there is $\alpha > 0$ such that $|f(t)| \le \alpha$ for all $t \in [a, b]$.

Let $c \in [a, b]$. Then for $x \in [a, b]$, by domain additivity (Proposition 6.8) and the convention stated in Remark 6.9, we see that

$$F(x) - F(c) = \int_a^x f(t)dt - \int_a^c f(t)dt = \int_c^x f(t)dt.$$

Hence by the basic inequality for Riemann integrals (Proposition 6.3),

$$|F(x) - F(c)| = \left| \int_c^x f(t)dt \right| \le \alpha\, |x - c|.$$

This implies that F is continuous at c.

Since x and c are arbitrary points in $[a, b]$, and α does not depend on them, we see that f satisfies a Lipschitz condition on $[a, b]$. □

Examples 6.23. (i) If $f : [-1, 1] \to \mathbb{R}$ is the integer part function defined by $f(x) := [x]$, then f is monotonically increasing, and hence integrable, but not continuous. Here

$$F(x) := \int_{-1}^x f(t)dt = \begin{cases} -1 - x & \text{if } x \in [-1, 0], \\ -1 & \text{if } x \in (0, 1]. \end{cases}$$

It is easily seen that F is continuous on $[-1, 1]$, and in fact, F satisfies the Lipschitz condition with $\alpha = 1$. (See Figure 6.4.)

(ii) If $f : [-1, 1] \to \mathbb{R}$ is the absolute value function defined by $f(x) := |x|$, then f is continuous and hence integrable, but not differentiable. Here

$$F(x) := \int_{-1}^x f(t)dt = \begin{cases} (1 - x^2)/2 & \text{if } x \in [-1, 0], \\ (1 + x^2)/2 & \text{if } x \in (0, 1]. \end{cases}$$

It is easily seen that F is differentiable on $[-1, 1]$. (See Figure 6.5.)

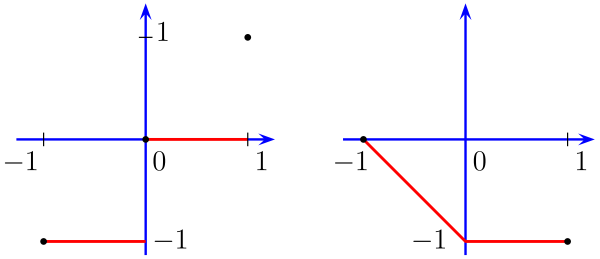

Fig. 6.4. Graphs of $f(x) = [x]$ and of $F(x) = \int_{-1}^{x} [t] dt$.

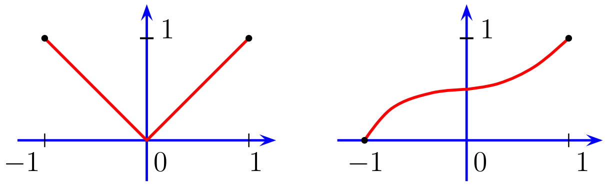

Fig. 6.5. Graphs of $f(x) = |x|$ and of $F(x) = \int_{-1}^{x} |t| dt$.

Proposition 6.22 shows that although an integrable function f may be discontinuous on $[a, b]$, the function $F : [a, b] \to \mathbb{R}$ obtained by integrating f from a to $x \in [a, b]$ is continuous on $[a, b]$. We shall see below in the first part of the FTC that if f happens to be continuous on $[a, b]$, then F is differentiable on $[a, b]$. Thus, unlike the process of differentiation, in which the derivative of a differentiable function may not be differentiable (Example 4.13 (ii)), integration is a smoothing process. This is illustrated by Examples 6.23.

Proposition 6.24 (Fundamental Theorem of Calculus). *Let $f : [a, b] \to \mathbb{R}$ be an integrable function.*

(i) *Define $F : [a, b] \to \mathbb{R}$ by*

$$F(x) := \int_a^x f(t)dt.$$

If f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$. Consequently, if f is continuous on $[a, b]$, then F is differentiable on $[a, b]$ and $F' = f$.

(ii) *If f is differentiable and f' is integrable on $[a, b]$, then*

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Proof. (i) Suppose f is continuous at $c \in [a, b]$. Define $F_1 : [a, b] \to \mathbb{R}$ by

$$F_1(x) := \begin{cases} \dfrac{F(x) - F(c)}{x - c} & \text{if } x \neq c, \\ f(c) & \text{if } x = c. \end{cases}$$

Then for $x \in [a, b]$ with $x \neq c$, by domain additivity,

$$F_1(x) - F_1(c) = \frac{1}{x - c}\left(\int_c^x f(t)dt\right) - f(c) = \frac{1}{x - c}\int_c^x \big(f(t) - f(c)\big)dt,$$

since $\int_c^x f(c)dt = f(c)(x - c)$. Let $\epsilon > 0$ be given. Since f is continuous at c, by Proposition 3.8, there is $\delta > 0$ such that

$$t \in [a, b] \text{ and } |t - c| < \delta \Longrightarrow |f(t) - f(c)| < \epsilon.$$

Hence if $x \in [a, b]$ with $0 < |x - c| < \delta$, then by the basic inequality for Riemann integrals (Proposition 6.3),

$$|F_1(x) - F_1(c)| \leq \frac{1}{|x - c|}\int_c^x \big|f(t) - f(c)\big|dt \leq \frac{\epsilon|x - c|}{|x - c|} = \epsilon.$$

This proves that F_1 is continuous at c. Thus from the Carathéodory Lemma (Proposition 4.2) and Remark 4.14, we conclude that F is differentiable at c and $F'(c) = f(c)$. This proves (i).

(ii) Suppose f is differentiable and f' is integrable on $[a, b]$. Consider a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$. By the MVT, there is $c_i \in (x_{i-1}, x_i)$ for each $i = 1, \ldots, n$ such that

$$f(b) - f(a) = \sum_{i=1}^{n}\big(f(x_i) - f(x_{i-1})\big) = \sum_{i=1}^{n} f'(c_i)(x_i - x_{i-1}).$$

Hence $L(P, f') \leq f(b) - f(a) \leq U(P, f')$. Since this holds for every partition P of $[a, b]$, we obtain $L(f') \leq f(b) - f(a) \leq U(f')$. But since f' is integrable, $L(f') = \int_a^b f'(x)dx = U(f')$, and so $\int_a^b f'(x)dx = f(b) - f(a)$. □

The following notion is motivated by the FTC. Let I be an interval containing more than one point and let $f : I \to \mathbb{R}$ be a function. We say that f **has an antiderivative** on I if there is a differentiable function $F : I \to \mathbb{R}$ such that $f = F'$. Such a function F is called an **antiderivative** or a **primitive** of f. It is clear from Corollary 4.23 that if an antiderivative of f exists, then it is unique up to addition of a constant. The existence of an antiderivative is ensured by part (i) of the FTC if $I = [a, b]$ and f is continuous on I. However, there exist integrable functions that have no antiderivative. Indeed, Proposition 4.16 shows that if f has an antiderivative on I, then f has the IVP on I. In particular, if $I = [a, b]$ with $b - a \geq 1$, then the integer part function on I is not the derivative of any function, since it does not have the IVP on I. On the other hand, a discontinuous function can have an antiderivative. An example is provided by the restriction to $[-1, 1]$ of the function f_0 considered in Proposition 7.19 with $f_0(0) = 0$.

Remarks 6.25. (i) In view of part (ii) of the FTC, if an integrable function $f : [a, b] \to \mathbb{R}$ has an antiderivative F, then F is called an **indefinite integral** of f, and it is denoted by $\int f(x)dx$. Note, however, that this notation is somewhat ambiguous, since an indefinite integral of f is unique only up to an additive constant. For this reason, one writes

$$\int f(x)dx = F(x) + C,$$

where C denotes an arbitrary constant. Notice that in this case,

$$\int_a^b f(x)dx = F(b) - F(a),$$

where the right side is independent of the choice of an indefinite integral. The right side of the above equality is sometimes denoted by

$$\big[F(x)\big]_a^b \quad \text{or} \quad F(x)\big|_a^b.$$

With this in mind, the Riemann integral of $f : [a, b] \to \mathbb{R}$ is sometimes referred to as the **definite integral** of f over $[a, b]$.

(ii) The converse of part (i) of the FTC is not true, that is to say, if $f : [a, b] \to \mathbb{R}$ is integrable and if F, defined as in part (i) of the FTC, is differentiable, then f need not be continuous. For example, if $f : [-1, 1] \to \mathbb{R}$ is defined by $f(0) := 1$ and $f(x) := 0$ for all $x \in [-1, 1]$ with $x \neq 0$, then it is clear that f is integrable, but not continuous, while $F(x) = \int_{-1}^{x} f(t)dt = 0$ for all $x \in [-1, 1]$ is differentiable on $[-1, 1]$.

(iii) Part (i) of the FTC shows that a continuous function on $[a, b]$ has an antiderivative. On a related note, several questions can be asked. For example, if $f : [a, b] \to \mathbb{R}$ has an antiderivative F on $[a, b]$, then

- must f be bounded?
- if f is bounded, then must it be integrable?
- if f is integrable, then must it be continuous?

The answers to all these questions are negative. Exercise 7.49 and Example 7.21 provide counterexamples for the first and the last questions. For the second question, we refer to Volterra's example given on pages 56–57 of [43].

(iv) Part (ii) of the FTC can be alternatively stated as follows. For an integrable function $f : [a, b] \to \mathbb{R}$ and a differentiable function $F : [a, b] \to \mathbb{R}$,

$$F' = f \Longrightarrow F(x) = F(a) + \int_a^x f(t)dt \;\text{ for all } x \in [a, b].$$

The reverse implication is not true. In other words, there are $f, F : [a, b] \to \mathbb{R}$ such that f is integrable on $[a, b]$ and $F(x) = F(a) + \int_a^x f(t)dt$ for all $x \in [a, b]$, but F is not differentiable. This is shown by Example 6.23 (i). $\diamond$

The two parts of the FTC can be combined to obtain a necessary and sufficient condition for a function to have a continuous derivative on $[a, b]$.

Corollary 6.26 (Fundamental Theorem of Riemann Integration). *Let $F : [a, b] \to \mathbb{R}$ be a function. Then F is continuously differentiable on $[a, b]$ if and only if there is a continuous function $f : [a, b] \to \mathbb{R}$ such that*

$$F(x) = F(a) + \int_a^x f(t)dt \quad \textit{for all } x \in [a, b].$$

In this case, $F'(x) = f(x)$ for all $x \in [a, b]$.

Proof. Suppose F is continuously differentiable on $[a, b]$. Then F' is integrable, and hence by part (ii) of the FTC,

$$\int_a^x F'(t)dt = F(x) - F(a) \quad \text{for all } x \in [a, b].$$

Letting $f := F'$, we obtain the desired representation of F.

Conversely, suppose there is a continuous function $f : [a, b] \to \mathbb{R}$ such that

$$F(x) = F(a) + \int_a^x f(t)dt \quad \text{for all } x \in [a, b].$$

Then by part (i) of the FTC, F is differentiable and $F'(x) = 0 + f(x) = f(x)$ for all $x \in [a, b]$. Since f is continuous, F is continuously differentiable. $\square$

As mentioned at the beginning of this section, the FTC shows that the processes of differentiation and integration are inverse to each other. The FTC is the major link between the so-called differential calculus and integral calculus. Also, part (ii) of the FTC provides the most widely used method of evaluating Riemann integrals. Of course, in order to employ it, one must be able to conjure up a function whose derivative is the given function f. It is not always easy to do so, but some corollaries of the FTC (Propositions 6.28 and 6.29) are useful in this regard. On the other hand, part (i) of the FTC can be used to construct a differentiable function whose derivative is equal to a given continuous function on an interval. We shall illustrate this powerful technique in Chapter 7 while introducing the logarithmic and arctangent functions.

Examples 6.27. (i) Let r be a rational number and suppose $r \neq -1$. Consider $a > 0$ and $f : [a, b] \to \mathbb{R}$ defined by $f(x) = x^r$. Then f is continuous on $[a, b]$, as we have seen in Example 3.7 (iii). Hence f is integrable. Also, it follows from Example 4.8 that if $F(x) := x^{r+1}/(r + 1)$ for $x \in [a, b]$, then $F' = f$. Hence part (ii) of the FTC shows that

$$\int_a^b x^r dx = F(b) - F(a) = \frac{b^{r+1} - a^{r+1}}{r + 1}.$$

(In Corollary 7.11, this result will be generalized to the case in which r is a real number other than -1.) It is easy to see that if r is a positive integer, then the above result holds even when $a \leq 0$, and if r is a negative integer $\neq -1$, then the above result also holds when $a \leq b < 0$.

(ii) Let $a, b \in \mathbb{R}$ with $a < 0 < b$, and define $f : [a, b] \to \mathbb{R}$ by $f(x) = x^2$ if $a \leq x \leq 0$ and $f(x) = x$ if $0 < x < b$. Then f is continuous on $[a, b]$, as we have seen in Example 3.7 (iv). Hence f is integrable. Let $F_1(x) = x^3/3$ for $x \in [a, 0]$ and $F_2(x) = x^2/2$ for $x \in [0, b]$. Then

$$\int_a^b f(x)dx = \int_a^0 f(x)dx + \int_0^b f(x)dx = 0 - \frac{a^3}{3} + \frac{b^2}{2} - 0 = \frac{b^2}{2} - \frac{a^3}{3}$$

by domain additivity and by (i) above. $\diamond$

Now we consider two important consequences of the FTC that yield the two most powerful methods of evaluating integrals. The first result is about the Riemann integral of the product of two functions.

Proposition 6.28 (Integration by Parts). *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable. Assume that $g : [a, b] \to \mathbb{R}$ is integrable and has an antiderivative G on $[a, b]$. Then*

$$\int_a^b f(x)g(x)dx = f(b)G(b) - f(a)G(a) - \int_a^b f'(x)G(x)dx.$$

Proof. Let $H := fG$. Then $H' = fG' + f'G = fg + f'G$, by part (iii) of Proposition 4.6. Since f and G are differentiable, they are continuous and hence integrable. Also, since f' and g are assumed to be integrable, it follows from parts (i) and (iii) of Proposition 6.16 that the function $H' = fg + f'G$ is integrable. Hence by part (ii) of the FTC,

$$\int_a^b \big(f(x)g(x) + f'(x)G(x)\big)dx = H(b) - H(a) = f(b)G(b) - f(a)G(a).$$

This together with part (i) of Proposition 6.16 proves the proposition. $\square$

In the notation of Remark 6.25 (i), the conclusion of the above proposition can be stated as follows:

$$\int_a^b f(x)g(x)dx = \left[f(x)\int g(x)dx\right]_a^b - \int_a^b \left(f'(x)\int g(x)dx\right)dx,$$

with the understanding that on the right side, $\int g(x)dx$ denotes the same indefinite integral of g at both places. (Recall that two indefinite integrals of g differ by an additive constant.)

Next, we consider the method of substitution for evaluating a Riemann integral. As in the last proposition, the proof is based on the FTC.

Proposition 6.29 (Integration by Substitution). *Let $f : [a, b] \to \mathbb{R}$ be continuous and let $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi([\alpha, \beta]) = [a, b]$. Assume that ϕ is differentiable and ϕ' is integrable on $[\alpha, \beta]$. Then $(f \circ \phi)\phi' : [\alpha, \beta] \to \mathbb{R}$ is integrable and*

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f(\phi(t))\phi'(t)dt.$$

In particular, if $\phi'(t) \neq 0$ for every $t \in (\alpha, \beta)$, then

$$\int_a^b f(x)dx = \int_{\alpha}^{\beta} f(\phi(t))|\phi'(t)|dt.$$

Proof. Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x f(u)du$. Part (i) of the FTC shows that F is differentiable and $F' = f$. Now consider the function $H : [\alpha, \beta] \to \mathbb{R}$ defined by $H := F \circ \phi$. Then by the Chain Rule (Proposition 4.10),

$$H'(t) = F'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t) \quad \text{for all } t \in [\alpha, \beta].$$

Since ϕ is differentiable, it is continuous, and since f is also continuous, the composite $f \circ \phi$ is continuous and hence integrable. Moreover, since ϕ' is integrable, by part (iii) of Proposition 6.16, the function $(f \circ \phi)\phi'$ is integrable. Also, $H := F \circ \phi$ and $H' = (f \circ \phi)\phi'$. Hence by part (ii) of the FTC,

$$\int_{\alpha}^{\beta} H'(t)dt = H(\beta) - H(\alpha) = \int_a^{\phi(\beta)} f(x)dx - \int_a^{\phi(\alpha)} f(x)dx = \int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx,$$

where the last equality follows from domain additivity (Proposition 6.8) and the convention stated in Remark 6.9. This proves the first assertion.

Now suppose $\phi'(t) \neq 0$ for every $t \in (\alpha, \beta)$. Then by the IVP of ϕ' on $[\alpha, \beta]$ (Proposition 4.16), we see that either $\phi'(t) > 0$ for all $t \in (\alpha, \beta)$, or $\phi'(t) < 0$ for all $t \in (\alpha, \beta)$. In the former case, ϕ is strictly increasing on $[\alpha, \beta]$, and so $\phi(\alpha) = a$, $\phi(\beta) = b$, and $|\phi'| = \phi'$, whereas in the latter case, ϕ is strictly decreasing on $[\alpha, \beta]$, and so $\phi(\alpha) = b$, $\phi(\beta) = a$, and $|\phi'| = -\phi'$. Thus

$$\int_a^b f(x)dx = \int_{\alpha}^{\beta} f(\phi(t))|\phi'(t)|dt,$$

thanks to the first assertion. □

If we let $f(x) := 1$ for all $x \in [a, b]$ in the above result, then we obtain

$$b - a = \int_{\alpha}^{\beta} |\phi'(t)|dt.$$

This shows that the absolute value of the derivative of the function ϕ acts as the change of scale factor in the method of substitution. For example, if p is a nonzero real number, $q \in \mathbb{R}$, and $\phi(t) := pt + q$ for all $t \in [\alpha, \beta]$, then the change of scale factor is the constant $|p|$.

Examples 6.30. (i) To evaluate $\int_0^1 x\sqrt{1-x}\,dx$, let $f(x) := x$ and $g(x) := \sqrt{1-x}$ for $x \in [0,1]$. Then $f'(x) = 1$ for $x \in [0,1]$. Also, if we let $G(x) := -(2/3)(1-x)^{3/2}$, then $G'(x) = g(x)$ for $x \in [0,1]$, that is, $G' = g$. Integrating by Parts (Proposition 6.28), we obtain

$$\int_0^1 x\sqrt{1-x}\,dx = 0 - 0 - \int_0^1 \left(-\frac{2}{3}\right)(1-x)^{3/2}dx = \frac{2}{3}\int_0^1 (1-x)^{3/2}dx.$$

If we let $F(x) := -(2/5)(1-x)^{5/2}$ for $x \in [0,1]$, then $F'(x) = (1-x)^{3/2}$ for $x \in [0,1]$, and hence

$$\int_0^1 x\sqrt{1-x}\,dx = \frac{2}{3}\big(F(1) - F(0)\big) = \frac{2}{3}\left(0 - \left(-\frac{2}{5}\right)\right) = \frac{4}{15}.$$

(ii) To evaluate $\int_0^1 t\sqrt{1-t^2}\,dt$, let $f(x) := \sqrt{x}$ for $x \in [0,1]$ and let us take $\phi(t) := 1 - t^2$ for $t \in [0,1]$. Note that f is continuous, $\phi(0) = 1$, $\phi(1) = 0$, and $\phi'(t) = -2t$ for all $t \in [0,1]$. In particular, $\phi'(t) \neq 0$ for all $t \in (0,1)$. Thus Integration by Substitution (Proposition 6.29) yields

$$\int_0^1 t\sqrt{1-t^2}\,dt = \frac{1}{2}\int_0^1 f(\phi(t))|\phi'(t)|dt = \frac{1}{2}\int_0^1 f(x)dx = \frac{1}{2}\int_0^1 \sqrt{x}\,dx.$$

Now if we let $F(x) := (2/3)x^{3/2}$ for $x \in [0,1]$, then

$$\frac{1}{2}\int_0^1 \sqrt{x}\,dx = \frac{1}{2}\big(F(1) - F(0)\big) = \frac{1}{2}\left(\frac{2}{3} - 0\right) = \frac{1}{3}.$$

Thus $\int_0^1 t\sqrt{1-t^2}dt = \frac{1}{3}$. $\diamond$

6.4 Riemann Sums

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. As a corollary of the Riemann condition, we have seen that f is integrable if and only if there is a sequence (P_n) of partitions of $[a,b]$ such that $U(P_n, f) - L(P_n, f) \to 0$. Moreover, in this case, $L(P_n, f) \to \int_a^b f(x)dx$ and $U(P_n, f) \to \int_a^b f(x)dx$. Although we have made good use of this corollary in Example 6.7 and Proposition 6.10, there are a number of difficulties in employing it to test the integrability of an arbitrary bounded function and, if such a function is found to be integrable, then to compute its Riemann integral. First, the calculation of $U(P, f)$ and $L(P, f)$, for a given partition P, involves finding the absolute maxima and minima of f over several subintervals of $[a,b]$. This task is rarely easy. Second, it is not clear how to go about choosing a partition P_n, $n \in \mathbb{N}$, so as to obtain $U(P_n, f) - L(P_n, f) \to 0$. Finally, when f is known to be integrable, how does one compute at least an approximate value of its integral without much difficulty? In this section, we shall address these questions.

To overcome the first difficulty mentioned above, namely of having to calculate several maxima and minima of f, we give an alternative approach. While calculating maxima and minima of f over several subintervals of $[a,b]$ may be difficult, evaluating f at several points of $[a,b]$ is relatively easy. With this in mind, we introduce the following concept. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$, and let $\mathcal{T}$ be a **tag set** associated with P, by which we mean a set $\mathcal{T} = \{s_1, \dots, s_n\}$, where s_i is a point in the ith subinterval $[x_{i-1}, x_i]$ for $i = 1, \dots, n$. Then

$$S(P, \mathcal{T}, f) := \sum_{i=1}^{n} f(s_i)(x_i - x_{i-1})$$

is called a **Riemann sum** for f corresponding to P and the tag set $\mathcal{T}$.

For example, the proof of Proposition 6.10 shows that $L(P,f)$ and $U(P,f)$ are Riemann sums $S(P, \mathcal{T}, f)$ for some $\mathcal{T}$ when f is monotonic or continuous.

Let us now address the second difficulty regarding the choice of a partition P so as to make $U(P,f) - L(P,f)$ small. The discussion at the beginning of this chapter suggests that we may start with any partition of $[a,b]$ and refine it successively. An important point to note here is the following: Mere addition of new points would not make the difference between the corresponding upper and lower sums tend to zero; the new points need to be so chosen that the length of the largest subinterval of the refined partition is smaller than the length of the largest subinterval of the given partition. This consideration leads us to the following notion.

For a partition P of $[a,b]$, we define the **mesh** of P to be the length of the largest subinterval of P. Thus, if $P = \{x_0, x_1, \dots, x_n\}$, then

$$\mu(P) := \max\{x_i - x_{i-1} : i = 1, \dots, n\}.$$

We prove an important result, which essentially says that upper sums $U(P,f)$ approximate the upper integral $U(f)$ and lower sums approximate the lower integral $L(f)$ if the mesh $\mu(P)$ of P is made small.

Proposition 6.31. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Given any $\epsilon > 0$, there is $\delta > 0$ such that for every partition P of $[a,b]$ with $\mu(P) < \delta$,*

$$0 \le L(f) - L(P,f) < \epsilon \quad \textit{and} \quad 0 \le U(P,f) - U(f) < \epsilon,$$

and consequently, for every tag set $\mathcal{T}$ associated with P,

$$L(f) - \epsilon < S(P, \mathcal{T}, f) < U(f) + \epsilon.$$

Proof. Let $\epsilon > 0$ be given. Since $U(f)$ is the infimum of the set of all upper sums for f and $L(f)$ is the supremum of the set of all lower sums for f, there are partitions P_1 and P_2 of $[a,b]$ such that $U(P_1, f) < U(f) + \epsilon/2$ and $L(P_2, f) > L(f) - \epsilon/2$. Let P_0 denote the common refinement of P_1 and P_2. Then by part (i) of Lemma 6.2,

$$U(P_0, f) < U(f) + \frac{\epsilon}{2} \quad \text{and} \quad L(P_0, f) > L(f) - \frac{\epsilon}{2}.$$

Let $\alpha > 0$ be such that $|f(x)| \le \alpha$ for all $x \in [a, b]$. If the partition P_0 contains n_0 points, define $\delta := \epsilon/4\alpha n_0$. Consider any partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ such that $\mu(P) < \delta$. Let P^* denote the common refinement of P and P_0. Again, by part (i) of Lemma 6.2,

$$U(P^*, f) \le U(P, f) \quad \text{and} \quad U(P^*, f) \le U(P_0, f).$$

We observe that positive contributions to the difference $U(P, f) - U(P^*, f)$ can arise only when a point x^* of the partition P_0 lies in an open interval (x_{i-1}, x_i) induced by the partition P. Further, any such contribution is at most $2\alpha\mu(P)$. This follows by noting that if $x^* \in (x_{i-1}, x_i)$, and if

$$M_\ell^* = \sup\{f(x) : x \in [x_{i-1}, x^*]\} \quad \text{and} \quad M_r^* = \sup\{f(x) : x \in [x^*, x_i]\},$$

then the contribution to $U(P, f) - U(P^*, f)$ arising from the point x^* is

$$\begin{aligned}
&M_i(f)(x_i - x_{i-1}) - M_\ell^*(x^* - x_{i-1}) - M_r^*(x_i - x^*) \\
&\quad = (M_i(f) - M_\ell^*)(x^* - x_{i-1}) + (M_i(f) - M_r^*)(x_i - x^*) \\
&\quad \le 2\alpha\big((x^* - x_{i-1}) + (x_i - x^*)\big) \\
&\quad = 2\alpha(x_i - x_{i-1}) \\
&\quad \le 2\alpha\mu(P).
\end{aligned}$$

Since the total number of points in the partition P_0 is n_0, we see that

$$U(P, f) - U(P^*, f) \le n_0 \cdot 2\alpha\mu(P) < 2\alpha n_0 \delta = \frac{\epsilon}{2}.$$

Thus for every partition P of $[a, b]$ with $\mu(P) < \delta$,

$$U(P, f) < U(P^*, f) + \frac{\epsilon}{2} \le U(P_0, f) + \frac{\epsilon}{2} < U(f) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = U(f) + \epsilon.$$

Similarly, we can show that for every partition P of $[a, b]$ with $\mu(P) < \delta$,

$$L(P, f) > L(f) - \epsilon.$$

Hence for every tag set $\mathcal{T}$ associated with a partition P of $[a, b]$ with $\mu(P) < \delta$,

$$L(f) - \epsilon < L(P, f) \le S(P, \mathcal{T}, f) \le U(P, f) < U(f) + \epsilon,$$

and thus $L(f) - \epsilon < S(P, \mathcal{T}, f) < U(f) + \epsilon$. □

The following consequence of Proposition 6.31 can be viewed as a refined version of the Riemann condition for integrability (Proposition 6.5).

Corollary 6.32. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\epsilon > 0$, there is $\delta > 0$ such that*

$$U(P, f) - L(P, f) < \epsilon \quad \textit{for every partition } P \textit{ of } [a, b] \textit{ with } \mu(P) < \delta.$$

Further, if f is integrable on $[a,b]$, then for every $\epsilon > 0$, there is $\delta > 0$ such that for every partition P of $[a,b]$ with $\mu(P) < \delta$ and every tag set $\mathcal{T}$ associated with P,

$$\left| S(P, \mathcal{T}, f) - \int_a^b f(x)dx \right| < \epsilon.$$

Proof. If f is integrable, then $L(f) = U(f) = \int_a^b f(x)dx$, and hence Proposition 6.31 implies both the assertions. The converse of the first assertion follows from the Riemann condition (Proposition 6.5). □

Corollary 6.33. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Let (P_n) be a sequence of partitions of $[a,b]$ such that $\mu(P_n) \to 0$. Then*

$$L(P_n, f) \to L(f) \quad \textit{and} \quad U(P_n, f) \to U(f).$$

In particular, if f is integrable on $[a,b]$, then

$$L(P_n, f) \to \int_a^b f(x)dx \quad \textit{and} \quad U(P_n, f) \to \int_a^b f(x)dx,$$

and moreover, if $\mathcal{T}_n$ is any tag set associated to P_n for $n \in \mathbb{N}$, then

$$S(P_n, \mathcal{T}_n, f) \to \int_a^b f(x)dx.$$

Proof. Let $\epsilon > 0$ be given. Then there is $\delta > 0$ satisfying the conclusion of Proposition 6.31. Since $\mu(P_n) \to 0$, there is $n_0 \in \mathbb{N}$ such that $0 \le \mu(P_n) < \delta$ for all $n \ge n_0$. Hence by Proposition 6.31, it follows that

$$|L(P_n, f) - L(f)| < \epsilon \text{ and } |U(P_n, f) - U(f)| < \epsilon \quad \text{for all } n \ge n_0,$$

So $L(P_n, f) \to L(f)$ and $U(P_n, f) \to U(f)$. If f is integrable, then $L(f) = U(f) = \int_a^b f(x)dx$, and also, $L(P_n, f) \le S(P_n, \mathcal{T}_n, f) \le U(P_n, f)$ for every tag set $\mathcal{T}_n$ associated with P_n for $n \in \mathbb{N}$. Thus the desired results follow. □

Remark 6.34. The above corollary answers the questions raised at the beginning of the section. One chooses partitions P_n whose mesh tends to zero as $n \to \infty$ and picks a suitable tag set for each of them. It may be emphasized that the only requirement here is that $\mu(P_n) \to 0$; the actual partition points and the points in the tag sets at which f is evaluated can be chosen with sole regard to the convenience of summation. This enables us to find approximations of the Riemann integral of f when we are not able to evaluate it exactly. For example, if f does not have an antiderivative, or if we are simply not able to think of an antiderivative of f, or if the evaluation of an antiderivative of f at a and b is impossible, then part (ii) of the FTC (Proposition 6.24) becomes inoperative as far as the evaluation of the Riemann integral of f is concerned, and we may resort to calculating it approximately. On the other hand, if the

Riemann integral of f can be evaluated by employing part (ii) of the FTC, then limits of certain Riemann sums for f can be found. In practice it will be convenient to use the partition P_n of the given interval into n equal parts and the tag sets consisting of the left endpoints or of the right endpoints or of the midpoints of the subintervals corresponding to the partition. ◇

Examples 6.35. (i) Let $f : [1, 2] \to \mathbb{R}$ be defined by $f(x) := 1/x$. Clearly, f is decreasing on $[1, 2]$, and hence f is integrable on $[1, 2]$. Let $n \in \mathbb{N}$ and let $\mathsf{P}_n := \{1, 1+(1/n), \ldots, 1+(n/n)\}$ be the partition of $[1, 2]$ into n equal parts. Let $\mathcal{T}_n := \{1 + (i/n) : i = 0, 1, \ldots, n-1\}$ be the tag set of the left endpoints of subintervals corresponding to P_n. Then $\mu(\mathsf{P}_n) = 1/n \to 0$ and

$$S(\mathsf{P}_n, \mathcal{T}_n, f) = \sum_{i=1}^{n} \frac{1}{1 + (i-1)/n}\left(\frac{i}{n} - \frac{i-1}{n}\right) = \sum_{i=1}^{n} \frac{1}{n+i-1}.$$

Hence by Corollary 6.33,

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} = \sum_{i=1}^{n} \frac{1}{n+i-1} \to \int_1^2 \frac{1}{x}dx.$$

(ii) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) := 1/(1+x^2)$. Clearly, f is continuous and hence integrable on $[0, 1]$. Let $n \in \mathbb{N}$ and let $\mathsf{P}_n := \{0, 1/n, \ldots, n/n\}$ be the partition of $[0, 1]$ into n equal parts. Let $\mathcal{T}_n := \{(i/n) : i = 1, \ldots, n\}$ be the tag set of the right endpoints of subintervals corresponding to P_n. Then $\mu(P_n) = 1/n \to 0$ and $S(\mathsf{P}_n, \mathcal{T}_n, f)$ is equal to

$$\sum_{i=1}^{n} \frac{1}{1 + (i/n)^2}\left(\frac{i}{n} - \frac{i-1}{n}\right) = \sum_{i=1}^{n} \frac{n^2}{n^2+i^2} \cdot \frac{1}{n} = \sum_{i=1}^{n} \frac{n}{n^2+i^2}.$$

Hence by Corollary 6.33,

$$\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2} = \sum_{i=1}^{n} \frac{n}{n^2+i^2} \to \int_0^1 \frac{1}{1+x^2}dx.$$

(iii) Consider

$$a_n := \sum_{i=1}^{n} \frac{1}{\sqrt{n^2+in}} = \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+2n}} + \cdots + \frac{1}{\sqrt{n^2+n^2}} \quad \text{for } n \in \mathbb{N}.$$

Then

$$a_n = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sqrt{1+(i/n)}} = \sum_{i=1}^{n} \frac{1}{\sqrt{1+(i/n)}}\left(\frac{i}{n} - \frac{i-1}{n}\right) \quad \text{for all } n \in \mathbb{N}.$$

We observe that $a_n = S(\mathsf{P}_n, \mathcal{T}_n, f)$, where $f(x) := 1/\sqrt{1+x}$ for $x \in [0, 1]$, $\mathsf{P}_n := \{0, 1/n, \ldots, n/n\}$, and $\mathcal{T}_n := \{(i/n) : i = 1, \ldots, n\}$. In this case,

f clearly has an antiderivative, namely $F(x) := 2\sqrt{1+x}$ for $x \in [0,1]$. Since $\mu(\mathsf{P}_n) = 1/n \to 0$, by Corollary 6.33 and part (ii) of the FTC (Proposition 6.24), we obtain

$$\sum_{i=1}^{n} \frac{1}{\sqrt{n^2+in}} \to \int_0^1 \frac{1}{\sqrt{1+x}}dx = F(1) - F(0) = 2(\sqrt{2}-1).$$

(iv) Let r be a nonnegative rational number and consider

$$a_n := \sum_{i=1}^{n} \frac{i^r}{n^{r+1}} = \frac{1^r + 2^r + \cdots + n^r}{n^{r+1}} \quad \text{for } n \in \mathbb{N}.$$

Then

$$a_n = \frac{1}{n}\sum_{i=1}^{n} \left(\frac{i}{n}\right)^r = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^r \left(\frac{i}{n} - \frac{i-1}{n}\right) \quad \text{for all } n \in \mathbb{N}.$$

As in the previous example, it follows that

$$\sum_{i=1}^{n} \frac{i^r}{n^{r+1}} \to \int_0^1 x^r dx = \frac{1}{r+1} \quad \text{as } n \to \infty. \qquad \diamond$$

We end this section with a result of theoretical interest. It is sometimes ascribed to Darboux, and it can be used to provide an alternative definition of the Riemann integral of a bounded function as a "limit of a sum". It is noteworthy that this characterization does not involve the notion of a mesh.

Proposition 6.36. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if there is $r \in \mathbb{R}$ satisfying the condition that for every $\epsilon > 0$, there is a partition P of $[a,b]$ such that*

$$|S(P, \mathcal{T}, f) - r| < \epsilon \quad \textit{for all tag sets } \mathcal{T} \textit{ associated with } P.$$

In this case, $r = \int_a^b f(x)dx$.

Proof. If f is integrable, then by the second assertion in Corollary 6.32, the stated condition holds with $r = \int_a^b f(x)dx$.

Conversely, suppose $r \in \mathbb{R}$ satisfies the stated condition. Let $\epsilon > 0$ be given. Then there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that

$$|S(P, \mathcal{T}, f) - r| < \epsilon \quad \text{for all tag sets } \mathcal{T} \text{ associated with } P.$$

For $i = 1, \ldots, n$, let us consider $m_i(f) := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ and $M_i(f) := \sup\{f(x) : x \in [x_{i-1}, x_i]\}$. Then there are $s_i, s_i' \in [x_{i-1}, x_i]$ such that

$$f(s_i) < m_i(f) + \epsilon \quad \text{and} \quad f(s_i') > M_i(f) - \epsilon \quad \text{for } i = 1, \ldots, n.$$

Consequently, if we let $\mathcal{T} = \{s_1, \ldots, s_n\}$ and $\mathcal{T}' = \{s'_1, \ldots, s'_n\}$, then

$$S(P, \mathcal{T}, f) < L(P, f) + \epsilon(b - a) \quad \text{and} \quad S(P, \mathcal{T}', f) > U(P, f) - \epsilon(b - a).$$

Since $L(P, f) \leq L(f) \leq U(f) \leq U(P, f)$, we obtain

$$S(P, \mathcal{T}, f) - \epsilon(b - a) < L(f) \leq U(f) < S(P, \mathcal{T}', f) + \epsilon(b - a).$$

But by the stated condition, $S(P, \mathcal{T}, f), S(P, \mathcal{T}', f) \in (r - \epsilon, r + \epsilon)$. Hence

$$r - \epsilon(1 + b - a) < L(f) \leq U(f) < r + \epsilon(1 + b - a).$$

Since $\epsilon > 0$ is arbitrary, we see that $r \leq L(f) \leq U(f) \leq r$, that is, $L(f) = U(f) = r$. Thus f is integrable and $\int_a^b f(x)dx = r$. □

6.5 Riemann Integrals over Bounded Sets

While it is both traditional and convenient to develop the theory of Riemann integration for real-valued functions defined on a closed and bounded interval in $\mathbb{R}$, it is not difficult to see that the theory extends to functions defined on arbitrary bounded subsets of $\mathbb{R}$. Such an extension will be worked out in this section. We shall also introduce the notion of a subset of $\mathbb{R}$ of (one-dimensional) content zero. It will be helpful in giving a useful sufficient condition for a function to be integrable on a bounded subset of $\mathbb{R}$. This condition generalizes the result in Corollary 6.12, where we showed that if f has only finitely many discontinuities in $[a, b]$, then f is integrable. We shall end this section with a formal definition of the classical concept of "length" of a bounded subset of $\mathbb{R}$.

Let D be a bounded subset of $\mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function. We say that f is **integrable** over D if f is a bounded function and if there are $a, b \in \mathbb{R}$ with $D \subseteq [a, b]$ such that the function $f^* : [a, b] \to \mathbb{R}$ defined by

$$f^*(x) := \begin{cases} f(x) & \text{if } x \in D, \\ 0 & \text{otherwise,} \end{cases}$$

is integrable on $[a, b]$. In this case, the **Riemann integral** of f over D is defined by

$$\int_D f(x)dx := \int_a^b f^*(x)dx.$$

Let us first observe that these notions are well-defined. In other words, the integrability of f and the value of its Riemann integral are independent of the choice of $a, b \in \mathbb{R}$ such that $D \subseteq [a, b]$. Indeed, since D is bounded, we can consider $m, M \in \mathbb{R}$ defined by

$$m := \inf\{f(x) : x \in D\} \quad \text{and} \quad M := \sup\{f(x) : x \in D\}.$$

Since $D \subseteq [a,b]$, we see that $a \le m$ and $M \le b$, that is, $[m,M] \subseteq [a,b]$; moreover, if $f^* : [a,b] \to \mathbb{R}$ is as above, then by domain additivity,

$$\int_a^b f^*(x)dx = \int_m^M f^*(x)dx,$$

since $f^*(x) = 0$ for all $x \in [a,b]$ with $x \notin [m,M]$.

The algebraic and order properties of Riemann integrals of functions on closed and bounded intervals in $\mathbb{R}$ extend to functions over the bounded set D. In other words, analogues of Propositions 6.16 and 6.19 hold for integrable functions on D. These analogues are straightforward consequences of the results proved earlier, except in the case of reciprocals of functions $f : D \to \mathbb{R}$ with $f(x) \neq 0$ for all $x \in D$, because the extended function f^* vanishes at points not in D. For this reason, we prove an analogue of part (iv) of Proposition 6.16. Subsequently, the algebraic and order properties of Riemann integrals over bounded sets may be used tacitly.

Proposition 6.37. *Let D be a bounded subset of $\mathbb{R}$ and let $f : D \to \mathbb{R}$ be integrable over D. Suppose there is $\delta > 0$ such that $|f(x)| \ge \delta$ for all $x \in D$. Then $1/f : D \to \mathbb{R}$ is integrable over D.*

Proof. Let $a, b \in \mathbb{R}$ be such that $D \subseteq [a,b]$. Write $h := 1/f$. Then $|h(x)| \le 1/\delta$ for all $x \in D$, and so h is a bounded function. Define $f^*, h^* : [a,b] \to \mathbb{R}$ by

$$f^*(x) := \begin{cases} f(x) & \text{if } x \in D, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad h^*(x) := \begin{cases} h(x) & \text{if } x \in D, \\ 0 & \text{otherwise.} \end{cases}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. We claim that

$$h^*(x) - h^*(y) \le \frac{1}{\delta^2}\big(M_i(f^*) - m_i(f^*)\big) \quad \text{for all } i = 1, \ldots, n \text{ and } x, y \in [x_{i-1}, x_i].$$

Fix $i \in \{1, \ldots, n\}$ and $x, y \in [x_{i-1}, x_i]$. If $x \in D$ and $y \in D$, then the claim follows exactly as in the proof of part (iv) of Proposition 6.16. If $x \notin D$ and $y \notin D$, then $h^*(x) = 0 = h^*(y)$, and the claim holds trivially. Suppose $x \in D$ and $y \notin D$. Then

$$h^*(x) - h^*(y) = \frac{1}{f(x)}.$$

In case $f(x) < 0$, the claim holds trivially. Suppose $f(x) \ge 0$. Then $f(x) \ge \delta$ and hence $M_i(f^*) \ge \delta$. Moreover, since $f^*(y) = 0$, we obtain $m_i(f^*) \le 0$. Thus in this case,

$$h^*(x) - h^*(y) = \frac{1}{f(x)} \le \frac{1}{\delta} \le \frac{M_i(f^*)}{\delta^2} \le \frac{1}{\delta^2}\big(M_i(f^*) - m_i(f^*)\big),$$

as claimed. When $x \notin D$ and $y \in D$, the claim is proved similarly. Now since the claim holds for all $x, y \in [x_{i-1}, x_i]$, by taking the supremum over x and the infimum over y, we obtain

$$M_i(h^*) - m_i(h^*) \le \frac{1}{\delta^2}\big(M_i(f^*) - m_i(f^*)\big) \quad \text{for all } i = 1, \ldots, n.$$

By summing from $i = 1$ to $i = n$, we see that

$$U(P, h^*) - L(P, h^*) \le \frac{1}{\delta^2}\big(U(P, f^*) - L(P, f^*)\big).$$

Thus the Riemann condition and the integrability of f^* imply that h^* is integrable on $[a, b]$, and so $h = 1/f$ is integrable over D. □

Sets of Content Zero

For a function $f : [a, b] \to \mathbb{R}$, we have given several sufficient conditions that ensure that f is integrable on $[a, b]$. For example, if f is continuous, or more generally, if f is bounded and has only finitely many discontinuities in $[a, b]$, then f is integrable. However, in the case of functions $f : D \to \mathbb{R}$ defined on a bounded subset D of $\mathbb{R}$, it is not only the nature of f but also of D that plays a role in the integrability of f. For example, if $D = [0, 1] \cap \mathbb{Q}$ and $f : D \to \mathbb{R}$ is defined by $f(x) := 1$ for all $x \in D$, then f is continuous on D, but f is not integrable over D, since its extension $f^* : [0, 1] \to \mathbb{R}$ is the Dirichlet function. In order to give sufficient conditions for integrability of functions over bounded sets, we shall now introduce a useful notion. Here by the **length** of a closed and bounded interval $[\alpha, \beta]$ in $\mathbb{R}$, we mean the nonnegative real number $\beta - \alpha$.

A bounded subset E of $\mathbb{R}$ is said to be of **(one-dimensional) content zero** if the following condition holds: For every $\epsilon > 0$, there is a finite number of closed and bounded intervals whose union contains E and the sum of whose lengths is less than ϵ.

Examples 6.38. (i) Every finite subset of $\mathbb{R}$ is of content zero. This is clear, because if $E := \{x_1, \ldots, x_n\} \subseteq \mathbb{R}$ and $\epsilon > 0$ is given, then we can cover E by the intervals $[x_i, x_i + (\epsilon/2n)]$ of length $\epsilon/2n$ each for $i = 1, \ldots, n$.

(ii) The infinite set $E := \{1/n : n \in \mathbb{N}\}$ is of content zero. To see this, let $\epsilon > 0$ be given. Then there is $n_0 \in \mathbb{N}$ such that $n_0 > 2/\epsilon$. Now $[0, \epsilon/2]$ contains $1/n$ for all $n \ge n_0$, while the interval $[1/n, 1/n + (\epsilon/2n_0)]$ contains $1/n$ for $n = 1, \ldots, n_0 - 1$. Thus the union of these n_0 intervals contains E, and the sum of the lengths of these intervals is less than ϵ.

(iii) Let $E := [a, b]$, where $a, b \in \mathbb{R}$ with $a < b$. Then E is not of content zero. This is intuitively obvious. To give a formal proof, let us suppose $E \subseteq [a_1, b_1] \cup \cdots \cup [a_n, b_n]$. Removing, if necessary, those $[a_j, b_j]$ that have no point in common with $E = [a, b]$, we may assume that $[a_j, b_j] \cap [a, b]$ is nonempty, that is, $a_j \le b$ and $b_j \ge a$, for all $j = 1, \ldots, n$. Now since $a \in [a_1, b_1] \cup \cdots \cup [a_n, b_n]$, there is some $j_1 \in \{1, \ldots, n\}$ such that $a_{j_1} \le a$. In case $b_{j_1} \ge b$, we obtain $(b_{j_1} - a_{j_1}) \ge (b - a)$. Otherwise, $b_{j_1} < b$, and so $b_{j_1} \in [a, b]$. Hence there is some $j_2 \in \{1, \ldots, n\}$ such that $a_{j_2} \le b_{j_1}$. Again, if $b_{j_2} \ge b$, then $j_2 \ne j_1$, and as indicated in Figure 6.6, we obtain

$$(b_{j_1} - a_{j_1}) + (b_{j_2} - a_{j_2}) \geq (b-a) + (b_{j_1} - a_{j_2}) \geq (b-a).$$

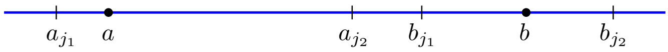

Fig. 6.6. Interval $[a, b]$ covered by overlapping intervals $[a_{j_1}, b_{j_1}]$, $[a_{j_2}, b_{j_2}]$.

Continuing in this way, we obtain distinct $j_1, \ldots, j_m \in \{1, \ldots, n\}$ such that $b_{j_m} \geq b$ and $a_{j_k} \leq b_{j_{k-1}}$ for $k = 2, \ldots, m$. Since $a_{j_1} \leq a$, we see that

$$\sum_{i=1}^{n} (b_i - a_i) \geq \sum_{k=1}^{m} (b_{j_k} - a_{j_k}) \geq (b-a) + \sum_{k=2}^{m} \left(b_{j_{k-1}} - a_{j_k}\right) \geq (b-a).$$

This shows that if $0 < \epsilon < b - a$, then $[a, b]$ cannot be covered by a finite union of closed and bounded intervals whose sum of lengths is less than ϵ. Thus $[a, b]$ is not of content zero.

(iv) The set $E := \mathbb{Q} \cap [0, 1]$ is not of content zero. To see this, let us suppose $E \subseteq [a_1, b_1] \cup \cdots \cup [a_n, b_n]$. Since $[a_1, b_1] \cup \cdots \cup [a_n, b_n]$ is a closed set, it must contain the limits of all convergent sequences with terms in $\mathbb{Q} \cap [0, 1]$, and therefore it contains $[0, 1]$. But then, as in (iii) above, the sum of the lengths of $[a_1, b_1], \ldots, [a_n, b_n]$ is ≥ 1.

More examples can be easily generated if we observe that a subset of a set of content zero is of content zero, and also that a finite union of sets of content zero is of content zero.

Proposition 6.39. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If the set of discontinuities of f is of content zero, then f is integrable.*

Proof. Let E denote the set of discontinuities of f. Since f is bounded, there is $\alpha \in \mathbb{R}$ with $|f(x)| \leq \alpha$ for all $x \in [a, b]$. Let $\epsilon > 0$ be given. Since E is of content zero, there are intervals $[a_1, b_1], \ldots, [a_m, b_m]$ such that

$$E \subseteq \bigcup_{k=1}^{m} [a_k, b_k] \quad \text{and} \quad \sum_{k=1}^{m} (b_k - a_k) < \frac{\epsilon}{2}.$$

Removing, if necessary, each interval $[a_k, b_k]$ that has no point in common with E, we may assume that $E \cap [a_k, b_k]$ is nonempty for each $k = 1, \ldots, m$. In particular, $[a, b] \cap [a_k, b_k]$ is a closed and bounded interval, and we may replace $[a_k, b_k]$ by $[a, b] \cap [a_k, b_k]$ and assume that $[a_k, b_k] \subseteq [a, b]$ for each $k = 1, \ldots, m$. Now for each $k = 1, \ldots, m$, let us enlarge $[a_k, b_k]$ slightly to $[c_k, d_k]$ so as to ensure that no point of E, except possibly a and b, is among the endpoints c_k, d_k and $[c_k, d_k] \subseteq [a, b]$ for all $k = 1, \ldots, m$, and further,

$$\sum_{k=1}^{m}(d_k - c_k) < \epsilon.$$

This is possible since E is of content zero, while each interval $[\alpha, \beta]$ with $\alpha < \beta$ is not of content zero. Clearly, $E \subseteq J$, where $J := \cup_{k=1}^{m}[c_k, d_k]$.

Now let P_ϵ be the partition of $[a, b]$ formed by the endpoints c_k, d_k for $k = 1, \ldots, m$ and a, b. Then each subinterval of $[a, b]$ induced by P_ϵ either is contained in J or is disjoint from E. Let E_0 denote the union of the subintervals of $[a, b]$ induced by P_ϵ that are disjoint from J. Then E_0 is closed and bounded, and f is continuous on E_0. Hence by Proposition 3.20, f is uniformly continuous on E_0. So by Proposition 3.22, there is $\delta > 0$ such that

$$x, y \in E_0 \text{ and } |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \epsilon.$$

Now let $P_\epsilon^* = \{x_0, x_1, \ldots, x_n\}$ be a refinement of P_ϵ such that $\mu(P_\epsilon^*) < \delta$. Then $U(P_\epsilon^*, f) - L(P_\epsilon^*, f)$ can be written as

$$\sum{}' \big(M_i(f) - m_i(f)\big)(x_i - x_{i-1}) + \sum{}'' \big(M_i(f) - m_i(f)\big)(x_i - x_{i-1}),$$

where $\sum'$ is the summation over those $i = 1, \ldots, n$ for which $[x_{i-1}, x_i]$ is contained in J, while $\sum''$ is the summation over the remaining $i = 1, \ldots, n$. By the uniform continuity of f on E_0,

$$\sum{}' \big(M_i(f) - m_i(f)\big)(x_i - x_{i-1}) \le \epsilon \sum{}'' (x_i - x_{i-1}) \le \epsilon\, (b - a).$$

On the other hand, all the subintervals $[x_{i-1}, x_i]$ corresponding to the second summation are contained in J, and hence their total length is at most $\sum_{k=1}^{m}(d_k - c_k)$, which is less than ϵ. Thus we obtain

$$\sum{}'' \big(M_i(f) - m_i(f)\big)(x_i - x_{i-1}) < \epsilon \sum{}'' \big(M_i(f) - m_i(f)\big) \le 2\alpha\epsilon.$$

It follows that $U(P_\epsilon^*, f) - L(P_\epsilon^*, f) \le (b - a + 2\alpha)\,\epsilon$. Hence by the Riemann condition, f is integrable over $[a, b]$. □

We can use the above result not only to check the integrability of various functions on closed and bounded intervals, but also to obtain a useful sufficient condition for a function on an arbitrary bounded subset of $\mathbb{R}$ to be integrable over that subset. Before proving this, let us recall (from Section 5.1) that a boundary point of a subset D of $\mathbb{R}$ is a real number c such that for every $r > 0$, the interval $(c - r, c + r)$ contains a point of D as well as a point not belonging to D. The **boundary** of D, denoted by ∂D, is the set of all boundary points of D. It is clear from the definition that if $c \in D$ is not a boundary point of D, then c is an interior point of D.

Corollary 6.40. *Let D be a bounded subset of $\mathbb{R}$ and let $f : D \to \mathbb{R}$ be a bounded function. If the boundary ∂D of D is of content zero and if the set of discontinuities of f is also of content zero, then f is integrable over D.*

In case D itself is of content zero, then f is integrable over D and the Riemann integral of f over D is equal to zero.

Proof. Let $[a, b]$ be a closed and bounded interval containing the set D and let $f^* : [a, b] \to \mathbb{R}$ be the extension of f obtained by putting $f^*(x) := 0$ for $x \in [a, b] \setminus D$. If E and E^* denote the sets of discontinuities of f and f^*, respectively, then it is clear that $E^* \subseteq E \cup \partial D$. Thus if both ∂D and E are of content zero, then so is E^*. Hence by Proposition 6.39, f^* is integrable on $[a, b]$, and so f is integrable over D.

In case D itself is of content zero, then for every $\epsilon > 0$, we can find a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that D is contained in the union of the subintervals $[x_{i-1}, x_i]$ induced by P whose total length is less than ϵ. Since f^* is zero outside D, it follows that

$$-\alpha\,\epsilon < L(P, f^*) \le U(P, f^*) < \alpha\,\epsilon, \quad \text{where} \quad \alpha := \sup\{|f(x)| : x \in D\}.$$

Hence by the Riemann condition, f^* is integrable on $[a, b]$ and $\int_a^b f^*(x)dx = 0$. Consequently, f is integrable over D and $\int_D f(x)dx = 0$. $\square$

Remark 6.41. The above corollary gives two conditions that together imply the integrability of a bounded function over a bounded subset D of $\mathbb{R}$. Neither of these conditions is necessary. For example, let $f(x) := 0$ for all $x \in D$. Then f is integrable over D even if ∂D is not of content zero. Next, let us suppose $D := [0, 1]$ and let $f : D \to \mathbb{R}$ be the Thomae function. As seen in Example 3.34, the set of discontinuities of f is $\mathbb{Q} \cap [0, 1]$, which is not of content zero. (See Example 6.38 (iv).) On the other hand, Example 6.14 (vii) shows that f is integrable. Note that the Thomae function also shows that the converse of Proposition 6.39 is not true.

Neither of the two conditions in Corollary 6.40 can be dropped from its hypotheses. For example, if $D := \mathbb{Q} \cap [0, 1]$ and $f(x) := 1$ for all $x \in D$, then f is continuous on D, but not integrable over D, since its extension f^* to $[0, 1]$ is the Dirichlet function. The Dirichlet function on $[0, 1]$ also provides an example of a function that is not integrable even though the boundary of $[0, 1]$ is of content zero. $\diamond$

Concept of Length of a Bounded Subset of $\mathbb{R}$

Let D be a bounded subset of $\mathbb{R}$. We say that D has **length** if the constant function

$$1_D : D \to \mathbb{R} \quad \text{defined by} \quad 1_D(x) := 1 \text{ for all } x \in D$$

is integrable over D. In this case, the length of D is defined by

$$\ell(D) := \int_D 1_D(x)dx.$$

For instance, if $D := [\alpha, \beta]$, then clearly 1_D is integrable on D and $\int_D 1_D(x)dx = \int_\alpha^\beta dx = \beta - \alpha$. Thus the above definition of length is consistent with the definition given in the beginning of this subsection of the length of a closed and bounded interval.

Proposition 6.42. *Let D be a bounded subset of $\mathbb{R}$. Then*

$$D \text{ has length} \Longleftrightarrow \partial D \text{ is of content zero.}$$

Furthermore,

$$D \text{ has length and } \ell(D) = 0 \Longleftrightarrow D \text{ is of content zero.}$$

Proof. Suppose ∂D is of content zero. Since the function 1_D is continuous on D, by Corollary 6.40, 1_D is integrable over D, that is, D has length.

Conversely, suppose D has length. Let $\epsilon > 0$ be given. Now $D \subseteq [a, b]$ for some $a, b \in \mathbb{R}$ with $a \le b$ and 1_D^* is integrable, where $1_D^* : [a, b] \to \mathbb{R}$ denotes the extension of 1_D obtained by putting $1_D^*(x) := 0$ for $x \in [a, b]$ with $x \notin D$. Hence there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that $U(P, 1_D^*) - L(P, 1_D^*) < \epsilon/2$. Note that $\partial D \subseteq [a, b]$ and consider the set

$$K := \{k \in \mathbb{N} : k \le n \text{ and } (x_{k-1}, x_k) \text{ contains a point of } \partial D\}.$$

Then for every $k \in K$, the open interval (x_{k-1}, x_k) must contain a point of D as well as a real number not in D. Hence

$$M_k(1_D^*) = 1 \quad \text{and} \quad m_k(1_D^*) = 0 \quad \text{for all } k \in K.$$

Consequently, $\sum_{k \in K} (x_k - x_{k-1})$ can be written as

$$\sum_{k \in K} \left(M_k(1_D^*) - m_k(1_D^*)\right) (x_k - x_{k-1}) \le U(P, 1_D^*) - L(P, 1_D^*) < \frac{\epsilon}{2}.$$

On the other hand, the finite set $\{x_0, x_1, \ldots, x_n\}$ can also be covered by finitely many closed and bounded intervals, the sum of whose lengths is less than $\epsilon/2$. But since $\partial D \subseteq [a, b]$, every point of ∂D is either in the finite set $\{x_0, x_1, \ldots, x_n\}$ or in (x_{k-1}, x_k) for some $k = 1, \ldots, n$, and this k is necessarily in K. It follows that ∂D can be covered by finitely many closed and bounded intervals, the sum of whose lengths is less than ϵ. Thus ∂D is of content zero.

Furthermore, suppose D has length and $\ell(D) = 0$. Then $U(1_D^*) = 0$. Hence for every $\epsilon > 0$, there is a partition P of $[a, b]$ with the property that $U(P, 1_D^*) < \epsilon$. Now D is a subset of the union of the subintervals induced by P that contain a point of D. The sum of lengths of these subintervals is evidently $U(P, 1_D^*)$, which is less than ϵ. This shows that D has content zero. Conversely, if D has content zero, then by Corollary 6.40, 1_D is integrable over D and $\int_D 1_D(x)dx = 0$, that is, D has length and $\ell(D) = 0$. □

We end this section with an analogue of domain additivity for Riemann integrals of functions defined over bounded subsets of $\mathbb{R}$. In Proposition 6.8, we have seen that if $f : [a, b] \to \mathbb{R}$ is integrable over subintervals $[a, c]$ and $[c, b]$ (where $c \in [a, b]$), then it is integrable over $[a, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$. Note that the intersection of these subintervals is "small", namely just the point $\{c\}$. In the version below, we will require the intersection of two subsets of a given bounded set to have content zero.

Proposition 6.43. *Let D_1 and D_2 be bounded subsets of $\mathbb{R}$ such that $D_1 \cap D_2$ is of content zero. Suppose $D := D_1 \cup D_2$ and $f : D \to \mathbb{R}$ is a bounded function such that f is integrable over D_1 and over D_2. Then f is integrable over D and*

$$\int_D f(x)dx = \int_{D_1} f(x)dx + \int_{D_2} f(x)dx.$$

Proof. Since $D_1 \cap D_2$ is of content zero, by Corollary 6.40, f is integrable over $D_1 \cap D_2$ and $\int_{D_1 \cap D_2} f(x)dx = 0$. Let f_1, f_2, and g denote the restrictions of f to D_1, D_2, and $D_1 \cap D_2$, respectively. Also, let $[a, b]$ be a closed and bounded interval containing D, and let f^*, f_1^*, f_2^*, and g^* denote the extensions of f, f_1, f_2, and g to $[a, b]$ obtained by defining them to be zero outside D, D_1, D_2, and $D_1 \cap D_2$, respectively. Then

$$f^*(x) = f_1^*(x) + f_2^*(x) - g(x) \quad \text{for all } x \in [a, b].$$

Moreover, since f_1^*, f_2^*, and g^* are integrable on $[a, b]$, it follows that f^* is integrable on $[a, b]$ and

$$\int_a^b f^*(x)dx = \int_a^b f_1^*(x)dx + \int_a^b f_2^*(x)dx - \int_a^b g^*(x)dx.$$

Since $\int_a^b g^*(x)dx = \int_{D_1 \cap D_2} f(x)dx = 0$, this yields the desired result. $\square$

We remark that if in Proposition 6.43, instead of assuming that $D_1 \cap D_2$ is of content zero, we assume that f is integrable over $D_1 \cap D_2$, then by a similar argument, we will see that f is integrable over $D_1 \cup D_2$ and

$$\int_{D_1 \cup D_2} f(x)dx = \int_{D_1} f(x)dx + \int_{D_2} f(x)dx - \int_{D_1 \cap D_2} f(x)dx.$$

Corollary 6.44. *Let D_1 and D_2 be bounded subsets of $\mathbb{R}$ such that each of D_1, D_2, and $D_1 \cap D_2$ has length, and moreover $\ell(D_1 \cap D_2) = 0$. Then $D_1 \cup D_2$ has length and*

$$\ell(D_1 \cup D_2) = \ell(D_1) + \ell(D_2).$$

Proof. By Proposition 6.42, $D_1 \cap D_2$ is of content zero. Hence by taking $D := D_1 \cup D_2$ and $f := 1_D$, the desired result follows from Proposition 6.43. $\square$

As before, if in Corollary 6.44, we assume that $D_1 \cap D_2$ has length, then we can show that $D_1 \cup D_2$ has length and the following general formula holds:

$$\ell(D_1 \cup D_2) = \ell(D_1) + \ell(D_2) - \ell(D_1 \cap D_2).$$

Notes and Comments

Given a bounded function $f : [a, b] \to \mathbb{R}$, we have produced two candidates, namely $U(f)$ and $L(f)$, for being designated the integral of f; when they coincide, we say that f is integrable and the common value is called its Riemann integral. While this approach demands patience and careful attention on the part of the reader to begin with, it is a natural way to formulate a plausible definition of "area". Hence we have preferred it to an alternative approach of defining integrability in terms of the existence of a "limit" of Riemann sums.

We have deduced all the essential features of the set of integrable functions from a single criterion called the Riemann condition. It is simple to state and easy to use. It does not involve the concept of a mesh of a partition. While we have given a number of sufficient conditions for a bounded function f on $[a, b]$ to be integrable, we have not discussed a characterization of integrability in terms of the nature of the set of points of $[a, b]$ at which f is discontinuous. This characterization involves the notion of (Lebesgue) measure, or at least the notion of a subset of $\mathbb{R}$ having (Lebesgue) measure zero. It can be stated as follows: A bounded function $f : [a, b] \to \mathbb{R}$ is integrable if and only if the set of discontinuities of f has (Lebesgue) measure zero. We refer the reader to Theorem 11.33 of Rudin [71] or to Theorem 7.34 of Goldberg [35] for this result. It can be used to derive some of the main properties of the Riemann integral rather neatly. (See, for example, Section 7.3 of Goldberg [35].) A weaker version of this condition, involving the notion of a subset of $\mathbb{R}$ having content zero, is treated in Section 6.5.

The Fundamental Theorem of Calculus (FTC) has two parts. In essence, the first part says that the integral of the derivative of a function gives back the function, whereas the second part says that the derivative of the integral of a function gives back the function. We have given proofs of these two parts that are independent of each other. When the given function is continuously differentiable, the first part can be easily deduced from the second. (See Exercise 6.13.) The methods of Integration by Parts and Integration by Substitution, which are particularly useful in evaluations of integrals, are shown as easy consequences of the FTC. Our statement of the result concerning integration by substitution includes a statement that can be viewed as a counterpart of the change of variables formula in multivariable calculus involving the absolute value of the Jacobian. For a statement of this formula for functions of two or three variables, we refer to Propositions 5.61 and 5.71 of [33] and the references therein.

In the section on Riemann sums, we have introduced the concept of the mesh of a partition and used it to obtain approximations of a Riemann integral and also to calculate limits of certain sequences, each term of which is a Riemann sum. We have carefully avoided any mention of "limits" as the mesh of a partition approaches zero, since that would involve a more general notion of a "limit of a net". The interested reader may consult the description of the "Riemann net" given on page 230 of the book of Joshi [44].

Exercises

Part A

6.1. Let $c \in (a, b)$ and $f, g : [a, b] \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} 0 & \text{if } a \le x \le c, \\ 1 & \text{if } c < x \le b, \end{cases} \quad \text{and} \quad g(x) := \begin{cases} (x-c)/(a-c) & \text{if } a \le x \le c, \\ (x-c)/(b-c) & \text{if } c < x \le b. \end{cases}$$

Show from first principles that both f and g are integrable on $[a, b]$. Also, prove that this follows from Corollary 6.12. (Hint: For $n \in \mathbb{N}$, consider the partitions of $[a, c]$ and $[c, b]$ into n equal parts each.)

6.2. Let $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} 1 + x & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$$

Is f integrable?

6.3. Let $f : [a, b] \to \mathbb{R}$ be integrable. Show that the Riemann integral of f is the unique real number r satisfying the following condition: For every $\epsilon > 0$, there is a partition P_ϵ of $[a, b]$ such that

$$r - \epsilon < L(P_\epsilon, f) \le r \le U(P_\epsilon, f) < r + \epsilon.$$

6.4. Let $f : [0, 3] \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 0 & \text{if } 0 \le x \le 1, \\ 2 & \text{if } 1 < x \le 2, \\ -1 & \text{if } 2 < x \le 3. \end{cases}$$

Show that f is neither monotonic nor continuous on [0,3], but f is integrable on $[0, 3]$. Find the Riemann integral of f.

6.5. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions. Show that

$$L(f) + L(g) \le L(f + g) \quad \text{and} \quad U(f + g) \le U(f) + U(g).$$

Hence conclude that if f and g are integrable, then so is $f + g$, and the Riemann integral of $f + g$ is equal to the sum of the Riemann integrals of f and g.

6.6. Give examples of bounded functions $f, g : [a, b] \to \mathbb{R}$ that are not integrable, but $|f|$, $f + g$, and fg are all integrable.

6.7. Let $f : [a, b] \to \mathbb{R}$ be a function. Show that f is integrable if (i) rf is integrable for some nonzero $r \in \mathbb{R}$, or (ii) if f is bounded, $f(x) \ne 0$ for all $x \in [a, b]$, and $1/f$ is integrable.

6.8. Let $f : [a,b] \to \mathbb{R}$ be any function. Suppose there is $r \in \mathbb{R}$ and for each $n \in \mathbb{N}$, there are integrable functions $g_n, h_n : [a,b] \to \mathbb{R}$ with $g_n \le f \le h_n$ such that $\int_a^b g_n(x)dx \to r$ and $\int_a^b h_n(x)dx \to r$ as $n \to \infty$. Show that f is integrable and the Riemann integral of f is equal to r.

6.9. Let $f : [a,b] \to \mathbb{R}$ be integrable and $f(x) \ge 0$ for all $x \in [a,b]$. Show that $\int_a^b f(x)dx \ge 0$. If, in addition, f is continuous and $\int_a^b f(x)dx = 0$, then show that $f(x) = 0$ for all $x \in [a,b]$. Give an example of an integrable function on $[a,b]$ such that $f(x) \ge 0$ for all $x \in [a,b]$ and $\int_a^b f(x)dx = 0$, but $f(x) \ne 0$ for some $x \in [a,b]$.

6.10. Evaluate the following limits.

(i) $\displaystyle\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} \frac{du}{u+\sqrt{u^2+1}}$, (ii) $\displaystyle\lim_{x\to 0} \frac{1}{x^3} \int_0^x \frac{t^2 dt}{t^4+1}$,

(iii) $\displaystyle\lim_{x\to 0} \frac{1}{x^6} \int_0^{x^2} \frac{t^2 dt}{t^6+1}$, (iv) $\displaystyle\lim_{x\to x_0} \frac{x}{x-x_0} \int_{x_0}^x f(t)dt$,

(v) $\displaystyle\lim_{x\to x_0} \frac{x}{x^2-x_0^2} \int_{x_0}^x f(t)dt$, provided f is continuous at x_0.

6.11. Let $a, b, c \in \mathbb{R}$ with $a < c < b$, and for $j = 1, 2, 3$, consider $f_j : [a,b] \to \mathbb{R}$ given by

(i) $f_1(x) := \begin{cases} 0 & \text{if } x \le c, \\ 1 & \text{if } c < x, \end{cases}$ (ii) $f_2(x) := \begin{cases} 0 & \text{if } x \ne c, \\ 1 & \text{if } x = c, \end{cases}$

(iii) $f_3(x) := \begin{cases} (x-c)/(a-c) & \text{if } x \le c, \\ (x-c)/(b-c) & \text{if } c < x. \end{cases}$

For $j = 1, 2, 3$, let $F_j(x) := \int_a^x f_j(t)dt, x \in [a,b]$. Find F_j for $j = 1, 2, 3$. Show the following.

(i) f_1 is discontinuous at c and F_1 is continuous but not differentiable at c.
(ii) f_2 is discontinuous at c and F_2 is differentiable at c, but $F_2'(c) \ne f_2(c)$.
(iii) f_3 is continuous at c, but not differentiable at c, while F_3 is differentiable at c and $F_3'(c) = f_3(c)$.

[Note: There exists an integrable function $f : [a,b] \to \mathbb{R}$ such that f is discontinuous at c, but the corresponding function $F : [a,b] \to \mathbb{R}$ is differentiable at c and $F'(c) = f(c)$. See Proposition 7.19.]

6.12. Let $n \in \mathbb{N}$. Find a function $f : [-1,1] \to \mathbb{R}$ for which $f^{(n)}(0)$ exists, but $f^{(n+1)}(0)$ does not. (Hint: Begin with the absolute value function and use part (ii) of Proposition 6.22 repeatedly.)

6.13. If $f : [a,b] \to \mathbb{R}$ is continuously differentiable, then prove part (ii) of the FTC using part (i) of the FTC. (Hint: If two differentiable functions have the same derivative on an interval, then they differ by a constant.)

6.14. Let $f : [a,b] \to \mathbb{R}$ be integrable and let $F : [a,b] \to \mathbb{R}$ be given by $F(x) := \int_a^x f(t)dt$ for $x \in [a,b]$. If f is nonnegative on $[a,b]$, then show that F is monotonically increasing on $[a,b]$. Also, if f is monotonically increasing on $[a,b]$, then show that F is convex on $[a,b]$. (Hint: To prove the convexity of F, note that $\big(F(x) - F(x_1)\big)/(x - x_1) \le f(x) \le \big(F(x_2) - F(x)\big)/(x_2 - x)$

whenever $a \le x_1 < x < x_2 \le b$, and proceed as in the proof of part (i) of Proposition 4.33, or alternatively, use Exercise 1.63.)

6.15. Let $f : [a, \infty) \to \mathbb{R}$ be a bounded function that is integrable on $[a, x]$ for every $x \ge a$. Let $F(x) := \int_a^x f(t)dt$ for $x \ge a$. Show that F satisfies a Lipschitz condition on $[a, \infty)$, and so F is uniformly continuous on $[a, \infty)$.

6.16. Let $f : [0, \infty) \to \mathbb{R}$ be continuous and $f(x) \ge 0$ for all $x \in [0, \infty)$. If for each $b > 0$, the area bounded by the x-axis, the lines $x = 0, x = b$, and the curve $y = f(x)$ is given by $\sqrt{b^2 + 1} - 1$, determine the function f.

6.17. Let p be a real number and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. (Such a function is said to be **periodic**.) Show that the integral $\int_a^{a+p} f(t)dt$ has the same value for every real number a. (Hint: Part (i) of Proposition 6.24.)

6.18. Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that for every $x \in [a, b]$,

$$\int_a^x \left(\int_a^u f(t)dt \right) du = \int_a^x (x - u)f(u)du.$$

6.19. Let $f : [a, b] \to \mathbb{R}$ be integrable. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) := \int_x^b f(t)dt \quad \text{for } x \in [a, b].$$

Show that F is continuous on $[a, b]$. Further, show that if f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = -f(c)$. (Hint: Propositions 6.8, 6.22, and 6.24.)

6.20. Let $g : [c, d] \to \mathbb{R}$ be such that $g([c, d]) \subseteq [a, b]$, and let $f : [a, b] \to \mathbb{R}$ be integrable. Define $F : [c, d] \to \mathbb{R}$ by

$$F(y) := \int_a^{g(y)} f(t)dt \quad \text{for } y \in [c, d].$$

If g is differentiable at $y_0 \in [c, d]$ and f is continuous at $g(y_0)$, then show that F is differentiable at y_0 and $F'(y_0) = f(g(y_0))g'(y_0)$.

6.21. Let $f, g : [a, b] \to \mathbb{R}$ be such that f is differentiable, f' is integrable, and g is continuous. If

$$G(x) := \int_a^x g(t)dt \quad \text{and} \quad \widetilde{G}(x) := \int_x^b g(t)dt \quad \text{for } x \in [a, b],$$

then show that

$$\int_a^b f(x)g(x)dx = f(b)G(b) - \int_a^b f'(x)G(x)dx = f(a)\widetilde{G}(a) + \int_a^b f'(x)\widetilde{G}(x)dx.$$

(Compare Proposition 6.28.)

6.22. **(Leibniz Rule for Integrals)** Let f be a continuous function on $[a,b]$ and let u,v be differentiable functions on $[c,d]$. If the ranges of u and v are contained in $[a,b]$, prove that

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

6.23. For $x \in \mathbb{R}$, let $F(x) := \displaystyle\int_1^{2x} \frac{1}{1+t^2}\,dt$ and $G(x) := \displaystyle\int_0^{x^2} \frac{1}{1+\sqrt{t}}\,dt$. Find F' and G'.

6.24. Let $f : [0,\infty) \to \mathbb{R}$ be continuous. Find $f(2)$ if for all $x \geq 0$,

(i) $\displaystyle\int_0^x f(t)dt = x^2(1+x)$, (ii) $\displaystyle\int_0^{f(x)} t^2 dt = x^2(1+x)$,

(iii) $\displaystyle\int_0^{x^2} f(t)dt = x^2(1+x)$, (iv) $\displaystyle\int_0^{x^2(1+x)} f(t)dx = x$.

6.25. Let $n,m \in \mathbb{N}$. Find $\displaystyle\lim_{m\to\infty}\int_0^1 \frac{x^n}{(1+x)^m}dx$ and $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{x^n}{(1+x)^m}dx$.

6.26. Find $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{nx^{n-1}}{1+x}dx$. (Hint: Proposition 6.28.)

6.27. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. If F is an antiderivative of f on $[a,b]$, then show that

$$\int_a^b f^2(x)dx = F(b)F'(b) - F(a)F'(a) - \int_a^b F(x)F''(x)dx.$$

6.28. Evaluate (i) $\displaystyle\int_0^{1/4} \frac{x}{\sqrt{1-4x^2}}dx$, (ii) $\displaystyle\int_1^8 x^{1/3}(x^{4/3}-1)^{1/2}dx$.
(Hint: Proposition 6.29.)

6.29. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that f' is continuous on $[a,b]$ and $f'(x) \neq 0$ for all $x \in [a,b]$. If $f([a,b]) = [c,d]$, then show that $f^{-1} : [c,d] \to \mathbb{R}$ is integrable and

$$\int_c^d f^{-1}(y)dy = f^{-1}(d)d - f^{-1}(c)c - \int_{f^{-1}(c)}^{f^{-1}(d)} f(x)dx.$$

(Hint: Propositions 6.28 and 6.29.)

6.30. Let $f : [a,b] \to \mathbb{R}$ be a bounded function and define $g : [-b,-a] \to \mathbb{R}$ by $g(t) := f(-t)$. Show that $L(g) = L(f)$ and $U(g) = U(f)$. Deduce that g is integrable on $[-b,-a]$ if and only if f is integrable on $[a,b]$ and in that case the Riemann integral of g is equal to the Riemann integral of f.

6.31. Assuming that f is integrable on $[0,1]$, show that

$$\lim_{n\to\infty}\frac{1}{n}\left(f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right) = \int_0^1 f(x)dx.$$

6.32. Consider the sequence whose nth term is given by the following. In each case, determine the limit of the sequence by expressing the nth term as a Riemann sum for a suitable function.

(i) $\displaystyle\frac{1}{n^{17}}\sum_{i=1}^{n} i^{16}$, (ii) $\displaystyle\frac{1}{n^{5/2}}\sum_{i=1}^{n} i^{3/2}$, (iii) $\displaystyle\sum_{i=1}^{n}\frac{1}{\sqrt{in+n^2}}$,

(iv) $\displaystyle\frac{1}{n}\left\{\sum_{i=1}^{n}\left(\frac{i}{n}\right)+\sum_{i=n+1}^{2n}\left(\frac{i}{n}\right)^{3/2}+\sum_{i=2n+1}^{3n}\left(\frac{i}{n}\right)^{2}\right\}$.

6.33. Do $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{i+n}}$ and $\displaystyle\lim_{n\to\infty}\frac{1}{n^{18}}\sum_{i=1}^{n} i^{16}$ exist? If yes, find them.

6.34. Find an approximate value of $1^{1/3}+2^{1/3}+\cdots+1000^{1/3}$.

6.35. If (a_n) is a convergent sequence of real numbers and $E:=\{a_n : n\in\mathbb{N}\}$ is the set of its terms, then show that E is of content zero. Also, show that the set of irrational numbers in $[0,1]$ is not of content zero.

Part B

6.36. Let $a,b\in\mathbb{R}$ with $0\le a<b$ and $m\in\mathbb{N}$, and let $f:[a,b]\to\mathbb{R}$ be defined by $f(x):=x^m$. Show from first principles that

$$\int_a^b f(x)dx=\frac{b^{m+1}-a^{m+1}}{m+1}.$$

(Hint: If $P=\{x_0,x_1,\dots,x_n\}$ is a partition of $[a,b]$, then for each $j=0,1,\dots,m$, observe that $L(P,f)\le\sum_{i=1}^{n}x_i^{m-j}x_{i-1}^{j}(x_i-x_{i-1})\le U(P,f)$, and also that $\sum_{j=0}^{m}\sum_{i=1}^{n}x_i^{m-j}x_{i-1}^{j}(x_i-x_{i-1})=b^{m+1}-a^{m+1}$.)

6.37. **(Domain Additivity of Lower/Upper Riemann Integrals)** Suppose $f:[a,b]\to\mathbb{R}$ is a bounded function. For $c\in(a,b)$, let f_1 and f_2 denote the restrictions of f to the subintervals $[a,c]$ and $[c,b]$. Prove that $L(f)=L(f_1)+L(f_2)$ and $U(f)=U(f_1)+U(f_2)$. (Compare Proposition 6.8.)

6.38. Let $f:[a,b]\to\mathbb{R}$ be integrable and let $g:[m(f),M(f)]\to\mathbb{R}$ be continuous. Show that $g\circ f:[a,b]\to\mathbb{R}$ is integrable. (Hint: Given $\epsilon>0$, find $\delta>0$ using the uniform continuity of g. There is a partition P of $[a,b]$ such that $U(P,f)-L(P,f)<\delta^2$. Divide the summands in $U(P,f)-L(P,f)$ into two parts depending on whether or not $M_i(f)-m_i(f)<\delta$. Use the Riemann condition for $g\circ f$.)

6.39. Let $f_1,\dots,f_m:[a,b]\to\mathbb{R}$ be integrable functions and let $r_j:=\int_a^b f_j(x)dx$ for $j=1,\dots,m$. Show that the function $\sqrt{f_1^2+\cdots+f_m^2}$ is integrable and

$$\sqrt{r_1^2+\cdots+r_m^2}\le\int_a^b\sqrt{f_1^2(x)+\cdots+f_m^2(x)}\,dx.$$

(Hint: Note that $\sum_{j=1}^{m}r_j^2=\sum_{j=1}^{m}r_j\int_a^b f_j(x)dx=\int_a^b\left(\sum_{j=1}^{m}r_jf_j(x)\right)dx$ and use Proposition 1.12.)

6.40. Let $m, n \in \mathbb{Z}$ with $m, n \geq 0$. Show that

$$\int_0^1 x^m (1-x)^n \, dx = \frac{m!\, n!}{(m+n+1)!}.$$

(Hint: If $n \in \mathbb{N}$ and $I_{m,n}$ denotes the given integral, then using Integration by Parts, $I_{m,n} = \big(n/(m+1)\big) I_{m+1,n-1}$, and $I_{m+n,0} = 1/(m+n+1)$.)

6.41. Let $a \in \mathbb{R}$ and $n \in \mathbb{Z}$ with $n \geq 0$. Show that

$$\int_0^a (a^2 - x^2)^n \, dx = \frac{(2^n n!)^2}{(2n+1)!} \cdot a^{2n+1}.$$

Deduce that

$$1 - \frac{1}{3}\binom{n}{1} + \frac{1}{5}\binom{n}{2} - \frac{1}{7}\binom{n}{3} + \cdots + \frac{(-1)^n}{2n+1}\binom{n}{n} = \frac{(2^n n!)^2}{(2n+1)!}.$$

(Hint: For $n \geq 0$, let I_n denote the given integral. Then $I_0 = a$ and $I_n = a^2 I_{n-1} - \int_0^a x g_n(x) dx$, where $g_n(x) := x(a^2 - x^2)^{n-1}$ for $n \in \mathbb{N}$. Use Integration by Parts to obtain $I_n = 2na^2 I_{n-1}/(2n+1)$ for $n \in \mathbb{N}$.)

6.42. **(Taylor Theorem with Integral Remainder)** Let n be a nonnegative integer and let $f : [a, b] \to \mathbb{R}$ be such that $f', f'', \ldots, f^{(n)}$ exist on $[a, b]$ and further, $f^{(n)}$ is continuously differentiable on $[a, b]$. Show that

$$f(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{1}{n!}\int_a^b (b-t)^n f^{(n+1)}(t) dt.$$

Further, show that the integral remainder is equal to

$$\frac{(b-a)^{n+1}}{n!} \int_0^1 (1-s)^n f^{(n+1)}(a + s(b-a)) ds$$

and that there is $c \in [a, b]$ such that the integral remainder is equal to

$$\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

(Hint: Induction on n, Integration by Parts, and IVP of $f^{(n+1)}$)
[Note: Unlike the Lagrange form of the remainder, the integral remainder does not involve an undetermined number $c \in (a, b)$.]

6.43. **(Taylor Theorem for Integrals)** Let $n \in \mathbb{N}$ and $f : [a, b] \to \mathbb{R}$ be such that $f', f'', \ldots, f^{(n-1)}$ exist on $[a, b]$, and further, $f^{(n-1)}$ is continuous on $[a, b]$ and differentiable on (a, b). Show that there is $c \in (a, b)$ such that

$$\int_a^b f(x) dx = f(a)(b-a) + \cdots + \frac{f^{(n-1)}(a)}{n!}(b-a)^n + \frac{f^{(n)}(c)}{(n+1)!}(b-a)^{n+1}.$$

(Hint: For $x \in [a, b]$, define $F(x) := \int_a^x f(t) dt$ and apply Proposition 4.25.)

6.44. Let $f : [a, b] \to \mathbb{R}$ be integrable. If $\phi : [\alpha, \beta] \to \mathbb{R}$ is a differentiable function such that ϕ' is integrable on $[\alpha, \beta]$, and $\phi'(t) \neq 0$ for every $t \in (\alpha, \beta)$, then show that the function $(f \circ \phi)|\phi'| : [\alpha, \beta] \to \mathbb{R}$ is integrable and

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))|\phi'(t)|dt.$$

(Hint: Either $\phi' > 0$ or $\phi' < 0$ on (α, β). Let $\psi := (f \circ \phi)|\phi|$ and let $\epsilon > 0$ be given. For any partition P of $[a, b]$, obtain a partition Q of $[\alpha, \beta]$ with $L(P, f) < L(Q, \psi) + \epsilon$. Hence $L(f) \leq L(\psi)$. Similarly, $U(f) \geq U(\psi)$.)
[Note: This gives a stronger version of Proposition 6.29.]

6.45. **(Bliss Theorem)** Let $f, g : [a, b] \to \mathbb{R}$ be any functions. For each $n \in \mathbb{N}$, consider a partition $P_n := \{x_{n,0}, x_{n,1}, \ldots, x_{n,k_n}\}$ of $[a, b]$, and for $i = 1, \ldots, k_n$, let $s_{n,i}$, $t_{n,i} \in [x_{n,i-1}, x_{n,i}]$, and let

$$\widetilde{S}_n := \sum_{i=1}^{k_n} f(s_{n,i})g(t_{n,i})(x_{n,i} - x_{n,i-1}).$$

Show that if f is integrable, g is continuous, and $\mu(P_n) \to 0$, then

$$\widetilde{S}_n \to \int_a^b f(x)g(x)dx.$$

(Hint: If $\mathcal{T}_n := \{s_{n,i} : i = 1, \ldots, k_n\}$, then $S(P_n, \mathcal{T}_n, fg) \to \int_a^b f(x)g(x)dx$. Also, g is uniformly continuous.)

6.46. Let $f : [a, b] \to \mathbb{R}$ be a monotonic function. If $G : [a, b] \to \mathbb{R}$ is differentiable and G' is continuous, then show that there is $c \in [a, b]$ such that

$$\int_a^b f(x)G'(x)dx = f(b)G(b) - f(a)G(a) - G(c)\big(f(b) - f(a)\big).$$

(Hint: Given any partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, consider the sum $\sum_{i=1}^n f(x_i)\big(G(x_i) - G(x_{i-1})\big)$. Write it as $f(b)G(b) - f(a)G(a) - \sum_{i=1}^n G(x_{i-1})\big(f(x_i) - f(x_{i-1})\big)$ and also as $\sum_{i=1}^n f(x_i)G'(s_i)(x_i - x_{i-1})$ for some $s_i \in [x_{i-1}, x_i]$. Use Exercise 6.45 and note $m(g)\big(f(b) - f(a)\big) \leq \sum_{i=1}^n G(x_{i-1})\big(f(x_i) - f(x_{i-1})\big) \leq M(g)\big(f(b) - f(a)\big)$.)

6.47. **(First Mean Value Theorem for Integrals)** Let $f : [a, b] \to \mathbb{R}$ be a continuous function and let $g : [a, b] \to \mathbb{R}$ be a nonnegative integrable function. Use the IVP of f to show that there is $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

Give examples to show that neither the continuity of f nor the nonnegativity of g can be omitted.
[Note: For another version of this result, see Exercise 8.72.]

6.48. **(Second Mean Value Theorem for Integrals)** Let $f : [a, b] \to \mathbb{R}$ be a monotonic function and let $g : [a, b] \to \mathbb{R}$ be either a nonnegative integrable function or a continuous function. Show that there is $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.$$

Give an example to show that the monotonicity of f cannot be omitted. (Hint: Without loss of generality, suppose f is (monotonically) increasing. Let $G(x) := \int_a^x g(t)dt$ for $x \in [a, b]$. If g is a nonnegative integrable function, then $f(a)G(b) \le \int_a^b f(x)g(x)dx \le f(b)G(b)$. If g is continuous, use Exercise 6.46.)

6.49. **(Cauchy Condition)** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there is a partition P of $[a, b]$ such that for all tag sets $\mathcal{T}$ and $\mathcal{T}'$ associated with P,

$$|S(P, \mathcal{T}, f) - S(P, \mathcal{T}', f)| < \epsilon.$$

(Hint: Argue as in the proof of Proposition 6.36.)

6.50. Let E be a bounded subset of $\mathbb{R}$ and let ∂E denote the boundary of E.
 (i) Show that if E is of content zero, then $\overline{E} := E \cup \partial E$ is of content zero.
 (ii) Show that E is of content zero if and only if E has no interior points and ∂E is of content zero.

6.51. Let $f : [a, b] \to \mathbb{R}$ be integrable, and let $g : [a, b] \to \mathbb{R}$ be a bounded function such that the set $\{x \in [a, b] : g(x) \ne f(x)\}$ is of content zero. Show that g is integrable and

$$\int_a^b g(x)dx = \int_a^b f(x)dx.$$

(Compare Proposition 6.13.)

7

Elementary Transcendental Functions

In this chapter, we shall use the theory of Riemann integration developed in Chapter 6 to introduce some classical functions, known as the logarithmic, exponential, and trigonometric functions. Collectively, these are called the **elementary transcendental functions**. In Sections 7.1 and 7.2 below we give formal definitions of these functions and derive several of their interesting properties. In this process, the important real numbers e and π will also be formally defined.

In the earlier chapters, we have scrupulously avoided any mention of the logarithmic, exponential, and trigonometric functions, since their very definitions had to be postponed. As a result, several interesting examples and counterexamples could not be given earlier. Many of these arise from the function obtained by taking the sine of the reciprocal of the identity function. These are discussed in Section 7.3.

The trigonometric functions enable us to introduce polar coordinates of a point in the plane other than the origin. This is done in Section 7.4, and in this context, we also give a formal definition of the angle between two line segments emanating from a point as well as of the angle between two intersecting curves.

In the final section of this chapter, we show that the elementary transcendental functions are indeed transcendental, that is, they are not algebraic functions.

In the section on exercises, we have given problems of theoretical importance as well as problems of practical use. The latter include several trigonometric results that are listed for ready reference. In addition to this section of exercises, we include a section devoted to revision exercises in which the reader will revisit many concepts considered earlier in this book in relation to the new supply of functions that is made available in this chapter.

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Undergraduate Texts in Mathematics, https://doi.org/10.1007/978-3-030-01400-1_7

7.1 Logarithmic and Exponential Functions

We have seen in Example 4.8 that if $r \in \mathbb{Q}$ and $g : (0, \infty) \to \mathbb{R}$ is the rth-power function defined by $g(x) := x^r$, then $g'(x) = rx^{r-1}$ for all $x \in (0, \infty)$. This implies that for every rational number s, except $s = -1$, the sth-power function can be integrated in terms of a similar function. In fact,

$$\frac{d}{dx}\left(\frac{x^{s+1}}{s+1}\right) = x^s \quad \text{provided } s \neq -1.$$

To deal with the exceptional case $s = -1$, a new function has to be introduced, and we shall do so in this section. This will in fact enable us to define and study real powers of positive real numbers.

The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(t) = 1/t$ is continuous. Hence it is Riemann integrable on every closed and bounded subinterval of $(0, \infty)$. For $x \in (0, \infty)$, we define the **natural logarithm** of x by

$$\ln x := \int_1^x \frac{1}{t} dt.$$

The function $\ln : (0, \infty) \to \mathbb{R}$ is known as the **logarithmic function**. We write $\ln x$, rather than $\ln(x)$, for the value of the logarithmic function at $x \in (0, \infty)$.

Clearly, $\ln 1 = 0$. Moreover, since $1/t \geq 0$ for all $t \in (0, \infty)$, we see that $\ln x \geq 0$ if $x > 1$, while $\ln x \leq 0$ if $0 < x < 1$.

Proposition 7.1 (Properties of the Logarithmic Function and the Definition of e). (i) $\ln$ *is a differentiable function on* $(0, \infty)$ *and*

$$(\ln)'x = \frac{1}{x} \quad \textit{for every } x \in (0, \infty).$$

(ii) $\ln$ *is strictly increasing as well as strictly concave on* $(0, \infty)$.
(iii) $\ln x \to \infty$ *as* $x \to \infty$, *whereas* $\ln x \to -\infty$ *as* $x \to 0^+$.
(iv) *For every* $y \in \mathbb{R}$, *there is a unique* $x \in (0, \infty)$ *such that* $\ln x = y$. *In other words,* $\ln : (0, \infty) \to \mathbb{R}$ *is a bijective function. In particular, there is a unique real number* e *such that* $\ln e = 1$. *Moreover,* $2 \leq e \leq 4$.

Proof. (i) Since the function $g : (0, \infty) \to \mathbb{R}$ given by $g(t) := 1/t$ is continuous, part (i) of the FTC (Proposition 6.24) shows that the function ln is differentiable on $(0, \infty)$ and $(\ln)'x = g(x) = 1/x$ at every $x \in (0, \infty)$.

(ii) Since the derivative of ln is positive on $(0, \infty)$, it follows from part (iii) of Proposition 4.29 that ln is strictly increasing on $(0, \infty)$. Further, since

$$(\ln)''x = -\frac{1}{x^2} < 0 \quad \text{for every } x \in (0, \infty),$$

part (iv) of Proposition 4.34 shows that ln is strictly concave on $(0, \infty)$.

(iii) Given any positive integer $n \geq 2$, we note that

$$\ln n = \int_1^n \frac{1}{t}dt = \sum_{k=2}^{n} \int_{k-1}^{k} \frac{1}{t}dt \ \geq \ \sum_{k=2}^{n} \int_{k-1}^{k} \frac{1}{k}dt = \sum_{k=2}^{n} \frac{1}{k},$$

since $(1/t) \geq (1/k)$ for all $0 < t \leq k$, whereas

$$\begin{aligned} \ln \frac{1}{n} &= -\int_{1/n}^{1} \frac{1}{t}dt = -\sum_{k=2}^{n} \int_{1/k}^{1/(k-1)} \frac{1}{t}dt \\ &\leq -\sum_{k=2}^{n} \int_{1/k}^{1/(k-1)} (k-1)dt = -\sum_{k=2}^{n} \frac{1}{k}, \end{aligned}$$

since $-(1/t) \leq -(k-1)$ for all $0 < t \leq 1/(k-1)$. Because $\sum_{k=2}^{n}(1/k) \to \infty$ by part (ii) of Example 2.14, we see that the function ln is neither bounded above nor bounded below on $(0, \infty)$. Also, since ln is (strictly) increasing on $(0, \infty)$, it follows from Proposition 3.44 that $\ln x \to \infty$ as $x \to \infty$ and $\ln x \to -\infty$ as $x \to 0^+$.

(iv) The function ln is one-one, since it is strictly increasing. Now let $y \in \mathbb{R}$. Since $\ln x \to -\infty$ as $x \to 0^+$, there is some $x_0 > 0$ such that $\ln x_0 < y$, and since $\ln x \to \infty$ as $x \to \infty$, there is some $x_1 > 0$ such that $y < \ln x_1$. But by part (i) above, the function ln is continuous on the interval $[x_0, x_1]$. Hence the IVP (Proposition 3.16) shows that there is some $x \in (x_0, x_1)$ such that $\ln x = y$. Thus the function $\ln : (0, \infty) \to \mathbb{R}$ is bijective. In particular, there is a unique real number e such that $\ln e = 1$. Moreover,

$$\ln 2 = \int_1^2 \frac{1}{t}dt \ \leq \ \int_1^2 1\, dt = 1$$

and

$$\ln 4 = \int_1^4 \frac{1}{t}dt = \int_1^2 \frac{1}{t}dt + \int_2^4 \frac{1}{t}dt \ \geq \ \int_1^2 \frac{1}{2}dt + \int_2^4 \frac{1}{4}dt = \frac{1}{2} + \frac{1}{2} = 1.$$

Thus $\ln 2 \leq 1 \leq \ln 4$. Since the function ln is bijective and increasing, it follows that $2 \leq e \leq 4$. □

Remark 7.2. The number e defined in the above proposition plays a significant role in analysis. We shall see alternative expressions for e in Corollary 7.7. Better lower and upper bounds for the number e can be obtained. (See, for example, Exercise 7.7.) The decimal expansion of e is given by

$$e = 2.71828182\ldots\ .$$

This is not a recurring decimal expansion. Indeed, e is an irrational number (Exercise 2.23), and in fact, e is transcendental, that is, it is not a root of any nonzero polynomial with rational coefficients. (See [8] for a proof.) ◇

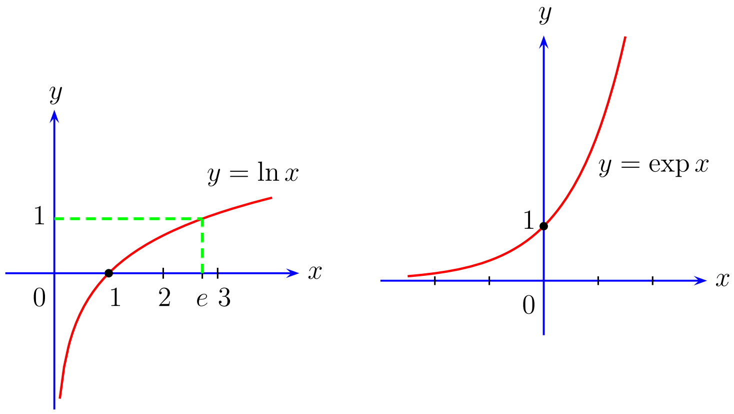

Fig. 7.1. Graphs of the logarithmic and exponential functions.

The geometric properties of the logarithmic function proved in Proposition 7.1 can be used to draw its graph as in Figure 7.1.

It follows from part (i) of Proposition 7.1 that the function ln is infinitely differentiable on $(0, \infty)$, and for $k = 1, 2, \ldots$,

$$(\ln)^{(k)}x = (-1)^{k-1}(k-1)!\, x^{-k}, \quad x \in (0, \infty).$$

Hence the nth Taylor polynomial for ln about 1 is given by

$$P_n(x) = \ln 1 + \sum_{k=1}^{n} \frac{(\ln)^{(k)}1}{k!}(x-1)^k = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k, \quad x \in \mathbb{R}.$$

In particular, the linear and the quadratic approximations of ln around 1 are given by

$$L(x) = P_1(x) = x - 1 \quad \text{and} \quad Q(x) = P_2(x) = (x-1) - \frac{(x-1)^2}{2}, \quad x \in \mathbb{R}.$$

Let us now turn to the inverse of the logarithmic function. The inverse of the bijective function $\ln : (0, \infty) \to \mathbb{R}$ is known as the **exponential function** and is denoted by $\exp : \mathbb{R} \to (0, \infty)$. We write $\exp x$, rather than $\exp(x)$, for the value of the exponential function at $x \in \mathbb{R}$. Thus for all $x \in \mathbb{R}$ and $y \in (0, \infty)$,

$$\exp x = y \Longleftrightarrow \ln y = x.$$

Note that by definition, $\exp x > 0$ for all $x \in \mathbb{R}$. Moreover, since $\ln 1 = 0$ and $\ln e = 1$, we see that $\exp 0 = 1$ and $\exp 1 = e$.

Proposition 7.3 (Properties of the Exponential Function). (i) exp *is a differentiable function on* $\mathbb{R}$ *and*

$$(\exp)' x = \exp x \quad \textit{for every } x \in \mathbb{R}.$$

(ii) exp *is strictly increasing as well as strictly convex on* $\mathbb{R}$.
(iii) $\exp x \to \infty$ *as* $x \to \infty$, *whereas* $\exp x \to 0$ *as* $x \to -\infty$.

Proof. (i) Let $x \in \mathbb{R}$ and $c \in (0, \infty)$ be such that $\ln c = x$. Since the function $\ln : (0, \infty) \to \mathbb{R}$ is differentiable at c and $(\ln)'c = 1/c \neq 0$, Proposition 4.12 shows that the inverse function $\exp : \mathbb{R} \to (0, \infty)$ is differentiable at $x = \ln c$ and

$$(\exp)'(x) = \exp'(\ln c) = \frac{1}{(\ln)'c} = c = \exp x.$$

(ii) Since the derivative of exp is positive on $\mathbb{R}$, it follows from part (iii) of Proposition 4.29 that exp is strictly increasing on $\mathbb{R}$.

Further, since

$$(\exp)''x = (\exp)'x = \exp x > 0 \quad \text{for all } x \in \mathbb{R},$$

it follows from part (iii) of Proposition 4.34 that exp is strictly convex on $\mathbb{R}$.

(iii) Since the range of the function exp is the domain of the function ln, namely the interval $(0, \infty)$, we see that exp is not bounded above on $\mathbb{R}$, whereas $\inf\{\exp x : x \in \mathbb{R}\} = 0$. Also, since exp is (strictly) increasing on $\mathbb{R}$, it follows from Proposition 3.44 that $\exp x \to \infty$ as $x \to \infty$, whereas $\exp x \to 0$ as $x \to -\infty$. □

The geometric properties of the exponential function proved in Proposition 7.3 can be used to draw its graph as in Figure 7.1.

It follows from part (i) of Proposition 7.3 that the function exp is infinitely differentiable on $\mathbb{R}$, and for $k = 1, 2, \ldots,$

$$(\exp)^{(k)}x = \exp x \quad \text{for all } x \in \mathbb{R}.$$

Hence the nth Taylor polynomial for exp about 0 is given by

$$P_n(x) = \exp 0 + \sum_{k=1}^{n} \frac{(\exp)^{(k)}(0)}{k!}(x - 0)^k = 1 + \sum_{k=1}^{n} \frac{x^k}{k!}, \quad x \in \mathbb{R}.$$

In particular, the linear and the quadratic approximations of exp around 0 are given by

$$L(x) = P_1(x) = 1 + x \quad \text{and} \quad Q(x) = P_2(x) = 1 + x + \frac{x^2}{2}, \quad x \in \mathbb{R}.$$

The logarithmic and the exponential functions have interesting behavior with respect to the multiplication and addition of real numbers. This is made precise in the following result.

Proposition 7.4. (i) *For all positive real numbers* x_1 *and* x_2,

$$\ln x_1 x_2 = \ln x_1 + \ln x_2.$$

(ii) *For all real numbers* x_1 *and* x_2*,*

$$\exp(x_1 + x_2) = (\exp x_1)(\exp x_2).$$

Proof. (i) Given any $x_1, x_2 \in (0, \infty)$, by domain additivity,

$$\ln x_1 x_2 = \int_1^{x_1 x_2} \frac{dt}{t} = \int_1^{x_1} \frac{dt}{t} + \int_{x_1}^{x_1 x_2} \frac{dt}{t} = \ln x_1 + \int_1^{x_2} \frac{ds}{s} = \ln x_1 + \ln x_2,$$

where the second inequality uses the convention in Remark 6.9 in case $x_2 < 1$, and the third equality uses the substitution $t = sx_1$. This proves (i).

(ii) Let x_1 and x_2 be real numbers. Define $y_1 := \exp x_1$ and $y_2 = \exp x_2$. Then y_1 and y_2 are positive real numbers, and by (i) above, we see that $\ln y_1 y_2 = \ln y_1 + \ln y_2 = x_1 + x_2$. Consequently, $\exp(x_1 + x_2) = y_1 y_2 = (\exp x_1)(\exp x_2)$. □

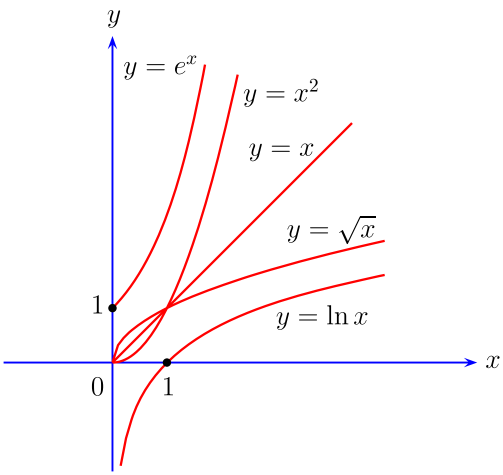

Fig. 7.2. Illustration of the growth rate: graphs of $f(x) = \ln x$, $f(x) = \sqrt{x}$, $f(x) = x$, $f(x) = x^2$, and $f(x) = e^x$ (for $x > 0$).

In the examples below, we consider some important limits involving the functions ln and exp.

Examples 7.5. (i) By the definition of derivative and the earlier observations that $\ln 1 = 0$, $(\ln)'1 = 1$, $\exp 0 = 1$, and $(\exp)'0 = 1$, we obtain

$$\lim_{h \to 0} \frac{\ln(1+h)}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{\exp h - 1}{h} = 1.$$

(ii) Since $(\ln)'x = 1/x$ for $x \in (0, \infty)$ and $\ln x \to \infty$ as $x \to \infty$, while $(\exp)'x = x$ for $x \in \mathbb{R}$ and $\exp x \to \infty$ as $x \to \infty$, L'Hôpital's Rule for $\frac{\infty}{\infty}$ indeterminate forms (Proposition 4.42) shows that

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{x}{\exp x} = 0.$$

In a similar manner, it can be easily seen that for every $k \in \mathbb{N}$,

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/k}} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{x^k}{\exp x} = 0.$$

These limits show that the growth rate of $\ln x$ is less than that of any root of x, while the growth rate of $\exp x$ is more than that of any positive integral power of x as x tends to ∞ (Remark 3.40). This can be illustrated by the graphs of the curves $y = \ln x$, $y = \sqrt{x}$, $y = x$, $y = x^2$ and $y = e^x$ (for $x > 0$) drawn in Figure 7.2. $\diamond$

Real Powers of Positive Numbers

The logarithmic and exponential functions enable us to define the bth power of a, where a is any positive real number and b is any real number. Recall that if r is any rational number, then we have defined in Chapter 1 the rth power a^r of every $a \in (0, \infty)$. We observe that

$$\ln a^r = r \ln a \quad \text{for all } a \in (0, \infty) \text{ and } r \in \mathbb{Q}.$$

To see this, consider the function $f : (0, \infty) \to \mathbb{R}$ given by

$$f(x) = \ln x^r - r \ln x.$$

Then by part (i) of Proposition 7.1 and the Chain Rule (Proposition 4.10),

$$f'(x) = \frac{1}{x^r} r x^{r-1} - r\frac{1}{x} = 0 \quad \text{for all } x \in (0, \infty),$$

and so $f(x) = f(1) = \ln 1^r - r \ln 1 = 0 - 0 = 0$ for all $x \in (0, \infty)$. In particular, the equation $f(a) = 0$ gives $\ln a^r = r \ln a$. Thus

$$a^r = \exp(r \ln a) \quad \text{for all } a \in (0, \infty) \text{ and } r \in \mathbb{Q}.$$

Since the number $\exp(r \ln a)$ is well-defined for every real (and not just rational) number r, we are naturally led to the following definition. Let a be a positive number and b a real number. The bth **power** of a is defined by

$$a^b := \exp(b \ln a).$$

Here a is called the **base** and b is called the **exponent**. If b is a rational number, this definition of a^b coincides with our earlier definition, as we have just seen. Clearly, the equality $a^b := \exp(b \ln a)$ is equivalent to the equality $\ln a^b = b \ln a$ for $a > 0$ and $b \in \mathbb{R}$. We note that $a^b > 0$ for all $a \in (0, \infty)$ and $b \in \mathbb{R}$.

Let us consider the special case $a = e$, that is, when the base is the unique positive real number satisfying $\ln e = 1$. Then

$$e^x = \exp(x \ln e) = \exp x \quad \text{for all } x \in \mathbb{R}.$$

We have thus found a short notation for $\exp x$, namely e^x, where $x \in \mathbb{R}$. From now on, we may employ this notation.

The following proposition gives an alternative way of determining e^x for $x \in \mathbb{R}$. As a corollary, we can relate our definition of e to the two limits considered in Examples 2.10 (i) and (ii).

Proposition 7.6. *For all* $x \in \mathbb{R}$*,*

$$\lim_{h \to 0} (1 + xh)^{1/h} = e^x.$$

In particular,

$$e = \lim_{h \to 0} (1 + h)^{1/h}.$$

Proof. The first assertion is obvious if $x = 0$. Suppose $x \in \mathbb{R}$ and $x \neq 0$. Now,

$$\lim_{h \to 0} \frac{\ln(1 + xh)}{xh} = \lim_{h \to 0} \frac{\ln(1 + h)}{h} = 1,$$

as seen in Example 7.5 (i). Consequently,

$$\lim_{h \to 0} \ln(1 + xh)^{1/h} = \lim_{h \to 0} \frac{\ln(1 + xh)}{h} = x \left(\lim_{h \to 0} \frac{\ln(1 + xh)}{xh} \right) = x.$$

Now since the exponential function is continuous at x, we obtain

$$\lim_{h \to 0} (1+xh)^{1/h} = \lim_{h \to 0} \exp \left(\ln(1 + xh)^{1/h} \right) = \exp \left(\lim_{h \to 0} \ln(1 + xh)^{1/h} \right) = \exp x.$$

This proves the first assertion. The particular case is obtained by considering $x = 1$. □

Corollary 7.7. *For all* $x \in \mathbb{R}$*,*

$$\lim_{n \to \infty} \left(1 + \frac{x}{n} \right)^n = e^x.$$

In particular,

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{k!}.$$

Proof. Consider the sequence (h_n) defined by $h_n = 1/n$ for $n \in \mathbb{N}$. Then $h_n \to 0$. Hence the first assertion follow from Proposition 7.6. Putting $x = 1$, we obtain the first equality in the second assertion, while the last equality here was shown earlier, in Example 2.10 (ii). □

Returning to the general power a^b with base $a \in (0, \infty)$ and exponent $b \in \mathbb{R}$, let us consider the following two functions. For a fixed $a \in (0, \infty)$, the **power function** $f_a : \mathbb{R} \to \mathbb{R}$ **with base** a, and for a fixed $b \in \mathbb{R}$, the **power function** $g_b : (0, \infty) \to \mathbb{R}$ **with exponent** b, are defined by

$$f_a(x) := a^x \quad \text{and} \quad g_b(x) := x^b.$$

Note that $f_a(x) > 0$ for all $x \in \mathbb{R}$ and $g_b(x) > 0$ for all $x \in (0, \infty)$. We study the basic properties of these functions in Propositions 7.8 and 7.10.

Proposition 7.8 (Properties of the Power Function with a Fixed Base). *Let a be a positive real number and $f_a : \mathbb{R} \to (0, \infty)$ the power function with base a given by $f_a(x) := a^x$. Then*

(i) *f_a is a differentiable function on $\mathbb{R}$ and*

$$f_a'(x) = (\ln a)a^x = (\ln a)f_a(x) \quad \textit{for every } x \in \mathbb{R}.$$

(ii) *If $a > 1$, then f_a is strictly increasing as well as strictly convex on $\mathbb{R}$. If $a < 1$, then f_a is strictly decreasing as well as strictly convex on $\mathbb{R}$. If $a = 1$, then $f_a(x) = 1$ for all $x \in \mathbb{R}$.*

(iii) *If $a \neq 1$, then f_a is not bounded above on $\mathbb{R}$. More precisely, if $a > 1$, then $f_a(x) \to \infty$ as $x \to \infty$, whereas $f_a(x) \to 0$ as $x \to -\infty$, and if $a < 1$, then $f_a(x) \to \infty$ as $x \to -\infty$, whereas $f_a(x) \to 0$ as $x \to \infty$.*

(iv) *If $a \neq 1$, then the function $f_a : \mathbb{R} \to (0, \infty)$ is bijective, and its inverse $f_a^{-1} : (0, \infty) \to \mathbb{R}$ is given by*

$$f_a^{-1}(x) = \frac{\ln x}{\ln a} \quad \textit{for } x \in (0, \infty).$$

(v) *For all x_1 and x_2 in $\mathbb{R}$,*

$$f_a(x_1 + x_2) = f_a(x_1)f_a(x_2), \quad \textit{that is,} \quad a^{x_1+x_2} = a^{x_1}a^{x_2}.$$

(vi) *For all x_1 and x_2 in $\mathbb{R}$,*

$$\big(f_a(x_1)\big)^{x_2} = f_a(x_1x_2) = \big(f_a(x_2)\big)^{x_1}, \quad \textit{that is,} \quad (a^{x_1})^{x_2} = a^{x_1x_2} = (a^{x_2})^{x_1}.$$

Proof. (i) Since $f_a(x) = \exp(x \ln a)$ for $x \in \mathbb{R}$ and $(\exp)' = \exp$, the Chain Rule (Proposition 4.10) shows that

$$f_a'(x) = (\ln a)\exp(x \ln a) = (\ln a)a^x = (\ln a)f_a(x).$$

(ii) Let $a > 1$. Then $\ln a > 0$, and hence the derivative f_a' of f_a is positive on $\mathbb{R}$. This shows that f_a is strictly increasing on $\mathbb{R}$. Further, since

$$f_a''(x) = (\ln a)f_a'(x) = (\ln a)^2 f_a(x) > 0 \quad \text{for all } x \in \mathbb{R},$$

it follows that f_a is strictly convex on $\mathbb{R}$. Similar arguments yield the desired results for f_a if $a < 1$. If $a = 1$, then $f_1(x) = 1^x = e^{x \ln 1} = e^0 = 1$ for all $x \in \mathbb{R}$.

(iii) If $a > 1$, then $\ln a > 0$, and so $f_a(x) = e^{(\ln a)x} \to \infty$ as $x \to \infty$, whereas $f_a(x) \to 0$ as $x \to -\infty$. Similarly, if $a < 1$, then $\ln a < 0$ and so $f_a(x) = e^{(\ln a)x} \to \infty$ as $x \to -\infty$, whereas $f_a(x) \to 0$ as $x \to \infty$. This shows that if $a \neq 1$, then f_a is not bounded above on $\mathbb{R}$.

(iv) Let $a \neq 1$. The bijectivity of $f_a : \mathbb{R} \to (0, \infty)$ follows from the bijectivity of the function $\exp : \mathbb{R} \to (0, \infty)$ and the fact that $\ln a \neq 0$. For $x \in (0, \infty)$,

$$f_a(\ln x/\ln a) = a^{\ln x/\ln a} = \exp\big((\ln x/\ln a)\ln a\big) = \exp(\ln x) = x.$$

Hence $f_a^{-1}(x) = \ln x/\ln a$ for $x \in (0, \infty)$.

(v) For all x_1 and x_2 in $\mathbb{R}$,

$$\exp\left((x_1 + x_2)\ln a\right) = \exp(x_1 \ln a + x_2 \ln a) = \exp(x_1 \ln a)\exp(x_2 \ln a),$$

and thus $f_a(x_1 + x_2) = f_a(x_1)f_a(x_2)$, as desired.

(vi) For all x_1 and x_2 in $\mathbb{R}$,

$$\big(f_a(x_1)\big)^{x_2} = (a^{x_1})^{x_2} = \exp(x_2 \ln a^{x_1}) = \exp(x_2 x_1 \ln a) = a^{x_2 x_1} = f_a(x_2 x_1).$$

Interchanging x_1 and x_2, we see that

$$\big(f_a(x_2)\big)^{x_1} = f_a(x_1 x_2).$$

Since $x_2 x_1 = x_1 x_2$, we obtain the desired result. $\square$

The geometric properties of the function f_a proved in Proposition 7.8 can be used to draw its graph as in Figure 7.3, when $a = 2$ and $a = \frac{1}{2}$.

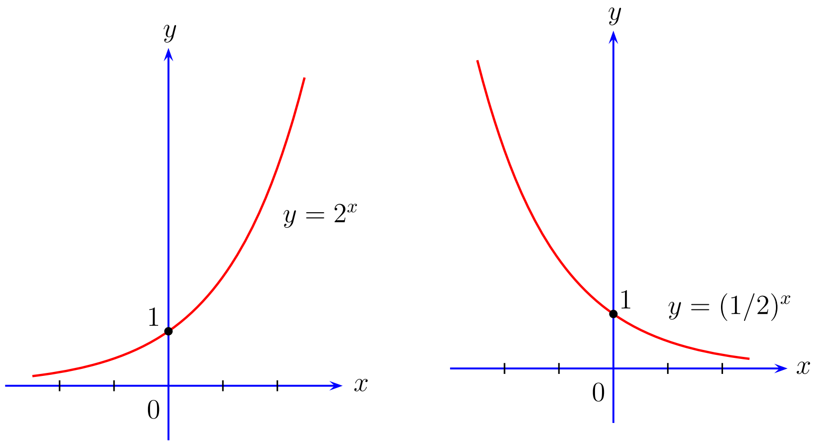

Fig. 7.3. Graphs of the power function f_a with base (i) $a = 2$, and (ii) $a = 1/2$.

Remark 7.9. Let $a > 0$ and $f_a(x) := a^x$ for $x \in \mathbb{R}$. If $a \neq 1$, then the inverse $f_a^{-1} : (0, \infty) \to \mathbb{R}$ of the function $f_a : \mathbb{R} \to (0, \infty)$ exists, and it is denoted by $\log_a$; this is known as the **logarithmic function with base** a. Thus

$$\log_a x = \frac{\ln x}{\ln a}, \quad x \in (0, \infty).$$

Clearly, if $a = e$, then $\log_a = \log_e = \ln$. For this reason, the number e is often referred to as the **base of the natural logarithm**. Another commonly used base is $a = 10$. The function $\log_{10}$ is often written simply as log. Thus

$$\log x = \frac{\ln x}{\ln 10}, \quad x \in (0, \infty).$$

The first few digits in the decimal expansion of $\ln 10$ are given by

$$\ln 10 = 2.30258509\ldots\ .$$

This enables us to compute the value of $\log x$ if we know $\ln x$, and conversely, for $x \in (0, \infty)$. $\diamond$

Proposition 7.10 (Properties of the Power Function with a Fixed Exponent). *Let b be a real number and let $g_b : (0, \infty) \to (0, \infty)$ be the power function with exponent b given by $g_b(x) := x^b$. Then*

(i) *g_b is a differentiable function on $(0, \infty)$ and*

$$g_b'(x) = bx^{b-1} = bg_{b-1}(x) \quad \text{for every } x \in (0, \infty).$$

(ii) *If $b > 0$, then g_b is strictly increasing, and if $b < 0$, then g_b is strictly decreasing on $(0, \infty)$. Further, if $b > 1$ or $b < 0$, then g_b is strictly convex, and if $0 < b < 1$, then g_b is strictly concave on $(0, \infty)$. If $b = 0$, then $g_b(x) = 1$, and if $b = 1$, then $g_b(x) = x$ for all $x \in (0, \infty)$.*
(iii) *If $b \neq 0$, then g_b is not bounded above on $(0, \infty)$. More precisely, if $b > 0$, then $g_b(x) \to \infty$ as $x \to \infty$, whereas $g_b(x) \to 0$ as $x \to 0$, and if $b < 0$, then $g_b(x) \to \infty$ as $x \to 0$, whereas $g_b(x) \to 0$ as $x \to \infty$.*
(iv) *If $b \neq 0$, then the function $g_b : (0, \infty) \to (0, \infty)$ is bijective and the function $g_{1/b} : (0, \infty) \to (0, \infty)$ is the inverse of g_b.*
(v) *For all x_1 and x_2 in $(0, \infty)$,*

$$g_b(x_1 x_2) = g_b(x_1) g_b(x_2), \quad \text{that is,} \quad (x_1 x_2)^b = x_1^b x_2^b.$$

Proof. (i) Since $g_b(x) = \exp(b \ln x)$ for $x \in (0, \infty)$, the Chain Rule (Proposition 4.10) shows that

$$g_b'(x) = \frac{b}{x} \exp(b \ln x) = bx^{b-1} = bg_{b-1}(x).$$

(ii) We note that for each $b \in \mathbb{R}$, the function g_b is positive on $(0, \infty)$. Hence by (i) above, the derivative g_b' of g_b is positive on $(0, \infty)$ or negative on $(0, \infty)$

according as $b > 0$ or $b < 0$. This shows that g_b is strictly increasing on $(0, \infty)$ or strictly decreasing on $(0, \infty)$ according as $b > 0$ or $b < 0$. Further,

$$g_b''(x) = bg_{b-1}'(x) = b(b-1)g_b(x) \quad \text{for all } x \in (0, \infty).$$

Hence it follows that g_b is strictly convex on $(0, \infty)$ or strictly concave on $(0, \infty)$ according as $b \in (-\infty, 0) \cup (1, \infty)$ or $b \in (0, 1)$. If $b = 0$, then $g_0(x) = e^{0 \ln x} = e^0 = 1$, and if $b = 1$, then $g_1(x) = e^{\ln x} = x$ for all $x \in (0, \infty)$.

(iii) If $b > 0$, then by the properties of the function ln, $g_b(x) = e^{b \ln x} \to \infty$ as $x \to \infty$, whereas $g_b(x) \to 0$ as $x \to 0$. Similarly, we obtain the desired results for g_b if $b < 0$. This shows that if $b \neq 0$, then g_b is not bounded above on $(0, \infty)$.

(iv) Let $b \neq 0$. The bijectivity of $g_b : (0, \infty) \to (0, \infty)$ follows from the bijectivity of the functions $\ln : (0, \infty) \to \mathbb{R}$ and $\exp : \mathbb{R} \to (0, \infty)$. Also,

$$g_{1/b}(g_b(x)) = g_{1/b}(x^b) = \exp\big((\ln x^b)/b\big) = \exp(\ln x) = x \qquad \text{for all } x \in (0, \infty).$$

Similarly, $g_b(g_{1/b}(x)) = x$ for all $x \in (0, \infty)$. Thus $g_{1/b}$ is the inverse of g_b.

(v) For all x_1, $x_2 \in (0, \infty)$,

$$\exp(b \ln x_1 x_2) = \exp b(\ln x_1 + \ln x_2) = \exp(b \ln x_1) \exp(b \ln x_2),$$

and thus, $g_b(x_1 x_2) = g_b(x_1) g_b(x_2)$, as desired. $\square$

The geometric properties of the function g_b proved in Proposition 7.10 can be used to draw its graph when $b = \sqrt{2}$ and $b = 1/\sqrt{2}$ as in Figure 7.4.

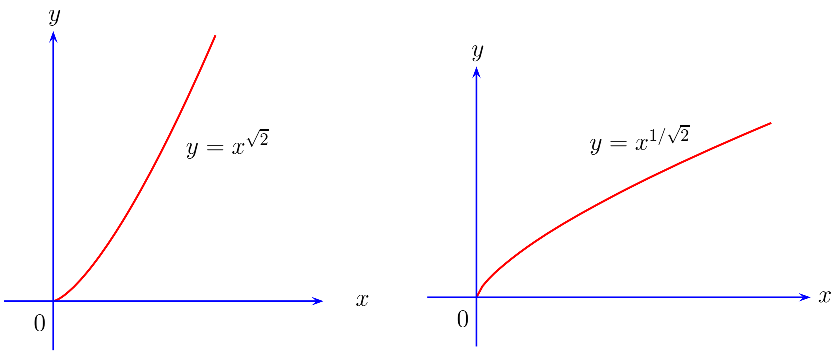

Fig. 7.4. Graphs of the power function g_b with exponent (i) $b = \sqrt{2}$, and (ii) $b = 1/\sqrt{2}$.

The following result is a generalization of Example 6.27 (i).

Corollary 7.11. *Let r be a real number such that $r \neq -1$. Let a and b be real numbers such that $0 < a < b$, and let $f : [a, b] \to \mathbb{R}$ be given by $f(x) := x^r$. Then f is integrable and*

$$\int_a^b f(x)dx = \frac{b^{r+1} - a^{r+1}}{r+1}.$$

Proof. Since f is continuous on $[a, b]$, it is integrable. Define $F : [a, b] \to \mathbb{R}$ by $F(x) := x^{r+1}/(r + 1)$. By part (i) of Proposition 7.10, F is differentiable and $F'(x) = x^r$ for all $x \in [a, b]$. Hence part (ii) of the FTC (Proposition 6.24) shows that

$$\int_a^b f(x)dx = F(b) - F(a) = \frac{b^{r+1} - a^{r+1}}{r+1},$$

as desired. □

Remark 7.12. Parts (v) and (vi) of Proposition 7.8 and part (v) of Proposition 7.10 are known as the **laws of exponents** or the **laws of indices**. We list them below:

(i) $a^{r+s} = a^r a^s$ for all $a \in (0, \infty)$ and $r, s \in \mathbb{R}$,
(ii) $(a^r)^s = a^{rs}$ for all $a \in (0, \infty)$ and $r, s \in \mathbb{R}$,
(iii) $(a_1 a_2)^r = (a_1)^r (a_2)^r$ for all $a_1, a_2 \in (0, \infty)$ and $r \in \mathbb{R}$.

For integral powers, we stated these earlier, in Section 1.1. ◇

Remark 7.13. Let $D \subseteq \mathbb{R}$ and $f, g : D \to \mathbb{R}$ be functions such that $f(x) > 0$ for all $x \in D$. Consider the function $h : D \to \mathbb{R}$ defined by

$$h(x) := f(x)^{g(x)}.$$

Properties of the function h can be studied by considering the function $k : D \to \mathbb{R}$ defined by

$$k(x) := \ln h(x) = g(x) \ln f(x).$$

For example, let $D := (0, \infty)$, and $f, g : D \to \mathbb{R}$ be defined by $f(x) := x$ and $g(x) := 1/x$. As we have seen in Example 7.5 (ii), $k(x) := (\ln x)/x \to 0$ as $x \to \infty$, and hence

$$\lim_{x\to\infty} x^{1/x} = 1.$$

We note that the indeterminate forms 0^0, ∞^0, and 1^∞ involving the functions f and g can be reduced to the indeterminate form $0 \cdot \infty$ (treated in Remark 4.46) involving the functions $\ln f$ and g, since

(i) $f(x) \to 0$ and $g(x) \to 0 \Longrightarrow \ln f(x) \to -\infty$ and $g(x) \to 0$,
(ii) $f(x) \to \infty$ and $g(x) \to 0 \Longrightarrow \ln f(x) \to \infty$ and $g(x) \to 0$,
(iii) $f(x) \to 1$ and $g(x) \to \infty \Longrightarrow \ln f(x) \to 0$ and $g(x) \to \infty$.

Revision Exercise R.22 given at the end of this chapter can be worked out using these considerations. ◇

7.2 Trigonometric Functions

Using the logarithmic function defined in Section 7.1, the reciprocal of any linear polynomial can be integrated. Indeed, up to a constant multiple, such a function is given by $1/(x-\alpha)$, where $\alpha \in \mathbb{R}$, and

$$\frac{d}{dx}\big(\ln(x-\alpha)\big) = \frac{1}{x-\alpha} \quad \text{for } x \in \mathbb{R},\ x > \alpha.$$

The next question that naturally arises is whether we can integrate the reciprocal of a quadratic polynomial, say $x^2 + ax + b$, where $a, b \in \mathbb{R}$. If this quadratic happens to be the square of a linear polynomial, say $(x-\alpha)^2$, then the answer is easy, because

$$\frac{d}{dx}\left(\frac{-1}{x-\alpha}\right) = \frac{1}{(x-\alpha)^2} \quad \text{for } x \in \mathbb{R},\ x \neq \alpha.$$

Further, if the quadratic factors into distinct linear factors, that is, if

$$x^2 + ax + b = (x-\alpha)(x-\beta) \quad \text{for some } \alpha,\ \beta \in \mathbb{R},\ \alpha > \beta,$$

then

$$\frac{1}{x^2+ax+b} = \frac{1}{\alpha-\beta}\left(\frac{1}{x-\alpha} - \frac{1}{x-\beta}\right) = \frac{1}{\alpha-\beta}\frac{d}{dx}\left(\ln\frac{x-\alpha}{x-\beta}\right) \quad \text{for } x > \alpha.$$

If, however, the quadratic x^2+ax+b has no real root, then we face a difficulty. The simplest example of this kind is the quadratic x^2+1. To be able to integrate the reciprocal of this polynomial, a new function has to be introduced and we shall do it in this section. In the Notes and Comments at the end of this chapter, we shall indicate how every rational function can be integrated using only this new function, the logarithmic function, and, of course, the rational functions themselves.

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) := 1/(1+t^2)$ is continuous. Hence it is Riemann integrable on every closed and bounded interval. For $x \in \mathbb{R}$, we define the **arctangent** of x by

$$\arctan x := \int_0^x \frac{1}{1+t^2}\,dt.$$

The function $\arctan : \mathbb{R} \to \mathbb{R}$ is known as the **arctangent function**. The reason for this terminology will be explained in Section 8.4 when we discuss the "length" of an arc of a circle. We write $\arctan x$, rather than $\arctan(x)$, for the value of the arctangent function at $x \in \mathbb{R}$.

Clearly, $\arctan 0 = 0$, and since $1/(1+t^2) \geq 0$ for all $t \in \mathbb{R}$, we see that $\arctan x \geq 0$ if $x > 0$, while $\arctan x \leq 0$ if $x < 0$.

Proposition 7.14 (Properties of the Arctangent Function and the Definition of π). (i) arctan *is a differentiable function on* $\mathbb{R}$ *and*

$$(\arctan)'x = \frac{1}{1+x^2} \quad \textit{for every } x \in \mathbb{R}.$$

(ii) arctan *is strictly increasing on* $\mathbb{R}$*, strictly convex on* $(-\infty, 0)$*, strictly concave on* $(0, \infty)$*, and* 0 *is a point of inflection for* arctan.

(iii) arctan *is an odd function. Also, it is bounded on* $\mathbb{R}$*. We define*

$$\pi := 2 \sup\{\arctan x : x \in (0, \infty)\}.$$

Then $\arctan x \to \pi/2$ *as* $x \to \infty$*, whereas* $\arctan x \to -\pi/2$ *as* $x \to -\infty$*. Moreover,* $2 \le \pi \le 4$.

(iv) *For every* $y \in (-\pi/2, \pi/2)$*, there is a unique* $x \in \mathbb{R}$ *such that* $\arctan x = y$*. In other words,* $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ *is a bijective function.*

Proof. (i) Since $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) := 1/(1+t^2)$ is continuous, part (i) of the Fundamental Theorem of Calculus (Proposition 6.24) shows that arctan is differentiable and $(\arctan)'x = f(x) = 1/(1+x^2)$ for every $x \in \mathbb{R}$.

(ii) Since the derivative of arctan is positive on $\mathbb{R}$, it follows from part (iii) of Proposition 4.29 that arctan is strictly increasing on $\mathbb{R}$. Further, since

$$(\arctan)''x = -\frac{2x}{(1+x^2)^2},$$

which is positive for all $x \in (-\infty, 0)$ and negative for all $x \in (0, \infty)$, parts (iii) and (iv) of Proposition 4.34 show that arctan is strictly convex on $(-\infty, 0)$ and strictly concave on $(0, \infty)$. Hence 0 is a point of inflection for arctan.

(iii) For $x \in \mathbb{R}$,

$$\arctan(-x) = \int_0^{-x} \frac{1}{1+t^2}\,dt = -\int_0^{x} \frac{1}{1+s^2}\,ds = -\arctan x,$$

by employing the substitution $s = -t$. Hence arctan is an odd function.

Next, we show that arctan is a bounded function. For $x \in (1, \infty)$,

$$\arctan x = \int_0^1 \frac{1}{1+t^2}\,dt + \int_1^x \frac{1}{1+t^2}\,dt,$$

and since $1 \ge t^2$ for $t \in [0, 1]$, while $t^2 \ge 1$ for $t \in [1, x]$, we see that

$$\int_0^1 \frac{1}{1+1}\,dt + \int_1^x \frac{1}{t^2+t^2}\,dt \le \arctan x \le \int_0^1 \frac{1}{1}\,dt + \int_1^x \frac{1}{t^2}\,dt.$$

The definite integrals above are easy to evaluate, and thus we obtain

$$1 - \frac{1}{2x} \le \arctan x \le 2 - \frac{1}{x} \quad \text{for all } x \in (1, \infty).$$

Since the function arctan is strictly increasing and odd, it follows that

$$-2 < \arctan x < 2 \quad \text{ for all } x \in \mathbb{R}.$$

Consequently, there is a well-defined real number $\pi \leq 4$ such that

$$\pi = 2\sup\{\arctan x : x \in (0,\infty)\}, \quad \text{that is,} \quad \frac{\pi}{2} = \sup\{\arctan x : x \in (0,\infty)\}.$$

Since arctan is increasing and since $\arctan x \geq 1 - (1/2x)$ for all $x \in (1,\infty)$,

$$\frac{\pi}{2} = \sup\{\arctan x : x \in (1,\infty)\} \geq \sup\left\{1 - \frac{1}{2x} : x \in (0,\infty)\right\} = 1.$$

Thus the real number π satisfies $2 \leq \pi \leq 4$.

(iv) The function arctan is one-one, since it is strictly increasing. Consider $y \in (-\pi/2, \pi/2)$. Since $\arctan x \to -\pi/2$ as $x \to -\infty$, there is some $x_0 \in \mathbb{R}$ such that $\arctan x_0 < y$, and since $\arctan x \to \pi/2$ as $x \to \infty$, there is some $x_1 \in \mathbb{R}$ such that $y < \arctan x_1$. But by part (i) above, the function arctan is continuous on the interval $[x_0, x_1]$. Hence the IVP (Proposition 3.16) shows that there is some $x \in (x_0, x_1)$ such that $\arctan x = y$. Thus the function $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ is bijective. ☐

Remark 7.15. We have seen that $\arctan 0 = 0$. It is not difficult to determine the value of $\arctan 1$. Indeed, for $x \in [1,\infty)$, the substitution $t = 1/s$ gives

$$\int_1^x \frac{1}{1+t^2}dt = \int_{1/x}^1 \frac{1}{1+s^2}ds \quad \text{and hence} \quad \lim_{x\to\infty} \int_1^x \frac{1}{1+t^2}dt = \int_0^1 \frac{1}{1+s^2}ds$$

by noting that $1/x \to 0$ as $x \to \infty$ and applying Proposition 6.22. Consequently,

$$\lim_{x\to\infty} (\arctan x - \arctan 1) = \lim_{x\to\infty} \int_1^x \frac{1}{1+t^2}dt = \arctan 1.$$

Now since $\arctan x \to \pi/2$ as $x \to \infty$, it follows that $\arctan 1 = \pi/4$.

The number π is traditionally defined as the area of a circular disk of radius 1 or as half the perimeter of a circle of radius 1. But these definitions presuppose the notions of area or length. In Chapter 8, we shall reconcile our definition of π with the traditional definitions after giving precise definitions of “area” and “length”.

We have seen already that $2 \leq \pi \leq 4$. Better lower and upper bounds for the number π can be obtained. (See, for example, Exercise 7.17.) The decimal expansion of π is given by

$$\pi = 3.14159265\ldots\ .$$

This is not a recurring decimal. Indeed, π is an irrational number (Exercise 7.59), and in fact, π is transcendental, that is, it is not a root of any nonzero polynomial with rational coefficients. (See [8] for a proof.) ◇

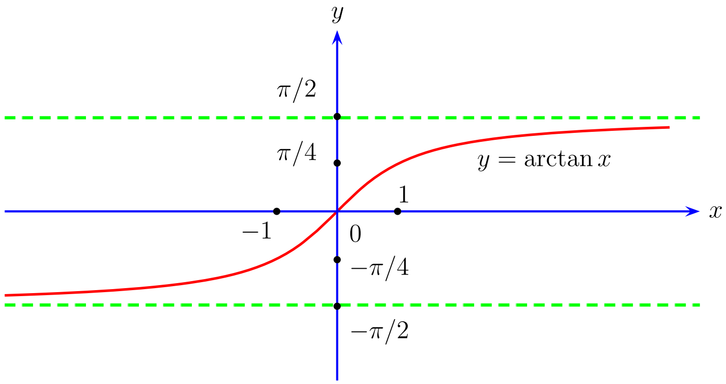

Fig. 7.5. Graph of the arctangent function.

The geometric properties of arctan obtained in Proposition 7.14, and the determination of $\arctan 1$ and the estimates for π given in Remark 7.15 can be used to draw the graph of the arctangent function as in Figure 7.5.

Let us now turn to the inverse of the arctangent function. The inverse of the bijective function $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ is known as the **tangent function** and is denoted by $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$. We write $\tan x$, rather than $\tan(x)$, for the value of the tangent function at $x \in (-\pi/2, \pi/2)$. The function tan is characterized by the following:

$$x \in (-\pi/2, \pi/2) \text{ and } \tan x = y \iff y \in \mathbb{R} \text{ and } \arctan y = x.$$

Note that $\tan x > 0$ for $x \in (0, \pi/2)$, while $\tan x < 0$ for $x \in (-\pi/2, 0)$. Moreover, since $\arctan 0 = 0$, we obtain $\tan 0 = 0$.

Proposition 7.16 (Properties of the Tangent Function). (i) tan *is a differentiable function on* $(-\pi/2, \pi/2)$ *and*

$$(\tan)'x = 1 + \tan^2 x \quad \textit{for every } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

(ii) tan *is strictly increasing on* $(-\pi/2, \pi/2)$. *Also,* tan *is strictly concave on* $(-\pi/2, 0)$, *strictly convex on* $(0, \pi/2)$, *and* 0 *is its point of inflection.*
(iii) tan *is an odd function on* $(-\pi/2, \pi/2)$. *Also,* $\tan x \to \infty$ *as* $x \to (\pi/2)^-$, *whereas* $\tan x \to -\infty$ *as* $x \to (-\pi/2)^+$.

Proof. (i) Let $x \in (-\pi/2, \pi/2)$ and $c \in \mathbb{R}$ be such that $\arctan c = x$. Since the function $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ is differentiable at c and $(\arctan)'c = 1/(1 + c^2) \neq 0$, Proposition 4.12 shows that the inverse function $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is differentiable at $x = \arctan c$ and

$$(\tan)'x = \frac{1}{(\arctan)'c} = 1 + c^2 = 1 + \tan^2 x.$$

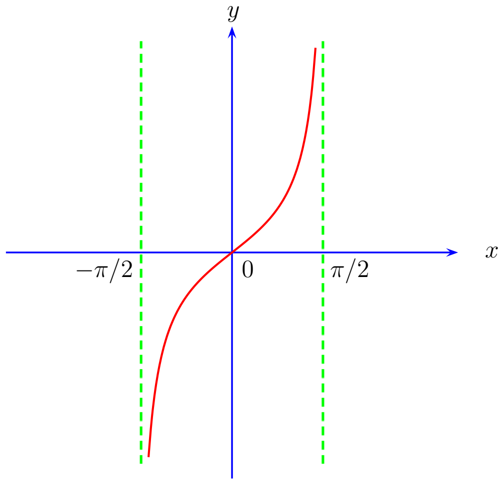

Fig. 7.6. Graph of the tangent function on $(-\pi/2,\,\pi/2)$.

(ii) Since the derivative of tan is positive on $(-\pi/2, \pi/2)$, it follows from part (iii) of Proposition 4.29 that tan is strictly increasing on $(-\pi/2, \pi/2)$. Further, since

$$(\tan)''x = \frac{d}{dx}\left(1 + \tan^2 x\right) = 2\tan x\left(1 + \tan^2 x\right) \qquad \text{for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

and since $\tan x > 0$ for $x \in (0, \pi/2)$, while $\tan x < 0$ for $x \in (-\pi/2, 0)$, it follows from parts (iii) and (iv) of Proposition 4.34 that tan is strictly concave on $(-\pi/2, 0)$ and it is strictly convex on $(0, \pi/2)$. So 0 is a point of inflection for tan.

(iii) Let $x \in (-\pi/2, \pi/2)$ and $y = \tan x$. Then

$$\tan(-x) = \tan(-\arctan y) = \tan(\arctan(-y)) = -y = -\tan x.$$

Thus tan is an odd function on $(-\pi/2, \pi/2)$.

Since the range of the function tan is the domain of the function arctan, namely $\mathbb{R}$, we see that tan is neither bounded above nor bounded below on $(-\pi/2, \pi/2)$. Also, since tan is (strictly) increasing on $(-\pi/2, \pi/2)$, it follows from Proposition 3.44 that $\tan x \to \infty$ as $x \to (\pi/2)^-$ and $\tan x \to -\infty$ as $x \to (-\pi/2)^+$. $\square$

The geometric properties of tan obtained in the above proposition can be used to draw its graph as in Figure 7.6.

Sine and Cosine Functions

To begin with, we define the **sine function** and the **cosine function** on the interval $(-\pi/2, \pi/2)$ by

$$\sin x := \frac{\tan x}{\sqrt{1+\tan^2 x}} \quad \text{and} \quad \cos x := \frac{1}{\sqrt{1+\tan^2 x}} \quad \text{for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

It is clear from the definition that

$$\tan x = \frac{\sin x}{\cos x} \quad \text{for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Further, the properties of the tangent function (Proposition 7.16) yield the following:

- $\sin 0 = 0$ and $\cos 0 = 1$.
- $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ for all $x \in (-\pi/2, \pi/2)$.
- $0 < \sin x < 1$ for all $x \in (0, \pi/2)$ and $-1 < \sin x < 0$ for all $x \in (-\pi/2, 0)$, while $0 < \cos x \le 1$ for all $x \in (-\pi/2, \pi/2)$.
- $\sin x \to 1$ as $x \to (\pi/2)^-$ and $\sin x \to -1$ as $x \to (-\pi/2)^+$, while $\cos x \to 0$ as $x \to (\pi/2)^-$ and as $x \to (-\pi/2)^+$.
- Both sin and cos are differentiable on $(-\pi/2, \pi/2)$ and satisfy

$$(\sin)'x = \cos x \quad \text{and} \quad (\cos)'x = -\sin x \quad \text{for all } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

It follows that sin is strictly increasing on $(-\pi/2, \pi/2)$, while cos is strictly increasing on $(-\pi/2, 0)$ and strictly decreasing on $(0, \pi/2)$. Also, since

$$(\sin)''x = (\cos)'x = -\sin x \;\text{ and }\; (\cos)''x = (-\sin)'x = -\cos x \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

we see that sin is strictly convex on $(-\pi/2, 0)$ and strictly concave on $(0, \pi/2)$, while cos is strictly concave on $(-\pi/2, \pi/2)$. In particular, 0 is a point of inflection for sin.

The geometric properties of sin and cos obtained above can be used to draw their graphs as in Figure 7.7.

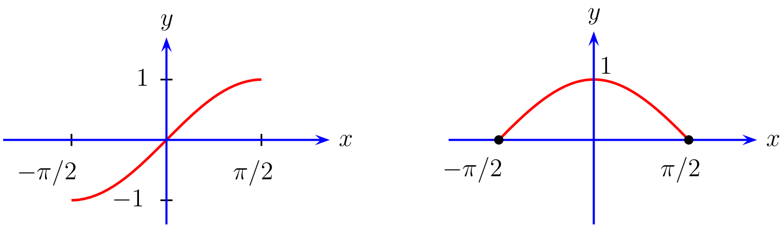

Fig. 7.7. Graphs of the sine function and the cosine function on $(-\pi/2, \pi/2)$.

By the definition of derivative and the earlier observations that $\sin 0 = 0$, $(\sin)'0 = 1$, $\cos 0 = 1$, and $(\cos)'0 = 0$, we obtain the following important limits:

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$$

We now define sin and cos at $\pi/2$ as follows:

$$\sin\frac{\pi}{2} := 1 \quad \text{and} \quad \cos\frac{\pi}{2} := 0.$$

Next, we extend sin and cos to $\mathbb{R}$ by requiring

$$\sin(x+\pi) := -\sin x \quad \text{and} \quad \cos(x+\pi) := -\cos x \quad \text{for } x \in \mathbb{R}.$$

It follows that

- $\sin(x+2\pi) = \sin x$ and $\cos(x+2\pi) = \cos x$ for all $x \in \mathbb{R}$.
- $\sin(-x) = \sin x$ and $\cos(-x) = \cos x$ for all $x \in \mathbb{R}$, that is, sin is an odd function and cos is an even function on $\mathbb{R}$.
- $\sin x = 0$ if and only if $x = k\pi$ for some $k \in \mathbb{Z}$, and $\cos x = 0$ if and only if $x = (2k+1)\pi/2$ for some $k \in \mathbb{Z}$.

Recalling that $\tan x = \sin x/\cos x$ for $x \in (-\pi/2, \pi/2)$, we extend the function tan to $\mathbb{R} \setminus \{(2k+1)\pi/2 : k \in \mathbb{Z}\}$ as follows:

$$\tan x := \frac{\sin x}{\cos x} \quad \text{for } x \in \mathbb{R} \setminus \{(2k+1)\pi/2 : k \in \mathbb{Z}\}.$$

Then

$$\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin x}{-\cos x} = \tan x \quad \text{for } x \in \mathbb{R}\setminus\{(2k+1)\pi/2 : k \in \mathbb{Z}\}.$$

Hence for each $k \in \mathbb{Z}$,

$$\tan x \to \infty \text{ as } x \to \frac{(2k+1)\pi}{2}^{-} \quad \text{and} \quad \tan x \to -\infty \text{ as } x \to \frac{(2k+1)\pi}{2}^{+}.$$

In view of the above remarks, the graphs of the (extended) sine, cosine, and tangent functions can be drawn as in Figures 7.8, 7.9, and 7.10, respectively.

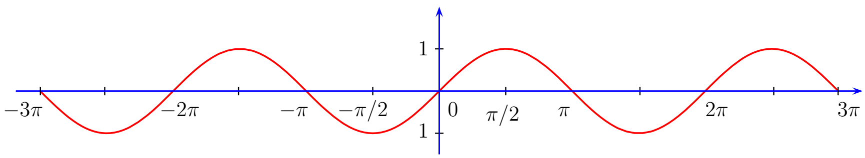

Fig. 7.8. Graph of the sine function on $[-3\pi, 3\pi]$.

We shall now consider the differentiability of the functions sin, cos, and tan.

Proposition 7.17. *The functions* sin *and* cos *are differentiable on* $\mathbb{R}$, *and*

$$(\sin)'x = \cos x \quad \textit{and} \quad (\cos)'x = -\sin x \quad \textit{for all } x \in \mathbb{R}.$$

Also, the function tan *is differentiable on* $\mathbb{R} \setminus \{(2k+1)\pi/2 : k \in \mathbb{Z}\}$, *and*

$$(\tan)'x = 1 + \tan^2 x \quad \textit{for all } \mathbb{R} \setminus \{(2k+1)\pi/2 : k \in \mathbb{Z}\}.$$

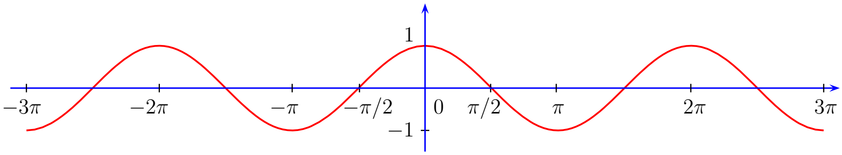

Fig. 7.9. Graph of the cosine function on $[-3\pi, 3\pi]$.

Proof. We have mentioned before that both sin and cos are differentiable on $(-\pi/2, \pi/2)$ and their derivatives satisfy the relations stated in the proposition. Also, from our extension of sin and cos to $\mathbb{R}$, it is clear that this holds at every $x \in \mathbb{R}$ for which $x \neq (2k+1)\pi/2$ for every $k \in \mathbb{Z}$.

Let us now prove that sin is differentiable at $\pi/2$ and its derivative at $\pi/2$ is 0. First, note that $\sin x \to 1$ and $\cos x \to 0$ as $x \to (\pi/2)^-$, and $\sin \pi/2 = 1$. Hence by L'Hôpital's Rule for $\frac{0}{0}$ indeterminate forms 4.39,

$$\lim_{x \to (\pi/2)^-} \frac{\sin x - \sin(\pi/2)}{x - (\pi/2)} = \lim_{x \to (\pi/2)^-} \frac{\cos x}{1} = 0.$$

Next, let $u = x - \pi$. Then $u \to (-\pi/2)^+$ as $x \to (\pi/2)^+$. Moreover, $\sin x = \sin(u + \pi) = -\sin u$. Also, $\sin u \to -1$ and $\cos u \to 0$ as $u \to (-\pi/2)^+$. Hence again by L'Hôpital's Rule, we obtain

$$\lim_{x \to (\pi/2)^+} \frac{\sin x - \sin(\pi/2)}{x - (\pi/2)} = \lim_{u \to (-\pi/2)^+} \frac{-\sin u - 1}{u + (\pi/2)} = \lim_{u \to (-\pi/2)^+} \frac{-\cos u}{1} = 0.$$

This proves $(\sin)'(\pi/2) = 0$. Similarly, it can be shown that for each $k \in \mathbb{Z}$, both sin and cos are differentiable at $(2k+1)\pi/2$, and

$$\begin{aligned}
(\sin)'\left((2k+1)\frac{\pi}{2}\right) &= 0 = \cos\left((2k+1)\frac{\pi}{2}\right), \\
(\cos)'\left((2k+1)\frac{\pi}{2}\right) &= -1 = -\sin\left((2k+1)\frac{\pi}{2}\right) \quad \text{when } k \text{ is even, and} \\
(\cos)'\left((2k+1)\frac{\pi}{2}\right) &= 1 = -\sin\left((2k+1)\frac{\pi}{2}\right) \quad \text{when } k \text{ is odd.}
\end{aligned}$$

Thus sin and cos are differentiable on $\mathbb{R}$, and their derivatives satisfy the relations stated in the proposition.

The differentiability of the function tan and the formula for its derivative follow from parts (iii) and (iv) of Proposition 4.6. ☐

The above proposition implies that sin and cos are infinitely differentiable on $\mathbb{R}$, and for $k \in \mathbb{N}$, their kth derivatives are given by

$$(\sin)^{(k)} x = \begin{cases} (-1)^{k/2} \sin x & \text{if } k \text{ is even,} \\ (-1)^{(k-1)/2} \cos x & \text{if } k \text{ is odd,} \end{cases}$$

and

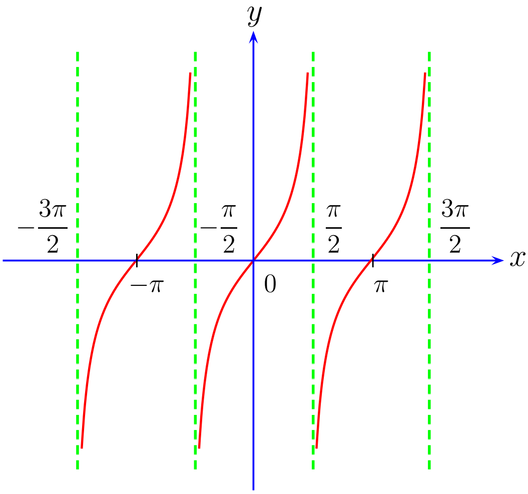

Fig. 7.10. Graph of the tangent function on $(-3\pi/2, \pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2)$.

$$(\cos)^{(k)} x = \begin{cases} (-1)^{k/2} \cos x & \text{if } k \text{ is even,} \\ (-1)^{(k+1)/2} \sin x & \text{if } k \text{ is odd.} \end{cases}$$

Hence the nth Taylor polynomial for sin about 0 is given by

$$P_n(x) = \begin{cases} x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots + (-1)^{(n-1)/2} \dfrac{x^n}{n!} & \text{if } n \text{ is odd,} \\[2ex] x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots + (-1)^{(n-2)/2} \dfrac{x^{n-1}}{(n-1)!} & \text{if } n \text{ is even.} \end{cases}$$

In particular, the linear as well as the quadratic approximation of sin around 0 is given by

$$L(x) = Q(x) = x, \quad x \in \mathbb{R}.$$

Similarly, the nth Taylor polynomial for cos about 0 is given by

$$P_n(x) = \begin{cases} 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots + (-1)^{n/2} \dfrac{x^n}{n!} & \text{if } n \text{ is even,} \\[2ex] 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots + (-1)^{(n-1)/2} \dfrac{x^{n-1}}{(n-1)!} & \text{if } n \text{ is odd.} \end{cases}$$

In particular, the linear and the quadratic approximations of cos around 0 are given by

$$L(x) = 1 \quad \text{and} \quad Q(x) = 1 - \frac{x^2}{2}, \quad x \in \mathbb{R}.$$

We now prove an important identity and the addition formulas for the sine and cosine functions.

Proposition 7.18. (i) *For all* $x \in \mathbb{R}$,

$$\sin^2 x + \cos^2 x = 1.$$

(ii) *For all* x_1 *and* x_2 *in* $\mathbb{R}$,

$$\sin(x_1 + x_2) = \sin x_1 \cos x_2 + \cos x_1 \sin x_2$$

and

$$\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2.$$

Proof. (i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \sin^2 x + \cos^2 x \quad \text{for } x \in \mathbb{R}.$$

Then $f'(x) = 2\sin x \cos x + 2\cos x(-\sin x) = 0$ for all $x \in \mathbb{R}$, and hence $f(x) = f(0) = \sin^2 0 + \cos^2 0 = 0 + 1 = 1$ for all $x \in \mathbb{R}$. This proves (i).

(ii) To derive the addition formulas, fix $x_2 \in \mathbb{R}$ and define $g, h : \mathbb{R} \to \mathbb{R}$ by

$$g(x) := \cos x \sin(x + x_2) - \sin x \cos(x + x_2) \quad \text{for } x \in \mathbb{R}$$

and

$$h(x) := \sin x \sin(x + x_2) + \cos x \cos(x + x_2) \quad \text{for } x \in \mathbb{R}.$$

Then it can be easily checked that $g'(x) = 0 = h'(x)$ for all $x \in \mathbb{R}$, and hence

$$g(x) = g(-x_2) = \sin x_2 \quad \text{and} \quad h(x) = h(-x_2) = \cos x_2.$$

Putting $x = x_1$ in these equations, we obtain

$$\cos x_1 \sin(x_1 + x_2) - \sin x_1 \cos(x_1 + x_2) = \sin x_2$$

and

$$\sin x_1 \sin(x_1 + x_2) + \cos x_1 \cos(x_1 + x_2) = \cos x_2.$$

Solving these two linear equations for $\sin(x_1+x_2)$ and $\cos(x_1+x_2)$, we obtain

$$\sin(x_1 + x_2) = \sin x_1 \cos x_2 + \cos x_1 \sin x_2$$

and

$$\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2,$$

as desired. □

We now consider the reciprocals of the functions sin, cos, and tan. The **cosecant function** and the **secant function** are defined by

$$\csc x := \frac{1}{\sin x} \quad \text{if } x \in \mathbb{R},\ x \neq k\pi \text{ for every } k \in \mathbb{Z},$$

and

$$\sec x := \frac{1}{\cos x} \quad \text{if } x \in \mathbb{R},\ x \neq (2k+1)\frac{\pi}{2} \text{ for every } k \in \mathbb{Z}.$$

The **cotangent function** is defined by

$$\cot x := \frac{\cos x}{\sin x} \quad \text{if } x \in \mathbb{R},\ x \neq k\pi \text{ for every } k \in \mathbb{Z}.$$

Thus $\cot x$ is the reciprocal of $\tan x$ if $x \neq k\pi/2$ for every $k \in \mathbb{Z}$.

The functions sin, cos, tan, csc, sec, and cot are known as the **trigonometric functions**. Several elementary results concerning these functions are given in Exercises 7.28–7.35. They follow from their definitions and Proposition 7.18.

Let us now consider the **inverse trigonometric functions**. The function $f : (-\pi/2, \pi/2) \to \mathbb{R}$ defined by $f(x) := \tan x$ is bijective. Its inverse is the function $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$, with which we started our discussion in this section. This function is also denoted by $\tan^{-1}$. Thus

$$\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$$

is the function characterized by the following:

$$y \in \mathbb{R} \text{ and } \tan^{-1} y = x \Longleftrightarrow x \in (-\pi/2, \pi/2) \text{ and } \tan x = y.$$

Also, as we have seen in part (i) of Proposition 7.14,

$$(\tan^{-1})'y = \frac{1}{1+y^2} \quad \text{for all } y \in \mathbb{R}.$$

The function $g : [-\pi/2, \pi/2] \to [-1, 1]$ defined by $g(x) := \sin x$ is bijective. Its inverse is denoted by $\sin^{-1}$ or by arcsin. Thus

$$\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$$

is the function characterized by the following:

$$y \in [-1, 1] \text{ and } \sin^{-1} y = x \Longleftrightarrow x \in [-\pi/2, \pi/2] \text{ and } \sin x = y.$$

By the Continuous Inverse Theorem (Proposition 3.17), the function $\sin^{-1}$ is continuous on $[-1, 1]$. Also, the derivative formula for the inverse function (Proposition 4.12) shows that for $y \in (-1, 1)$ and $y = \sin x$ with $x \in (-\pi/2, \pi/2)$,

$$(\sin^{-1})'y = \frac{1}{f'(x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - \sin^2 x}} = \frac{1}{\sqrt{1 - y^2}}.$$

Thus $\sin^{-1}$ is differentiable on $(-1, 1)$.

Similarly, the function $h : [0, \pi] \to [-1, 1]$ defined by $h(x) := \cos x$ is bijective. Its inverse is denoted by $\cos^{-1}$ or by arccos. Thus

$$\cos^{-1} : [-1, 1] \to [0, \pi]$$

is the function characterized by the following:

$$y \in [-1, 1] \text{ and } \cos^{-1} y = x \iff x \in [0, \pi] \text{ and } \cos x = y.$$

By Proposition 3.17, the function $\cos^{-1}$ is continuous on $[-1, 1]$. Also, as before, for $y \in (-1, 1)$ and $y = \cos x$ with $x \in (0, \pi)$,

$$(\cos^{-1})'y = \frac{1}{g'(x)} = \frac{1}{-\sin x} = \frac{-1}{\sqrt{1 - \cos^2 x}} = \frac{-1}{\sqrt{1 - y^2}}.$$

Thus $\cos^{-1}$ is differentiable on $(-1, 1)$.

The graphs of the inverse trigonometric functions $\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$ and $\cos^{-1} : [-1, 1] \to [0, \pi]$ are shown in Figure 7.11.

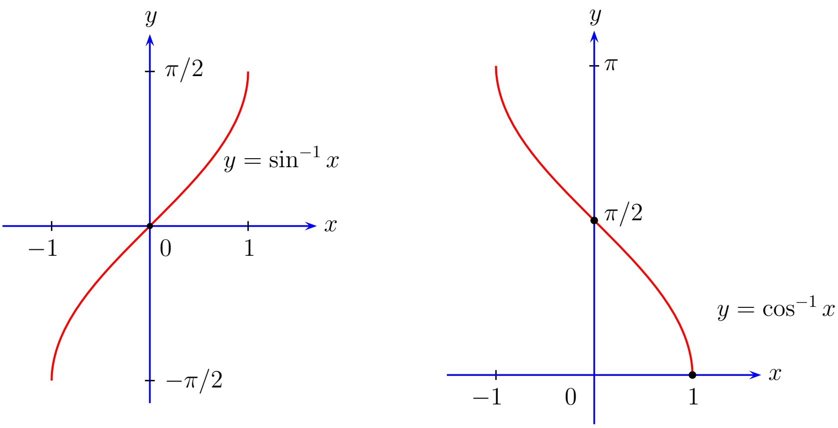

Fig. 7.11. Graphs of $\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$ and $\cos^{-1} : [-1, 1] \to [0, \pi]$.

The function $f_1 : (0, \pi) \to \mathbb{R}$ defined by $f_1(x) := \cot x$ is bijective. Its inverse is denoted by $\cot^{-1}$. Thus

$$\cot^{-1} : \mathbb{R} \to (0, \pi)$$

is the function characterized by the following:

$$y \in \mathbb{R} \text{ and } \cot^{-1} y = x \iff x \in (0, \pi) \text{ and } \cot x = y.$$

The function $g_1 : [-\pi/2, 0) \cup (0, \pi/2] \to (-\infty, -1] \cup [1, \infty)$ defined by $g_1(x) := \csc x$ is bijective. Its inverse is denoted by $\csc^{-1}$. Thus

$$\csc^{-1} : (-\infty, -1] \cup [1, \infty) \to [-\pi/2, 0) \cup (0, \pi/2]$$

is the function characterized by the following:

$$y \in \mathbb{R},\ |y| \geq 1 \text{ and } \csc^{-1} y = x \iff x \in \mathbb{R},\ 0 < |x| \leq \pi/2 \text{ and } \csc x = y.$$

The function $h_1 : [0, \pi/2) \cup (\pi/2, \pi] \to (-\infty, -1] \cup [1, \infty)$ defined by $h_1(x) := \sec x$ is bijective. Its inverse is denoted by $\sec^{-1}$. Thus

$$\sec^{-1} : (-\infty, -1] \cup [1, \infty) \to [0, \pi/2) \cup (\pi/2, \pi]$$

is the function characterized by the following:

$$y \in \mathbb{R},\ |y| \geq 1 \text{ and } \sec^{-1} y = x \iff x \in [0, \pi],\ x \neq \pi/2 \text{ and } \sec x = y.$$

The formulas for the derivatives of the functions $\cot^{-1}$, $\csc^{-1}$, and $\sec^{-1}$ are given in Exercise 7.42. For various relations involving the inverse trigonometric functions, see Exercises 7.36–7.41.

7.3 Sine of the Reciprocal

In this section we study the composition of the reciprocal function $x \mapsto 1/x$ and the sine function. We also study some related functions. To begin with, consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

$$f(x) := \sin \frac{1}{x}, \quad x \neq 0.$$

The reason for paying special attention to this function is that it has many interesting properties, and it provides, along with other associated functions, several examples and counterexamples for various statements in calculus and analysis. We have deferred these examples till now, since the trigonometric functions were introduced only in Section 7.2.

Properties of the sine function developed earlier yield the following:

- f is an odd function.
- f is a bounded function. In fact, $-1 \leq f(x) \leq 1$ for all $x \in \mathbb{R} \setminus \{0\}$.
- $f(x) = 0$ if and only if $x = 1/k\pi$ for some nonzero $k \in \mathbb{Z}$, while $f(x) = 1$ if and only if $x = 2/(4k+1)\pi$ for some $k \in \mathbb{Z}$, and $f(x) = -1$ if and only if $x = 2/(4k-1)\pi$ for some $k \in \mathbb{Z}$.
  The graph of the function f drawn in Figure 7.12 shows that it oscillates between 1 and -1 more and more rapidly as we approach 0 from the left or from the right.
- f is continuous on $\mathbb{R} \setminus \{0\}$, since the function given by $x \mapsto 1/x$ is continuous on $\mathbb{R} \setminus \{0\}$ and the sine function is continuous on $\mathbb{R}$ (Proposition 3.5).

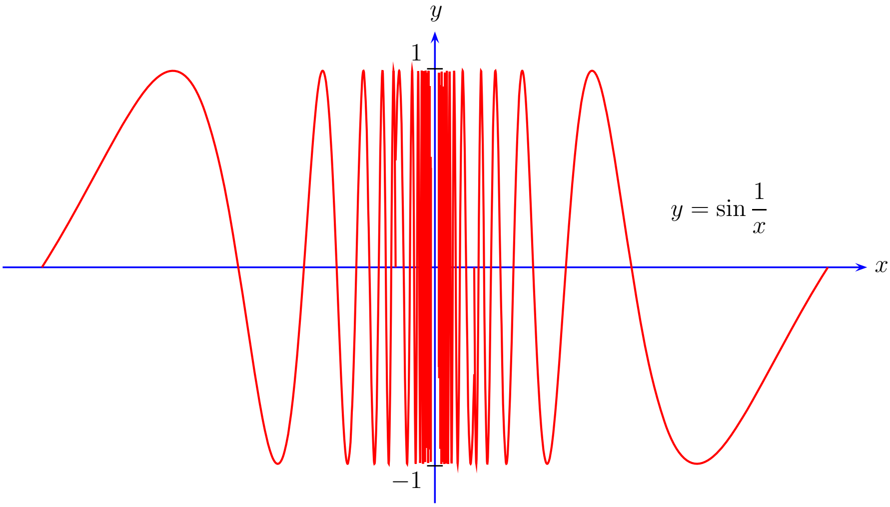

Fig. 7.12. Graph of $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) := \sin(1/x)$.

- f is not uniformly continuous on $(0, \delta)$ for any $\delta > 0$. This follows by noting that if $x_n = 1/n\pi$ and $y_n = 2/(4n + 1)\pi$ for $n \in \mathbb{N}$, then $x_n - y_n \to 0$ and $x_n, y_n \in (0, \delta)$ for all large $n \in \mathbb{N}$, but since $f(x_n) = 0$ and $f(y_n) = 1$, we see that $f(x_n) - f(y_n) \not\to 0$. Similarly, f is not uniformly continuous on $(-\delta, 0)$ for any $\delta > 0$.
  However, if $D \subseteq \mathbb{R}$ and there exists $\delta > 0$ such that $D \subseteq (-\infty, -\delta] \cup [\delta, \infty)$, then f is uniformly continuous on D. This can be seen as follows. Let (x_n) and (y_n) be any sequences in D such that $x_n - y_n \to 0$. Then

$$\begin{aligned}\sin\frac{1}{x_n} - \sin\frac{1}{y_n} &= 2\cos\frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_n}\right)\sin\frac{1}{2}\left(\frac{1}{x_n} - \frac{1}{y_n}\right)\\ &= -2\cos\left(\frac{x_n + y_n}{2x_n y_n}\right)\sin\left(\frac{x_n - y_n}{2x_n y_n}\right) \quad \text{for all } n \in \mathbb{N}.\end{aligned}$$

  (See Exercise 7.29.) Since $|x_n| \geq \delta$ and $|y_n| \geq \delta$ for all $n \in \mathbb{N}$, we see that $(x_n - y_n)/2x_n y_n \to 0$ and hence

$$|f(x_n) - f(y_n)| = \left|\sin\frac{1}{x_n} - \sin\frac{1}{y_n}\right| \leq 2\left|\sin\left(\frac{x_n - y_n}{2x_n y_n}\right)\right| \to 0 \quad \text{as} \quad n \to \infty.$$

- f is infinitely differentiable at every nonzero $x \in \mathbb{R}$, thanks to the Chain Rule (Proposition 4.10). In particular,

$$f'(x) = -\frac{1}{x^2}\cos\frac{1}{x} \quad \text{and} \quad f''(x) = \frac{2}{x^3}\cos\frac{1}{x} - \frac{1}{x^4}\sin\frac{1}{x} \quad \text{for } x \in \mathbb{R} \setminus \{0\}.$$

  It is clear that for all $\delta > 0$, f' and f'' are not bounded on $(0, \delta)$ as well as on $(-\delta, 0)$.

- For all $\delta > 0$, f is not monotonic on $(0, \delta)$ as well as on $(-\delta, 0)$. To see this, let $x_k := 1/k\pi$ for nonzero $k \in \mathbb{Z}$; note that $f'(x_k) = (-1)^{k+1}k^2\pi^2$ and apply part (i) of 4.30.
- For all $\delta > 0$, f is neither convex nor concave on $(0, \delta)$ as well as on $(-\delta, 0)$. To see this, let $x_k := 1/k\pi$ for nonzero $k \in \mathbb{Z}$; note that $f''(x_k) = (-1)^k k^3\pi^3$ and apply part (i) of Corollary 4.35.

We remark that similar properties are possessed by the real-valued function $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

$$g(x) := \cos\frac{1}{x}, \quad x \neq 0.$$

(See Exercise 7.45.)

Let $r_0 \in \mathbb{R}$, and consider the function $f_0 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_0(x) := \begin{cases} f(x) & \text{if } x \neq 0, \\ r_0 & \text{if } x = 0. \end{cases}$$

It follows from Corollary 6.12 that for every $a, b \in \mathbb{R}$ with $a < b$, the function f_0 is Riemann integrable on $[a, b]$. Consider now $F_0 : \mathbb{R} \to \mathbb{R}$ defined by

$$F_0(x) := \int_0^x f_0(t)dt.$$

Observe that the function F_0 does not depend on the choice of $r_0 \in \mathbb{R}$, thanks to Proposition 6.13. Let $c \in \mathbb{R}$, $c \neq 0$. Since f_0 is continuous at c, part (i) of the Fundamental Theorem of Calculus (Proposition 6.24) shows that the function F_0 is differentiable at c and $F_0'(c) = f_0(c) = f(c)$. However, it is not clear how the functions f_0 and F_0 behave near 0. We shall now analyze this.

Proposition 7.19. *Let f_0 and F_0 be as defined above.*

(i) *f_0 is not continuous at 0; in fact, neither $\lim_{x\to 0^+} f_0(x)$ nor $\lim_{x\to 0^-} f_0(x)$ exists.*
(ii) *F_0 is differentiable at 0 and $F_0'(0) = 0$, that is,*

$$\lim_{x\to 0} \frac{1}{x}\int_0^x \sin\frac{1}{t}dt = 0.$$

Proof. (i) Let $x_n := 1/n\pi$ and $y_n := 2/(4n + 1)\pi$ for $n \in \mathbb{N}$. Then (x_n) and (y_n) are sequences of positive real numbers such that $x_n \to 0$ and $y_n \to 0$, but $f(x_n) = 0$ and $f(y_n) = 1$ for all $n \in \mathbb{N}$, and so $f(x_n) \to 0$, whereas $f(y_n) \to 1$. Thus it follows that $\lim_{x\to 0^+} f(x)$ does not exist. Since f is an odd function, $\lim_{x\to 0^-} f(x)$ does not exist. Finally, since $f_0(x) = f(x)$ for all nonzero $x \in \mathbb{R}$, (i) is proved.

(ii) Let $x \in \mathbb{R}$ with $x > 0$. By Proposition 6.22,

$$F_0(x) - F_0(0) = \int_0^x \sin\frac{1}{t}\,dt = \lim_{r\to 0^+}\int_r^x \sin\frac{1}{t}\,dt.$$

Fix $r \in \mathbb{R}$ such that $0 < r \leq x$. Substituting $t = 1/u$ and then integrating by parts (that is, using Propositions 6.29 and 6.28), we obtain

$$\int_r^x \sin\frac{1}{t}\,dt = \int_{1/x}^{1/r} (\sin u)\frac{1}{u^2}\,du = x^2\cos\frac{1}{x} - r^2\cos\frac{1}{r} - 2\int_{1/x}^{1/r}\frac{\cos u}{u^3}\,du.$$

Since

$$0 \leq \int_{1/x}^{1/r}\left|\frac{\cos u}{u^3}\right|du \leq \int_{1/x}^{1/r}\frac{1}{u^3}\,du = \frac{1}{2}(x^2 - r^2),$$

we see that

$$\left|\int_0^x \sin\frac{1}{t}\,dt\right| \leq \lim_{r\to 0^+}\left|\int_r^x \sin\frac{1}{t}\,dt\right| \leq \lim_{r\to 0^+}(x^2 + r^2 + x^2 - r^2) = 2x^2.$$

Thus for every $x \in (0,\infty)$,

$$\left|\frac{1}{x}\int_0^x \sin\frac{1}{t}\,dt\right| \leq 2x, \quad \text{and so} \quad \lim_{x\to 0^+}\frac{F_0(x) - F_0(0)}{x - 0} = \lim_{x\to 0^+}\frac{1}{x}\int_0^x \sin\frac{1}{t}\,dt = 0.$$

Replacing x by $-x$ and noting that sin is an odd function, we also obtain

$$\lim_{x\to 0^-}\frac{F_0(x) - F_0(0)}{x - 0} = \lim_{x\to 0^-}\frac{1}{x}\int_0^x \sin\frac{1}{t}\,dt = 0.$$

Hence the desired result follows by Proposition 3.37. ☐

We remark that a similar result holds for the cosine of the reciprocal. (See Exercise 7.46.)

We shall now study some functions associated with the function f_0. They are obtained by multiplying f_0 by the identity function and by the square of the identity function.

Example 7.20. Consider the function $f_1 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_1(x) := \begin{cases} x\sin(1/x) & \text{if } x \neq 0,\\ 0 & \text{if } x = 0.\end{cases}$$

Properties of the function f developed earlier yield the following:

- f_1 is an even function.
- f_1 is a bounded function. In fact, $-1 < f_1(x) < 1$ for all $x \in \mathbb{R}$. To see this, note that $f_1(x) = \sin(1/x)/(1/x)$ if $x \neq 0$, and $-y < \sin y < y$ for all nonzero $y \in \mathbb{R}$ (as can be seen by a simple application of the MVT). Since $\lim_{h\to 0}\sin h/h = 1$, it follows that $f_1(x) \to 1$ as $x \to \infty$ and as $x \to -\infty$.

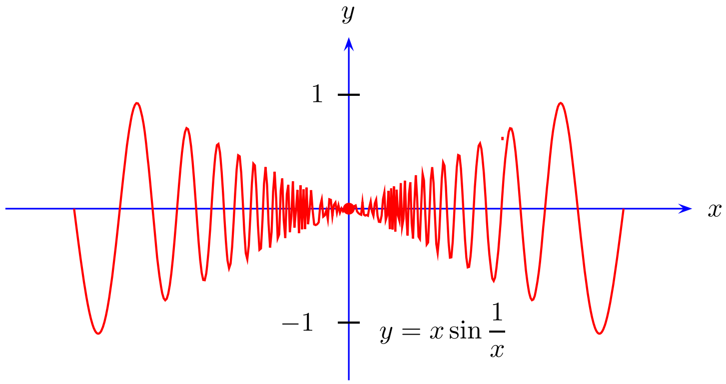

Fig. 7.13. Illustration of damped oscillations: Graph of $f_1 : \mathbb{R} \to \mathbb{R}$ given by $f_1(0) := 0$ and $f_1(x) := x \sin(1/x)$ for $x \neq 0$.

- The oscillations of the function f_1, inherited from the function f, are "damped" near 0, because $|f_1(x)| \leq |x|$ for all $x \in \mathbb{R}$. This behavior of f_1 is shown in Figure 7.13.
- f_1 is continuous on $\mathbb{R}$. To see this, we note that f_1 is a product of two functions each of which is continuous on $\mathbb{R} \setminus \{0\}$, and moreover, f_1 is continuous at 0, because if (x_n) is any sequence such that $x_n \to 0$, then $|f_1(x_n)| \leq |x_n|$ implies that $f_1(x_n) \to 0$.
- f_1 is infinitely differentiable at every nonzero $x \in \mathbb{R}$, thanks to part (iv) of Proposition 4.6. In particular,

$$f_1'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \quad \text{and} \quad f_1''(x) = -\frac{1}{x^3} \sin \frac{1}{x} \quad \text{for } x \in \mathbb{R} \setminus \{0\}.$$

  It is clear that for every $\delta > 0$, both f_1' and f_1'' are not bounded on $(0, \delta)$ as well as on $(-\delta, 0)$.
- For all $\delta > 0$, the function f_1 is not monotonic on $(0, \delta)$ as well as on $(-\delta, 0)$. To see this, let $x_k := 1/k\pi$ for nonzero $k \in \mathbb{Z}$; note that $f_1'(x_k) = (-1)^{k+1} k\pi$ and apply part (i) of Corollary 4.30.
- For all $\delta > 0$, the function f_1 is neither convex nor concave on $(0, \delta)$ as well as on $(-\delta, 0)$. To see this, let $y_k := 2/(2k+1)\pi$ for nonzero $k \in \mathbb{Z}$; note that $f_1''(y_k) = (-1)^{k+1}(k + (1/2))^3 \pi^3$ and apply part (i) of Corollary 4.35.
- The right and left derivatives of f_1 at 0 do not exist, that is, the limits

$$\lim_{x \to 0^+} \frac{f_1(x) - f_1(0)}{x - 0} = \lim_{x \to 0^+} \sin \frac{1}{x} \quad \text{and} \quad \lim_{x \to 0^-} \frac{f_1(x) - f_1(0)}{x - 0} = \lim_{x \to 0^-} \sin \frac{1}{x}$$

  do not exist, as we have seen in part (i) of Proposition 7.19.

The function $|f_1|$ has an absolute minimum (although it is not a strict minimum) at 0. However, there is no $\delta > 0$ such that f is decreasing on $(-\delta, 0]$ and f is increasing on $[0, \delta)$, because $|f_1|(1/k\pi) = 0$ for all nonzero $k \in \mathbb{Z}$, while $|f_1|(2/(2k+1)\pi) = 2/|2k+1|\pi$ for all $k \in \mathbb{Z}$. This phenomenon was earlier illustrated in Example 1.20 of infinitely many zigzags.

The function f_1 can be used to conclude that the converse of L'Hôpital's Rule for $\frac{\infty}{\infty}$ indeterminate forms is not true. For this purpose, consider functions $h_1, g_1 : (0, \infty) \to \mathbb{R}$ defined by

$$h_1(x) := \frac{1 + f_1(x)}{x} \quad \text{and} \quad g_1(x) := \frac{1}{x}.$$

Then $g_1(x) \to \infty$ as $x \to 0^+$, $g'(x) = -1/x^2 \neq 0$ for all $x \in (0, \infty)$, and

$$\lim_{x \to 0^+} \frac{h_1(x)}{g_1(x)} = \lim_{x \to 0^+} \big(1 + f_1(x)\big) = 1 + 0 = 0,$$

but

$$\lim_{x \to 0^+} \frac{h_1'(x)}{g_1'(x)} = \lim_{x \to 0^+} \frac{\big(-1 - \cos(1/x)\big)/x^2}{-1/x^2} = \lim_{x \to 0^+} \big(1 + \cos(1/x)\big)$$

does not exist, because $\lim_{x \to 0^+} \cos(1/x)$ does not exist. $\diamond$

Example 7.21. Consider the function $f_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_2(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Properties of the functions f and f_1 developed earlier yield the following:

- f_2 is an odd function.
- For all $a \in \mathbb{R}$, the function f_2 is not bounded above on (a, ∞). This follows by noting that $\sin(1/x)/(1/x) \to 1$ as $x \to \infty$, so that

$$f_2(x) = x \frac{\sin(1/x)}{(1/x)} \to \infty \quad \text{as } x \to \infty.$$

  Being an odd function, f_2 is not bounded below on $(-\infty, b)$ for any $b \in \mathbb{R}$.
- The oscillations of the function f_2, inherited from the function f, are doubly damped near 0, because $|f_2(x)| \leq |x|^2$ for all $x \in \mathbb{R}$. This behavior of f_2 is shown in Figure 7.14.
- f_2 is continuous on $\mathbb{R}$, since $f_2(x) = x f_1(x)$ for all $x \in \mathbb{R}$ and the function f_1 is continuous on $\mathbb{R}$.
- f_2 is infinitely differentiable at every nonzero $x \in \mathbb{R}$. In particular, for every $x \in \mathbb{R} \setminus \{0\}$,

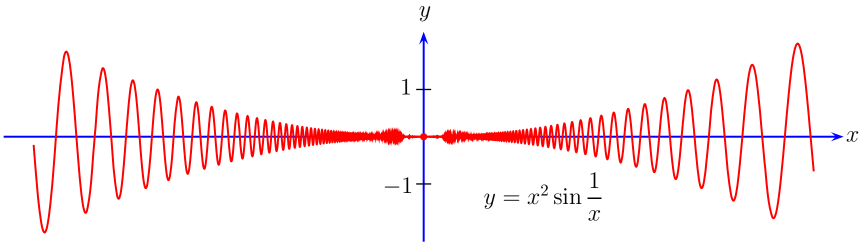

Fig. 7.14. Illustration of doubly damped oscillations: the graph of $f_2 : \mathbb{R} \to \mathbb{R}$ given by $f_2(0) := 0$ and $f_2(x) := x^2 \sin(1/x)$ for $x \neq 0$.

$$f_2'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \quad \text{and} \quad f_2''(x) = 2 \sin \frac{1}{x} - \frac{2}{x} \cos \frac{1}{x} - \frac{1}{x^2} \sin \frac{1}{x}.$$

It is clear that f_2' is bounded on $x \in \mathbb{R} \setminus \{0\}$, but for every $\delta > 0$, the second derivative f_2'' is not bounded on $(0, \delta)$ as well as on $(-\delta, 0)$.

- For every $\delta > 0$, f_2 is not monotonic on $(0, \delta)$ as well as on $(-\delta, 0)$. To see this, let $x_k := 1/k\pi$ for nonzero $k \in \mathbb{Z}$; note that $f_2'(x_k) = (-1)^{k+1}$ and apply part (i) of Corollary 4.30.
- For every $\delta > 0$, f_2 is neither convex nor concave on $(0, \delta)$ as well as on $(-\delta, 0)$. To see this, let $y_k := 1/k\pi$ for nonzero $k \in \mathbb{Z}$; note that $f_2''(y_k) = (-1)^{k+1} 2k\pi$ and apply part (i) of Corollary 4.35.
- f_2 is differentiable at 0. In fact,
$$f_2'(0) = \lim_{x \to 0} \frac{f_2(x) - f_2(0)}{x - 0} = \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$
But f_2' is not continuous at 0, because $\lim_{x \to 0} f_2'(x)$ does not exist. This follows because $\lim_{x \to 0} \cos(1/x)$ does not exist.
- Although $f_2'(0) = 0$, the function f_2 does not have a local extremum at 0, nor is 0 a point of inflection for f, because of the oscillatory nature of f_2 and f_2' around 0.

The function f_2 can be used to conclude that the converse of L'Hôpital's Rule for $\frac{0}{0}$ indeterminate forms is not true. For this purpose, let $g_2(x) := \sin x$ for $x \in \mathbb{R}$. Then $\lim_{x \to 0} f_2(x) = 0 = \lim_{x \to 0} g_2(x)$ and

$$\lim_{x \to 0} \frac{f_2(x)}{g_2(x)} = \left(\lim_{x \to 0} \frac{x}{\sin x} \right) \left(\lim_{x \to 0} x \sin \frac{1}{x} \right) = (1)(0) = 0,$$

but

$$\lim_{x \to 0} \frac{f_2'(x)}{g_2'(x)} = \lim_{x \to 0} \frac{2x \sin(1/x) - \cos(1/x)}{\cos x}$$

does not exist, because $\lim_{x \to 0} \cos(1/x)$ does not exist, but on the other hand, $\lim_{x \to 0} 2x \sin(1/x) = 0$ and $\lim_{x \to 0} \cos x = 1$. ◇

For further examples, see Exercises 7.47, 7.50, and 7.62.

7.4 Polar Coordinates

Having defined the trigonometric functions and the number π, we are in a position to describe an alternative and useful way of representing points in the plane $\mathbb{R}^2$ by their polar coordinates. Roughly speaking, the polar coordinates of a point $P = (x, y) \in \mathbb{R}^2$ are the numbers r and θ satisfying the equations

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Geometrically speaking, the number r represents the distance from P to the origin $O = (0, 0)$, whereas θ can be interpreted as the "angle" from the positive x-axis to the line segment OP. However, there is a certain ambiguity if we define r and θ simply by the above equations. Indeed, if (r, θ) satisfy these equations, then so do $(r, \theta + 2\pi)$, $(r, \theta - 2\pi)$, $(-r, \theta + \pi)$, etc.; the special case $P = O$ is even worse, because we can take $r = 0$ and θ to be any real number. To avoid such ambiguities and to enable us to give a precise definition of polar coordinates, we first prove the following proposition. In the sequel, we shall also give a formal definition of the notion of *angle*, and clarify the geometric interpretation of polar coordinates.

Proposition 7.22. *If $x, y \in \mathbb{R}$ are such that $(x, y) \neq (0, 0)$, then r and θ defined by*

$$r := \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta := \begin{cases} \cos^{-1}\left(\dfrac{x}{r}\right) & \text{if } y \geq 0, \\ -\cos^{-1}\left(\dfrac{x}{r}\right) & \text{if } y < 0, \end{cases}$$

satisfy the following properties:

$$r, \theta \in \mathbb{R}, \quad r > 0, \quad \theta \in (-\pi, \pi], \quad x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta.$$

Conversely, if $r, \theta \in \mathbb{R}$ are such that $r > 0$ and $\theta \in (-\pi, \pi]$, then $x := r\cos\theta$ and $y := r\sin\theta$ are real numbers such that $(x, y) \neq (0, 0)$, $r = \sqrt{x^2 + y^2}$, and θ equals $\cos^{-1}(x/r)$ or $-\cos^{-1}(x/r)$ according as $y \geq 0$ or $y < 0$.

Proof. Let $x, y \in \mathbb{R}$ with $(x, y) \neq (0, 0)$ be given. Define r and θ by the formulas displayed above. Then $(x, y) \neq (0, 0)$ implies that $r > 0$. Also, since $|x/r| \leq 1$ and since $\cos^{-1}$ is a map from $[-1, 1]$ to $[0, \pi]$, we see that θ is well-defined and $\theta \in [-\pi, \pi]$. Further, if $y < 0$, then $|x/r| < 1$, and therefore $\cos^{-1}(x/r) \in (0, \pi)$. Thus $\theta \in (-\pi, \pi]$. Moreover, since $\cos(-\theta) = \cos\theta$, it follows that $\cos\theta = x/r$, that is, $x = r\cos\theta$. Consequently, $y^2 = r^2(1 - \cos^2\theta)$, and hence $y = \pm r\sin\theta$. But from the definition of θ, it is clear that $y \geq 0$ if and only if $0 \leq \theta \leq \pi$. It follows that $y = r\sin\theta$. This proves that r and θ satisfy the desired properties.

Conversely, let $r, \theta \in \mathbb{R}$ be given such that $r > 0$ and $\theta \in (-\pi, \pi]$. Define $x := r\cos\theta$ and $y := r\sin\theta$. Since $r > 0$ and $\cos^2\theta + \sin^2\theta = 1$, it is clear that $r = \sqrt{x^2 + y^2}$, and in particular, $(x, y) \neq (0, 0)$. Also, since $\theta \in (-\pi, \pi]$ and $x/r = \cos\theta$, it follows that if $\theta \in [0, \pi]$, then $\theta = \cos^{-1}(x/r)$, whereas if $\theta \in (-\pi, 0)$, then $-\theta \in (0, \pi)$ and $\cos(-\theta) = \cos\theta = x/r$, and consequently, $-\theta = \cos^{-1}(x/r)$. Moreover, $y = r\sin\theta \geq 0$ if and only if $\theta \in [0, \pi]$, and thus we see that x and y satisfy the desired properties. □

In view of the above proposition, we define the **polar coordinates** of a point $P = (x, y)$ in $\mathbb{R}^2$, different from the origin, to be the pair (r, θ) defined by the formulas displayed above. Equivalently, r and θ are real numbers determined by the conditions $r > 0$, $\theta \in (-\pi, \pi]$, $x = r\cos\theta$, and $y = r\sin\theta$.

For example, the polar coordinates of the points $(1, 0)$, $(3, 4)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$, and $(3, -4)$ are $(1, 0)$, $(5, \cos^{-1}(3/5))$, $(1, \pi/2)$, $(1, \pi)$, $(1, -\pi/2)$, and $(5, -\cos^{-1}(3/5))$, respectively. The polar coordinates of the origin $(0, 0)$ are not defined.

For a point $P = (x, y) \in \mathbb{R}^2$, we sometimes call the pair (x, y) the **Cartesian coordinates** or the **rectangular coordinates** of P.

Remarks 7.23. (i) In the definition of polar coordinates, we have required that θ should lie in the interval $(-\pi, \pi]$. This is actually a matter of convention. Alternative conditions are possible and can sometimes be found in books on calculus. For example, a commonly used alternative is to require that θ should lie in the interval $[0, 2\pi)$. In this case, we have to change $-\cos^{-1}(x/r)$ to $2\pi - \cos^{-1}(x/r)$ in the formula for θ in Proposition 7.22. Yet another alternative is to let r take positive as well as negative values but restrict θ to the interval $[0, \pi)$. In this case, we have to set r equal to $\sqrt{x^2 + y^2}$ or $-\sqrt{x^2 + y^2}$, according as $y \geq 0$ or $y < 0$, and set θ equal to $\cos^{-1}(x/r)$ (regardless of the sign of y) in Proposition 7.22. In any case, the key equations remain $x = r\cos\theta$ and $y = r\sin\theta$. In fact, some books disregard the questions of uniqueness and define the polar coordinates of the point (x, y) to be *any* pair (r, θ) of real numbers satisfying $x = r\cos\theta$ and $y = r\sin\theta$. We shall, however, prefer that a change of coordinates be determined unambiguously and adhere to the definition above.

(ii) It is more common to describe the "inverse formula" for $\theta \in (-\pi, \pi]$ satisfying $x = r\cos\theta$ and $y = r\sin\theta$ in terms of the arctangent function, namely $\theta = \tan^{-1}(y/x)$. However, this is correct only when $x > 0$. For a comprehensive "inverse formula", one has to consider four other cases separately. Indeed, $\theta = \tan^{-1}(y/x) + \pi$ if $x < 0$ and $y \geq 0$; $\theta = \tan^{-1}(y/x) - \pi$ if $x < 0$ and $y < 0$; $\theta = \pi/2$ if $x = 0$ and $y > 0$; finally, $\theta = -\pi/2$ if $x = 0$ and $y < 0$. To avoid this, we have used $\cos^{-1}$ in Proposition 7.22, and as a result, it suffices to consider only two cases.

(iii) In classical geometry, the polar coordinates are described as follows. In the plane choose a point O, called the pole, and a ray emanating from O, called the polar axis. The polar coordinates of a point P are now (r, θ), where r is the distance of P from the pole O, and θ is (any) angle from the polar axis to the line joining O and P. In our approach, the plane comes equipped with a (Cartesian) coordinate system, and we have fixed the pole O to be the origin and the polar axis to be the positive x-axis. ◇

We have seen earlier that an equation in x and y determines a curve in the plane. Similarly, an equation in r and θ determines a curve in the plane, namely, the curve consisting of points in the plane whose polar coordinates satisfy this equation. In case the equation is satisfied when $r = 0$, we regard the origin as a point on the curve. Frequently, the equations we come across are of the form $r = p(\theta)$, where p is a real-valued function defined on some subset of $(-\pi, \pi]$. If no domain for p is specified, then this may be assumed to be $(-\pi, \pi]$. For ease of reference, we may use the terms **Cartesian equation** and **polar equation** to mean an equation (of a curve in the plane) in Cartesian coordinates and in polar coordinates, respectively.

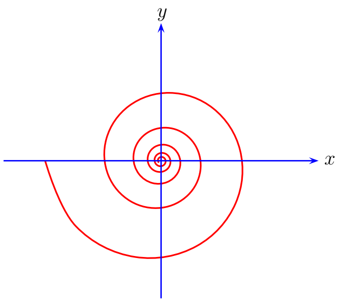

Fig. 7.15. Spiral of Archimedes $r = \theta$, and the cardioid $r = 2(1 + \cos\theta)$.

Often, a curve can be described by a Cartesian equation as well as by a polar equation. Sometimes, the latter can be simpler. For example, a circle of radius 2 centered at the origin can be described by the Cartesian equation $x^2 + y^2 = 4$, whereas its polar equation is simply $r = 2$. On the other hand, to see what the curve given by the polar equation $r = 2\sin\theta$ might look like, it may be easier to first convert it to a Cartesian equation. To do so, note that the polar equation is equivalent to $r^2 = 2r\sin\theta$, and hence the Cartesian equation is given by $x^2 + y^2 = 2y$, that is, $x^2 + (y - 1)^2 = 1$. Thus, the curve with the polar equation $r = 2\sin\theta$ is a circle of radius 1 centered at the point $(0, 1)$ on the y-axis.

We describe below a few classical examples of curves that admit a nice description in polar coordinates.

Examples 7.24. (i) (**Spiral**) The graph of an equation of the type $r = a\theta$ looks like a curve that winds around the origin, and is known as a spiral (of Archimedes). The graph of $r = \theta$ is shown in Figure 7.15.

(ii) (**Cardioid**) A polar equation of the type $r = a(1 + \cos\theta)$ gives rise to a heart-shaped curve, known as a cardioid. This curve can also be described as the locus of a point on the circumference of a circle rolling round the circumference of another circle of equal radius. A sketch of $r = 2(1+\cos\theta)$ is shown in Figure 7.15.

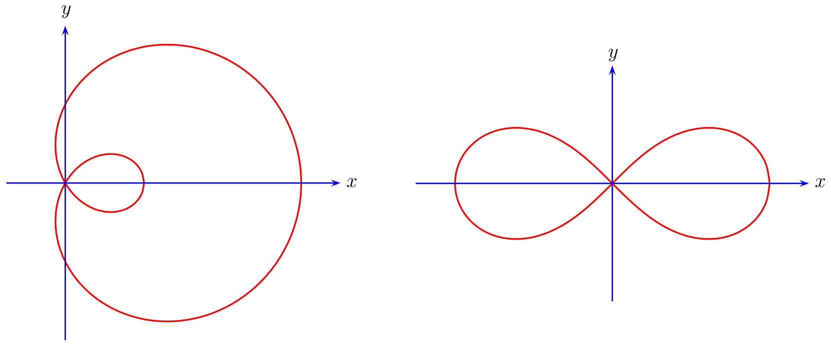

Fig. 7.16. Limaçon $r = 1 + 2\cos\theta$, and the lemniscate $r^2 = 2\cos 2\theta$.

(iii) (**Limaçon**) A polar equation of the form $r = b + a\cos\theta$, which is more general than that of a cardioid, traces a curve known as a limaçon[1] (of Pascal). It looks similar to a cardioid except that instead of a cusp, it has an inner loop (provided $b < a$). A picture of the limaçon $r = 1 + 2\cos\theta$ is shown in Figure 7.16.

(iv) (**Lemniscate**) A polar equation of the form $r^2 = 2a^2\cos 2\theta$ gives rise to a curve shaped like a figure 8 or a bow tied in a ribbon, which is called a lemniscate[2] (of Bernoulli). A graph of a lemniscate with $a = 1$ is shown in Figure 7.16, and it may be observed that it displays a great deal of symmetry.

(v) (**Rose**) Polar equations of the type $r = a\cos n\theta$ or $r = a\sin n\theta$ give rise to floral-shaped curves, known as **rhodonea curves**, or simply roses. Typically, if n is an odd integer, then it has n petals, whereas if n is an even integer, then it has $2n$ petals. Graphs of roses with $a = 1$ and with $n = 4, 5$ are shown in Figure 7.17. The configurations for which n is not an integer are also interesting, albeit more complicated. For example, if n is irrational, then there are infinitely many petals. Varying the values of a, we can obtain different petal lengths. $\diamond$

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[1] The name *limaçon* comes from the Latin word limax, which means a snail.
[2] The name *lemniscate* comes from the Latin word lemniscus, meaning a ribbon.

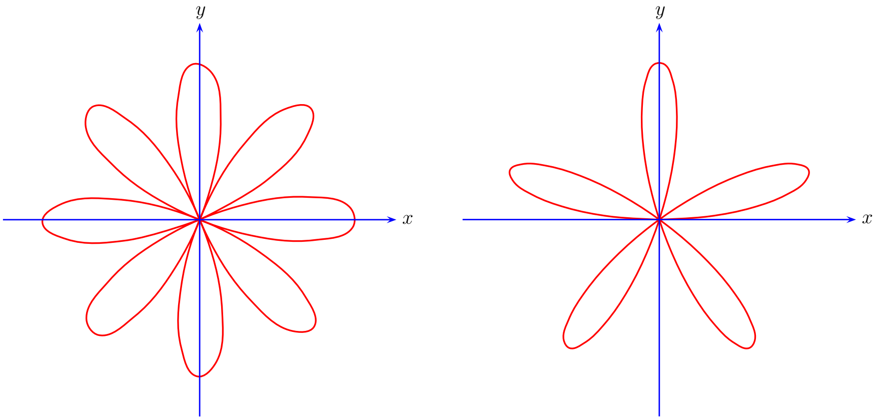

Fig. 7.17. Roses $r = \cos n\theta$ for $n = 4$ (with $2n$ petals) and $n = 5$ (with n petals).

Notion of an Angle

In this subsection, we define the basic notion of an angle in various contexts, and also relate it to polar coordinates discussed above. The formal definition will use the inverse trigonometric functions that are defined in Section 7.2.

To begin with, we consider line segments OP_1 and OP_2 emanating from a common point $O = (x_0, y_0)$. If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are different from O, then we define the **angle** between OP_1 and OP_2 to be the real number

$$\cos^{-1} \frac{(x_1 - x_0)(x_2 - x_0) + (y_1 - y_0)(y_2 - y_0)}{\left(\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}\right)\left(\sqrt{(x_2 - x_0)^2 + (y_2 - y_0)^2}\right)}.$$

This angle is denoted by $\angle(OP_1, OP_2)$ or by $\angle P_1OP_2$. Note that by the Cauchy–Schwarz inequality (Proposition 1.12),

$$\left| \frac{(x_1 - x_0)(x_2 - x_0) + (y_1 - y_0)(y_2 - y_0)}{\left(\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}\right)\left(\sqrt{(x_2 - x_0)^2 + (y_2 - y_0)^2}\right)} \right| \le 1.$$

Thus, $\angle(OP_1, OP_2)$ is a well-defined real number that lies between 0 and π. We say that the angle between the line segments OP_1 and OP_2 is (i) **acute** if $0 \le \angle(OP_1, OP_2) < \pi/2$, (ii) **obtuse** if $\pi/2 < \angle(OP_1, OP_2) \le \pi$, and (iii) a **right angle** if $\angle(OP_1, OP_2) = \pi/2$. Note that the angle between the line segments OP_1 and OP_2 is acute, obtuse, or a right angle according as the number $(x_1 - x_0)(x_2 - x_0) + (y_1 - y_0)(y_2 - y_0)$ is positive, negative, or zero.

Remark 7.25. In classical geometry, the notion of angle is regarded as self-evident and synonymous with the configuration formed by two line segments emanating from a point. One assigns the degree measure to angles in such a way that the degree measure of the angle between the line segments OP_1 and OP_2 is $180°$ (read: 180 degrees) if O, P_1, and P_2 are collinear and O lies between P_1 and P_2. With this approach, it is far from obvious that every "angle" is capable of a precise measurement (by real numbers). Such an assumption is also implicit when one "defines" the trigonometric functions by drawing right-angled triangles and looking at ratios of sides. We have bypassed these difficulties by opting to define the trigonometric functions and the notion of angle by analytic means. In our setup, the degree measure is also easy to define. One simply identifies $180°$ with π, so that $1°$ becomes equivalent to the real number $\pi/180$. Thus, the **degree** measure of the angle between the line segments OP_1 and OP_2, denoted by $\angle P_1OP_2$, is $(180\alpha/\pi)°$ if $\alpha = \angle(OP_1, OP_2)$. To make a distinction, one sometimes says that α is the **radian**[3] measure of $\angle P_1OP_2$. For example, $\pi/2$, $\pi/3$, $\pi/4$, and $\pi/6$ correspond, in the degree measure, to $90°$, $60°$, $45°$, and $30°$, respectively. $\diamond$

To relate the notion of angle to polar coordinates, let us consider the special case in which O is the origin $(0,0)$, P_1 is the point $A := (1,0)$ on the x-axis, and P_2 is an arbitrary point $P = (x,y)$ other than the origin. Let (r,θ) be the polar coordinates of P. We have seen that $r = \sqrt{x^2+y^2}$ represents the distance from P to the origin O. On the other hand, the angle between OA and OP is given by

$$\angle(OA, OP) = \cos^{-1} \frac{x \cdot 1 + y \cdot 0}{\left(\sqrt{x^2+y^2}\right)\left(\sqrt{1^2+0^2}\right)} = \cos^{-1} \frac{x}{r}.$$

It follows that θ can be interpreted, in analogy with "signed area" defined in Remark 6.21, as the "**signed angle**" from OA to OP, namely

$$\theta = \begin{cases} \angle(OA, OP) & \text{if } P \text{ is in the upper half-plane or the } x\text{-axis } (y \geq 0), \\ -\angle(OA, OP) & \text{if } P \text{ is in the lower half-plane } (y < 0). \end{cases}$$

The notion of "signed angle" is illustrated in Figure 7.18. It may be noted that the sign depends on the "orientation", that is, it is positive if we move from A to P in the counterclockwise direction (when P is above the x-axis) and negative if we move from A to P in the clockwise direction (when P is below the x-axis).

Now we shall consider a variant of the notion of angle, which enables us to talk about the angle between two lines rather than two line segments emanating from a common point. Intuitively, it is clear that two intersecting lines give rise to two distinct angles, which are *complementary* in the sense that their sum is π. As a convention, we shall give preference to the acute angle among these two angles. Formally, we proceed as follows.

[3] The word *radian*, derived from radius, has the following dictionary meaning: an angle subtended at the center of a circle by an arc whose length is equal to the radius. We can reconcile our current usage of the word with this meaning when the notion of length of an arc is formally defined in Chapter 8.

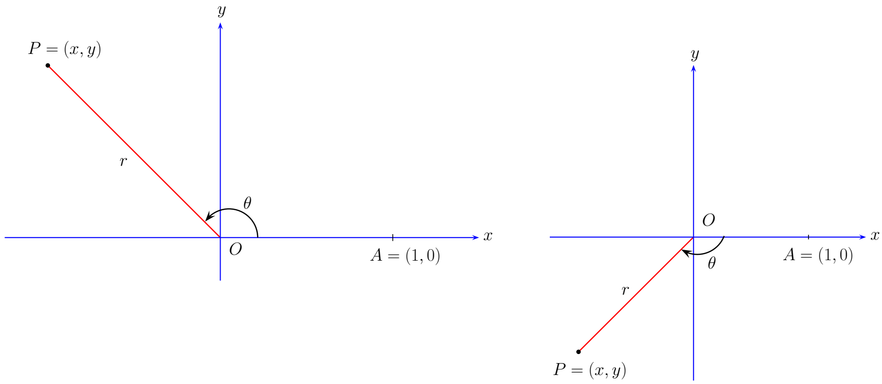

Fig. 7.18. Illustration of the "signed angle" θ from OA to OP.

Let L_1 and L_2 be any lines in the plane $\mathbb{R}^2$. If $L_1 \parallel L_2$, that is, if L_1 and L_2 are parallel (in particular, if $L_1 = L_2$), then we define the **(acute) angle** between L_1 and L_2, denoted by $\measuredangle(L_1, L_2)$, to be 0. If $L_1 \nparallel L_2$, that is, if L_1 and L_2 are not parallel, then they intersect in a unique point, say $O = (x_0, y_0)$. Now, choose any point $P_1 = (x_1, y_1)$ on L_1 such that $P_1 \neq O$ and any point $P_2 = (x_2, y_2)$ on L_2 such that $P_2 \neq O$. Define

$$\measuredangle(L_1, L_2) := \cos^{-1} \frac{|(x_1 - x_0)(x_2 - x_0) + (y_1 - y_0)(y_2 - y_0)|}{\left(\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}\right)\left(\sqrt{(x_2 - x_0)^2 + (y_2 - y_0)^2}\right)}.$$

From the Cauchy–Schwarz inequality (and the conditions when equality holds), it follows that the fraction in the above expression is < 1, and in view of the absolute value in the numerator of this fraction, we see that $0 < \measuredangle(L_1, L_2) \leq \pi/2$. In general, that is, regardless of whether or not $L_1 \parallel L_2$, we see that $\measuredangle(L_1, L_2) \in [0, \pi/2]$; also, since $\cos(\pi - \alpha) = -\cos\alpha$ for all $\alpha \in \mathbb{R}$,

$$\measuredangle(L_1, L_2) = \begin{cases} \angle(OP_1, OP_2) & \text{if } \angle(OP_1, OP_2) \text{ is acute,} \\ \pi - \angle(OP_1, OP_2) & \text{if } \angle(OP_1, OP_2) \text{ is obtuse.} \end{cases}$$

However, it remains to be seen that when $L_1 \nparallel L_2$, then the above definition of $\measuredangle(L_1, L_2)$ does not depend on the choice of the points P_1, P_2, other than O, on L_1, L_2, respectively. To this end, let us first note that either L_i is the vertical line $x = x_0$, or else it has a well-defined slope, say m_i, for $i = 1, 2$. Now, if neither L_1 nor L_2 is vertical, then $x_i \neq x_0$ and $m_i = (y_i - y_0)/(x_i - x_0)$ for $i = 1, 2$. So in this case, dividing the numerator and the denominator of the fraction in the definition of $\measuredangle(L_1, L_2)$ by $|(x_1 - x_0)(x_2 - x_0)|$, we obtain

$$\measuredangle(L_1, L_2) = \cos^{-1} \frac{|1 + m_1 m_2|}{\sqrt{1 + m_1^2}\sqrt{1 + m_2^2}}.$$

In case m_1 is not defined, that is, if $x_1 = x_0$, then $y_1 \neq y_0$ (since $P_1 \neq O$) and $x_2 \neq x_0$ (since $L_1 \nparallel L_2$), and hence $|y_1 - y_0| \neq 0$, and m_2 is defined; thus, dividing the numerator and the denominator of the fraction in the definition of $\measuredangle(L_1, L_2)$ by $|x_2 - x_0|$, we obtain

$$\measuredangle(L_1, L_2) = \cos^{-1} \frac{|m_2|}{\sqrt{1 + m_2^2}}.$$

Similarly, if m_2 is not defined, that is, if $x_2 = x_0$, then $y_2 \neq y_0$ and $x_1 \neq x_0$, and hence $|y_2 - y_0| \neq 0$, and m_1 is defined; thus in this case,

$$\measuredangle(L_1, L_2) = \cos^{-1} \frac{|m_1|}{\sqrt{1 + m_1^2}}.$$

This proves that our definition of $\measuredangle(L_1, L_2)$ is independent of the choice of P_1, P_2, different from O, on L_1, L_2, respectively. In the process, we also obtained alternative expressions for $\measuredangle(L_1, L_2)$. These show in particular that

$$\begin{aligned} \measuredangle(L_1, L_2) = \pi/2 &\iff (x_1 - x_0)(x_2 - x_0) + (y_1 - y_0)(y_2 - y_0) = 0 \\ &\iff \text{(i) } m_1 \text{ and } m_2 \text{ are both defined and } m_1 m_2 = -1, \text{ or} \\ &\quad\;\;\;\, \text{(ii) } m_1 \text{ is not defined and } m_2 = 0, \text{ or vice versa.} \end{aligned}$$

If any of these equivalent conditions hold, then we shall say that L_1 and L_2 are **perpendicular lines** and write $L_1 \perp L_2$. As usual, we may write $L_1 \not\perp L_2$ to indicate that the lines L_1 and L_2 are not perpendicular.

In a special case, we can obtain another expression for $\measuredangle(L_1, L_2)$ as described below.

Proposition 7.26. *Suppose L_1 and L_2 are nonvertical lines in the plane with slopes m_1 and m_2, respectively. Assume that $L_1 \not\perp L_2$ (so that $m_1 m_2 \neq -1$). Then*

$$\measuredangle(L_1, L_2) = \tan^{-1} \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|.$$

Proof. If $L_1 \parallel L_2$, then $\measuredangle(L_1, L_2) = 0$ and $m_1 = m_2$. So the desired equality is clearly true in this case. Now assume that $L_1 \nparallel L_2$. Then L_1 and L_2 intersect in a unique point, say $O = (x_0, y_0)$, and we may choose points $P_i = (x_i, y_i)$ on L_i such that $P_i \neq O$, for $i = 1, 2$. Let $\alpha = \measuredangle(L_1, L_2)$. Then

$$\cos\alpha = \frac{|(x_1 - x_0)(x_2 - x_0) + (y_1 - y_0)(y_2 - y_0)|}{\left(\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}\right)\left(\sqrt{(x_2 - x_0)^2 + (y_2 - y_0)^2}\right)}.$$

Now an easy computation shows that

$$\sin^2\alpha = 1 - \cos^2\alpha = \frac{\big((x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)\big)^2}{\big((x_1 - x_0)^2 + (y_1 - y_0)^2\big)\big((x_2 - x_0)^2 + (y_2 - y_0)^2\big)}.$$

Since $\alpha \in (0, \pi/2]$, it follows that $\sin\alpha > 0$, and thus

$$\sin\alpha = \frac{|(x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)|}{\left(\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}\right)\left(\sqrt{(x_2 - x_0)^2 + (y_2 - y_0)^2}\right)}.$$

Since $L_1 \not\perp L_2$, we see that $\alpha \neq \pi/2$, and so $\cos\alpha \neq 0$. Thus,

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{|(x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)|}{|(x_1 - x_0)(x_2 - x_0) + (y_1 - y_0)(y_2 - y_0)|}.$$

In other words,

$$\alpha = \measuredangle(L_1, L_2) = \tan^{-1}\left|\frac{(x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)}{(x_1 - x_0)(x_2 - x_0) + (y_1 - y_0)(y_2 - y_0)}\right|.$$

Since both L_1 and L_2 are nonvertical, by multiplying and dividing the last fraction by $(x_1 - x_0)(x_2 - x_0)$, we obtain the desired equality. □

The notion of angle between lines can be extended to curves as follows. Suppose C_1 and C_2 are curves in the plane that intersect at a point P. Assume that the tangent, say L_i, to C_i at P is defined for each $i = 1, 2$. Then the **angle** at P between the curves C_1 and C_2, denoted by $\measuredangle(C_1, C_2; P)$, is defined to be $\measuredangle(L_1, L_2)$. In case $\measuredangle(C_1, C_2; P) = \pi/2$, the curves C_1 and C_2 are said to intersect **orthogonally** at the point P.

Examples 7.27. (i) Consider the curves C_1 and C_2 defined by the equations $y = x^2$ and $y = 2 - x^3$. These intersect at the point $P = (1, 1)$. Considering the derivatives at $x = 1$, we see that the slopes of the tangents to C_1 and C_2 at P are given by $m_1 = 2$ and $m_2 = -3$, respectively. Hence using Proposition 7.26, we obtain

$$\measuredangle(C_1, C_2; P) = \tan^{-1}\left|\frac{2 - (-3)}{1 + 2(-3)}\right| = \tan^{-1}|-1| = \tan^{-1} 1 = \frac{\pi}{4}.$$

Thus the angle between the two curves at P is $\pi/4$.

(ii) Consider the curves C_1 and C_2 defined by the equations $y = e^x$ and $y^2 - 2y + 1 - x = 0$, respectively. These intersect at the point $P = (0, 1)$. The tangent L_1 to C_1 at P is given by the line $y - 1 = x$, whereas the tangent L_2 to C_2 at P is given by the vertical line $x = 0$. Note that to determine the latter, it is more convenient to look at the derivatives with respect to y than with respect to x. Thus the slope m_1 of L_1 equals 1, whereas the slope m_2 of L_2 is not defined. Hence the formula in Proposition 7.26 cannot be used. But we can directly use the definition of $\measuredangle(L_1, L_2)$ or the formula $\cos^{-1}\left(|m_1|/\sqrt{1 + m_1^2}\right)$ applicable when m_2 is not defined, to conclude that $\measuredangle(C_1, C_2; P) = \cos^{-1}\left(1/\sqrt{2}\right) = \pi/4$. ◇

7.5 Transcendence

The functions discussed in this chapter, namely, the logarithmic, exponential, and trigonometric functions, are often called **elementary transcendental functions**. As we have seen in Chapter 1, the term **transcendental function** has a definite meaning attached to it. It is therefore natural that we justify this terminology and show that the logarithmic, exponential, and trigonometric functions are indeed transcendental. To do so is the aim of this section.

Let us begin by recalling that given a subset D of $\mathbb{R}$, a function $f : D \to \mathbb{R}$ is said to be a **transcendental function** if it is not an algebraic function, that is, if there is no polynomial

$$P(x, y) = p_n(x)y^n + p_{n-1}(x)y^{n-1} + \cdots + p_1(x)y + p_0(x),$$

where $n \in \mathbb{N}$ and $p_0(x), p_1(x), \ldots, p_n(x)$ are polynomials in x with real coefficients, such that $p_n(x)$ is a nonzero polynomial, and

$$P(c, f(c)) = 0 \quad \text{for all } c \in D.$$

In this case, we refer to $P(x, y)$ as a **polynomial satisfied by** $y = f(x)$ and the positive integer n as the y**-degree** of $P(x, y)$.

Our first goal is to show that the logarithmic function is transcendental. We shall achieve this in two steps. First, we prove a simpler result that the logarithmic function is not a rational function. Next, we will use this to prove that the logarithmic function is not an algebraic function.

Lemma 7.28. *The logarithmic function* $\ln : (0, \infty) \to \mathbb{R}$ *is not a rational function. More precisely, there do not exist polynomials* $p(x), q(x)$ *and an open interval* $I \subseteq (0, \infty)$ *such that*

$$q(x) \neq 0 \quad \textit{for all } x \in I \quad \textit{and} \quad \ln x = \frac{p(x)}{q(x)} \quad \textit{for all } x \in I.$$

Proof. Suppose to the contrary that there are polynomials $p(x), q(x)$ and an open interval $I \subseteq (0, \infty)$ such that $q(x) \neq 0$ for all $x \in I$ and $\ln x = p(x)/q(x)$ for all $x \in I$. Canceling common factors, if any, we may assume that the polynomials $p(x)$ and $q(x)$ are not divisible by any nonconstant polynomial in x. Since $q(x) \neq 0$ for all $x \in I$, taking derivatives on both sides of the equation

$$\ln x = \frac{p(x)}{q(x)},$$

we obtain for all $x \in I$,

$$\frac{1}{x} = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2}, \text{ that is, } q(x)^2 = x\big(p'(x)q(x) - p(x)q'(x)\big).$$

Both sides of the last equation are polynomials, and the equation is satisfied at infinitely many points; hence it is an identity of polynomials. Consequently, the polynomial x divides the polynomial $q(x)$. Now let us write $q(x) = x^k q_1(x)$, where $k \in \mathbb{N}$ and $q_1(x)$ is a polynomial in x that is not divisible by x, that is, $q_1(0) \neq 0$. Then, $q'(x) = kx^{k-1}q_1(x) + x^k q_1'(x)$, and therefore,

$$x^{2k}q_1(x)^2 = x^{k+1}p'(x)q_1(x) - kx^k p(x)q_1(x) - x^{k+1}p(x)q_1'(x).$$

Dividing throughout by x^k and rearranging terms, we obtain the identity

$$kp(x)q_1(x) = x\big(p'(x)q_1(x) - p(x)q_1'(x) - x^{k-1}q_1(x)^2\big).$$

This implies that the polynomial x divides the polynomial $p(x)$, which is a contradiction, since $p(x)$ and $q(x)$ were assumed to have no nonconstant common factor. This completes the proof. $\square$

Proposition 7.29. *The logarithmic function* $\ln : (0, \infty) \to \mathbb{R}$ *is a transcendental function.*

Proof. Assume the contrary, that is, suppose ln is an algebraic function. Let

$$P(x, y) = p_n(x)y^n + p_{n-1}(x)y^{n-1} + \cdots + p_1(x)y + p_0(x)$$

be a polynomial of y-degree n satisfied by $y = \ln x$ such that $n \in \mathbb{N}$ is the least among the y-degrees of all polynomials satisfied by $y = \ln x$. Let us write

$$q_j(x) := \frac{p_j(x)}{p_n(x)} \quad \text{for } j = 0, 1, \ldots, n-1 \quad \text{and} \quad Q(x, y) := y^n + \sum_{j=0}^{n-1} q_j(x)y^j.$$

Further, let $D := \{c \in (0, \infty) : p_n(c) \neq 0\}$. It is clear that D contains all except finitely many points of $(0, \infty)$, each $q_j(x)$ is defined on D, and $Q(c, \ln c) = 0$ for all $c \in D$. Also, note that every point of D is its interior point. Moreover, each $q_j(x)$ is differentiable on D and its derivative $q_j'(x)$ is a rational function defined on D. Thus, using the Chain Rule (Proposition 4.10), we see that the derivative of $Q(x, \ln x)$ is equal to

$$\begin{aligned} &n(\ln x)^{n-1}\left(\frac{1}{x}\right) + \sum_{j=0}^{n-1}\left(q_j'(x)(\ln x)^j + jq_j(x)(\ln x)^{j-1}\left(\frac{1}{x}\right)\right) \\ &= \left(q_{n-1}'(x) + \frac{n}{x}\right)(\ln x)^{n-1} + \sum_{j=0}^{n-2}\left(q_j'(x) + \frac{j+1}{x}q_{j+1}(x)\right)(\ln x)^j. \end{aligned}$$

Since $q_j'(x) = \big(p_j'(x)p_n(x) - p_j(x)p_n'(x)\big)/p_n(x)^2$, taking common denominators, we see that the derivative of $Q(x, \ln x)$ is equal to $\widetilde{P}(x, \ln x)/xp_n(x)^2$, where $\widetilde{P}(x, y)$ is a polynomial in y whose coefficients are polynomials in x. Since $Q(c, \ln c) = 0$ for all $c \in D$, we obtain $\widetilde{P}(c, \ln c) = 0$ for all $c \in D$. Moreover, since $\widetilde{P}(x, \ln x)$ is defined at every $x \in (0, \infty)$ and gives a continuous function from $(0, \infty)$ to $\mathbb{R}$, which vanishes on D, it follows that $\widetilde{P}(c, \ln c) = 0$

for all $c \in (0,\infty)$. Also, the leading coefficient of $\widetilde{P}(x,y)$, that is, the coefficient of y^{n-1} in $\widetilde{P}(x,y)$, is a nonzero polynomial (in x). For if this leading coefficient were zero, then $q'_{n-1}(t) = -n/t$ for all $t \in D$. Now, since D misses only finitely many points of $(0,\infty)$, in view of Proposition 6.13 and the FTC, we may integrate both sides from $t = 1$ to $t = x$ and obtain $q_{n-1}(x) - q_{n-1}(1) = -n \ln x$ for all $x \in D$, and consequently, $\ln x$ is a rational function (on D), which is impossible by Lemma 7.28. Thus $\widetilde{P}(x,y)$ would be a polynomial satisfied by $y = \ln x$ and its y-degree is $n-1$. This contradicts the minimality of n. Hence $\ln : (0,\infty) \to \mathbb{R}$ is a transcendental function. □

Corollary 7.30. *The exponential function* $\exp : \mathbb{R} \to \mathbb{R}$ *is a transcendental function.*

Proof. Assume the contrary, that is, suppose exp is an algebraic function. Let

$$P(x,y) = p_n(x)y^n + p_{n-1}(x)y^{n-1} + \cdots + p_1(x)y + p_0(x)$$

be a polynomial satisfied by $y = \exp x$, where $n \in \mathbb{N}$ and $p_n(x)$ is a nonzero polynomial. Let us write $P(x,y)$ as a polynomial in x whose coefficients are polynomials in y:

$$P(x,y) = \widetilde{p}_m(y)x^m + \widetilde{p}_{m-1}(y)x^{m-1} + \cdots + \widetilde{p}_1(y)x + \widetilde{p}_0(y),$$

where m is a nonnegative integer so chosen that $\widetilde{p}_m(y)$ is a nonzero polynomial. Note that m is, in fact, positive, because otherwise, $P(x,y) = \widetilde{p}_0(y)$ would be a nonzero polynomial in one variable with infinitely many roots, namely, $y = \exp c$ for every $c \in \mathbb{R}$. Now let $\widetilde{P}(x,y) := P(y,x)$. Then $\widetilde{P}(x,y)$ is a polynomial in two variables with positive y-degree. Moreover, since $P(c, \exp c) = 0$ for all $c \in \mathbb{R}$, and also since $\exp : \mathbb{R} \to (0,\infty)$ is bijective with its inverse given by $\ln : (0,\infty) \to \mathbb{R}$, it follows that $P(\ln d, d) = 0$ for all $d \in (0,\infty)$. In other words, $\widetilde{P}(d, \ln d) = 0$ for all $d \in (0,\infty)$. Thus, $\ln : (0,\infty) \to \mathbb{R}$ would be an algebraic function, which contradicts Proposition 7.29. □

Now let us turn to trigonometric functions. As we shall see below, it is easier to prove that these are transcendental. The key property used in the proof is that trigonometric functions have infinitely many zeros.

Proposition 7.31. *The trigonometric functions* $\sin : \mathbb{R} \to \mathbb{R}$*,* $\cos : \mathbb{R} \to \mathbb{R}$*, and* $\tan : \mathbb{R} \setminus \{(2m+1)\pi/2 : m \in \mathbb{Z}\} \to \mathbb{R}$ *are transcendental functions.*

Proof. Assume the contrary, that is, suppose any one of them, say, $\sin : \mathbb{R} \to \mathbb{R}$, is an algebraic function. Then there is a polynomial

$$P(x,y) = p_n(x)y^n + p_{n-1}(x)y^{n-1} + \cdots + p_1(x)y + p_0(x)$$

that is satisfied by $y = \sin x$, and we may assume that $n \in \mathbb{N}$ is the least possible y-degree of such a polynomial. We claim that $p_0(x) = P(x, 0)$ is necessarily a nonzero polynomial in x. To see this, suppose $p_0(x)$ is the zero polynomial. Then $n > 1$. Indeed, if we had $n = 1$, then $p_1(x) \neq 0$, and since $p_0(x) = 0$, we obtain $P(x, y) = p_1(x)y$, and hence $p_1(c)\sin c = 0$ for all $c \in \mathbb{R}$. Consequently, $p_1(c) = 0$ for all $c \in \mathbb{R}$ for which $\sin c \neq 0$, that is, for all $c \in \mathbb{R} \setminus \{m\pi : m \in \mathbb{Z}\}$. Hence $p_1(x)$ is the zero polynomial, which is a contradiction. Thus $n > 1$, and so if we let

$$P_1(x, y) = p_n(x)y^{n-1} + p_{n-1}(x)y^{n-2} + \cdots + p_2(x)y + p_1(x),$$

then the polynomial $P_1(x, y)$ has positive y-degree. Also, $P(x, y) = yP_1(x, y)$, and hence $(\sin c)P_1(c, \sin c) = 0$ for all $c \in \mathbb{R}$. Consequently, $P_1(c, \sin c) = 0$ for all $c \in \mathbb{R} \setminus \{m\pi : m \in \mathbb{Z}\}$. But the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $P_1(x, \sin x)$ is continuous. So it follows that $P_1(c, \sin c) = 0$ for all $c \in \mathbb{R}$. Thus $y = \sin x$ satisfies the polynomial $P_1(x, y)$ of y-degree $n - 1 \in \mathbb{N}$, which contradicts the minimality of n. Thus $p_0(x)$ is a nonzero polynomial in x, and consequently it has only finitely many roots. But $p_0(m\pi) = P(m\pi, 0) = P(m\pi, \sin m\pi) = 0$ for all $m \in \mathbb{Z}$, and so we obtain a contradiction. This proves that $\sin : \mathbb{R} \to \mathbb{R}$ is transcendental.

The proof in the case of the cosine and tangent functions is similar, since each of them has infinitely many zeros (namely, $(2m + 1)\pi/2$ for $m \in \mathbb{Z}$ and $m\pi$ for $m \in \mathbb{Z}$, respectively). $\square$

Remark 7.32. Having justified the term *transcendental* in "elementary transcendental function", one may wonder whether the term *elementary* should also be justified in the same way. To this effect, we remark that no intrinsic definition of the term *elementary function* appears to be known. In fact, an **elementary function** is usually "defined" as a function built up from algebraic, exponential, logarithmic, and trigonometric functions and their inverses by a finite combination of the operations of addition, multiplication, division, and root extraction (which are called the elementary operations), and the operation of repeated compositions. $\diamond$

Notes and Comments

The idea of integration, which can be traced back to the work of Archimedes around 225 BC, is one of the oldest and the most fundamental in calculus. The quest for evaluating integrals of known functions is a fruitful way of inventing new functions. When a known function is Riemann integrable (for example, if it is continuous), we can abstractly define its antiderivative. If this cannot be determined in terms of known functions, we obtain, nevertheless, a nice new function waiting to be understood better! As explained in Sections 7.1 and 7.2, $1/x$ *and* $1/(1+x^2)$ *are the simplest rational functions whose integrals pose such a problem. This leads to the introduction of the logarithmic function* ln *and the arctangent function* arctan. *With these at our disposal, we can integrate* *every rational function! This follows from the method of partial fractions, whereby every rational function can be decomposed as a sum of simpler rational functions of the form* $A/(ax+b)^m$ *or* $(Bx+C)/(ax^2+bx+c)^m$*, and these can be integrated in terms of rational functions and the functions* ln *and* arctan.

The inverses of the logarithmic and arctangent functions lead to even nicer functions, namely the exponential function and the tangent function. Other classical trigonometric functions are easily defined using the tangent function.

The approach outlined above gives not only a precise definition of the logarithmic, exponential, and trigonometric functions, but also a genuine motivation for introducing the same. In most texts on calculus, the trigonometric functions are "defined" by drawing triangles and mentioning that angles are "measured" in radians. The main problem with this approach is succinctly described by Hardy [39, §163], who writes, "The whole difficulty lies in the question, what is the x *which occurs in* $\cos x$ *and* $\sin x$*." Hardy also describes different methods to develop an analytic theory (cf. [39, §224]), and the approach we have chosen is one of them.*

The logarithmic and exponential functions also help us to give a "natural" definition of the important number e*. Likewise, the trigonometric functions help us give a precise definition of the important number* π*. The numbers* e *and* π*, and to a lesser extent, the Euler constant* γ *(defined in Exercise 7.2 below), have fascinated mathematicians and amateurs alike for centuries. For more on these, see the books of Maor [64], Arndt and Haenel [5], and Havil [42], which are devoted to* e*,* π*, and* γ*, respectively.*

Failure to be able to integrate a function has often led to interesting developments in mathematics. For example, a rich and fascinating theory of the so-called elliptic integrals and elliptic functions arises in this way. We will comment more on this in the next chapter, where the notion of arc length will be defined. As another example, we cite the theory of differential equations. Indeed, seeking an antiderivative of a function f *may be viewed as the problem of finding a solution* $y = F(x)$ *of the equation* $y' = f$*. A differential equation is, more generally, an equation such as* $y' = f$ *with* y' *replaced by a combination of* $y, y', y'', \ldots, y^{(n)}$*. Attempts to "solve" differential equations have led to newer classes of functions. To wit, functions known as the Legendre function, Bessel function, and Gauss hypergeometric function arise in this way. These functions enrich the realm of functions beyond algebraic and elementary transcendental functions, and they are sometimes called special functions or higher transcendental functions. For an introduction to these topics, see the books of Simmons [74] and Forsyth [31].*

In the last section of this chapter, we prove that the functions ln, exp, sin, cos, *and* tan *are transcendental. This section is inspired by the article of Hamming [38].*

Exercises

Part A

7.1. For every $x \in \mathbb{R}$ with $x > 1$, show that

$$\sum_{k=1}^{[x]} \frac{1}{k} - \frac{[x]}{x} \leq \ln x \leq \sum_{k=2}^{[x]-1} \frac{1}{k} + \frac{[x]}{x},$$

where $[x]$ denotes the integer part of $[x]$. In particular, show that

$$\frac{13}{22} \leq \ln 2.2 \leq \frac{11}{10} \quad \text{and} \quad 1 \leq \ln 3.6 \leq \frac{17}{10}.$$

7.2. Consider the sequence (c_n) defined by

$$c_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \quad \text{for } n \in \mathbb{N}.$$

Show that (c_n) is convergent. (Hint: (c_n) is monotonically decreasing and $c_n \geq 0$ for all $n \in \mathbb{N}$.)
[Note: The limit of the sequence (c_n) is known as the **Euler constant**. It is usually denoted by γ. Approximately, $\gamma = 0.5772156649\ldots$, but it is not known whether γ is rational or irrational.]

7.3. Let $a > 0$ and $r \in \mathbb{Q}$. Show that $\ln ax^r = \ln a + r \ln x$ for all $x \in (0, \infty)$, assuming only that $(\ln)'x = 1/x$ for all $x \in (0, \infty)$.

7.4. Show that for all $x > 0$,

$$x - \frac{x^2}{2} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}.$$

7.5. Let $\alpha \in \mathbb{R}$ and let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that $f'(x) = \alpha/x$ for all $x \in (0, \infty)$ and $f(1) = 0$. Show that $f(x) = \alpha \ln x$ for all $x \in (0, \infty)$. (Compare Exercise 4.5.)

7.6. Let $f : (0, \infty) \to \mathbb{R}$ be continuous and satisfy

$$\int_1^{xy} f(t)dt = y \int_1^x f(t)dt + x \int_1^y f(t)dt \quad \text{for all } x, y \in (0, \infty).$$

Show that $f(x) = f(1)\,(1+\ln x)$ for all $x \in (0, \infty)$. (Hint: Consider $F(x) := \left(\int_1^x f(t)dt\right)/x$ for $x \in (0, \infty)$ and use Exercise 7.5.)

7.7. Show that $2.5 < e < 3$. (Hint: Divide $[1, 2.5]$ and $[1, 3]$ into subintervals of length $\frac{1}{4}$.)

7.8. Show that

(i) $\displaystyle\int_a^b \ln x \, dx = b(\ln b - 1) - a(\ln a - 1)$ for all $a, b \in (0, \infty)$,

(ii) $\displaystyle\int_a^b \exp x \, dx = \exp b - \exp a$ for all $a, b \in \mathbb{R}$.

7.9. Let $\alpha \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f' = \alpha f$ and $f(0) = 1$. Show that $f(x) = e^{\alpha x}$ for all $x \in \mathbb{R}$. (Compare Exercise 4.6.)

7.10. The **hyperbolic sine** and **hyperbolic cosine** functions from $\mathbb{R}$ to $\mathbb{R}$ are defined by

$$\sinh x := \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x := \frac{e^x + e^{-x}}{2} \quad \text{for } x \in \mathbb{R}.$$

Show that for every $t \in \mathbb{R}$, the point $(\cosh t, \sinh t)$ is on the hyperbola $x^2 - y^2 = 1$. Also, show that

(i) $\sinh 0 = 0$, $\cosh 0 = 1$ and $\cosh^2 - \sinh^2 = 1$ for all $x \in \mathbb{R}$.

(ii) $(\sinh)' x = \cosh x$ and $(\cosh)' x = \sinh x$ for all $x \in \mathbb{R}$.

(iii) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$ and $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$ for all $x, y \in \mathbb{R}$.

Sketch the graphs of the functions sinh and cosh.

7.11. Let $a, b \in (0, \infty)$.

(i) Consider the functions $f, g : (0, \infty) \to \mathbb{R}$ defined by $f(x) := \log_a x$ and $g(x) := \log_b x$. Show that f and g have the same rate as $x \to \infty$.

(ii) Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := a^x$ and $g(x) := b^x$. Show that the growth rate of f is less than that of g as $x \to \infty$ if and only if $a < b$.

7.12. For $b \in \mathbb{R}$, consider the function $g_b : (0, \infty) \to (0, \infty)$ defined by $g_b(x) = x^b$. Show that $g_{b_1} \circ g_{b_2} = g_{b_1 b_2} = g_{b_2} \circ g_{b_1}$ for all $b_1, b_2 \in \mathbb{R}$.

7.13. Let $f : (0, \infty) \to \mathbb{R}$ satisfy $f(xy) = f(x)f(y)$ for all $x, y \in (0, \infty)$. If f is continuous at 1, show that either $f(x) = 0$ for all $x \in (0, \infty)$, or there exists $r \in \mathbb{R}$ such that $f(x) = x^r$ for all $x \in (0, \infty)$. (Hint: If $f(1) \neq 0$, then $f(x) > 0$ for all $x \in (0, \infty)$, and so we can consider $g = \ln \circ f \circ \exp : \mathbb{R} \to \mathbb{R}$ and use Exercise 3.4) (Compare Exercises 1.17 (ii) and 3.6.)

7.14. Let $r \in \mathbb{R}$ be positive and consider the function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = x^r$. Show that the growth rate of $\ln x$ is less than that of f, while the growth rate of $\exp x$ is more than that of f as $x \to \infty$.

7.15. If $a > 0$ and $f, g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) := e^{-1/x^2}$ and $g(x) := ax^2$, then show that $f(x) \to 0$ and $g(x) \to 0$, but $f(x)^{g(x)} \to e^{-a}$ as $x \to 0$.

7.16. Show that

$$\frac{x}{1 + x^2} < \arctan x < x \quad \text{for all } x \in (0, 1]$$

and

$$1 - \frac{1}{2x} < \arctan x < 2 - \frac{1}{x} \quad \text{for all } x \in (1, \infty).$$

7.17. Show that $2.88 < \pi < 3.39$ by dividing the interval $[0, 1]$ into subintervals of length $1/4$. (Hint: $\arctan 1 = \pi/4$.)

7.18. Let D and E be the unions of open intervals defined as follows.

$$D = \bigcup_{k \in \mathbb{Z}} \left(\frac{(4k-1)\pi}{2}, \frac{(4k+1)\pi}{2} \right) \quad \text{and} \quad E = \bigcup_{k \in \mathbb{Z}} \left(\frac{(4k-3)\pi}{2}, \frac{(4k-1)\pi}{2} \right).$$

Show that

$$\sin x = \begin{cases} \dfrac{\tan x}{\sqrt{1+\tan^2 x}} & \text{if } x \in D, \\ -\dfrac{\tan x}{\sqrt{1+\tan^2 x}} & \text{if } x \in E, \end{cases} \qquad \cos x = \begin{cases} \dfrac{1}{\sqrt{1+\tan^2 x}} & \text{if } x \in D, \\ -\dfrac{1}{\sqrt{1+\tan^2 x}} & \text{if } x \in E. \end{cases}$$

7.19. Show from first principles that the function cos is differentiable at $\pi/2$ and its derivative at $\pi/2$ is -1.

7.20. Show that $0 < x\cos x < \sin x$ for all $x \in (0, \pi/2)$ and $\sin x < x \cos x < 0$ for all $x \in (-\pi/2, 0)$. Deduce that $x < \tan x$ for all $x \in (0, \pi/2)$ and $\tan x < x$ for all $x \in (-\pi/2, 0)$.

7.21. Show that for $x \in (0, \pi/2)$,

$$\frac{2x}{\pi} < \sin x < \min\{1, x\} \quad \text{and} \quad 1 - \frac{2x}{\pi} < \cos x < \min\left\{1, \frac{\pi}{2} - x\right\},$$

whereas for $x \in (-\pi/2, 0)$,

$$\max\{-1, x\} < \sin x < \frac{2x}{\pi} \quad \text{and} \quad 1 + \frac{2x}{\pi} < \cos x < \min\left\{1, \frac{\pi}{2} + x\right\}.$$

7.22. Prove that $|\sin x - \sin y| \le |x - y|$ and $|\cos x - \cos y| \le |x - y|$ for all $x, y \in \mathbb{R}$.

7.23. Show that

$$\int_a^b \sin x \, dx = \cos a - \cos b \ \text{ and } \ \int_a^b \cos x \, dx = \sin b - \sin a \ \text{ for all } a, b \in \mathbb{R}.$$

Deduce that $\int_{-\pi}^{\pi} \sin x \, dx = 0 = \int_{-\pi}^{\pi} \cos x \, dx$.

7.24. Let $\beta \in \mathbb{R}$. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable functions such that

$$f' = \beta g, \quad g' = -\beta f, \quad f(0) = 0, \quad \text{and} \quad g(0) = 1.$$

Show that $f(x) = \sin \beta x$ and $g(x) = \cos \beta x$ for all $x \in \mathbb{R}$. (Hint: Consider $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) := (f(x) - \sin \beta x)^2 + (g(x) - \cos \beta x)^2$. Find h'.) (Compare Exercise 4.8.)

7.25. Let $\alpha, \beta \in \mathbb{R}$. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable functions such that

$$f' = \alpha f + \beta g, \quad g' = \alpha g - \beta f, \quad f(0) = 0, \quad \text{and} \quad g(0) = 1.$$

Show that $f(x) = e^{\alpha x} \sin \beta x$ and $g(x) = e^{\alpha x} \cos \beta x$ for all $x \in \mathbb{R}$. (Hint: Consider $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) := (f(x) - e^{\alpha x} \sin \beta x)^2 + (g(x) - e^{\alpha x} \cos \beta x)^2$. Find h'.) (Compare Exercise 4.7.)

7.26. Let $\alpha, \beta \in \mathbb{R}$. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable functions such that

$$f' = \alpha f + \beta g, \quad g' = \alpha g + \beta f, \quad f(0) = 0, \quad \text{and} \quad g(0) = 1.$$

Show that $f(x) = e^{\alpha x} \sinh \beta x$ and $g(x) = e^{\alpha x} \cosh \beta x$ for all $x \in \mathbb{R}$.

7.27. Show that $\lim_{x\to 0}(\sin x)/|x|$ does not exist.

7.28. Prove the following for all $x \in \mathbb{R}$:
$\sin(\pi - x) = \sin x, \quad \sin((\pi/2) - x) = \cos x, \quad \sin((\pi/2) + x) = \cos x,$
$\cos(\pi - x) = -\cos x, \quad \cos((\pi/2) - x) = \sin x, \quad \cos((\pi/2) + x) = -\sin x.$

7.29. Prove the following for all $x_1, x_2 \in \mathbb{R}$:
(i) $\sin x_1 + \sin x_2 = 2\sin((x_1 + x_2)/2)\cos((x_1 - x_2)/2)$,
(ii) $\sin x_1 - \sin x_2 = 2\cos((x_1 + x_2)/2)\sin((x_1 - x_2)/2)$,
(iii) $\cos x_1 + \cos x_2 = 2\cos((x_1 + x_2)/2)\cos((x_1 - x_2)/2)$,
(iv) $\cos x_1 - \cos x_2 = 2\sin((x_1 + x_2)/2)\sin((x_2 - x_1)/2)$.

7.30. Prove the following for all $x_1, x_2 \in \mathbb{R}$:

$$\begin{aligned} 2\sin x_1 \cos x_2 &= \sin(x_1 + x_2) + \sin(x_1 - x_2), \\ 2\cos x_1 \cos x_2 &= \cos(x_1 + x_2) + \cos(x_1 - x_2), \\ 2\sin x_1 \sin x_2 &= \cos(x_1 - x_2) - \cos(x_1 + x_2). \end{aligned}$$

Deduce the following **orthogonality relations**: For every $k, j \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} (\sin kx)(\cos jx)dx = 0,$$

and if $k \neq j$, then

$$\int_{-\pi}^{\pi} (\cos kx)(\cos jx)dx = 0 = \int_{-\pi}^{\pi} (\sin kx)(\sin jx)dx.$$

7.31. Prove the following for all $x \in \mathbb{R}$:
(i) $\sin 2x = 2\sin x \cos x$,
(ii) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$,
(iii) $\sin 3x = 3\sin x - 4\sin^3 x$,
(iv) $\cos 3x = 4\cos^3 x - 3\cos x$.
Deduce that

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}, \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}, \quad \cos\frac{\pi}{3} = \frac{1}{2} = \sin\frac{\pi}{6}.$$

7.32. Prove the following for all $x_1, x_2 \in \mathbb{R}$:
(i) $\sin x_1 = \sin x_2 \iff x_2 = m\pi + (-1)^m x_1$, where $m \in \mathbb{Z}$.
(ii) $\cos x_1 = \cos x_2 \iff x_2 = 2m\pi \pm x_1$, where $m \in \mathbb{Z}$.
(iii) $\sin x_1 = \sin x_2$ and $\cos x_1 = \cos x_2 \iff x_2 = 2m\pi + x_1$, where $m \in \mathbb{Z}$.
(Hint: Exercise 7.29 and solutions of the equations $\sin x = 0$, $\cos x = 0$.)

7.33. If $x \in \mathbb{R}$ with $x \neq (2k+1)\pi/2$ for every $k \in \mathbb{Z}$, then show that

$$1 + \tan^2 x = \sec^2 x, \quad (\tan)'x = \sec^2 x, \quad \text{and} \quad (\sec)'x = \sec x \tan x.$$

7.34. If $x \in \mathbb{R}$ with $x \neq k\pi$ for every $k \in \mathbb{Z}$, then show that

$$1 + \cot^2 x = \csc^2 x, \quad (\cot)'x = -\csc^2 x, \quad \text{and} \quad (\csc)'x = -\csc x \cot x.$$

7.35. If $x_1, x_2 \in \mathbb{R}$ are such that none of x_1, x_2, and $x_1 + x_2$ equals $(2k+1)\pi/2$ for any $k \in \mathbb{Z}$, then show that

$$\tan(x_1 + x_2) = \frac{\tan x_1 + \tan x_2}{1 - (\tan x_1)(\tan x_2)}.$$

7.36. Prove the following for all $y_1, y_2 \in \mathbb{R}$:

(i) $\tan^{-1} y_1 + \tan^{-1} y_2 = \tan^{-1}\left(\dfrac{y_1 + y_2}{1 - y_1 y_2}\right)$ if $y_1 y_2 < 1$,

(ii) $\tan^{-1}|y_1| + \tan^{-1}|y_2| = \dfrac{\pi}{2}$ if $y_1 y_2 = 1$,

(iii) $\tan^{-1}|y_1| + \tan^{-1}|y_2| = \tan^{-1}\left(\dfrac{|y_1| + |y_2|}{1 - y_1 y_2}\right)$ if $y_1 y_2 > 1$.

7.37. Prove the following:

(i) $\sin\left(\sin^{-1} y\right) = y$ for all $y \in [-1, 1]$ and

$$\sin^{-1}(\sin x) = \begin{cases} x & \text{if } x \in [-\pi/2, \pi/2], \\ \pi - x & \text{if } x \in (\pi/2, 3\pi/2]. \end{cases}$$

(ii) $\cos\left(\cos^{-1} y\right) = y$ for all $y \in [-1, 1]$ and $\cos^{-1}(\cos x) = |x|$ for all $x \in [-\pi, \pi]$.

7.38. If $y \in (-1, 1)$, then show that

$$\sin^{-1} y = \int_0^y \frac{1}{\sqrt{1-t^2}}\, dt \quad \text{and} \quad \cos^{-1} y = \frac{\pi}{2} - \int_0^y \frac{1}{\sqrt{1-t^2}}\, dt.$$

Deduce that

$$\lim_{y \to 1^-} \int_0^y \frac{1}{\sqrt{1-t^2}}\, dt = \frac{\pi}{2}.$$

7.39. If $y \in (1, \infty)$, then show that

$$\sec^{-1} y = \lim_{a \to 1^+} \int_a^y \frac{1}{t\sqrt{t^2-1}}\, dt \quad \text{and} \quad \csc^{-1} y = \frac{\pi}{2} - \lim_{a \to 1^+} \int_a^y \frac{1}{t\sqrt{t^2-1}}\, dt.$$

7.40. Prove the following:

(i) $\cot^{-1} y = \dfrac{\pi}{2} - \tan^{-1} y$ for all $y \in \mathbb{R}$,

(ii) $\csc^{-1} y = \sin^{-1}(1/y)$ for all $y \in \mathbb{R}$ with $|y| \geq 1$,

(iii) $\sec^{-1} y = \cos^{-1}(1/y)$ for all $y \in \mathbb{R}$ with $|y| \geq 1$.

(Hint: $\tan^{-1}|y| + \tan^{-1}|1/y| = \pi/2$ for all $y \in \mathbb{R}$ with $y \neq 0$.)

7.41. For all $y \in [-1, 1]$, show that

$$\sin^{-1} y + \sin^{-1}(-y) = 0, \; \cos^{-1} y + \cos^{-1}(-y) = \pi, \; \sin^{-1} y + \cos^{-1}(y) = \frac{\pi}{2},$$

and for all $y \in \mathbb{R}$ with $|y| \geq 1$, show that

$$\csc^{-1} y + \sec^{-1} y = \frac{\pi}{2}.$$

7.42. Prove the following:

(i) $(\cot^{-1})'y = -\dfrac{1}{1+y^2}$ for all $y \in \mathbb{R}$,

(ii) $(\csc^{-1})'y = -\dfrac{1}{|y|\sqrt{y^2-1}}$ for all $y \in \mathbb{R}$ with $|y| > 1$,

(iii) $(\sec^{-1})'y = \dfrac{1}{|y|\sqrt{y^2-1}}$ for all $y \in \mathbb{R}$ with $|y| > 1$.

7.43. Let $r_0 \in \mathbb{R}$, and consider the function $f_0 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_0(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ r_0 & \text{if } x = 0. \end{cases}$$

(i) Show that f_0 is not continuous at 0. Conclude that the function given by $x \mapsto \sin(1/x)$ for $x \in \mathbb{R} \setminus \{0\}$ cannot be extended to $\mathbb{R}$ as a continuous function.

(ii) If I is an interval and $I \subset \mathbb{R} \setminus \{0\}$, then show that f_0 has the IVP on I. If I an interval such that $0 \in I$, then show that f_0 has the IVP on I if and only if $|r_0| \leq 1$.

7.44. Consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) := \begin{cases} |x| + |x \sin(1/x)| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that h has a strict absolute minimum at 0, but for every $\delta > 0$, the function h is neither decreasing on $(-\delta, 0)$ nor increasing on $(0, \delta)$.

7.45. Consider the function $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $g(x) := \cos(1/x)$. Prove the following:

(i) g is an even function.

(ii) $\lim_{x\to 0} g(x)$ does not exist, but $\lim_{x\to 0^+} \big(g(x) - g(-x)\big)$ exists. Also, g cannot be extended to $\mathbb{R}$ as a continuous function.

(iii) For every $\delta > 0$, the function g is not uniformly continuous on $(0, \delta)$ as well as on $(-\delta, 0)$, but it is uniformly continuous on $(\infty, -\delta] \cup [\delta, \infty)$.

(iv) For every $\delta > 0$, the function g is not monotonic, not convex, and not concave on $(0, \delta)$ as well as on $(-\delta, 0)$.

7.46. Let $r_0 \in \mathbb{R}$ and consider the function $g_0 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

$$g_0(x) := \begin{cases} \cos(1/x) & \text{if } x \neq 0, \\ r_0 & \text{if } x = 0. \end{cases}$$

Show that g_0 is not continuous at 0. Define $G_0 : \mathbb{R} \to \mathbb{R}$ by $G_0(x) := \int_0^x \cos(1/t)dt$. Show that G_0 is differentiable at 0 and $G_0'(0) = 0$, that is,

$$\lim_{x\to 0} \frac{1}{x} \int_0^x \cos\frac{1}{t}\,dt = 0.$$

7.47. Consider the functions g_1, $g_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$g_1(x) := \begin{cases} x\cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \quad \text{and} \quad g_2(x) := \begin{cases} x^2\cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Establish properties of g_1 and g_2 similar to those of the functions f_1 and f_2 given in Example 7.20 and Example 7.21, respectively.

7.48. Consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = (\sin(1/x))/x$. Show that the amplitude of the oscillation of the function f increases without bound as x tends to 0.

7.49. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x^2\sin(1/x^2) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is differentiable on $\mathbb{R}$, but for every $\delta > 0$, the function f' is not bounded on $[-\delta, \delta]$. Thus f' has an antiderivative on the interval $[-1, 1]$, but f' is not Riemann integrable on $[-1, 1]$.

7.50. Let $n \in \mathbb{N}$ and consider the function $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) := \begin{cases} x^n\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove the following: (i) If n is odd and $k := (n-1)/2$, then $f_n^{(k)}$ exists and is continuous on $\mathbb{R}$, but $f_n^{(k+1)}$ does not exist at 0. (ii) If n is even and $k := n/2$, then $f_n^{(k)}$ exists on $\mathbb{R}$, but it is not continuous at 0. (Compare Exercise 4.13.)

7.51. Find the polar coordinates of the points in $\mathbb{R}^2$ whose Cartesian coordinates are as follows:
(i) $(1, 1)$, (ii) $(0, 3)$, (iii) $(2, 2\sqrt{3})$, (iv) $(2\sqrt{3}, 2)$.

7.52. If $x, y \in \mathbb{R}$ are not both zero and (r, θ) are the polar coordinates of (x, y), then determine the polar coordinates of (i) (y, x), and (ii) (tx, ty), where t is any positive real number.

7.53. Let r be a positive real number and let $\theta \in (-\pi, \pi]$ and $\alpha \in \mathbb{R}$ be such that $\theta + \alpha \in (-\pi, \pi]$. If P and P_α denote the points with polar coordinates (r, θ) and $(r, \theta + \alpha)$, respectively, then find the Cartesian coordinates of P_α in terms of the Cartesian coordinates of P.
[Note: The transformation $P \mapsto P_\alpha$ corresponds to a rotation of the plane by the angle α.]

7.54. Find the angle(s) between the curves $x^2 + y^2 = 16$ and $y^2 = 6x$ at their point(s) of intersection.

7.55. Determine whether the following functions are algebraic or transcendental:
(i) $f(x) = \pi x^{11} + \pi^2 x^5 + 9$ for $x \in \mathbb{R}$,
(ii) $f(x) = \dfrac{ex^2 + \pi}{\pi x^2 + e}$ for $x \in \mathbb{R}$,

(iii) $f(x) = \ln_{10} x$ for $x > 0$,
(iv) $f(x) = x^{\pi}$ for $x > 0$.

7.56. Is it possible that

$$\ln x = \left(\sqrt[3]{ex^2 + (\pi - 2e)x + e - \pi} + \sqrt{\pi x^2 + (\sqrt{2} - 2\pi)x + \pi - \sqrt{2}} \right)^{1/17}$$

for all $x > 0$? Justify your answer.

Part B

7.57. Let $p,\ q \in (1, \infty)$ be such that $(1/p) + (1/q) = 1$.
(i) If $f : [0, \infty) \to \mathbb{R}$ is defined by $f(x) := (1/q) + (1/p)\, x - x^{1/p}$, then show that $f(x) \geq f(1)$ for all $x \in [0, \infty)$.
(ii) Show that $ab \leq (a^p/p) + (b^q/q)$ for all $a, b \in [0, \infty)$. (Hint: If $b \neq 0$, let $x := a^p/b^q$ in (i).)
(iii) **(Generalized AM-GM Inequality)** Use (ii) to show that

$$a^{1/p} b^{1/q} \leq \frac{a + b}{(p^{1/p})(q^{1/q})} \quad \text{for all } a, b \in [0, \infty).$$

(iv) **(Hölder Inequality for Sums)** Given any $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{R}$, prove that

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q \right)^{1/q}.$$

Deduce the Cauchy–Schwarz inequality as a special case.
(v) **(Hölder Inequality for Integrals)** Given any continuous functions $f, g : [a, b] \to \mathbb{R}$, prove that

$$\int_a^b |f(x) g(x)| dx \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} \left(\int_a^b |g(x)|^q dx \right)^{1/q}.$$

(vi) **(Minkowski Inequality for Sums)** Given any $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{R}$, prove that

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p \right)^{1/p}.$$

(Hint: The pth power of the expression on the left can be written as $\sum_{i=1}^{n} |a_i| \left(|a_i + b_i| \right)^{p-1} + \sum_{i=1}^{n} |b_i| \left(|a_i + b_i| \right)^{p-1}$; now use (iii).)
(vii) **(Minkowski Inequality for Integrals)** Given any continuous functions $f, g : [a, b] \to \mathbb{R}$, prove that

$$\left(\int_a^b |f(x) + g(x)|^p dx \right)^{1/p} \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} + \left(\int_a^b |g(x)|^p dx \right)^{1/p}.$$

7.58. Let $n \in \mathbb{N}$. By applying L'Hôpital's Rule n times, prove the following:

(i) $$\lim_{x\to 0} \frac{\exp x - \sum\limits_{k=0}^{n} x^k/k!}{x^{n+1}} = \frac{1}{(n+1)!},$$

(ii) $$\lim_{x\to 1} \frac{\ln x - \sum\limits_{k=1}^{n} (-1)^k (x-1)^k/k}{(x-1)^{n+1}} = \frac{(-1)^n}{(n+1)},$$

(iii) $$\lim_{x\to 0} \frac{\sin x - \sum\limits_{k=0}^{\lceil (n-2)/2\rceil} (-1)^k x^{2k+1}/(2k+1)!}{x^{n+1}} = \begin{cases} \dfrac{(-1)^{n/2}}{(n+1)!} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

(iv) $$\lim_{x\to 0} \frac{\cos x - \sum\limits_{k=0}^{\lfloor n/2\rfloor} (-1)^k x^{2k}/(2k)!}{x^{n+1}} = \begin{cases} \dfrac{(-1)^{(n+1)/2}}{(n+1)!} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

7.59. Let $p, q \in \mathbb{N}$. For $n \in \mathbb{N}$, consider the function $f_n : [0, p/q] \to \mathbb{R}$ defined by $f_n(x) := x^n(p - qx)^n/n!$. Prove the following results:

(i) $f_n(0) = 0 = f_n(p/q)$. Also, $f_n^{(k)}(0) = -f_n^{(k)}(p/q) \in \mathbb{Z}$ for each $k \in \mathbb{N}$; in fact, $f_n^{(k)}(0) = 0 = f_n^{(k)}(p/q)$ if $k \le n$ or $k > 2n$.

(ii) $\max\{f_n(x) : x \in [0, p/q]\} = f_n(p/2q)$, and $f_n(p/2q) \to 0$ as $n \to \infty$.

(iii) Let, if possible, $\pi = p/q$, and consider $a_n := \int_0^\pi f_n(x) \sin x\, dx$. Then $a_n \in \mathbb{Z}$ for each $n \in \mathbb{N}$ (by repeated use of Integration by Parts), whereas $0 < a_n < 1$ for all large $n \in \mathbb{N}$.

(iv) π is irrational.

7.60. (i) Show that for every $n \in \mathbb{N}$, $\displaystyle\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx.$

(ii) Show that for every $k \in \mathbb{N}$,

$$\int_0^{\pi/2} \sin^{2k} x\, dx = \frac{(2k-1)(2k-3)\cdots 3\cdot 1}{(2k)(2k-2)\cdots 4\cdot 2} \cdot \frac{\pi}{2} = \frac{(2k)!}{(2^k k!)^2} \cdot \frac{\pi}{2}$$

and

$$\int_0^{\pi/2} \sin^{2k+1} x\, dx = \frac{2k(2k-2)\cdots 4\cdot 2}{(2k+1)(2k-1)\cdots 3\cdot 1} = \frac{(2^k k!)^2}{(2k+1)!}.$$

(iii) For $k \in \mathbb{N}$, let $\mu_k := \left(\int_0^{\pi/2} \sin^{2k} x\, dx\right) / \left(\int_0^{\pi/2} \sin^{2k+1} x\, dx\right)$. Show that $1 \le \mu_k \le (2k+1)/2k$ for each $k \in \mathbb{N}$ and consequently that $\mu_k \to 1$ as $k \to \infty$. Deduce that

$$\sqrt{\pi} = \lim_{k\to\infty} \frac{(k!)^2 2^{2k}}{(2k)!\sqrt{k}}.$$

(Hint: $\sin^{2k+1} x \le \sin^{2k} x \le \sin^{2k-1} x$ for all $x \in [0, \pi/2]$.)
[Note: This result is known as the **Wallis Formula**. It provides an approximation of π, namely, $(k!)^4 2^{4k}/\big((2k)!\big)^2 k$, for large $k \in \mathbb{N}$.]

7.61. (i) Show that for every $n \in \mathbb{N}$,

$$\frac{1}{n+\frac{1}{2}} \le \int_n^{n+1} \frac{dx}{x} \le \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right).$$

(ii) Let (a_n) be the sequence defined by $a_n := n!e^n/n^n\sqrt{n}$ for $n \in \mathbb{N}$. Show that

$$\ln\left(\frac{a_n}{a_{n+1}}\right) = \left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right) - 1,$$

and hence

$$1 \le \frac{a_n}{a_{n+1}} \le \exp\left(\frac{1}{4}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right) \quad \text{for all } n \in \mathbb{N}.$$

Deduce that (a_n) is a monotonically decreasing sequence of positive real numbers and it is convergent. Let $\alpha := \lim_{n\to\infty} a_n$.

(iii) Use the inequalities in (ii) to show that

$$1 \le \frac{a_n}{a_{n+k}} \le \exp\left(\frac{1}{4}\left(\frac{1}{n} - \frac{1}{n+k}\right)\right) \quad \text{for all } n, k \in \mathbb{N}.$$

Taking the limit as $k \to \infty$, deduce that $\alpha > 0$ and furthermore, $1 \le (a_n/\alpha) \le \exp(1/4n)$ for all $n \in \mathbb{N}$.

(iv) Show that the Wallis formula given in Exercise 7.60 can be written as $\sqrt{2\pi} = \lim\limits_{n\to\infty} a_n^2/a_{2n}$. Deduce that $\alpha = \sqrt{2\pi}$.

(v) Use (iii) and (iv) to show that for all $n \in \mathbb{N}$,

$$(\sqrt{2\pi})n^{n+\frac{1}{2}}e^{-n} \le n! \le (\sqrt{2\pi})n^{n+\frac{1}{2}}e^{-n+(1/4n)}$$

and conclude that

$$\lim_{n\to\infty} \frac{n!}{(\sqrt{2\pi n})n^n e^{-n}} = 1.$$

[Note: This result is known as the **Stirling Formula**. It provides an approximation of $n!$, namely, $(\sqrt{2\pi})n^{n+\frac{1}{2}}e^{-n}$, for large $n \in \mathbb{N}$.]

7.62. Let r, $s \in \mathbb{R}$ with $s > 0$ and let $F : [0,1] \to \mathbb{R}$ be the function defined by

$$F(x) := \begin{cases} x^r \sin(1/x^s) & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove the following:

(i) F is continuous $\iff r > 0$, (ii) F is differentiable $\iff r > 1$,
(iii) F' is bounded $\iff r \ge 1+s$, (iv) F' is continuous $\iff r > 1+s$,
(v) F is twice differentiable $\iff r > 2+s$,
(vi) F'' is bounded $\iff r \ge 2+2s$, (vii) F'' is continuous $\iff r > 2+2s$.

7.63. Prove that the secant function, the cosecant function, and the cotangent function are transcendental.

Revision Exercises

R.1. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x \sin x$ and $g(x) := x + \sin x$. State whether f and g are bounded.

R.2. Consider the sequence whose nth term is given below. Examine whether it is convergent. In case it is convergent, find its limit.
(i) $\dfrac{n!}{10^n}$, (ii) $\left(\dfrac{n}{n+1}\right)^n$, (iii) $\dfrac{\ln n}{n^{1/n}}$.

R.3. Show that

$$\lim_{m\to\infty} \lim_{n\to\infty} |(\cos m!\pi x)^n| = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

R.4. Suppose $a \in \mathbb{R}$ or $a = \infty$, and (a_n) is a sequence of positive real numbers such that $a_n \to a$. Show that $(a_1 \cdots a_n)^{1/n} \to a$. Also, show that the converse does not hold. (Hint: Proposition 2.15 and Exercise 2.24.)

R.5. Consider the function $f : (-\pi/2, \pi/2) \to \mathbb{R}$ defined by $f(x) = \tan x$. Show that f is not uniformly continuous on $[0, \pi/2)$, but for every $\delta > 0$, the function f is uniformly continuous on $[-(\pi/2) + \delta, (\pi/2) - \delta]$.

R.6. For $x \in \mathbb{R}$, let $f(x) := x(\sin x + 2)$ and $g(x) := x(\sin x + 1)$. Show that $f(x) \to \infty$, but $g(x) \not\to \infty$ as $x \to \infty$.

R.7. Find $f'(x)$ if (i) $f(x) := x^x$ for $x > 0$, (ii) $f(x) := (x^x)^x$ for $x > 0$, (iii) $f(x) := x^{(x^x)}$ for $x > 0$, (iv) $f(x) := (\ln x)^x / x^{\ln x}$ for $x > 1$.

R.8. Let $a > 0$. Show that $-x^a \ln x < 1/ae$ for all $x \in (0, 1), x \neq e^{-1/a}$.

R.9. Let $r, s, t \in \mathbb{R}$ and $x \in (0, \infty)$. If $r > 1$, then show that $(1 + x)^r > 1 + x^r$. Deduce that if $0 < s < t$, then $(1 + x^s)^t > (1 + x^t)^s$.

R.10. Let $f : [0, \pi/2] \to \mathbb{R}$ be a continuous function.
(i) If f satisfies $f'(x) = 1/(1 + \cos x)$ for all $x \in (0, \pi/2)$ and if $f(0) = 3$, then find an estimate for $f(\pi/2)$.
(ii) If f satisfies $f'(x) = 1/(1 + x \sin x)$ for all $x \in (0, \pi/2)$ and if $f(0) = 1$, then find an estimate for $f(\pi/2)$.

R.11. Prove that $(\pi/15) < \tan(\pi/4) - \tan(\pi/5) < (\pi/10)$. Hence conclude that

$$\frac{10 - \pi}{10} < \tan \frac{\pi}{5} < \frac{15 - \pi}{15}.$$

R.12. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x + \sin x$. Show that f is strictly increasing on $\mathbb{R}$ although f' vanishes at infinitely many points. Find intervals of convexity/concavity, and points of inflection for f.

R.13. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := x^2 - 2\cos x$. Show that g is strictly convex on $\mathbb{R}$ although g'' vanishes at infinitely many points. Find intervals of increase/decrease and local extrema of g. Does g have an absolute minimum?

R.14. Locate intervals of increase/decrease, intervals of convexity/concavity, local maxima/minima, and the points of inflection for $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) := (\ln x)/x$. Sketch the curve $y = f(x)$.

R.15. Locate intervals of increase/decrease of the following functions:
(i) $f(x) := x^{1/x}$, $x \in (0, \infty)$, (ii) $f(x) := \left(1 + \dfrac{1}{x}\right)^x$, $x \in (0, \infty)$.

R.16. Determine which of the two numbers e^π and π^e is greater. (Hint: Find the absolute minimum of the function defined by $f(x) := x^{1/x}$ for $x \in (0, \infty)$ and put $x = \pi$; alternatively, find the absolute minimum of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := e^x - 1 - x$ and put $x := (\pi/e) - 1$.)

R.17. Consider the functions $f, g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) := \sin(1/x)$ and $g(x) := \cos(1/x)$. Locate intervals of increase/decrease, intervals of convexity/concavity, local maxima/minima, and the points of inflection for f and g.

R.18. Evaluate the following limits:
(i) $\lim_{x \to 0^+} x \ln x$, (ii) $\lim_{x \to 0^+} \dfrac{\ln x}{x}$, (iii) $\lim_{x \to \infty} (x - \ln x)$, (iv) $\lim_{x \to \infty} \dfrac{\ln(\ln x)}{\ln x}$,
(v) $\lim_{x \to \infty} \dfrac{x^5}{e^x}$, (vi) $\lim_{x \to \infty} \dfrac{2^x - 1}{2^x + 3}$, (vii) $\lim_{x \to 0} \dfrac{3^{\sin x} - 1}{x}$.

R.19. Evaluate the following limits:
(i) $\lim_{x \to 0} \dfrac{x - \sin^{-1} x}{\sin^3 x}$, (ii) $\lim_{x \to \pi/2} (\sec x - \tan x)$, (iii) $\lim_{x \to 0} \dfrac{x - \tan x}{x - \sin x}$,
(iv) $\lim_{x \to 0} \dfrac{x \cot x - 1}{x^2}$, (v) $\lim_{x \to \pi/2} \dfrac{\tan 3x}{\tan x}$, (vi) $\lim_{x \to 1} (1 - x) \tan(\pi x/2)$,
(vii) $\lim_{x \to 0} \sin^{-1} x \cot x$, (viii) $\lim_{x \to 0} \dfrac{\cos x - 1 + (x^2/2)}{x^4}$, (ix) $\lim_{x \to 0} \dfrac{\tan x}{\sec x}$,
(x) $\lim_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$, (xi) $\lim_{x \to 0} \dfrac{\sin 2x}{2x^2 + x}$, (xii) $\lim_{x \to 0} \dfrac{\sin x - x}{x}$,
(xiii) $\lim_{x \to 0} \dfrac{\sin x - x}{x^2}$, (xiv) $\lim_{x \to 0} \dfrac{\sin x - x}{x^3}$.

R.20. Discuss whether $\lim_{x \to c} f(x)/g(x)$ and $\lim_{x \to c} f'(x)/g'(x)$ exist if
(i) $c := 0$, $f(x) := x^2 \sin(1/x)$, $g(x) := \sin x$ for $x \in \mathbb{R}$, $x \neq 0$,
(ii) $c := 0$, $f(x) := x \sin(1/x)$, $g(x) := \sin x$ for $x \in \mathbb{R}$, $x \neq 0$,
(iii) $c := \infty$, $f(x) := x(2 + \sin x)$, $g(x) := x^2 + 1$ for $x \in \mathbb{R}$,
(iv) $c := \infty$, $f(x) := x(2 + \sin x)$, $g(x) := x + 1$ for $x \in \mathbb{R}$.

R.21. Let $a > 0$ and let $f : [a, \infty) \to \mathbb{R}$ be a differentiable function. Assume that $f(x) + f'(x) \to \ell$ as $x \to \infty$, where $\ell \in \mathbb{R}$ or $\ell = \infty$ or $\ell = -\infty$. Show that $f(x) \to \ell$ as $x \to \infty$. In the case $\ell \in \mathbb{R}$, show that $f'(x) \to 0$ as $x \to \infty$. (Hint: Use Proposition 4.42 for the functions $g, h : [a, \infty) \to \mathbb{R}$ defined by $g(x) := f(x)e^x$ and $h(x) := e^x$.)

R.22. Evaluate the following limits:
(i) $\lim_{x \to 0^+} (\sin x)^{\tan x}$, (ii) $\lim_{x \to \infty} (x^2)^{1/\sqrt{x}}$, (iii) $\lim_{x \to (\pi/2)^-} (\sin x)^{\tan x}$.

R.23. For $x \in \mathbb{R}$, let $f(x) := 2x + \sin 2x$ and $g(x) := f(x)/(2 + \sin x)$. Do

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} \quad \text{and} \quad \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

exist? Explain in view of L'Hôpital's Rules.

R.24. For $x \in (0, \infty)$, let $f(x) := \ln x$ and $g(x) := x$. Show that

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \frac{f'(x)}{g'(x)} = \infty.$$

Explain in view of L'Hôpital's Rules.

R.25. Arrange the following functions in descending order of their growth rates as $x \to \infty$:

$$2^x, e^x, x^x, (\ln x)^x, e^{x/2}, x^{1/2}, \log_2 x, \ln(\ln x), (\ln x)^2, x^e, x^2, \ln x, (2x)^x, x^{2x}.$$

R.26. Let $n \in \mathbb{N}$ and let $a_1, \ldots, a_n$ be positive real numbers. Prove that

$$\lim_{x \to 0} \left(\frac{a_1^x + \cdots + a_n^x}{n} \right)^{1/x} = (a_1 \cdots a_n)^{1/n}.$$

(Hint: Apply the logarithm and use L'Hôpital's Rule.)

R.27. Let $n \in \mathbb{N}$ and let $a_1, \ldots, a_n$ be positive real numbers. For every $p \in \mathbb{R}$ such that $p \neq 0$, define

$$M_p = \left(\frac{a_1^p + \cdots + a_n^p}{n} \right)^{1/p}.$$

In view of Exercise R.26 above, define $M_0 = (a_1 \cdots a_n)^{1/n}$. Prove that if $p, q \in \mathbb{R}$ are such that $p < q$, then $M_p \leq M_q$, and equality holds if and only if $a_1 = \cdots = a_n$. (Hint: Use part (ii) of Proposition 7.10 and the Jensen Inequality stated in Exercise 1.32.)

[Note: As mentioned in Exercise 1.51, the above inequality is called the **power mean inequality** and it includes the A.M.-G.M. inequality and the G.M.-H.M. inequality as special cases. This inequality is also valid for $p = -\infty$ and $q = \infty$ if we set $M_{-\infty} := \min\{a_1, \ldots, a_n\}$ and $M_\infty := \max\{a_1, \ldots, a_n\}$.]

R.28. Let $D \subseteq \mathbb{R}$ and let c be an interior point of D. Also, let $f : D \to \mathbb{R}$ be a function that is differentiable at c. If $f''(c)$ exists, then show that there is a function $f_2 : D \to \mathbb{R}$ such that f_2 is continuous at c and

$$f(x) = f(c) + (x - c)f'(c) + (x - c)^2 f_2(x) \quad \text{for all } x \in D,$$

and then $f_2(c) = f''(c)/2$. Give an example to show that the converse is not true. (Hint: Use L'Hôpital's Rule and the function $f_n : \mathbb{R} \to \mathbb{R}$ given in Exercise 7.50 with $n = 3$.) (Compare Proposition 4.2.)

R.29. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := e^{-1/x}$ if $x > 0$ and $f(x) = 0$ if $x \leq 0$. Show that f is infinitely differentiable at 0, and for each $n \in \mathbb{N}$, the nth Taylor polynomial of f around 0 is the zero polynomial. (Compare Example 9.32 (iii).)

R.30. Find the absolute maximum of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := (\sin x - \cos x)^2$.

R.31. Prove the following estimates for the errors $e_1(x) := \ln x - P_1(x)$ and $e_2(x) := \ln x - P_2(x)$, $x \in (0,2)$, in the linear and the quadratic approximations of the function ln around 1:

$$|e_1(x)| \le \frac{(x-1)^2}{2}, \quad |e_2(x)| \le \frac{(x-1)^3}{3} \quad \text{if } 1 < x < 2,$$

and

$$|e_1(x)| \le \frac{1}{2}\left(\frac{1}{x}-1\right)^2, \quad |e_2(x)| \le \frac{1}{3}\left(\frac{1}{x}-1\right)^3 \quad \text{if } 0 < x < 1.$$

R.32. Prove the following estimates for the errors $e_1(x) := \exp x - P_1(x)$ and $e_2(x) := \exp x - P_2(x)$, $x \in (-1,1)$, in the linear and the quadratic approximations of the function exp around 0:

$$|e_1(x)| \le \frac{e\,x^2}{2} \quad \text{and} \quad |e_2(x)| \le \frac{e\,x^3}{6} \quad \text{if } 0 < x < 1,$$

and

$$|e_1(x)| \le \frac{x^2}{2} \quad \text{and} \quad |e_2(x)| \le -\frac{x^3}{6} \quad \text{if } -1 < x < 0.$$

R.33. Show that each of the following functions maps the given interval I into itself and has a unique fixed point in that interval. Also, show that if x_0 belongs to this interval, then the Picard sequence with initial point x_0 converges to the unique fixed point of the function.
(i) $g(x) := \sqrt{\sin x}$, $I = [\pi/4, \pi/2]$, (ii) $g(x) := 1 + (\sin x)/2$, $I = [0,2]$.

R.34. (i) Show that 0 is the only fixed point of the function $\sin : \mathbb{R} \to \mathbb{R}$.
(ii) Show that the function $\cos : \mathbb{R} \to \mathbb{R}$ has a unique fixed point c^*. Assuming $\pi > 3$, show that the function cos maps the interval $[\pi/8, 1]$ into itself. Deduce that $0.375 < c^* < 0.925$.
[Note: When a calculator is in radian mode, if we key in any number and press the "sin" key repeatedly, then eventually we reach 0, and if we press the "cos" key repeatedly, then eventually we reach 0.7390851. A similar phenomenon occurs when a calculator is in degree mode.]

R.35. For each of the following functions, show that the equation $f(x) = 0$ has a unique solution in the given interval I. Use Newton's method with the given initial point x_0 to find an approximate value of this root.
(i) $f(x) := x - \cos x$, $I = [\cos 1, 1]$, $x_0 = 1$,
(ii) $f(x) := x - 1 - (\sin x)/2$, $I = [0,2]$, $x_0 = 1.5$. (Compare these iterates with those of the Picard method obtained in Exercise R.33 (ii).)

R.36. For all $h \in \mathbb{R}$ and $n = 1, 2, \ldots$, show that

$$2\sin\frac{h}{2}\,(\sin h + \sin 2h + \cdots + \sin nh) = \cos\frac{h}{2} - \cos\left(n+\frac{1}{2}\right)h.$$

Hence find $\int_0^{\pi/2} \sin x\,dx$ without using the FTC.

R.37. Let $f : [a,b] \to \mathbb{R}$ be differentiable and assume that f' is Riemann integrable on $[a,b]$. If $f(x) > 0$ for all $x \in [a,b]$, then show that

$$\int_a^b \frac{f'(x)}{f(x)} dx = \ln f(b) - \ln f(a).$$

(Hint: Apply part (ii) of the FTC to $\ln f$.)

R.38. Let $a, b, \alpha, \beta \in \mathbb{R}$. Prove the following:

(i) $\displaystyle\int_a^b \frac{1}{x-\alpha} dx = \ln \frac{b-\alpha}{a-\alpha}$, provided $a, b > \alpha$,

(ii) $\displaystyle\int_a^b \frac{2x+\alpha}{x^2+\alpha x+\beta} dx = \ln \frac{b^2+\alpha b+\beta}{a^2+\alpha a+\beta}$, provided $\alpha^2 < 4\beta$.

R.39. Prove the following:

(i) $\displaystyle\int_0^b \tan x \, dx = \ln \sec b$, provided $b \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

(ii) $\displaystyle\int_0^b \sec x \, dx = \ln(\sec b + \tan b)$, provided $b \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

(iii) $\displaystyle\int_b^{\pi/2} \cot x \, dx = \ln \csc b$, provided $b \in (0, \pi)$,

(iv) $\displaystyle\int_b^{\pi/2} \csc x \, dx = \ln(\csc b + \cot b)$, provided $b \in (0, \pi)$.

In particular, show that

$$\int_0^{\pi/4} \tan x \, dx = \ln \sqrt{2}, \quad \int_0^{\pi/4} \sec x \, dx = \ln(1+\sqrt{2}),$$
$$\int_{\pi/4}^{\pi/2} \cot x \, dx = \ln \sqrt{2}, \quad \int_{\pi/4}^{\pi/2} \csc x \, dx = \ln(1+\sqrt{2}).$$

R.40. Let $\alpha, \beta \in \mathbb{R}$ be such that $-\pi < \alpha < \beta < \pi$ and let $P(x,y)$, $Q(x,y)$ be polynomials such that $Q(\sin\theta, \cos\theta) \neq 0$ for all $\theta \in [\alpha, \beta]$. Show that

$$\int_\alpha^\beta \frac{P(\sin\theta, \cos\theta)}{Q(\sin\theta, \cos\theta)} d\theta = \int_{\tan(\alpha/2)}^{\tan(\beta/2)} \frac{P\left(2t/(1+t^2), (1-t^2)/(1+t^2)\right)}{Q\left(2t/(1+t^2), (1-t^2)/(1+t^2)\right)} \frac{2}{1+t^2} dt.$$

R.41. Evaluate the following integrals:

(i) $\displaystyle\int_0^{\pi/2} \frac{1}{2+\cos\theta} d\theta$, (ii) $\displaystyle\int_{\pi/2}^{2\pi/3} \frac{\cot\theta}{1+\cos\theta} d\theta$, (iii) $\displaystyle\int_\alpha^\beta \sec\theta \, d\theta$,

where $\alpha, \beta \in \mathbb{R}$ with $-\frac{\pi}{2} < \alpha < \beta < \frac{\pi}{2}$. (Hint: Substitute $t = \tan(\theta/2)$.) [Note: The substitution $t = \tan(\theta/2)$ converts the integral of any rational function in trigonometric functions (in the parameter θ) to an integral of a rational function (in the variable t). The latter can, in general, be evaluated using partial fractions. Therefore, the integral of every rational function in trigonometric functions can be evaluated.]

R.42. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R},\ \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) := \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x - t) dt.$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.

R.43. Find the linear and quadratic approximations to the function f around 1, where $f : [0, \infty) \to \mathbb{R}$ is defined by

$$f(x) := 1 + \int_1^x \frac{10}{1 + \sqrt{t}} dt.$$

R.44. Prove the following:

(i) For $x \in \mathbb{R}$, $\displaystyle\int_0^x \frac{1}{\sqrt{1+t^2}}\, dt = \ln\left(x + \sqrt{1+x^2}\right)$,

(ii) For $x \in \mathbb{R}$, $\displaystyle\int_0^x \sqrt{1+t^2}\, dt = \frac{1}{2}\left(x\sqrt{1+x^2} + \ln\left(x + \sqrt{1+x^2}\right)\right)$,

(iii) For $x \in [-1, 1]$, $\displaystyle\int_0^x \sqrt{1-t^2}\, dt = \frac{1}{2}\left(x\sqrt{1-x^2} + \sin^{-1} x\right)$.

R.45. Find (i) $\displaystyle\int_4^9 \frac{1}{x - \sqrt{x}}\, dx$, (ii) $\displaystyle\int_1^3 \frac{1}{\sqrt{x}(x+1)}\, dx$, (iii) $\displaystyle\int_1^{1/\sqrt{2}} \frac{1}{x\sqrt{4x^2-1}}\, dx$.

R.46. Evaluate the following limits:

(i) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \cos\frac{i\pi}{n}$, (ii) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^n \frac{n}{i^2+n^2}$, (iii) $\displaystyle\lim_{n\to\infty} \left(\sin\frac{1}{n}\right) \sum_{i=1}^n \frac{n^2}{i^2+n^2}$.

8

Applications and Approximations of Riemann Integrals

In this chapter, we shall consider some geometric applications of Riemann integrals. They deal with defining and finding the areas of certain planar regions, volumes of certain solid bodies including solid bodies generated by revolving planar regions about a line, lengths of "piecewise smooth" curves, and areas of surfaces generated by revolving such planar curves about a line. Subsequently, we show how to find the "centroids" of the geometric objects considered earlier. The coordinates of a centroid are in some sense the averages of the coordinate functions. In the last section of this chapter, we give a number of methods for evaluating Riemann integrals approximately. We also establish error estimates for these approximations. This procedure would be useful, in particular, if we needed to find approximations of arc lengths, areas, and volumes of various geometric objects whenever exact evaluation of the Riemann integrals involved therein is either difficult or impossible.

8.1 Area of a Region Between Curves

In this section, we shall show how "areas" of certain planar regions that lie between two curves can be found using Riemann integrals. It may be remarked that the general concept of the area of a planar region is usually defined using double integrals, which are studied in a course in multivariable calculus. (See, for example, [33, p. 241].) The definitions of areas of special planar regions given in this section can be reconciled with the general concept.

Recall that in Section 6.1, we began our discussion of Riemann integrals by *assuming* that the area of a rectangle $[x_1, x_2] \times [y_1, y_2]$ is $(x_2 - x_1)(y_2 - y_1)$. Let $f : [a, b] \to \mathbb{R}$ be a bounded nonnegative function. The concept of a Riemann integral was motivated by an attempt to give a meaning to the "area" of the region lying under the graph of f. We have defined the area of the region $R := \{(x, y) \in \mathbb{R}^2 : a \le x \le b \text{ and } 0 \le y \le f(x)\}$ to be

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$$\text{Area }(R) := \int_a^b f(x)dx.$$

This naturally leads us to the following definition. Let $f_1, f_2 : [a, b] \to \mathbb{R}$ be integrable functions such that $f_1 \leq f_2$. Then the **area** of the region between the curves given by $y = f_1(x)$, $y = f_2(x)$ and between the (vertical) lines given by $x = a$, $x = b$, that is, of the region

$$R := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } f_1(x) \leq y \leq f_2(x)\},$$

is defined to be

$$\text{Area }(R) := \int_a^b \big(f_2(x) - f_1(x)\big)dx.$$

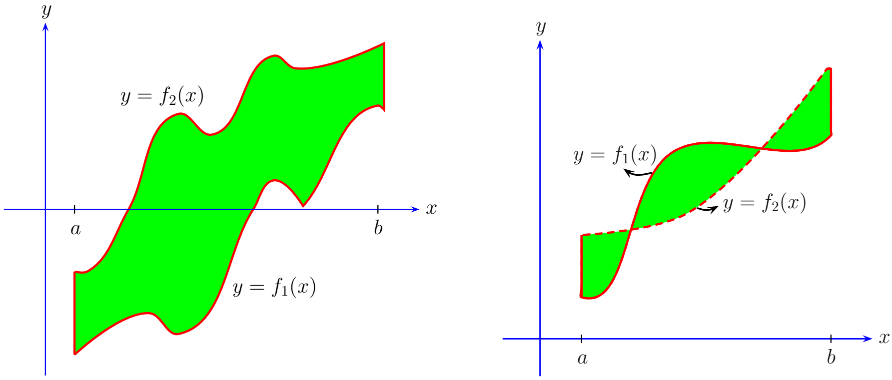


Fig. 8.1. Region between the curves $y = f_1(x)$, $y = f_2(x)$ and the lines $x = a$, $x = b$ when the curves do not cross each other, and when they cross each other.

If a planar region R can be divided into a finite number of nonoverlapping subregions of the types considered above, then the area of R is defined to be the sum of the areas of these subregions. For example, if curves given by $y = f_1(x)$, $y = f_2(x)$, where $f_1, f_2 : [a, b] \to \mathbb{R}$ are continuous functions, cross each other at a finite number of points, then the area of the region bounded by these curves and the lines given by $x = a$, $x = b$ turns out to be equal to

$$\int_a^b |f_2(x) - f_1(x)|dx.$$

Similarly, if $g_1, g_2 : [c, d] \to \mathbb{R}$ are integrable functions such that $g_1 \leq g_2$, then the **area** of the region between the curves given by $x = g_1(y)$, $x = g_2(y)$ and between the (horizontal) lines given by $y = c$, $y = d$, that is, of the region

$$R := \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } g_1(y) \leq x \leq g_2(y)\},$$

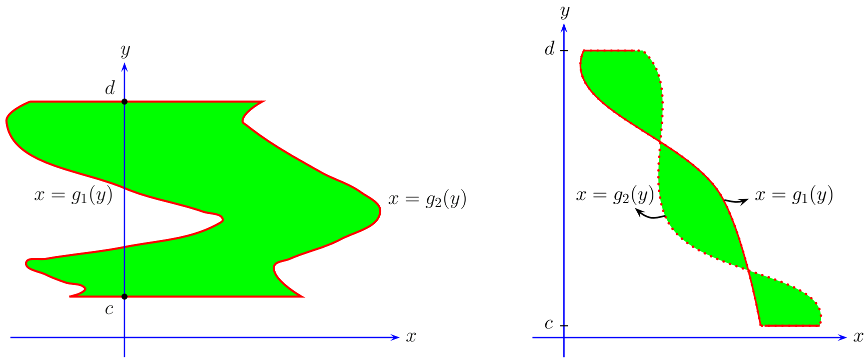


Fig. 8.2. Region between the curves $x = g_1(y)$, $x = g_2(y)$, and the lines $y = c$, $y = d$, when the curves do not cross each other, and when they cross each other.

is defined to be

$$\text{Area } (R) := \int_c^d \big(g_2(y) - g_1(y)\big)dy.$$

Also, if curves given by $x = g_1(y)$, $x = g_2(y)$, where $g_1, g_2 : [c, d] \to \mathbb{R}$ are continuous functions, cross each other at a finite number of points, then the area of the region bounded by these curves and the lines given by $y = c$, $y = d$ turns out to be equal to

$$\int_c^d |g_2(y) - g_1(y)|dy.$$

Examples 8.1. (i) Let $0 < a < b$ and consider the triangular region enclosed by the lines given by $y = hx/a$, $y = h(x-b)/(a-b)$, and the x-axis. These lines form a triangle with base b and height h. We show that the area of this region is equal to $bh/2$. The perpendicular from the vertex (a, h) to the x-axis divides the triangular region into two triangular subregions having bases a and $b - a$, and both having height h. The area of the given triangular region is then equal to the sum of the areas of these subregions. (See Figure 8.3.) The first subregion is the region between the curves $y = hx/a$, $y = 0$ and between the lines given by $x = 0$, $x = a$. Hence its area is equal to

$$\int_0^a \Big(\frac{hx}{a} - 0\Big)dx = \frac{h}{a} \cdot \frac{a^2}{2} = \frac{ha}{2}.$$

Likewise, the area of the second subregion is equal to $h(b - a)/2$. Hence the required area is $(ha/2) + (h(b - a)/2) = hb/2$.

(ii) The region enclosed by the loop of the curve given by $y^2 = x(1 - x)^2$ is the region between the curves given by $y = \sqrt{x}(1 - x)$, $y = -\sqrt{x}(1 - x)$ and between the lines given by $x = 0$, $x = 1$. Hence its area is equal to

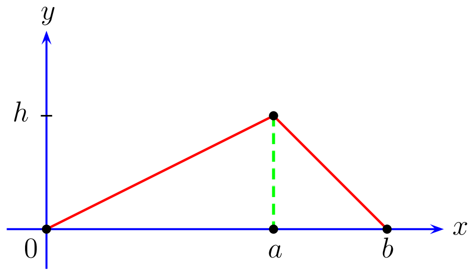


Fig. 8.3. Triangular region in Example 8.1 (i) with its two triangular subregions.

$$\int_0^1 \left(\sqrt{x}(1-x) - \left(-\sqrt{x}(1-x)\right)\right) dx = 2\int_0^1 \left(x^{1/2} - x^{3/2}\right) dx = \frac{8}{15}.$$

(iii) The area of the region bounded by the curves $x = y^3$, $x = y^5$ and the lines given by $y = -1$, $y = 1$ is equal to

$$\int_{-1}^1 |y^5 - y^3| dy = \int_{-1}^0 (y^5 - y^3) dy + \int_0^1 (y^3 - y^5) dy = \frac{1}{6}.$$

(iv) To determine the area of the region in the strip $\{(x, y) \in \mathbb{R}^2 : -2 \le x \le 1\}$ bounded by the parabolas $x = -2y^2$ and $x = 1 - 3y^2$, we first find their points of intersection. Now $-2y^2 = 1 - 3y^2$ implies $y = \pm 1$ and $1 - 3y^2 \ge -2y^2$ for all $y \in [-1, 1]$. Hence

$$\int_{-1}^1 \left((1-3y^2) - (-2y^2)\right) dy = \int_{-1}^1 \left(1 - y^2\right) dy = \frac{4}{3}$$

is the required area. ◇

We shall now calculate the area enclosed by an ellipse. As a special case, this will give us the area enclosed by a circle and lead us to an important classical formula for π.

Proposition 8.2. *Let a, b be positive real numbers.*

(i) *The area of the region enclosed by an ellipse given by $(x^2/a^2)+(y^2/b^2) = 1$ is equal to πab.*
(ii) *The area of a circular disk enclosed by a circle given by $x^2 + y^2 = a^2$ is equal to πa^2. In other words, if D denotes this disk, then*

$$\pi = \frac{\textit{Area of } D}{(\textit{Radius of } D)^2}.$$

(iii) *For $\varphi \in [0, \pi]$, the area of the sector of a disk of radius a that subtends an angle φ at the center, that is, the area of the planar region given by $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le a^2$ and $0 \le \theta(x, y) \le \varphi\}$, is equal to $a^2\varphi/2$.*

Proof. (i) The area enclosed by the given ellipse is four times the area between the curves given by $y = b\sqrt{a^2 - x^2}/a$, $y = 0$ and between the lines given by $x = 0$, $x = a$. Hence it is equal to

$$4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\cdot a^2 \int_0^{\pi/2} \cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta = \pi ab.$$

(ii) Letting $b = a$ in (i) above, we see that the area of a disk of radius a is equal to πa^2. The desired formula for π is then immediate.

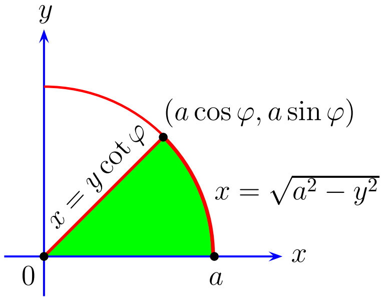


Fig. 8.4. Sector marked by the points $(0, 0)$, $(a, 0)$, and $(a\cos\varphi, a\sin\varphi)$.

(iii) If $\varphi = 0$, then the sector reduces to a line segment, and its area is clearly equal to 0. Now let $\varphi \in (0, \pi/2]$. The sector marked by the points $(0, 0)$, $(a, 0)$, and $(a\cos\varphi, a\sin\varphi)$ is the region between the curves $x = (\cot\varphi)y$, $x = \sqrt{a^2 - y^2}$ and between the lines given by $y = 0$, $y = a\sin\varphi$. Hence its area is equal to

$$\int_0^{a\sin\varphi}\left(\sqrt{a^2 - y^2} - (\cot\varphi)y\right)dy = \left(a^2\int_0^{\varphi}\cos^2 t\,dt\right) - \cot\varphi\frac{a^2\sin^2\varphi}{2} = \frac{a^2\varphi}{2}.$$

By symmetry, the formula holds for $\varphi \in (\pi/2, \pi]$ as well. This can be seen as follows. Let $\psi := \pi - \varphi$. Then $\psi \in [0, \pi/2)$, and by what we have already proved, the area of the desired sector is equal to

$$\frac{\pi a^2}{2} - \frac{a^2\psi}{2} = \frac{\pi a^2}{2} - \frac{a^2(\pi - \varphi)}{2} = \frac{a^2\varphi}{2},$$

as before. □

The formulas given in the above proposition are of fundamental importance. In part (iii) of Proposition 7.14, we have defined π as two times the supremum of the set $\{\arctan x : x \in (0, \infty)\}$. Now the same real number turns out to be the area of a circular disk divided by the square of the radius of the disk. This formula for π makes it plain that the ratio of the area of a (circular) disk to the square of its radius is independent of the radius. We have thus proved a fact that is usually taken for granted when π is introduced in high-school geometry.

Curves Given by Polar Equations

The formula for the area of a sector of a disk given in part (iii) of Proposition 8.2 enables us to define areas of planar regions between curves given by certain polar equations.

Let us consider a curve given by a polar equation of the form $r = p(\theta)$. Let $\alpha, \beta \in \mathbb{R}$ be such that either $-\pi < \alpha < \beta \leq \pi$ or $\alpha = -\pi$, $\beta = \pi$. Consider a nonnegative integrable function $p : [\alpha, \beta] \to \mathbb{R}$ and assume that $p(\pi) = p(-\pi)$ if $\alpha = -\pi$, $\beta = \pi$. Let

$$R := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 : \alpha \leq \theta \leq \beta \text{ and } 0 \leq r \leq p(\theta)\}$$

denote the region bounded by the curve given by $r = p(\theta)$ and the rays given by $\theta = \alpha$, $\theta = \beta$. Let $(r(x, y), \theta(x, y))$ denote the polar coordinates of $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. By Proposition 7.22, it follows that

$$R\setminus\{(0,0)\} = \left\{(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} : \alpha \leq \theta(x, y) \leq \beta \text{ and } r(x, y) \leq p(\theta(x, y))\right\}.$$

If $\{\theta_0, \theta_1, \ldots, \theta_n\}$ is a partition of $[\alpha, \beta]$, then the planar region R gets divided into n subregions

$$\{(0,0)\}\cup\left\{(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} : \theta_{i-1} \leq \theta(x, y) \leq \theta_i \text{ and } r(x, y) \leq p(\theta(x, y))\right\},$$

where $i = 1, \ldots, n$. (See Figure 8.5.) For each i, let us choose $\gamma_i \in [\theta_{i-1}, \theta_i]$ and replace the ith subregion by the sector

$$\{(0,0)\} \cup \left\{(x, y) \in \mathbb{R}^2 : \theta_{i-1} \leq \theta(x, y) \leq \theta_i \text{ and } r(x, y) \leq p(\gamma_i)\right\}$$

of the disk of radius $p(\gamma_i)$ with center at $(0, 0)$. By part (iii) of Proposition 8.2, the area of this sector is equal to

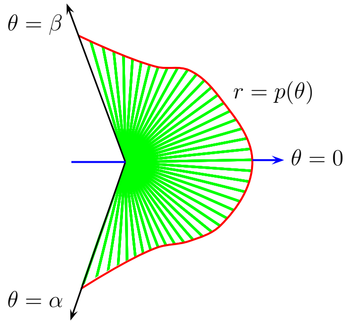
$$p(\gamma_i)^2(\theta_i - \theta_{i-1})/2, \quad i = 1, \ldots, n.$$

With this in view, the area of the region R is defined to be

$$\text{Area } (R) := \frac{1}{2}\int_\alpha^\beta p(\theta)^2 d\theta.$$

Further, if $p_1, p_2 : [\alpha, \beta] \to \mathbb{R}$ are integrable functions such that $0 \leq p_1 \leq p_2$ and $p_i(\pi) = p_i(-\pi)$, $i = 1, 2$, in case $\alpha = -\pi$, $\beta = \pi$, then the area of the region R between the curves given by $r = p_1(\theta)$, $r = p_2(\theta)$ and between the rays given by $\theta = \alpha$, $\theta = \beta$ is defined to be

$$\text{Area } (R) := \frac{1}{2}\int_\alpha^\beta \left(p_2(\theta)^2 - p_1(\theta)^2\right) d\theta.$$



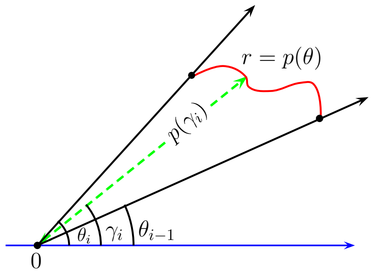


Fig. 8.5. Region bounded by the polar curve $r = p(\theta)$ and rays $\theta = \alpha$, $\theta = \beta$, and its "ith subregion".

Examples 8.3. (i) Let $a, \alpha, \beta \in \mathbb{R}$ be such that $a > 0$ and $-\pi < \alpha < \beta \le \pi$. Consider $p_1, p_2 : [\alpha, \beta] \to \mathbb{R}$ given by $p_1(\theta) := 0$ and $p_2(\theta) := a$. Then the area of the sector

$$\{(x, y) \in \mathbb{R}^2 : \alpha \le \theta(x, y) \le \beta,\ 0 \le r(x, y) \le a\}$$

of the disk of radius a is equal to

$$\frac{1}{2}\int_{\alpha}^{\beta} p(\theta)^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} a^2 d\theta = \frac{a^2(\beta - \alpha)}{2},$$

as it should be in view of part (iii) of Proposition 8.2.

(ii) Let $a \in \mathbb{R}$ with $a > 0$. The area of the region enclosed by the cardioid $r = a(1 + \cos\theta)$ is equal to

$$\frac{1}{2}\int_{-\pi}^{\pi} \big(a(1 + \cos\theta)\big)^2 d\theta = \frac{a^2}{2}\int_{-\pi}^{\pi} \left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta = \frac{3a^2\pi}{2}.$$

(iii) The area of the region between the circle given by $r = 2$ and the spiral given by $r = \theta$ lying between the rays given by $\theta = 0$, $\theta = \pi/2$ is equal to

$$\frac{1}{2}\int_{0}^{\pi/2} \big(2^2 - \theta^2\big) d\theta = \pi - \frac{\pi^3}{48}.$$

Note that $\theta \le 2$ for all $\theta \in [0, \pi/2]$. ◇

Area between curves given by polar equations of the form $\theta = \alpha(r)$ is treated in Exercises 8.17 and 8.18.

We conclude this section by mentioning again that the definitions of areas of various kinds of regions discussed here can be unified with the help of double integrals. (See, for example, the first two subsections of Section 6.2 of [33].) This would also show that the areas of a region calculated using different definitions given in this section must turn out to be the same!

8.2 Volume of a Solid

In this section we shall show how volumes of certain solid bodies can be found using Riemann integrals. It may be remarked that the general concept of the volume of a solid body is usually introduced in a course in multivariable calculus with the help of triple integrals. (See, for example, [33, p. 274].) The definitions of volumes of special solid bodies given in this section can be reconciled with the general concept.

Let us consider volumes of solid bodies that can be thought to be made up of cross-sections taken in one of the following ways:

1. Cross-sections by planes perpendicular to a fixed line,
2. Cross-sections by right circular cylinders having a fixed axis.

Slicing by Planes Perpendicular to a Fixed Line

Let D be a bounded subset of $\mathbb{R}^3 := \{(x, y, z) : x, y, z \in \mathbb{R}\}$ lying between two parallel planes and let L denote a line perpendicular to these planes. A cross-section of D by a plane is called a **slice** of D. Let us assume that we are able to determine the "area" of a slice of D by any plane perpendicular to L.

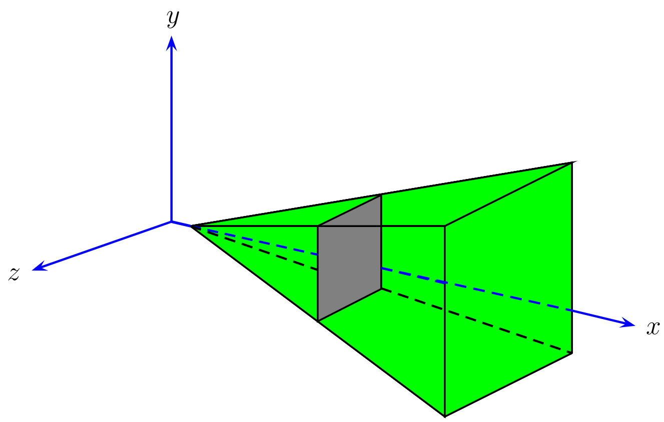


Fig. 8.6. Slicing a solid by planes perpendicular to a fixed line.

For the sake of concreteness, let the line L be the x-axis and assume that D lies between the planes given by $x = a$ and $x = b$, where $a, b \in \mathbb{R}$ with $a < b$. For $s \in [a, b]$, let $A(s)$ denote the area of the slice $\{(x, y, z) \in D : x = s\}$ obtained by intersecting D with the plane given by $x = s$. If $\{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, then the solid D gets divided into n subsolids

$$\{(x, y, z) \in D : x_{i-1} \le x \le x_i\}, \quad i = 1, \ldots, n.$$

Let us choose $s_i \in [x_{i-1}, x_i]$ and replace the ith subsolid by a rectangular slab having volume equal to $A(s_i)(x_i - x_{i-1})$ for $i = 1, \ldots, n$. Then it is natural to consider

$$\sum_{i=1}^{n} A(s_i)(x_i - x_{i-1})$$

as an approximation of the desired volume of D. We therefore define the volume of D to be

$$\text{Vol } (D) := \int_a^b A(x)dx,$$

provided the "area function" $A : [a, b] \to \mathbb{R}$ is integrable.

Similarly, if $D \subseteq \{(x, y, z) \in \mathbb{R}^3 : c \le y \le d\}$ for some $c, d \in \mathbb{R}$ with $c < d$, and for $t \in [c, d]$, if $A(t)$ denotes the area of the slice $\{(x, y, z) \in D : y = t\}$ obtained by intersecting D with the plane given by $y = t$, then we define the volume of D to be

$$\text{Vol } (D) := \int_c^d A(y)dy,$$

provided the "area function" $A : [c, d] \to \mathbb{R}$ is integrable.

Likewise, if $D \subseteq \{(x, y, z) \in \mathbb{R}^3 : p \le z \le q\}$ for some $p, q \in \mathbb{R}$ with $p < q$, and for $u \in [p, q]$, if $A(u)$ denotes the area of the slice $\{(x, y, z) \in D : z = u\}$ obtained by intersecting D with the plane given by $z = u$, then we define the volume of D to be

$$\text{Vol } (D) := \int_p^q A(z)dz,$$

provided the "area function" $A : [p, q] \to \mathbb{R}$ is integrable.

Examples 8.4. (i) Let $a, b, c, d, p, q \in \mathbb{R}$ and consider the cuboid

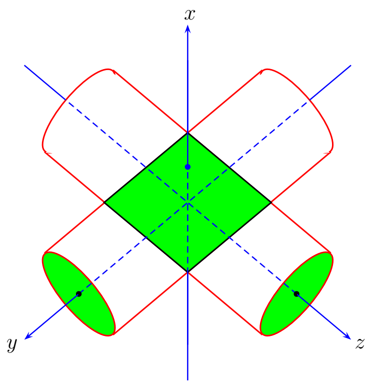
$$D := \{(x, y, z) \in \mathbb{R}^3 : a \le x \le b,\ c \le y \le d,\ p \le z \le q\}.$$

Then for each fixed $s \in [a, b]$, the area of the slice $\{(x, y, z) \in D : x = s\}$ of D is $A(s) := (d - c)(q - p)$, and hence

$$\text{Vol } (D) = \int_a^b A(x)dx = (d - c)(q - p)(b - a).$$

Alternatively, for each fixed $t \in [c, d]$, we may consider the area $A(t) := (b - a)(q - p)$ of the slice $\{(x, y, z) \in D : y = t\}$ of D, or for each fixed $u \in [p, q]$, we may consider the area $A(u) := (b - a)(d - c)$ of the slice $\{(x, y, z) \in D : z = u\}$ of D for finding the volume of D.

(ii) Let $a \in \mathbb{R}$ with $a > 0$. Let us find the volume of the solid D enclosed by the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. (See Figure 8.7.) The solid D lies between the planes $x = -a$ and $x = a$, and for a fixed $s \in [-a, a]$, the slice $\{(x, y, z) \in D : x = s\}$ is given by



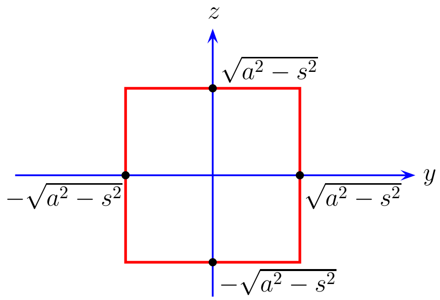


Fig. 8.7. Solid enclosed by two cylinders and a slice resulting in a square region.

$$\left\{(s, y, z) \in \mathbb{R}^3 : |y| \leq \sqrt{a^2 - s^2} \text{ and } |z| \leq \sqrt{a^2 - s^2}\right\}.$$

This slice is a square region of side $2\sqrt{a^2 - s^2}$, and its area is equal to

$$A(s) := (2\sqrt{a^2 - s^2})^2 = 4(a^2 - s^2).$$

Hence

$$\int_{-a}^{a} A(x)dx = 4\int_{-a}^{a}(a^2 - x^2)dx = 8\int_{0}^{a}(a^2 - x^2)dx = 8\Big(a^3 - \frac{a^3}{3}\Big) = \frac{16a^3}{3}$$

is the required volume. $\diamond$

We shall now calculate the volume enclosed by an ellipsoid, and as a special case, the volume enclosed by a sphere. It will lead us to another important classical formula for π.

Proposition 8.5. (i) *The volume of a solid enclosed by an ellipsoid given by $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$, where $a, b, c > 0$, is equal to $4\pi abc/3$.*

(ii) *The volume of a spherical ball enclosed by the sphere given by $x^2 + y^2 + z^2 = a^2$ is equal to $4\pi a^3/3$. In other words, if B denotes a spherical ball, then*

$$\pi = \frac{3}{4}\frac{\textit{Volume of } B}{(\textit{Radius of } B)^3}.$$

(iii) *Let $a > 0$. For $\varphi \in [0, \pi]$, the volume of the (solid) spherical cone*

$$\left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2 \text{ and } 0 \leq \cos^{-1}\left(x/\sqrt{x^2 + y^2 + z^2}\right) \leq \varphi\right\}$$

is equal to $2\pi a^3(1 - \cos\varphi)/3$.

Proof. (i) The given ellipsoid lies between the planes given by $x = -a$ and $x = a$. Also, for $s \in (-a, a)$, the area $A(s)$ of its slice

$$\left\{(s, y, z) \in \mathbb{R}^3 : \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 - \frac{s^2}{a^2}\right\}$$

by the plane given by $x = s$ is the area enclosed by the ellipse

$$\frac{y^2}{b^2\big(1 - (s^2/a^2)\big)} + \frac{z^2}{c^2\big(1 - (s^2/a^2)\big)} = 1,$$

and hence by part (i) of Proposition 8.2,

$$A(s) = \pi \left(b\sqrt{1 - (s^2/a^2)}\right)\left(c\sqrt{1 - (s^2/a^2)}\right) = \pi bc\Big(1 - \frac{s^2}{a^2}\Big).$$

Thus the volume enclosed by the ellipsoid is equal to

$$\int_{-a}^{a} A(x)dx = \pi bc \int_{-a}^{a} \Big(1 - \frac{x^2}{a^2}\Big)dx = \pi bc\Big(2a - \frac{2a^3}{3a^2}\Big) = \frac{4}{3}\pi abc.$$

(ii) Letting $b = a$ and $c = a$ in (i) above, we see that the volume of the spherical ball of radius a is equal to $4\pi a^3/3$. The desired formula for π is then immediate.

(iii) If $\varphi = 0$, then the (solid) spherical cone reduces to the line segment $\{(x, 0, 0) \in \mathbb{R}^3 : 0 \le x \le a\}$, and its volume is clearly equal to 0. Also, if $\varphi = \pi/2$, then the (solid) spherical cone is easily seen to be the half spherical ball $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le a^2 \text{ and } x \ge 0\}$, and by (ii) above, its volume is equal to $2\pi a^3/3$. Now let $\varphi \in (0, \pi/2)$. For $s \in [0, a\cos\varphi]$, the slice of the (solid) spherical cone by the plane given by $x = s$ is a disk of radius $s\tan\varphi$, and so its area $A(s)$ is equal to $\pi s^2 \tan^2\varphi$, whereas for $t \in (a\cos\varphi, a]$, the slice of the (solid) spherical cone by the plane given by $x = t$ is a disk of radius $\sqrt{a^2 - t^2}$, and so its area $A(t)$ is equal to $\pi(t^2 - a^2)$. (See Figure 8.8.) Hence the volume of the (solid) spherical cone is equal to

$$\begin{aligned}
&\int_0^{a\cos\varphi} \pi x^2 \tan^2\varphi\, dx + \int_{a\cos\varphi}^{a} \pi(a^2 - x^2)dx \\
&= \pi\tan^2\varphi\frac{a^3\cos^3\varphi}{3} + \pi\Big(a^3 - \frac{a^3}{3} - a^3\cos\varphi + \frac{a^3\cos^3\varphi}{3}\Big) \\
&= \frac{\pi a^3}{3}\Big(\sin^2\varphi\cos\varphi + 2 - 3\cos\varphi + \cos^3\varphi\Big) = \frac{2\pi a^3}{3}(1 - \cos\varphi).
\end{aligned}$$

By symmetry, the formula holds for $\varphi \in (\pi/2, \pi]$ as well. This can be seen as follows. Let $\psi := \pi - \varphi$. Then $\psi \in [0, \pi/2)$, and by what we have already proved, the volume of the desired spherical cone is equal to

$$\frac{4\pi a^3}{3} - \frac{2\pi a^3}{3}(1 - \cos\psi) = \frac{2\pi a^3}{3} + \frac{2\pi a^3}{3}\cos(\pi - \varphi) = \frac{2\pi a^3}{3}(1 - \cos\varphi),$$

as before. □

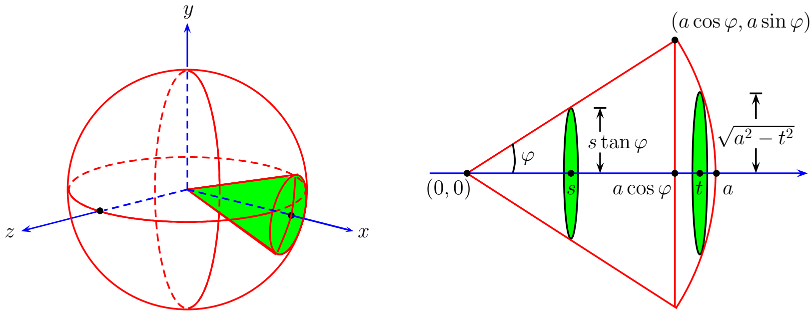


Fig. 8.8. A solid spherical cone inside a sphere and its slices by planes $x = s$, $x = t$.

The formula for π given in part (ii) of the above proposition makes it plain that the ratio of the volume of a spherical ball to the cube of its radius is independent of the radius.

Slivering by Coaxial Right Circular Cylinders

Suppose that a bounded solid D lies between two cylinders having a given line L as their common axis. A cross-section of D by a cylinder is called a **sliver** of D. Let us assume that we are able to determine the "surface area" of a sliver of D by any cylinder having L as its axis.

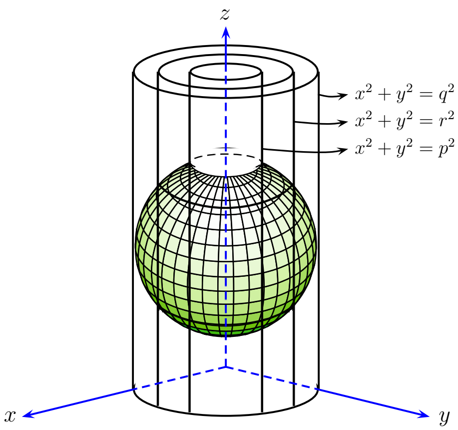


Fig. 8.9. Slivering a solid lying between the cylinders $x^2 + y^2 = p^2$ and $x^2 + y^2 = q^2$ by right coaxial cylinders $x^2 + y^2 = r^2$ for $r \in [p, q]$.

For the sake of concreteness, let the given line L be the z-axis, let $p, q \in \mathbb{R}$ with $0 \le p < q$, and assume that D lies between the cylinders given by

$x^2 + y^2 = p^2$ and $x^2 + y^2 = q^2$, and for $r \in [p, q]$, let $A(r)$ denote the surface area of the sliver

$$\{(x, y, z) \in D : x^2 + y^2 = r^2\}$$

of D obtained by intersecting it with the cylinder given by $x^2 + y^2 = r^2$. (See Figure 8.9.) If $\{r_0, r_1, \ldots, r_n\}$ is a partition of $[p, q]$, then the solid D gets divided into n subsolids

$$\left\{(x, y, z) \in D : r_{i-1} \le \sqrt{x^2 + y^2} \le r_i\right\}, \quad i = 1, \ldots, n.$$

Let us choose $s_i \in [r_{i-1}, r_i]$ and replace the ith subsolid by a cylindrical solid having volume equal to $A(s_i)(r_i - r_{i-1})$ for $i = 1, \ldots, n$. Then it is natural to consider

$$\sum_{i=1}^{n} A(s_i)(r_i - r_{i-1})$$

as an approximation of the desired volume of D. We therefore define the volume of D to be

$$\text{Vol } (D) := \int_p^q A(r)dr,$$

provided the "surface area function" $A : [p, q] \to \mathbb{R}$ is integrable.

We now address the question of finding the surface area $A(r)$ of the sliver

$$\{(x, y, z) \in D : x^2 + y^2 = r^2\}$$

of D for a fixed $r \in [p, q]$. Let

$$E_r := \{(\theta, z) \in [-\pi, \pi] \times \mathbb{R} : (r \cos \theta, r \sin \theta, z) \in D\}$$

denote the **parameter domain** for the sliver. Then the **surface area** $A(r)$ of this sliver is defined to be r times the area $B(r)$ of the planar region E_r. Thus the volume of D is equal to

$$\text{Vol } (D) = \int_p^q rB(r)dr,$$

where $B(r)$ is the planar area of the parameter domain E_r given above for each $r \in [p, q]$.

Similar considerations hold if the given line L is the y-axis and there exist $a, b \in \mathbb{R}$ with $0 \le a < b$ such that D lies between the cylinders given by $z^2 + x^2 = a^2$ and $z^2 + x^2 = b^2$, or if the given line L is the x-axis and there exist $c, d \in \mathbb{R}$ with $0 \le c < d$ such that D lies between the cylinders given by $y^2 + z^2 = c^2$ and $y^2 + z^2 = d^2$.

Examples 8.6. (i) Let $p, q, h \in \mathbb{R}$ with $0 < p < q$ and $h > 0$, and consider the cylindrical shell

$$D := \{(x, y, z) \in \mathbb{R}^3 : p \le \sqrt{x^2 + y^2} \le q \text{ and } 0 \le z \le h\}.$$

For a fixed $r \in [p, q]$, the sliver

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r^2 \text{ and } 0 \le z \le h\}$$

of D, obtained by intersecting D with the cylinder given by $x^2 + y^2 = r^2$, has the parameter domain

$$E_r := \{(\theta, z) \in \mathbb{R}^2 : -\pi \le \theta \le \pi \text{ and } 0 \le z \le h\}.$$

Since the area $B(r)$ of the rectangular region E_r is equal to

$$\big(\pi - (-\pi)\big) \cdot (h - 0) = 2\pi h$$

for each $r \in [p, q]$, we see that the volume of D is equal to

$$\int_p^q rB(r)dr = \int_p^q r(2\pi h)dr = \pi h(q^2 - p^2).$$

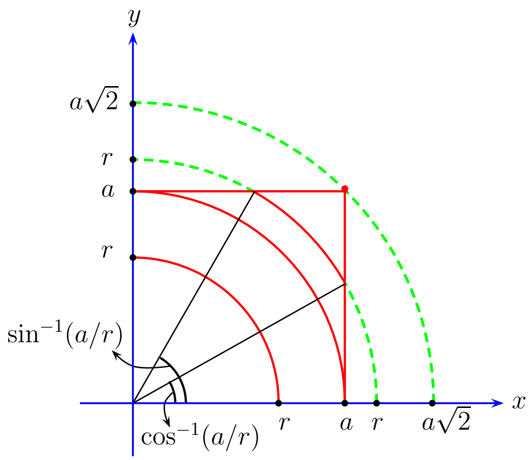


Fig. 8.10. Projections on the xy-plane of slivers of a cube by right coaxial cylinders.

(ii) Let $a \in \mathbb{R}$ with $a > 0$, and consider the cube

$$D := \{(x, y, z) \in \mathbb{R}^3 : 0 \le x, y, z \le a\}$$

of side a. It lies inside the cylinder $x^2 + y^2 = (a\sqrt{2})^2 = 2a^2$. For a fixed $0 \le r \le a\sqrt{2}$, consider the sliver

$$\big\{(x, y, z) \in \mathbb{R}^3 : 0 \le x, y, z \le a \text{ and } x^2 + y^2 = r^2\big\}$$

of D obtained by intersecting it with the cylinder given by $x^2 + y^2 = r^2$. The projections of these slivers on the xy-plane are depicted in Figure 8.10. It is clear that if $0 \le r \le a$, then the sliver is given by

$$\left\{(x,y,z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ x^2+y^2 = r^2 \text{ and } 0 \leq z \leq a\right\},$$

and its parameter domain

$$E_r := \{(\theta, z) \in \mathbb{R}^2 : \theta \in [0, \pi/2] \text{ and } 0 \leq z \leq a\}$$

has area $B(r) = (\pi/2)a = a\pi/2$. On the other hand, if $a < r \leq a\sqrt{2}$, then the sliver is given by

$$\left\{(r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 : \cos^{-1}\frac{a}{r} \leq \theta \leq \sin^{-1}\frac{a}{r} \text{ and } 0 \leq z \leq a\right\},$$

and its parameter domain

$$E_r = \left\{(\theta, z) \in \mathbb{R}^2 : \cos^{-1}\frac{a}{r} \leq \theta \leq \sin^{-1}\frac{a}{r} \text{ and } 0 \leq z \leq a\right\}$$

has area $B(r) = \left(\sin^{-1}(a/r) - \cos^{-1}(a/r)\right)a$. Thus the volume of D is equal to

$$\int_0^{a\sqrt{2}} rB(r)dr = \int_0^a r\frac{a\pi}{2}dr + \int_a^{a\sqrt{2}} r\left(\sin^{-1}\frac{a}{r} - \cos^{-1}\frac{a}{r}\right)a\,dr.$$

Substituting $r = a\csc\theta$, and then integrating by parts, we obtain

$$\begin{aligned}\int_a^{a\sqrt{2}} r\sin^{-1}\frac{a}{r}\,dr &= a^2\int_{\pi/4}^{\pi/2} \theta\csc^2\theta\cot\theta d\theta \\ &= -\frac{a^2}{2}\left(\theta\cot^2\theta\Big|_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2}\cot^2\theta d\theta\right) = \frac{a^2}{2},\end{aligned}$$

while substituting $r = a\sec\theta$, and then integrating by parts, we obtain

$$\begin{aligned}\int_a^{a\sqrt{2}} r\cos^{-1}\frac{a}{r}\,dr &= a^2\int_0^{\pi/4} \theta\sec^2\theta\tan\theta d\theta \\ &= \frac{a^2}{2}\left(\theta\tan^2\theta\Big|_0^{\pi/4} - \int_0^{\pi/4}\tan^2\theta d\theta\right) = \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right).\end{aligned}$$

Hence we can conclude that the volume of D is equal to

$$a\frac{\pi}{2}\cdot\frac{a^2}{2} + a\cdot\frac{a^2}{2} - a\cdot\frac{a^2}{2}\left(\frac{\pi}{2} - 1\right) = a^3,$$

as expected. $\diamond$

Solids of Revolution

A subset of $\mathbb{R}^3$ that can be generated by revolving a planar region about an axis is known as a **solid of revolution**. For example, the spherical ball

$\{(x, y, z) : x^2 + y^2 + z^2 \le a^2\}$ of radius a can be generated by revolving the semidisk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le a^2$ and $y \ge 0\}$ about the x-axis, or by revolving the semidisk $\{(x, y) \in \mathbb{R}^2 : x^2+y^2 \le a^2$ and $x \ge 0\}$ about the y-axis. Likewise, the cylindrical solid $\{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 \le a^2$ and $0 \le x \le h\}$ can be generated by revolving the rectangle $[0, h] \times [0, a]$ about the x-axis.

If the planar region being revolved is bounded and the axis of revolution is one of the coordinate axes, then the volume of the corresponding solid of revolution can be found using one of the definitions of volume given earlier in this section. It may be remarked that the case in which a general plane domain is revolved about an arbitrary line in its plane can be treated in a course on multivariable calculus with the help of triple integrals.

First let us consider slices of a solid of revolution by planes perpendicular to the axis of revolution. In general, each such slice is a circular "washer", or in other words, an annulus (that is, a disk from which a smaller disk with the same center is removed). If the region touches the axis of revolution at the point of slicing, then the slice is simply a disk. (See Figure 8.11.) For this reason, this method of finding the volume of a solid of revolution is known as the **washer method** or the **disk method**.

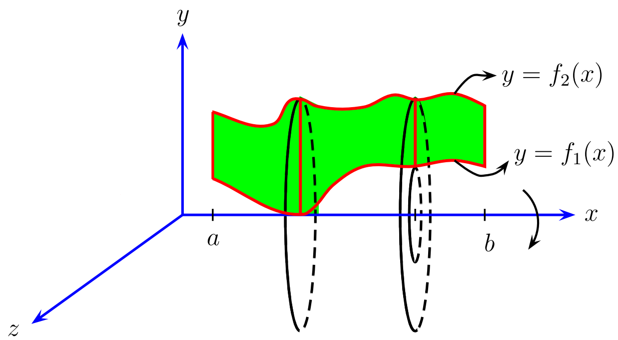


Fig. 8.11. Illustration of the washer method, or the disk method.

For the sake of concreteness, let $f_1, f_2 : [a, b] \to \mathbb{R}$ be integrable functions such that $0 \le f_1 \le f_2$, and suppose that the region between the curves given by $y = f_1(x)$, $y = f_2(x)$ and between the lines given by $x = a$, $x = b$ is revolved about the x-axis. Let D denote the corresponding solid of revolution. Then for $s \in [a, b]$, the area $A(s)$ of the annular slice of D by the plane given by $x = s$ is equal to

$$\pi f_2(s)^2 - \pi f_1(s)^2 = \pi \left(f_2(s)^2 - f_1(s)^2 \right).$$

Hence the volume of D is equal to

$$\text{Vol } (D) = \pi \int_a^b \left(f_2(x)^2 - f_1(x)^2 \right) dx.$$

Similarly, if $g_1, g_2 : [c, d] \to \mathbb{R}$ are integrable functions such that $0 \le g_1 \le g_2$, and the region between the curves given by $x = g_1(y)$, $x = g_2(y)$ and between the lines given by $y = c$, $y = d$ is revolved about the y-axis, then the volume of the solid D of revolution is equal to

$$\text{Vol }(D) = \pi \int_c^d \big(g_2(y)^2 - g_1(y)^2\big)\, dy.$$

Next, let us consider slivers of a solid of revolution by right circular cylinders whose axis is the same as the axis of revolution. In general, each such sliver is a cylindrical shell. For this reason, this method of finding the volume of a solid of revolution is known as the **shell method**. Note that if the radius of a cylindrical shell is r and its height is h, then the corresponding parameter domain is $[-\pi, \pi] \times [0, h]$. The latter has area $2\pi h$, and hence the surface area of the sliver is $r \cdot 2\pi h = 2\pi r h$.

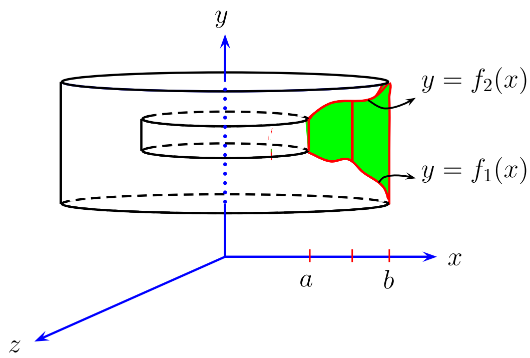


Fig. 8.12. Illustration of the shell method.

For the sake of concreteness, let $f_1, f_2 : [a, b] \to \mathbb{R}$ be integrable functions such that $f_1 \le f_2$ and assume that $a \ge 0$. Suppose the region between the curves given by $y = f_1(x)$, $y = f_2(x)$ and between the lines given by $x = a$, $x = b$ is revolved about the y-axis to generate a solid D. Consider $s \in [a, b]$ and the sliver $\{(x, y, z) \in D : z^2 + x^2 = s^2\}$ of D by the cylinder given by $z^2 + x^2 = s^2$. Its parameter domain is $E_s := [-\pi, \pi] \times [f_1(s), f_2(s)]$. Since the area of E_s is equal to $B(s) := 2\pi[f_2(s) - f_1(s)]$, we see that the area of the sliver is equal to

$$sB(s) := 2\pi s\big(f_2(s) - f_1(s)\big).$$

Hence the volume of D is equal to

$$\text{Vol }(D) = 2\pi \int_a^b x\big(f_2(x) - f_1(x)\big)dx.$$

Similarly, if $g_1, g_2 : [c,d] \to \mathbb{R}$ are integrable functions such that $g_1 \le g_2$ with $c \ge 0$, and if the region between the curves given by $x = g_1(y)$, $x = g_2(y)$ and between the lines given by $y = c$, $y = d$ is revolved about the x-axis, then the volume of the solid D of revolution is equal to

$$\text{Vol }(D) = 2\pi \int_c^d y\big(g_2(y) - g_1(y)\big)dy.$$

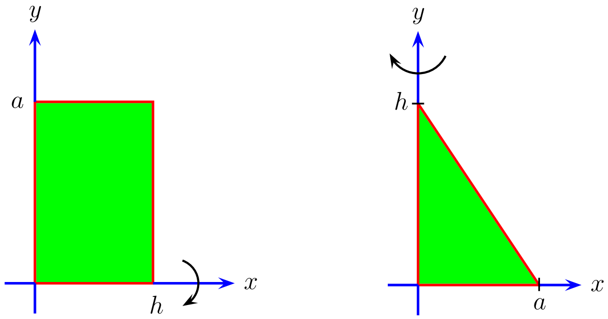


Fig. 8.13. Rectangular and triangular regions in Example 8.7 (i) and (ii).

Examples 8.7. (i) Let a and h be positive real numbers. A right circular cylindrical solid D of radius a and height h is obtained by revolving the rectangular region bounded by the lines given by $f_2(x) = a$, $f_1(x) = 0$, $x = 0$, and $x = h$ about the x-axis. (See Figure 8.13.) By the disk method, the volume of D is

$$\text{Vol }(D) = \pi \int_0^h a^2 dx = \pi a^2 h.$$

Alternatively, by the shell method,

$$\text{Vol }(D) = 2\pi \int_0^a yh\, dy = \pi a^2 h.$$

(ii) Let a and h be positive real numbers. A right circular conical solid D of radius a and height h is obtained by revolving the triangular region bounded by the lines given by $x = 0$, $y = 0$, and $(x/a) + (y/h) = 1$ about the y-axis. (See Figure 8.13.) By the disk method, the volume of D is

$$\text{Vol }(D) = \pi \int_0^h a^2 \Big(1 - \frac{y}{h}\Big)^2 dy = \pi a^2 \int_0^1 hu^2 du = \frac{1}{3}\pi a^2 h.$$

Alternatively, by the shell method,

$$\text{Vol }(D) = 2\pi \int_0^a xh\Big(1 - \frac{x}{a}\Big)dx = 2\pi h\Big(\frac{a^2}{2} - \frac{1}{a}\frac{a^3}{3}\Big) = \frac{1}{3}\pi a^2 h.$$

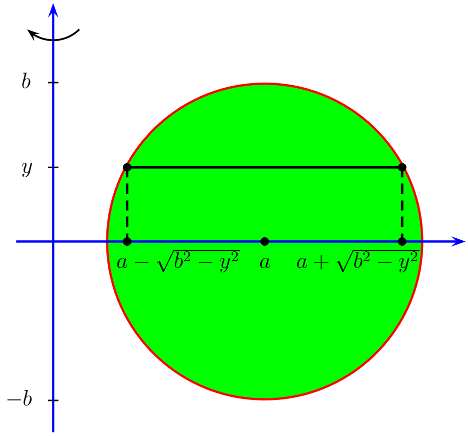


Fig. 8.14. The disk in Example 8.7 (iii) being revolved about the y-axis.

(iii) Let $a, b \in \mathbb{R}$ with $0 < b < a$. If the disk $\{(x,y) \in \mathbb{R}^2 : (x-a)^2 + y^2 \le b^2\}$ is revolved about the y-axis, we obtain a solid torus D. (See Figure 8.14.) By the washer method, the volume of D is

$$\begin{aligned}\text{Vol }(D) &= \pi \int_{-b}^{b} \left(\left(a + \sqrt{b^2-y^2}\right)^2 - \left(a - \sqrt{b^2-y^2}\right)^2 \right) dy \\ &= \pi \int_{-b}^{b} 4a\sqrt{b^2-y^2}dy = 8\pi ab^2 \int_0^1 \sqrt{1-u^2}du \\ &= 8\pi ab^2 \frac{\sin^{-1} 1}{2} = 2\pi^2 ab^2.\end{aligned}$$

(See Revision Exercise R.44 (iii) given at the end of Chapter 7.)

(iv) Let R denote the region in the first quadrant between the parabolas given by $y = x^2$ and $y = 2 - x^2$. (See Figure 8.15.) Consider the solid D_1 generated by revolving the region R about the x-axis. By the washer method, the volume of D_1 is

$$\text{Vol }(D_1) = \pi \int_0^1 \left((2-x^2)^2 - (x^2)^2\right) dx = \pi \int_0^1 \left(4 - 4x^2\right) = 4\pi\left(1 - \frac{1}{3}\right).$$

Thus Vol $(D_1) = 8\pi/3$. Alternatively, by the shell method,

$$\begin{aligned}\text{Vol }(D_1) &= 2\pi \int_0^1 y\sqrt{y}\, dy + 2\pi \int_1^2 y\sqrt{2-y}\, dy \\ &= 2\pi \frac{2}{5} + 2\pi \int_0^1 (2-u)\sqrt{u}\, du = 2\pi\left(\frac{2}{5} + \frac{4}{3} - \frac{2}{5}\right) = \frac{8\pi}{3}.\end{aligned}$$

Consider next the solid D_2 generated by revolving the region R about the y-axis. By the disk method, the volume of D_2 is

$$\text{Vol }(D_2) = \pi \int_0^1 (\sqrt{y})^2\, dy + \pi \int_1^2 \left(\sqrt{2-y}\right)^2 dy = \pi\Big(\frac{1}{2} + \frac{1}{2}\Big) = \pi.$$

Alternatively, by the shell method,

$$\text{Vol }(D_2) = 2\pi \int_0^1 x\left((2-x^2) - x^2\right) dx = 4\pi \int_0^1 \left(x - x^3\right) dx = 4\pi\Big(\frac{1}{2} - \frac{1}{4}\Big).$$

Thus, as before, Vol $(D_2) = \pi$.

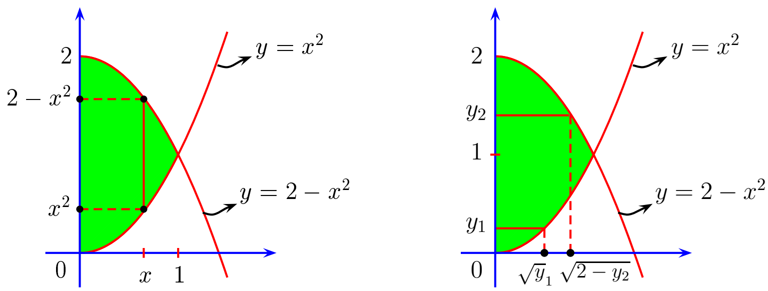


Fig. 8.15. Region in the first quadrant bounded by the parabolas $y = x^2$, $y = 2-x^2$.

It may be observed that depending on the shape of a region relative to the axis of revolution, we may decide whether the washer method or the shell method turns out to be easier than the other. In any case, since both methods must give the same answer, one of them can be used as a check on the calculations for the other. ◇

We conclude this section by mentioning again that the definitions of volumes of various kinds of solids discussed here can all be unified in a course in multivariable calculus with the help of triple integrals. (See, for example, the last four subsections of Section 6.1 of [33].) This would show that the volumes of a solid calculated using different definitions given in this section must turn out to be the same!

8.3 Arc Length of a Curve

In this section, we shall discuss how to measure the distance covered while going along a curve, that is, how to calculate the "length" of a curve. We shall base our discussion only on the *assumption* that the (Euclidean) distance between two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, which is in accordance with the Pythagorean theorem of elementary geometry. Since our treatment here is in the form of an application of Riemann integration, we shall consider only those curves whose "length" can be determined using Riemann integrals. The more general notion of a "rectifiable curve" is treated in Exercise 8.70.

Let us first consider a special situation. Suppose $x^\circ, y^\circ, a_1, a_2$ are real numbers, and a curve is given by $(\phi_1(t), \phi_2(t))$, $t \in [\alpha, \beta]$, where

$$\phi_1(t) := x^\circ + a_1 t \quad \text{and} \quad \phi_2(t) := y^\circ + a_2 t \quad \text{for } t \in [\alpha, \beta].$$

The image of this curve is the line segment from the point $(x^\circ + a_1\alpha,\ y^\circ + a_2\alpha)$ to the point $(x^\circ + a_1\beta,\ y^\circ + a_2\beta)$, and its length is

$$\sqrt{\big((x^\circ + a_1\beta) - (x^\circ + a_1\alpha)\big)^2 + \big((y^\circ + a_2\beta) - (y^\circ + a_2\alpha)\big)^2},$$

which is equal to $(\beta - \alpha)\sqrt{a_1^2 + a_2^2}$. Note that $a_1 = \phi_1'(t)$ and $a_2 = \phi_2'(t)$ for all $t \in [\alpha, \beta]$. This observation is crucial in developing the notion of the length of a curve, because any "nice" curve can be approximated locally by a line segment. To explain this, let t_0 be an interior point of an interval $[\alpha, \beta]$ and consider a curve C given by $(x(t), y(t))$, $t \in [\alpha, \beta]$, where the functions x and y are differentiable at t_0. Let

$$\phi_1(t) := x(t_0) + x'(t_0)(t - t_0) \quad \text{and} \quad \phi_2(t) := y(t_0) + y'(t_0)(t - t_0) \quad \text{for } t \in [\alpha, \beta].$$

Then by Proposition 5.11, we see that

$$x(t) - \phi_1(t) \to 0 \quad \text{and} \quad y(t) - \phi_2(t) \to 0 \quad \text{as } t \to t_0.$$

Thus the line segment given by $(\phi_1(t), \phi_2(t))$, $t \in [\alpha, \beta]$, approximates the curve C around t_0. It is therefore reasonable to expect that if $[\alpha, \beta]$ is a small interval about the point t_0, then the "length" of the curve C should be approximately equal to the length of this line segment, which is equal to

$$\sqrt{\phi_1'(t_0)^2 + \phi_2'(t_0)^2}\,(\beta - \alpha) = \sqrt{x'(t_0)^2 + y'(t_0)^2}\,(\beta - \alpha).$$

We observe that this line segment is tangent to the curve C at $(x(t_0), y(t_0))$.

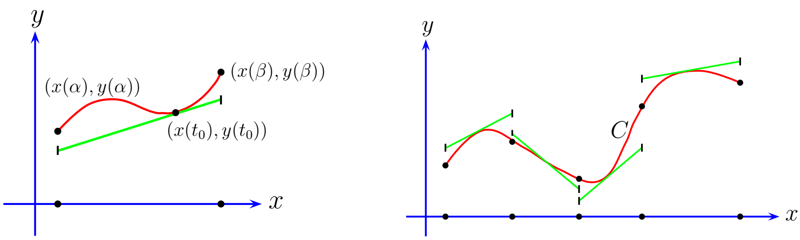


Fig. 8.16. Finding the arc length by considering the tangents to a curve.

Keeping the above motivation in mind, we proceed as follows. A parametrically defined curve C in $\mathbb{R}^2$ given by $(x(t), y(t))$, $t \in [\alpha, \beta]$, is said to be

smooth if the functions x and y are differentiable and their derivatives are continuous on $[\alpha, \beta]$. In this case, the **arc length** of C is defined to be

$$\ell(C) := \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Note that the arc length of C is well-defined, because by parts (i), (iii), and (v) of Proposition 3.2, the function $\sqrt{(x')^2 + (y')^2}$ is continuous, and hence by part (ii) of Proposition 6.10, it is integrable.

We emphasize that the arc length of a curve C is defined in terms of its given specific parametrization. The curve C should not be confused with its image $\{(x(t), y(t)) \in \mathbb{R}^2 : t \in [\alpha, \beta]\}$. For example, the curve C_1 given by $(\cos t, \sin t)$, $t \in [-\pi, \pi]$, and the curve C_2 given by $(\cos 2t, \sin 2t)$, $t \in [-\pi, \pi]$, have the same domain $[-\pi, \pi]$ and the same image $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, but they are obviously different curves, since C_1 winds around the origin $(0, 0)$ once, while C_2 winds around the origin $(0, 0)$ twice! We now show that the arc length of a curve does not change under certain "reparametrizations".

Proposition 8.8. *Let C be a smooth curve given by $(x(t), y(t))$, $t \in [\alpha, \beta]$. Suppose $\phi : [\gamma, \delta] \to \mathbb{R}$ is a differentiable function such that ϕ' is integrable, $\phi([\gamma, \delta]) = [\alpha, \beta]$, and $\phi'(u) \neq 0$ for every $u \in (\gamma, \delta)$. Let $\widetilde{C}$ denote the parametrically defined curve given by $(\tilde{x}(u), \tilde{y}(u))$, $u \in [\gamma, \delta]$, where $\tilde{x}, \tilde{y} : [\gamma, \delta] \to \mathbb{R}$ are given by $\tilde{x} := x \circ \phi$, $\tilde{y} := y \circ \phi$. Then $\widetilde{C}$ is a smooth curve, and*

$$\ell(\widetilde{C}) = \ell(C).$$

Proof. Consider the function $t \longmapsto \sqrt{x'(t)^2 + y'(t)^2}$ from $[\alpha, \beta]$ to $\mathbb{R}$. Since x' and y' are continuous on $[\alpha, \beta]$, it follows from parts (i), (iii), and (v) of Proposition 3.2 that this function is continuous (and hence integrable) on $[\alpha, \beta]$. Now Proposition 6.29 shows that

$$\begin{aligned}
\ell(C) &= \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2} dt \\
&= \int_{\gamma}^{\delta} \sqrt{x'(\phi(u))^2 + y'(\phi(u))^2}\, |\phi'(u)| du \\
&= \int_{\gamma}^{\delta} \sqrt{\tilde{x}'(u)^2 + \tilde{y}'(u)^2}\, du,
\end{aligned}$$

since $\tilde{x}'(u) = x'(\phi(u))\phi'(u)$ and $\tilde{y}'(u) = y'(\phi(u))\phi'(u)$ for all $u \in [\gamma, \delta]$ by the Chain Rule (Proposition 4.10). Thus $\ell(\widetilde{C}) = \ell(C)$. $\square$

Let us consider some important special cases of parametrically defined curves, namely curves defined by a Cartesian equation of the form $y = f(x)$ or of the form $x = g(y)$, and curves defined by a polar equation of the form $r = p(\theta)$ or of the form $\theta = \alpha(r)$.

First, let $a, b \in \mathbb{R}$ with $a < b$, $f : [a, b] \to \mathbb{R}$, and let a smooth curve C be given by $y = f(x)$, $x \in [a, b]$. Then the arc length of C is equal to

$$\ell(C) = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

This follows by considering the Cartesian coordinate x as a parameter and $[a, b]$ as the parameter interval. Similarly, if $c, d \in \mathbb{R}$ with $c < d$, $g : [c, d] \to \mathbb{R}$, and a smooth curve C is given by $x = g(y)$, $y \in [c, d]$, then the arc length of C is equal to

$$\ell(C) = \int_c^d \sqrt{1 + g'(y)^2}\, dy.$$

Next, let $\alpha, \beta \in \mathbb{R}$, $p : [\alpha, \beta] \to [0, \infty)$, and let a smooth curve C be given by $r = p(\theta)$, $\theta \in [\alpha, \beta]$. Then the arc length of C is equal to

$$\ell(C) = \int_\alpha^\beta \sqrt{p(\theta)^2 + p'(\theta)^2}\, d\theta.$$

This follows by considering the polar coordinate θ as a parameter with $[\alpha, \beta]$ as the parameter interval, so that C is parametrically defined by

$$x(\theta) = p(\theta)\cos\theta \quad \text{and} \quad y(\theta) = p(\theta)\sin\theta,$$

which show that for all $\theta \in [\alpha, \beta]$,

$$\begin{aligned} x'(\theta)^2 + y'(\theta)^2 &= \big(p'(\theta)\cos\theta - p(\theta)\sin\theta\big)^2 + \big(p'(\theta)\sin\theta + p(\theta)\cos\theta\big)^2 \\ &= p(\theta)^2 + p'(\theta)^2. \end{aligned}$$

Arc length of a curve given by a polar equation of the form $\theta = \alpha(r)$ is treated in Exercises 8.31 and 8.32.

Proposition 8.9. (i) *For $\varphi \in [0, \pi]$, the length of the arc of a circle given by $x := a\cos t$, $y := a \sin t$, $0 \le t \le \varphi$ (which subtends an angle φ at the center), is equal to $a\varphi$.*

(ii) *The perimeter of the circle given by $x^2 + y^2 = a^2$ is equal to $2\pi a$. In other words, if C denotes a circle, then*

$$\pi = \frac{1}{2}\,\frac{\textit{Perimeter of } C}{\textit{Radius of } C}.$$

Proof. (i) The circular arc is given by the polar equation $r = p(\theta)$, where $p(\theta) := a$ for all $\theta \in [0, \varphi]$. Hence the length of the arc is equal to

$$\int_0^\varphi \sqrt{a^2 + 0^2}\, d\theta = a\varphi.$$

(ii) The perimeter of a circle is twice the arc length of its semicircle. Letting $\varphi = \pi$ in (i) above, we see that the perimeter of the circle is equal to $2\pi a$. The desired formula for π is then immediate. $\square$

The formula for π given in part (ii) of the above proposition makes it plain that the ratio of the perimeter of a circle to its diameter is independent of the radius. This fact is usually taken for granted when π is introduced in high-school geometry.

Part (i) of the above proposition says that the length of an arc of a semicircle is equal to the radius of the circle times the angle between 0 and π (in radian measure) that the arc subtends at the center. This explains the dictionary meaning of the word “radian”, namely an angle subtended at the center by an arc whose length is equal to the radius. Thus if the radius of a circle is 1, then the length of an arc of its semicircle is equal to the angle the arc subtends at the center. This also explains the use of the name “arctangent” of the function whose inverse is the function $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$. Indeed, $y = \arctan x$ for $x \in (0, \infty)$ if y is the length of an arc of the unit circle subtending an angle at the center whose tangent is x. For example, $\pi/3 = \arctan \sqrt{3}$ means that an arc of length $\pi/3$ of the unit circle subtends an angle θ at the center such that $\tan \theta = \sqrt{3}$.

Before considering some illustrative examples, we remark that the notion of the length of a smooth curve can be extended to slightly more general curves as follows. A parametrically defined curve C in $\mathbb{R}^2$ given by $(x(t), y(t))$, $t \in [\alpha, \beta]$, is said to be **piecewise smooth** if the functions x and y are continuous on $[\alpha, \beta]$ and if there is a finite number of points $\gamma_0 < \gamma_1 < \cdots < \gamma_n$ in $[\alpha, \beta]$, with $\gamma_0 = \alpha$ and $\gamma_n = \beta$, such that for each $i = 1, \ldots, n$, the curve given by $(x(t), y(t))$, $t \in [\gamma_{i-1}, \gamma_i]$, is smooth. If the curve C is piecewise smooth, then the **length** of C is defined to be

$$\ell(C) := \sum_{i=1}^{n} \int_{\gamma_{i-1}}^{\gamma_i} \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

In view of Propositions 6.8 and 6.13, we may write

$$\ell(C) := \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2}\, dt \quad \text{if } C \text{ is piecewise smooth.}$$

For example, if $x(t) := t$ and $y(t) := |t|$ for $t \in [-1, 1]$, then the curve given by $(x(t), y(t))$, $t \in [-1, 1]$, is piecewise smooth.

For a parametrically defined curve C in $\mathbb{R}^3$ given by $(x(t), y(t), z(t))$, $t \in [\alpha, \beta]$, we may define the concepts of “smoothness” and “piecewise smoothness” analogously, and if C is a piecewise smooth curve, the **arc length** of C is defined to be

$$\ell(C) := \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

Examples 8.10. (i) Let $m, c \in \mathbb{R}$ and consider the line segment given by $y = mx + c$, $x \in [0, 1]$, from the point $(0, c)$ to the point $(1, m + c)$. Its length is equal to

$$\int_0^1 \sqrt{1+m^2}\,dx = \sqrt{1+m^2},$$

which is equal to the distance between the points $(0,c)$ and $(1,m+c)$.

(ii) Let $a \in \mathbb{R}$ and consider the parabolic curve given by $y = ax^2$, $x \in [0,1]$. Its arc length is equal to

$$\int_0^1 \sqrt{1+(2ax)^2}\,dx = \frac{1}{2a}\int_0^{2a} \sqrt{1+u^2}du$$
$$= \frac{1}{2}\sqrt{1+4a^2} + \frac{1}{4a}\ln\left(2a+\sqrt{1+4a^2}\right).$$

(See Revision Exercise R.44 (ii) given at the end of Chapter 7.)

(iii) Consider the curve given by $y = (2x^6+1)/8x^2$, $x \in [1,2]$. Its arc length is equal to

$$\int_1^2 \sqrt{1+\left(x^3-\frac{1}{4x^3}\right)^2}dx = \int_1^2 \left(x^3+\frac{1}{4x^3}\right)dx = \frac{123}{32}.$$

(iv) Let $a \in \mathbb{R}$ with $a > 0$, and consider the upper half of the cardioid given by $r = a(1+\cos\theta)$, $\theta \in [0,\pi]$. Its arc length is equal to

$$\int_0^\pi \sqrt{a^2(1+\cos\theta)^2 + a^2(-\sin\theta)^2}\,d\theta = \int_0^\pi \sqrt{2a^2(1+\cos\theta)}\,d\theta$$
$$= 2a\int_0^\pi \cos(\theta/2)d\theta = 4a.$$

(v) Consider a **helix** in $\mathbb{R}^3$ given by the parametric equations

$$x(t) = a\cos t, \quad y(t) = a\sin t, \quad \text{and} \quad z(t) = bt + c, \quad t \in \mathbb{R},$$

where $a, b, c \in \mathbb{R}$ with $a > 0$ and $b \neq 0$. It lies on the cylinder given by $x^2+y^2=a^2$. See Figure 8.17.) For $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, let C denote a part of the helix given by $(x(t), y(t), z(t))$, $t \in [\alpha, \beta]$. Then

$$\ell(C) = \int_\alpha^\beta \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}\,dt = (\beta-\alpha)\sqrt{a^2+b^2}.$$

Consider points $P_1 := (x_1, y_1, z_1)$ and $P_2 := (x_2, y_2, z_2)$ on the cylinder given by $x^2+y^2=a^2$. Then $x_1^2+y_1^2 = a^2 = x_2^2+y_2^2$. Let us first assume that $(x_1,y_1) \neq (x_2,y_2)$, that is, P_1 does not lie vertically above or below P_2, and also that $z_1 \neq z_2$, that is, P_1 and P_2 do not lie in a plane parallel to the xy-plane. If (a,θ_1) and (a,θ_2) denote the polar coordinates of (x_1,y_1) and (x_2,y_2) respectively, then $\theta_1 \neq \theta_2$, since $(x_1,y_1) \neq (x_2,y_2)$, and the helix given by the equations

$$x(t) = a\cos t,\ y(t) = a\sin t,\ z(t) = \frac{z_2-z_1}{\theta_2-\theta_1}(t-\theta_1)+z_1, \quad t \in \mathbb{R},$$

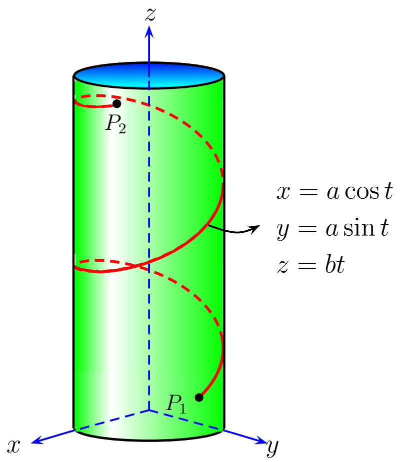


Fig. 8.17. A helix lying on the cylinder $x^2 + y^2 = a^2$.

lies on the cylinder and passes through the points P_1 and P_2. We may assume that $\theta_1 < \theta_2$ without loss of generality. Letting $\alpha = \theta_1$ and $\beta = \theta_2$, it follows from what we have seen above that the arc length of the part of this helix from P_1 to P_2 is equal to

$$(\theta_2 - \theta_1)\sqrt{a^2 + \frac{(z_2 - z_1)^2}{(\theta_2 - \theta_1)^2}} = \sqrt{a^2(\theta_2 - \theta_1)^2 + (z_2 - z_1)^2}.$$

If we slit the cylinder vertically along a straight line parallel to the z-axis and open it up, then the points on the cylinder may be represented by the strip $S := \{(s, z) \in \mathbb{R}^2 : -a\pi < s \le a\pi\}$. In fact, a point $P = (x, y, z)$ on the cylinder corresponds to the point $Q := (a\theta, z)$ in S, where (a, θ) are the polar coordinates of (x, y). Let points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ on the cylinder correspond to points $Q_1 := (a\theta_1, z_1)$ and $Q_2 := (a\theta_2, z_2)$ in S respectively. Then the part of the above-mentioned helix from P_1 to P_2 corresponds to the line segment from Q_1 to Q_2 in S. We note that the (Euclidean) distance between Q_1 and Q_2 is the same as the arc length of the part of this helix from P_1 to P_2. This is expressed by saying that if points P_1 and P_2 on a cylinder are neither one above the other nor at the same height, then the **geodesic**, that is, the shortest path, on the cylinder from P_1 to P_2 is a helix.

It is clear that if $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ lie one above the other, that is, if $(x_1, y_1) = (x_2, y_2)$, then the geodesic on the cylinder from P_1 to P_2 is the line segment given by

$$x(t) = x_1, \quad y(t) = y_1, \quad z(t) = (z_2 - z_1)t + z_1, \quad t \in [0, 1].$$

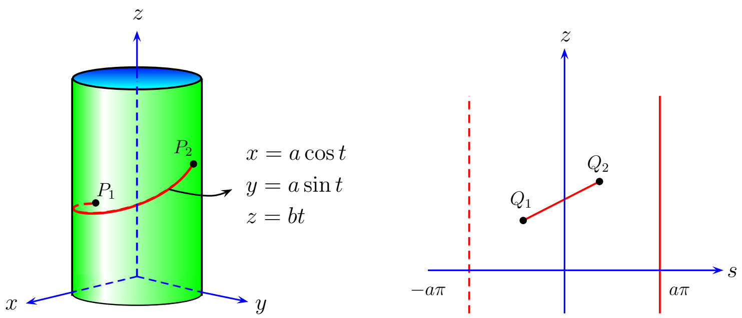


Fig. 8.18. Helix as a geodesic on the cylinder and the corresponding line segment when the cylinder is slit vertically and opened up.

Also, it can be argued that if P_1 and P_2 are at the same height, that is, if $z_1 = z_2$, then the geodesic on the cylinder from P_1 to P_2 is the circular arc given by $x(t) = a\cos t$, $y(t) = a\sin t, z(t) = z_1$, $t \in [\theta_1, \theta_2]$.
Likewise, if P_1 and P_2 are points on a sphere, then the geodesic on the sphere from P_1 to P_2 is an arc of the great circle passing through them. (A **great circle** is the intersection of the sphere with a plane passing through the center of the sphere.) To see this, one may rotate the sphere, if necessary, and assume that the great circle passing through P_1 and P_2 is in fact the equator of the sphere. $\diamond$

Remark 8.11. Let us recall how we found the area of a circular disk of radius a in Section 8.1. We first found the area enclosed by the ellipse given by the equation $(x^2/a^2) + (y^2/b^2) = 1$, and then considered the special case $b = a$, which corresponds to a circle of radius a. Let us try to adopt a similar procedure and find the arc length of an ellipse. Consider an ellipse C parametrically given by $x(t) = a\cos t$, $y(t) = b\sin t$, $t \in [-\pi, \pi]$. Then

$$\ell(C) = \int_{-\pi}^{\pi} \sqrt{(-a\sin t)^2 + (b\cos t)^2}\, dt = 2\int_0^{\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt.$$

If $b = a$, then we easily get $\ell(C) = 2a\pi$, which gives the perimeter of a circle of radius a. On the other hand, if $b \neq a$, then the above integral cannot be evaluated in terms of known functions.

A similar situation occurs if we attempt to calculate the arc length of a lemniscate C given by the Cartesian equation $(x^2 + y^2)^2 = x^2 - y^2$, or by the polar equation $r^2 = \cos 2\theta$. Considering its parametric equations $x(t) := \left(\cos t\sqrt{1+\cos^2 t}\right)/\sqrt{2}$, $y(t) := (\cos t \sin t)/\sqrt{2}$, $t \in [-\pi, \pi]$, we obtain

$$\ell(C) = 2\int_0^{\pi} \frac{1}{\sqrt{1+\cos^2 t}} dt := 2\varpi, \text{ say.}$$

A special case of a formula of Gauss says that π/ϖ is equal to the arithmetic-geometric mean of $\sqrt{2}$ and 1. (See Exercise 2.12 for the definition.) A simple proof of Gauss's formula is given in [67].) In the study of lemniscates, the number ϖ plays a role very similar to the role played by the number π in the study of circles. Notice, for example, that just as the length of a circle given by $x^2 + y^2 = 1$ is 2π, the length of the lemniscate C is 2ϖ. ◇

8.4 Area of a Surface of Revolution

A surface of revolution is generated when a curve is revolved about a line. In this section, we shall define the area of such a surface and calculate it in several special cases. It may be remarked that the concept of the area of a general surface is usually developed in a course on multivariable calculus with the help of double integrals. (See, for example, [33, p. 314].) The definition of surface area for surfaces of revolution given in this section can be reconciled with the general concept.

Let C be a parametrically defined curve in $\mathbb{R}^2$ given by $(x(t), y(t))$, $t \in [\alpha, \beta]$, and let L be a line in $\mathbb{R}^2$ given by the equation $ax+by+c = 0$, where $a, b, c \in \mathbb{R}$ and not both a and b are equal to zero. Let us begin by considering the curve C to be a line segment P_1P_2 with endpoints $P_1 := (x_1, y_1)$ and $P_2 := (x_2, y_2)$. Thus C is parametrically given by $x(t) := (x_2-x_1)t+x_1$, $y(t) := (y_2-y_1)t+y_1$, $t \in [0, 1]$. Let us assume that the line segment P_1P_2 does not cross the line L. Further, let d_1 and d_2 denote the distances of P_1 and P_2 from L, and let λ denote the length of the line segment P_1P_2. We show that the "area" of the surface generated by revolving P_1P_2 about L is equal to

$$\pi(d_1 + d_2)\lambda.$$

To this end, first note that if $P_1P_2 \perp L$, then $\lambda = |d_1 - d_2|$, and the surface of revolution is a circular washer with radii d_1 and d_2. (See the picture on the left in Figure 8.19.) Thus its area is equal to

$$|\pi d_1^2 - \pi d_2^2| = \pi(d_1 + d_2)|d_1 - d_2| = \pi(d_1 + d_2)\lambda.$$

Next, if $P_1P_2 \parallel L$, then $d_1 = d_2$. Write $d := d_1 = d_2$. In this case, the surface of revolution is a right circular cylinder with radius d and length λ. (See the picture on the right in Figure 8.19.) If we slit open this cylinder along a straight line parallel to its axis, we obtain a rectangle of sides $2\pi d$ and λ. Hence its area is equal to

$$2\pi d\lambda = \pi(d_1 + d_2)\lambda.$$

Assume now that $P_1P_2 \not\perp L$ and $P_1P_2 \nparallel L$. Then the surface of revolution is a frustum (that is, a piece) of a right circular cone with base radii d_1 and d_2, and slant height λ. In order to find the area of this frustum, let us first find the area of a right circular cone with base radius d and slant height ℓ.

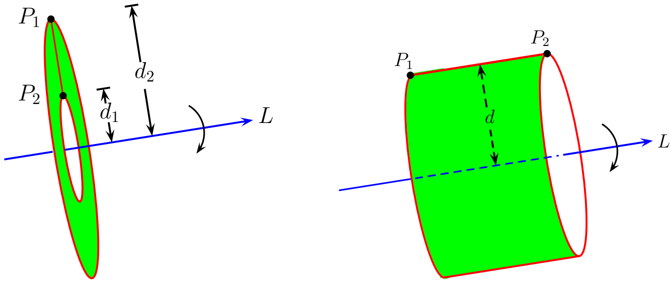


Fig. 8.19. Revolving the segment P_1P_2 about L when $P_1P_2 \perp L$ and when $P_1P_2 \parallel L$.

If we slit open the cone along a straight line from its vertex to a point in its base, we obtain a sector of a disk of radius ℓ such that the length of its arc is equal to $\ell\,\varphi = 2\pi d$. (See the picture on the left in Figure 8.20.) By part (iii) of Proposition 8.2, the area of this sector is equal to

$$\frac{1}{2}\ell^2\varphi = \frac{1}{2}\ell^2\left(\frac{2\pi d}{\ell}\right) = \pi\ell d,$$

which is therefore the surface area of a right circular cone with base radius d and slant height ℓ.

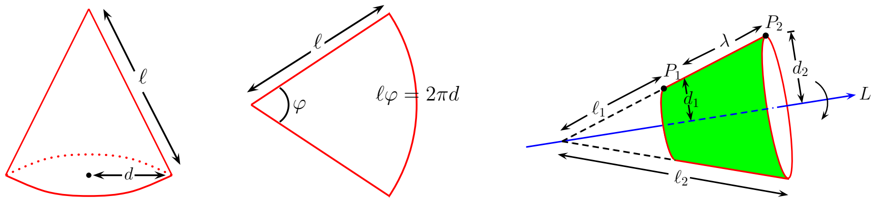


Fig. 8.20. A right circular cone, the sector of a disk obtained by slitting it open, and the frustum of the cone.

To find the surface area of the frustum of the right circular cone with base radii d_1 and d_2, and slant height λ, we may assume without loss of generality that $d_1 < d_2$. Then this frustum is obtained by removing from a cone of radius d_2 and slant height ℓ_2 a smaller cone of radius d_1 and slant height ℓ_1, where $\ell_1, \ell_2 \in \mathbb{R}$ satisfy $\ell_2 > \ell_1 > 0$ and $\ell_2 - \ell_1 = \lambda$. See the picture on the right in Figure 8.20.) Using similarity of triangles, $d_1\ell_2 = d_2\ell_1$. Hence the area of the surface of the frustum is equal to

$$\pi d_2\ell_2 - \pi d_1\ell_1 = \pi(d_2\ell_2 - d_2\ell_1 + d_1\ell_2 - d_1\ell_1) = \pi(d_1+d_2)(\ell_2-\ell_1) = \pi(d_1+d_2)\lambda,$$

as desired.

Consider now the general case in which C is a parametrically defined curve given by $(x(t), y(t))$, $t \in [\alpha, \beta]$, and let $\{t_0, t_1, \ldots, t_n\}$ be a partition of $[\alpha, \beta]$. Let us replace the piece $(x(t), y(t))$, $t \in [t_{i-1}, t_i]$, of the curve C by the line segment $P_{i-1}P_i$ for $i = 1, \ldots, n$, where $P_i := (x(t_i), y(t_i))$. (See Figure 8.21.) Then the sum of the areas of the frustums of the cones generated by these line segments is equal to

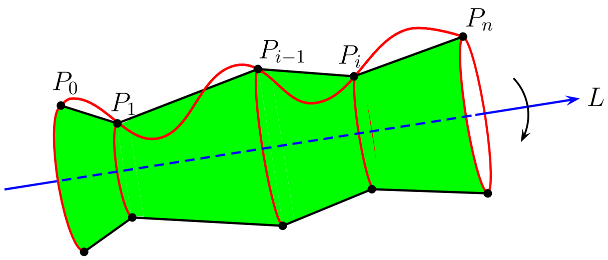


Fig. 8.21. A surface of revolution approximated by frustums of right circular cones.

$$\sum_{i=1}^{n} \pi(d_{i-1} + d_i)\lambda_i,$$

where d_i is the distance of P_i from the line L for $i = 0, 1, \ldots, n$ and λ_i is the length of the line segment $P_{i-1}P_i$ for $i = 1, \ldots, n$. This sum can be considered an approximation of the conceived area of the surface obtained by revolving the curve C about the line L. Now $d_i = |ax(t_i) + by(t_i) + c|/\sqrt{a^2 + b^2}$ for $i = 0, 1, \ldots, n$, whereas

$$\lambda_i = \sqrt{\big(x(t_i) - x(t_{i-1})\big)^2 + \big(y(t_i) - y(t_{i-1})\big)^2} \quad \text{for } i = 1, \ldots, n.$$

If the functions x and y are continuous on $[t_{i-1}, t_i]$ and differentiable on (t_{i-1}, t_i), then by the MVT, there exist $s_i, u_i \in (t_{i-1}, t_i)$ such that

$$\lambda_i = \sqrt{x'(s_i)^2 + y'(u_i)^2}\,(t_i - t_{i-1}) \quad \text{for } i = 1, \ldots, n,$$

and hence the sums $\sum_{i=1}^{n} d_{i-1}\lambda_i$ and $\sum_{i=1}^{n} d_i\lambda_i$ may be considered approximations of the integral

$$\int_{\alpha}^{\beta} \frac{|ax(t) + by(t) + c|}{\sqrt{a^2 + b^2}} \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

These considerations lead to the following definition of the area of a surface of revolution.

Let C be a piecewise smooth curve given by $(x(t), y(t))$, $t \in [\alpha, \beta]$. Consider a line L given by $ax + by + c = 0$, where not both a and b are equal to zero.

Assume that the line L does not cross the curve C. For $t \in [\alpha, \beta]$, let $\rho(t)$ denote the distance of the point $(x(t), y(t))$ on C from the line L, that is,

$$\rho(t) = \frac{|ax(t) + by(t) + c|}{\sqrt{a^2 + b^2}}.$$

Then the area of the surface S of revolution obtained by revolving the curve C about the line L is defined to be

$$\text{Area } (S) := 2\pi \int_\alpha^\beta \rho(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

It may be observed that since the line L does not cross the curve C, either $ax(t) + by(t) + c \geq 0$ for all $t \in [\alpha, \beta]$, or $ax(t) + by(t) + c \leq 0$ for all $t \in [\alpha, \beta]$. We consider some important special cases.

First, if a piecewise smooth curve C given by $y = f(x)$, $x \in [a, b]$, where $f \geq 0$ on $[a, b]$ or $f \leq 0$ on $[a, b]$, is revolved about the x-axis, then the area of the surface S of revolution so generated is equal to

$$\text{Area } (S) = 2\pi \int_a^b |f(x)|\sqrt{1 + f'(x)^2}\, dx.$$

Similarly, if a piecewise smooth curve C given by $x = g(y)$, $y \in [c, d]$, where $g(y) \geq 0$ for all $y \in [c, d]$ or $g(y) \leq 0$ for all $y \in [c, d]$, is revolved about the y-axis, then the area of the surface S of revolution so generated is equal to

$$\text{Area } (S) = 2\pi \int_c^d |g(y)|\sqrt{1 + g'(y)^2}\, dy.$$

Next, let a piecewise smooth curve C be given by $r = p(\theta)$, $\theta \in [\alpha, \beta]$, where $p \geq 0$ on $[\alpha, \beta]$. If L denotes a line through the origin containing a ray given by $\theta = \gamma$, where $\gamma \in (-\pi, \pi]$, and not crossing the curve C, then the area of the surface S obtained by revolving the curve C about the line L is equal to

$$\text{Area } (S) = 2\pi \int_\alpha^\beta p(\theta)|\sin(\theta - \gamma)|\sqrt{p(\theta)^2 + p'(\theta)^2}\, d\theta\, .$$

This follows by considering the polar coordinate θ as a parameter and noting, as in Section 8.3, that $x'(\theta)^2 + y'(\theta)^2 = p(\theta)^2 + p'(\theta)^2$, $\theta \in [\alpha, \beta]$, and also that the distance of the point $(p(\theta)\cos\theta, p(\theta)\sin\theta)$ from the line L is equal to $p(\theta)|\sin(\theta - \gamma)|$ for all $\theta \in [\alpha, \beta]$. (See Figure 8.22.)

Area of a surface generated by revolving a curve given by a polar equation of the form $\theta = \alpha(r)$ is treated in Exercises 8.41 and 8.42.

Proposition 8.12. (i) *Let $\varphi \in [0, \pi]$ and let C denote the arc of a circle given by $x := a\cos t$, $y := a\sin t$, $0 \leq t \leq \varphi$ (which subtends an angle φ at the*

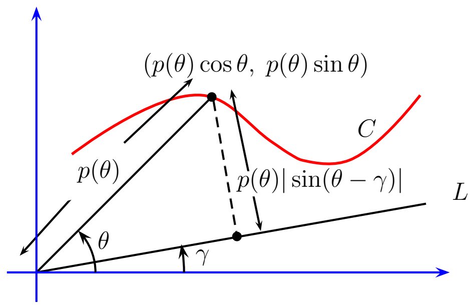


Fig. 8.22. Revolving a polar curve C about the line making an angle γ with the positive x-axis.

center). Then the surface area of the spherical cap generated by revolving C about the x-axis is equal to $2\pi a^2(1-\cos\varphi)$.

(ii) *The surface area of a sphere given by $x^2+y^2+z^2=a^2$ is equal to $4\pi a^2$. In other words, if S denotes this sphere, then*

$$\pi = \frac{1}{4}\frac{\textit{Surface Area of } S}{(\textit{Radius of } S)^2}.$$

Proof. (i) The arc C is given by the polar equation $r=p(\theta)$, where $p(\theta):=a$ for $\theta\in[0,\varphi]$. If it is revolved about the x-axis, then the area of the spherical cap so generated is equal to

$$2\pi\int_0^{\varphi} a|\sin\theta|\sqrt{a^2+0^2}\,d\theta = 2\pi a^2\int_0^{\varphi}\sin\theta\,d\theta = 2\pi a^2(1-\cos\varphi).$$

(ii) A sphere of radius a is obtained by revolving a semicircle of radius a about the line containing its diameter. Letting $\varphi=\pi$ in the formula obtained in (i) above, we see that the surface area of a sphere of radius a is equal to $2\pi a^2(1-(-1))=4\pi a^2$. □

The formula for π given in part (ii) of the above proposition makes it plain that the ratio of the surface area of a sphere to the square of its radius is independent of the radius.

Remark 8.13. Let S be a surface lying on a sphere of radius a. Then S is said to subtend a **solid angle** Θ at the center of the sphere, where Θ is equal to the "surface area" of S divided by a^2. For example, let $\varphi\in[0,\pi]$ and consider the arc of the circle of radius a given by $x:=a\cos t$, $y:=a\sin t$, $0\le t\le\varphi$, which subtends an angle φ at the center of the circle. Then by part (i) of the above proposition, the spherical cap generated by revolving this arc about the x-axis subtends a solid angle $\Theta_\varphi=2\pi(1-\cos\varphi)$ at the center of the sphere. By part (iii) of Proposition 8.5, the volume of the corresponding (solid) spherical cone is equal to $2\pi a^3(1-\cos\varphi)/3=a^3\Theta_\varphi/3$.

In particular, letting $\varphi = \pi/2$ and $\varphi = \pi$ in part (i) of the above proposition, we see that a hemisphere subtends a solid angle 2π at the center and the entire sphere subtends a solid angle 4π at the center. (Note that the volume of the solid enclosed by the entire sphere of radius a, that is, of the ball of radius a, is equal to $4\pi a^3/3$, in conformity with part (ii) of Proposition 8.5.) The standard unit of a solid angle is known as **steradian**. Hence the largest solid angle is of 4π steradians, that is, approximately 12.566 steradians. $\diamond$

Examples 8.14. (i) Consider the line segment given by $(x/a) + (y/h) = 1$, $x \in [0, a]$, where $a, h > 0$. The surface area of the cone S of radius a and height h generated by revolving this line segment about the y-axis is

$$\text{Area }(S) = 2\pi \int_0^h a\left(1 - \frac{y}{h}\right)\sqrt{1 + \left(\frac{a}{h}\right)^2}\,dy = 2\pi a \frac{\sqrt{a^2 + h^2}}{h}\left(h - \frac{h}{2}\right),$$

which is equal to $\pi a\sqrt{a^2 + h^2}$, as expected.

(ii) Consider the spheroid S generated by revolving the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, where $a, b > 0$, $a \neq b$, about the x-axis. It is given by the Cartesian equation $(x^2/a^2)+(y^2/b^2)+(z^2/b^2) = 1$. To find its surface area, it suffices to consider the curve C given by $y = (b/a)\sqrt{a^2 - x^2}$, $x \in [-a, a]$. We obtain

$$\begin{aligned}\text{Area }(S) &= 2\pi \int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\sqrt{1 + \frac{b^2x^2}{a^2(a^2 - x^2)}}\,dx \\ &= \frac{2\pi b}{a}\int_{-a}^{a}\sqrt{a^2 + \frac{(b^2 - a^2)x^2}{a^2}}\,dx.\end{aligned}$$

First consider the case $a < b$. If $c := \sqrt{b^2 - a^2}/a$, then $c > 0$ and

$$\begin{aligned}\text{Area }(S) &= \frac{2\pi b}{a}\cdot 2a\int_0^a \sqrt{1 + \left(\frac{cx}{a}\right)^2}\,dx = \frac{4\pi ab}{c}\int_0^c \sqrt{1 + t^2}\,dt \\ &= \frac{2\pi ab}{c}\left(c\sqrt{1 + c^2} + \ln\left(c + \sqrt{1 + c^2}\right)\right).\end{aligned}$$

(See Revision Exercise R.44 (ii) given at the end of Chapter 7.) Next, consider the case $a > b$. If $c := \sqrt{a^2 - b^2}/a$, then $0 < c < 1$ and

$$\begin{aligned}\text{Area }(S) &= \frac{2\pi b}{a}\cdot 2a\int_0^a \sqrt{1 - \left(\frac{cx}{a}\right)^2}\,dx = \frac{4\pi ab}{c}\int_0^c \sqrt{1 - t^2}\,dt \\ &= \frac{2\pi ab}{c}\left(c\sqrt{1 - c^2} + \sin^{-1} c\right).\end{aligned}$$

(See Revision Exercise R.44 (iii) given at the end of Chapter 7.) In both cases, if $b \to a$, then L'Hôpital's Rule for $\frac{0}{0}$ indeterminate forms (Proposition 4.39) shows that the surface area tends to $4\pi a^2$, which is the surface area of a sphere of radius a, as seen in part (ii) of Proposition 8.12.

We remark that the surface of the ellipsoid given by $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$, where a, b, c are distinct positive numbers, is not a surface of revolution. In fact, the calculation of the surface area of such an ellipsoid involves the so-called elliptic integrals.

(iii) Consider the torus S obtained by revolving the circle given by $(x-a)^2 + y^2 = b^2$, where $0 < b < a$, about the y-axis. To find its area, we use the parametric equations $x(t) := a + b\cos t$, $y(t) := b\sin t, t \in [-\pi, \pi]$. Hence

$$\begin{aligned}\text{Area }(S) &= 2\pi \int_{-\pi}^{\pi} (a + b\cos t)\sqrt{(-b\sin t)^2 + (b\cos t)^2}\,dt \\ &= 2\pi b \int_{-\pi}^{\pi} (a + b\cos t)dt = 4\pi^2 ab\end{aligned}$$

is the required area. ◇

We conclude this section by mentioning again that the definition of surface area of a surface of revolution discussed here is a special case of the general concept of surface area that is usually defined with the help of double integrals or surface integrals. (See, for example, [33, Prop. 6.16] or [65, § 6.3, 6.4].)

8.5 Centroids

Before introducing the concept of a centroid of a geometric object, let us consider the notion of the average of a function. For this purpose, we recall that given $n \in \mathbb{N}$ and a function $f : \{1, \ldots, n\} \to \mathbb{R}$, the average of the values of f at the points $1, \ldots, n$, that is, the **average** of f, is given by

$$\mathrm{Av}(f) := \frac{f(1) + \cdots + f(n)}{n}.$$

In general, there can be repetition among the values $f(1), \ldots, f(n)$ of f. If $y_1, \ldots, y_k$ are the distinct values of f, and if for each $j = 1, \ldots, k$, the function f assumes the value y_j at a total number of n_j points, that is, the set $\{i \in \mathbb{N} : 1 \le i \le n \text{ and } f(i) = y_j\}$ has n_j elements, then $n_1 + \cdots + n_k = n$ and we can also write

$$\mathrm{Av}(f) = \frac{n_1 y_1 + \cdots + n_k y_k}{n_1 + \cdots + n_k}.$$

Simple examples show that $\mathrm{Av}(f)$ need not be any of the values of f.

Next, consider a function $f : \mathbb{N} \to \mathbb{R}$. How can we define the average of f? One possibility is

$$\mathrm{Av}(f) := \lim_{n\to\infty} \frac{f(1) + \cdots + f(n)}{n},$$

if the limit exists. Note that by Proposition 2.15, if the sequence $(f(n))$ is convergent, then this limit exists and is equal to $\lim_{n\to\infty} f(n)$.

Let us now consider a closed interval $[a, b]$ in $\mathbb{R}$ and a real-valued function $f : [a, b] \to \mathbb{R}$. How should we define the average of f? Suppose $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ and we choose $s_i \in (x_{i-1}, x_i)$ for $i = 1, \ldots, n$. If f were to assume the value $f(s_i)$ at every point of the subinterval (x_{i-1}, x_i), then we could define

$$\mathrm{Av}(f) := \frac{f(s_1)(x_1 - x_0) + \cdots + f(s_n)(x_n - x_{n-1})}{(x_1 - x_0) + \cdots + (x_n - x_{n-1})} = \frac{1}{b-a}\sum_{i=1}^{n} f(s_i)(x_i - x_{i-1})$$

in analogy with the discrete case considered earlier.

Now assume that f is integrable on $[a, b]$. In view of the result of Darboux about Riemann sums (Proposition 6.36), we define the **average** of f by

$$\mathrm{Av}(f) := \frac{1}{b-a}\int_a^b f(x)dx.$$

As in the case of a function defined on $\{1, \ldots, n\}$, the average of f need not be any of the values of f. For example, consider $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) := 1$ if $x \ge 0$ and $f(x) := -1$ if $x < 0$. Then

$$\begin{aligned}\mathrm{Av}(f) = \frac{1}{1-(-1)}\int_{-1}^{1} f(x)dx &= \frac{1}{2}\left(\int_{-1}^{0} f(x)dx + \int_0^1 f(x)dx\right)\\ &= \frac{1}{2}(-1+1) = 0,\end{aligned}$$

but f does not assume the value 0 at any point. If, however, f is continuous, then $\mathrm{Av}(f)$ is always one of the values of f. (See Exercises 8.72 and 6.47.)

"Weighted" averages arise when we wish to give either more or less importance to some of the values of a function. Given $n \in \mathbb{N}$ and $f : \{1, \ldots, n\} \to \mathbb{R}$, let $w(1), \ldots, w(n)$ be nonnegative numbers such that $w(1) + \cdots + w(n) \ne 0$. If we decide to assign weights $w(1), \ldots, w(n)$ to the values $f(1), \ldots, f(n)$ respectively, then the **weighted average** of f with respect to these weights is given by

$$\mathrm{Av}(f; w) := \frac{w(1)f(1) + \cdots + w(n)f(n)}{w(1) + \cdots + w(n)}.$$

With this in mind, we make the following definitions. An integrable function $w : [a, b] \to \mathbb{R}$ is called a **weight function** if w is nonnegative and $\int_a^b w(x)dx \ne 0$. Let $f : [a, b] \to \mathbb{R}$ be an integrable function and $w : [a, b] \to \mathbb{R}$ a weight function. Then the **weighted average** of f with respect to w is defined by

$$\mathrm{Av}(f; w) := \frac{1}{W}\int_a^b f(x)w(x)dx, \quad \text{where} \quad W := \int_a^b w(x)dx.$$

Note that the product fw of the functions f and w is integrable by part (iii) of Proposition 6.16. Observe that if $w(x) = 1$ for all $x \in [a, b]$, then $\mathrm{Av}(f; w) = \mathrm{Av}(f)$.

Roughly speaking, the **centroid** of a set is a point whose coordinates are the averages (or the weighted averages) of the corresponding coordinate functions defined on the set. Thus the x-coordinate (or the first coordinate) of the centroid of a subset D of $\mathbb{R}^3$ is the average value of the function $f : D \to \mathbb{R}$ given by $f(x, y, z) = x$. Similar comments hold for the other coordinates. The difficulty in defining a centroid at this stage lies in the fact that so far we have defined the average of a function defined only on an *interval* $[a, b]$. To be able to deal with centroids of more general sets, we would have to extend the notion of an average to functions defined on them. This is usually done in a course on multivariable calculus. (See, for example, [33, p. 323].) At present, we shall see how centroids of a limited variety of sets can still be defined using Riemann integrals. These definitions turn out to be special cases of the general treatment given in a course on multivariable calculus.

Curves and Surfaces

To begin with, let us consider a line segment P_1P_2 in $\mathbb{R}^2$ with endpoints $P_1 := (x_1, y_1)$ and $P_2 := (x_2, y_2)$. Let $x, y : [0, 1] \to \mathbb{R}$ be defined by

$$x(t) := (x_2 - x_1)t + x_1 \quad \text{and} \quad y(t) := (y_2 - y_1)t + y_1 \quad \text{for } t \in [0, 1].$$

As t runs over the parameter interval $[0, 1]$, we obtain all the points $(x(t), y(t))$ on the line segment P_1P_2. The **centroid** of P_1P_2 is defined to be the point $(\overline{x}, \overline{y})$ given by

$$\overline{x} = \frac{1}{1-0}\int_0^1 x(t)dt = \frac{x_1 + x_2}{2} \quad \text{and} \quad \overline{y} = \frac{1}{1-0}\int_0^1 x(t)dt = \frac{y_1 + y_2}{2}.$$

Thus the centroid of P_1P_2 is its midpoint.

More generally, consider a piecewise smooth curve C given by $(x(t), y(t))$, $t \in [\alpha, \beta]$. While defining the centroid of C, we shall use weighted averages with the weight function $w : [\alpha, \beta] \to \mathbb{R}$ given by $w(t) := \sqrt{x'(t)^2 + y'(t)^2}$. This is reasonable, since the length of the curve C is given by

$$\ell(C) = \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

If $\ell(C) \neq 0$, the **centroid** $(\overline{x}, \overline{y})$ of C is defined by

$$\overline{x} = \frac{1}{\ell(C)}\int_\alpha^\beta x(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt$$

and

$$\overline{y} = \frac{1}{\ell(C)}\int_\alpha^\beta y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Note that $\overline{x} = \mathrm{Av}(x; w)$ and $\overline{y} = \mathrm{Av}(y; w)$.

Let us now consider a surface S generated by revolving the above-mentioned curve C about a line L given by $ax+by+c=0$ that does not cross C. As in the previous section, let $\rho(t) := |ax(t)+by(t)+c|/\sqrt{a^2+b^2}$ denote the distance of the point $(x(t), y(t))$, $t \in [\alpha, \beta]$, on the curve C from this line. Assume that

$$\text{Area }(S) := 2\pi \int_\alpha^\beta \rho(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt$$

is not equal to zero. If the line L is the x-axis, then we define the **centroid** $(\overline{x}, \overline{y}, \overline{z})$ of S by $\overline{y} := 0$, $\overline{z} := 0$ (because of symmetry), and

$$\overline{x} := \frac{2\pi}{\text{Area }(S)} \int_\alpha^\beta x(t)|y(t)|\sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

On the other hand, if the line L is the y-axis, then we define the **centroid** $(\overline{x}, \overline{y}, \overline{z})$ of S by $\overline{x} := 0$, $\overline{z} := 0$ (because of symmetry), and

$$\overline{y} := \frac{2\pi}{\text{Area }(S)} \int_\alpha^\beta y(t)|x(t)|\sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Note that $\overline{x} = \text{Av}(x; w)$, $\overline{y} = \text{Av}(y; w)$, and $\overline{z} = \text{Av}(z; w)$, where the weight function $w : [\alpha, \beta] \to \mathbb{R}$ is given by $w(t) := \rho(t)\sqrt{x'(t)^2 + y'(t)^2}$.

Examples 8.15. (i) Let $a \in \mathbb{R}$ with $a > 0$. Consider a semicircle C of radius a given by $x(t) := a\cos t$, $y(t) := a \sin t$, $t \in [0, \pi]$. Then

$$\ell(C) = \int_0^\pi \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt = a\pi.$$

Hence

$$\overline{x} = \frac{1}{a\pi}\int_0^\pi a\cos t \cdot a\, dt = 0 \quad \text{and} \quad \overline{y} = \frac{1}{a\pi}\int_0^\pi a\sin t \cdot a\, dt = \frac{2a}{\pi}.$$

We note that the centroid $(0, 2a/\pi)$ of the semicircle C does not lie on C. Also, we could have directly obtained the x-coordinate $\overline{x}$ of the centroid to be equal to 0 by the symmetry of the semicircle about the y-axis.

(ii) Consider a cycloid C given by $x(t) := t - \sin t$ and $y(t) := 1 - \cos t$, $t \in [0, 2\pi]$. Then

$$\begin{aligned}\ell(C) &= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt \\ &= \int_0^{2\pi} 2\sin\frac{t}{2}\, dt = 8.\end{aligned}$$

Hence

$$\overline{x} = \frac{1}{8}\int_0^{2\pi} (t - \sin t)\cdot 2\sin\frac{t}{2}\, dt = \pi \quad \text{and} \quad \overline{y} = \frac{1}{8}\int_0^{2\pi} (1-\cos t)\cdot 2\sin\frac{t}{2}\, dt = \frac{4}{3}.$$

Thus the centroid of C is at $\left(\pi, \frac{4}{3}\right)$.

(iii) Let $a \in \mathbb{R}$ with $a > 0$. Consider the surface S generated by revolving an arc of the circle given by $x(t) := a\cos t$, $y(t) := a\sin t$, $t \in [0, \pi/2]$, about the line $y = -a$. Since the distance of $(x(t), y(t))$ from this line is equal to $a + a\sin t$ for $t \in [0, \pi]$, we obtain

$$\text{Area } (S) = 2\pi \int_0^{\pi/2} a(1+\sin t)\sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt = a^2\pi(\pi+2).$$

It can be seen that $\overline{y} = -a$ and $\overline{z} = 0$ by symmetry, whereas

$$\overline{x} = \frac{1}{a^2\pi(\pi+2)} \cdot 2\pi \int_0^{\pi/2} (a\cos t)a(1+\sin t)a\, dt = \frac{3\pi a^3}{a^2\pi(\pi+2)} = \frac{3a}{\pi+2}.$$

Thus the centroid of S is at $(3a/(\pi+2),\, -a,\, 0)$.

(iv) Let a and h be positive real numbers. The surface S of a right circular cone of base radius a and height h is generated by revolving the line segment given by $(x/a) + (y/h) = 1$, $x \in [0, a]$, about the y-axis. As we have seen in Example 8.14 (i), Area $(S) = \pi a\sqrt{a^2+h^2}$. Then $\overline{x} = 0 = \overline{z}$ and

$$\overline{y} = \frac{2\pi}{\pi a\sqrt{a^2+h^2}} \int_0^h ya\left(1 - \frac{y}{h}\right)\sqrt{1 + \frac{a^2}{h^2}}\, dy = \frac{h}{3},$$

as one may expect. ◇

Planar Regions and Solid Bodies

Let us first consider the centroids of certain planar regions that we dealt with in Section 8.1. Let $f_1, f_2 : [a, b] \to \mathbb{R}$ be integrable functions such that $f_1 \le f_2$, and consider the region

$$R := \{(x, y) \in \mathbb{R}^2 : a \le x \le b \text{ and } f_1(x) \le y \le f_2(x)\}.$$

Recall that the area of R is defined to be

$$\text{Area } (R) := \int_a^b \big(f_2(x) - f_1(x)\big)dx.$$

Let us assume that Area $(R) \neq 0$. In view of our comments in the introduction of this section, we define the x-coordinate of the **centroid** of R by

$$\overline{x} := \frac{1}{\text{Area } (R)} \int_a^b x\big(f_2(x) - f_1(x)\big)dx.$$

In order to define the y-coordinate of the centroid of R, note that for $x \in [a, b]$, the vertical slice of the region R at x has length $f_2(x) - f_1(x)$, and its centroid is its midpoint $\big(x, \big(f_1(x)+f_2(x)\big)/2\big)$. Since $\big(f_2(x)-f_1(x)\big)\cdot\big(f_1(x)+f_2(x)\big)/2 =$

$\big(f_2(x)^2 - f_1(x)^2\big)/2$ for all $x \in [a, b]$, it is reasonable to define the y-coordinate of the **centroid** of R by

$$\overline{y} := \frac{1}{2\text{Area }(R)} \int_a^b \big(f_2(x)^2 - f_1(x)^2\big)dx.$$

It may be noted that $\overline{x} = \text{Av}(f; w)$ and $\overline{y} = \text{Av}(g; w)$, where the functions $f, g, w : [a, b] \to \mathbb{R}$ are given by $f(x) := x$, $g(x) := \big(f_1(x) + f_2(x)\big)/2$, $w(x) := \big(f_2(x) - f_1(x)\big)$.

Similarly, if $g_1, g_2 : [c, d] \to \mathbb{R}$ are integrable functions such that $g_1 \le g_2$, then the **centroid** $(\overline{x}, \overline{y})$ of the region

$$R := \{(x, y) \in \mathbb{R}^2 : c \le y \le d \text{ and } g_1(y) \le x \le g_2(y)\}$$

is defined by

$$\overline{x} = \frac{1}{2\text{Area }(R)} \int_c^d \big(g_2(y)^2 - g_1(y)^2\big)dy$$

and

$$\overline{y} = \frac{1}{\text{Area }(R)} \int_c^d y\big(g_2(y) - g_1(y)\big)dy,$$

provided the area

$$\text{Area }(R) := \int_d^c \big(g_2(y) - g_1(y)\big)dy$$

of R is not zero.

Finally, we shall consider the centroids of certain solid bodies that we dealt with in Section 8.2. First suppose that a bounded solid D lies between the planes given by $x = a$ and $x = b$, where $a, b \in \mathbb{R}$ with $a < b$, and for $x \in [a, b]$, let $A(x)$ denote the area of the slice of D at x by a plane perpendicular to the x-axis. Assume that

$$\text{Vol }(D) := \int_a^b A(x)dx$$

is not equal to zero. For each $x \in [a, b]$, the x-coordinate of the centroid of the slice of D at x is x itself. In view of this, we define the x-coordinate of the **centroid** of D by

$$\overline{x} := \frac{1}{\text{Vol }(D)} \int_a^b xA(x)dx.$$

Further, if for each $x \in [a, b]$, the y-coordinate and the z-coordinate of the centroid of the slice of D at x are given by $\tilde{y}(x)$ and $\tilde{z}(x)$, then we define the y-coordinate and the z-coordinate of the **centroid** of D by

$$\overline{y} := \frac{1}{\text{Vol }(D)} \int_a^b \tilde{y}(x)A(x)dx \quad \text{and} \quad \overline{z} := \frac{1}{\text{Vol }(D)} \int_a^b \tilde{z}(x)A(x)dx.$$

Note that $\overline{x} = \text{Av}(f; A)$, $\overline{y} = \text{Av}(g; A)$, and $\overline{z} = \text{Av}(h; A)$, where the functions $f, g, h : [a, b] \to \mathbb{R}$ are given by $f(x) := x$, $g(x) := \tilde{y}(x)$, $h(x) := \tilde{z}(x)$.

In particular, let the solid D be generated by revolving the region

$$\{(x, y) \in \mathbb{R}^2 : a \le x \le b \text{ and } f_1(x) \le y \le f_2(x)\}$$

about the x-axis, where $f_1, f_2 : [a, b] \to \mathbb{R}$ are integrable functions such that $0 \le f_1 \le f_2$. Recall that by the washer method, $A(x) = \pi\big(f_2(x)^2 - f_1(x)^2\big)$ for all $x \in [a, b]$ and

$$\text{Vol }(D) = \pi \int_b^a \big(f_2(x)^2 - f_1(x)^2\big)dx.$$

Thus we see that

$$\overline{x} = \frac{\pi}{\text{Vol }(D)} \int_a^b x\big(f_2(x)^2 - f_1(x)^2\big)dx,$$

whereas $\overline{y} = 0 = \overline{z}$, since $\tilde{y}(x) = 0 = \tilde{z}(x)$ for all $x \in [a, b]$ by symmetry.

Similar considerations hold for a solid whose volume is given by

$$\int_c^d A(y)dy \quad \text{or} \quad \int_p^q A(z)dz,$$

as described in Section 8.2.

Next, suppose that a bounded solid D lies between the cylinders given by $x^2 + y^2 = p^2$ and $x^2 + y^2 = q^2$, where $p, q \in \mathbb{R}$ with $0 \le p < q$. For $r \in [p, q]$, consider the sliver $\{(x, y, z) \in D : x^2 + y^2 = r^2\}$ of D by the cylinder given by $x^2 + y^2 = r^2$; let $E_r := \{(\theta, z) \in [-\pi, \pi] \times \mathbb{R} : (r\cos\theta, r\sin\theta, z) \in D\}$ denote the parameter domain for this sliver and let $B(r)$ denote the area of the planar region E_r. Assume that

$$\text{Vol }(D) := \int_p^q rB(r)dr$$

is not equal to zero. For each $r \in [p, q]$, if $B(r) \ne 0$ and $(\tilde{x}(r), \tilde{y}(r), \tilde{z}(r))$ denotes the centroid of the sliver of D at r, then the **centroid** $(\overline{x}, \overline{y}, \overline{z})$ of the solid D is defined by

$$\overline{x} := \frac{1}{V} \int_p^q \tilde{x}(r)rB(r)dr, \quad \overline{y} := \frac{1}{V} \int_p^q \tilde{y}(r)rB(r)dr, \quad \overline{z} := \frac{1}{V} \int_p^q \tilde{z}(r)rB(r)dr,$$

where $V := \text{Vol}(D)$. Note that $\overline{x} = \text{Av}(f; A)$, $\overline{y} = \text{Av}(g; A)$, and $\overline{z} = \text{Av}(h; A)$, where the functions $A, f, g, h : [p, q] \to \mathbb{R}$ are given by $A(r) := rB(r)$, $f(r) := \tilde{x}(r)$, $g(r) := \tilde{y}(r)$, $h(r) := \tilde{z}(r)$.

Similar considerations hold for a solid whose volume is given by

$$\int_a^b xB(x)dx \quad \text{or} \quad \int_c^d yB(y)dy,$$

as described in Section 8.2.

In particular, let the solid D be generated by revolving the region

$$R := \{(x, y) \in \mathbb{R}^2 : a \le x \le b \text{ and } f_1(x) \le y \le f_2(x)\}$$

about the y-axis, where $0 \le a < b$ and $f_1, f_2 : [a, b] \to \mathbb{R}$ are integrable functions such that $f_1 \le f_2$. Recall that by the shell method, the volume of D is given by

$$\text{Vol }(D) = 2\pi \int_a^b x\big(f_2(x) - f_1(x)\big)dx.$$

As observed before, for each $x \in [a, b]$, the centroid of the vertical cut of the region R at $x \in [a, b]$ is at $(x, \big(f_1(x) + f_2(x)\big)/2)$, and hence the y-coordinate of the centroid of the sliver of D at x is given by $\big(f_1(x) + f_2(x)\big)/2$. Thus

$$\begin{aligned}\overline{y} &:= \frac{2\pi}{\text{Vol }(D)} \int_a^b \frac{\big(f_1(x) + f_2(x)\big)}{2} x\big(f_2(x) - f_1(x)\big)dx \\ &= \frac{\pi}{\text{Vol }(D)} \int_a^b x\big(f_2(x)^2 - f_1(x)^2\big)dx,\end{aligned}$$

whereas $\overline{x} = 0 = \overline{z}$ by symmetry.

Examples 8.16. (i) Let b and h be positive real numbers. Consider the planar region enclosed by the right-angled triangle whose vertices are at $(0, 0)$, $(b, 0)$, and $(0, h)$. The area of this triangular region is $bh/2$. Hence the coordinates of its centroid are given by

$$\overline{x} = \frac{2}{bh} \int_0^b x\Big[\frac{h}{b}(b - x) - 0\Big]dx = \frac{2}{b^2} \int_0^b (bx - x^2)dx = \frac{2}{b^2}\left(\frac{b^3}{2} - \frac{b^3}{3}\right) = \frac{b}{3}$$

and

$$\overline{y} = \frac{1}{2} \cdot \frac{2}{bh} \int_0^b \Big[\Big(\frac{h}{b}(b - x)\Big)^2 - 0^2\Big]dx = \frac{h}{b^3} \int_0^b (b - x)^2 dx = \frac{h}{b^3} \cdot \frac{b^3}{3} = \frac{h}{3}.$$

(ii) Let b and h be positive real numbers. Consider the planar region enclosed by the rectangle whose vertices are at $(0, 0)$, $(b, 0)$, $(0, h)$, and (b, h). The area of this rectangular region is bh. Hence the coordinates of its centroid are given by

$$\overline{x} = \frac{1}{bh} \int_0^b xh\, dx = \frac{b}{2} \quad \text{and} \quad \overline{y} = \frac{1}{2} \cdot \frac{1}{bh} \int_0^b (h^2 - 0^2)dx = \frac{h}{2}.$$

Thus the centroid of the rectangular region is at $(b/2, h/2)$.

(iii) Let $a \in \mathbb{R}$ with $a > 0$. Consider the semicircular region

$$\{(x, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } x^2 + y^2 \leq a^2\}.$$

Its area is $\pi a^2/2$, as seen in part (iii) of Proposition 8.2. Then $\overline{x} = 0$ by symmetry and

$$\overline{y} = \frac{2}{\pi a^2} \cdot \frac{1}{2} \int_{-a}^{a} \left(\left(\sqrt{a^2 - x^2}\right)^2 - 0^2 \right) dx = \frac{4a}{3\pi}.$$

Thus the centroid of the semicircular region is at $(0,\, 4a/3\pi)$.

(iv) Consider the region bounded by the curves given by $x = 2y - y^2$ and $x = 0$. Since $2y - y^2 = x = 0$ implies that $y = 0$ or $y = 2$, and since $2y - y^2 \geq 0$ for all $y \in [0, 2]$, the region is given by

$$\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 2 \text{ and } 0 \leq x \leq 2y - y^2\}.$$

The area of this region is $\int_0^2 (2y - y^2)dy = 4/3$. The curve given by $x = 2y - y^2$ is in fact the parabola given by $(y-1)^2 = 1 - x$. Thus $\overline{y} = 1$ by symmetry and

$$\overline{x} = \frac{1}{2(4/3)} \int_0^2 \left((2y - y^2)^2 - 0^2\right) dy = \frac{1}{2(4/3)} \frac{16}{15} = \frac{2}{5}.$$

Thus the centroid of the region is at $(\frac{2}{5},\, 1)$.

(v) Let b and h be positive real numbers. A right circular conical solid of base radius a and height h is generated by revolving the triangular region bounded by the lines given by $x = 0$, $y = 0$, and $(x/a) + (y/h) = 1$ about the y-axis. The volume of this solid cone is $\pi a^2 h/3$, as seen in Example 8.7 (ii). Hence

$$\overline{y} = \frac{\pi}{\pi a^2 h/3} \int_0^h y a^2 \left(1 - \frac{y}{h}\right)^2 dy = \frac{\pi}{(\pi a^2 h/3)} \frac{a^2 h^2}{12} = \frac{h}{4}$$

and $\overline{x} = 0 = \overline{z}$ by symmetry. Thus the centroid of the conical solid is at $(0, h/4, 0)$.

(vi) Let R denote the planar region in the first quadrant between the parabolas given by $y = x^2$ and $y = 2 - x^2$. Consider the solid D_1 obtained by revolving the region R about the x-axis. Its volume is $8\pi/3$, as seen in Example 8.7 (iv). Hence

$$\overline{x} = \frac{\pi}{8\pi/3} \int_0^1 x\Big(\left(2 - x^2\right)^2 - \left(x^2\right)^2 \Big) dx = \frac{\pi}{(8\pi/3)} \cdot 1 = \frac{3}{8}$$

and $\overline{y} = 0 = \overline{z}$ by symmetry. Thus the centroid of the solid E_1 is at $\left(\frac{3}{8},\, 0,\, 0\right)$. Next, consider the solid D_2 obtained by revolving the region R about the y-axis. Its volume is π, as seen in Example 8.7 (iv). Hence

$$\overline{y} = \frac{2\pi}{\pi}\int_0^1 x\left(\frac{x^2+(2-x^2)}{2}\right)(2-x^2-x^2)dx = \frac{2\pi}{\pi}\frac{1}{2} = 1$$

and $\overline{x} = 0 = \overline{z}$ by symmetry. Thus the centroid of the solid D_2 is at $(0, 1, 0)$. We could also have obtained this by symmetry. ◇

As indicated earlier, the definitions of centroids of planar regions, surfaces of revolution, and solid bodies given here are special cases of the more general definitions of centroids given in multivariable calculus using multiple integrals. (See, for example, the last four subsections of Section 6.3 of [33].)

Theorems of Pappus

The following result relates the centroid of a curve to the area of the surface of revolution generated by it.

Proposition 8.17 (Pappus Theorem for Surfaces of Revolution). *Let C be a piecewise smooth curve in $\mathbb{R}^2$ and let L be a line in $\mathbb{R}^2$ that does not cross C. If C is revolved about L, then the area of the surface so generated is equal to the product of the arc length of C and the distance traveled by the centroid of C. Symbolically,*

Area of Surface of Revolution = Arc length × Distance Traveled by Centroid.

Proof. Let the curve C be given by $(x(t), y(t))$, $t \in [\alpha, \beta]$, and let the line L be given by $ax + by + c = 0$, where $a^2 + b^2 \neq 0$. Recall that the arc length of C is

$$\ell(C) := \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Also, the area of the surface S generated by revolving C about L is

$$A(S) := 2\pi \int_\alpha^\beta \rho(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt,$$

where $\rho(t)$ is the distance of the point $(x(t), y(t))$, $t \in [\alpha, \beta]$, from the line L.

On the other hand, the centroid $(\overline{x}, \overline{y})$ of C is given by

$$\overline{x} := \frac{\displaystyle\int_\alpha^\beta x(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt}{\ell(C)} \quad \text{and} \quad \overline{y} := \frac{\displaystyle\int_\alpha^\beta y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt}{\ell(C)}.$$

Further, the distance d traveled by $(\overline{x}, \overline{y})$ about the line L is 2π times its distance from the line L, which is

$$\begin{aligned} 2\pi\frac{|a\overline{x} + b\overline{y} + c|}{\sqrt{a^2+b^2}} &= \frac{2\pi}{\ell(C)\sqrt{a^2+b^2}}\left|\int_\alpha^\beta \big(ax(t) + by(t) + c\big)\sqrt{x'(t)^2 + y'(t)^2}\, dt\right| \\ &= \frac{2\pi}{\ell(C)}\int_\alpha^\beta \rho(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt, \end{aligned}$$

because $ax(t) + by(t) + c \geq 0$ for all $t \in [\alpha, \beta]$ or $ax(t) + by(t) + c \leq 0$ for all $t \in [\alpha, \beta]$. Thus

$$\text{Area } (S) = \ell(C) \times d.$$

This proves the theorem. □

The next result relates the centroid of a planar region lying between two curves with the volume of the solid obtained by revolving the region about the x-axis or the y-axis. We remark that an extension of this result involving planar regions of a more general kind is given in [33, Prop. 6.22]

Proposition 8.18 (Pappus Theorem for Solids of Revolution). *Let R be a planar region given by*

$$\{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x)\},$$

where $f_1, f_2 : [a, b] \to \mathbb{R}$ are integrable functions such that $f_1 \leq f_2$. If either $f_1(x) \geq 0$ and the region R is revolved about the x-axis, or $a \geq 0$ and the region R is revolved about the y-axis, then the volume of the solid so generated is equal to the product of the area of R and the distance traveled by the centroid of R. Symbolically,

Volume of Solid of Revolution = Area × Distance Traveled by Centroid.

Proof. We note that the area of the region R is

$$\text{Area } (R) := \int_a^b \big(f_2(x) - f_1(x)\big)dx.$$

Let $(\overline{x}, \overline{y})$ denote the centroid of R.

First assume that $f_1(x) \geq 0$ for all $x \in [a, b]$ and that the region R is revolved about the x-axis. Then by the washer method, the volume of the solid D so generated is

$$\text{Vol } (D) := \pi \int_a^b \big(f_2(x)^2 - f_1(x)^2\big)dx.$$

On the other hand,

$$\begin{aligned} \overline{y} &= \frac{1}{\text{Area } (R)} \int_a^b \frac{\big(f_1(x) + f_2(x)\big)}{2} \big(f_2(x) - f_1(x)\big)dx \\ &= \frac{1}{2\text{Area } (R)} \int_a^b \big(f_2(x)^2 - f_1(x)^2\big)dx. \end{aligned}$$

Further, the distance d traveled by $(\overline{x}, \overline{y})$ about the x-axis is equal to 2π times its distance from the x-axis, that is,

$$d = 2\pi\overline{y}.$$

Thus we see that

$$\text{Vol }(D) = \text{Area }(R) \times d.$$

This proves the desired result in the case that the planar region R is revolved about the x-axis.

Next, assume that $a \geq 0$ and the region R is revolved about the y-axis. Then by the shell method, the volume of the solid so generated is

$$\text{Vol }(D) := 2\pi \int_a^b x\big(f_2(x) - f_1(x)\big)dx.$$

On the other hand,

$$\overline{x} = \frac{1}{\text{Area }(R)} \int_a^b x\big(f_2(x) - f_1(x)\big)dx.$$

Further, the distance d traveled by $(\overline{x}, \overline{y})$ about the y-axis is equal to 2π times its distance from the y-axis, that is,

$$d = 2\pi\overline{x}.$$

Thus again, we see that

$$\text{Vol }(D) = \text{Area }(R) \times d.$$

This proves the desired result in the case that the planar region R is revolved about the y-axis. □

If we know any two of the three quantities (i) the length of a planar curve, (ii) the distance of its centroid from a line in its plane, and (iii) the area of the surface obtained by revolving the curve about the line, then the result of Pappus allows us to find the remaining quantity easily. In case the curve is symmetric in some way, we can in fact determine its centroid. This also holds for the area of a planar region, the distance of its centroid from a line in its plane, and the volume of the solid obtained by revolving the region about the line. The following examples illustrate these comments.

Examples 8.19. We verify the conclusions of the theorems of Pappus in several special cases. We present them in tabular form for easy verification of the results of Pappus.

(i) Let $\ell(C)$ denote the length of a piecewise smooth curve C. If $(\overline{x}, \overline{y})$ denotes the centroid of C, then its distance from the y-axis is $\overline{x}$. Let S denote the surface obtained by revolving C about the y-axis. By Proposition 8.17,

$$\text{Area }(S) = \ell(C) \times 2\pi\overline{x}.$$

Curve C	Surface S	$\ell(C)$	$\overline{x}$	Area (S)
Line segment $(x/a)+(y/h)=1,\ x\geq 0,\ y\geq 0$	Cone	$\sqrt{a^2+h^2}$	$\dfrac{a}{2}$	$\pi a\sqrt{a^2+h^2}$
Line segment $x=a,\ 0\leq y\leq h$	Cylinder	h	a	$2\pi ah$
Semicircle $x^2+y^2=a^2,\ x\geq 0$	Sphere	πa	$\dfrac{2a}{\pi}$	$4\pi a^2$
Circle $(x-a)^2+y^2=b^2,\ 0<b<a$	Torus	$2\pi b$	a	$4\pi^2 ab$

(ii) Let R be a planar region. If $(\overline{x},\overline{y})$ denotes the centroid of R, then its distance from the y-axis is $\overline{x}$. Let D denote the surface obtained by revolving R about the y-axis. By Proposition 8.18,

$$\text{Vol }(D)=\text{Area }(R)\times 2\pi\overline{x}.$$

Region R	Solid D	Area (R)	$\overline{x}$	Vol (D)
Triangle enclosed by the lines $x=0,\ y=0,\ (x/a)+(y/h)=1$	Cone	$\dfrac{ah}{2}$	$\dfrac{a}{3}$	$\dfrac{\pi a^2h}{3}$
Rectangle enclosed by the lines $x=0,\ y=0,\ x=a,\ y=h$	Cylinder	ah	$\dfrac{a}{2}$	πa^2h
Semidisk $x^2+y^2\leq a^2,\ x\geq 0$	Ball	$\dfrac{\pi a^2}{2}$	$\dfrac{4a}{3\pi}$	$\dfrac{4\pi a^3}{3}$
Disk $(x-a)^2+y^2=b^2,\ 0<b<a$	Torus	πb^2	a	$2\pi^2ab^2$

In various examples given in this chapter, we have independently calculated all the quantities mentioned in the above tables. ◇

8.6 Quadrature Rules

In Chapter 6, we have given various criteria for deciding the integrability of a bounded function $f:[a,b]\to\mathbb{R}$. The actual evaluation of the Riemann integral, however, can pose serious difficulties. If f is integrable and has an antiderivative F, then the Fundamental Theorem of Calculus tells us that $\int_a^b f(x)dx=F(b)-F(a)$. But an integrable function f need not have an antiderivative, and even if it has one, it may not be useful in evaluating $\int_a^b f(x)dx$ in terms of known functions. For example, if $f(x):=1/x$ for $x\in[1,2]$, then the function f has an antiderivative, namely, $F(x) := \int_1^x (1/t)dt$, $x \in [1, 2]$. But it is hardly useful in evaluating $\int_1^2 f(x)dx$. A similar comment holds for the function given by $f(x) := 1/(1+x^2)$ for $x \in [0, 1]$. Sometimes integration by parts and integration by substitution are helpful in evaluating Riemann integrals, but the scope of such techniques is very limited. In fact, it is not possible to evaluate Riemann integrals of many of the functions that occur in practice. As we have mentioned in Section 6.4, the Riemann sums can then be employed to find approximate values of a Riemann integral. In the present section, we shall describe a number of efficient procedures for evaluating a Riemann integral approximately. These are known as **quadrature rules**. A quadrature rule for $[a, b]$ associates to an integrable function $f : [a, b] \to \mathbb{R}$ a real number

$$Q(f) := \sum_{i=1}^{n} w_i f(s_i),$$

where $n \in \mathbb{N}$, $w_i \in \mathbb{R}$, and $s_i \in [a, b]$ for $i = 1, \ldots, n$. The real numbers $w_1, \ldots, w_n$ are known as the **weights**, and the points $s_1, \ldots, s_n$ are known as the **nodes** of the quadrature rule Q. For example, if $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ and $s_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$, then the Riemann sum

$$S(P, f) := \sum_{i=1}^{n} f(s_i)(x_i - x_{i-1})$$

is an example of a quadrature rule whose nodes are $s_1, \ldots, s_n$ and whose weights are $x_1 - x_0, \ldots, x_n - x_{n-1}$. We shall now construct some simple quadrature rules by replacing the function f by a polynomial function of degree 0, 1, or 2, and by considering the "signed area" under the curve given by the polynomial function.

1. Let us fix $s \in [a, b]$ and let us replace the function f by the polynomial function p_0 of degree 0 that is equal to the value $f(s)$ of f at s. The "signed area" under the curve given by $y = p_0(x)$, whose graph is a horizontal line segment, is equal to the "area" of the rectangle with base $[a, b]$ and "height" $f(s)$. This gives the **Rectangular Rule**, which associates to f the number

$$R(f) := (b - a)f(s).$$

In particular, if s is the midpoint $(a + b)/2$ of $[a, b]$, then we obtain the **Midpoint Rule**, which associates to f the number

$$M(f) := (b - a)f\Big(\frac{a + b}{2}\Big).$$

2. Let us replace the function f by the polynomial function p_1 of degree 1 whose values at a and b are equal to $f(a)$ and $f(b)$. The "signed area" under the curve given by $y = p_1(x)$, whose graph is an inclined line segment, is equal to the "area" of the trapezoid with base $[a, b]$ and the "lengths" of the two

parallel sides equal to $f(a)$ and $f(b)$ respectively. This gives the **Trapezoidal Rule**, which associates to f the number

$$T(f) := \frac{(b-a)}{2}\big(f(a) + f(b)\big).$$

3. Let us replace the function f by the polynomial function p_2 of degree 2 whose values at a, $(a+b)/2$, and b are equal to $f(a)$, $f((a+b)/2)$, and $f(b)$ respectively. The "signed area" under the curve given by $y = p_2(x)$, whose graph is, in general, a parabola, gives the **Simpson Rule**, which associates to f the number

$$S(f) := \frac{(b-a)}{6}\Big(f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b)\Big).$$

The defining expressions for $S(f)$, $T(f)$, $M(f)$, and $R(f)$ are motivated by the following result on the integrals of polynomial functions of degree ≤ 2.

Proposition 8.20. *Let $c_0, c_1, c_2 \in \mathbb{R}$ and let $f : [a,b] \to \mathbb{R}$ be the polynomial function defined by $f(x) := c_2x^2 + c_1x + c_0$ for $x \in [a,b]$. Then*

$$\int_a^b f(x)dx = \frac{(b-a)}{6}\left(f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b)\right).$$

Moreover, in case $c_2 = 0$, then

$$\int_a^b f(x)dx = \frac{(b-a)}{2}\big(f(a) + f(b)\big) = (b-a)f\Big(\frac{a+b}{2}\Big).$$

In case $c_2 = c_1 = 0$, then $\int_a^b f(x)dx = (b-a)f(s)$ for all $s \in [a,b]$.

Proof. Evaluating the Riemann integral of f over $[a,b]$ and using the identities $(b^2 - a^2) = (b-a)(b+a)$ and $(b^3 - a^3) = (b-a)(b^2 + ba + a^2)$, we obtain

$$\begin{aligned}\int_a^b f(x)dx &= \frac{c_2}{3}(b^3 - a^3) + \frac{c_1}{2}(b^2 - a^2) + c_0(b-a)\\ &= \frac{(b-a)}{6}\Big(2c_2(a^2 + ab + b^2) + 3c_1(a+b) + 6c_0\Big)\\ &= \frac{(b-a)}{6}\Big(f(a) + c_2(a+b)^2 + 2c_1(a+b) + 4c_0 + f(b)\Big).\end{aligned}$$

The last expression implies the first assertion. Moreover, putting $c_2 = 0$, we obtain the second assertion as well. The case $c_2 = c_1 = 0$ is trivial. □

The simple quadrature rules given above can be expected to yield only rough approximations of a Riemann integral of a function on $[a,b]$. To obtain more precise approximations, we may partition the interval $[a,b]$ into smaller intervals and apply the above quadrature rules to the function f restricted to each subinterval and then sum the "signed areas" so obtained. It is often convenient and also efficient to consider partitions of $[a, b]$ into equal parts.

For $n \in \mathbb{N}$, let $\mathsf{P_n} := \{x_{0,n}, x_{1,n}, \ldots, x_{n,n}\}$ denote the partition of $[a, b]$ into n equal parts. For the sake of simplicity of notation, we denote $x_{i,n}$ by x_i for $i = 0, 1, \ldots, n$. Let

$$h_n := \frac{b-a}{n} \quad \text{and} \quad y_i = f(x_i) \quad \text{for } i = 0, 1, \ldots, n.$$

Note that $x_i - x_{i-1} = h_n$ for $i = 1, \ldots, n$.

1. For $i = 1, \ldots, n$, let s_i be a point in $[x_{i-1}, x_i]$ and let us replace the curve given by $y = f(x)$ on the ith subinterval $[x_{i-1}, x_i]$ by a horizontal line segment passing through the point $(s_i, f(s_i))$. Since the "signed area" of the rectangle with base $x_i - x_{i-1}$ and "height" $f(s_i)$ is $h_n f(s_i)$, we obtain the **Compound Rectangular Rule**, which associates to f the number

$$R_n(f) := h_n \sum_{i=1}^{n} f(s_i).$$

Since $R_n(f)$ is a Riemann sum for f corresponding to P_n and $\mu(P_n) = h_n \to 0$ as $n \to \infty$, it follows that $R_n(f) \to \int_a^b f(x)dx$ as $n \to \infty$. (See Remark 6.34.)

In case s_i is the midpoint $\overline{x}_i := (x_{i-1} + x_i)/2$ of the ith subinterval, we obtain the **Compound Midpoint Rule**, which associates to f the number

$$M_n(f) := h_n \sum_{i=1}^{n} f(\overline{x}_i).$$

(See Figure 8.23.)

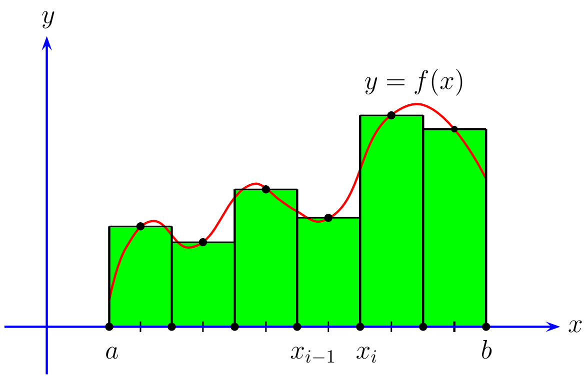


Fig. 8.23. Illustration of the Compound Midpoint Rule.

2. For $i = 1, \ldots, n$, let us replace the curve given by $y = f(x)$ on the ith subinterval $[x_{i-1}, x_i]$ by a line segment joining the points (x_{i-1}, y_{i-1}) and

(x_i, y_i). Since the "signed area" of the trapezoid with base $x_i - x_{i-1}$ and parallel sides of "lengths" y_{i-1} and y_i is

$$\frac{h_n}{2}(y_{i-1} + y_i) = \frac{h_n}{2}\big(f(x_{i-1}) + f(x_i)\big),$$

we obtain the **Compound Trapezoidal Rule**, which associates to f the number

$$\begin{aligned} T_n(f) &:= \frac{h_n}{2}\sum_{i=1}^{n}(y_{i-1} + y_i) = \frac{h_n}{2}(y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n) \\ &= \frac{h_n}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right). \end{aligned}$$

(See Figure 8.24.) We observe that

$$T_n(f) = \frac{1}{2}\big(R_n^{\ell}(f) + R_n^{r}(f)\big),$$

where

$$R_n^{\ell}(f) := \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) \quad \text{and} \quad R_n^{r}(f) := \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}).$$

Since $R_n^{\ell}(f) \to \int_a^b f(x)dx$ and $R_n^{r}(f) \to \int_a^b f(x)dx$ as $n \to \infty$, we see that

$$T_n(f) \to \frac{1}{2}\left(\int_a^b f(x)dx + \int_a^b f(x)dx\right) = \int_a^b f(x)dx.$$

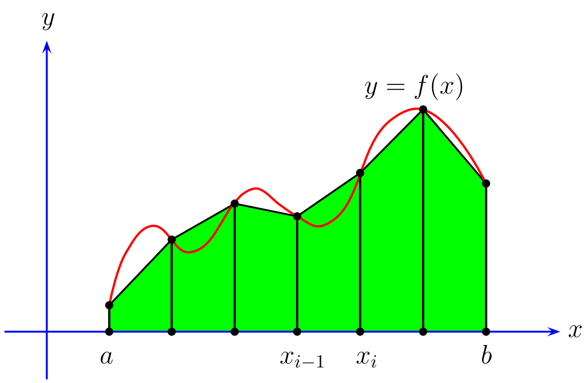


Fig. 8.24. Illustration of the Compound Trapezoidal Rule.

3. Assume that n is even. For $i = 1, 3, \ldots, n-1$, let us replace the curve given by $y = f(x)$ on the subinterval $[x_{i-1}, x_{i+1}]$ by a parabola passing through the points (x_{i-1}, y_{i-1}), (x_i, y_i), and (x_{i+1}, y_{i+1}). By Proposition 8.20, we see that the "signed area" under this quadratic curve is

$$\frac{2h_n}{6}(y_{i-1} + 4y_i + y_{i+1}) = \frac{h_n}{3}\big(f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\big),$$

and thus we obtain the **Compound Simpson Rule**, which associates to f the number

$$\begin{aligned} S_n(f) &:= \frac{h_n}{3} \sum_{i=1,\, i \,\mathrm{odd}}^{n-1} (y_{i-1} + 4y_i + y_{i+1}) \\ &= \frac{h_n}{3}\big(y_0 + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2}) + y_n\big) \\ &= \frac{h_n}{3}\left(f(x_0) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{(n/2)-1} f(x_{2i}) + f(x_n)\right). \end{aligned}$$

(See Figure 8.25.) We note that if $k := n/2$, then $h_k = 2h_n$ and

$$S_n(f) = \frac{h_k}{6}\sum_{j=1}^{k} \big(f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\big).$$

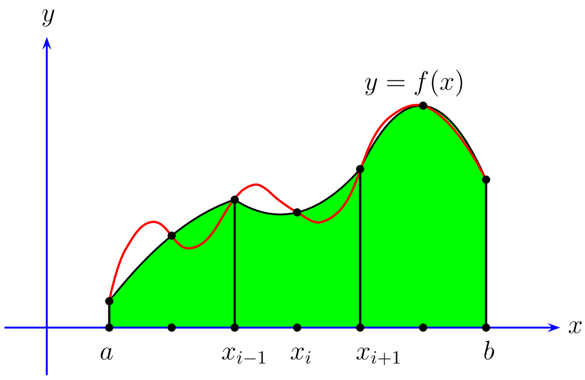


Fig. 8.25. Illustration of the Compound Simpson Rule.

Observe that each of the three sums $h_k \sum_{j=1}^{k} f(x_{2j-2})$, $h_k \sum_{j=1}^{k} f(x_{2j-1})$, and $h_k \sum_{j=1}^{k} f(x_{2j})$ is a Riemann sum for f corresponding to the partition $\mathsf{Q_k} := \{x_0, x_2, \ldots, x_{2k-2}, x_{2k}\}$. Hence as $n \to \infty$,

$$S_n(f) \to \frac{1}{6}\left(\int_a^b f(x)dx + 4\int_a^b f(x)dx + \int_a^b f(x)dx\right) = \int_a^b f(x)dx.$$

If the function $f : [a,b] \to \mathbb{R}$ is sufficiently smooth, it is possible to obtain error estimates for the approximations $R_n(f)$, $M_n(f)$, $T_n(f)$, and $S_n(f)$ of $\int_a^b f(x)dx$. We shall show that such an error is $O(1/n)$ for $R_n(f)$ in general, while it is $O(1/n^2)$ for $M_n(f)$ and $T_n(f)$, and $O(1/n^4)$ for $S_n(f)$.

We first consider the Compound Rectangular Rule and the Compound Midpoint Rule. In the following result, we shall estimate the difference between the "signed area" under the curve given by $y = f(x)$, $a \le x \le b$, and the "signed area" obtained by replacing the function f by a constant function on the entire interval $[a,b]$. The key idea is to use the Taylor Theorem for the functions F, $G : [a,b] \to \mathbb{R}$ given by $F(x) = \int_a^x f(t)dt$ and $G(x) = \int_x^b f(t)dt$.

Lemma 8.21. *Consider a function $f : [a,b] \to \mathbb{R}$.*

(i) *Let f be continuous on $[a,b]$ and differentiable on (a,b). Then given any $c \in [a,b]$, there exist ξ, $\eta \in (a,b)$ such that*

$$\int_a^b f(x)dx = (b-a)f(c) + \frac{1}{2}\left((b-c)^2 f'(\xi) - (a-c)^2 f'(\eta)\right).$$

(ii) *Let f be continuously differentiable on $[a,b]$ and let f'' exist on (a,b). Then there exists $\zeta \in (a,b)$ such that*

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24} f''(\zeta).$$

Proof. (i) Consider the functions F, $G : [a,b] \to \mathbb{R}$ defined by

$$F(x) := \int_a^x f(t)dt \quad \text{and} \quad G(x) := \int_x^b f(t)dt.$$

Then by domain additivity (Proposition 6.8),

$$F(x) + G(x) = \int_a^b f(t)dt \quad \text{and hence} \quad F'(x) = -G'(x) \quad \text{for all } x \in [a,b].$$

Thus by part (i) of the FTC (Proposition 6.24),

$$F'(x) = f(x) = -G'(x) \quad \text{for all } x \in [a,b].$$

Further, F'' and G'' exist on (a,b) and

$$F''(x) = f'(x) = -G''(x) \quad \text{for all } x \in (a,b).$$

Let $c \in [a,b]$ be given. By the Taylor Theorem (Proposition 4.25) for the function F on the interval $[c,b]$ and $n = 1$, there exists $\xi \in (c,b)$ such that

$$F(b) = F(c) + (b-c)F'(c) + \frac{(b-c)^2}{2}F''(\xi),$$

that is,

$$\int_a^b f(t)dt = \int_a^c f(t)dt + (b-c)f(c) + \frac{(b-c)^2}{2}f'(\xi).$$

Also, by the version of the Taylor Theorem for right endpoints (Remark 4.26 (ii)), there exists $\eta \in (a,c)$ such that

$$G(a) = G(c) + (a-c)G'(c) + \frac{(a-c)^2}{2}G''(\eta),$$

that is,

$$\int_a^b f(t)dt = \int_c^b f(t)dt + (c-a)f(c) - \frac{(c-a)^2}{2}f'(\eta).$$

Adding the two equations for $\int_a^b f(t)dt$ given above, we obtain by domain additivity,

$$2\int_a^b f(t)dt = \int_a^b f(t)dt + (b-a)f(c) + \frac{(b-c)^2}{2}f'(\xi) - \frac{(c-a)^2}{2}f'(\eta),$$

that is,

$$\int_a^b f(t)dt = (b-a)f(c) + \frac{1}{2}\left((b-c)^2 f'(\xi) - (c-a)^2 f'(\eta)\right)],$$

as desired.

(ii) Let F, G, and c be as in part (i) above. Since f'' exists on (a,b),

$$F'''(x) = f''(x) = -G'''(x) \quad \text{for all } x \in (a,b).$$

By the Taylor Theorem for $n=2$, there are $\xi \in (c,b)$ and $\eta \in (a,c)$ such that

$$F(b) = F(c) + (b-c)F'(c) + \frac{(b-c)^2}{2}F''(c) + \frac{(b-c)^3}{6}F'''(\xi),$$
$$G(a) = G(c) + (a-c)G'(c) + \frac{(a-c)^2}{2}G''(c) + \frac{(a-c)^3}{6}G'''(\eta),$$

that is,

$$\int_a^b f(t)dt = \int_a^c f(t)dt + (b-c)f(c) + \frac{(b-c)^2}{2}f'(c) + \frac{(b-c)^3}{6}f''(\xi),$$
$$\int_a^b f(t)dt = \int_c^b f(t)dt + (c-a)f(c) - \frac{(c-a)^2}{2}f'(c) + \frac{(c-a)^3}{6}f''(\eta).$$

Adding the above two equations and letting $c = (a+b)/2$, we obtain

$$2\int_a^b f(t)dt = \int_a^b f(t)dt + (b-a)f\Big(\frac{a+b}{2}\Big) + \frac{(b-a)^3}{48}\big(f''(\xi)+f''(\eta)\big).$$

By the Intermediate Value Property of f'' (Proposition 4.16), we see that there exists ζ between ξ and η such that $\big(f''(\xi)+f''(\eta)\big)/2 = f''(\zeta)$. Hence

$$\int_a^b f(t)dt = (b-a)f\Big(\frac{a+b}{2}\Big) + \frac{(b-a)^3}{24}f''(\zeta),$$

as desired. □

The above proof shows how the factor $(b-a)^3$ (in place of the factor $(b-a)^2$) arises in the remainder term when c is the midpoint $(a+b)/2$ of the interval $[a,b]$ and the given function f is twice differentiable. This is not possible for any other point c in $[a,b]$.

To obtain error estimates for $R_n(f)$ and $M_n(f)$, we apply the results of the above lemma with the interval $[a,b]$ replaced by the subintervals induced by the partition $\mathsf{P_n} := \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ into n equal parts, and then take the sum.

Proposition 8.22. *Consider a function $f : [a,b] \to \mathbb{R}$ and $n \in \mathbb{N}$.*

(i) *Let f be continuous on $[a,b]$ and differentiable on (a,b). Suppose there exists $\alpha \in \mathbb{R}$ such that $|f'(x)| \le \alpha$ for all $x \in (a,b)$. Then*

$$\left|\int_a^b f(x)dx - R_n(f)\right| \le \frac{(b-a)^2\alpha}{2n}.$$

(ii) *Let f be continuously differentiable on $[a,b]$ and let f'' exist on (a,b). Suppose there exists $\beta \in \mathbb{R}$ such that $|f''(x)| \le \beta$ for all $x \in (a,b)$. Then*

$$\left|\int_a^b f(x)dx - M_n(f)\right| \le \frac{(b-a)^3\beta}{24n^2}.$$

Proof. Let $\mathsf{P_n} := \{x_0, x_1, \ldots, x_n\}$ denote the partition of $[a,b]$ into n equal parts, so that $x_i - x_{i-1} = (b-a)/n$ for $i = 1, \ldots, n$.

(i) Let $s_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$ and

$$R_n(f) = \sum_{i=1}^n f(s_i)(x_i - x_{i-1}).$$

By domain additivity (Proposition 6.8),

$$\int_a^b f(x)dx - R_n(f) = \sum_{i=1}^n \Big(\int_{x_{i-1}}^{x_i} f(x)dx - (x_i - x_{i-1})f(s_i)\Big).$$

By part (i) of Lemma 8.21 applied to the function f on the interval $[x_{i-1}, x_i]$ and with $c = s_i$, we see that the ith summand on the right-hand side of the above equation is

$$\frac{1}{2}\left((x_i - s_i)^2 f'(\xi_i) - (x_{i-1} - s_i)^2 f'(\eta_i)\right)$$

for some ξ_i, $\eta_i \in (x_{i-1}, x_i)$. Since

$$(x_i - s_i)^2 + (x_{i-1} - s_i)^2 \leq \big((x_i - s_i) + (s_i - x_{i-1})\big)^2 = (x_i - x_{i-1})^2 = \frac{(b-a)^2}{n^2}$$

and since $|f'(\xi_i)|$, $|f'(\eta_i)| \leq \alpha$ for $i = 1, \ldots, n$, we obtain

$$\left|\int_a^b f(x)dx - R_n(f)\right| \leq \frac{1}{2}\frac{(b-a)^2\alpha}{n^2} \cdot n = \frac{(b-a)^2\alpha}{2n},$$

as desired.

(ii) Again by domain additivity,

$$\int_a^b f(x)dx - M_n(f) = \sum_{i=1}^{n}\left[\int_{x_{i-1}}^{x_i} f(x)dx - (x_i - x_{i-1}) f\Big(\frac{x_{i-1} + x_i}{2}\Big)\right].$$

By part (ii) of Lemma 8.21 applied to the function f on the interval $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$, we see that the ith summand on the right-hand side of the above equation is

$$\frac{(x_i - x_{i-1})^3}{24} f''(\zeta_i)$$

for some ζ_i in (x_{i-1}, x_i). Since $x_i - x_{i-1} = (b-a)/n$ and $|f''(\zeta_i)| \leq \beta$ for $i = 1, \ldots, n$, we obtain

$$\left|\int_a^b f(x)dx - M_n(f)\right| \leq \frac{(b-a)^3\beta}{24n^3} \cdot n = \frac{(b-a)^3\beta}{24n^2},$$

as desired. □

We proceed to derive error estimates for the Compound Trapezoidal Rule and the Compound Simpson Rule. As before, we shall estimate the difference between the "signed area" under the curve given by $y = f(x)$, $a \leq x \leq b$, and the "signed area" obtained by replacing the function f by a polynomial function of degree at most one (for the Trapezoidal Rule) as well as by a polynomial function of degree at most two (for the Simpson Rule) on the entire interval $[a, b]$. Then we apply the results with the interval $[a, b]$ replaced by the subintervals induced by the partition $\mathsf{P}_n := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ into n equal parts, and take the sum.

Lemma 8.23. *Consider a function* $f : [a, b] \to \mathbb{R}$.

(i) *Let* f *be continuously differentiable on* $[a, b]$ *and let* f'' *exist on* (a, b). *Then there exists* $\xi \in (a, b)$ *such that*

$$\int_a^b f(x)dx = \frac{(b-a)}{2}\big(f(a) + f(b)\big) - \frac{(b-a)^3}{12} f''(\xi).$$

(ii) *Let* f', f'', f''' *exist and be continuous on* $[a, b]$, *and let* $f^{(4)}$ *exist on* (a, b). *Then there exists* $\eta \in (a, b)$ *such that*

$$\int_a^b f(x)dx = \frac{(b-a)}{6}\left(f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b)\right) - \frac{(b-a)^5}{2880} f^{(4)}(\eta).$$

Proof. (i) Consider the function $F : [a, b] \to$ defined by

$$F(x) := \int_a^x f(t)dt - \frac{x-a}{2}\big(f(a) + f(x)\big) \quad \text{for } x \in [a, b].$$

Then $F(a) = 0$ and

$$F(b) = \int_a^b f(t)dt - \frac{b-a}{2}\big(f(a) + f(b)\big).$$

In order to express $F(b)$ as $-(b-a)^3 f''(\xi)/12$ for some $\xi \in (a, b)$, we consider the function $G : [a, b] \to \mathbb{R}$ defined by

$$G(x) := F(x) - \frac{(x-a)^3}{(b-a)^3} F(b).$$

Then $G(a) = 0 = G(b)$, and by part (i) of the FTC (Proposition 6.24),

$$\begin{aligned} G'(x) &= f(x) - \frac{1}{2}\big(f(a) + f(x)\big) - \frac{x-a}{2} f'(x) - \frac{3(x-a)^2}{(b-a)^3} F(b) \\ &= \frac{f(x) - f(a)}{2} - \frac{x-a}{2} f'(x) - \frac{3(x-a)^2}{(b-a)^3} F(b) \quad \text{for } x \in [a, b]. \end{aligned}$$

Hence $G'(a) = 0$ and also

$$\begin{aligned} G''(x) &= \frac{f'(x)}{2} - \frac{f'(x)}{2} - \frac{x-a}{2} f''(x) - \frac{6(x-a)}{(b-a)^3} F(b) \\ &= -\frac{x-a}{2}\left(f''(x) + \frac{12}{(b-a)^3} F(b)\right) \quad \text{for } x \in (a, b). \end{aligned}$$

Using the Taylor Theorem (Proposition 4.25) for the function G with $n = 1$, we see that there exists $\xi \in (a, b)$ such that

$$G(b) = G(a) + G'(a)(b-a) + G''(\xi)\frac{(b-a)^2}{2},$$

that is, $G''(\xi) = 0$. Since $\xi \neq a$, it follows that

$$F(b) = -\frac{(b-a)^3}{12} f''(\xi),$$

as desired.

(ii) Consider the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) := \int_{a+(b-x)/2}^{(b+x)/2} f(t)dt - \frac{x-a}{6}\left(f\left(a + \frac{b-x}{2}\right) + 4f\left(\frac{a+b}{2}\right) + f\left(\frac{b+x}{2}\right)\right).$$

Then $F(a) = 0$ and

$$F(b) = \int_a^b f(t)dt - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).$$

In order to express $F(b)$ as $-(b-a)^5 f^{(4)}(\eta)/2880$ for some $\eta \in (a, b)$, consider the function $G : [a, b] \to \mathbb{R}$ defined by

$$G(x) := F(x) - \frac{(x-a)^5}{(b-a)^5} F(b).$$

Then $G(a) = 0 = G(b)$, and by part (i) of the FTC as well as the Chain Rule (Proposition 4.10), we see that for all $x \in [a, b]$,

$$\begin{aligned} G'(x) = {} & \frac{1}{2} f\left(\frac{b+x}{2}\right) + \frac{1}{2} f\left(a + \frac{b-x}{2}\right) \\ & -\frac{1}{6}\left(f\left(a + \frac{b-x}{2}\right) + 4f\left(\frac{a+b}{2}\right) + f\left(\frac{b+x}{2}\right)\right) \\ & -\frac{x-a}{6}\left(-\frac{1}{2} f'\left(a + \frac{b-x}{2}\right) + \frac{1}{2} f'\left(\frac{b+x}{2}\right)\right) - \frac{5(x-a)^4}{(b-a)^5} F(b). \end{aligned}$$

Hence $G'(a) = 0$. It can be easily verified that $G''(a) = 0$ and for $x \in [a, b]$,

$$G'''(x) = -\frac{x-a}{48}\left(f'''\left(\frac{b+x}{2}\right) - f'''\left(a + \frac{b-x}{2}\right)\right) - \frac{60(x-a)^2}{(b-a)^5} F(b).$$

By the Taylor Theorem for G and $n = 2$, there exists $\xi \in (a, b)$ such that

$$G(b) = G(a) + G'(a)(b-a) + G''(a)\frac{(b-a)^2}{2} + G'''(\xi)\frac{(b-a)^3}{6},$$

that is, $G'''(\xi) = 0$. Since $\xi \neq a$, it follows that

$$f'''\left(\frac{b+\xi}{2}\right) - f'''\left(a + \frac{b-\xi}{2}\right) = -\frac{2880(\xi - a)}{(b-a)^5} F(b).$$

Now by the MVT (Proposition 4.20) applied to the function f''' on the interval $[a+(b-\xi)/2, (b+\xi)/2]$, there exists $\eta \in (a+(b-\xi)/2, (b+\xi)/2) \subseteq (a,b)$ such that

$$f'''\Big(\frac{b+\xi}{2}\Big) - f'''\Big(a+\frac{b-\xi}{2}\Big) = \Big(\frac{b+\xi}{2} - a - \frac{b-\xi}{2}\Big) f^{(4)}(\eta) = (\xi - a) f^{(4)}(\eta).$$

Again, since $\xi \neq a$, it follows that

$$F(b) = -\frac{(b-a)^5}{2880} f^{(4)}(\eta),$$

as desired. □

Proposition 8.24. *Consider a function $f : [a,b] \to \mathbb{R}$ and $n \in \mathbb{N}$.*

(i) *Let f be continuously differentiable on $[a,b]$ and let f'' exist on (a,b). Suppose there exists $\beta \in \mathbb{R}$ such that $|f''(x)| \le \beta$ for all $x \in (a,b)$. Then*

$$\left| \int_a^b f(x)dx - T_n(f) \right| \le \frac{(b-a)^3 \beta}{12n^2}.$$

(ii) *Let f', f'', f''' exist and be continuous on $[a,b]$, and let $f^{(4)}$ exist on (a,b). Suppose there exists $\gamma \in \mathbb{R}$ such that $|f^{(4)}(x)| \le \gamma$ for all $x \in (a,b)$, and let $n \in \mathbb{N}$ be even. Then*

$$\left| \int_a^b f(x)dx - S_n(f) \right| \le \frac{(b-a)^4 \gamma}{180n^4}.$$

Proof. Let $\mathsf{P_n} := \{x_0, x_1, \dots, x_n\}$ denote the partition of $[a,b]$ into n equal parts, so that $x_i - x_{i-1} = (b-a)/n$ for $i = 1, \dots, n$.

(i) By domain additivity,

$$\int_a^b f(x)dx - T_n(f) = \sum_{i=1}^n \Big(\int_{x_{i-1}}^{x_i} f(x)dx - \frac{(x_i - x_{i-1})}{2} \big(f(x_{i-1}) + f(x_i)\big) \Big).$$

By part (i) of Lemma 8.23 applied to the function f on the interval $[x_{i-1}, x_i]$ for $i = 1, \dots, n$, we see that the ith summand on the right side of the above equation is equal to

$$-\frac{(x_i - x_{i-1})^3}{12} f''(\xi_i)$$

for some ξ_i in (x_{i-1}, x_i). Since $x_i - x_{i-1} = (b-a)/n$ and $|f''(\xi_i)| \le \beta$ for $i = 1, \dots, n$, we obtain

$$\left| \int_a^b f(x)dx - T_n(f) \right| \le \frac{(b-a)^3 \beta}{12n^3} \cdot n = \frac{(b-a)^3 \beta}{12n^2},$$

as desired.

(ii) Consider the partition $Q_n := \{x_0, x_2, \ldots, x_{n-2}, x_n\}$ of $[a, b]$. By domain additivity,

$$\int_a^b f(x)dx - S_n(f)$$

is equal to the sum of the terms

$$\int_{x_{i-1}}^{x_{i+1}} f(x)dx - \frac{(x_{i+1} - x_{i-1})}{6}\left(f(x_{i-1}) + 4f\left(\frac{x_{i-1} + x_{i+1}}{2}\right) + f(x_{i+1})\right)$$

for $i = 1, 3, \ldots, n-1$. By part (ii) of Lemma 8.23 applied to the function f on the interval $[x_{i-1}, x_{i+1}]$ for $i = 1, 3, \ldots, n-1$, we see that the ith term given above is

$$-\frac{(x_{i+1} - x_{i-1})^5}{2880} f^{(4)}(\eta_i)$$

for some $\eta_i \in (x_{i-1}, x_{i+1})$. Since $x_{i+1} - x_{i-1} = 2(b-a)/n$ and $|f^{(4)}(\eta_i)| \leq \gamma$ for $i = 1, 3, \ldots, n-1$, we obtain

$$\left|\int_a^b f(x)dx - S_n(f)\right| \leq \frac{2^5(b-a)^5\gamma}{2880n^5} \cdot \frac{n}{2} = \frac{(b-a)^5\gamma}{180n^4},$$

as desired. □

If we wish to approximate $\int_a^b f(x)dx$ by $R_n(f)$, $M_n(f)$, $T_n(f)$, or $S_n(f)$ with an error less than or equal to a given small positive number (such as 10^{-3}, 10^{-4}, etc.), then Propositions 8.22 and 8.24 can be used to find how large n must be taken, provided we know an upper bound for $|f'|$, $|f''|$, or $|f^{(4)}|$ as the case may be.

The important point to be noted here is that the approximations $R_n(f)$, $M_n(f)$, $T_n(f)$, and $S_n(f)$ are available for use if we know the values of the function f only at a certain n equally spaced points in $[a, b]$.

Example 8.25. Consider the function $f : [1, 2] \to \mathbb{R}$ defined by $f(x) := 1/x$. We know from Chapter 7 that $\int_1^2 f(x)dx = \ln 2$, and to estimate this value, we may apply the quadrature rules discussed above. For $n \in \mathbb{N}$, consider the partition $\mathsf{P_n} := \{x_0, x_1, \ldots, x_n\}$ of the interval $[1, 2]$ into n equal parts. Then

$$x_i = 1 + \frac{i}{n}, \quad i = 0, \ldots, n,$$

and so $h_n = 1/n$. If we use right endpoints of the subintervals for calculating $R_n(f)$, then

$$R_n(f) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+(i/n)} = \sum_{i=1}^{n}\frac{1}{n+i},$$

$$M_n(f) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{(x_{i-1}+x_i)/2} = \frac{1}{n}\sum_{i=1}^{n}\frac{2}{2+\big((2i-1)/n\big)} = 2\sum_{i=1}^{n}\frac{1}{2(n+i)-1},$$

$$T_n(f) = \frac{1}{2n}\left(\frac{1}{1}+2\left(\frac{1}{1+(1/n)}+\frac{1}{1+(2/n)}+\cdots+\frac{1}{2-(1/n)}\right)+\frac{1}{2}\right)$$
$$= \frac{3}{4n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1},$$

and if n is even, then

$$S_n(f) = \frac{1}{3n}\left(\frac{1}{1}+4\left(\frac{1}{1+(1/n)}+\frac{1}{1+(3/n)}+\cdots+\frac{1}{2-(1/n)}\right)\right.$$
$$\left.+2\left(\frac{1}{1+(2/n)}+\frac{1}{1+(4/n)}+\cdots+\frac{1}{2-(2/n)}\right)+\frac{1}{2}\right)$$
$$= \frac{1}{2n}+\frac{4}{3}\left(\frac{1}{n+1}+\frac{1}{n+3}+\cdots+\frac{1}{2n-1}\right)$$
$$+\frac{2}{3}\left(\frac{1}{n+2}+\frac{1}{n+4}+\cdots+\frac{1}{2n-2}\right).$$

Now note that for all $x \in (1,2)$,

$$|f'(x)| = \left|\frac{-1}{x^2}\right| \le 1, \quad |f''(x)| = \left|\frac{2}{x^3}\right| \le 2 \text{ and } \left|f^{(4)}(x)\right| = \left|\frac{24}{x^5}\right| \le 24.$$

Hence Proposition 8.22 shows that for each $n \in \mathbb{N}$,

$$\left|\int_1^2 \frac{1}{x}dx - R_n(f)\right| \le \frac{1}{2n} \quad \text{and} \quad \left|\int_1^2 \frac{1}{x}dx - M_n(f)\right| \le \frac{1}{12n^2},$$

while Proposition 8.24 shows that for each $n \in \mathbb{N}$,

$$\left|\int_1^2 \frac{1}{x}dx - T_n(f)\right| \le \frac{1}{6n^2} \quad \text{and if } n \text{ is even, then} \quad \left|\int_1^2 \frac{1}{x}dx - S_n(f)\right| \le \frac{2}{15n^4}.$$

To approximate the Riemann integral $\int_1^2 (1/x)dx$ with an error less than 10^{-3}, we must choose

(i) $n \ge 501$ if we use $R_n(f)$, so as to have $\dfrac{1}{2n} < 10^{-3}$,

(ii) $n \ge 10$ if we use $M_n(f)$, so as to have $\dfrac{1}{12n^2} < 10^{-3}$,

(iii) $n \ge 13$ if we use $T_n(f)$, so as to have $\dfrac{1}{6n^2} < 10^{-3}$, and

(iv) $n \ge 4$ if we use $S_n(f)$, so as to have $\dfrac{2}{15n^4} < 10^{-3}$. ◇

Notes and Comments

We have given in this chapter a systematic development of the notion of area of a region between two curves given by Cartesian equations of the form $y = f(x)$ *or* $x = g(y)$, *or by polar equations of the form* $r = p(\theta)$ *or* $\theta = \alpha(r)$. *Two methods of finding the volume of a solid body are described in this chapter: (i) by considering the slices of the solid body by planes perpendicular to a given line and (ii) by considering the slivers of the solid body by cylinders having a common axis. For solids obtained by revolving planar regions about a line, these two methods specialize to the washer method and the shell method.*

We have motivated the definition of the length of a smooth curve by considering the tangent line approximations of such a curve. The following alternative motivation is often given. If a curve C *is given by* $(x(t), y(t))$, $t \in [\alpha, \beta]$, *consider a partition* $\{t_0, t_1, \ldots, t_n\}$ *of the interval* $[\alpha, \beta]$. *The sum*

$$\sum_{i=1}^{n} \sqrt{\big(x(t_i) - x(t_{i-1})\big)^2 + \big(y(t_i) - y(t_{i-1})\big)^2}$$

of the lengths of the line segments joining the points $(x(t_{i-1}), y(t_{i-1}))$ *and* $(x(t_i), y(t_i))$ *for* $i = 1, \ldots, n$ *can be considered an approximation of the "length" of the curve* C. *If the functions* x *and* y *are continuous on* $[\alpha, \beta]$ *and are differentiable on* (α, β), *then by the MVT this sum can be written as*

$$\sum_{i=1}^{n} \sqrt{x'(s_i)^2 + y'(u_i)^2}\,(t_i - t_{i-1}),$$

where $s_i, u_i \in (t_{i-1}, t_i)$ *for* $i = 1, \ldots, n$. *We are then naturally led to the definition of the length of* C *given in the text. We have opted for a motivation based on tangent lines because this consideration extends analogously to a motivation for the definition of the "area of a smooth surface" based on tangent planes given in a course on multivariable calculus. (See, for example, [33, pp. 311–313].) In contrast, the analogue of the limit of the sums of the lengths of chords, namely the limit of the areas of inscribed polyhedra formed of triangles, may not exist even for a simple-looking surface such as a cylinder. See, for example, Appendix A.4 of Chapter 4 in Volume II of the book by Courant and John [21].*

We have extended the notion of length to a piecewise smooth curve using domain additivity. In fact, the length of any "rectifiable" curve can be defined. However, we have relegated this to an exercise, since the present chapter deals with applications of integration, and rectifiability is defined without any reference to integration. (See Exercise 8.70.)

The basic idea behind the definition of the area of a surface generated by revolving a curve about a line is to approximate the curve by a piecewise linear curve and to consider the areas of the frustums of cones generated by revolving the line segments that approximate the curve.

For $\varphi \in [0, \pi]$*, the sector* $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2$ *and* $0 \leq \theta(x, y) \leq \varphi\}$ *of a disk of radius a subtends an angle* φ *at the center. We have shown that the area of this sector is* $a^2\varphi/2$*, and if this sector is revolved about the x-axis, then it generates a (solid) spherical cone whose volume is* $2a^3(1 - \cos\varphi)/3$*, while the surface area of the spherical cap so generated is* $2\pi a^2(1 - \cos\varphi)$*. Letting* $\varphi = \pi$*, we may obtain the area of a disk of radius a, the volume of a ball of radius a, and the surface area of a sphere of radius a.*

We have calculated the area enclosed by an ellipse and the volume enclosed by an ellipsoid. However, it is not possible to calculate the arc length of an ellipse or the surface area of an ellipsoid in terms of algebraic functions and elementary transcendental functions. The same holds for the arc length of a lemniscate. To find these, one is led to the so-called "elliptic integrals" and "lemniscate integrals". Inverting functions defined by elliptic integrals gives rise to a new class of functions known as "elliptic functions", just as inverting the function arctan *defined by*

$$\arctan x = \int_0^x \frac{1}{1+t^2}dt \quad \textit{for } x \in \mathbb{R}$$

led us to the tangent function in Section 7.2. The study of elliptic functions, and their geometric counterpart, the elliptic curves, is rich and fascinating, and it connects many branches of mathematics. For an early history of elliptic integrals, see the articles of Sridharan [77], and for a relatively accessible introduction to elliptic curves, see the book of Silverman and Tate [73].

The results given in this chapter show that the real number π *introduced in Section 7.2 is equal to each of the following:*

$$\frac{\textit{Area}(D)}{\textit{Radius}(D)^2}, \quad \frac{3}{4}\frac{\textit{Volume}(B)}{\textit{Radius}(B)^3}, \quad \frac{1}{2}\frac{\textit{Perimeter}(C)}{\textit{Radius}(C)}, \quad \frac{1}{4}\frac{\textit{Surface Area}(S)}{\textit{Radius}(S)^2},$$

where D*,* B*,* C*, and* S *denote a disk, a ball, a circle, and a sphere respectively. These formulas are often used in high-school geometry without any proofs.*

The results of Pappus regarding the centroids of surfaces of revolution and of solids of revolution are truly remarkable, especially since they were conceived as early as the fourth century AD. They reduce the calculations of areas of surfaces of revolution and volumes of solids of revolution to the calculations of arc lengths and planar areas respectively.

We have pointed out in this chapter that one needs the notions of multiple integrals to introduce the general concepts of area and volume. We have provided specific references in which it is shown that the definitions given in this chapter are special cases of the general treatment. These show, for example, that the volume of a solid body calculated by the washer method and by the shell method must come out to be the same!

In the section on quadrature rules, our proofs of error estimates do not use divided differences; they are based only on the Fundamental Theorem of Calculus and the Taylor Theorem. These proofs are inspired by the treatment *on pages 328–330 of Hardy's book [39]. Admittedly, they are quite involved, but they display the power of the Taylor Theorem. If f is an infinitely differentiable function and the "Taylor series" of f converges to f, then these error estimates can be obtained more easily, as indicated in Exercise 9.53.*

Exercises

Part A

8.1. Find the average of the function $f : [1,2] \to \mathbb{R}$ defined by $f(x) := 1/x$.

8.2. Given a circle of radius a and a diameter AB of the circle, chords are drawn perpendicular to AB intercepting equal segments at each point of AB. Find the average length of these chords.

8.3. Given a circle of radius a and a diameter AB of the circle, for each $n \in \mathbb{N}$, n chords are drawn perpendicular to AB so as to intercept equal arcs along the circumference of the circle. Find the limit of the average length of these n chords as $n \to \infty$.

8.4. Let $f : [a,b] \to \mathbb{R}$ be differentiable such that f' is integrable on $[a,b]$. Show that the average of f' is equal to the average rate of change of f on $[a,b]$, namely $\big(f(b) - f(a)\big)/(b-a)$.

8.5. Let a, b be positive real numbers. If $f(x) := (b/a)\sqrt{a^2 - x^2}$ and $w(x) := x$ for $0 \le x \le a$, find the average of
(i) f^2 with respect to w, (ii) f with respect to w^2.

8.6. If $f, g : [a,b] \to \mathbb{R}$ are integrable functions, then show that $\text{Av}(f+g) = \text{Av}(f) + \text{Av}(g)$, but $\text{Av}(fg)$ may not be equal to $\text{Av}(f)\text{Av}(g)$.

8.7. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) := x$. Find $\text{Av}(f; w)$ and $\text{Av}(w; f)$ if $w : [0,1] \to \mathbb{R}$ is defined by
(i) $w(x) := x$, (ii) $w(x) := x^2$, (iii) $w(x) := 1 - x$, (iv) $w(x) := x(1-x)$.

8.8. Find the area of the region bounded by the given curves in each of the following cases:
(i) $y = 0$, $y = 2x + 3$, $x = 0$, and $x = 1$, (ii) $y = 4 - x^2$ and $y = 0$,
(iii) $\sqrt{x} + \sqrt{y} = 1$, $x = 0$, and $y = 0$, (iv) $y = x^4 - 2x^2$ and $y = 2x^2$,
(v) $y = 3x^5 - x^3$, $x = -1$, and $x = 1$, (vi) $x = y^3$ and $x = y^2$,
(vii) $y = 2 - (x-2)^2$ and $y = x$, (viii) $x = 3y - y^2$ and $x + y = 3$.

8.9. Find the area of the region bounded on the right by the line given by $x + y = 2$, on the left by the parabola given by $y = x^2$, and below by the x-axis.

8.10. Let $a \in \mathbb{R}$. Define $f(x) := x - x^2$ and $g(x) := ax$ for $x \in \mathbb{R}$. Determine a such that the region above the graph of g and below the graph of f has area equal to $\frac{9}{2}$.

8.11. Show that the area of a triangular region with vertices at $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) is equal to $|\Delta|/2$, where Δ is the 3×3 determinant

$$\Delta = \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}.$$

8.12. Show that the area of the elliptical region given by $ax^2 + 2bxy + cy^2 \le 1$, where $a, b, c \in \mathbb{R}$, $c > 0$, and $ac - b^2 > 0$, is equal to $\pi/\sqrt{ac - b^2}$.

8.13. Let $\alpha, \beta \in \mathbb{R}$ with $\beta \ne 0$. Show that the areas $A_0, A_1, A_2, \ldots$ of the regions bounded by the x-axis and the half-waves of the curve $y = e^{\alpha x} \sin \beta x$, $x \ge 0$, form a geometric progression with the common ratio $e^{\alpha\pi/\beta}$.

8.14. Let $a \in \mathbb{R}$ with $a > 0$. Find the area of the region enclosed by the lemniscate given by the polar equation $r = a\sqrt{2\cos 2\theta}$ and the rays $\theta = 0$, $\theta = \pi/4$.

8.15. Let $a \in \mathbb{R}$ with $a > 0$. Find the area of the region inside the circle given by $r = 6a\cos\theta$ and outside the cardioid given by $r = 2a(1 + \cos\theta)$.

8.16. Let $a \in \mathbb{R}$ with $a > 0$. Find the area of the region enclosed by the loop of the **folium of Descartes** given by $x^3 + y^3 = 3axy$.

8.17. Let $p, q \in \mathbb{R}$ satisfy $0 \le p < q$ and let $\alpha_1, \alpha_2 : [p, q] \to \mathbb{R}$ be integrable functions such that $-\pi \le \alpha_1 \le \alpha_2 \le \pi$. Let $R := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 : p \le r \le q \text{ and } \alpha_1(r) \le \theta \le \alpha_2(r)\}$ denote the region between the curves given by $\theta = \alpha_1(r)$, $\theta = \alpha_2(r)$ and between the circles given by $r = p$, $r = q$. Define

$$\text{Area } (R) := \int_p^q r\big(\alpha_2(r) - \alpha_1(r)\big)dr.$$

Give a motivation for the above definition along the lines of the motivation given in the text for the definition of the area of the region between curves given by polar equations of the form $r = p(\theta)$.

8.18. (i) Let $p, q \in \mathbb{R}$ be such that $0 \le p < q$ and $\varphi \in [0, \pi]$. Using the formula given in Exercise 8.17, show that the area of the circular strip $\{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 : p \le r \le q \text{ and } 0 \le \theta \le \varphi\}$ is $(q^2 - p^2)\varphi/2$.

(ii) Let $\alpha : [1, 2] \to \mathbb{R}$ be given by $\alpha(r) := 4\pi(r - 1)(2 - r)$, and let $R := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 : 1 \le r \le 2 \text{ and } 0 \le \theta \le \alpha(r)\}$ Show that the area of R is equal to π.

(iii) Let $R := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 : 1 \le r \le 2 \text{ and } r \le \theta \le r\sqrt{r}\}$. Find Area (R).

8.19. Let $a \in \mathbb{R}$ with $a > 0$. The base of a certain solid body is the disk given by $x^2 + y^2 \le a^2$. Each of its slices by a plane perpendicular to the x-axis is an isosceles right-angled triangular region with one of the two equal sides in the base of the solid body. Find the volume of the solid body.

8.20. A solid body lies between the planes given by $y = -2$ and $y = 2$. Each of its slices by a plane perpendicular to the y-axis is a disk with a diameter extending between the curves given by $x = y^2$ and $x = 8 - y^2$. Find the volume of the solid body.

8.21. A twisted solid is generated as follows. A fixed line L in 3-space and a square of side s in a plane perpendicular to L are given. One vertex of the square is on L. As this vertex moves a distance h along L, the square turns through a full revolution with L as the axis. Find the volume of the solid generated by this motion. What would the volume be if the square had turned through two full revolutions in moving the same distance along the line L?

8.22. Let $a, b \in \mathbb{R}$ with $0 \le a < b$. Suppose that a planar region R lies between the lines given by $x = a$ and $x = b$, and for each $s \in [a, b]$, the line given by $x = s$ intersects R in a finite number of line segments whose total length is $\ell(s)$. If the function $\ell : [a, b] \to \mathbb{R}$ is integrable, then show that the volume of the solid body obtained by revolving the region R about the y-axis is equal to

$$2\pi \int_a^b x\,\ell(x)dx.$$

8.23. Find the volume of the solid of revolution obtained by revolving the region bounded by the curves given by $y = 3 - x^2$ and $y = -1$ about the line given by $y = -1$ by both the washer method and the shell method.

8.24. The disk given by $x^2 + (y - b)^2 \le a^2$, where $0 < a < b$, is revolved about the x-axis to generate a solid torus. Find the volume of this solid torus by both the washer method and the shell method.

8.25. A round hole of radius $\sqrt{3}$ cm is bored through the center of a solid ball of radius 2 cm. Find the volume cut out.

8.26. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the curves given by $y = x^3$ and $y = 4x$ about the x-axis by both the washer method and the shell method.

8.27. Let $f : [0, \infty) \to [0, \infty)$ be a continuous function. If for each $a > 0$, the volume of the solid obtained by revolving the region under the curve $y = f(x)$, $0 \le x \le a$, about the x-axis is equal to $a^2 + a$, determine f.

8.28. Find the volume of the solid generated by revolving the region bounded by the curves given by $y = \sqrt{x}$, $y = 2$, and $x = 0$ about the x-axis by both the washer method and the shell method. If the region is revolved about the line given by $x = 4$, what is the volume of the solid so generated?

8.29. If the region bounded by the curves given by $y = \tan x$, $y = 0$, and $x = \pi/3$ is revolved about the x-axis, find the volume of the solid so generated.

8.30. Find the arc length of each of the curves mentioned below.

(i) the cuspidal cubic given by $y^2 = x^3$ between the points $(0, 0)$ and $(4, 8)$,

(ii) the cycloid given by $x = t - \sin t,\ y = 1 - \cos t,\ -\pi \le t \le \pi$,

(iii) the curve given by $(y + 1)^2 = 4x^3,\ 0 \le x \le 1$,

(iv) the curve given by $y = \displaystyle\int_0^x \sqrt{\cos 2t}\, dt,\ 0 \le x \le \pi/4$.

8.31. Let $p, q \in \mathbb{R}$ with $0 \le p < q$ and $\alpha : [p, q] \to \mathbb{R}$. If a piecewise smooth curve C is given by $\theta = \alpha(r)$, $r \in [p, q]$. Show that its arc length is

$$\ell(C) = \int_p^q \sqrt{1 + r^2\alpha'(r)^2}\, dr.$$

(Hint: If $x(r) := r\cos\alpha(r)$ and $y(r) := r\sin\alpha(r)$ for $r \in [p,q]$, then $x'(r)^2 + y'(r)^2 = 1 + r^2\alpha'(r)^2$.)

8.32. Show that the arc length of the spiral given by $\theta = r$, $r \in [0,\pi]$, is

$$\frac{1}{2}\pi\sqrt{1+\pi^2} + \frac{1}{2}\ln\left(\pi + \sqrt{1+\pi^2}\right).$$

(Hint: Revision Exercise R.44 (ii) given at the end of Chapter 7.)

8.33. For each of the following curves, find the arc length as well as the area of the surface generated by revolving the curve about the x-axis.
 (i) the asteroid given by $x = a\cos^3\theta,\ y = a\sin^3\theta,\ -\pi \le \theta \le \pi$,
 (ii) the loop of the curve given by $9x^2 = y(3-y)^2$, $0 \le y \le 3$.

8.34. For each of the following curves, find the arc length as well as the area of the surface generated by revolving the curve about the line given by $y = -1$.
 (i) $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$, $1 \le x \le 3$, (ii) $x = \dfrac{3}{5}y^{5/3} - \dfrac{3}{4}y^{1/3}$, $1 \le y \le 8$.

8.35. Find the arc length of the curve given by

$$y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}, \quad 1 \le x \le 4,$$

and find the area of the surface generated by revolving the curve about the x-axis.

8.36. Show that the surface area of the torus obtained by revolving the circle given by $x^2 + (y-b)^2 = a^2$, where $0 < a < b$, about the x-axis is equal to $4\pi^2 ab$. (Compare Example 8.14 (iii).)

8.37. For each of the following curves, find the area of the surface generated by revolving the curve about the y-axis.
 (i) $y = (x^2+1)/2,\ 0 \le x \le 1$,
 (ii) $x = t+1,\ y = (t^2/2) + t,\ 0 \le t \le 1$.

8.38. Let $a \in \mathbb{R}$ with $a > 0$. An arc of the catenary given by $y = a\cosh(x/a)$ whose endpoints have abscissas 0 and a is revolved about the x-axis. Show that the surface area A and the volume V of the solid thus generated are related by the formula $A = 2V/a$.

8.39. How accurately should we measure the radius of a ball in order to calculate its surface area within 3 percent of its exact value?

8.40. Given a right circular cone of base radius a and height h, find the radius and the height of the right circular cylinder having the largest lateral surface area that can be inscribed in the cone.

8.41. Let $p, q \in \mathbb{R}$ with $0 \le p < q$ and $\alpha : [p,q] \to \mathbb{R}$. Suppose a piecewise smooth curve given by $\theta = \alpha(r)$, $r \in [p,q]$, is revolved about a line through the origin containing a ray given by $\theta = \gamma$, and not crossing the curve. If S denotes the surface so generated, then show that

$$\text{Area } (S) = 2\pi \int_p^q r|\sin(\alpha(r) - \gamma)|\sqrt{1 + r^2\alpha'(r)^2}\,dr.$$

(Hint: Compare Exercise 8.31 and note that for $r \in [p, q]$, the distance of the point $(r\cos\alpha(r), r\sin\alpha(r))$ from the line L is equal to $r|\sin(\alpha(r)-\gamma)|$.)

8.42. Let $\ell, \phi \in \mathbb{R}$ with $\ell > 0$. Consider the line segment given by $\theta = \alpha(r)$, where $\alpha(r) := \varphi$ for $r \in [0, \ell]$. If this line segment is revolved about the x-axis, show that the area of the cone S so generated is equal to $\pi\ell^2|\sin\varphi|$. [Note: Since the right circular cone S has slant height ℓ and base radius $\ell|\sin\varphi|$, the result matches the earlier calculation of the surface area of a right circular cone done by splitting it open.]

8.43. If a piecewise smooth curve C is given by $y = f(x)$, $x \in [a, b]$, and $\ell(C) = \int_a^b \sqrt{1 + f'(x)^2}\,dx \neq 0$, then show that the centroid $(\overline{x}, \overline{y})$ of C is given by

$$\overline{x} = \frac{1}{\ell(C)} \int_a^b x\sqrt{1 + f'(x)^2}\,dx \quad \text{and} \quad \overline{y} = \frac{1}{\ell(C)} \int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx.$$

8.44. If a piecewise smooth curve C is given by $r = p(\theta)$, $\theta \in [\alpha, \beta]$, and if its length $\ell(C)$ is nonzero, then show that the centroid $(\overline{x}, \overline{y})$ of C is given by

$$\overline{x} = \frac{1}{\ell(C)} \int_\alpha^\beta p(\theta)\cos\theta\sqrt{p(\theta)^2 + p'(\theta)^2}\,d\theta$$

and

$$\overline{y} = \frac{1}{\ell(C)} \int_\alpha^\beta p(\theta)\sin\theta\sqrt{p(\theta)^2 + p'(\theta)^2}\,d\theta.$$

8.45. Let $a > 0$ and $\varphi \in [0, \pi]$. Find the centroid of the arc of the circle given by the polar equation $r = a$, $0 \le \theta \le \varphi$.

8.46. By choosing a suitable coordinate system, find the centroids of (i) a hemisphere of radius a and (ii) a cylinder of radius a and height h.

8.47. Let $a \in \mathbb{R}$ with $a > 0$. Find the centroid of the region bounded by the curves given by $y = -a$, $x = a$, $x = -a$, and $y = \sqrt{a^2 - x^2}$.

8.48. Find the centroid of the region enclosed by the curves given by $y^2 = 8x$ and $y = x^2$.

8.49. Find the centroid of the region in the first quadrant bounded by the curves given by $4y = x^2$, $x = 0$, and $y = 4$.

8.50. Find the centroid of the region in the first quadrant bounded by the curves given by $4x^2 + 9y^2 = 36$ and $x^2 + y^2 = 9$.

8.51. Let $a \in \mathbb{R}$ with $a > 0$. Show that the centroid of the ball $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le a^2\}$ is $(0, 0, 0)$.

8.52. Let $a \in \mathbb{R}$ with $a > 0$. Find the centroid of the hemispherical solid body generated by revolving the region under the curve given by $y = \sqrt{a^2 - x^2}$, $0 \le x \le a$.

8.53. Find the centroid of the region bounded by the curves given by $x = y^2 - y$ and $x = y$. If this region is revolved about the x-axis, find the centroid of the solid body so generated.

8.54. The region bounded by the curves given by $y = 0$, $x = 3$, and $y = x^2$ is revolved about the x-axis. Find the centroid of the solid body so generated.

8.55. Let $a > 0$. Use a result of Pappus to find the centroid of the region bounded by the curves given by $y = \sqrt{a^2 - x^2}$, $y = 0$, and $x = 0$. (Hint: Revolve the given region about a coordinate axis to generate a hemispherical solid.)

8.56. Let $a > 0$. Use a result of Pappus to find the centroid of the semicircular region bounded by the curves given by $y = \sqrt{a^2 - x^2}$ and $y = 0$. If this region is revolved about the line given $y = -a$, find the volume of the solid so generated.

8.57. Let $a > 0$. Use a result of Pappus to find the centroid of the semicircular arc $y = \sqrt{a^2 - x^2}$. If this arc is revolved about the line given by $y = a$, find the surface area so generated.

8.58. Let a and b be positive real numbers such that $a < b$. Find the y-coordinate of the centroid of the region bounded by curves given by $y = \sqrt{a^2 - x^2}$, $y = \sqrt{b^2 - x^2}$, and $y = 0$.

8.59. Use a result of Pappus to find (i) the volume of a cylinder with height h and radius a, (ii) the volume of a cone with height h and base radius a.

8.60. Use a result of Pappus to show that the lateral surface area of a cone of base radius a and slant height ℓ is $\pi \ell a$.

8.61. If $f : [a, b] \to \mathbb{R}$ is a polynomial function of degree at most 3, then show that for every $n \in \mathbb{N}$,

$$S_n(f) = \int_a^b f(x)dx.$$

(Compare part (ii) of Proposition 8.24.)

8.62. Let $f : [a, b] \to \mathbb{R}$ be convex. Show that for every $n \in \mathbb{N}$, the error

$$\int_a^b f(x)dx - T_n(f)$$

in using $T_n(f)$ as an approximation of $\int_a^b f(x)dx$ is nonpositive, and if f is a concave function, then it is nonnegative.

8.63. Let $f : [a, b] \to \mathbb{R}$ be any function. Let $n \in \mathbb{N}$ be even and let $\mathsf{P_n} := \{x_0, x_1, \ldots, x_n\}$ be the partition of $[a, b]$ into n equal parts. If $k := n/2$ and $\mathsf{Q_k} := \{x_0, x_2, \ldots, x_{2k-2}, x_{2k}\}$, then show that

$$S_n(f) = \frac{1}{3}\big(T_k(f) + 2M_k(f)\big),$$

where $S_n(f)$ is defined with respect to $\mathsf{P_n}$ and $T_k(f)$, $M_k(f)$ are defined with respect to $\mathsf{Q_k}$. Deduce that if f is integrable, then

$$S_n(f) \to \int_a^b f(x)dx \quad \text{as} \quad n \to \infty.$$

8.64. If f is continuous on $[a, b]$, f' exists on (a, b), and there exists $\alpha \in \mathbb{R}$ such that $|f'(x)| \le \alpha$ for all $x \in (a, b)$, then show that

$$\left| \int_a^b f(x)dx - M_n(f) \right| \le \frac{(b-a)^2\alpha}{4n}.$$

(Compare parts (i) and (ii) of Proposition 8.22.)

8.65. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) := 1/(1+x^2)$. Find $R_n(f)$, $M_n(f)$, and $T_n(f)$ for $n \in \mathbb{N}$, and $S_n(f)$ for even $n \in \mathbb{N}$. Prove that

$$\left|\int_0^1 f(x)dx - R_n(f)\right| \leq \frac{1}{n}, \quad \left|\int_0^1 f(x)dx - M_n(f)\right| \leq \frac{1}{6n^2},$$

while

$$\left|\int_0^1 f(x)dx - T_n(f)\right| \leq \frac{1}{3n^2} \text{ and } \left|\int_0^1 f(x)dx - S_n(f)\right| \leq \frac{2}{15n^4} \ (n \text{ even}).$$

Find how large n must be taken if we wish to approximate $\int_0^1 f(x)dx$ with an error less than 10^{-4} using each of $R_n(f)$, $M_n(f)$, $T_n(f)$, and $S_n(f)$.

8.66. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) := (1-x^2)^{3/2}$. Find $R_n(f)$, $M_n(f)$, $T_n(f)$, and $S_n(f)$ for $n = 4$ and $n = 6$. Also, find the corresponding error estimates.

8.67. Consider the **error function** $\operatorname{erf} : \mathbb{R} \to \mathbb{R}$ defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Use the Compound Simpson Rule with $n = 4$ to find an approximation α to $\operatorname{erf}(1)$ in terms of π and e. Show that $|\operatorname{erf}(1) - \alpha| \leq 19/5760$.

8.68. Consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = xe^{-x^2}$. Find $T_n(f)$ and $S_n(f)$ with $n = 2$ and $n = 4$. Obtain the corresponding error estimates, and compare them with the actual errors

$$\int_0^1 f(x)dx - T_n(f) \quad \text{and} \quad \int_0^1 f(x)dx - S_n(f).$$

Part B

8.69. Let $h > 0$. For each $x \in [0,h]$, the area of the slice at x of a solid body by a plane perpendicular to the x-axis is given by $A(x) := ax^2 + bx + c$. If $B_1 := A(0) = c$, $M := A(h/2) = (ah^2 + 2bh + 4c)/4$, and $B_2 := A(h) = ah^2 + bh + c$, then show that the volume of the solid body is equal to $(B_1 + 4M + B_2)/6$.
[Note: This formula is known as the **Prismoidal Formula**.]

8.70. Let a curve C in $\mathbb{R}^2$ be given by $(x(t), y(t))$, $t \in [\alpha, \beta]$. For a partition $\{t_0, t_1, \ldots, t_n\}$ of $[\alpha, \beta]$, let

$$\ell(C,P) := \sum_{i=1}^{n} \sqrt{\big(x(t_i) - x(t_{i-1})\big)^2 + \big(y(t_i) - y(t_{i-1})\big)^2}.$$

If the set $\{\ell(C,P) : P \text{ is a partition of } [\alpha,\beta]\}$ is bounded above, then the curve C is said to be **rectifiable**, and the **length** of C is defined to be

$$\ell(C) := \sup\{\ell(C,P) : P \text{ is a partition of } [\alpha,\beta]\}.$$

(i) If $\gamma \in (\alpha, \beta)$, and the curves C_1 and C_2 are given by $(x(t), y(t))$, $t \in [\alpha, \gamma]$, and by $(x(t), y(t))$, $t \in [\gamma, \beta]$, respectively, then show that C is rectifiable if and only if C_1 and C_2 are rectifiable.

(ii) Suppose that the functions x and y are differentiable on $[\alpha, \beta]$, and one of the derivatives x' and y' is continuous on $[\alpha, \beta]$, while the other is integrable on $[\alpha, \beta]$. Show that the curve C is rectifiable and

$$\ell(C) = \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

(Hint: Propositions 4.20, 6.36, and 3.20 and Exercise 6.39.) (Compare Exercise 6.45.)

(iii) Show that the conclusion in (ii) above holds if the functions x and y are continuous on $[\alpha, \beta]$ and if there exist a finite number of points $\gamma_0 < \gamma_1 < \cdots < \gamma_n$ in $[\alpha, \beta]$, where $\gamma_0 = \alpha$ and $\gamma_n = \beta$, such that the assumptions made in (ii) above about the functions x and y hold on each of the subintervals $[\gamma_{i-1}, \gamma_i]$ for $i = 1, \ldots, n$.

[Note: The result in (iii) above shows that the definition of the length of a piecewise smooth curve given in Section 8.3 is consistent with the definition of the length of a rectifiable curve given above. Analogous definitions and results hold for a curve in $\mathbb{R}^3$.]

8.71. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = x^2 \sin(\pi/x^2)$ for $x \in (0, 1]$. Given any $n \in \mathbb{N}$, consider the partition

$$P_n := \left\{0,\ n^{-1/2},\ \left(n - \frac{1}{2}\right)^{-1/2},\ (n-1)^{-1/2}, \ldots,\ (3/2)^{-1/2},\ 1\right\}$$

of $[0, 1]$ and write $P_n := \{x_0, x_1, \ldots, x_{2n-2}\}$. Show that

$$\sum_{i=1}^{2n-2} \sqrt{\left(x_i - x_{i-1}\right)^2 + \left(f(x_i) - f(x_{i-1})\right)^2} \geq \left(\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right).$$

Deduce that the curve $y = f(x)$, $0 \le x \le 1$, is not rectifiable even though the function f is differentiable. (Hint: Exercise 2.10.)

8.72. Let $f : [a, b] \to \mathbb{R}$ be a bounded function that is continuous on (a, b), and let $w : [a, b] \to \mathbb{R}$ be a weight function that is continuous and positive on (a, b). Show that there exists $c \in (a, b)$ such that $\mathrm{Av}(f; w) = f(c)$. (Hint: Apply the Cauchy Mean Value Theorem (Proposition 4.38) to the functions F, $G : [a, b] \to \mathbb{R}$ defined by $F(x) := \int_a^x f(t)w(t)dt$ and $G(x) := \int_a^x w(t)dt$.)

8.73. Let $f : [a, b] \to \mathbb{R}$ be a bounded function that is continuous on (a, b). If the range of f is contained in (α, β) and $\phi : [\alpha, \beta] \to \mathbb{R}$ is a convex function that is continuous at α and β, then show that $\mathrm{Av}(f) \in (\alpha, \beta)$, the function $\phi \circ f : [a, b] \to \mathbb{R}$ is integrable, and $\phi(\mathrm{Av}(f)) \le \mathrm{Av}(\phi \circ f)$. (Hint: Considering partitions of $[a, b]$ into equal parts, use Exercises 8.72, 6.38, and 3.44, and also Proposition 6.36.)

9

Infinite Series and Improper Integrals

If $a_1, \ldots, a_n$ are any real numbers, then we can add them together and form their sum $a_1 + \cdots + a_n$. In this chapter, we shall investigate whether we can "add" infinitely many real numbers. In other words, if (a_k) is a sequence of real numbers, then we ask whether we can give a meaning to a symbol such as $a_1 + a_2 + \cdots$ or $\sum_{k=1}^{\infty} a_k$. This leads us to consider what is known as an infinite series, or simply a series, of real numbers. The study of infinite series is taken up in the first three sections of this chapter, and this may be viewed as a sequel to the theory of sequences developed in Chapter 2. In Section 9.1 below, we define the notion of convergence of a series and thus give a precise meaning to the idea of forming the sum of infinitely many real numbers. A number of useful tests for the convergence of series are given in Section 9.2. In Section 9.3, we study a special kind of series, known as power series. We also discuss here the Taylor series, which is a natural analogue of the notion studied in Chapter 4 of the Taylor polynomial of a function.

In the last three sections of this chapter, we develop the theory of improper integrals, which are a continuous analogue of infinite series and which extend the theory of integration developed in Chapter 6. The notion of convergence of improper integrals and some basic properties are discussed in Section 9.4. A number of useful tests for the convergence of improper integrals are given in Section 9.5. In Section 9.6, we discuss some "integrals" that are related to improper integrals of the kind studied in the previous sections. We also discuss here the beta function and the gamma function, which are quite important and useful in analysis.

9.1 Convergence of Series

An **infinite series**, or for short, a **series**, of real numbers is an ordered pair $\big((a_k), (A_n)\big)$ of sequences (a_k) and (A_n) of real numbers such that

$$A_n = a_1 + \cdots + a_n \quad \text{for all } n \in \mathbb{N}.$$

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Undergraduate Texts in Mathematics, https://doi.org/10.1007/978-3-030-01400-1_9

Equivalently, it is an ordered pair $\big((a_k),(A_n)\big)$ of sequences in $\mathbb{R}$ such that

$$a_k = A_k - A_{k-1} \quad \text{for all } k \in \mathbb{N}, \quad \text{where} \quad A_0 := 0, \text{ by convention.}$$

The first sequence (a_k) is called the **sequence of terms**, and the second sequence (A_n) is called the **sequence of partial sums** of the (infinite) series $\big((a_k),(A_n)\big)$. For simplicity and brevity, we shall use the notation $\sum_{k\geq 1} a_k$ for the infinite series $\big((a_k),(A_n)\big)$. In this notation, prominence is given to the first sequence (a_k), but the second sequence (A_n) is just as important. At any rate, the two sequences (a_k) and (A_n) determine each other uniquely.

In some cases, it is convenient to consider the sequence (a_k) of terms indexed as $a_0, a_1, a_2, \ldots,$ or more generally, as $a_m, a_{m+1}, \ldots$ for some $m \in \mathbb{Z}$. The corresponding series will be denoted by $\sum_{k\geq 0} a_k$, or more generally, by $\sum_{k\geq m} a_k$. In such cases, the sequence (A_n) of partial sums will be indexed as $A_0, A_1, A_2, \ldots$ or more generally, as $A_m, A_{m+1}, \ldots$ for some $m \in \mathbb{Z}$. Accordingly, the convention $A_0 := 0$ is replaced by $A_{-1} := 0$ or more generally, $A_{m-1} := 0$. Sometimes, the indexing of (a_k) will be clear from the context, and we may simply use the notation $\sum_k a_k$ in place of the more elaborate $\sum_{k\geq 1} a_k$, or $\sum_{k\geq 0} a_k$, or $\sum_{k\geq m} a_k$. In the sequel, a statement dependent on $k \in \mathbb{N}$ is said to be true **for all large** $k \in \mathbb{N}$ if there exists $k_0 \in \mathbb{N}$ such that it is true for all $k \geq k_0$.

We say that a series $\sum_{k\geq 1} a_k$ is **convergent** if the sequence (A_n) of its partial sums is convergent, that is, if there exists $A \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ satisfying

$$|A_n - A| < \epsilon \quad \text{for all } n \geq n_0.$$

In this case, by part (i) of Proposition 2.2, the real number A is unique, and it is called the **sum** of the series $\sum_{k\geq 1} a_k$, and we denote it by $\sum_{k=1}^{\infty} a_k$. Henceforth when we write

$$\sum_{k=1}^{\infty} a_k = A,$$

we mean that A is a real number, the series $\sum_{k\geq 1} a_k$ is convergent, and its sum is equal to A. In this case we may also say that $\sum_{k\geq 1} a_k$ **converges** to A. An infinite series is said to be **divergent** if it is not convergent. In particular, we say that the series **diverges** to ∞ or to $-\infty$ according as its sequence of partial sums tends to ∞ or to $-\infty$. It is useful to keep in mind that the convergence of a series is not affected by changing a finite number of its terms, although its sum may change by doing so. (See Exercise 9.2.)

Given a series $\sum_{k\geq 1} a_k$ and $n \in \mathbb{N}$, by the **tail of the series** $\sum_{k\geq 1} a_k$ **after** n **terms**, we mean the series $\sum_{k\geq 1} b_k$, where $b_k := a_{k+n}$ for $k \in \mathbb{N}$; we shall denote this tail by $\sum_{k>n} a_k$. It is clear that for $n \in \mathbb{N}$ and $A \in \mathbb{R}$, the series $\sum_{k\geq 1} a_k$ converges to A if and only if its tail after n terms converges to $A - a_1 - \cdots - a_n$. This implies that a series is convergent if and only if its tail after n terms converges to 0 as $n \to \infty$.

In Chapter 2 we have considered many sequences that are in fact sequences of partial sums of some important series. We list them here for convenience.

Examples 9.1. (i) **(Geometric Series)** Let $a \in \mathbb{R}$. Define $a_0 := 1$ and $a_k := a^k$ for $k \in \mathbb{N}$. If $a \neq 1$, then for $n = 0, 1, 2, \ldots$,

$$A_n := a_0 + a_1 + \cdots + a_n = 1 + a + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}.$$

Suppose $|a| < 1$. We have seen in Example 2.7 (i) that $A_n \to 1/(1-a)$. Thus $\sum_{k \geq 0} a_k$ is convergent, and its sum is equal to $1/(1-a)$, that is,

$$1 + \sum_{k=1}^{\infty} a^k = \frac{1}{1-a} \quad \text{for } a \in \mathbb{R} \text{ with } |a| < 1.$$

This is perhaps the most important example of a convergent series. Its special feature is that we are able to give a simple closed-form formula for each of its partial sums as well as its sum. On the other hand, if $|a| \geq 1$, then $\sum_{k \geq 0} a_k$ is divergent, and this can be seen as follows. If $a \geq 1$, then $A_n \geq n + 1$ for $n = 0, 1, 2, \ldots$, and so $A_n \to \infty$. Thus, in this case $\sum_{k \geq 0} a_k$ diverges to ∞. Next, if $a = -1$, then $A_{2n} = 1$ and $A_{2n+1} = 0$ for all $n = 0, 1, 2, \ldots$, and so $\sum_{k \geq 0} a_k$ is divergent. Finally, if $a < -1$, then $A_{2n} \to \infty$, whereas $A_{2n+1} \to -\infty$, and hence $\sum_{k \geq 0} a_k$ is divergent.

(ii) **(Exponential Series)** For $k = 0, 1, 2, \ldots$, define $a_k := 1/k!$. Then for $n = 0, 1, 2, \ldots$,

$$A_n := a_0 + a_1 + \cdots + a_n = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!}.$$

We have seen in Example 2.10 (i) that (A_n) is convergent. Moreover, Example 2.10 (ii) and Corollary 7.7 show that $A_n \to e$. Thus $\sum_{k \geq 0} a_k$ is convergent and its sum is equal to e. More generally, given any $x \in \mathbb{R}$, if we define $a_0 := 0$ and $a_k := x^k/k!$ for $k \in \mathbb{N}$, then we shall see in Example 9.34 (ii) that $\sum_{k \geq 0} a_k$ is convergent and its sum is e^x, that is,

$$1 + \sum_{k=1}^{\infty} \frac{x^k}{k!} = e^x \quad \text{for } x \in \mathbb{R}.$$

(iii) **(Harmonic Series and Its Variants)** As seen in Example 2.10 (iii),

$$\sum_{k \geq 1} \frac{1}{k} \text{ diverges to } \infty.$$

The divergent series $\sum_{k \geq 1} (1/k)$ is called the **harmonic series**. Replacing k by its powers, we obtain important and useful variants of this series. Let p be a real number. Arguments given in Example 2.10 (v) show that

$$\sum_{k\geq 1} \frac{1}{k^p} \text{ diverges to } \infty \text{ if } p \leq 1, \text{ but converges if } p > 1.$$

For an alternative proof of the above fact, see Example 9.43 (i).

(iv) **(Alternating Series)** We have also seen in Example 2.10 (iii) that

$$\sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \text{ is convergent.}$$

Series such as this, whose successive terms alternate signs, are called **alternating series**. A test for the convergence of certain alternating series, called Leibniz Test, is given in Corollary 9.22. On the other hand, for showing the divergence of certain (alternating) series, the kth Term Test, given in Proposition 9.8, is quite useful. Using these tests, we shall see in Examples 9.9 (i) and 9.24 (i) that

$$\sum_{k\geq 1} \frac{(-1)^{k-1}}{k^p} \text{ diverges if } p \leq 0, \text{ but converges if } p > 0. \qquad \diamond$$

Since the convergence of a series is defined in terms of the convergence of a particular sequence, namely the sequence of its partial sums, many results about the convergence of series follow from the corresponding results given in Chapter 2 for the convergence of sequences. We mention them below without giving detailed proofs.

- The sequence of partial sums of a convergent series is bounded. This follows from part (ii) of Proposition 2.2.
- Sums and scalar multiples of convergent series are convergent. In fact, if $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

  $$\sum_{k=1}^{\infty}(a_k + b_k) = A + B \quad \text{and} \quad \sum_{k=1}^{\infty}(ra_k) = rA \text{ for every } r \in \mathbb{R}.$$

  Further, if $a_k \leq b_k$ for all $k \in \mathbb{N}$, then $A \leq B$. This follows from parts (i) and (ii) of Proposition 2.3 and part (iii) of Proposition 2.4. (For products, see Exercises 9.1 and 9.45.)
- **(Sandwich Theorem)** If (a_k), (b_k), and (c_k) are sequences of real numbers such that $a_k \leq c_k \leq b_k$ for each $k \in \mathbb{N}$, and further, $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = A$, then $\sum_{k=1}^{\infty} c_k = A$. This follows from Proposition 2.5.
- **(Cauchy Criterion)** A series $\sum_k a_k$ is convergent if and only if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

  $$\left| \sum_{k=n+1}^{m} a_k \right| < \epsilon \quad \text{for all } m > n \geq n_0.$$

  This follows from Proposition 2.22 by noting that the sequence (A_n) of partial sums satisfies $A_m - A_n = \sum_{k=n+1}^{m} a_k$ for all $m > n$.

Remark 9.2. As a simple application of the geometric series and the second property above, we can strengthen Exercise 2.33. Indeed, if the decimal expansion of $y \in [0, 1)$ is finite or recurring and $b_1, b_2, \ldots$ denote the digits of y, then there exist $i, j \in \mathbb{N}$ with $i \le j$ such that

$$y = \sum_{k=0}^{i-1} \frac{b_k}{10^k} + \left(\frac{b_i}{10^i} + \frac{b_{i+1}}{10^{i+1}} + \cdots + \frac{b_{j-1}}{10^{j-1}} \right) A,$$

where

$$A = \sum_{k=0}^{\infty} \frac{1}{10^{(j-i)k}} = \frac{10^{(j-i)}}{10^{(j-i)} - 1}.$$

Consequently, y is a rational number. Thus we can conclude that $y \in [0, 1)$ is a rational number if and only if its decimal expansion is finite or recurring. A similar result holds for every real number. $\diamond$

Telescoping Series and Series with Nonnegative Terms

If (b_k) is a sequence of real numbers, the series $\sum_{k \ge 1}(b_k - b_{k+1})$ is known as a **telescoping series**. We have the following result regarding its convergence.

Proposition 9.3. *A telescoping series $\sum_{k \ge 1}(b_k - b_{k+1})$ is convergent if and only if the sequence (b_k) is convergent, and in this case,*

$$\sum_{k=1}^{\infty}(b_k - b_{k+1}) = b_1 - \lim_{k \to \infty} b_k.$$

Proof. Note that for every $n \in \mathbb{N}$,

$$\sum_{k=1}^{n}(b_k - b_{k+1}) = b_1 - b_{n+1}.$$

This yields the desired result. $\square$

It may be noted that every series $\sum_{k \ge 1} a_k$ can be written as a telescoping series. In fact, if A_n is the nth partial sum of the series $\sum_{k \ge 1} a_k$, then letting $b_1 := 0$ and $b_k := -A_{k-1}$ for $k \ge 2$, we obtain $a_k = b_k - b_{k+1}$ for all $k \in \mathbb{N}$. But then determining whether the sequence (b_k) is convergent is the same as determining the convergence of the given series $\sum_{k \ge 1} a_k$. In some special cases, however, it is possible to write $a_k = b_k - b_{k+1}$ for all $k \in \mathbb{N}$ without considering the partial sums A_n. In these cases, we can determine the convergence of the series and find its sum using Proposition 9.3. For example, consider the series $\sum_{k \ge 1} 1/k(k+1)$. Note that

$$a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} = b_k - b_{k+1} \quad \text{for all } k \in \mathbb{N},$$

where $b_k := 1/k$ for $k \in \mathbb{N}$. Since $b_k \to 0$, we see that $\sum_{k\geq 1} 1/k(k+1)$ is convergent and

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty}(b_k - b_{k+1}) = b_1 - \lim_{k\to\infty} b_k = 1 - 0 = 1.$$

Our next result is a characterization of the convergence of a series with nonnegative terms. An interesting application of this result, known as the **Cauchy Condensation Test**, is given in Exercise 9.6.

Proposition 9.4. *Let (a_k) be a sequence such that $a_k \geq 0$ for all $k \in \mathbb{N}$. Then $\sum_{k\geq 1} a_k$ is convergent if and only if the sequence (A_n) of its partial sums is bounded above, and in this case,*

$$\sum_{k=1}^{\infty} a_k = \sup\{A_n : n \in \mathbb{N}\}.$$

If (A_n) is not bounded above, then $\sum_{k\geq 1} a_k$ diverges to ∞.

Proof. Since $a_k \geq 0$ for all $k \in \mathbb{N}$, we see that $A_{n+1} = A_n + a_{n+1} \geq A_n$ for all $n \in \mathbb{N}$, that is, the sequence (A_n) of the partial sums of $\sum_{k\geq 1} a_k$ is monotonically increasing. By part (ii) of Proposition 2.2 and part (i) of Proposition 2.8, we see that the sequence (A_n) is convergent if and only if it is bounded above, and in this case,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} A_n = \sup\{A_n : n \in \mathbb{N}\}.$$

Also, if (A_n) is not bounded above, then by Proposition 2.13, $A_n \to \infty$, that is, $\sum_{k\geq 1} a_k$ diverges to ∞. $\square$

A result similar to Proposition 9.4 holds if $a_k \leq 0$ for all $k \in \mathbb{N}$. (See Exercise 9.5.) More generally, if a_k has the same sign for all large $k \in \mathbb{N}$, then $\sum_k a_k$ is convergent if and only if (A_n) is bounded. However, if there is no $k_0 \in \mathbb{N}$ such that a_k is of the same sign for all $k \geq k_0$, then the series $\sum_k a_k$ may diverge even though the sequence of its partial sums is bounded. This is illustrated by the series $\sum_{k\geq 1}(-1)^k$, for which the sequence (A_n) of partial sums is given by $A_{2n-1} := -1$ and $A_{2n} := 0$ for all $n \in \mathbb{N}$.

If each term a_k of a series $\sum_k a_k$ is either equal to 0 or has the same sign, then clearly, the series $\sum_k a_k$ is convergent if and only if the series $\sum_k |a_k|$ is convergent. This may not hold if the terms a_k are of mixed signs. Thus we are led to the following concept. A series $\sum_k a_k$ is said to be **absolutely convergent** if the series $\sum_k |a_k|$ is convergent. We now give an important result about absolutely convergent series of real numbers.

Proposition 9.5. *An absolutely convergent series is convergent.*

Proof. Let $\sum_k a_k$ be an absolutely convergent series. Then $\sum_k 2|a_k|$ is a convergent series with nonnegative terms. Moreover, $0 \le a_k + |a_k| \le 2|a_k|$ for all $k \in \mathbb{N}$. Hence from Proposition 9.4, it follows that $\sum_k a_k + |a_k|$ is convergent. Since $\sum_k |a_k|$ is convergent, this implies that $\sum_k a_k$ is convergent. □

An alternative proof of the above result using the Cauchy Criterion (Proposition 2.22), and in fact, a proof of the equivalence of this result with the Cauchy completeness of $\mathbb{R}$, is outlined in Exercise 9.48.

The converse of Proposition 9.5 does not hold, as can be seen by considering the series $\sum_{k\ge 1}(-1)^{k-1}/k$, which is convergent but not absolutely convergent. A convergent series that is not absolutely convergent is said to be **conditionally convergent**.

We end this section by discussing a weaker form of convergence of series, known as Cesàro convergence or Cesàro summability, in analogy with the corresponding notion for sequences discussed toward the end of Section 2.1. In the context of Cesàro convergence for series, it is traditional to use series whose terms a_k are indexed from $k = 0$ onward, and we shall do so here.

A series $\sum_{k\ge 0} a_k$ is said to be **Cesàro convergent** or **Cesàro summable** if the sequence $(A_n)_{n\ge 0}$ of its partial sums is Cesàro convergent, that is,

$$\lim_{n\to\infty} B_n \text{ exists,} \quad \text{where} \quad B_n := \frac{A_0 + A_1 + \cdots + A_n}{n+1} \quad \text{for } n \ge 0.$$

We shall refer to (B_n) as the sequence of **Cesàro means** of the series $\sum_{k\ge 0} a_k$. If (B_n) is convergent, then $\lim_{n\to\infty} B_n$ is called the **Cesàro sum** of $\sum_{k\ge 0} a_k$.

Proposition 9.6. *Let $\sum_{k\ge 0} a_k$ be a series of real numbers. If $\sum_{k\ge 0} a_k$ is convergent, then it is Cesàro convergent, and its Cesàro sum is equal to its sum. Conversely, if $\sum_{k\ge 0} a_k$ is Cesàro convergent, then*

$$\sum_{k\ge 0} a_k \textit{ is convergent} \iff \frac{a_1 + 2a_2 + \cdots + na_n}{n+1} \to 0.$$

Proof. Both the assertions follow by applying Proposition 2.15 to the sequence (A_n) of partial sums of $\sum_{k\ge 0} a_k$, and by noting that the sequence of Cesàro means of $\sum_{k\ge 0} a_k$ is precisely the sequence of arithmetic means of (A_n) and

$$\sum_{k=0}^{n}(A_n - A_k) = \sum_{k=0}^{n}(a_{k+1} + a_{k+2} + \cdots + a_n) = a_1 + 2a_2 + \cdots + na_n$$

for each $n \ge 0$. □

The converse of the first assertion in the above result is not true, in general. For example, if $a_k := (-1)^k$ for $k \ge 0$, then it is easy to see that for $n \ge 0$, the partial sums A_n and the Cesàro means B_n of the series $\sum_{k\ge 0} a_k$ are given by

$$A_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \quad \text{and} \quad B_n = \begin{cases} (n+2)/2(n+1) & \text{if } n \text{ is even,} \\ 1/2 & \text{if } n \text{ is odd.} \end{cases}$$

Thus $\sum_{k\geq 0} a_k$ is Cesàro convergent (with Cesàro sum equal to $1/2$), but not convergent. On the other hand, if $a_k := 1$ for all $k \geq 0$, then $\sum_{k\geq 0} a_k$ is clearly not convergent, and it is also not Cesàro convergent, since its Cesàro means B_n are given by $B_n = (n+1)(n+2)/2(n+1) = (n+2)/2$ for all $n \geq 0$.

The last assertion in Proposition 9.6 yields the following useful sufficient condition for a Cesàro convergent series to be convergent.

Corollary 9.7. *Let $(a_k)_{k\geq 0}$ be a sequence in $\mathbb{R}$ such that $a_k = o(1/k)$, that is, $ka_k \to 0$. If $\sum_{k\geq 0} a_k$ is Cesàro convergent, then it is convergent.*

Proof. Since $ka_k \to 0$, by Proposition 2.15, the sequence of the arithmetic means of (ka_k) converges to 0, that is, $(a_1 + 2a_2 + \cdots + na_n)/(n+1) \to 0$. Thus the desired result follows from Proposition 9.6. □

9.2 Convergence Tests for Series

In this section we shall consider several practical tests that enable us to check the convergence or the divergence of a wide variety of series. We begin with a simple result on which most of the tests for divergence of a series are based.

Proposition 9.8 (kth Term Test). *If $\sum_k a_k$ is convergent, then $a_k \to 0$ as $k \to \infty$. In other words, if $a_k \not\to 0$, then $\sum_k a_k$ is divergent.*

Proof. Let $\sum_k a_k$ be a convergent series. If (A_n) is its sequence of partial sums and A is its sum, then $a_k = A_k - A_{k-1} \to A - A = 0$. □

Examples 9.9. (i) If $p \in \mathbb{R}$ with $p \leq 0$, then $|(-1)^{k-1}k^{-p}| \geq 1$ for all $k \in \mathbb{N}$. Hence by the kth Term Test (Proposition 9.8),

$$\sum_{k\geq 1} \frac{(-1)^{k-1}}{k^p} \text{ is divergent if } p \leq 0.$$

(ii) The converse of the kth Term Test (Proposition 9.8) does not hold, as can be seen by considering the harmonic series $\sum_{k\geq 1} 1/k$. ◇

Remark 9.10. A variant of the kth Term Test (Proposition 9.8), known as the **Abel kth Term Test**, is given in Exercise 9.7. This variant can also be useful in establishing the divergence of a series. ◇

Tests for Absolute Convergence

We shall now give a variety of tests to determine the absolute convergence (and hence the convergence) of a series.

Proposition 9.11 (Comparison Test). *Let (a_k) and (b_k) be sequences of real numbers such that $|a_k| \le b_k$ for all large $k \in \mathbb{N}$. If $\sum_k b_k$ is convergent, then $\sum_k a_k$ is absolutely convergent.*

Proof. Let $k_0 \in \mathbb{N}$ be such that $|a_k| \le b_k$ for all $k \ge k_0$. Suppose $\sum_k b_k$ is convergent. Let $\epsilon > 0$ be given. Then by the Cauchy Criterion, there is $n_0 \in \mathbb{N}$ such that $\left|\sum_{k=n+1}^m b_k\right| < \epsilon$ for all $m > n \ge n_0$. Let $n_1 := \max\{n_0, k_0\}$. Then

$$\sum_{k=n+1}^{m} |a_k| \le \sum_{k=n+1}^{m} b_k = \left|\sum_{k=n+1}^{m} b_k\right| < \epsilon \quad \text{for all } m > n \ge n_1.$$

Hence by the Cauchy Criterion, $\sum_k a_k$ is absolutely convergent. □

It follows from the above result that if $a_k = O(b_k)$ and $b_k \ge 0$ for all large $k \in \mathbb{N}$, then the convergence of $\sum_k b_k$ implies the absolute convergence of $\sum_k a_k$. The above result can also be stated as follows. If $|a_k| \le b_k$ for all large $k \in \mathbb{N}$ and $\sum_k |a_k|$ diverges to ∞, then $\sum_k b_k$ also diverges to ∞.

As seen below, the geometric series and the series $\sum_{k\ge1} 1/k^p$, where $p \in \mathbb{R}$, are often useful in employing the Comparison Test.

Examples 9.12. (i) For $k = 0, 1, 2, \ldots$, let $a_k := (2^k + k)/(3^k + k)$. If we let $b_k := (2/3)^k$, then $\sum_{k\ge0} b_k$ is convergent and

$$|a_k| = \frac{2^k + k}{3^k + k} \le \frac{2^k + 2^k}{3^k} = 2\left(\frac{2}{3}\right)^k = 2b_k \quad \text{for all } k \ge 0.$$

Hence by the Comparison Test, $\sum_{k\ge0} a_k$ is convergent.

(ii) Let $a_k := 1/(1+k^2+k^4)^{1/3}$ for $k \in \mathbb{N}$. If we let $b_k := 1/k^{4/3}$, then $\sum_{k\ge0} b_k$ is convergent and

$$|a_k| = \frac{1}{(1 + k^2 + k^4)^{1/3}} \le \frac{1}{k^{4/3}} = b_k \quad \text{for all } k \in \mathbb{N}.$$

Hence by the Comparison Test, $\sum_{k=0}^{\infty} a_k$ is convergent. ◇

Given a series $\sum_k a_k$, it may be difficult to look for a convergent series $\sum_k b_k$ such that $|a_k| \le b_k$ for all large $k \in \mathbb{N}$. It is often easier to find a convergent series $\sum_k b_k$ of nonzero terms such that the ratio a_k/b_k approaches a limit as $k \to \infty$. In these cases, the following result is useful.

Corollary 9.13 (Limit Comparison Test). *Let (a_k) and (b_k) be sequences of real numbers such that $a_k > 0$ and $b_k > 0$ for all large $k \in \mathbb{N}$. Suppose there exists $\ell \in \mathbb{R} \cup \{\infty\}$ such that $a_k/b_k \to \ell$.*

(i) *If* $\ell \neq 0$ *and* $\ell \neq \infty$, *then*

$$\sum_k a_k \textit{ is convergent } \iff \sum_k b_k \textit{ is convergent.}$$

(ii) *If* $\ell = 0$ *and* $\sum_k b_k$ *is convergent, then* $\sum_k a_k$ *is convergent.*

(iii) *If* $\ell = \infty$ *and* $\sum_k a_k$ *is convergent, then* $\sum_k b_k$ *is convergent.*

Proof. First, suppose $\ell \neq 0$ and $\ell \neq \infty$. Then $\ell > 0$, since $a_k/b_k > 0$ for all large $k \in \mathbb{N}$. Choose $\epsilon \in \mathbb{R}$ such that $0 < \epsilon < \ell$. Then there exists $k_0 \in \mathbb{N}$ such that $a_k > 0$, $b_k > 0$, and $|(a_k/b_k) - \ell| < \epsilon$ for all $k \geq k_0$. This implies that

$$0 < a_k \leq (\ell + \epsilon) b_k \quad \text{and} \quad 0 < b_k \leq \frac{a_k}{\ell - \epsilon} \quad \text{for all } k \geq k_0.$$

Hence by applying the Comparison Test (Proposition 9.11) to (a_k) and (b_k), we obtain (i). In case $\ell = 0$ and we choose $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$, then the first inequality above still holds, and so the Comparison Test also yields (ii). Finally, if $\ell = \infty$, then we choose $k_0 \in \mathbb{N}$ such that $b_k > 0$, and $(a_k/b_k) > 1$ for all $k \geq k_0$. Hence $0 < b_k \leq a_k$ for all $k \geq k_0$, and so the Comparison Test yields (iii). □

Examples 9.14. (i) Let $a_k := (2^k + k)/(3^k - k)$ for all $k \in \mathbb{N}$. If we let $b_k := (2/3)^k$, then $a_k > 0$ and $b_k > 0$ for all $k \in \mathbb{N}$. Moreover,

$$\frac{a_k}{b_k} = \frac{1 + (k/2^k)}{1 - (k/3^k)} \to 1.$$

Since $\sum_{k \geq 0} b_k$ is convergent, by the Limit Comparison Test, or more precisely, by part (i) of Corollary 9.13, we see that $\sum_{k \geq 0} a_k$ is convergent.

(ii) Let $a_k := \sin(1/k)$ for all $k \in \mathbb{N}$. If we let $b_k := 1/k$, then $a_k > 0$ and $b_k > 0$ for all $k \in \mathbb{N}$. Moreover,

$$\frac{a_k}{b_k} = \frac{\sin(1/k)}{(1/k)} \to 1.$$

Since $\sum_{k \geq 1} b_k$ is divergent, by the Limit Comparison Test, or more precisely, by part (i) of Corollary 9.13, we see that $\sum_{k \geq 1} a_k$ is divergent.

(iii) Let $p \in \mathbb{R}$ and $a_k := (\ln k)/k^p$ for $k \in \mathbb{N}$. First assume that $p > 1$ and let $q := (p+1)/2$. Then $1 < q < p$. If we let $b_k := 1/k^q$ for $k \in \mathbb{N}$, then $b_k > 0$ for all $k \in \mathbb{N}$, and by L'Hôpital's Rule,

$$\frac{a_k}{b_k} = \frac{(\ln k)/k^p}{1/k^q} = \frac{\ln k}{k^{p-q}} \to 0.$$

Since $\sum_{k \geq 1} b_k$ is convergent, by part (ii) of Corollary 9.13, we see that $\sum_{k \geq 1} a_k$ is convergent. On the other hand, if $p \leq 1$, then

$$a_k = \frac{\ln k}{k^p} \geq \frac{1}{k^p} \quad \text{for } k \geq 2, \quad \text{and} \quad \sum_k \frac{1}{k^p} \text{ is divergent.}$$

Hence we see that $\sum_{k\geq 1} a_k$ is divergent. Thus we can conclude that

$$\sum_{k\geq 1} \frac{\ln k}{k^p} \quad \text{converges if } p > 1 \text{ and diverges if } p \leq 1.$$

(iv) Let $p > 0$ and $a_k := 1/(\ln k)^p$ for $k \in \mathbb{N}$ with $k \geq 2$. If we let $b_k := 1/k$ for $k = 2, 3, \ldots$, then $b_k > 0$ for $k \geq 2$, and by L'Hôpital's Rule,

$$\frac{a_k}{b_k} = \frac{1/(\ln k)^p}{1/k} = \frac{k}{(\ln k)^p} \to \infty.$$

Since $a_k > 0$ for $k \geq 2$ and $\sum_{k\geq 2} b_k$ is divergent, by part (iii) of Corollary 9.13, we see that $\sum_{k\geq 2} a_k$ is divergent. $\diamond$

For an infinite series $\sum_k a_k$ with positive terms, the quotients a_{k+1}/a_k of consecutive terms are sometimes called the **ratios** of $\sum_k a_k$. The following result is a slightly more general version of an important and useful fact that if the ratios of a series $\sum_k a_k$ with positive terms do not exceed the ratios of a convergent series with positive terms, then $\sum_k a_k$ is convergent.

Proposition 9.15 (Ratio Comparison Test). *Let (a_k) and (b_k) be sequences in $\mathbb{R}$ with $a_k \neq 0$ and $b_k > 0$ for all large $k \in \mathbb{N}$.*

(i) *If $|a_{k+1}/a_k| \leq b_{k+1}/b_k$ for all large $k \in \mathbb{N}$ and $\sum_{k\geq 1} b_k$ is convergent, then $\sum_{k\geq 1} a_k$ is absolutely convergent.*
(ii) *If $|a_{k+1}/a_k| \geq b_{k+1}/b_k$ for all large $k \in \mathbb{N}$ and $\sum_{k\geq 1} b_k$ is divergent, then $\sum_{k\geq 1} a_k$ is not absolutely convergent.*

Proof. (i) Suppose $k_0 \in \mathbb{N}$ is such that $a_k \neq 0$, $b_k > 0$, and moreover, $|a_{k+1}/a_k| \leq b_{k+1}/b_k$ for all $k \geq k_0$. Then for every $k \geq k_0$,

$$|a_k| \leq |a_{k-1}|\frac{b_k}{b_{k-1}} \leq |a_{k-2}|\frac{b_{k-1}}{b_{k-2}} \cdot \frac{b_k}{b_{k-1}} \leq \cdots \leq |a_{k_0}| \left(\frac{b_{k_0+1}}{b_{k_0}} \cdots \frac{b_{k-1}}{b_{k-2}} \cdot \frac{b_k}{b_{k-1}}\right).$$

Thus $|a_k| \leq (|a_{k_0}|/b_{k_0})\, b_k$ for all $k \geq k_0$. Hence by applying the Comparison Test to $\sum_k a_k$ and the series $\sum_k (|a_{k_0}|/b_{k_0})\, b_k$ with positive terms, we see that $\sum_{k\geq 1} a_k$ is absolutely convergent.

(ii) Suppose $k_0 \in \mathbb{N}$ is such that $a_k \neq 0$, $b_k > 0$, and $|a_{k+1}/a_k| \geq b_{k+1}/b_k$ for all $k \geq k_0$. Then as in the proof of (i), we see that $|a_k| \geq (|a_{k_0}|/b_{k_0})\, b_k$ for all $k \geq k_0$. Since $\sum_{k\geq 1} b_k$ is divergent, the Comparison Test implies that $\sum_{k\geq 1} |a_k|$ is divergent, that is, $\sum_{k\geq 1} a_k$ is not absolutely convergent. $\square$

A corollary of the Ratio Comparison Test is the **D'Alembert Ratio Test**, or simply the **Ratio Test**, which is given below. It is one of the most widely employed tests to determine the convergence of a series.

Proposition 9.16 (Ratio Test). *Let (a_k) be a sequence of real numbers such that $a_k \neq 0$ for all large $k \in \mathbb{N}$.*

(i) *If there exists $\alpha \in \mathbb{R}$ with $\alpha < 1$ such that $|a_{k+1}/a_k| \leq \alpha$ for all large $k \in \mathbb{N}$, then $\sum_{k \geq 1} a_k$ is absolutely convergent.*
(ii) *If $|a_{k+1}/a_k| \geq 1$ for all large $k \in \mathbb{N}$, then $\sum_{k \geq 1} a_k$ is divergent.*

Hence if $\overline{\ell} := \limsup_{k\to\infty} |a_{k+1}/a_k|$ and $\underline{\ell} := \liminf_{k\to\infty} |a_{k+1}/a_k|$, then

$$\sum_{k\geq 1} a_k \text{ is absolutely convergent if } \overline{\ell} < 1, \text{ and it is divergent if } \underline{\ell} > 1.$$

In particular, if the sequence $(|a_{k+1}/a_k|)$ of ratios of $\sum_{k \geq 1} a_k$ is convergent, then the above statement holds with $\overline{\ell} = \underline{\ell} = \lim_{k\to\infty} |a_{k+1}/a_k|$.

Proof. (i) Supose there exists $\alpha \in \mathbb{R}$ with $\alpha < 1$ such that $|a_{k+1}/a_k| \leq \alpha$ for all large $k \in \mathbb{N}$. Then $0 < \alpha < 1$, and hence the geometric series $\sum_k \alpha^k$ is convergent. Thus the desired result follows from part (i) of the Ratio Comparison Test by taking $b_k := \alpha^k$ for all $k \in \mathbb{N}$.

(ii) If $k_0 \in \mathbb{N}$ is such that $a_k \neq 0$ and $|a_{k+1}/a_k| \geq 1$ for all $k \geq k_0$, then $|a_k| \geq |a_{k-1}| \geq \cdots \geq |a_{k_0}| > 0$ for all $k \geq k_0$. Hence $a_k \not\to 0$. Thus by the kth Term Test (Proposition 9.8), $\sum_{k\geq 1} a_k$ is divergent.

To deduce the assertion involving $\overline{\ell}$ and $\underline{\ell}$, it suffices to observe the following. Suppose (c_k) is any sequence in $\mathbb{R}$. Then $\limsup_{k\to\infty} c_k < 1$ if and only if there exists $\alpha \in \mathbb{R}$ with $\alpha < 1$ such that $c_k \leq \alpha$ for all large $k \in \mathbb{N}$. On the other hand, if $\liminf_{k\to\infty} c_k > 1$, then $c_k \geq 1$ for all large $k \in \mathbb{N}$. These observations follow easily from Proposition 2.25 and Corollary 2.18. □

The following result, known as the **Cauchy Root Test**, or simply the **Root Test**, is one of the most basic tests to determine the convergence of a series.

Proposition 9.17 (Root Test). *Let (a_k) be a sequence of real numbers.*

(i) *If there exists $\alpha \in \mathbb{R}$ with $\alpha < 1$ such that $|a_k|^{1/k} \leq \alpha$ for all large k, then $\sum_{k\geq 1} a_k$ is absolutely convergent.*
(ii) *If $|a_k|^{1/k} \geq 1$ for infinitely many $k \in \mathbb{N}$, then $\sum_{k\geq 1} a_k$ is divergent.*

Consequently, if $\overline{\ell} := \limsup_{k\to\infty} |a_k|^{1/k}$, then

$$\sum_{k\geq 1} a_k \text{ is absolutely convergent when } \overline{\ell} < 1, \text{ and it is divergent when } \overline{\ell} > 1.$$

This holds, in particular, if $\left(|a_k|^{1/k}\right)$ is convergent and $\overline{\ell} = \lim_{k\to\infty} |a_k|^{1/k}$.

Proof. (i) Suppose $\alpha < 1$ and $k_0 \in \mathbb{N}$ is such that $|a_k|^{1/k} \leq \alpha$ for all $k \geq k_0$. Then $\alpha \geq 0$. Also, if we let $b_k := \alpha^k$ for $k \in \mathbb{N}$, then $|a_k| \leq b_k$ for all $k \geq k_0$.

Since $\sum_{k\geq 1} b_k$ is convergent, the Comparison Test (Proposition 9.11) shows that $\sum_{k\geq 1} a_k$ is absolutely convergent.

(ii) If $|a_k|^{1/k} \geq 1$ for infinitely many $k \in \mathbb{N}$, then $|a_k| \geq 1$ for infinitely many $k \in \mathbb{N}$ and therefore $a_k \not\to 0$ as $k \to \infty$. Hence the kth Term Test (Proposition 9.8) shows that $\sum_{k\geq 1} a_k$ is divergent.

To deduce the assertion involving $\overline{\ell}$, it suffices to use the characterization mentioned in the proof of the Ratio Test for the condition $\limsup_{k\to\infty} c_k < 1$, and observe that if $\limsup_{k\to\infty} c_k > 1$, then $c_k > 1$ for infinitely many $k \in \mathbb{N}$. The last observation follows easily from Proposition 2.25. □

Remarks 9.18. (i) Both the Root Test and the Ratio Test deduce absolute convergence of a series by comparing it with a geometric series. The Ratio Test is often simpler to use than the Root Test because usually it is easier to calculate ratios than roots. But the Root Test has a wider applicability than the Ratio Test in the following sense. Whenever the Ratio Test gives (absolute) convergence of a series, so does the Root Test (Exercise 9.49), and moreover, the Root Test can yield (absolute) convergence of a series for which the Ratio Test is inconclusive (Example 9.19 (iv)).

(ii) Both the Root Test and the Ratio Test deduce divergence of a series by appealing to the kth Term Test (Proposition 9.8). It may be observed that for deducing the divergence of a series $\sum_{k\geq 1} a_k$, the Root Test requires $|a_k| \geq 1$ for infinitely many $k \in \mathbb{N}$, while the Ratio Test requires $|a_{k+1}| \geq |a_k|$ for all large $k \in \mathbb{N}$. The series $\sum_k a_k$ may not diverge if we have $|a_{k+1}| \geq |a_k|$ only for infinitely many $k \in \mathbb{N}$. For example, consider

$$a_1 := 1, \quad a_{2k} := \frac{1}{(k+1)^2}, \quad \text{and} \quad a_{2k+1} := \frac{1}{k^2} \quad \text{for all } k \in \mathbb{N}.$$

Then $|a_{2k+1}| \geq |a_{2k}|$ for all $k \in \mathbb{N}$, but $\sum_{k\geq 1} a_k$ is convergent because it has positive terms and for $k \in \mathbb{N}$,

$$a_3 + a_5 + \cdots + a_{2k+1} = \sum_{j=1}^{k} \frac{1}{j^2} \quad \text{and} \quad a_1 + a_2 + a_4 + \cdots + a_{2k} = \sum_{j=1}^{k+1} \frac{1}{j^2}.$$

Hence the sequence of partial sums of the series $\sum_{k\geq 1} a_k$ is bounded.

(iii) If $a_k \neq 0$ for all large k and $|a_{k+1}/a_k| \to 1$ as $k \to \infty$, then the Ratio Test is inconclusive in deducing the convergence or divergence of $\sum_k a_k$. In this case, $|a_k|^{1/k} \to 1$ as well (Exercise 9.50). Hence the Root Test is also inconclusive. In this event, $\sum_{k\geq 1} a_k$ may be divergent or convergent, as the examples $\sum_{k\geq 1}(1/k)$ and $\sum_{k\geq 1}(1/k^2)$ show.

Using the Ratio Comparison Test (Exercise 9.15) in conjunction with the series $\sum_{k\geq 1} 1/k^p$, where $p > 0$, one can obtain a result known as the **Raabe Test**, which is useful when $|a_{k+1}/a_k| \to 1$. See Exercises 9.11, 9.12, and 9.13.

Another test for the convergence for a series of nonnegative terms, known as the Integral Test, which involves "improper integrals", will be given in Proposition 9.42. ◇

Examples 9.19. (i) Let $a_k := k^2/2^k$ for $k \in \mathbb{N}$. Then for each $k \in \mathbb{N}$,

$$\frac{|a_{k+1}|}{|a_k|} = \frac{(k+1)^2}{2^{k+1}} \frac{2^k}{k^2} = \frac{1}{2}\left(1 + \frac{1}{k}\right)^2.$$

Hence $|a_{k+1}|/|a_k| \to 1/2$ as $k \to \infty$. So by the Ratio Test, $\sum_{k\geq 1} a_k$ is (absolutely) convergent. Alternatively, we may use the Root Test. Indeed, by Example 2.7 (iv), $k^{1/k} \to 1$, and hence $|a_k|^{1/k} = (k^{1/k})^2/2 \to \frac{1}{2}$.

(ii) Let $a_k := k!/2^k$ for $k \in \mathbb{N}$. Then for each $k \in \mathbb{N}$,

$$\frac{|a_{k+1}|}{|a_k|} = \frac{(k+1)!}{2^{k+1}} \frac{2^k}{k!} = \frac{k+1}{2}.$$

Hence $|a_{k+1}|/|a_k| \to \infty$ as $k \to \infty$. So by the Ratio Test, $\sum_{k\geq 1} a_k$ is divergent. Alternatively, we may use the Root Test. To this end, note that $|a_k|^{1/k} = (k!)^{1/k}/2$ for all $k \in \mathbb{N}$. Since $k! \geq 2^k$ for all $k \geq 4$, we see that $|a_k|^{1/k} \geq 1$ for all $k \geq 4$. Hence $\sum_{k\geq 1} a_k$ is divergent. (In fact, Exercise 2.11 shows that $|a_k|^{1/k} \to \infty$.)

(iii) Let $a_k := k!/k^k$ for $k \in \mathbb{N}$. Then for each $k \in \mathbb{N}$,

$$\frac{|a_{k+1}|}{|a_k|} = \frac{(k+1)!}{(k+1)^{k+1}} \frac{k^k}{k!} = \frac{k+1}{k+1}\left(\frac{k}{k+1}\right)^k = \frac{1}{(1+1/k)^k}.$$

Hence $|a_{k+1}|/|a_k| \to 1/e$ by Corollary 7.7. So by the Ratio Test, $\sum_{k\geq 1} a_k$ is (absolutely) convergent.

(iv) For $k \in \mathbb{N}$, let $a_{2k-1} := 1/4^k$ and $a_{2k} := 1/9^k$. Since

$$\frac{|a_{2k}|}{|a_{2k-1}|} = \left(\frac{4}{9}\right)^k < \frac{4}{9} < 1 \quad \text{and} \quad \frac{|a_{2k+1}|}{|a_{2k}|} = \frac{1}{4}\left(\frac{9}{4}\right)^k \geq 1 \quad \text{for all } k \geq 2,$$

the Ratio Test is inconclusive. On the other hand,

$$|a_{2k-1}|^{1/(2k-1)} = \frac{1}{2}\left(\frac{1}{2}\right)^{1/(2k-1)} \quad \text{and} \quad |a_{2k}|^{1/2k} = \frac{1}{3} \quad \text{for all } k \geq 1,$$

and hence $|a_k|^{1/k} \leq \frac{1}{2}$ for all $k \geq 1$. Consequently, by the Root Test, $\sum_{k\geq 1} a_k$ is (absolutely) convergent. $\diamond$

Tests for Conditional Convergence

The tests considered so far give either the absolute convergence or the divergence of a series. We now consider some tests that give conditional convergence. They are based on the following simple result, which may be compared with Exercise 1.13.

Proposition 9.20 (Partial Summation Formula). *Consider $a_k, b_k \in \mathbb{R}$ for $k \in \mathbb{N}$. Let $n, m \in \mathbb{N}$ with $m \geq n$, and define $B_{n,m} := \sum_{k=n}^{m} b_k$. Then*

$$\sum_{k=n}^{m} a_k b_k = a_m B_{n,m} + \sum_{k=n}^{m-1} (a_k - a_{k+1}) B_{n,k}.$$

Proof. Since $b_n = B_{n,n}$ and $b_k = B_{n,k} - B_{n,k-1}$ for $n < k \leq m$, we see that

$$\sum_{k=n}^{m} a_k b_k = a_n B_{n,n} + a_{n+1}(B_{n,n+1} - B_{n,n}) + \cdots + a_m(B_{n,m} - B_{n,m-1}).$$

The expression on the right can be written as

$$(a_n - a_{n+1})B_{n,n} + (a_{n+1} - a_{n+2})B_{n,n+1} + \cdots + (a_{m-1} - a_m)B_{n,m-1} + a_m B_{n,m}.$$

This yields the desired formula. □

Proposition 9.21 (Dirichlet Test). *Let (a_k) and (b_k) be sequences in $\mathbb{R}$ such that (a_k) is monotonic, $a_k \to 0$, and the sequence (B_n) defined by $B_n := \sum_{k=1}^{n} b_k$ for $n \in \mathbb{N}$ is bounded. Then the series $\sum_{k \geq 1} a_k b_k$ is convergent. Further, for $n \in \mathbb{N}$, let $\beta_n := \sup\{|\sum_{k=n}^{m} b_k| : m \in \mathbb{N}$ and $m \geq n\}$. Then*

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq |a_1|\beta_1 \quad \text{and} \quad \left|\sum_{k=n}^{\infty} a_k b_k\right| \leq |a_n|\beta_n \leq 2|a_n|\beta_1 \quad \text{for all } n \in \mathbb{N}.$$

Proof. Since the sequence (B_n) is bounded, β_1 is well-defined and $\beta_1 = \sup\{|B_m| : m \in \mathbb{N}\}$. Let $m, n \in \mathbb{N}$ with $m \geq n$. Then

$$\left|\sum_{k=n}^{m} b_k\right| = \left|\sum_{k=1}^{m} b_k - \sum_{k=1}^{n-1} b_k\right| \leq \left|\sum_{k=1}^{m} b_k\right| + \left|\sum_{k=1}^{n-1} b_k\right| \leq 2\beta_1.$$

This shows that β_n is well-defined and $\beta_n \leq 2\beta_1$.

First assume that the sequence (a_k) is decreasing. Then $a_m \geq 0$, since $a_k \to 0$. By the Partial Summation Formula (Proposition 9.20), we obtain

$$\left|\sum_{k=n}^{m} a_k b_k\right| \leq \beta_n\left(|a_m| + \sum_{k=n}^{m-1} |a_k - a_{k+1}|\right) \leq \beta_n\left(a_m + \sum_{k=n}^{m-1} (a_k - a_{k+1})\right) = a_n\beta_n.$$

Next, if the sequence (a_k) is increasing, then the sequence $(-a_k)$ is decreasing, and $-a_k \to 0$. Hence $|\sum_{k=n}^{m}(-a_k)b_k| \leq -a_n\beta_n$. Thus

$$\left|\sum_{k=n}^{m} a_k b_k\right| \leq |a_n|\beta_n \leq 2|a_n|\beta_1$$

in either case. Since $|a_n| \to 0$, the Cauchy Criterion shows that the series $\sum_{k \geq 1} a_k b_k$ is convergent. Further, given $n \in \mathbb{N}$, we may let $m \to \infty$ in the above inequality to obtain $|\sum_{k=n}^{\infty} a_k b_k| \leq |a_n|\beta_n \leq 2|a_n|\beta_1$, as desired. □

A similar result, known as the **Abel Test for Series**, is given in Exercise 9.15.

Corollary 9.22 (Leibniz Test). *Let* (a_k) *be a monotonic sequence such that* $a_k \to 0$. *Then* $\sum_{k\geq 1}(-1)^{k-1}a_k$ *is convergent. Moreover,*

$$\left|\sum_{k=n}^{\infty}(-1)^{k-1}a_k\right| \leq |a_n| \quad \textit{for all } n \in \mathbb{N}.$$

Proof. Define $b_k := (-1)^{k-1}$ for $k \in \mathbb{N}$ and for $n \in \mathbb{N}$, let B_n and β_n be as in Proposition 9.21. Then $B_n = 1$ if n is odd and $B_n = 0$ if n is even. Thus the sequence (B_n) is bounded. Moreover, $\beta_n = 1$ for all $n \in \mathbb{N}$. Hence the Dirichlet Test shows that the series $\sum_{k\geq 1}(-1)^{k-1}a_k$ is convergent, and the absolute value of its tail after n terms is at most $|a_n|$ for each $n \in \mathbb{N}$. □

Corollary 9.23 (Convergence Test for Trigonometric Series). *Let* (a_k) *be a monotonic sequence such that* $a_k \to 0$. *Then*

(i) $\sum_{k\geq 1} a_k \sin kx$ *is convergent for each* $x \in \mathbb{R}$.
(ii) $\sum_{k\geq 1} a_k \cos kx$ *is convergent for each* $x \in \mathbb{R}$ *with* $x \neq 2m\pi$ *for every* $m \in \mathbb{Z}$.

Proof. (i) Let $x \in \mathbb{R}$. Define $b_k := \sin kx$ for $k \in \mathbb{N}$ and $B_n := \sum_{k=1}^{n} b_k$ for $n \in \mathbb{N}$. Now

$$2\sin kx \sin(x/2) = \cos\big(kx - (x/2)\big) - \cos\big(kx + (x/2)\big) \quad \text{for each } k \in \mathbb{N},$$

and hence

$$2B_n \sin\frac{x}{2} = \sum_{k=1}^{n}\left(\cos\frac{(2k-1)x}{2} - \cos\frac{(2k+1)x}{2}\right) = \cos\frac{x}{2} - \cos\frac{(2n+1)x}{2}.$$

If $\sin(x/2) = 0$, that is, if $x = 2m\pi$ for some $m \in \mathbb{Z}$, then $b_k = 0$ for each $k \in \mathbb{N}$, and so $B_n = 0$ for each $n \in \mathbb{N}$. If $\sin(x/2) \neq 0$, then for each $n \in \mathbb{N}$,

$$|2B_n \sin(x/2)| \leq 2 \quad \text{and hence} \quad |B_n| \leq \frac{1}{|\sin(x/2)|}.$$

Thus the sequence (B_n) is bounded in all cases. Hence the desired result follows from the Dirichlet Test (Proposition 9.21).

(ii) Let $x \in \mathbb{R}$ with $x \neq 2m\pi$ for all $m \in \mathbb{Z}$. Define $b_k := \cos kx$ for $k \in \mathbb{N}$ and $B_n := \sum_{k=1}^{n} b_k$ for $n \in \mathbb{N}$. Now

$$2\cos kx \sin(x/2) = \sin\big(kx + (x/2)\big) - \sin\big(kx - (x/2)\big) \quad \text{for each } k \in \mathbb{N},$$

and hence

$$2B_n \sin\frac{x}{2} = \sum_{k=1}^{n}\left(\sin\frac{(2k+1)x}{2} - \sin\frac{(2k-1)x}{2}\right) = \sin\frac{(2n+1)x}{2} - \sin\frac{x}{2}.$$

Since $\sin(x/2) \neq 0$, the desired result follows as in (i) above. □

It may be observed that Corollary 9.22 is a special case of part (ii) of Corollary 9.23 with $x = \pi$.

Examples 9.24. (i) Let $p > 0$ and $a_k := 1/k^p$ for $k \in \mathbb{N}$. Then (a_k) is monotonic and $a_k \to 0$. Hence by the Leibniz Test, the series

$$\sum_{k\geq 1} \frac{(-1)^{k-1}}{k^p}$$

is convergent. If $p > 1$, then it is absolutely convergent, and if $p \leq 1$, then it is conditionally convergent (Example 9.1 (iii)).

(ii) Let $p > 0$ and $a_k := 1/(\ln k)^p$ for $k \in \mathbb{N}$ with $k \geq 2$. Then (a_k) is monotonic and $a_k \to 0$. Hence by the Leibniz Test, the series

$$\sum_{k\geq 2} \frac{(-1)^{k-1}}{(\ln k)^p}$$

is convergent. In fact, it is conditionally convergent (Example 9.14 (iv)).

(iii) Even if the signs of the terms of a sequence (a_k) alternate and $a_k \to 0$, the series $\sum_{k\geq 1} a_k$ may not converge, that is, the monotonicity assumption in the Leibniz Test (Corollary 9.22) cannot be omitted. For example, consider the series

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{2^2} - \frac{1}{4} + \frac{1}{2^3} - \frac{1}{5} + \frac{1}{2^4} - \frac{1}{6} + \cdots .$$

Since the sequence of partial sums of the series $\sum_{k\geq 1} 1/2^k$ is bounded and the sequence of partial sums of the series $\sum_{k=3}^{\infty} 1/k$ is unbounded, it follows that the sequence of partial sums of the series displayed above is unbounded. Hence it is divergent, although the signs of its terms alternate and the kth term tends to zero as $k \to \infty$.

(iv) Let $p > 0$ and $x \in \mathbb{R}$. Then the series $\sum_{k\geq 1} (\sin kx)/k^p$ is convergent. This follows by letting $a_k := 1/k^p$ for $k \in \mathbb{N}$ in the Convergence Test for Trigonometric Series (part (i) of Corollary 9.23). Similarly, if $x \neq 2m\pi$ for all $m \in \mathbb{Z}$, then the series $\sum_{k\geq 1} (\cos kx)/k^p$ is convergent. On the other hand, if $x = 2m\pi$ for some $m \in \mathbb{Z}$, then the series

$$\sum_{k\geq 1} \frac{\cos k(2m\pi)}{k^p} = \sum_{k\geq 1} \frac{1}{k^p}$$

is divergent if $p \leq 1$ and convergent if $p > 1$. ◇

9.3 Power Series

For $k = 0, 1, 2, \ldots$, let $c_k \in \mathbb{R}$. The series

$$\sum_{k\geq 0} c_k x^k := c_0 + \sum_{k\geq 1} c_k x^k, \quad \text{where } x \in \mathbb{R},$$

is called a **power series**, and the numbers $c_0, c_1, c_2, \ldots$ are called its **coefficients**.It is clear that if $x = 0$, then for any choice of $c_0, c_1, c_2, \ldots$, the power series $\sum_{k\geq 0} c_k x^k$ is convergent and its sum is equal to c_0. Also, if there exists $k_0 \in \mathbb{N}$ such that $c_k = 0$ for all $k > k_0$, then for every $x \in \mathbb{R}$, the power series $\sum_{k\geq 0} c_k x^k$ is convergent and its sum is equal to $c_0 + c_1 x + \cdots + c_{k_0} x^{k_0}$. More generally, if $a \in \mathbb{R}$, then the series $\sum_{k\geq 0} c_k (x-a)^k$, where $x \in \mathbb{R}$, is called a **power series** around a. The treatment of such a series can be reduced to a power series around 0 by letting $\tilde{x} := x - a$. Here are some simple but nontrivial examples of power series.

Examples 9.25. (i) Let $c_0 := 1$ and $c_k := k^k$ for $k \in \mathbb{N}$. Given any $x \in \mathbb{R}$ with $x \neq 0$, it is clear that $|c_k x^k| > 1$, for all $k \in \mathbb{N}$ satisfying $k > 1/|x|$. Hence $c_k x^k \not\to 0$ as $k \to \infty$. So by the kth Term Test, $\sum_{k\geq 0} c_k x^k$ is divergent for every nonzero $x \in \mathbb{R}$.

(ii) For $k = 0, 1, 2, \ldots$, let $c_k := 1/k!$. Given any $x \in \mathbb{R}$,

$$\frac{|c_{k+1} x^{k+1}|}{|c_k x^k|} = \frac{|x|}{k+1} \to 0 \quad \text{as} \quad k \to \infty.$$

So by the Ratio Test, $\sum_{k\geq 0} c_k x^k$ is absolutely convergent for every $x \in \mathbb{R}$.

(iii) For all $k = 0, 1, 2, \ldots$, let $c_k := 1$. Then $\sum_{k\geq 0} c_k x^k$ is the geometric series $1 + x + x^2 + \cdots$, and we have seen in Example 9.1 (i) that it is convergent if $|x| < 1$ and its sum is $1/(1-x)$, while it is divergent if $|x| \geq 1$. ◇

The above examples are typical as far as the convergence of a power series $\sum_{k\geq 0} c_k x^k$ for various values of x is concerned. The general phenomenon is described by the following basic result.

Lemma 9.26 (Abel Lemma). *Let x_0 and $c_0, c_1, c_2, \ldots$ be real numbers. If the set $\{c_k x_0^k : k \in \mathbb{N}\}$ is bounded, then the power series $\sum_{k\geq 0} c_k x^k$ is absolutely convergent for every $x \in \mathbb{R}$ with $|x| < |x_0|$. In particular, if $\sum_{k\geq 0} c_k x_0^k$ is convergent, then the power series $\sum_{k\geq 0} c_k x^k$ is absolutely convergent for every $x \in \mathbb{R}$ with $|x| < |x_0|$.*

Proof. If $x_0 = 0$, then there is nothing to prove. Suppose $x_0 \neq 0$. Let $\alpha \in \mathbb{R}$ be such that $|c_k x_0^k| \leq \alpha$ for all $k \in \mathbb{N}$. Given any $x \in \mathbb{R}$ with $|x| < |x_0|$, let $\beta := |x|/|x_0|$. Then $|c_k x^k| = |c_k x_0^k| \beta^k \leq \alpha \beta^k$ for all $k \in \mathbb{N}$. Since $|\beta| < 1$, the geometric series $\sum_k \beta^k$ is convergent. So by the Comparison Test, it follows that $\sum_{k\geq 0} c_k x^k$ is absolutely convergent. In case $\sum_{k\geq 0} c_k x_0^k$ is convergent, then by the kth Term Test, $c_k x_0^k \to 0$, and hence the sequence $\left(c_k x_0^k\right)$ is bounded, that is, the set $\{c_k x_0^k : k \in \mathbb{N}\}$ is bounded. □

Proposition 9.27. *A power series $\sum_{k\geq 0} c_k x^k$ is either absolutely convergent for all $x \in \mathbb{R}$, or there is a unique nonnegative real number r such that the series is absolutely convergent for each $x \in \mathbb{R}$ with $|x| < r$ and is divergent for each $x \in \mathbb{R}$ with $|x| > r$.*

Proof. Let $E := \left\{x \in \mathbb{R} : \sum_{k\geq 0} c_k x^k \text{ is convergent}\right\}$. Then $0 \in E$. If E is not bounded above, then given $x \in \mathbb{R}$, we may find $x_0 \in E$ such that $|x| < |x_0|$, and then $\sum_{k\geq 0} c_k x^k$ is absolutely convergent by Lemma 9.26. Next, suppose E is bounded above and let $r := \sup\{|x| : x \in E\}$. If $x \in \mathbb{R}$ and $|x| < r$, then by the definition of a supremum, we may find $x_0 \in E$ such that $|x| < |x_0|$, and so by Lemma 9.26, we see that $\sum_{k\geq 0} c_k x^k$ is absolutely convergent. If $x \in \mathbb{R}$ and $|x| > r$, then $x \notin E$, and so the power series $\sum_{k\geq 0} c_k x^k$ is divergent. This proves the existence of the nonnegative real number r with the desired properties. The uniqueness of r is obvious. $\square$

We say that the **radius of convergence** of a power series is ∞ if the power series is absolutely convergent for all $x \in \mathbb{R}$; otherwise, it is defined to be the unique nonnegative real number r such that the power series is absolutely convergent for each $x \in \mathbb{R}$ with $|x| < r$ and divergent for each $x \in \mathbb{R}$ with $|x| > r$. If r is the radius of convergence of a power series, then the open interval $\{x \in \mathbb{R} : |x| < r\}$ is called the **interval of convergence** of that power series; note that the interval of convergence is the empty set if $r = 0$ and is $\mathbb{R}$ if $r = \infty$. Given a power series $\sum_{k\geq 0} c_k x^k$, the set

$$S := \left\{x \in \mathbb{R} : \sum_{k\geq 0} c_k x^k \text{ is convergent}\right\}$$

may be distinct from the interval of convergence. In fact, S is always nonempty, and it can equal $\{0\}$ or $\mathbb{R}$ or an interval of the form $(-r, r)$, $[-r, r]$, $[-r, r)$, $(-r, r]$ for some $r > 0$. In the following table we illustrate various possibilities for the set S.

	Power series	Radius of convergence	S
(i)	$\sum_{k\geq 1} k^k x^k$	0	$\{0\}$
(ii)	$\sum_{k\geq 0} x^k/k!$	∞	$(-\infty, \infty)$
(iii)	$\sum_{k\geq 0} x^k$	1	$(-1, 1)$
(iv)	$\sum_{k\geq 1} x^k/k^2$	1	$[-1, 1]$
(v)	$\sum_{k\geq 1} x^k/k$	1	$[-1, 1)$
(vi)	$\sum_{k\geq 1} (-1)^k x^k/k$	1	$(-1, 1]$

The entries in (i), (ii), and (iii) above follow from Examples 9.25 (i), (ii), and (iii) respectively. For the power series in (iv) above, we note that

$$\frac{|x^{k+1}|}{(k+1)^2}\frac{k^2}{|x^k|} = \left(\frac{k}{k+1}\right)^2 |x| \to |x| \ \text{ as } \ k \to \infty.$$

So by the Ratio Test, the series is absolutely convergent if $|x| < 1$ and it is divergent if $|x| > 1$. Letting $p = 2$ in Example 9.1 (iii), we see that it is convergent if $x = 1$. Further, its convergence for $x = -1$ follows from Proposition 9.5. Similarly, the entries in (v) and (vi) above follow from the Ratio Test, Example 9.1 (iii) with $p = 1$, and Example 9.1 (iv).

The following result characterizes the radius of convergence of a power series $\sum_{k\geq 0} c_k x^k$ in terms of the sequence $(|c_k|^{1/k})$.

Proposition 9.28. *Let $\sum_{k\geq 0} c_k x^k$ be a power series with coefficients in $\mathbb{R}$, and let $\overline{\ell} := \limsup_{k\to\infty} |c_k|^{1/k}$. Then the radius of convergence of $\sum_{k\geq 0} c_k x^k$ is $1/\overline{\ell}$, with the convention that $1/\overline{\ell} := \infty$ if $\overline{\ell} = 0$ and $1/\overline{\ell} := 0$ if $\overline{\ell} = \infty$.*

Proof. Given any $x \in \mathbb{R}$, let $a_k := c_k x^k$ for $k \in \mathbb{N}$ and observe that

$$\limsup_{k\to\infty} |a_k|^{1/k} = |x| \limsup_{k\to\infty} |c_k|^{1/k} = \overline{\ell}\,|x|.$$

Now the desired result follows from the Root Test and Proposition 9.27. □

A result similar to Proposition 9.28 involving the ratios $|c_{k+1}|/|c_k|$, $k \in \mathbb{N}$, in place of the roots $|c_k|^{1/k}$, $k \in \mathbb{N}$, is given in Exercise 9.51. The following result is useful in calculating the radius of convergence of a power series.

Corollary 9.29. *Let $\sum_{k\geq 0} c_k x^k$ be a power series and let r be its radius of convergence. If $|c_k|^{1/k} \to \ell$, then $r = 1/\ell$. In case $c_k \neq 0$ for all large $k \in \mathbb{N}$ and $|c_{k+1}|/|c_k| \to \ell$, then $r = 1/\ell$. Here we continue to adopt the convention in Proposition 9.28 that $1/\ell := \infty$ if $\ell = 0$ and $1/\ell := 0$ if $\ell = \infty$.*

Proof. The first assertion is an immediate consequence of Proposition 9.28, because if $|c_k|^{1/k} \to \ell$, then $\ell = \limsup_{k\to\infty} |c_k|^{1/k}$. The second assertion follows from Proposition 9.27 and the Ratio Test applied to sequences (a_k), where $a_k := c_k x^k$ for $x \in \mathbb{R}$ with $x \neq 0$, because if $c_k \neq 0$ for all large $k \in \mathbb{N}$ and $|c_{k+1}|/|c_k| \to \ell$, then $\liminf_{k\to\infty} |c_{k+1}|/|c_k| = \ell = \limsup_{k\to\infty} |c_{k+1}|/|c_k|$, and so $\liminf_{k\to\infty} |a_{k+1}|/|a_k| = \ell|x| = \limsup_{k\to\infty} |a_{k+1}|/|a_k|$. □

Examples 9.30. (i) For $k = 1, 2, \ldots$, let

$$c_{2k-1} = \frac{1}{4^k} \quad \text{and} \quad c_{2k} = \frac{1}{9^k}.$$

Then $|c_j|^{1/j}$ for $j = 1, 2, \ldots$ are given by

$$\frac{1}{4}, \frac{1}{3}, \frac{1}{4^{2/3}}, \frac{1}{3}, \frac{1}{4^{3/5}}, \frac{1}{3}, \frac{1}{4^{4/7}}, \ldots.$$

We note that $(1/4^k)^{1/(2k-1)} \to 1/2$, whereas $(1/9^k)^{1/2k} \to 1/3$. Thus it can be seen that

$$\limsup_{k\to\infty} |c_k|^{1/k} = \frac{1}{2} \quad \text{and} \quad \liminf_{k\to\infty} |c_k|^{1/k} = \frac{1}{3}.$$

It follows from Proposition 9.28 that the radius of convergence of the power series $\sum_{k\geq 0} c_k x^k$ is 2. Note that taking ratios of consecutive coefficients of the power series is not helpful here.

(ii) For $k = 0, 1, 2, \ldots$, let $c_k := k^3/3^k$. Since

$$\frac{|c_{k+1}|}{|c_k|} = \frac{(k+1)^3}{3^{k+1}} \frac{3^k}{k^3} = \frac{1}{3}\left(1 + \frac{1}{k}\right)^3 \to \frac{1}{3},$$

the radius of convergence of $\sum_{k\geq 0} c_k x^k$ is 3.

(iii) For $k = 0, 1, 2, \ldots$, let $c_k := 2^k/k$. In view of Example 2.7 (iv),

$$|c_k|^{1/k} = \frac{2}{k^{1/k}} \to \frac{2}{1} = 2.$$

Hence the radius of convergence of $\sum_{k\geq 0} c_k x^k$ is 2. ◇

The above examples, as well as the last five examples in the table given before Proposition 9.28, provide us with a good collection of convergent power series. More examples can be generated using the result below, which essentially says that if a power series is convergent in the interval $(-r, r)$, then the power series obtained by term-by-term differentiation or by term-by-term integration is also convergent in the same interval.

Proposition 9.31. *Let $\sum_{k\geq 0} c_k x^k$ be a power series and let r be its radius of convergence. Then the radius of convergence of each of the series $\sum_{k\geq 1} k c_k x^{k-1}$ and $\sum_{k\geq 0} c_k x^{k+1}/(k+1)$ is also r.*

Proof. Observe that for every $k \in \mathbb{N}$ with $k \geq 2$, we can write

$$|kc_k|^{1/(k-1)} = \left(k^{1/k}|c_k|^{1/k}\right)^{k/(k-1)} \quad \text{and} \quad \left|\frac{c_{k-1}}{k}\right|^{1/k} = \frac{\left(|c_{k-1}|^{1/(k-1)}\right)^{(k-1)/k}}{k^{1/k}}.$$

By Example 2.7 (iv), $k^{1/k} \to 1$, and clearly $k/(k-1) \to 1$. Hence in view of Proposition 2.25, we see that

$$\limsup_{k\to\infty} |kc_k|^{1/(k-1)} = \limsup_{k\to\infty} |c_k|^{1/k} = \limsup_{k\to\infty} \left|\frac{c_{k-1}}{k}\right|^{1/k}.$$

Thus it follows from Proposition 9.28 that the radius of convergence of the series $\sum_{k\geq 1} k c_k x^{k-1}$ and $\sum_{k\geq 0} c_k x^{k+1}/(k+1)$ is r. □

Suppose a power series $\sum_{k\geq 0} c_k x^k$ has a positive radius of convergence r. We shall see in Chapter 10 that the resulting function $f : (-r, r) \to \mathbb{R}$ defined by $f(x) := \sum_{k=0}^{\infty} c_k x^k$ is differentiable and $f'(x) = \sum_{k\geq 1} k c_k x^{k-1}$ for $x \in (-r, r)$. (See part (iii) of Proposition 10.29.) By repeated application of this along with Proposition 9.31, we see that f is infinitely differentiable and

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)c_k x^{k-n} \quad \text{for } n \in \mathbb{N} \text{ and } x \in (-r, r).$$

In particular, $f^{(n)}(0) = n!c_n$ and hence $c_n = f^{(n)}(0)/n!$ for all $n \geq 0$. More generally, if $a \in \mathbb{R}$ and if the power series $\sum_{k\geq 0} c_k (x-a)^k$ around a has a positive radius of convergence r, then $f : (a-r, a+r) \to \mathbb{R}$ defined by $f(x) := \sum_{k\geq 0} c_k (x-a)^k$ is infinitely differentiable, and

$$c_n = \frac{f^{(n)}(a)}{n!} \quad \text{for all } n = 0, 1, 2, \ldots \quad \text{and thus} \quad f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

In the next subsection, we will turn around these considerations so as to associate to an infinitely differentiable function a power series around a given point of its domain.

Taylor Series

Let $a \in \mathbb{R}$ and let I be an interval in $\mathbb{R}$ such that a is an interior point of I. Suppose $f : I \to \mathbb{R}$ is an infinitely differentiable function. The power series around the point a given by

$$\sum_{k\geq 0} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the **Taylor series** of f around a. In the special case that $a = 0$, the Taylor series of f around a is sometimes called the **Maclaurin series** of f.

The observations made toward the end of the last subsection show that if $f : I \to \mathbb{R}$ is given by a power series around an interior point a of I that is convergent in an open interval containing a, then this power series is necessarily the Taylor series of f around a. On the other hand, the Taylor series of f around a is always defined as long as $f^{(n)}(a)$ exists for each $n \in \mathbb{N}$. In general, the Taylor series of f around a is obviously convergent at $x = a$. But at points $x \in I$ with $x \neq a$, it may be convergent or divergent; also, when it is convergent at $x \in I$, it may not converge to $f(x)$. This is illustrated by the following examples.

Examples 9.32. (i) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a polynomial function given by $f(x) := c_n x^n + \cdots + c_1 x + c_0$. Then it is clear that $f^{(k)}(0) = k!c_k$ for $k = 0, 1, \ldots, n$ and $f^{(k)}(0) = 0$ for $k > n$. Thus the Taylor series of f

around 0 converges to $f(x)$ for each $x \in \mathbb{R}$. More generally, for every $a \in \mathbb{R}$, by writing $x = (x-a) + a$, and using the Binomial Theorem, we obtain $f(x) = b_n(x-a)^n + \cdots + b_1(x-a) + b_0$ for unique $b_0, b_1, \ldots, b_n \in \mathbb{R}$, and this implies in a similar way that the Taylor series of f around a converges to $f(x)$ for each $x \in \mathbb{R}$.

(ii) Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := |x|$. Then f is infinitely differentiable at $x = 1$, and the Taylor series of f around 1 is easily seen to be $1 + (x-1)$. Thus the Taylor series of f around 1 is convergent on all of $\mathbb{R}$, but it does not converge to $f(x)$ when $x < 0$.

(iii) Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Note that $f'(x) = (2/x^3)e^{-1/x^2}$ for all $x \in \mathbb{R} \setminus \{0\}$. More generally, there is a polynomial $p_n(x)$ for each $n \in \mathbb{N}$ such that

$$f^{(n)}(x) = p_n(1/x)e^{-1/x^2} \quad \text{for all } x \in \mathbb{R} \setminus \{0\} \text{ and } n \in \mathbb{N}.$$

To see this, use induction on n. The case $n = 1$ has been noted already (where $p_1(x) := 2x^3$). If $f^{(n)}$ is given by the above formula for some $n \in \mathbb{N}$, then $f^{(n)}(x)$ is a sum of terms of the form $c_k e^{-1/x^2}/x^k$, where $c_k \in \mathbb{R}$ and k varies over a finite set of nonnegative integers and $x \in \mathbb{R} \setminus \{0\}$. The derivative of each such term exists at each $x \in \mathbb{R} \setminus \{0\}$, and is given by

$$c_k \frac{(2/x^3)e^{-1/x^2}x^k - kx^{k-1}e^{-1/x^2}}{x^{2k}} = c_k \left(\frac{2}{x^{k+3}} - \frac{k}{x^{k+1}} \right) e^{-1/x^2}.$$

It follows that $f^{(n+1)}(x)$ exists and is given by $p_{n+1}(1/x)e^{-1/x^2}$ for some polynomial $p_{n+1}(x)$ for $x \in \mathbb{R} \setminus \{0\}$. Next, we claim that $f^{(n)}(0)$ exists and is 0 for each $n \geq 0$. This is clear from the definition of f if $n = 0$. If $f^{(n)}(0) = 0$ for some $n \geq 0$, then the substitution $y = 1/x$ shows that

$$\lim_{x \to 0^+} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{y \to \infty} y f^{(n)}(1/y) = \lim_{y \to \infty} \frac{y p_n(y)}{e^{y^2}}.$$

Note that by successive applications of L'Hôpital's Rule (Remark 4.43),

$$\lim_{y \to \infty} \frac{y^k}{e^{y^2}} = 0 \quad \text{for every } k \in \mathbb{N}.$$

Since $p_n(y)$ is a finite sum of the form $\sum_{k=0}^{d} c_k y^k$, it follows that the right derivative $f_+^{(n+1)}(0)$ exists and is equal to 0. In a similar way (where we take rhe limit as $y \to -\infty$), we see that the left derivative $f_-^{(n+1)}(0)$ exists and is equal to 0. Thus by induction on n, we conclude that $f^{(n)}(0) = 0$ for all $n \geq 0$. Consequently, the Taylor series of f around 0 is simply the zero function. Clearly, this Taylor series is convergent at each $x \in \mathbb{R}$, but its sum is never equal to $f(x)$, unless $x = 0$. ◇

In Exercise 10.61, we give an example of a real-valued function on $\mathbb{R}$ whose Taylor series around 0 is defined (that is, the function is infinitely differentiable at 0), but this Taylor series converges only at 0.

We will now take up the general question of convergence of the Taylor series of a function around a point. Let $f : I \to \mathbb{R}$ be an infinitely differentiable function, where I is an interval in $\mathbb{R}$, and let $a \in I$. Recall that for $n = 0, 1, \ldots$, the nth Taylor polynomial of f around a is the polynomial given by

$$P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Let $R_n(x) := f(x) - P_n(x)$ for $x \in I$. If $x \in I$ is such that $R_n(x) \to 0$ as $n \to \infty$, then clearly the Taylor series of f around a converges to $f(x)$. This is obviously the case if $x = a$. In general, for all $x \in I$, by the Taylor Formula,

$$R_n(x) = \frac{f^{(n+1)}(c_{x,n})}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c_{x,n} \text{ between } a \text{ and } x.$$

As mentioned in Chapter 4, the above expression for $R_n(x)$ is known as the Lagrange form of the remainder in the Taylor Formula. It gives the following sufficient condition for the convergence of the Taylor series of a function.

Proposition 9.33. *Let $a \in \mathbb{R}$ and let I be an interval containing a. Suppose $f : I \to \mathbb{R}$ is an infinitely differentiable function. If there is $\alpha > 0$ such that*

$$\left|f^{(n)}(x)\right| \le \alpha^n \qquad \text{for all } n \in \mathbb{N} \text{ and } x \in I,$$

then the Taylor series of f converges to $f(x)$ for each $x \in I$.

Proof. Suppose there is $\alpha > 0$ such that $\left|f^{(n)}(x)\right| \le \alpha^n$ for all $n \in \mathbb{N}$ and $x \in I$. Then the Lagrange form of the remainder in the Taylor Formula implies that

$$|R_n(x)| \le \frac{|\alpha(x-a)|^{n+1}}{(n+1)!} \quad \text{for each } x \in I.$$

Let $x \in I$. From Example 2.7 (ii) (with $|\alpha(x-a)|$ in place of a), we see that $|\alpha(x-a)|^n/n! \to 0$. Consequently, $R_n(x) \to 0$ as $n \to \infty$. □

We shall use the above result tacitly in the examples below, in which we determine the Taylor series of some classical functions.

Examples 9.34. (i) Let $a := 0$, $I := \mathbb{R}$, and let $f : I \to \mathbb{R}$ be the sine function given by $f(x) := \sin x$. Then $|f^{(n)}(x)| \le 1$ for all $n \in \mathbb{N}$ and $x \in I$. Using the Taylor polynomials of f around 0 found in Section 7.2, we see that the Taylor series of f is convergent for $x \in I$ and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^s}{5!} - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} \quad \text{for } x \in \mathbb{R}.$$

For the Taylor series of the cosine function, see Exercise 9.21.

(ii) Let $a := 0$, $\beta > 0$, $I := (-\beta, \beta)$, and let $f : I \to \mathbb{R}$ be the exponential function given by $f(x) := e^x$. Then $|f^{(n)}(x)| = e^x \le e^\beta$ for all $n \in \mathbb{N}$ and $x \in I$. Using the Taylor polynomials for f found in Section 7.1, we see that the Taylor series of f is convergent for $x \in I$ and

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for } x \in (-\beta, \beta).$$

Since $\beta > 0$ is arbitrary, we see that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for all } x \in \mathbb{R}.$$

(iii) Let $a := 0$, $I := (-1, 1]$, and let $f : I \to \mathbb{R}$ be given by $f(x) := \ln(1+x)$. Then $f(0) = 0$, and for each $k \in \mathbb{N}$,

$$f^{(k)}(x) = (-1)^{k-1} \frac{(k-1)!}{(1+x)^k} \quad \text{for } x \in (-1, 1].$$

In particular, $f^{(k)}(0) = (-1)^{k-1}(k-1)!$ for each $k \in \mathbb{N}$. Hence the Taylor series of f around 0 is given by

$$\sum_{k \ge 1} \frac{(-1)^{k-1} x^k}{k}, \quad x \in \mathbb{R}.$$

Since it is not easy to show that $R_n(x) \to 0$ as $n \to \infty$ for $x \in (-1, 1]$ using the Lagrange form of the remainder, we proceed as follows. By the definition of the logarithmic function,

$$f(x) = \int_1^{1+x} \frac{dt}{t} = \int_0^x \frac{ds}{1+s} \quad \text{for } x \in (-1, 1].$$

Now for each $s \in \mathbb{R}$ with $s \ne -1$ and each $n \in \mathbb{N}$, we note that

$$\frac{1}{1+s} = \frac{1}{1-(-s)} = 1 - s + s^2 + \cdots + (-1)^{n-1} s^{n-1} + (-1)^n \frac{s^n}{1+s}.$$

Given any $x \in (-1, 1]$, we integrate both sides from 0 to x and obtain

$$f(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1} x^k}{k} + (-1)^n \int_0^x \frac{s^n}{1+s} ds.$$

If $x = 0$, this gives $f(0) = 1$. Suppose $x \in (0, 1)$. Then

$$\left| \int_0^x \frac{s^n}{1+s} ds \right| \le \int_0^x s^n ds = \frac{x^{n+1}}{n+1},$$

which tends to 0 as $n \to \infty$. Next, suppose $x \in (-1, 0)$. Substituting $u = -s$, we obtain

$$\left| \int_0^x \frac{s^n}{1+s} ds \right| = \left| \int_0^{-x} \frac{(-1)^{n+1} u^n}{1-u} du \right| \le \int_0^{-x} \frac{u^n}{1+x} du = \frac{(-x)^{n+1}}{(1+x)(n+1)},$$

which tends to 0 as $n \to \infty$. Hence we see that the Taylor series of f is convergent for $x \in (-1, 1]$ and

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k} \quad \text{for } x \in (-1, 1].$$

In particular, the sum of the alternating series $\sum_{k\ge 1} (-1)^{k-1}/k$ is $\ln 2$.

(iv) Let $a := 0$, $I = (-1, 1)$, $r \in \mathbb{R}$, and $f : I \to \mathbb{R}$ be given by $f(x) := (1+x)^r$. Then $f(0) = 1$ and $f^{(k)}(0) = r(r-1)\cdots(r-k+1)$ for $k \in \mathbb{N}$. If r is a nonnegative integer, then $f^{(k)}(0) = 0$ for all $k > r + 1$. Hence

$$f(x) = 1 + \sum_{k=1}^{r} \binom{r}{k} x^k \quad \text{for } x \in I.$$

(Note that we obtain the same result by the Binomial Theorem.) Suppose now that r is not a nonnegative integer. Then $f^{(k)}(0) \ne 0$ for each $k \ge 0$. Thus for each $n \in \mathbb{N}$, the nth Taylor polynomial of f around 0 is given by

$$P_n(x) := 1 + \sum_{k=1}^{n} \frac{r(r-1)\cdots(r-k+1)}{k!} x^k.$$

Using the Cauchy Form of Remainder (Exercise 4.51), it can be shown that $f(x) - P_n(x) \to 0$ as $n \to \infty$ for each $x \in (-1, 1)$. (See Exercise 9.52.) Hence we see that the Taylor series of f is convergent for $x \in I$ and

$$(1+x)^r = 1 + \sum_{k=1}^{\infty} \frac{r(r-1)\cdots(r-k+1)}{k!} x^k \quad \text{for } x \in (-1, 1).$$

This series is known as the **binomial series**. In particular, taking $r = -1$, we obtain

$$\frac{1}{1+x} = 1 + \sum_{k=1}^{\infty} \frac{(-1)(-2)\cdots(-k)}{k!} x^k = \sum_{k=0}^{\infty} (-1)^k x^k \quad \text{for } x \in (-1, 1).$$

Replacing x by $-x$, we thus recover the formula

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{for } x \in (-1, 1),$$

for the sum of the geometric series. ◇

Remark 9.35. Let I be an open interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a function. Then f is said to be **real analytic** if f is locally given by a convergent power series, that is, for each $a \in I$, there exists $r > 0$ and a power series $\sum_{k\geq 0} c_k(x-a)^k$ that converges to $f(x)$ for all $x \in I$ with $|x-a| < r$. It is clear that if the Taylor series of f around each $a \in I$ converges to f in an open interval containing a, then f is real analytic. The converse is also true, as we shall see in part (iii) of Proposition 10.29. In particular, real analytic functions are infinitely differentiable on $\mathbb{R}$. However, as seen in Example 9.32 (iii), there exist infinitely differentiable functions on $\mathbb{R}$ that are not real analytic on $\mathbb{R}$. We remark that such a phenomenon is absent for complex-valued functions of a complex variable, and the notions of a complex analytic function and of an infinitely differentiable function of a complex variable coincide. See, for example, Theorems 10.6 and 10.16 of Rudin [72]. $\diamond$

9.4 Convergence of Improper Integrals

In Chapter 6, we considered the Riemann integral of a bounded function defined on a closed and bounded interval in $\mathbb{R}$. In this and the next two sections, we shall extend the process of integration to functions defined on a semi-infinite interval or the doubly infinite interval $\mathbb{R}$ that are bounded on each closed and bounded subinterval, and also to unbounded functions defined on bounded or unbounded intervals.

We begin by considering functions defined on a semi-infinite interval of the form $[a, \infty)$, where $a \in \mathbb{R}$. Our treatment will run parallel to that of infinite series given in Sections 9.1 and 9.2. In analogy with an infinite series, we shall first give a formal (and pedantic) definition of an improper integral and then adopt suitable conventions in order to simplify our treatment. In the sequel, a statement dependent on $t \in [a, \infty)$ is said to be true **for all large** $t \in [a, \infty)$ if there exists $t_0 \in [a, \infty)$ such that it is true for all $t \geq t_0$.

Let $a \in \mathbb{R}$. An **improper integral** on $[a, \infty)$ is an ordered pair (f, F) of real-valued functions f and F defined on $[a, \infty)$ such that f is integrable on $[a, x]$ for every $x \geq a$ and

$$F(x) = \int_a^x f(t)dt \quad \text{for all } x \in [a, \infty).$$

Note that in view of the Fundamental Theorem of Calculus (Proposition 6.24), we obtain the following. If (f, F) is an improper integral on $[a, \infty)$, then F is continuous with $F(a) = 0$, and moreover, if f is continuous, then F is differentiable with $f = F'$. Conversely, if $f, F : [a, \infty) \to \mathbb{R}$ are such that f is integrable and F is differentiable with $F(a) = 0$ and $f = F'$, then (f, F) is an improper integral on $[a, \infty)$. For simplicity and brevity, we shall use the notation $\int_{t\geq a} f(t)dt$ for the improper integral (f, F) on $[a, \infty)$. In this notation, prominence is given to the first function f, but the second function

F is just as important. At any rate, F is uniquely determined by f, and if f is continuous, then f is uniquely determined by F. We shall refer to the function F as the **partial integral function** of f and for any $x \in [a, \infty)$, we may refer to $F(x) = \int_a^x f(t)dt$ as a **partial integral** of f.

Let $a \in \mathbb{R}$ and let $f : [a, \infty) \to \mathbb{R}$ be any function. We say that the improper integral $\int_{t \geq a} f(t)dt$ is **convergent** if f is integrable on $[a, x]$ for every $x \in [a, \infty)$ and the limit

$$\lim_{x \to \infty} \int_a^x f(t)dt$$

exists. It is clear that if this limit exists, then it is unique, and we shall denote it by $\int_a^\infty f(t)dt$, and call it the **value** of the improper integral $\int_{t \geq a} f(t)dt$. Usually, when we write

$$\int_a^\infty f(t)dt = I,$$

we mean that $I \in \mathbb{R}$ and the improper integral $\int_{t \geq a} f(t)dt$ is convergent with I as its value. In this case we may also say that $\int_{t \geq a} f(t)dt$ **converges** to I. An improper integral is said to be **divergent** if it is not convergent. In particular, if $\int_a^x f(t)dt \to \infty$ or $\int_a^x f(t)dt \to -\infty$ as $x \to \infty$, then we say that the improper integral **diverges** to ∞ or to $-\infty$, as the case may be.

Given an improper integral $\int_{t \geq a} f(t)dt$ and $x \in [a, \infty)$, by the **tail of the improper integral** $\int_{t \geq a} f(t)dt$ **from** x **onward**, we mean the improper integral $\int_{t \geq x} f(t)dt$. It is clear that for $x \in [a, \infty)$ and $I \in \mathbb{R}$, the improper integral $\int_{t \geq a} f(t)dt$ converges to I if and only if its tail from x onward converges to $I - \int_a^x f(t)dt$. This implies that an improper integral over $[a, \infty)$ is convergent if and only if its tail from x onward converges to 0 as $x \to \infty$.

Examples 9.36. (i) Let a and α be positive real numbers, and let us consider the improper integral $\int_{t \geq a} \alpha^t \, dt$. Given any $x \in [a, \infty)$,

$$\int_a^x \alpha^t \, dt = \begin{cases} (\alpha^x - \alpha^a)/\ln \alpha & \text{if } \alpha \neq 1, \\ x - a & \text{if } \alpha = 1. \end{cases}$$

Thus, in view of part (iii) of Proposition 7.8, it follows that if $\alpha < 1$, then $\int_a^\infty \alpha^t \, dt = -(\alpha^a/\ln \alpha)$, while if $\alpha \geq 1$, then $\int_{t \geq a} \alpha^t \, dt$ diverges to ∞.

(ii) The improper integral $\int_{t \geq 0} te^{-t^2} dt$ converges to $1/2$, since

$$\int_0^x te^{-t^2} dt = \frac{1}{2} \int_0^{x^2} e^{-s} ds = \frac{1}{2} \left(1 - e^{-x^2}\right) \to \frac{1}{2} \quad \text{as } x \to \infty.$$

(iii) Let $p \in \mathbb{R}$ and consider the improper integral $\int_{t \geq 1} (1/t^p)dt$. Given any $x \in [1, \infty)$, it is clear that

$$\int_1^x \frac{1}{t^p}\,dt = \begin{cases} (x^{1-p}-1)/(1-p) & \text{if } p \neq 1, \\ \ln x & \text{if } p = 1. \end{cases}$$

It follows that if $p > 1$, then $\int_{t\geq 1}(1/t^p)dt$ converges to $1/(p-1)$, while if $p \leq 1$, then it diverges to ∞.

(iv) The improper integral $\int_{t\geq 0} dt/(1+t^2)$ converges to $\pi/2$, since

$$\int_0^x \frac{dt}{1+t^2} = \arctan x \to \frac{\pi}{2} \quad \text{as } x \to \infty.$$

(v) The improper integral $\int_{t\geq 0} \cos t\,dt$ is divergent, since $\int_0^x \cos t\,dt = \sin x$ for all $x \in \mathbb{R}$ and $\lim_{x\to\infty} \sin x$ does not exist. $\diamond$

It may be observed that there is a remarkable analogy between the definition of an infinite series $\sum_{k\geq 1} a_k$ and the definition of an improper integral $\int_{t\geq a} f(t)dt$. The sequence of terms (a_k) corresponds to the function $f : [a,\infty) \to \mathbb{R}$, and a partial sum $A_n := \sum_{k=1}^n a_k$, where $n \in \mathbb{N}$, corresponds to a partial integral $F(x) := \int_a^x f(t)dt$, where $x \in [a,\infty)$. The convention that A_0, being the empty sum, is 0 corresponds to the initial condition $F(a) = 0$. Further, the difference quotient

$$a_k = A_k - A_{k-1} = \frac{A_k - A_{k-1}}{k-(k-1)}, \quad \text{where } k \in \mathbb{N},$$

corresponds to the derivative

$$f(t) = F'(t) = \lim_{s\to t} \frac{F(s)-F(t)}{s-t}, \quad \text{where } t \in [a,\infty).$$

This analogy will become more and more apparent as we develop the theory of improper integrals. At the same time, we will point out instances in which the analogy breaks down. (See Example 9.44 and Remark 9.52.)

The following results follow from the corresponding results for limits of functions of a real variable as the variable tends to infinity, just as similar results in the case of infinite series followed from the corresponding results for limits of sequences. In what follows, we have let $a \in \mathbb{R}$ and f, g, h denote real-valued functions on $[a,\infty)$ that are assumed to be integrable on $[a,x]$ for every $x \in [a,\infty)$.

- If $\int_{t\geq a} f(t)dt$ is convergent, then the set $\left\{\int_a^x f(t)dt : x \in [a,\infty)\right\}$ of partial integrals is bounded. (To see this, let $F(x) := \int_a^x f(t)dt$ for $x \in [a,\infty)$ and note that since $\int_{t\geq a} f(t)dt$ is convergent, there exists $x_0 \geq a$ such that $|F(x)| \leq 1 + \left|\int_a^\infty f(t)dt\right|$ for all $x \geq x_0$, and moreover, if $|f(t)| \leq \alpha$ for all $t \in [a, x_0]$, then $|F(x)| \leq \alpha|x-a| \leq \alpha|x_0 - a|$ for all $x \in [a, x_0]$.)

- Let $\int_a^\infty f(t)dt = I$ and $\int_a^\infty g(t)dt = J$. Then
$$\int_a^\infty (f(t)+g(t))dt = I+J \quad \text{and} \quad \int_a^\infty (rf)(t)dt = rI \quad \text{for every } r \in \mathbb{R}.$$
Further, if $f(t) \le g(t)$ for all $t \in [a,\infty)$, then $I \le J$.
- **(Sandwich Theorem)** If $f(t) \le h(t) \le g(t)$ for each $t \in [a,\infty)$, and if $\int_a^\infty f(t)dt = I = \int_a^\infty g(t)dt$, then $\int_a^\infty h(t)dt = I$.
- **(Cauchy Criterion)** An improper integral $\int_{t\ge a} f(t)dt$ is convergent if and only if for every $\epsilon > 0$, there exists $x_0 \in [a,\infty)$ such that
$$\left|\int_x^y f(t)dt\right| < \epsilon \quad \text{for all } y \ge x \ge x_0.$$
(To see this, let F denote the partial integral function of f, and write $F(y) - F(x) = \int_x^y f(t)dt$ for all $y \ge x \ge x_0$, and use Proposition 3.39.)

Integrals of Derivatives and of Nonnegative Functions

We now consider two results about improper integrals of functions of a special kind. The following result is an analogue of the result about the convergence of a telescoping series (Proposition 9.3).

Proposition 9.37. *Let $g : [a,\infty) \to \mathbb{R}$ be a differentiable function such that its derivative g' is integrable on $[a,x]$ for every $x \ge a$. Then $\int_{t\ge a} g'(t)dt$ is convergent if and only if $\lim_{x\to\infty} g(x)$ exists, and in this case,*
$$\int_a^\infty g'(t)dt = \lim_{x\to\infty} g(x) - g(a).$$

Proof. By part (ii) of the FTC (Proposition 6.24),
$$\int_a^x g'(t)dt = g(x) - g(a) \quad \text{for all } x \in [a,\infty).$$
This implies the desired result. $\square$

It may be noted that if a function $f : [a,\infty) \to \mathbb{R}$ is continuous and if we define $g : [a,\infty) \to \mathbb{R}$ by
$$g(x) := \int_a^x f(t)dt,$$
that is, if g is the "partial integral" of $\int_{t\ge a} f(t)dt$, then by part (ii) of the FTC, the improper integral $\int_{t\ge a} f(t)dt$ can be written as $\int_{t\ge a} g'(t)dt$. But then determining whether $\lim_{x\to\infty} g(x)$ exists is the same as determining whether the given improper integral $\int_{t\ge a} f(t)dt$ is convergent. In some special cases, however, it is possible to find an antiderivative g of the function f without

considering any "partial integral" of f. In these cases, we can determine the convergence of the improper integral $\int_{t\geq a} f(t)dt$ using Proposition 9.37. In fact, Examples 9.36 (i)–(v) illustrate this technique.

Our next result is regarding the convergence of an improper integral of a nonnegative function. It is an analogue of the result about the convergence of a series of nonnegative terms (Proposition 9.4).

Proposition 9.38. *Let $f : [a, \infty) \to \mathbb{R}$ be a nonnegative function, and let $F : [a, \infty) \to \mathbb{R}$ be the partial integral function of $\int_{t\geq a} f(t)dt$. Then $\int_{t\geq a} f(t)dt$ is convergent if and only if F is bounded above, and in this case,*

$$\int_a^\infty f(t)dt = \sup\{F(x) : x \in [a, \infty)\}.$$

If the function F is not bounded above, then $\int_{t\geq a} f(t)dt$ diverges to ∞.

Proof. Since $f(t) \geq 0$ for all $t \in [a, \infty)$, using the domain additivity of Riemann integrals (Proposition 6.8), we see that

$$F(y) = \int_a^x f(t)dt + \int_x^y f(t)dt \geq \int_a^x f(t)dt = F(x) \quad \text{for all } y \geq x \geq a.$$

Hence the function F is monotonically increasing. By part (i) of Proposition 3.44 with $b = \infty$, we see that $\lim_{x\to\infty} F(x)$ exists if and only if F is bounded above, and in this case,

$$\int_a^\infty f(t)dt = \lim_{x\to\infty} F(x) = \sup\{F(x) : x \in [a, \infty)\}.$$

Also, by part (i) of Proposition 3.44 with $b = \infty$, if F is not bounded above, then $F(x) \to \infty$ as $x \to \infty$, that is, $\int_{t\geq a} f(t)dt$ diverges to ∞. □

A result similar to the one above holds if $f(t) \leq 0$ for all $t \in [a, \infty)$. (See Exercise 9.25.) More generally, if $f(t)$ has the same sign for all large $t \in [a, \infty)$, then $\int_{t\geq a} f(t)dt$ is convergent if and only if F is bounded. However, if there is no $t_0 \in [a, \infty)$ such that $f(t)$ is of the same sign for all $t \in \mathbb{R}$ with $t \geq t_0$, then the improper integral $\int_{t\geq a} f(t)dt$ may diverge even though F is bounded. This is illustrated by the improper integral $\int_1^\infty f(t)dt$, where $f : [1, \infty) \to \mathbb{R}$ is defined by $f(s) := (-1)^{[s]}$. Here, the partial integral function $F : [1, \infty) \to \mathbb{R}$ is given by $F(x) = -1 + x - [x]$ if $[x]$ is even and $F(x) = -x + [x]$ if $[x]$ is odd. Clearly, F is bounded, but since $F(2n-1) = 0$ and $F(2n) = -1$ for all $n \in \mathbb{N}$, we see that $\lim_{x\to\infty} F(x)$ does not exist, that is, $\int_1^\infty f(t)dt$ is divergent.

Example 9.39. Consider $f : [0, \infty) \to \mathbb{R}$ defined by $f(t) := (1+\sin t)/(1+t^2)$. Then $f(t) \geq 0$ for all $t \in [0, \infty)$ and

$$F(x) = \int_0^x \frac{1 + \sin t}{1 + t^2} dt \leq \int_0^x \frac{2}{1 + t^2} dt = 2 \arctan x \leq \pi \quad \text{for all } x \in [0, \infty).$$

Hence $\int_{t\geq 0} f(t)dt$ is convergent. On the other hand, consider $g : [0,\infty) \to \mathbb{R}$ defined by $g(t) := (2+\cos t)/t$. Then $g(t) \geq 0$ for all $t \in [1,\infty)$ and

$$G(x) = \int_1^x \frac{2+\cos t}{t}dt \geq \int_1^x \frac{1}{t}dt = \ln x \quad \text{for all } x \in [1,\infty).$$

Since $\ln x \to \infty$ as $x \to \infty$, it follows that $\int_{t\geq 1} g(t)dt$ diverges to ∞. $\diamond$

An improper integral $\int_{t\geq a} f(t)dt$ is said to be **absolutely convergent** if the improper integral $\int_{t\geq a} |f(t)|dt$ is convergent. The following result is an analogue of Proposition 9.5, and the proof is quite similar.

Proposition 9.40. *An absolutely convergent improper integral is convergent.*

Proof. Let $a \in \mathbb{R}$ and let $\int_{t\geq a} f(t)dt$ be an absolutely convergent improper integral.Then $\int_{t\geq a} 2|f(t)|dt$ is convergent. Also, $0 \leq f(t)+|f(t)| \leq 2|f(t)|$ for all $t \in [a,\infty)$, Hence from Proposition 9.38, it follows that $\int_{t\geq a} (f(t)+|f(t)|)\, dt$ is convergent. Consequently, $\int_{t\geq a} f(t)dt$ is convergent. $\square$

An alternative proof of the above result can easily be given using the Cauchy Criterion for improper integrals. Example 9.41 below shows that the converse of the result in Proposition 9.40 does not hold. A convergent improper integral that is not absolutely convergent is said to be **conditionally convergent**. Another example of a conditionally convergent improper integral (which is modeled on the conditionally convergent infinite series $\sum_{k\geq 1}(-1)^{k-1}/k$) is given in Exercise 9.37.

Example 9.41. Consider the improper integral $\int_{t\geq 1}(\cos t/t)dt$. Integrating by parts, we see that

$$\int_1^x \frac{\cos t}{t}dt = \frac{\sin x}{x} - \sin 1 + \int_1^x \frac{\sin t}{t^2}dt \quad \text{for all } x \geq 1.$$

Further, $(\sin x)/x \to 0$ as $x \to \infty$ and also

$$\int_1^x \left|\frac{\sin t}{t^2}\right| dt \leq \int_1^x \frac{1}{t^2}dt = 1 - \frac{1}{x} \to 1 \quad \text{as } x \to \infty.$$

Hence by Proposition 9.38, the improper integral $\int_{t\geq 1} |(\sin t)/t^2|dt$ is convergent, that is, $\int_{t\geq 1}(\sin t)/t^2 dt$ is absolutely convergent, and so by Proposition 9.40, it is convergent. Consequently, $\int_{t\geq 1}(\cos t/t)dt$ is convergent and

$$\int_1^\infty \frac{\cos t}{t}dt = -\sin 1 + \int_1^\infty \frac{\sin t}{t^2}dt.$$

On the other hand, for each $n \in \mathbb{N}$ with $n \geq 2$,

$$\int_{\pi}^{n\pi} \left|\frac{\cos t}{t}\right| dt = \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\cos t|}{t} dt \geq \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\cos t|}{k\pi} dt = \sum_{k=2}^{n} \frac{2}{k\pi}.$$

Since the series $\sum_{k\geq 2} 1/k$ diverges to ∞, it follows from Proposition 9.38 that the improper integral

$$\int_{t\geq 1} \left|\frac{\cos t}{t}\right| dt$$

diverges to ∞. Thus $\int_{t\geq 1} (\cos t/t)dt$ is conditionally convergent. $\diamond$

Let us now discuss whether the convergence of an improper integral $\int_{t\geq 1} f(t)dt$ of a nonnegative function f is related to the convergence of the infinite series $\sum_{k\geq 1} f(k)$. Consider $f : [1,\infty) \to \mathbb{R}$ given by $f(t) := 1$ if $t \in \mathbb{N}$ and $f(t) := 0$ if $t \notin \mathbb{N}$. Then it is easy to see that $\int_{t\geq a} f(t)dt$ is convergent but $\sum_{k\geq 1} f(k)$ is divergent. On the other hand, if we let $g := 1 - f$, then $g \geq 0$ and it is easily seen that $\int_{t\geq 1} g(t)dt$ is divergent, but $\sum_{k\geq 1} g(k)$ is convergent. Thus, in general, the convergence of $\int_{t\geq 1} f(t)dt$ is independent of the convergence of $\sum_{k\geq 1} f(k)$. In view of this, the following result is noteworthy.

Proposition 9.42 (Integral Test). *Let $f : [1,\infty) \to \mathbb{R}$ be a nonnegative monotonically decreasing function. Then the improper integral $\int_{t\geq 1} f(t)dt$ is convergent if and only if the infinite series $\sum_{k\geq 1} f(k)$ is convergent, and in this case,*

$$\sum_{k=2}^{\infty} f(k) \leq \int_{1}^{\infty} f(t)dt \leq \sum_{k=1}^{\infty} f(k),$$

or equivalently,

$$\int_{1}^{\infty} f(t)dt \leq \sum_{k=1}^{\infty} f(k) \leq f(1) + \int_{1}^{\infty} f(t)dt.$$

Also, the improper integral $\int_{t\geq 1} f(t)dt$ diverges to ∞ if and only if the infinite series $\sum_{k\geq 1} f(k)$ diverges to ∞.

Proof. First, note that since f is monotonic, by part (i) of Proposition 6.10, f is integrable on $[1,x]$ for every $x \in [1,\infty)$. Define $F : [1,\infty) \to \mathbb{R}$ by $F(x) := \int_1^x f(t)dt$. Since f is nonnegative, the function F is monotonically increasing. Hence Proposition 9.38 implies that $\int_{t\geq 1} f(t)dt$ is convergent if and only if the set $\{F(n) : n \in \mathbb{N}\}$ is bounded above, and in this case,

$$\int_{1}^{\infty} f(t)dt = \sup\{F(n) : n \in \mathbb{N}\} = \lim_{n\to\infty} F(n).$$

Also, using Proposition 9.38 and the fact that F is monotonically increasing, we see that

$$\int_{t\geq 1} f(t)dt \text{ diverges to } \infty \Longleftrightarrow F(n) \to \infty \ \text{ as } \ n \to \infty.$$

Define

$$a_k := \int_k^{k+1} f(t)dt \quad \text{for } k \in \mathbb{N} \quad \text{and} \quad A_n := \sum_{k=1}^{n} a_k \quad \text{for } n \in \mathbb{N}.$$

Then $A_n = F(n+1)$ for all $n \in \mathbb{N}$. Further, since $a_k \geq 0$ for all $k \in \mathbb{N}$, it follows from Proposition 9.4 that the series $\sum_{k\geq 1} a_k$ is convergent if and only if the sequence $(F(n))$ is bounded above, that is, the improper integral $\int_{t\geq 1} f(t)dt$ is convergent. Also, it follows that $\int_{t\geq 1} f(t)dt$ diverges to ∞ if and only if $\sum_{k\geq 1} a_k$ diverges to ∞.

Now since f is monotonically decreasing,

$$f(k+1) \leq a_k \leq f(k) \quad \text{for all } k \in \mathbb{N}.$$

Thus the Comparison Test for series (Proposition 9.11) shows that $\sum_{k\geq 1} a_k$ is convergent if and only if $\sum_{k\geq 1} f(k)$ is convergent, and also that $\sum_{k\geq 1} a_k$ diverges to ∞ if and only if $\sum_{k\geq 1} f(k)$ diverges to ∞. Finally, since

$$\sum_{k=2}^{n+1} f(k) = \sum_{k=1}^{n} f(k+1) \leq A_n \leq \sum_{k=1}^{n} f(k) \quad \text{for all } n \in \mathbb{N},$$

we see that

$$\sum_{k=2}^{\infty} f(k) \leq \lim_{n\to\infty} A_n = \int_1^{\infty} f(t)dt \leq \sum_{k=1}^{\infty} f(k)$$

whenever $\int_{t\geq 1} f(t)dt$ is convergent. □

The above result can be extremely useful in deducing the convergence or the divergence of infinite series. Further, it can be used to obtain lower bounds and upper bounds for the partial sums of a series. These yield, in case the series converges, a lower bound and an upper bound for the sum of the series. These remarks are illustrated by Examples 9.43 below.

If in the Integral Test, the hypothesis that the function $f : [1, \infty) \to \mathbb{R}$ is nonnegative and monotonically decreasing is not satisfied, but if f is differentiable and f' is integrable on $[1, x]$ for every $x \geq 1$, then the convergence of $\int_{t\geq 1} f(t)dt$ and $\sum_{k\geq 1} f(k)$ can be related by what is known as the **Euler Summation Formula**. We refer the interested reader to pages 74–75 of [78].

Examples 9.43. (i) Let $p > 0$ and $f(t) := 1/t^p$ for $t \in [1, \infty)$. Then f is a nonnegative monotonically decreasing function. We have seen in Example 9.36 (iii) that $\int_{t\geq 1} f(t)dt$ is convergent if $p > 1$ and it diverges to ∞ if $p \leq 1$. Hence by Proposition 9.42, we see that $\sum_{k\geq 1}(1/k^p)$ is convergent if $p > 1$ and it diverges to ∞ if $p \leq 1$. We thus obtain an alternative proof the result given in Example 9.1 (iii). Further, if $p > 1$, we can estimate the sum using Proposition 9.42 as follows:

$$\frac{1}{p-1} = \int_1^\infty \frac{1}{t^p} dt \le \sum_{k=1}^{\infty} \frac{1}{k^p} \le 1 + \int_1^\infty \frac{1}{t^p} dt = \frac{p}{p-1}.$$

(ii) Let $p > 0$ and consider the infinite series

$$\sum_{k\ge 2} \frac{1}{k(\ln k)^p} = \sum_{k\ge 1} \frac{1}{(k+1)\,(\ln(k+1))^p}.$$

If $f : [1, \infty) \to \mathbb{R}$ is defined by

$$f(t) := \frac{1}{(t+1)\,(\ln(t+1))^p} \quad \text{for } t \in [1, \infty),$$

then f is clearly nonnegative and monotonically decreasing. For $x \in [1, \infty)$,

$$\int_1^x f(t)dt = \int_{\ln 2}^{\ln(x+1)} \frac{1}{s^p}\, ds = \begin{cases} \dfrac{(\ln(x+1))^{1-p} - (\ln 2)^{1-p}}{1-p} & \text{if } p \ne 1, \\ \ln\left(\ln(x+1)\right) - \ln(\ln 2) & \text{if } p = 1. \end{cases}$$

Letting $x \to \infty$, we see that

$$\int_1^\infty f(t)dt = \frac{1}{(p-1)(\ln 2)^{p-1}} \quad \text{if } p > 1,$$

while

$$\int_{t\ge 1} f(t)dt \text{ diverges to } \infty \quad \text{if } p \le 1.$$

This shows that the infinite series $\sum_{k\ge 2} 1/k(\ln k)^p$ is convergent if $p > 1$ and it diverges to ∞ if $p \le 1$. Further, if $p > 1$, then

$$\frac{1}{(p-1)(\ln 2)^{p-1}} \le \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p} \le \frac{1}{(\ln 2)^{p-1}} \left(\frac{1}{2\ln 2} + \frac{1}{p-1} \right).$$

The upper and lower bounds on the sums of the series in (i) and (ii) above are noteworthy. $\diamond$

9.5 Convergence Tests for Improper Integrals

In this section we shall consider several tests that enable us to conclude the convergence or divergence of improper integrals. We begin by pointing out that even if an improper integral $\int_{t\ge a} f(t)dt$ is convergent, $f(t)$ may not tend to 0 as $t \to \infty$. In other words, the kth Term Test for series (Proposition 9.8) does not have an analogue for improper integrals.

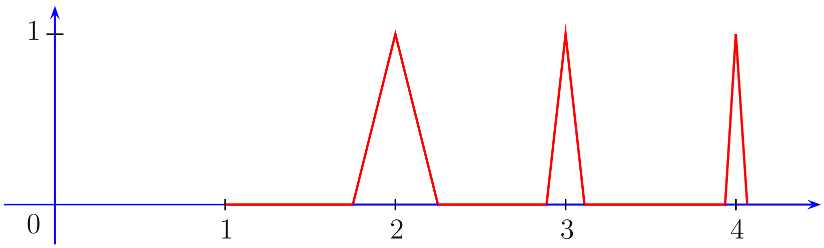

Fig. 9.1. Graph of the piecewise linear continuous function in Example 9.44.

The function $f : [1, \infty) \to \mathbb{R}$ defined by $f(t) := 1$ if $t \in \mathbb{N}$ and $f(t) := 0$ if $t \in [1, \infty) \setminus \mathbb{N}$ provides a simple example of a nonnegative function for which $\int_{t \geq 1} f(t)dt$ is convergent, but $f(t) \not\to 0$ as $t \to \infty$. An example of a continuous nonnegative function $f : [1, \infty) \to \mathbb{R}$ with a similar property is given below.

Example 9.44. Let $f : [1, \infty) \to \mathbb{R}$ be the piecewise linear function whose graph is as in Figure 9.1. Formally, f is defined as follows. Let $f(t) := 0$ if $1 \leq t < 2 - (1/2)^2$. Moreover, for any $k \in \mathbb{N}$ with $k \geq 2$, let

$$f(t) := \begin{cases} k^2 t - k^3 + 1 & \text{if } k - \dfrac{1}{k^2} \leq t \leq k, \\ -k^2 t + k^3 + 1 & \text{if } k < t \leq k + \dfrac{1}{k^2}, \\ 0 & \text{if } k + \dfrac{1}{k^2} < t < (k+1) - \dfrac{1}{(k+1)^2} \end{cases}$$

Note that the function f is continuous. Let $F(x) := \int_1^x f(t)dt$ for $x \in [1, \infty)$. Since the area of a triangle having height 1 and base $2/k^2$ is equal to $1/k^2$, we see that for each $n \in \mathbb{N}$ with $n \geq 2$,

$$F(x) = \sum_{k=2}^{n} \frac{1}{k^2} \quad \text{if } n + \frac{1}{n^2} \leq x < (n+1) - \frac{1}{(n+1)^2}.$$

Also, since the series $\sum_{k \geq 2} 1/k^2$ is convergent, it follows that the function F is bounded. So by Proposition 9.38, $\int_{t \geq 1} f(t)dt$ is convergent. However, since $f(k) = 1$ for each $k \in \mathbb{N}$ with $k \geq 2$, we see that $f(t) \not\to 0$ as $t \to \infty$. $\diamond$

By modifying the function in the above example, one can obtain a continuous nonnegative function $g : [1, \infty) \to \mathbb{R}$ such that $\int_{t \geq 1} g(t)dt$ is convergent, but $(g(k))$ is in fact an unbounded sequence. (See Exercise 9.24.) For more examples of this type, see Exercise 9.27. On the other hand, suppose $f : [a, \infty) \to \mathbb{R}$ is differentiable. If $\int_{t \geq a} f(t)dt$ and $\int_{t \geq a} f'(t)dt$ are convergent, then it can be shown that $f(t) \to 0$ as $t \to \infty$. (See Exercise 9.28.)

Remarks 9.45. (i) If a series $\sum_{k \geq 1} a_k$ is convergent, then clearly

$$a_k = A_k - A_{k-1} \to 0 \text{ as } k \to \infty,$$

where A_k is the kth partial sum of the series. In a similar way, if $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ are such that $\int_{t \geq a} f(t)dt$ is convergent, then clearly

$$\int_{x-1}^{x} f(t)dt = F(x) - F(x-1) \to 0 \text{ as } x \to \infty,$$

where F is the partial integral function of f. However, as the above examples show, this does not imply that $f(t) \to 0$ as $t \to \infty$.

(ii) Let $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ be such that $\int_{t \geq a} f(t)dt$ is convergent. Suppose there exists $r \in \mathbb{R}$ such that $f(t) \to r$ as $t \to \infty$. Then r must be equal to 0. Indeed, if $r > 0$, then there exists $t_0 \in [a, \infty)$ such that $f(t) > r/2$ for all $t \geq t_0$. This implies that $\int_{t \geq a} f(t)dt$ diverges to ∞. Likewise, if $r < 0$, then $\int_{t \geq a} f(t)dt$ diverges to $-\infty$. ◇

Tests for Absolute Convergence of Improper Integrals

We shall now consider, wherever possible, analogues of the tests for absolute convergence of infinite series in the case of improper integrals.

Proposition 9.46 (Comparison Test for Improper Integrals). *Suppose $a \in \mathbb{R}$ and $f, g : [a, \infty) \to \mathbb{R}$ are such that both f and g are integrable on $[a, x]$ for every $x \geq a$ and $|f(t)| \leq g(t)$ for all large $t \in [a, \infty)$. If $\int_{t \geq a} g(t)dt$ is convergent, then $\int_{t \geq a} f(t)dt$ is absolutely convergent.*

Proof. Let $t_0 \in [a, \infty)$ be such that $|f(t)| \leq g(t)$ for all $t \in [t_0, \infty)$. Suppose $\int_{t \geq a} g(t)dt$ is convergent. Let $\epsilon > 0$ be given. Then by the Cauchy Criterion, there exists $x_0 \in [a, \infty)$ such that $\left| \int_x^y g(t)dt \right| < \epsilon$ for all $y \geq x \geq x_0$. Now let $x_1 := \max\{t_0, x_0\}$. Then

$$\int_a^x |f(t)|dt \leq \int_x^y g(t)dt = \left| \int_x^y g(t)dt \right| < \epsilon \quad \text{for all } y \geq x \geq x_1.$$

Hence by the Cauchy Criterion, $\int_{t \geq a} f(t)dt$ is absolutely convergent. □

Examples 9.47. Arguing as in Examples 9.12, but using Proposition 9.46 (instead of Proposition 9.11) and Example 9.36 (iii), we obtain the following:

(i) $\displaystyle\int_{t \geq 0} \frac{2^t + t}{3^t + t}\, dt$ is convergent.

(ii) $\displaystyle\int_{t \geq 1} \frac{1}{(1 + t^2 + t^4)^{1/3}}\, dt$ is convergent. ◇

The following result is often more useful in practice than the Comparison Test for improper integrals (Proposition 9.46).

Corollary 9.48 (Limit Comparison Test for Improper Integrals). *Let $a \in \mathbb{R}$ and let $f, g : [a, \infty) \to \mathbb{R}$ be such that both f and g are integrable on $[a, x]$ for every $x \geq a$. Suppose $f(t) > 0$ and $g(t) > 0$ for all large $t \in [a, \infty)$. Also suppose there exists $\ell \in \mathbb{R} \cup \{\infty\}$ such that $f(t)/g(t) \to \ell$ as $t \to \infty$.*

(i) *If $\ell \neq 0$ and $\ell \neq \infty$, then*

$$\int_{t \geq a} f(t)dt \text{ is convergent} \iff \int_{t \geq a} g(t)dt \text{ is convergent.}$$

(ii) *If $\ell = 0$ and $\int_{t \geq a} g(t)dt$ is convergent, then $\int_{t \geq a} f(t)dt$ is convergent.*

(iii) *If $\ell = \infty$ and $\int_{t \geq a} f(t)dt$ is convergent, then $\int_{t \geq a} g(t)dt$ is convergent.*

Proof. The desired results can be deduced from Proposition 9.46 in a similar way as the results in Corollary 9.13 were deduced from Proposition 9.11. □

Examples 9.49. The first four examples are analogues of the corresponding examples considered earlier for infinite series. The assertions in (i), (ii), (iii), and (iv) below follow from Corollary 9.48 in the same manner as those in Example 9.14 followed from Corollary 9.13. We also give an additional example.

(i) $\displaystyle\int_{t \geq 0} \frac{2^t + t}{3^t - t}\, dt$ is convergent.

(ii) $\displaystyle\int_{t \geq 1} \sin \frac{1}{t}\, dt$ is divergent.

(iii) $\displaystyle\int_{t \geq 1} \frac{\ln t}{t^p}\, dt$ is convergent if $p > 1$ and it is divergent if $p \leq 1$.

(iv) $\displaystyle\int_2^\infty \frac{1}{(\ln t)^p}\, dt$ is divergent if $p > 0$.

(v) Let $q \in \mathbb{R}$ and let $f : [1, \infty) \to \mathbb{R}$ be given by $f(t) := e^{-t} t^q$. Then $\int_{t \geq 1} f(t)dt$ is convergent. To see this, choose $k \in \mathbb{N}$ with $k > q + 1$, and define $g : [1, \infty) \to \mathbb{R}$ by $g(t) := t^{q-k}$. Then $f(t) > 0$ and $g(t) > 0$ for all $t \in [1, \infty)$, and moreover, by L'Hôpital's Rule,

$$\frac{f(t)}{g(t)} = \frac{t^k}{e^t} \to 0 \quad \text{as} \quad t \to \infty.$$

Since $k - q > 1$, we see that $\int_{t \geq 1} g(t)dt$ is convergent. Hence by part (i) of Corollary 9.48, we conclude that $\int_{t \geq 1} e^{-t} t^q\, dt$ is convergent. ◇

The following result is an analogue of the (Cauchy) Root Test for infinite series (Proposition 9.17).

Proposition 9.50 (Root Test for Improper Integrals). *Let $a \in \mathbb{R}$ and let $f : [a, \infty) \to \mathbb{R}$ be a function that is integrable on $[a, x]$ for every $x \geq a$.*

(i) *If there exists* $\alpha \in \mathbb{R}$ *with* $\alpha < 1$ *such that* $|f(t)|^{1/t} \le \alpha$ *for all large* $t \in [a, \infty)$, *then* $\int_{t\ge a} f(t)dt$ *is absolutely convergent.*

(ii) *If there exists* $\delta \in \mathbb{R}$ *with* $\delta > 0$ *such that* $f(t) \ge \delta$ *for all large* $t \in [a, \infty)$, *then* $\int_{t\ge a} f(t)dt$ *diverges to* ∞.

In particular, if

$$|f(t)|^{1/t} \to \ell \ \ \text{as} \ \ t \to \infty, \qquad \text{where } \ell \in \mathbb{R} \text{ or } \ell = \infty,$$

then $\int_{t\ge a} f(t)dt$ *is absolutely convergent when* $\ell < 1$, *and it diverges to* ∞ *when* f *is nonnegative and* $\ell > 1$.

Proof. The assertion in (i) follows by letting $g(t) := \alpha^t$ for $t \in [a, \infty)$ and using the Comparison Test (Proposition 9.46). For the assertion in (ii), let $\delta > 0$ and let $t_1 \in [a, \infty)$ be such that $f(t) \ge \delta$ for all $t \ge t_1$. Then

$$\int_a^x f(t)dt \ge \int_a^{t_1} f(t)dt + \delta(x - t_1) \quad \text{for all } x \in [t_1, \infty).$$

Hence $\int_a^x f(t)dt \to \infty$ as $x \to \infty$.

To prove the last assertion, suppose there exists $\ell \in \mathbb{R} \cup \{\infty\}$ such that $|f(t)|^{1/t} \to \ell$ as $t \to \infty$. Note that $\ell \ge 0$. In case $\ell < 1$, we can find $\epsilon > 0$ such that $\ell + \epsilon < 1$ and there is $t_0 \in [a, \infty)$ such that $|f(t)|^{1/t} < \ell + \epsilon$ for all $t \ge t_0$. So using (i) with $\alpha := \ell + \epsilon$, we see that $\int_{t\ge a} f(t)dt$ is absolutely convergent. On the other hand, if f is nonnegative and $\ell > 1$, then we can find $\epsilon > 0$ such that $\ell - \epsilon > 1$ and there is $t_0 \in [a, \infty)$ such that $|f(t)|^{1/t} > \ell - \epsilon$ for all $t \ge t_0$. Now if $t_1 = \max\{t_0, 1\}$, then $f(t) = |f(t)| \ge (\ell - \epsilon)^t \ge \ell - \epsilon$ for all $t \ge t_1$. So using (ii) with $\delta := \ell - \epsilon$, we see that $\int_{t\ge a} f(t)dt$ is divergent. □

Examples 9.51. (i) Let $f(t) := t^2/2^t$ for $t \in [1, \infty)$. In view of Remark 7.13, $|f(t)|^{1/t} = (t^{1/t})^2/2 \to 1/2$ as $t \to \infty$. Hence $\int_{t\ge 1} f(t)dt$ is (absolutely) convergent. On the other hand, if $g(t) := 2^t/t^2$ for $t \in [1, \infty)$, then g is nonnegative and $|g(t)|^{1/t} \to 2$ as $t \to \infty$. Hence $\int_{t\ge 1} g(t)dt$ diverges to ∞.

(ii) If $f : [a, \infty) \to \mathbb{R}$ and $|f(t)|^{1/t} \to 1$ as $t \to \infty$, then $\int_{t\ge a} f(t)dt$ may be convergent or divergent. For example, if $f(t) := 1/t$ and $g(t) := 1/t^2$ for $t \in [1, \infty)$, then $|f(t)|^{1/t} \to 1$ and $|g(t)|^{1/t} = \left(|f(t)|^{1/t}\right)^2 \to 1$ as $t \to \infty$. However, $\int_{t\ge a} f(t)dt$ is divergent, whereas $\int_{t\ge a} g(t)dt$ is convergent. ◇

Remark 9.52. The Ratio Test for infinite series does not have a meaningful analogue for improper integrals partly because there is no notion of consecutive terms of a function $f(t)$ of a real variable t. ◇

Tests for Conditional Convergence of Improper Integrals

We shall now consider some tests that give conditional convergence of an improper integral. They are based on the following formula for Integration by

Parts (Proposition 6.28), which can be considered an analogue of the Partial Summation Formula (Proposition 9.20).

Let $f, g : [a, b] \to \mathbb{R}$ be such that f is differentiable and g is continuous. If f' is integrable and $G(x) := \int_a^x g(t)dt$ for $x \in [a, b]$, then

$$\int_a^b f(t)g(t)dt = f(b)G(b) - \int_a^b f'(t)G(t)dt.$$

Proposition 9.53 (Dirichlet Test for Improper Integrals). *Let $a \in \mathbb{R}$ and let $f, g : [a, \infty) \to \mathbb{R}$ be such that f is monotonic, $f(s) \to 0$ as $s \to \infty$, f is differentiable, f' is integrable on $[a, x]$ for every $x \geq a$, while g is continuous, and the function $G : [a, \infty) \to \mathbb{R}$ defined by $G(x) := \int_a^x g(t)dt$ is bounded. Then the improper integral $\int_{t \geq a} f(t)g(t)dt$ is convergent. Further, let*

$$\beta_x := \sup\left\{\left|\int_x^y g(t)dt\right| : y \in [x, \infty)\right\} \quad \text{for } x \in [a, \infty).$$

Then for each $x \in [a, \infty)$,

$$\left|\int_a^x f(t)g(t)dt\right| \leq f(a)\beta_a \quad \text{and} \quad \left|\int_x^\infty f(t)g(t)dt\right| \leq |f(x)|\beta_x \leq 2|f(x)|\beta_a.$$

Proof. Since G is bounded, $\beta_a = \sup\{|G(y)| : y \in [a, \infty)\}$ is well-defined. Let $x \in [a, \infty)$, and define $G_x : [x, \infty) \to \mathbb{R}$ by $G_x(y) := \int_x^y g(t)dt$. Then

$$|G_x(y)| = |G(y) - G(x)| \leq |G(y)| + |G(x)| \leq 2\beta_a \quad \text{for all } y \in [x, \infty).$$

This shows that $\beta_x = \sup\{|G_x(y)| : y \in [x, \infty)\}$ is well-defined and $\beta_x \leq 2\beta_a$.

Now consider $y \in [x, \infty)$. Integrating by parts, we obtain

$$\int_x^y f(t)g(t)dt = f(y)G_x(y) - \int_x^y f'(t)G_x(t)dt.$$

First assume that the function f is decreasing. Then $f' \leq 0$ on $[a, \infty)$. Also, since $f(t) \to 0$ as $t \to \infty$, we see that $f(y) \geq 0$. Hence

$$\left|\int_x^y f(t)g(t)dt\right| \leq \beta_x\left(|f(y)| + \int_x^y |f'(t)|dt\right) \leq \beta_x\big(f(y) - f(y) + f(x)\big) = f(x)\beta_x.$$

Next, if the function f is increasing, then the function $-f$ is decreasing, and $(-f)(t) \to 0$ as $t \to \infty$. Hence $\left|\int_x^y (-f)(t)g(t)dt\right| \leq -f(x)\beta_x$. Thus

$$\left|\int_x^y f(t)g(t)dt\right| \leq |f(x)|\beta_x \leq 2|f(x)|\beta_a$$

in either case. Since $f(x) \to 0$ as $x \to \infty$, the Cauchy Criterion shows that the improper integral $\int_{t \geq a} f(t)dt$ is convergent. Further, we may let $y \to \infty$ in the above inequality to obtain $\left|\int_x^\infty f(t)g(t)dt\right| \leq |f(x)|\beta_x \leq 2|f(x)|\beta_a$. □

A similar result, known as the **Abel Test for Improper Integrals**, is given in Exercise 9.32. While the Leibniz Test (Corollary 9.22) has no straightforward analogue for improper integrals, the Convergence Test for Trigonometric Series (Corollary 9.23) admits the following analogue.

Corollary 9.54 (Convergence Test for Fourier Integrals). *Let $a \in \mathbb{R}$ and let $f : [a, \infty) \to \mathbb{R}$ be a monotonic function such that $f(t) \to 0$ as $t \to \infty$. Suppose f is differentiable and f' is integrable on $[a, x]$ for every $x \ge a$. Then*

(i) $\int_{t \ge a} f(t) \sin ut\, dt$ *is convergent for each $u \in \mathbb{R}$.*

(ii) $\int_{t \ge a} f(t) \cos ut\, dt$ *is convergent for each $u \in \mathbb{R}$ with $u \neq 0$.*

Proof. (i) Let $u \in \mathbb{R}$. Define $g : [a, \infty) \to \mathbb{R}$ by $g(t) := \sin ut$. Then g is a continuous function. For $x \in [a, \infty)$, let $G(x) := \int_a^x g(t)dt$. If $u = 0$, then $g = 0$, and so $G = 0$. If $u \neq 0$, then by part (ii) of the FTC (Proposition 6.24),

$$|G(x)| = \frac{|\cos ua - \cos ux|}{|u|} \le \frac{2}{|u|} \quad \text{for all } x \ge a.$$

Thus in any event, the function G is bounded. Hence the convergence of $\int_{t \ge a} f(t) \sin ut\, dt$ for each $u \in \mathbb{R}$ follows from Proposition 9.53.

(ii) Suppose $u \in \mathbb{R}$ with $u \neq 0$. By part (ii) of the FTC (Proposition 6.24),

$$\left| \int_a^x \cos ut\, dt \right| = \frac{|\sin ux - \sin ua|}{|u|} \le \frac{2}{|u|} \quad \text{for all } x \ge a.$$

Thus as in (i) above, the convergence of $\int_{t \ge a} f(t) \cos ut\, dt$ for each $u \in \mathbb{R}$ with $u \neq 0$ follows from Proposition 9.53. □

Example 9.55. Let $p \in (0, 1]$ and $u \in \mathbb{R}$. Then the improper integral

$$\int_{t \ge 1} \frac{\sin ut}{t^p}\, dt$$

is convergent. This follows by applying Corollary 9.54 to $f : [1, \infty) \to \mathbb{R}$ defined by $f(t) := 1/t^p$. Similarly, if $u \in \mathbb{R}$, then the improper integral

$$\int_{t \ge 1} \frac{\cos ut}{t^p}\, dt$$

is convergent if $u \neq 0$. On the other hand, if $u = 0$, then the improper integral

$$\int_{t \ge 1} \frac{\cos 0}{t^p} dt = \int_{t \ge 1} \frac{1}{t^p}\, dt$$

is divergent. ◇

9.6 Related Improper Integrals

In the previous two sections we have considered convergence of an improper integral $\int_{t\geq a} f(t)dt$, where $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ is integrable on $[a, x]$ for all $x \geq a$. We shall now show that this treatment can be used to discuss the convergence of other types of "improper integrals".

Suppose $b \in \mathbb{R}$ and $f : (-\infty, b] \to \mathbb{R}$ is integrable on $[x, b]$ for every $x \leq b$. Define $\tilde{f} : [-b, \infty) \to \mathbb{R}$ by $\tilde{f}(u) := f(-u)$. Then for every $x \leq b$, that is, for every $-x \geq -b$, by considering $\phi : [-b, -a] \to \mathbb{R}$ given by $\phi(t) := -t$, we see from Proposition 6.29 that

$$\int_x^b f(t)dt = \int_{-b}^{-x} \tilde{f}(u)du.$$

We say that $\int_{t\leq b} f(t)dt$ is **convergent** if the improper integral $\int_{u\geq -b}^{\infty} \tilde{f}(u)du$ is convergent, that is, if the limit

$$\lim_{y\to\infty} \int_{-b}^{y} \tilde{f}(u)du = \lim_{x\to-\infty} \int_x^b f(t)dt$$

exists; in this case, the limit will be denoted by $\int_{-\infty}^b f(t)dt$, and called the **value** of $\int_{t\leq b} f(t)dt$. Otherwise, we say that $\int_{t\leq b} f(t)dt$ is **divergent**.

Next, let $f : \mathbb{R} \to \mathbb{R}$ be a function that is integrable on $[a, b]$ for all $a, b \in \mathbb{R}$ with $a \leq b$. We say that $\int_{t\in\mathbb{R}} f(t)dt$ is **convergent** if both $\int_{t\geq 0} f(t)dt$ and $\int_{t\leq 0} f(t)dt$ are convergent, that is, if the limits

$$\lim_{x\to\infty} \int_0^x f(t)dt \quad \text{and} \quad \lim_{x\to-\infty} \int_x^0 f(t)dt$$

both exist. In this case, the sum of these two limits is denoted by $\int_{-\infty}^{\infty} f(t)dt$, and called the **value** of $\int_{t\in\mathbb{R}} f(t)dt$. If any one of these limits does not exist, we say that $\int_{t\in\mathbb{R}} f(t)dt$ is **divergent**.

If the limit

$$\lim_{x\to\infty} \int_{-x}^{x} f(t)dt$$

exists, then this limit is called the **Cauchy principal value** of the integral of f on $\mathbb{R}$. If $\int_{t\in\mathbb{R}} f(t)dt$ is convergent, then since

$$\int_{-x}^{x} f(t)dt = \int_{-x}^{0} f(t)dt + \int_0^x f(t)dt \quad \text{for all } x \geq 0,$$

the Cauchy principal value of the integral of f on $\mathbb{R}$ exists and is equal to $\int_{-\infty}^{\infty} f(t)dt$. But the Cauchy principal value of the integral of f on $\mathbb{R}$ may exist even when $\int_{t\in\mathbb{R}} f(t)dt$ is divergent. For example, consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) := \sin t$. For every $x \geq 0$,

$$\int_0^x \sin t \, dt = 1 - \cos x = -\int_{-x}^0 \sin t \, dt \quad \text{and so} \quad \int_{-x}^x \sin t \, dt = 0.$$

Hence $\lim_{x\to\infty} \int_{-x}^x f(t)dt = 0$, but neither of the two limits $\lim_{x\to\infty} \int_0^x f(t)dt$ and $\lim_{x\to\infty} \int_{-x}^0 f(t)dt$ exists.

However, if $f : \mathbb{R} \to \mathbb{R}$ is a nonnegative function and the Cauchy principal value of the integral of f on $\mathbb{R}$ exists, then $\int_{t\in\mathbb{R}} f(t)dt$ is convergent. This can be seen as follows. For $x \geq 0$, let $F_1(x) := \int_0^x f(t)dt$, $F_2(x) := \int_{-x}^0 f(t)dt$, and $\ell := \lim_{x\to\infty} \int_{-x}^x f(t)dt$. Then F_1 and F_2 are monotonically increasing functions with $F_1(x) \leq \ell$ and $F_2(x) \leq \ell$ for all $x \geq 0$. Hence by part (i) of Proposition 3.44 with $b = \infty$, both $\lim_{x\to\infty} F_1(x)$ and $\lim_{x\to\infty} F_2(x)$ exist, that is, $\int_{t\in\mathbb{R}} f(t)dt$ is convergent.

Integrals of the type $\int_{t\geq a} f(t)dt$, $\int_{t\leq b} f(t)dt$, and $\int_{t\in\mathbb{R}} f(t)dt$ are sometimes known as **improper integrals of the first kind**, in contrast to those of the second kind, which we now describe.

Improper Integrals of Second Kind

Let $a, b \in \mathbb{R}$ with $a < b$. An **improper integral of the second kind** on $(a, b]$ is an ordered pair (f, F) of functions $f : (a, b] \to \mathbb{R}$ and $F : (a, b] \to \mathbb{R}$ such that f is unbounded on $(a, b]$ but integrable on $[x, b]$ for each $x \in (a, b]$ and

$$F(x) = \int_x^b f(t)dt \quad \text{for } x \in (a, b].$$

For simplicity and brevity, we use the notation $\int_{a<t\leq b} f(t)dt$ for the improper integral (f, F) on $(a, b]$. We say that $\int_{a<t\leq b} f(t)dt$ is **convergent** if f is integrable on $[x, b]$ for every $x \in (a, b]$ and the right limit

$$\lim_{x\to a^+} F(x) = \lim_{x\to a^+} \int_x^b f(t)dt$$

exists. In this case, the right limit is denoted by $\int_{a+}^b f(t)dt$, and called the **value** of the improper integral $\int_{a<t\leq b} f(t)dt$. If $\int_{a<t\leq b} f(t)dt$ is not convergent, then it is said to be **divergent**.

The study of improper integrals of the second kind can be reduced to the theory discussed in Sections 9.4 and 9.5 as follows. Given an improper integral of the second kind $\int_{a<t\leq b} f(t)dt$, define $c \in \mathbb{R}$ and $\tilde{f} : [c, \infty) \to \mathbb{R}$ by

$$c := \frac{1}{b-a} \quad \text{and} \quad \tilde{f}(u) := \frac{1}{u^2} f\left(a + \frac{1}{u}\right) \quad \text{for } u \in [c, \infty).$$

Then for every $x \in (a, b]$,

$$\int_x^b f(t)dt = \int_c^v \tilde{f}(u)du, \quad \text{where} \quad v := \frac{1}{x-a}.$$

Moreover, $x \to a^+$ if and only if $v \to \infty$. Consequently,

$$\int_{a<t\le b} f(t)dt \text{ is convergent} \iff \int_{u\ge c} \tilde{f}(u)du \text{ is convergent},$$

and in this case, $\int_{a^+}^{b} f(t)dt = \int_c^\infty \tilde{f}(u)du$.

A variant of the above is an improper integral $\int_{a\le t<b} f(t)dt$ of an unbounded function $f : [a,b) \to \mathbb{R}$ that is integrable on $[a,y]$ for every $y \in [a,b)$. The convergence of such an integral and the notion of its value are defined analogously. Moreover, if we define $c \in \mathbb{R}$ and $\tilde{f} : [c,\infty) \to \mathbb{R}$ by

$$c := \frac{1}{b-a} \quad \text{and} \quad \tilde{f}(u) := \frac{1}{u^2} f\left(b - \frac{1}{u}\right) \quad \text{for } u \in [c,\infty),$$

then for every $y \in [a,b)$,

$$\int_a^y f(t)dt = \int_c^v \tilde{f}(u)du, \quad \text{where} \quad v := \frac{1}{b-y}.$$

Moreover, $y \to b^-$ if and only if $v \to \infty$. Consequently,

$$\int_{a\le t<b} f(t)dt \text{ is convergent} \iff \int_{u\ge c} \tilde{f}(u)du \text{ is convergent},$$

and in this case $\int_a^{b^-} f(t)dt = \int_c^\infty \tilde{f}(u)du$.

Finally, consider an unbounded function $f : (a,b) \to \mathbb{R}$ that is integrable on $[x,y]$ for all $x,y \in (a,b)$ with $x \le y$. Let $c := (a+b)/2$. We say that $\int_{a<t<b} f(t)dt$ is **convergent** if both $\int_{a<t\le c} f(t)dt$ and $\int_{c\le t<b} f(t)dt$ are convergent, that is, if

$$\lim_{x\to a^+} \int_x^c f(t)dt \quad \text{and} \quad \lim_{x\to b^-} \int_c^x f(t)dt$$

both exist. In this case, the sum of these two limits is denoted by $\int_{a^+}^{b^-} f(t)dt$, and called the **value** of $\int_{a<t<b} f(t)dt$. We say that $\int_{a<t<b} f(t)dt$ is **divergent** if it is not convergent.

If the right limit

$$\lim_{\epsilon\to 0^+} \int_{a+\epsilon}^{b-\epsilon} f(t)dt$$

exists, then it is called the **Cauchy principal value** of the integral of f on (a,b). If $\int_{a<t<b} f(t)dt$ is convergent, then for every $\epsilon > 0$ with $a+\epsilon \le c \le b-\epsilon$,

$$\int_{a+\epsilon}^{c} f(t)dt + \int_c^{b-\epsilon} f(t)dt = \int_{a+\epsilon}^{b-\epsilon} f(t)dt.$$

Hence the Cauchy principal value of the integral of f on (a,b) is $\int_{a^+}^{b^-} f(t)dt$. However, the converse is not true. In other words, the Cauchy principal value of f on (a,b) may exist even when $\int_{a<t<b} f(t)dt$ is divergent. For example, let $f(t) := t/(t^2-1)$ for $t \in (-1,1)$. Then

$$f(t) = \frac{1}{2}\left(\frac{1}{1+t} - \frac{1}{1-t}\right) \quad \text{for } t \in (-1,1).$$

For each $\epsilon \in \mathbb{R}$ satisfying $0 < \epsilon < 1$,

$$\int_0^{1-\epsilon} f(t)dt = \frac{1}{2}\big(\ln(2-\epsilon)+\ln\epsilon\big) \quad \text{and} \quad \int_{-1+\epsilon}^{0} f(t)dt = -\frac{1}{2}\big(\ln\epsilon+\ln(2-\epsilon)\big).$$

Thus we see that $\lim_{\epsilon\to 0^+}\int_{-1+\epsilon}^{1-\epsilon} f(t)dt = 0$, but neither of the two limits $\lim_{\epsilon\to 0^+}\int_0^{1-\epsilon} f(t)dt$ and $\lim_{\epsilon\to 0^+}\int_{-1+\epsilon}^{0} f(t)dt$ exists. In general, it can be shown that if $f : (a,b) \to \mathbb{R}$ is a nonnegative function and the Cauchy principal value of the integral of f on (a,b) exists, then $\int_{a<t<b} f(t)dt$ is convergent. The proof is similar to the proof given earlier for the Cauchy principal value of the integral of a nonnegative function on $\mathbb{R}$.

Examples 9.56. (i) Let $f : (-\infty, 0] \to \mathbb{R}$ be given by $f(t) := e^t$. Since

$$\int_x^0 f(t)dt = 1 - e^x \to 1 \quad \text{as} \quad x \to -\infty,$$

we see that $\int_{-\infty}^0 f(t)dt$ is convergent.

(ii) Let $f : (-\infty,\infty) \to \mathbb{R}$ be given by $f(t) := e^{-t^2}$. For $x \geq 1$,

$$\begin{aligned} 0 \leq \int_0^x f(t)dt &= \int_0^1 e^{-t^2}dt + \int_1^x e^{-t^2}dt \leq \int_0^1 e^{-t^2}dt + \int_1^x e^{-t}dt \\ &= \int_0^1 e^{-t^2}dt + e^{-1} - e^{-x} \leq \int_0^1 e^{-t^2}dt + e^{-1}. \end{aligned}$$

Hence $\int_{t\geq 0} f(t)dt$ is convergent. Also, if $\tilde{f} : [0,\infty) \to \mathbb{R}$ is defined by $\tilde{f}(u) := f(-u)$, then the improper integral

$$\int_{u\geq 0} \tilde{f}(u)du = \int_{u\geq 0} e^{-u^2}du$$

is convergent, that is, $\int_{t\leq 0} f(t)dt$ is convergent. Thus we conclude that $\int_{t\in\mathbb{R}} e^{-t^2}dt$ is convergent.

(iii) Let $p \in \mathbb{R}$, $b \in (0,\infty)$, and $f(t) := 1/t^p$ for $t \in (0,b]$. If $c \in \mathbb{R}$ and $\tilde{f} : [c,\infty) \to \mathbb{R}$ is defined by

$$c := \frac{1}{b} \quad \text{and} \quad \tilde{f}(u) = \frac{1}{u^2} f\left(\frac{1}{u}\right),$$

then

$$\int_{u\geq c} \tilde{f}(u)du = \int_{u\geq c} \frac{1}{u^{2-p}}du.$$

As we have seen in Example 9.36 (iii), $\int_{u\geq c} u^{p-2}du$ is convergent if and only if $2-p>1$, that is, $p<1$. Consequently, the improper integral

$$\int_{0<t\leq b} \frac{1}{t^p}dt \text{ is convergent if and only if } p<1.$$

If $p<0$, then the function f is bounded on $(0,b]$, and if we define $f(0):=0$, then $f:[0,b]\to\mathbb{R}$ is in fact continuous, and therefore integrable, on $[0,b]$. Alternatively, for $x\in(0,b]$,

$$\int_x^1 \frac{1}{t^p}dt = \begin{cases} (1-x^{1-p})/(1-p) & \text{if } p\neq 1, \\ -\ln x & \text{if } p=1. \end{cases}$$

This shows that $\int_{0<t\leq 1}(1/t^p)dt$ is convergent if and only if $p<1$, and when $p<1$, it converges to $1/(1-p)$.

(iv) Let $f(t):=\ln t$ for $t\in(0,1]$. For $x\in(0,1]$,

$$\int_x^1 f(t)dt = (t\ln t - t))\Big|_x^1 = x-1-x\ln x.$$

Since $x\ln x\to 0$ as $x\to 0^+$, we see that $\int_{0<t\leq 1}\ln t\, dt$ is convergent and its value is -1. ◇

Definitions of absolute and conditional convergence as well as tests for the convergence of $\int_{t\leq b} f(t)dt$, $\int_{a<t\leq b} f(t)dt$, and $\int_{a\leq t<b} f(t)dt$ can be obtained by reducing these to improper integrals of the type $\int_{t\geq a} f(t)dt$, which were studied in Sections 9.4 and 9.5. Alternatively, such tests can be developed independently along similar lines. To illustrate these two procedures, let us consider the comparison test for the improper integral $\int_{a<t\leq b} f(t)dt$.

Proposition 9.57 (Comparison Test for Improper Integrals of the Second Kind). *Let $a,b\in\mathbb{R}$ with $a<b$ and let $f,g:(a,b]\to\mathbb{R}$ be such that both f and g are integrable on $[x,b]$ for every $x\geq a$ and there exists $t_0\in(a,b]$ such that $|f(t)|\leq g(t)$ for all $t\in(a,t_0]$. If $\int_{a<t\leq b} g(t)dt$ is convergent, then $\int_{a<t\leq b} f(t)dt$ is absolutely convergent.*

Proof. Let $c:=1/(b-a)$. Define $\tilde{f},\tilde{g}:[c,\infty)\to\mathbb{R}$ and $u_0\in[c,\infty)$ by

$$\tilde{f}(u):=\frac{1}{u^2}f\left(a+\frac{1}{u}\right), \quad \tilde{g}(u):=\frac{1}{u^2}g\left(a+\frac{1}{u}\right), \quad \text{and} \quad u_0:=\frac{1}{t_0-a}.$$

Then $|\tilde{f}(u)|\leq\tilde{g}(u)$ for all $u\in[u_0,\infty)$. Also, $\tilde{f}$ is integrable on $[c,y]$ for each $y\in[c,\infty)$. Suppose $\int_{a<t\leq b} g(t)dt$ is convergent, that is, the improper

integral $\int_{u\geq c} \tilde{g}(u)du$ is convergent. Then by Proposition 9.46, it follows that $\int_{u\geq c} |\tilde{f}(u)|du$ is convergent, that is, the improper integral $\int_{a<t\leq b} |f(t)|dt$ is convergent. This proves the desired result.

Alternatively, we can give a proof from first principles as follows. Consider $G : (a, b] \to \mathbb{R}$ defined by $G(x) := \int_x^b g(t)dt$. Let $\epsilon > 0$ be given. By the Cauchy Criterion (Proposition 3.35 and Remark 3.36), there is $x_0 \in (a, b]$ such that

$$G(y) - G(x) = \int_x^y g(t)dt < \epsilon \quad \text{whenever } a < x \leq y \leq x_0.$$

Let $x_1 := \min\{t_0, x_0\}$. Then $x_1 \in (a, b]$ and

$$\int_x^y |f(t)|dt \leq \int_x^y g(t)dt < \epsilon \quad \text{whenever } a < x \leq y \leq x_1.$$

Hence using once again the Cauchy Criterion, we see that the improper integral $\int_{a<t\leq b} f(t)dt$ is absolutely convergent. □

In turn, Proposition 9.57 can be used to deduce a Limit Comparison Test for improper integrals of the second kind, analogous to Corollary 9.48. This time, we leave the formulation of the statement and a proof to the reader. For treating conditional convergence of improper integrals of the second kind, an analogue of the Dirichlet Test (Proposition 9.53) is given in Exercise 9.42.

We can also consider a combination of improper integrals of the first kind and the second kind, that is, integrals of unbounded functions on unbounded intervals. The notion of convergence can be readily defined as follows using the theory we developed earlier. Let $a \in \mathbb{R}$ and let $f : (a, \infty) \to \mathbb{R}$ be an unbounded function that is integrable (and in particular bounded) on $[x, y]$ for all $x, y \in \mathbb{R}$ such that $a < x < y$. We say that $\int_{t>a} f(t)dt$ is **convergent** if both $\int_{a<t\leq a+1} f(t)dt$ and $\int_{t\geq a+1} f(t)dt$ are convergent. In this case, we define

$$\int_{a^+}^{\infty} f(t)dt := \int_{a^+}^{a+1} f(t)dt + \int_{a+1}^{\infty} f(t)dt.$$

Similarly, if $b \in \mathbb{R}$ and $f : (-\infty, b) \to \mathbb{R}$ is an unbounded function such that f is integrable on $[x, y]$ for all $x, y \in \mathbb{R}$ with $x < y < b$, then we say that $\int_{t<b} f(t)dt$ is **convergent** if both $\int_{t\leq b-1} f(t)dt$ and $\int_{b-1\leq t<b} f(t)dt$ are convergent. In this case, we define

$$\int_{-\infty}^{b^-} f(t)dt := \int_{-\infty}^{b-1} f(t)dt + \int_{b-1}^{b^-} f(t)dt.$$

It is clear that the study of such integrals easily reduces to that of improper integrals of the first and second kinds.

Examples 9.58. (i) Let $f(t) := 1/\sqrt{t}(t+1)$ for $t \in (0, \infty)$. Define $g(t) := 1/\sqrt{t}$ for $t \in (0, 1]$ and $h(t) := 1/t\sqrt{t}$ for $t \in [1, \infty)$. Observe that

$$|f(t)| \le g(t) \text{ for all } t \in (0,1] \quad \text{and} \quad |f(t)| \le h(t) \text{ for all } t \in [1,\infty).$$

We have seen in Example 9.56 (iii) that $\int_{0<t\le 1} g(t)dt$ is convergent, and so Proposition 9.57 shows that $\int_{0<t\le 1} f(t)dt$ is convergent. Also, from Example 9.43 (i), $\int_{t\ge 1} h(t)dt$ is convergent, and so Proposition 9.57 shows that $\int_{t\ge 1} f(t)dt$ is convergent. It follows that $\int_{t>0} f(t)dt$ is convergent.

(ii) Let $f_1(t) := 1/t^2$ and $f_2(t) = 1/\sqrt{t}$ for $t \in (0,\infty)$. Since $\int_{0<t\le 1} f_1(t)dt$ and $\int_{t\ge 1} f_2(t)dt$ are divergent, it follows that both $\int_{t>0} f_1(t)dt$ and $\int_{t>0} f_2(t)dt$ are divergent. $\diamond$

Remark 9.59. It may be noted that an improper integral of the second kind such as $\int_{a<t\le b} f(t)dt$ or $\int_{a\le t<b} f(t)dt$ or $\int_{a<t<b} f(t)dt$ converges to $\int_a^b \tilde{f}(t)dt$ in case f extends to an integrable function $\tilde{f} : [a,b] \to \mathbb{R}$. In this case, f is necessarily a bounded function on (a,b). $\diamond$

The Beta and Gamma Functions

We shall now consider two general examples of improper integrals, which lead to certain important functions in analysis.

To begin with, let p, q be any real numbers and let $f : (0,1) \to \mathbb{R}$ be the function defined by $f(t) := t^{p-1}(1-t)^{q-1}$. Consider the improper integrals

$$\int_{0<t\le 1/2} f(t)dt \quad \text{and} \quad \int_{1/2\le t<1} f(t)dt.$$

First, note that for all $x \in (0,1/2]$, the function f is continuous and hence integrable on $[x,1/2]$. Also, it is clear that

$$\frac{1}{2^{q-1}} t^{p-1} \le \left| t^{p-1}(1-t)^{q-1} \right| < t^{p-1} \quad \text{for all } t \in (0,1/2].$$

Further, since Example 9.56 (iii) shows that $\int_{0<t\le 1/2} t^{p-1}dt$ is convergent if and only if $1-p<1$, that is, $p>0$, it follows from the Comparison Test (Proposition 9.57) that $\int_{0<t\le 1/2} f(t)dt$ is convergent if and only if $p>0$.

Next, for all $x \in [1/2,1)$, the function f is continuous and hence integrable on $[1/2,x]$. Moreover, if we let $y := 1-x$, then $y \in (0,1/2]$, and the substitution $u = 1-t$ shows that

$$\int_{1/2}^{x} f(t)dt = \int_{y}^{1/2} u^{q-1}(1-u)^{p-1}du.$$

Hence using the result in the previous paragraph, we see that $\int_{1/2\le t<1} f(t)dt$ is convergent if and only if $q>0$. Consequently,

$$\int_{0<t<1} f(t)dt \text{ is convergent} \iff p>0 \text{ and } q>0.$$

In case $p \geq 1$ and $q \geq 1$, then Remark 9.59 applies because f extends to an integrable (and in fact, continuous) function $\tilde{f} : [0,1] \to \mathbb{R}$ by setting $\tilde{f}(0) := 0$ and $\tilde{f}(1) := 0$. But in general, $\int_{0<t<1} f(t)dt$ is an improper integral. In any case, we obtain a well-defined function $\beta : (0,\infty) \times (0,\infty) \to \mathbb{R}$ given by

$$\beta(p,q) := \int_{0^+}^{1^-} t^{p-1}(1-t)^{q-1}dt \quad \text{ for } (p,q) \in (0,\infty) \times (0,\infty).$$

This is known as the **beta function**. The substitution $t = 1 - s$ shows that $\beta(p,q) = \beta(q,p)$ for all $p > 0$ and $q > 0$. Also, the substitution $t = \sin^2\theta$ shows that

$$\beta(1/2,\, 1/2) = \int_{0^+}^{1^-} \frac{1}{\sqrt{t}}\frac{1}{\sqrt{1-t}}dt = \int_0^{\pi/2} 2\, d\theta = \pi,$$

which is a transcendental number, as mentioned in Section 7.2. In general, if p, q are positive nonintegral rational numbers, then by a theorem of Schneider, $\beta(p,q)$ is a transcendental number. For a proof, we refer to Section 6.2 of [8].

We shall now proceed to define another important function, which is a neat generalization of the factorial function in the sense that it extends the real-valued function on $\mathbb{N}$ given by $n \longmapsto (n-1)!$ to the set $(0,\infty)$ of all positive real numbers. To begin with, let us fix $u \in \mathbb{R}$ and consider $g : (0,\infty) \to \mathbb{R}$ given by $g(t) := e^{-t}t^{u-1}$, and the two improper integrals

$$\int_{0<t\leq 1} g(t)dt \quad \text{and} \quad \int_{t\geq 1} g(t)dt.$$

First, note that for all $x \in (0,1]$, the function g is continuous and hence integrable on $[x,1]$. Also, since $(1/e) \leq e^{-t} < 1$ whenever $t \in (0,1]$, we obtain

$$\frac{1}{e}t^{u-1} \leq \left|e^{-t}\, t^{u-1}\right| < t^{u-1} \quad \text{for all } t \in (0,1].$$

Further, since Example 9.56 (iii) shows that $\int_{0<t\leq 1} t^{u-1}dt$ is convergent if and only if $1 - u < 1$, that is, $u > 0$, it follows from the Comparison Test (Proposition 9.57) that $\int_{0<t\leq 1} g(t)dt$ is convergent if and only if $u > 0$.

Next, for all $x \in [1,\infty)$, the function g is continuous and hence integrable on $[1,x]$. Choose $n \in \mathbb{N}$ such that $n > u$. Then $n - u + 1 > 1$, and so in view of Example 9.36 (iii) and Example 7.5 (ii), we see that

$$\int_{t\geq 1} t^{u-n-1}dt \text{ is convergent} \quad \text{and} \quad \lim_{t\to\infty} \frac{e^{-t}t^{u-1}}{t^{u-n-1}} = \lim_{t\to\infty} \frac{t^n}{e^t} = 0.$$

Hence by the Limit Comparison Test (part (ii) of Proposition 9.48), it follows that $\int_{t\geq 1} g(t)dt$ is convergent for every $u \in \mathbb{R}$. Consequently, the improper integral $\int_{t>0} g(t)dt$ is convergent if and only if $u > 0$. Thus we obtain a well-defined function $\Gamma : (0,\infty) \to \mathbb{R}$ given by

$$\Gamma(u) := \int_{0^+}^{\infty} e^{-t}t^{u-1}dt \quad \text{ for } u \in (0,\infty).$$

This is known as the **gamma function**.

Proposition 9.60 (Properties of the Gamma Function).

(i) $\Gamma(u) > 0$ *for all* $u \in (0,\infty)$ *and* $\Gamma(u) \to \infty$ *as* $u \to 0^+$.
(ii) $\Gamma(u+1) = u\,\Gamma(u)$ *for all* $u \in (0,\infty)$.
(iii) $\Gamma(n) = (n-1)!$ *for all* $n \in \mathbb{N}$.

Proof. (i) For $u \in (0,\infty)$,

$$\Gamma(u) \geq \int_{0^+}^{1} e^{-t}t^{u-1}dt \geq e^{-1}\int_{0^+}^{1} t^{u-1}dt = \frac{e^{-1}}{u}.$$

This shows that $\Gamma(u) > 0$ for all $u > 0$ and $\Gamma(u) \to \infty$ as $u \to 0^+$.

(ii) Fix $u \in (0,\infty)$. For $\delta > 0$ and $x \geq \delta$, integration by parts yields

$$\begin{aligned}\int_{\delta}^{x} e^{-t}t^{u}dt &= -e^{-t}t^{u}\Big|_{\delta}^{x} + u\int_{\delta}^{x} e^{-t}t^{u-1}dt \\ &= \left(e^{-\delta}\delta^{u} - e^{-x}x^{u}\right) + u\int_{\delta}^{x} e^{-t}t^{u-1}dt.\end{aligned}$$

Now $e^{-\delta} \to 1$ as $\delta \to 0^+$, and by part (iii) of Proposition 7.10, $\delta^u \to 0$ as $\delta \to 0^+$. Thus $e^{-\delta}\delta^u \to 0$ as $\delta \to 0^+$. Consequently,

$$\int_{0^+}^{1} e^{-t}t^{u}dt = \lim_{\delta\to 0^+}\int_{\delta}^{1} e^{-t}t^{u}dt = \left(0 - e^{-1}\right) + u\left(\lim_{\delta\to 0^+}\int_{\delta}^{1} e^{-t}t^{u-1}dt\right).$$

Also, if we choose $n \in \mathbb{N}$ such that $n > u$, then $0 \leq e^{-x}x^u < e^{-x}x^n$ for all $x \geq 1$. From Example 7.5 (ii) we know that $e^{-x}x^n \to 0$ as $x \to \infty$. Thus $e^{-x}x^u \to 0$ as $x \to \infty$. Consequently,

$$\int_{1}^{\infty} e^{-t}t^{u}dt = \lim_{x\to\infty}\int_{1}^{x} e^{-t}t^{u}dt = \left(e^{-1} - 0\right) + u\left(\lim_{x\to\infty}\int_{1}^{x} e^{-t}t^{u-1}dt\right).$$

Combining these, we obtain $\Gamma(u+1) = u\,\Gamma(u)$, as desired.

(iii) From (ii) above, $\Gamma(n+1) = n\,\Gamma(n)$ for all $n \in \mathbb{N}$. Also, it is easy to see that $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$. Thus using induction on n, we obtain $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$. □

Remark 9.61. While $\Gamma(u)$ is easy to calculate when $u \in \mathbb{N}$, it is often difficult to determine the value of $\Gamma(u)$ at other positive real numbers u. In part (i) of Proposition 10.74, we prove $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and then calculate $\Gamma(n+\frac{1}{2})$ for all $n \in \mathbb{N}$. In fact, in Section 10.7, we shall obtain several additional properties of the gamma function, including the fact that $\Gamma(u) \to \infty$ as $u \to \infty$.

A crucial property of the gamma function is that it is a **log-convex** function, that is, the function $\Gamma_\ell : (0, \infty) \to \mathbb{R}$ defined by $\Gamma_\ell(s) := \ln \Gamma(s)$ is convex on $(0, \infty)$. A proof of this result is outlined in Exercise 9.56.

Exercise 10.66 gives an interesting relationship between the beta and gamma functions, namely,

$$\beta(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad \text{for all } p > 0 \text{ and } q > 0.$$

We refer the reader to [6] for more on the gamma function. ◇

Notes and Comments

Logically, the theory of infinite series is a particular case of the theory of sequences. In fact, the two are equivalent. However, from a pedagogical and historical point of view, it seems preferable to treat infinite series separately and at a stage when tools from the theory of integration are at our disposal. Also, it appears natural to treat improper integrals alongside infinite series.

We have followed Apostol [2] to define an infinite series as a pair of sequences, the first comprising the terms of the series and the second formed by the partial sums of the series. This might seem pedantic, but it avoids "defining" an infinite series as an expression or a symbol. Similar considerations apply to improper integrals. We have also made a slight notational distinction between the infinite series and its sum. Typically, the former is denoted by $\sum_{k\geq 1} a_k$ *or simply by* $\sum_k a_k$, *and the latter by* $\sum_{k=1}^{\infty} a_k$. *Likewise, the improper integral of* $f : [a, \infty) \to \mathbb{R}$ *is denoted by* $\int_{t\geq a} f(t)dt$, *and its value is denoted by* $\int_a^\infty f(t)dt$ *when the improper integral is convergent. .*

The treatments of the infinite series in Sections 9.1 and 9.2, and of the improper integrals in Sections 9.3 and 9.5, run parallel. Our development brings home the fact that they are the discrete and the continuous representations of the same theory. For example, the partial sum $A_n := \sum_{k=1}^{n} a_k$ *of an infinite series* $\sum_{k\geq 1} a_k$ *is analogous to the partial integral* $F(x) := \int_a^x f(t)dt$ *of the improper integral* $\int_{t\geq a} f(t)dt$. *Further, the counterparts of* $A_0 = 0$ *and* $a_n = A_n - A_{n-1}$ *for all* $n \geq 1$ *are* $F(a) = 0$ *and* $f(x) = F'(x)$ *for all* $x \geq a$, *provided* f *is continuous at* x. *Tests of convergence for the two are based on the same principles. However, there are a few exceptions. The* k*th Term Test and the Ratio Test for infinite series fail to have analogues for improper integrals.*

While the convergence of an infinite series can usually be determined using one of the several tests, finding the sum is often far more difficult. The only cases in which we have actually found the sum of an infinite series are those involving a geometric series, or in which a series can be written as a genuine telescoping series or the tail of a Taylor series can be shown to tend to zero. In fact, essentially the only series whose partial sums have a "closed form expression" is the geometric series. The situation for actually evaluating *improper integrals is similar. But if a function $f : [a, \infty) \to \mathbb{R}$ is integrable on $[a, x]$ for all $x \geq a$ and equals the derivative of a known function g, then the partial integral function F of the improper integral $\int_{t \geq a} f(t)dt$ is given by $F(x) = g(x) - g(a)$ for $x \geq a$. As a result, to evaluate the improper integral $\int_{t \geq a} f(t)dt$, one only needs to find $\lim_{x \to \infty} g(x)$ (Proposition 9.37). Needless to say, this procedure can be carried through in only a limited number of cases.*

Besides the usual notions of convergence and absolute convergence of infinite series, we have also discussed the notion of Cesàro convergence. This will be useful in the next chapter when we discuss Fourier series of functions. The sufficient condition given in Corollary 9.7 is due to Alfred Tauber (1897), and this result can be viewed as the simplest instance of what are now called Tauberian theorems. A powerful generalization of Corollary 9.7 wherein the condition $a_k = o(1/k)$ is replaced by the weaker condition $a_k = O(1/k)$ was proved by Hardy in 1910. A proof can be found in Section 6 of the first chapter of the treatise on Tauberian theory by Korevaar [51].

While we have given a number of tests for convergence of infinite series in the text and also in the exercises, the list is not meant to be comprehensive. A wealth of material, including a large number of convergence tests, can be found in old classics on infinite series such as the books of Bromwich [16] and Knopp [49]. See also the more specialized books of Dienes [25] and Hardy [40].

Power series are an important class of infinite series whose terms depend on a parameter. Their peculiar convergence behavior is brought out in Lemma 9.26. This result allows us to introduce the concept of the radius of convergence of a power series without any reference to the terms of the series. Of course, the calculation of the radius of convergence of a given power series will be based on the Root Test or the Ratio Test, for which either the roots of the absolute values of the terms of the series or the ratios of the successive terms of the series are needed. Taylor series form a special class of power series. Their convergence can be proved by showing that the remainder after the nth term tends to zero. This process does not use the Root Test or the Ratio Test and, when successful, yields also the sum of the series. Many classical functions admit a Taylor series, which can be effectively used to understand and study these functions. Conversely, new functions can be introduced by means of convergent power series, just as we introduced the logarithmic and arctangent functions by means of integrals of rational functions. It may be interesting to note that every power series is the Taylor series of some function. See the article of Meyerson [66] for a proof. Apart from power series and Taylor series, an important class of infinite series whose terms depend on a parameter is that of Fourier series. These are series of the form $a_0 + \sum_{k \geq 1} (a_k \cos kx + b_k \sin kx)$. The study of Fourier series is a rich and fascinating topic, and for more on this, we suggest the book of Stein and Shakarchi [79].

In order to retain the parallelism between infinite series and improper integrals, we have restricted the definition of an improper integral to cover only the "integrals" of the type $\int_{t \geq a} f(t)dt$, where $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ is a *function that is integrable on $[a, x]$ for each $x \geq a$. In doing so, there is no real loss of generality, since the treatment of improper integrals of other kinds can be reduced to that of improper integrals of the above type. As an application of improper integrals of the second kind, we have defined the beta function and the gamma function. We have restricted ourselves to discussing only the most rudimentary properties of these functions. The article of Davis [23] gives a very readable introduction to the gamma function. A lucid development of the gamma function and the beta function can be found in the book of Artin [6].*

Exercises

Part A

9.1. Give examples to show that if $\sum_k a_k$ and $\sum_k b_k$ are convergent series of real numbers, then the series $\sum_k a_k b_k$ may not be convergent. Also show that if $\sum_k a_k = A$ and $\sum_k b_k = B$, then $\sum_k a_k b_k$ may be convergent, but its sum may not be equal to AB.

9.2. Consider a series $\sum_k a_k$ and for each $n = 0, 1, 2, \ldots$, let $b_{n,k} := a_{n+k}$. Show that the series $\sum_k a_k$ is convergent if and only if for some $n = 0, 2, \ldots$, the series $\sum_k b_{n,k}$ is convergent. In this case, prove that the series $\sum_k b_{n,k}$ is convergent for every $n = 0, 1, 2, \ldots$, and $\sum_k b_{n,k} = \sum_k a_k - A_n$, where A_n is the nth partial sum of the series $\sum_k a_k$.

9.3. Show that the series $\sum_{k \geq 1} 2/(k + 1)(2k + 1)$ is convergent and its sum is less than or equal to 1. (Hint: Compare the given series with the series $\sum_{k \geq 1} 1/k(k + 1)$.)

9.4. Let $a \in \mathbb{R}$ with $a > 1$. Show that the series $\sum_{k \geq 1} \left(1/a^{k!}\right)$ is convergent.

9.5. Let $a_k \in \mathbb{R}$ with $a_k \leq 0$ for all $k \in \mathbb{N}$. Show that the series $\sum_{k \geq 1} a_k$ is convergent if and only if the sequence (A_n) of its partial sums is bounded below, and in this case, $\sum_{k=1}^{\infty} a_k = \inf\{A_n : n \in \mathbb{N}\}$. If (A_n) is not bounded below, then show that $\sum_{k \geq 1} a_k$ diverges to $-\infty$.

9.6. **(Cauchy Condensation Test)** Let (a_k) be a monotonically decreasing sequence of nonnegative real numbers. Show that the series $\sum_{k \geq 1} a_k$ is convergent if and only if the series $\sum_{k \geq 0} 2^k a_{2^k}$ is convergent. (Hint: Proposition 9.4.) Deduce the convergence and divergence of the series $\sum_{k \geq 1} 1/k^p$ and $\sum_{k \geq 2} 1/k(\ln k)^p$, where $p \in \mathbb{R}$. (Compare Example 9.43.)

9.7. **(Abel kth Term Test)** Suppose (a_k) is a monotonically decreasing sequence of nonnegative real numbers. If the series $\sum_k a_k$ is convergent, then show that $ka_k \to 0$ as $k \to \infty$. (Hint: Exercise 9.6.) Also, show that the converse of this result does not hold.

9.8. A sequence (a_k) is said to be of **bounded variation** if $\sum_{k \geq 1} |a_k - a_{k+1}|$ is convergent. Prove the following:

(i) A sequence of bounded variation is convergent.

(ii) Let (a_k) and (b_k) be of bounded variation and let $r \in \mathbb{R}$. Then (a_k+b_k), (ra_k), and $(a_k b_k)$ are of bounded variation. If $a_k \neq 0$ for all $k \in \mathbb{N}$, is $(1/a_k)$ of bounded variation?

(iii) Every bounded monotonically increasing sequence is of bounded variation. Further, if (b_k) and (c_k) are bounded monotonically increasing sequences and we define $a_k := b_k - c_k$ for $k \in \mathbb{N}$, then the sequence (a_k) is of bounded variation.

(iv) If (a_k) is of bounded variation, then there are bounded monotonically increasing sequences (b_k) and (c_k) such that $a_k = b_k - c_k$ for $k \in \mathbb{N}$. (Hint: Let $a_0 := 0$ and $v_k := |a_1| + |a_1 - a_2| + \cdots + |a_{k-1} - a_k|$ for $k \in \mathbb{N}$. Define $b_k := (v_k + a_k)/2$ and $c_k := (v_k - a_k)/2$ for $k \in \mathbb{N}$.)

9.9. Let $a, b \in \mathbb{R}$ be such that $0 < a < b$. For $k \in \mathbb{N}$, define

$$a_{2k-1} := a^{k-1}b^{k-1} \quad \text{and} \quad a_{2k} := a^k b^{k-1}.$$

Consider the series $\sum_{k\geq 1} a_k = 1 + a + ab + a^2b + a^2b^2 + a^3b^2 + \cdots$.

(i) Use the Ratio Test to show that $\sum_{k\geq 1} a_k$ is convergent if $b < 1$, and it is divergent if $a \geq 1$.

(ii) Use the Root Test to show that $\sum_{k\geq 1} a_k$ is convergent if $ab < 1$, and it is divergent if $ab > 1$.

9.10. For $k \in \mathbb{N}$, let $a_{2k-1} := 4^{k-1}/9^{k-1}$ and $a_{2k} := 4^{k-1}/9^k$. Show that $|a_{2k}/a_{2k-1}| = \frac{1}{9}$ and $|a_{2k+1}/a_{2k}| = 4$ for all $k \in \mathbb{N}$, and so the Ratio Test for the convergence of $\sum_{k\geq 1} a_k$ is inconclusive. Prove that $|a_k|^{1/k} \to \frac{2}{3}$ as $k \to \infty$ and use the Root Test to conclude that $\sum_{k\geq 1} a_k$ is convergent.

9.11. **(Raabe Test)** Let (a_k) be a sequence of real numbers. If there is $p > 1$ such that

$$|a_{k+1}| \leq \left(1 - \frac{p}{k}\right)|a_k| \quad \text{for all large } k \in \mathbb{N},$$

then show that $\sum_{k\geq 1} a_k$ is absolutely convergent. On the other hand, if

$$|a_{k+1}| \geq \left(1 - \frac{1}{k}\right)|a_k| > 0 \quad \text{for all large } k \in \mathbb{N},$$

then show that $\sum_{k\geq 1} |a_k|$ is divergent. (Hint: If $p > 1$ and $x \in [0, 1]$, then $1 - px \leq (1 - x)^p$. Use Exercise 9.15.)

9.12. (i) If $a_1 = a_2 = 1$ and $a_{k+1} := (k-1)a_k/(k+1)$ for $k \geq 2$, then show that $\sum_{k\geq 1} a_k$ is convergent.

(ii) If $a_1 = 1$ and $a_{k+1} := ka_k/(k+1)$ for $k \in \mathbb{N}$, then show that $\sum_{k\geq 1} a_k$ diverges to ∞.

(Hint: Exercise 9.11.)

9.13. **(Hypergeometric Series)** Let $\alpha, \beta, \gamma \in \mathbb{R}$ be positive. If $a_0 := 1$ and

$$a_k := \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)\beta(\beta+1)\cdots(\beta+k-1)}{\gamma(\gamma+1)\cdots(\gamma+k-1)k!} \quad \text{for } k \in \mathbb{N},$$

then show that the series $\sum_{k\geq 0} a_k$ is convergent if and only if $\gamma > \alpha + \beta$. (Hint: Exercise 9.11.)

9.14. Suppose the partial sums of a series $\sum_{k\geq 1} b_k$ are bounded. If $p > 0$ and $x \in (0,1)$, then show that the series $\sum_{k\geq 1} b_k/k^p$, $\sum_{k\geq 1} b_k/(\ln k)^p$, and $\sum_{k\geq 1} b_k x^k$ are convergent. (Hint: Proposition 9.21.)

9.15. **(Abel Test for Series)** If (a_k) is a bounded monotonic sequence and $\sum_{k\geq 1} b_k$ is a convergent series, then show that the series $\sum_{k\geq 1} a_k b_k$ is convergent. Further, show that for $n \in \mathbb{N}$,

$$\left|\sum_{k=n}^{\infty} a_k b_k\right| \leq (|a| + |a_n - a|)\, \beta_n,$$

where $a := \lim_{k\to\infty} a_k$ and $\beta_n := \sup\left\{\left|\sum_{k=n}^{m} b_k\right| : m \in \mathbb{N} \text{ and } m \geq n\right\}$. (Compare Proposition 9.21.)

9.16. Let $\sum_{k\geq 1} b_k$ be a convergent series. Show that the series $\sum_{k\geq 1} k^{1/k} b_k$ and $\sum_{k\geq 1} \big(1 + (1/k)\big)^k b_k$ are convergent. (Hint: Exercises 9.15, 2.7, and 2.8.)

9.17. Let $p \in \mathbb{R}$ with $p > 1$. Show that

$$\frac{1}{(p-1)(\ln 2)^{p-1}} \leq \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p} \leq \frac{p-1+2\ln 2}{2(p-1)(\ln 2)^p}.$$

(Hint: Proposition 9.42 and Example 9.43 (ii).)

9.18. Test the series $\sum_{k\geq 1} a_k$ for absolute/conditional convergence if for $k \in \mathbb{N}$, a_k is given as follows. In (vii)–(x) below, q, r, p, and θ are real numbers.

(i) $(-1)^k \dfrac{k}{3k-2}$, (ii) $\dfrac{k!}{2^k}$, (iii) ke^{-k}, (iv) $\dfrac{1}{\sqrt{1+k^3}}$,

(v) $(-1)^{k-1} \dfrac{k}{k^2+1}$, (vi) $(-1)^{k-1} \dfrac{1}{\ln(\ln k)}$, (vii) $\dfrac{k^q}{1+k^q}$,

(viii) $\dfrac{r^k}{1+r^{2k}}$, (ix) $(-1)^{k-1} \sin\left(\dfrac{1}{k^p}\right)$, (x) $\dfrac{\cos k\theta}{\sqrt{k}}$.

9.19. Find the radius of convergence of the power series $\sum_{k\geq 0} c_k x^k$ whose coefficients are defined by $c_{2k-1} := 3^{-k}$ and $c_{2k} := 2^k 5^{-k}$ for $k \in \mathbb{N}$.

9.20. Find the radius of convergence of the power series $\sum_{k\geq 0} c_k x^k$ if for $k \in \mathbb{N}$, the coefficient c_k is given as follows:

(i) $k!$, (ii) k^2, (iii) $\dfrac{k}{k^2+1}$, (iv) ke^{-k}, (v) c^{k^2}, where $c \in \mathbb{R}$,

(vi) $\dfrac{k^k}{k!}$, (vii) $\dfrac{2^k}{k^2}$, (viii) $\dbinom{k+m}{k}$, where $m \in \mathbb{N}$.

9.21. Let $f(x) := \cos x$ for $x \in \mathbb{R}$. Show that the Taylor series of f is convergent for $x \in \mathbb{R}$. Deduce that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad \text{for } x \in \mathbb{R}.$$

9.22. Let $a \in \mathbb{R}$, let I be an open interval containing a, and let the Taylor series of $f : I \to \mathbb{R}$ around a be $\sum_{k\geq 0} c_k (x-a)^k$. Show that there are infinitely many functions $g : I \to \mathbb{R}$ that have the same Taylor series around a.

9.23. Let (a_k) be a monotonic sequence of real numbers such that $a_k \to 0$. Also, let $x \in \mathbb{R}$ be such that $x \neq 2m\pi$ for every $m \in \mathbb{Z}$. Show that for each $n \in \mathbb{N}$,

$$\left|\sum_{k=n}^{\infty} a_k \sin kx\right| \leq \frac{2|a_n|}{|\sin(x/2)|} \quad \text{and} \quad \left|\sum_{k=n}^{\infty} a_k \cos kx\right| \leq \frac{2|a_n|}{|\sin(x/2)|}.$$

(Hint: Corollary 9.23.)

9.24. Modify the function given in Example 9.44 to obtain a piecewise linear nonnegative continuous function $g : [1, \infty) \to \mathbb{R}$ such that $g(1) = 0$, while

$$g(k) = \sqrt{k} \quad \text{and} \quad g\left(k - \frac{1}{k^2\sqrt{k}}\right) = 0 = g\left(k + \frac{1}{k^2\sqrt{k}}\right) \quad \text{for } k \geq 2.$$

Show that $\int_{t\geq 1} g(t)dt$ is convergent, but g is not bounded.

9.25. Let $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ be such that $f(t) \leq 0$ for all $t \geq a$ and f is integrable on $[a, x]$ for each $x \geq a$. Show that $\int_{t\geq a} f(t)dt$ is convergent if and only if its partial integral $F : [a, \infty) \to \mathbb{R}$ defined by $F(x) := \int_a^x f(t)dt$ is bounded below, and in this case, $\int_a^\infty f(t)dt = \inf\{F(t) : t \in [a, \infty)\}$. If F is not bounded below, then show that $\int_{t\geq a} f(t)dt$ diverges to $-\infty$.

9.26. Show that if $\sum_{k\geq 1} a_k$ is a convergent series with nonnegative terms, then the sequence (a_n) of its terms is bounded. Give an example to show that if $f : [1, \infty) \to \mathbb{R}$ is a nonnegative function such that $\int_{t\geq 1} f(t)dt$ is convergent, then f need not be bounded.

9.27. Let $f, g : [2, \infty) \to \mathbb{R}$ be defined by

$$f(t) := \begin{cases} 1 & \text{if } k \leq t < k + (1/k^2) \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(t) := \begin{cases} k & \text{if } k \leq t < k + (1/k^3) \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\int_{t\geq 2} f(t)dt$ and $\int_{t\geq 2} g(t)dt$ are convergent, $f(k) = 1$ for each $k \in \mathbb{N}$ with $k \geq 2$, and $g(k) \to \infty$ as $k \to \infty$.

9.28. Let $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ be such that f is integrable on $[a, x]$ for all $x \geq a$. Prove the following:

(i) If $\int_{t\geq a} f(t)dt$ is convergent and $f(x) \to \ell$ as $x \to \infty$, then $\ell = 0$.

(ii) If f is differentiable and $\int_{t\geq a} f'(t)dt$ is convergent, then there is $\ell \in \mathbb{R}$ such that $f(x) \to \ell$ as $x \to \infty$. (Hint: Use part (ii) of Proposition 6.24.)

(iii) If f is differentiable and both $\int_{t\geq a} f(t)dt$ and $\int_{t\geq a} f'(t)dt$ are convergent, then $f(x) \to 0$ as $x \to \infty$.

(iv) Deduce that the improper integral $\int_{t\geq 0} t \sin t^2 \, dt$ is divergent.

9.29. Show that $\int_{t\geq 1}(\cos t/t^p)dt$ and $\int_{t\geq 1}(\sin t/t^p)dt$ are absolutely convergent if $p > 1$ and that they are conditionally convergent if $0 < p \leq 1$.

9.30. Let $f : [1, \infty) \to \mathbb{R}$ be such that f is integrable on $[1, x]$ for every $x \geq 1$. Prove the following using Proposition 9.46:
 (i) If there exist $p > 1$ and $\ell \in \mathbb{R}$ such that $t^p f(t) \to \ell$ as $t \to \infty$, then $\int_{t\geq 1} f(t)dt$ is absolutely convergent.
 (ii) Suppose $f(t) > 0$ for all $t \in [1, \infty)$. If there exist $p \leq 1$ and $\ell \neq 0$ such that $t^p f(t) \to \ell$ as $t \to \infty$, then $\int_{t\geq 1} f(t)dt$ is divergent.

9.31. Let $g : [1, \infty) \to \mathbb{R}$ be a continuous real-valued function such that the function $G : [a, \infty) \to \mathbb{R}$ defined by $G(x) := \int_a^x g(t)dt$ is bounded. If $p \in \mathbb{R}$ with $p > 0$ and $x \in (0, 1)$, then show that the improper integrals

$$\int_{t\geq 1} \frac{g(t)}{t^p}dt, \quad \int_{t\geq 2} \frac{g(t)}{(\ln t)^p}dt, \quad \text{and} \quad \int_{t\geq 1} x^t g(t)dt$$

are convergent. (Hint: Proposition 9.53.)

9.32. **(Abel Test for Improper Integrals)** Let $a \in \mathbb{R}$ and $f, g : [a, \infty) \to \mathbb{R}$ be such that f is bounded, monotonic, and differentiable, f' is integrable on $[a, x]$ for every $x \geq a$, while g is continuous and the improper integral $\int_{t\geq a} g(t)dt$ is convergent. Show that the improper integral $\int_{t\geq a} f(t)g(t)dt$ is convergent. Further, show that for $x \in [a, \infty)$,

$$\left| \int_x^\infty f(t)g(t)dt \right| \leq (|\ell| + |f(x) - \ell|)\, \beta_x,$$

where $\ell := \lim_{t\to\infty} f(t)$ and $\beta_x := \sup\left\{ \left| \int_x^y g(t)dt \right| : y \in [x, \infty) \right\}$. (Hint: Proposition 9.53.)

9.33. Let $\int_{t\geq 1} g(t)dt$ be a convergent improper integral. Show that the improper integrals $\int_{t\geq 1} t^{1/t} g(t)dt$ and $\int_{t\geq 1} \left(1+\frac{1}{t}\right)^t g(t)dt$ are also convergent. (Hint: Exercise 9.32, and Revision Exercise R.15 given at the end of Chapter 7.)

9.34. Let $p \in \mathbb{R}$ with $p > 1$. Show that the improper integrals $\int_{t\geq 1} \sin t^p\, dt$ and $\int_{t\geq 1} \cos t^p\, dt$ are convergent. (Hint: Put $s = t^p$ and use Corollary 9.54.)

9.35. Let $p \in \mathbb{R}$ with $p > 0$. Show that the improper integrals $\int_{t\geq 2} (\sin t)/(\ln t)^p dt$ and $\int_{t\geq 2} (\cos t)/(\ln t)^p dt$ are conditionally convergent. (Hint: Corollary 9.54 and Exercise 9.29.)

9.36. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(t) := (1+t)/(1+t^2)$ for $t \in \mathbb{R}$. Show that the improper integral $\int_{\mathbb{R}} f(t)dt$ is divergent, but the Cauchy principal value of this improper integral exists and is equal to π.

9.37. Let $f : [1, \infty) \to \mathbb{R}$ be defined as follows. If $t \in [1, \infty)$ and $k \leq t < k + 1$ with $k \in \mathbb{N}$, let $f(t) := (-1)^{k-1}/k$. Show that $\int_{t\geq a} f(t)dt$ is conditionally convergent. (Hint: $\sum_{k\geq 1}(-1)^{k-1}/k$ is conditionally convergent.)

9.38. Show that the improper integral $\int_{t\geq 0} e^{t^2} dt$ is divergent, but $\int_{t\geq 0} e^{-t^2} dt$ is convergent. (Hint: Comparison with $\int_{t\geq 0} e^t dt$ and $\int_{t\geq 0} e^{-t} dt$.)

9.39. Let $p(t)$ and $q(t)$ be polynomials of degrees m and n respectively. Suppose $q(t) \neq 0$ for all $t \geq a$ and let $f : [a, \infty) \to \mathbb{R}$ be defined by $f(t) := p(t)/q(t)$.

Show that $\int_{t\geq a} f(t)dt$ is absolutely convergent if $n \geq m+2$, and $\int_{t\geq a} f(t)dt$ is divergent if $n < m+2$.

9.40. Let $f : (a, b] \to \mathbb{R}$ be a nonnegative function that is integrable on $[x, b]$ for every $x \in (a, b]$. Prove the following:
(i) If there exists $p \in (0, 1)$ such that $(t-a)^p f(t) \to \ell$ for some $\ell \in \mathbb{R}$, then the improper integral $\int_{a<t\leq b} f(t)dt$ is convergent.
(ii) If there exists $p \geq 1$ such that $(t-a)^p f(t) \to \ell$ for some $\ell \neq 0$, then the improper integral $\int_{a<t\leq b} f(t)dt$ is divergent.
(Hint: Comparison with $g(t) := 1/(t-a)^p$ for $t \in (a, b]$.)

9.41. Show that the improper integral $\int_{1<t\leq 2}(\sqrt{t}/\ln t)dt$ is divergent. (Hint: Exercise 9.40.)

9.42. **(Dirichlet Test for Convergence of Improper Integrals of the Second Kind)** Let $a, b \in \mathbb{R}$ with $a < b$, and let $f, g : (a, b] \to \mathbb{R}$ be such that f is monotonic, $f(t) \to 0$ as $t \to a^+$, f is differentiable, and f' is integrable on $[x, b]$ for each $x \in (a, b]$, while g is continuous on $(a, b]$, and the function $G : (a, b] \to \mathbb{R}$ defined by $G(x) := \int_x^b g(t)dt$ is bounded. Show that the improper integral $\int_{a<t\leq b} f(t)g(t)dt$ is convergent. Further, if $\beta_x := \sup\left\{\left|\int_y^x g(t)dt\right| : y \in (a, x]\right\}$ for $x \in (a, b]$, then show that

$$\left|\int_x^b f(t)g(t)dt\right| \leq |f(b)|\beta_b \quad \text{and} \quad \left|\int_{a^+}^x f(t)g(t)dt\right| \leq |f(x)|\beta_x \leq 2|f(x)|\beta_b.$$

(Compare Proposition 9.53. Hint: Exercise 6.21 and the Cauchy Criterion)

9.43. Let $(p, q) \in (0, \infty) \times (0, \infty)$. Show that

$$\beta(p, q) = \int_{0^+}^{\infty} \frac{u^{p-1}}{(1+u)^{p+q}}\,du = \int_{0^+}^{1} \frac{v^{p-1}+v^{q-1}}{(1+v)^{p+q}}dv.$$

(Hint: Substitute $t = u/(1+u)$ and then $v = 1/u$.)

9.44. Let $q \in \mathbb{R}$. Test the following improper integrals for convergence.

(i) $\displaystyle\int_{t\geq 1} \frac{1}{\sqrt{1+t^3}}\,dt$, (ii) $\displaystyle\int_{t\geq 1} \frac{t^q}{1+t^q}\,dt$, (iii) $\displaystyle\int_{t\geq 2} \frac{1}{\ln t}\,dt$,

(iv) $\displaystyle\int_{0<t\leq 1} \sin\left(\frac{1}{t}\right)dt$, (v) $\displaystyle\int_{0<t\leq 1} e^{1/t}t^q\,dt$, (vi) $\displaystyle\int_{0<t<1} \frac{1}{t\ln t}\,dt$.

Part B

9.45. **(Cauchy Product)** Suppose one of the series $\sum_{k\geq 0} a_k$ and $\sum_{k\geq 0} b_k$ is absolutely convergent and the other is convergent. Let A and B denote their respective sums. For each $k = 0, 1, \ldots$, let $c_k := \sum_{j=0}^{k} a_j b_{k-j}$. Show that the series $\sum_{k\geq 0} c_k$ is convergent and its sum is equal to AB. Give an example to show that the result may not hold if both $\sum_{k\geq 0} a_k$ and $\sum_{k\geq 0} b_k$ are conditionally convergent.
[Note: If $\sum_{k\geq 0} a_k$ and $\sum_{k\geq 0} b_k$ are convergent, and if the series $\sum_{k\geq 0} c_k$ is convergent, then its sum must be AB. This can be proved using a result of Abel given in Exercise 10.44 (i).]

9.46. **(Grouping of Terms)** Let $m_0 := 0$ and let $m_1 < m_2 < \cdots$ be natural numbers. Given a series $\sum_{k\geq 1} a_k$, define $b_k := a_{m_{k-1}+1} + \cdots + a_{m_k}$ for $k \in \mathbb{N}$. If the series $\sum_{k\geq 1} a_k$ is convergent, then show that the series $\sum_{k\geq 1} b_k$ is convergent and has the same sum. Give an example to show that $\sum_{k\geq 1} b_k$ may be convergent although $\sum_{k\geq 1} a_k$ is divergent.

9.47. **(Rearrangement of Terms)** Let $k \longmapsto j(k)$ be a bijection from $\mathbb{N}$ to $\mathbb{N}$. Given a series $\sum_{k\geq 1} a_k$, consider the series $\sum_{k\geq 1} b_k$, where $b_k := a_{j(k)}$. Then the series $\sum_{k\geq 1} b_k$ is called a **rearrangement** of the series $\sum_{k\geq 1} a_k$. Show that a series $\sum_{k\geq 1} a_k$ is absolutely convergent if and only if every rearrangement of it is convergent. In this case, the sum of a rearrangement is unchanged.

9.48. Use the triangle inequality and the Cauchy Criterion (Propositions 1.8 and 2.22) to conclude that if a series of real numbers is absolutely convergent, then it is convergent. Conversely, assuming that every absolutely convergent series of real numbers is convergent, deduce the Cauchy Criterion. (Hint: Given a Cauchy sequence (A_n) of real numbers, inductively construct a subsequence (A_{n_k}) such that $|A_{n_{k+1}} - A_{n_k}| \leq 1/k^2$ for all $k \in \mathbb{N}$ and consider $a_k := A_{n_{k+1}} - A_{n_k}$.)

9.49. For $k \in \mathbb{N}$, let $a_k \in \mathbb{R}$ with $a_k > 0$. Show that

$$\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} \leq \liminf_{k\to\infty} a_k^{1/k} \quad \text{and} \quad \limsup_{k\to\infty} a_k^{1/k} \leq \limsup_{k\to\infty} \frac{a_{k+1}}{a_k}.$$

9.50. For $k \in \mathbb{N}$, let $a_k \in \mathbb{R}$ with $a_k \neq 0$. If $|a_{k+1}|/|a_k| \to \ell$ as $k \to \infty$, then show that $|a_k|^{1/k} \to \ell$ as $k \to \infty$.

9.51. Let $\sum_{k\geq 0} c_k x^k$ be a power series with $c_k \neq 0$ for all $k \in \mathbb{N}$ and let r denote its radius of convergence Also, let $\overline{\ell} = \limsup_{k\to\infty} |c_{k+1}|/|c_k|$ and $\underline{\ell} = \liminf_{k\to\infty} |c_{k+1}|/|c_k|$. Prove the following:
 (i) If $\underline{\ell} = \infty$, then $r = 0$, whereas if $0 < \underline{\ell} < \infty$, then $r \leq 1/\underline{\ell}$.
 (ii) If $\overline{\ell} = 0$, then $r = \infty$, whereas if $0 < \overline{\ell} < \infty$, then $r \geq 1/\overline{\ell}$.

9.52. **(Binomial Series)** Let $r \in \mathbb{R}$ be such that $r \notin \{0, 1, 2, \ldots\}$, and define $f : (-1, 1) \to \mathbb{R}$ by $f(x) = (1 + x)^r$. Show that

$$f(x) = 1 + \sum_{k=1}^{\infty} \frac{r(r-1)\cdots(r-k+1)}{k!} x^k \quad \text{for } x \in (-1, 1).$$

(Hint: If $x \in (-1, 1)$, $n \in \mathbb{N}$, and $R_n(x)$ denotes the Cauchy form of the remainder as given in Exercise 4.51, then there exists c between 0 and x such that $|R_n(x)| \leq \left|r(r-1)\left(\frac{r}{2}-1\right)\cdots\left(\frac{r}{n}-1\right)\right| (1+c)^{r-1}|x|^{n+1}$.)

9.53. Let $f : [a, b] \to \mathbb{R}$ be an infinitely differentiable function. Assume that the Taylor series of f around a converges to $f(x)$ at every $x \in [a, b]$, that is,

$$f(x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad x \in [a, b].$$

Also, assume that the series obtained by integrating the above series term by term converges to $\int_a^b f(x)dx$, that is,

$$\int_a^b f(x)dx = f(a)(b-a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{(n+1)!}(b-a)^{n+1}.$$

If $M(f)$, $T(f)$, and $S(f)$ denote the Midpoint Rule, Trapezoidal Rule, and Simpson Rule for f, show that there exist α_n, β_n, γ_n in $\mathbb{R}$ such that the series $\sum_{n=4}^{\infty} \alpha_n(b-a)^n$, $\sum_{n=4}^{\infty} \beta_n(b-a)^n$, and $\sum_{n=6}^{\infty} \gamma_n(b-a)^n$ converge and

(i) $\displaystyle\int_a^b f(x)dx - M(f) = \frac{f''(a)}{24}(b-a)^3 + \sum_{n=4}^{\infty} \alpha_n(b-a)^n,$

(ii) $\displaystyle\int_a^b f(x)dx - T(f) = -\frac{f''(a)}{12}(b-a)^3 + \sum_{n=4}^{\infty} \beta_n(b-a)^n,$

(iii) $\displaystyle\int_a^b f(x)dx - S(f) = -\frac{f^{(4)}(a)}{2880}(b-a)^5 \sum_{n=6}^{\infty} \beta_n(b-a)^n.$

(Compare Lemmas 8.21 and 8.23, and the subsequent error estimates.)

9.54. Let $f : [1,\infty) \to \mathbb{R}$ be a nonnegative monotonically decreasing function such that $\int_{t\geq 1} f(t)dt$ is convergent. For $n \in \mathbb{N}$, let $B_n := \sum_{k=1}^{n} f(k)$ denote the nth partial sum of the convergent series $\sum_{k\geq 1} f(k)$. Show that

$$B_n + \int_{n+1}^{\infty} f(t)dt \leq \sum_{k=1}^{\infty} f(k) \leq B_n + \int_{n+1}^{\infty} f(t)dt + f(n+1).$$

Use this result to show that

$$\sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{n+1} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq \sum_{k=1}^{n} \frac{1}{k^2} \leq \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{n+1} + \frac{1}{(n+1)^2}.$$

Further, show that if $n \geq 31$, then

$$\left| \sum_{k=n+1}^{\infty} \frac{1}{k^2} - \frac{1}{n+1} \right| < \frac{1}{1000}.$$

9.55. Let $f : [1,\infty) \to \mathbb{R}$ be a nonnegative monotonically decreasing function. For $n \in \mathbb{N}$, define $c_n := \sum_{k=1}^{n} f(k) - \int_1^n f(t)dt$. Show that $\lim_{n\to\infty} c_n$ exists and

$$0 \leq f(1) - \int_1^2 f(t)dt \leq \lim_{n\to\infty} c_n \leq f(1).$$

Use this result to show that if $c_n := 1 + (1/2) + \cdots + (1/n) - \ln n$, then $c_n \to \gamma$, where γ satisfies $1 - \ln 2 < \gamma < 1$. (Compare Exercise 7.2.)

9.56. **(Log-Convexity of the Gamma Function)** Let $\Gamma_\ell : (0,\infty) \to \mathbb{R}$ be defined by $\Gamma_\ell(u) := \ln \Gamma(u)$. Show that Γ_ℓ is a convex function. (Hint: Use Exercise 7.57 (v) to show that if $p, q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $\Gamma((u/p) + (v/q)) \leq \Gamma(u)^{1/p}\Gamma(v)^{1/q}$ for all $u, v \in (0,\infty)$.)

10

Sequences and Series of Functions, Integrals Depending on a Parameter

In Chapter 9, we studied infinite series of real numbers as well as their continuous analogues, namely improper integrals of the first kind. In this chapter, we take this study further by considering infinite series of real-valued functions and improper integrals depending on a parameter. In this context, several natural questions arise. For example, if every term of a convergent series of real-valued functions is continuous, will the sum function be continuous? Answers to such questions involve an interchange of two limiting processes, as we shall point out.

Just as questions about series of real numbers go back to the behavior of sequences of real numbers studied in Chapter 2, questions about series of real-valued functions go back to the behavior of sequences of real-valued functions. In Section 10.1, we shall see that mere convergence of a sequence of functions at every point in the domain of their definition is insufficient to allow an interchange of the two limiting processes. This leads us to the concept of uniform convergence of a sequence of functions in Section 10.2, and of a series of functions in Section 10.3. In Section 10.4, we address a reverse question, namely whether an arbitrary continuous function can be uniformly approximated by continuous functions of a special kind. We prove two celebrated theorems of Weierstrass; the first states that for every continuous function f on a closed and bounded interval in $\mathbb{R}$, there is a sequence of polynomial functions that converges uniformly to f, while the second involves approximation of f by trigonometric functions. We give proofs of these results, one due to Bernstein and the other to Fejér, that yield important constructions of a uniformly convergent sequence and of a uniformly convergent series.

Another mode of convergence of a sequence of functions, known as bounded convergence, is considered in Section 10.5. While the treatment of this mode of convergence is admittedly more involved than the treatment of uniform convergence, we reap rich dividends in the form of stronger results in several cases. This is especially borne out in Section 10.6, in which we consider the Riemann integral as a continuous analogue of a finite sum. Several properties of the Riemann integral of a function depending on a parameter are obtained.

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They include boundedness, continuity, integrability, and differentiability. This section prepares us to study, in Section 10.7, improper integrals of functions that depend on a parameter. Here the Riemann integrals studied in the previous section serve as partial integrals. We illustrate the results proved in this chapter by considering trigonometric series, Dirichlet series, Fourier integrals, and Laplace integrals. We also give a fairly extensive treatment of the gamma function, and treat the beta function in the exercises.

10.1 Pointwise Convergence of Sequences

Let E be a set and (f_n) a sequence of real-valued functions defined on E. We say that (f_n) **converges pointwise** on E if for each $x \in E$, the sequence $(f_n(x))$ converges in $\mathbb{R}$. If (f_n) converges pointwise on E, then there is a unique real-valued function f on E such that $f_n(x) \to f(x)$ for every $x \in E$; in this case, we write $f_n \to f$ on E and call f the **pointwise limit** of (f_n).

Suppose $f_n \to f$ on E. We ask the following questions:

(i) Suppose each f_n is bounded on E. Must f be bounded on E? Further, suppose $\alpha_n := \sup\{f_n(x) : x \in E\}$ and $\beta_n := \inf\{f_n(x) : x \in E\}$ for $n \in \mathbb{N}$. Must the sequences (α_n) and (β_n) converge in $\mathbb{R}$? If so, must (α_n) converge to $\sup\{f(x) : x \in E\}$, and (β_n) to $\inf\{f(x) : x \in E\}$?

(ii) Suppose E is a subset of $\mathbb{R}$ and each f_n is continuous on E. Must f be continuous on E?

(iii) Suppose $E := [a, b]$, where $a, b \in \mathbb{R}$ with $a < b$, and each f_n is integrable on E. Must f be integrable on E? Must the sequence $\big(\int_a^b f_n(x)dx\big)$ converge in $\mathbb{R}$? If so, must it converge to $\int_a^b f(x)dx$?

(iv) Suppose $E := [a, \infty)$, where $a \in \mathbb{R}$, and each improper integral $\int_{x\geq a} f_n(x)dx$ is convergent. Must the improper integral $\int_{x\geq a} f(x)dx$ be convergent? Must the sequence $\big(\int_a^\infty f_n(x)dx\big)$ converge in $\mathbb{R}$? If so, must it converge to $\int_a^\infty f(x)dx$?

(v) Suppose E is an interval in $\mathbb{R}$ and each f_n is differentiable on E. Must f be differentiable on E? Must there exist a function g on E such that $f_n' \to g$? If so, must $g = f'$ on E?

Answers to these questions involve interchange of two processes, one of which is taking the limit as $n \to \infty$, as shown below:

(i) $\displaystyle \lim_{n\to\infty} \sup_{x\in E} f_n(x) \overset{?}{=} \sup_{x\in E} \lim_{n\to\infty} f_n(x), \qquad \lim_{n\to\infty} \inf_{x\in E} f_n(x) \overset{?}{=} \inf_{x\in E} \lim_{n\to\infty} f_n(x).$

(ii) $\displaystyle \lim_{n\to\infty} \lim_{m\to\infty} f_n(x_m) \overset{?}{=} \lim_{m\to\infty} \lim_{n\to\infty} f_n(x_m)$ for a convergent sequence (x_m) in E.

(iii) $\displaystyle \lim_{n\to\infty} \int_a^b f_n(x)dx \overset{?}{=} \int_a^b \Big(\lim_{n\to\infty} f_n(x)\Big)dx.$

(iv) $\displaystyle\lim_{n\to\infty} \int_a^\infty f_n(x)dx \stackrel{?}{=} \int_a^\infty \Big(\lim_{n\to\infty} f_n(x)\Big)dx.$

(v) $\displaystyle\lim_{n\to\infty} \frac{df_n}{dx}\Big|_{x=c} \stackrel{?}{=} \frac{d}{dx}\Big(\lim_{n\to\infty} f_n\Big)\Big|_{x=c}$, where $c \in E$.

In (ii) above, and hereinafter, by convergent sequence in E, we mean a sequence in a set E that converges to a point in E.

Examples 10.1. Answers to all the questions raised above are negative.

(i) Let $E := (0, 1]$.
   (a) For $n \in \mathbb{N}$, define $f_n : E \to \mathbb{R}$ by $f_n(x) := 1/x$ if $(1/n) < x \le 1$ and $f_n(x) := 0$ otherwise. Then each f_n is bounded on E. (In fact, $0 \le f_n \le n$ on E.) Also, $f_n \to f$ on E, where $f(x) := 1/x$ for $x \in E$. Clearly, f is not bounded on E.
   (b) For $n \in \mathbb{N}$ with n even, define $f_n : E \to \mathbb{R}$ by $f_n(x) := 1/(nx + 1)$ and for $n \in \mathbb{N}$ with n odd, define $f_n(x) := 0$. Clearly, $f_n \to f$ on E, where $f := 0$. Also, each f_n as well as f is bounded on E. Further, if $\alpha_n := \sup\{f_n(x) : x \in E\}$ for $n \in \mathbb{N}$, then we see that $\alpha_n = 0$ if n is odd and $\alpha_n = 1$ if n is even. Thus the sequence (α_n) is not convergent.
   (c) For $n \in \mathbb{N}$, define $f_n : E \to \mathbb{R}$ by $f_n(x) := 1/(nx+1)$. Clearly, $f_n \to f$ on E, where $f := 0$. Also, each f_n as well as f is bounded on E. Further, if $\alpha_n := \sup\{f_n(x) : x \in E\}$ for $n \in \mathbb{N}$, and $\alpha := \sup\{f(x) : x \in E\}$, then we see that $\alpha_n = 1$ for $n \in \mathbb{N}$, and $\alpha = 0$. Thus the sequence (α_n) is convergent, but not to α.

(ii) Let $E := [0, 1]$ and for $n \in \mathbb{N}$, define $f_n : E \to \mathbb{R}$ by $f_n(x) := nx$ if $0 \le x \le (1/n)$ and $f_n(x) := 1$ otherwise. Then each f_n is continuous on E. Also, $f_n \to f$ on E, where $f(0) := 0$ and $f(x) := 1$ if $0 < x \le 1$. Clearly, f is not continuous on E.

(iii) Let $E := [0, 1]$.
   (a) Let $r_1, r_2, \ldots$ be an enumeration of $\mathbb{Q} \cap E$. For $n \in \mathbb{N}$, define $f_n : E \to \mathbb{R}$ by $f_n(x) := 1$ if $x \in \{r_1, \ldots, r_n\}$ and $f_n(x) := 0$ otherwise. Then each f_n is integrable on E, since it is discontinuous at only a finite number of points, and in fact, $\int_0^1 f_n(x)dx = 0$. Also, $f_n \to f$ on E, where $f(x) := 1$ if $x \in \mathbb{Q} \cap E$ and $f(x) := 0$ if $x \in E \setminus \mathbb{Q}$. But the Dirichlet function f is not integrable, as we have seen in Example 6.4 (ii).
   (b) For $n \in \mathbb{N}$, define $f_n : E \to \mathbb{R}$ by $f_n(x) := n^2$ if $0 < x < 1/n$ and $f_n(x) := 0$ otherwise. Then each f_n is integrable on E, since it is discontinuous only at two points of E. Now $f_n(0) = 0$ for all $n \in \mathbb{N}$. Also, if $x \in (0, 1]$, then $f_n(x) = 0$ for all $n \ge (1/x)$. Hence $f_n \to f$ on E, where $f := 0$, and so $\int_0^1 f(x)dx = 0$. But $\int_0^1 f_n(x)dx = n$ for each $n \in \mathbb{N}$. Thus the sequence $\big(\int_0^1 f_n(x)dx\big)$ is not convergent.
   (c) For $n \in \mathbb{N}$, define $f_n : E \to \mathbb{R}$ by $f_n(x) := n$ if $0 < x < 1/n$ and $f_n(x) := 0$ otherwise. As in (b) above, each f_n is integrable on E, and $f_n \to f$ on E, where $f := 0$. Now $\int_0^1 f(x)dx = 0$, while $\int_0^1 f_n(x)dx = 1$

for $n \in \mathbb{N}$. Thus the sequence $\left(\int_0^1 f_n(x)dx\right)$ converges, but not to $\int_0^1 f(x)dx$.

(iv) Let $E := [0, \infty)$.

(a) For $n \in \mathbb{N}$, define $f_n : E \to \mathbb{R}$ by $f_n(x) := 1/x$ if $1 \le x \le n$ and $f_n(x) := 0$ otherwise. Then $\int_0^\infty f_n(x)dx = \ln n$ for each $n \in \mathbb{N}$. Also, $f_n \to f$ on E, where $f(x) := 0$ if $x \in [0, 1)$ and $f(x) := 1/x$ if $x \in [1, \infty)$. But the improper integral $\int_{x \ge 0} f(x)dx$ does not converge, as seen in Example 9.36 (iii).

(b) For $n \in \mathbb{N}$, define $f_n : E \to \mathbb{R}$ by $f_n(x) := 1/n$ if $0 < x \le n^2$ and $f_n(x) := 0$ otherwise. Then $\int_0^\infty f_n(x)dx = n$ for each $n \in \mathbb{N}$. Also, $f_n \to f$ on E, where $f := 0$, and so $\int_0^\infty f(x)dx = 0$. Thus each $\int_{x \ge 0} f_n(x)dx$ and $\int_{x \ge 0} f(x)dx$ are convergent improper integrals, but the sequence $\left(\int_0^\infty f_n(x)dx\right)$ is not convergent.

(c) For $n \in \mathbb{N}$, define $f_n : E \to \mathbb{R}$ by $f_n(x) := 1/n$ if $0 < x \le n$ and $f_n(x) := 0$ otherwise. Then $\int_0^\infty f_n(x)dx = 1$ for each $n \in \mathbb{N}$. Also, $f_n \to f$ on E, where $f := 0$, and so $\int_0^\infty f(x)dx = 0$. Thus, the sequence $\left(\int_0^\infty f_n(x)dx\right)$ converges, but not to $\int_0^\infty f(x)dx$.

(v) Let $E := (-1, 1)$.

(a) For $n \in \mathbb{N}$, define $f_n : E \to \mathbb{R}$ by $f_n(x) := \sqrt{x^2 + (1/n^2)}$. Then each f_n is differentiable on E and $f_n \to f$ on E, where $f(x) := |x|$ for $x \in E$. Clearly, f is not differentiable on E.

(b) For $n \in \mathbb{N}$, define $f_n : E \to \mathbb{R}$ by $f_n(x) := (\sin 2n\pi x)/n$. Then each f_n is differentiable, $f_n \to f$ on E, where $f := 0$. But since $f_n'(x) = 2\pi \cos 2n\pi x$ for $x \in E$, the sequence (f_n') does not converge pointwise on E. For example, $f_n'(1/2) = (-1)^n 2\pi$ for $n \in \mathbb{N}$, and so the sequence $(f_n'(1/2))$ is not convergent.

(c) For $n \in \mathbb{N}$, define $f_n : E \to \mathbb{R}$ by

$$f_n(x) := \begin{cases} \dfrac{2 - (1+x)^n}{n} & \text{if } -1 < x < 0, \\ \dfrac{(1-x)^n}{n} & \text{if } 0 \le x < 1. \end{cases}$$

Then each f_n is differentiable on E. Moreover, $f_n \to f$ on E, where $f := 0$. Since

$$f_n'(x) := \begin{cases} -(1+x)^{n-1} & \text{if } -1 < x < 0, \\ -(1-x)^{n-1} & \text{if } 0 \le x < 1, \end{cases}$$

it follows that $f_n' \to g$ on $(-1, 1)$, where $g(0) := -1$ and $g(x) := 0$ for $0 < |x| < 1$ by Example 2.7 (i). Clearly, $f' \ne g$. ◇

10.2 Uniform Convergence of Sequences

In an attempt to obtain affirmative answers to the questions posed at the beginning of Section 10.1, we introduce a stronger concept of convergence.

Let E be a set, and let (f_n) be a sequence of real-valued functions defined on E. We say that (f_n) **converges uniformly** on E if there is a function $f : E \to \mathbb{R}$ such that for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ satisfying

$$n \geq n_0 \text{ and } x \in E \Longrightarrow |f_n(x) - f(x)| < \epsilon.$$

Note that the natural number n_0 in this definition is independent of $x \in E$, although it may depend on the given sequence (f_n) of functions and on the given $\epsilon > 0$. Clearly, if such a function f exists, then it is unique. In this case, we write $f_n \to f$ uniformly on E, or we write $f_n(x) \to f(x)$ uniformly for $x \in E$, and call f the **uniform limit** of (f_n). Evidently, uniform convergence implies pointwise convergence, that is, if $f_n \to f$ uniformly, then $f_n \to f$.

It is easy to see that a sequence (f_n) of real-valued functions defined on E converges uniformly on E if and only if there exist a function $f : E \to \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that the function $|f_n - f|$ is bounded on E for each $n \in \mathbb{N}$ with $n \geq n_0$, and $\alpha_n \to 0$, where $\alpha_n := \sup\{|f_n(x) - f(x)| : x \in E\}$ for $n \geq n_0$.

Examples 10.2. (i) Pointwise convergence may not imply uniform convergence. For example, let $E := [0, 1]$, and $f_n(x) := 1/(nx + 1)$ for $x \in E$. Then $f_n \to f$ on E, where $f(0) := 1$ and $f(x) := 0$ for all $x \in (0, 1]$. But (f_n) does not converge uniformly to f on E, since $|f_n(1/n) - f(1/n)| = 1/2$ for every $n \in \mathbb{N}$.

(ii) For $n \in \mathbb{N}$, let $f_n(x) := x^n$, and let $f(x) := 0$ for $x \in (-1, 1)$. Then $f_n \to f$ on $(-1, 1)$. If $0 < r < 1$ and $E := [-r, r]$, then $f_n \to f$ uniformly on E, since $\alpha_n := \sup\{|f_n(x) - f(x)| : x \in [-r, r]\} = r^n \to 0$, but if $E := [0, 1)$, then (f_n) does not converge uniformly to f on E, since $\alpha_n := \sup\{|f_n(x) - f(x)| : x \in [0, 1)\} = 1 \not\to 0$. ◇

The following test is useful for checking uniform convergence of a sequence of functions when its pointwise limit is known.

Proposition 10.3 (Test for Uniform Convergence of Sequences). *Let (f_n) be a sequence of real-valued functions on a set E, and let $f_n \to f$ on E. Suppose there is a sequence (β_n) in $\mathbb{R}$ such that $\big|f_n(x) - f(x)\big| \leq \beta_n$ for all large $n \in \mathbb{N}$ and all $x \in E$. If $\beta_n \to 0$, then $f_n \to f$ uniformly on E.*

Proof. Let $\alpha_n := \sup\{|f_n(x) - f(x)| : x \in E\}$ for all large $n \in \mathbb{N}$. Then $0 \leq \alpha_n \leq \beta_n$ for all large $n \in \mathbb{N}$. If $\beta_n \to 0$, then $\alpha_n \to 0$, that is, $f_n \to f$ uniformly on E. □

We shall now give a criterion for checking uniform convergence of a sequence of functions without knowing its pointwise convergence. A sequence

(f_n) of real-valued functions defined on a set E is said to be **uniformly Cauchy** on E if for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ satisfying

$$m, n \geq n_0 \text{ and } x \in E \Longrightarrow |f_m(x) - f_n(x)| < \epsilon.$$

Proposition 10.4 (Cauchy Criterion for Uniform Convergence of Sequences). *Let (f_n) be a sequence of real-valued functions defined on a set E. Then (f_n) is uniformly convergent on E if and only if (f_n) is uniformly Cauchy on E.*

Proof. Suppose $f_n \to f$ uniformly on E. Let $\epsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \text{ and } x \in E \Longrightarrow |f_n(x) - f(x)| < \frac{\epsilon}{2}.$$

Hence for all $m, n \geq n_0$ and all $x \in E$, we obtain

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conversely, suppose (f_n) is uniformly Cauchy on E. Then $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$ for each $x \in E$. For $x \in E$, by Proposition 2.22, $(f_n(x))$ converges to a real number, which we denote by $f(x)$. Let $\epsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that

$$m, n \geq n_0 \text{ and } x \in E \Longrightarrow |f_m(x) - f_n(x)| < \epsilon.$$

Fix $n \geq n_0$ and let $m \to \infty$. Thus we obtain $|f(x) - f_n(x)| \leq \epsilon$ for all $x \in E$. This implies that $f_n \to f$ uniformly on E. □

There will be several occasions for us to use the two results proved above.

Uniform Convergence and Boundedness

We shall show that the limit function of a uniformly convergent sequence of bounded functions is bounded and also that such a sequence has the following important property. We say that a sequence (f_n) of real-valued functions defined on a set E is **uniformly bounded** on E if there exists $\alpha \in \mathbb{R}$ such that $|f_n(x)| \leq \alpha$ for all $n \in \mathbb{N}$ and all $x \in E$.

Proposition 10.5. *Let (f_n) be a sequence of real-valued functions defined on a set E such that $f_n \to f$ uniformly on E. If each f_n is bounded on E, then f is bounded on E and the sequence (f_n) is uniformly bounded on E. Moreover, in this case,* $\sup\{f_n(x) : x \in E\} \to \sup\{f(x) : x \in E\}$ *and* $\inf\{f_n(x) : x \in E\} \to \inf\{f(x) : x \in E\}$*. Conversely, if f is bounded on E, then there exists $\alpha \in \mathbb{R}$ such that $|f_n(x)| \leq \alpha$ for all large n and all $x \in E$.*

Proof. Since $f_n \to f$ uniformly on E, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \text{ and } x \in E \Longrightarrow |f_n(x) - f(x)| < 1.$$

In particular, $|f(x)| < 1 + |f_{n_0}(x)|$ and $|f_n(x)| < 1 + |f(x)| < 2 + |f_{n_0}(x)|$ for all $x \in E$ and $n \geq n_0$. Now suppose each f_n is bounded on E. Then the above inequalities show that f is bounded on E and the sequence (f_n) is uniformly bounded on E. Next, let $\epsilon > 0$ be given. Then there exists $n_\epsilon \in \mathbb{N}$ such that

$$f(x) - \epsilon < f_n(x) < f(x) + \epsilon \quad \text{for all } n \geq n_\epsilon \text{ and } x \in E.$$

Hence for each $n \geq n_\epsilon$,

$$\sup\{f(x) : x \in E\} - \epsilon \leq \sup\{f_n(x) : x \in E\} \leq \sup\{f(x) : x \in E\} + \epsilon.$$

This shows that $\sup\{f_n(x) : x \in E\} \to \sup\{f(x) : x \in E\}$. In a similar manner, we see that $\inf\{f_n(x) : x \in E\} \to \inf\{f(x) : x \in E\}$.

Conversely, suppose f is bounded on E. Then there exists $\alpha \in \mathbb{R}$ such that $|f(x)| \leq \alpha$ for all $x \in E$. But then $|f_n(x)| < 1 + \alpha$ for all $n \geq n_0$ and all $x \in E$, as desired. □

The above proposition shows that in parts (a), (b), and (c) of Example 10.1 (i), the convergence of (f_n) to f is nonuniform. On the other hand, there can be a uniformly bounded sequence (f_n) that converges pointwise to a bounded function f, and further, $\sup f_n \to \sup f$ and $\inf f_n \to \inf f$, but (f_n) does not converge to f uniformly. To see this, consider f_n and f defined on $[0, 1]$ as in Example 10.1 (ii). Here (f_n) does not converge uniformly to f on $[0, 1]$, since $|f_n(1/2n) - f(1/2n)| = 1/2$ for every $n \in \mathbb{N}$.

Uniform Convergence and Continuity

Proposition 10.6. *Let (f_n) be a sequence of real-valued continuous functions defined on a subset E of $\mathbb{R}$ such that $f_n \to f$ uniformly on E. Then f is continuous on E, and so if (x_m) is a convergent sequence in E, then*

$$\lim_{n\to\infty} \lim_{m\to\infty} f_n(x_m) = \lim_{m\to\infty} \lim_{n\to\infty} f_n(x_m).$$

Proof. Let $\epsilon > 0$ be given. Then there is $n_0 \in \mathbb{N}$ such that $|f_{n_0}(x) - f(x)| < \epsilon$ for all $x \in E$. Consider $c \in E$. Since f_{n_0} is continuous at c, there exists $\delta > 0$ such that $|f_{n_0}(x) - f_{n_0}(c)| < \epsilon$ whenever $x \in E$ and $|x - c| < \delta$, and in turn,

$$|f(x) - f(c)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(c)| + |f_{n_0}(c) - f(c)| < 3\epsilon.$$

Thus f is continuous at each $c \in E$.

Let $x_m \to c$ in E. For a fixed $n \in \mathbb{N}$, the continuity of f_n at c implies that $\lim_{m\to\infty} f_n(x_m) = f_n(c)$. Also, for a fixed $m \in \mathbb{N}$, since $f_n \to f$ on E, we see that $\lim_{n\to\infty} f_n(x_m) = f(x_m)$. Now since f is continuous at c, we obtain

$$\lim_{n\to\infty} \lim_{m\to\infty} f_n(x_m) = \lim_{n\to\infty} f_n(c) = f(c) = \lim_{m\to\infty} f(x_m) = \lim_{m\to\infty} \lim_{n\to\infty} f_n(x_m),$$

as desired. □

Proposition 10.6 can be used to show that some pointwise convergent sequences are not uniformly convergent. For instance, in Example 10.1(ii), $f_n \to f$ on E, each f_n is continuous on E, but the convergence is nonuniform since f is not continuous on E. As another example, let $E := (-1, 1]$, and for $n \in \mathbb{N}$, let $f_n(x) := x^n$, $x \in E$. Then $f_n \to f$ on E, where $f(x) := 0$ for $x \in (-1, 1)$ and $f(1) := 1$. The convergence is nonuniform, since each f_n is continuous on E, but f is not.

On the other hand, let $E := [0, 1]$ and $f_n(x) := nxe^{-nx}$ for $n \in \mathbb{N}$ and $x \in E$, and let $f := 0$ on E. Then each f_n and f are continuous on the closed and bounded subset E of $\mathbb{R}$. Now $f_n(0) = 0$ for all $n \in \mathbb{N}$. Also, for each fixed $x \in (0, 1]$, L'Hôpital's Rule for $\frac{\infty}{\infty}$ indeterminate forms (Proposition 4.42) shows that $f_n(x) \to 0$ as $n \to \infty$. Thus $f_n \to f$ on E. But (f_n) does not converge uniformly to f on E, since $|f_n(1/n) - f(1/n)| = e^{-1}$ for every $n \in \mathbb{N}$. Thus the converse of Proposition 10.6 is not true.

However, the following partial converse of Proposition 10.6 holds for a particular kind of sequence of functions. Recall from Chapter 1 that for real-valued functions f and g defined on a set E, we write $f \le g$ if $f(x) \le g(x)$ for all $x \in E$. We say that a sequence (f_n) of real-valued functions defined on a set E is **monotonic** if $f_n \ge f_{n+1}$ for all $n \in \mathbb{N}$, or if $f_n \le f_{n+1}$ for all $n \in \mathbb{N}$.

Proposition 10.7 (Dini Theorem). *Let (f_n) be a monotonic sequence of real-valued continuous functions on a closed and bounded subset E of $\mathbb{R}$. Let $f_n \to f$ on E, where f is continuous on E. Then $f_n \to f$ uniformly on E.*

Proof. First suppose $f_n \ge f_{n+1}$ for all $n \in \mathbb{N}$. Then $f_n - f \ge f_{n+1} - f$, and $f_n - f$ is continuous on E for each $n \in \mathbb{N}$. Replacing the sequence (f_n) by the sequence $(f_n - f)$ if necessary, we assume without loss of generality that $f = 0$ and $f_n \ge 0$ on E for all $n \in \mathbb{N}$.

Suppose for a moment that (f_n) does not converge uniformly to 0 on E. Then there exist $\epsilon > 0$, natural numbers $n_1 < n_2 < \cdots$, and $x_{n_1}, x_{n_2}, \ldots$ in E such that $f_{n_k}(x_{n_k}) \ge \epsilon$ for each $k \in \mathbb{N}$. Since E is a bounded subset of $\mathbb{R}$, the sequence (x_{n_k}) in $\mathbb{R}$ is bounded. By the Bolzano–Weierstrass theorem (Proposition 2.17), it has a convergent subsequence, which we shall denote by (x_{m_j}). Let $x_{m_j} \to x$ in $\mathbb{R}$. Since E is a closed subset of $\mathbb{R}$, we see that $x \in E$.

Now fix $i \in \mathbb{N}$, and consider $j \ge i$. Since $m_i \le m_j$ and $f_n \ge f_{n+1}$ for all $n \in \mathbb{N}$, we obtain $f_{m_i}(x_{m_j}) \ge f_{m_j}(x_{m_j}) \ge \epsilon$. Using the continuity of the function f_{m_i} at x, we see that $f_{m_i}(x_{m_j}) \to f_{m_i}(x)$ as $j \to \infty$. Hence the above inequality yields $f_{m_i}(x) \ge \epsilon$ for every $i \in \mathbb{N}$. But this contradicts the assumption that $f_n(x) \to 0$. Thus we conclude that $f_n \to f$ uniformly on E.

Next suppose $f_n \le f_{n+1}$ for all $n \in \mathbb{N}$. Then $-f_n \ge -f_{n+1}$ for all $n \in \mathbb{N}$. By the argument given above, $-f_n \to -f$ uniformly on E, which implies that $f_n \to f$ uniformly on E. □

Examples 10.8. (i) Let $E := [-1, 1]$ and $f_n(x) := \sqrt{x^2 + (1/n^2)}$ for $n \in \mathbb{N}$ and $x \in E$. Then $f_n \to f$ on E, where $f(x) := \sqrt{x^2} = |x|$ for $x \in E$.

Note that $[-1,1]$ is a closed and bounded subset of $\mathbb{R}$, $f_n \geq f_{n+1}$ for all $n \in \mathbb{N}$ on E, and each f_n is continuous on E, and so is f. Hence by Proposition 10.7, $f_n \to f$ uniformly on E.

(ii) We give examples to show that none of the hypotheses in the Dini Theorem for sequences of continuous functions can be omitted.

(a) Let $E := [0,2]$, and for each $n \in \mathbb{N}$, let $f_n(x) := 1 - |2nx - 3|$ if $(1/n) \leq x \leq (2/n)$ and $f_n(x) := 0$ otherwise. In this example, the set E is closed and bounded, but the sequence (f_n) is not monotonic.

(b) Let $E := (0,1]$, and for each $n \in \mathbb{N}$, let $f_n(x) := 1/(nx+1)$ for $x \in E$. In this example, the sequence (f_n) is monotonic and the set E is bounded but E is not closed.

(c) Let $E := [1,\infty)$, and for each $n \in E$, let $f_n(x) := x/(x+n)$ for $x \in E$. In this example, the sequence (f_n) is monotonic and the set E is closed, but E is not bounded.

(d) Let $E := [0,1]$, and for $n \in E$, let $f_n(x) := 1 - nx$ if $0 \leq x \leq (1/n)$ and $f_n(x) := 0$ otherwise. Here the set E is closed and bounded, the sequence (f_n) is monotonic, and it converges to a discontinuous function $f : E \to \mathbb{R}$ given by $f(0) := 1$ and $f(x) := 0$ if $x \in (0,1]$.

In (a), (b), and (c) above, $f_n \to f$ on E, where $f := 0$ on E, but the sequence (f_n) does not converge to f uniformly, since $\sup\{|f_n(x) - f(x)| : x \in E\} = 1$ for each $n \in \mathbb{N}$. In (d) above, (f_n) does not converge uniformly to f, since $|f_n(1/2n) - f(1/2n)| = 1/2$ for each $n \in \mathbb{N}$. $\diamond$

We remark that the Dini Theorem is one of the few results in which pointwise convergence of a sequence of functions is shown to be uniform. Another such result due to Pólya is given in Exercise 10.43.

Uniform Convergence and Integration

In the following and subsequent results of this chapter, when we write $[a,b]$, it will be tacitly assumed that $a,b \in \mathbb{R}$ and $a \leq b$.

Proposition 10.9. *Let (f_n) be a sequence of real-valued integrable functions defined on $[a,b]$ such that $f_n \to f$ uniformly on $[a,b]$. Then f is integrable on $[a,b]$ and*

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

Further, if we let $F_n(x) := \int_a^x f_n(t)dt$ and $F(x) := \int_a^x f(t)dt$ for $x \in [a,b]$, then $F_n \to F$ uniformly on $[a,b]$.

Proof. Since each f_n is bounded, Proposition 10.5 shows that f is bounded on $[a,b]$. Let $n \in \mathbb{N}$, and let $\alpha_n := \sup\{|f_n(x) - f(x)| : x \in [a,b]\}$. Then

$$f_n(x) - \alpha_n \leq f(x) \leq f_n(x) + \alpha_n \quad \text{for all } x \in [a,b].$$

Recalling the definitions of the lower Riemann integral and the upper Riemann integral, we obtain

$$L(f_n) - \alpha_n(b-a) \le L(f) \le U(f) \le U(f_n) + \alpha_n(b-a).$$

Since f_n is integrable, $L(f_n) = \int_a^b f_n(x)dx = U(f_n)$, and therefore we see that $0 \le U(f) - L(f) \le 2\alpha_n(b-a)$. Also, since $f_n \to f$ uniformly on $[a,b]$, we obtain $\alpha_n \to 0$. Hence $L(f) = U(f)$, that is, f is integrable on $[a,b]$. Moreover,

$$\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| \le \int_a^b |f_n(x) - f(x)|dx \le \alpha_n(b-a).$$

Thus $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$.

Further, let $F_n(x) := \int_a^x f_n(t)dt$ and $F(x) := \int_a^x f(t)dt$ for $x \in [a,b]$. Then

$$|F_n(x) - F(x)| \le \int_a^x |f_n(t) - f(t)|dt \le \alpha_n(x-a) \le \alpha_n(b-a) \quad \text{for } x \in [a,b].$$

Hence Proposition 10.3 shows that $F_n \to F$ uniformly on $[a,b]$. □

The above result shows that in parts (a), (b), and (c) of Example 10.1 (iii), the convergence of (f_n) to f is nonuniform. On the other hand, for (f_n) and f as in Example 10.2 (i), we see that $f_n \to f$, each f_n and f are integrable on $[0,1]$, and

$$\int_0^1 f_n(x)dx = \frac{\ln(nx+1)}{n}\Big|_0^1 = \frac{\ln(1+n)}{n} \to 0 = \int_0^1 f(x)dx,$$

but (f_n) does not converge to f uniformly on $[0,1]$, as we have already seen. This shows that the converse of Proposition 10.9 is not true.

Examples 10.10. (i) For $n \in \mathbb{N}$, let $f_n(x) := \sqrt{x^2 + (1/n^2)}$ for $x \in [-1,1]$. We have seen in Example 10.8 (i) that $f_n \to f$ uniformly on E, where $f(x) := |x|$ for $x \in [-1,1]$. By Proposition 10.9,

$$\int_{-1}^1 \sqrt{x^2 + (1/n^2)}\, dx = \int_{-1}^1 f_n(x)dx \to \int_{-1}^1 f(x)dx = \int_{-1}^1 |x|dx = 1.$$

(ii) For $n \in \mathbb{N}$, let $f_n(0) := 0$ and $f_n(x) := ne^{-nx}$ for $x \in (0,1]$. Then each f_n is integrable on $[0,1]$ and $f_n \to f$ on $[0,1]$, where $f := 0$. But $\int_0^1 f_n(x)dx = 1 - e^{-n} \to 1$, and so $\int_0^1 f_n(x)dx \not\to \int_0^1 f(x)dx$. Thus from Proposition 10.9, we deduce that (f_n) does not converge to f uniformly on $[0,1]$. ◇

The following result is a consequence of the Dini Theorem (Proposition 10.7). In Section 10.5, we shall prove a stronger result by replacing the assumption of the continuity of functions made here by their integrability.

Corollary 10.11 (Monotone Convergence Theorem for Continuous Functions). *Let (f_n) be a monotonic sequence of real-valued continuous functions on $[a,b]$. Let $f_n \to f$ on $[a,b]$, where f is continuous on $[a,b]$. Then*

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

Proof. Since $[a,b]$ is a closed and bounded subset of $\mathbb{R}$ and since every continuous function on $[a,b]$ is integrable on $[a,b]$ by part (ii) of Proposition 6.10, the desired result follows immediately from Propositions 10.7 and 10.9. $\square$

We now obtain an analogue of Proposition 10.9 for improper integrals. We notice that in parts (a), (b), and (c) of Examples 10.1 (iv), the convergence of (f_n) to f is in fact uniform. This follows since $\sup\{|f_n(x)-f(x)| : x \in E\} \to 0$ in each case. Yet in all these cases, $\int_0^\infty f_n(x)dx \not\to \int_0^\infty f(x)dx$. The following result, therefore, is noteworthy.

Proposition 10.12 (Dominated Convergence Theorem). *Let (f_n) be a sequence of real-valued functions defined on $[a,\infty)$ such that for each $n \in \mathbb{N}$, the improper integral $\int_{x\geq a} f_n(x)dx$ is convergent. Suppose there is a real-valued function g defined on $[a,\infty)$ such that $|f_n| \leq g$ on $[a,\infty)$ for all $n \in \mathbb{N}$ and the improper integral $\int_{x\geq a} g(x)dx$ is convergent. Further, suppose there exists $f : [a,\infty) \to \mathbb{R}$ such that $f_n \to f$ uniformly on $[a,b]$ for each $b \in [a,\infty)$. Then the improper integral $\int_{x\geq a} f(x)dx$ is convergent and*

$$\int_a^\infty f_n(x)dx \to \int_a^\infty f(x)dx.$$

Proof. Let $b \in [a,\infty)$. Proposition 10.9 shows that f is integrable on $[a,b]$. Also, $|f(x)| = \lim_{n\to\infty} |f_n(x)| \leq g(x)$ for all $x \in [a,\infty)$. Hence by the Comparison Test for Improper Integrals (Proposition 9.46), $\int_{x\geq a} f(x)dx$ is convergent.

Let $\epsilon > 0$ be given. Since $\int_{x\geq a} g(x)dx$ is convergent, there exists $b_0 \in [a,\infty)$ such that

$$\int_{b_0}^\infty g(x)dx = \int_a^\infty g(x)dx - \int_a^{b_0} g(x)dx < \epsilon.$$

By Proposition 10.9, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\int_a^{b_0} f_n(x)dx - \int_a^{b_0} f(x)dx\right| < \epsilon \quad \text{for all } n \geq n_0.$$

Also, $|f_n - f| \leq |f_n| + |f| \leq 2g$ on $[b_0,\infty)$ for all $n \in \mathbb{N}$, and so

$$\left|\int_{b_0}^\infty f_n(x)dx - \int_{b_0}^\infty f(x)dx\right| \leq \int_{b_0}^\infty |f_n(x) - f(x)|dx \leq 2\int_{b_0}^\infty g(x)dx < 2\epsilon$$

for all $n \geq n_0$. It follows that $\left|\int_a^\infty f_n(x)dx - \int_a^\infty f(x)dx\right| < \epsilon + 2\epsilon = 3\epsilon$ for all $n \geq n_0$. Thus $\int_a^\infty f_n(x)dx \to \int_a^\infty f(x)dx$. $\square$

Uniform Convergence and Differentiation

Under the hypothesis of uniform convergence of a sequence of real-valued functions, we obtained affirmative answers to questions posed in Section 10.1 for boundedness, continuity, and Riemann integrability of the uniform limit. For differentiation, uniform convergence is not sufficient to obtain affirmative answers. This can be seen by noting that in parts (a), (b), and (c) of Example 10.1 (v), the convergence of the sequence (f_n) is in fact uniform. However, if the convergence of the "derived sequence" (f_n') is uniform, and if (f_n) converges at least at one point, then we obtain affirmative answers.

Proposition 10.13. *Let $a, b \in \mathbb{R}$ with $a < b$, and let (f_n) be a sequence of real-valued continuously differentiable functions on $[a, b]$ such that (f_n) converges at a point $c \in [a, b]$, and (f_n') converges uniformly on $[a, b]$. Then there is a continuously differentiable function f on $[a, b]$ such that $f_n \to f$ uniformly on $[a, b]$, and $f_n' \to f'$ uniformly on $[a, b]$. In fact, f is given by*

$$f(x) = \lim_{n\to\infty} f_n(c) + \int_c^x \left(\lim_{n\to\infty} f_n'(t)\right) dt \quad \textit{for } x \in [a, b].$$

Proof. Let $r \in \mathbb{R}$ be such that $f_n(c) \to r$ in $\mathbb{R}$. Since each f_n' is continuous, it is integrable on $[a, b]$, and so part (ii) of Proposition 6.24 (Fundamental Theorem of Calculus) shows that

$$f_n(x) = f_n(c) + \int_c^x f_n'(t)dt \quad \text{for each } n \in \mathbb{N} \text{ and all } x \in [a, b].$$

Let $g : [a, b] \to \mathbb{R}$ be such that $f_n' \to g$ uniformly on $[a, b]$. Since each f_n' is continuous on $[a, b]$, by Proposition 10.6, g is continuous on $[a, b]$. Now define

$$f(x) := r + \int_c^x g(t)dt \quad \text{for } x \in [a, b].$$

By part (i) of Proposition 6.24 (Fundamental Theorem of Calculus), f is differentiable and $f' = g$ on $[a, b]$. It follows that f is continuously differentiable on $[a, b]$, and $f_n' \to f'$ uniformly on $[a, b]$.

Let $n \in \mathbb{N}$ be given. Define

$$G_n(x) := \int_a^x f_n'(t)dt \quad \text{and} \quad G(x) := \int_a^x f'(t)dt \quad \text{for } x \in [a, b].$$

Consider $x \in [a, b]$. By domain additivity (Proposition 6.8),

$$f_n(x) = f_n(c) + G_n(x) - \int_a^c f_n'(t)dt \quad \text{and} \quad f(x) = r + G(x) - \int_a^c f'(t)dt.$$

Now by Proposition 10.9, $\int_a^c f_n'(t)dt \to \int_a^c f'(t)dt$, and further, (G_n) converges uniformly to G on $[a, b]$. It follows that $f_n \to f$ uniformly on $[a, b]$. □

Let $f_n(x) := (nx+1)/ne^{nx}$ for $n \in \mathbb{N}$ and $x \in [0,1]$. Also let $f := 0$ on $[0,1]$. Then each f_n and f are continuously differentiable functions on $[0,1]$. Since $f_n'(x) = -nxe^{-nx} \le 0$ for all $x \in [0,1]$ and $n \in \mathbb{N}$, we see that each f_n is a monotonically decreasing function on $[0,1]$. Hence

$$0 \le f_n(x) \le f_n(0) = \frac{1}{n} \quad \text{for all } n \in \mathbb{N} \text{ and } x \in [0,1],$$

and so $f_n \to f$ uniformly on $[0,1]$ by Proposition 10.3. Also, we have seen in the subsection "Uniform Convergence and Continuity" that $f_n' \to f'$ on $[0,1]$, but the convergence is not uniform. Thus the converse of Proposition 10.13 is not true.

Remarks 10.14. (i) The hypothesis that (f_n) converges at one point of $[a,b]$ cannot be dropped from Proposition 10.13, as the following example shows. For $n \in \mathbb{N}$ and $x \in [0,1]$, let $f_n(x) := x + n$. Then $f_n'(x) = 1$ for all $n \in \mathbb{N}$ and $x \in [0,1]$, but (f_n) diverges at every point of $[0,1]$.

(ii) In Proposition 10.13, the continuity of the derivative of each f_n enabled us to use the Fundamental Theorem of Calculus, and also Proposition 10.9 on Riemann integration. It is possible to drop this continuity requirement and prove a similar result. See [71, Theorem 7.17]. ◇

Remark 10.15. We summarize below, without using any mathematical notation, the results in Propositions 10.5, 10.6, 10.9, 10.12, and 10.13 concerning uniform convergence of a sequence.

(i) The uniform limit of a sequence of real-valued bounded functions defined on an arbitrary set is a bounded function, and the given sequence is uniformly bounded. Further, the supremum (respectively, infimum) of the limit is the limit of the sequence of the suprema (respectively, infima) of the terms.

(ii) The uniform limit of a sequence of real-valued continuous functions defined on a subset of $\mathbb{R}$ is continuous.

(iii) The uniform limit of a sequence of real-valued integrable functions defined on a closed and bounded interval in $\mathbb{R}$ is integrable, and its Riemann integral is the limit of the sequence of Riemann integrals of the terms.

(iv) An analogue of part (iii) above holds for improper integrals if the functions are dominated by a function whose improper integral is convergent.

(v) If a sequence of real-valued continuously differentiable functions defined on a closed and bounded interval is convergent at one point and if the "derived" sequence is uniformly convergent on that interval, then the given sequence of functions converges uniformly and the uniform limit is continuously differentiable, and further, its derivative is the limit of the sequence of derivatives of the terms. ◇

Let us mention that the results about the boundedness, continuity, integrability, and differentiability of the limit function given in the above theorem hold if only a subsequence converges uniformly on E, there being no need for the entire sequence to converge uniformly on E.

10.3 Uniform Convergence of Series

Let (f_k) be a sequence of real-valued functions defined on a set E, and let

$$S_n := f_1 + \cdots + f_n = \sum_{k=1}^{n} f_k \quad \text{for } n \in \mathbb{N}.$$

Then S_n is called the nth **partial sum function** of the series $\sum_{k\geq 1} f_k$. We say that the series $\sum_{k\geq 1} f_k$ **converges pointwise** on E if the sequence (S_n) of its partial sum functions converges pointwise on E. In this case, we define

$$S(x) := \lim_{n\to\infty} S_n(x) = \sum_{k=1}^{\infty} f_k(x) \quad \text{for } x \in E,$$

and call it the **sum** of the series at x. The resulting function $S : E \to \mathbb{R}$ is called the **sum function** of the series $\sum_{k\geq 1} f_k$, and we write $S = \sum_{k=1}^{\infty} f_k$.

The sequence (S_n) of partial sum functions of an infinite series of functions determines the sequence (f_k) of functions, where $f_k := S_k - S_{k-1}$ is the kth term of the series for $k \in \mathbb{N}$, and $S_0 := 0$ on E. Hence Examples 10.1 can be used to show that various properties of the partial sum functions may not carry over to the sum function of a pointwise convergent series of functions. This motivates us, as in the case of a sequence of functions, to consider a stronger concept of convergence of a series of functions.

We say that a series $\sum_{k\geq 1} f_k$ of real-valued functions on a set E **converges uniformly** on E if the sequence (S_n) of its partial sum functions converges uniformly on E, that is, if there is a function $S : E \to \mathbb{R}$ such that for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ satisfying

$$n \geq n_0 \text{ and } x \in E \Longrightarrow |S_n(x) - S(x)| < \epsilon.$$

In this case, we may also say that the series $\sum_{k\geq 1} f_k(x)$ **converges uniformly** for $x \in E$. Note that the natural number n_0 mentioned above is independent of $x \in E$, although it may depend on the given functions $f_1, f_2, \ldots$ and on ϵ.

Clearly, a series $\sum_{k\geq 1} f_k$ of real-valued functions defined on E converges uniformly on E if and only if there exist a function $S : E \to \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that the function $|S_n - S|$ is bounded on E for each $n \in \mathbb{N}$ with $n \geq n_0$, and $\alpha_n \to 0$, where $\alpha_n := \sup\{|S_n(x) - S(x)| : x \in E\}$ for $n \geq n_0$.

Results involving uniform convergence of a sequence of functions carry over to the corresponding results involving uniform convergence of a series of functions. We illustrate this observation in the following two propositions.

The test given below is useful for checking uniform convergence of a pointwise convergent series of functions. In effect, it says that an infinite series of functions converges uniformly if its tail is uniformly small.

Proposition 10.16 (Test for Uniform Convergence of Series). *Let $\sum_{k\geq 1} f_k$ be a pointwise convergent series of real-valued functions on a set E. Suppose there is a sequence (β_n) in $\mathbb{R}$ such that $\left|\sum_{k=n+1}^{\infty} f_k(x)\right| \leq \beta_n$ for all large $n\in\mathbb{N}$ and all $x \in E$. If $\beta_n \to 0$, then $\sum_{k\geq 1} f_k$ converges uniformly on E.*

Proof. Let (S_n) be the sequence of partial sum functions of the series, and let S denote the pointwise sum of the series. Then $|S_n(x) - S(x)| = \left|\sum_{k=n+1}^{\infty} f_k(x)\right|$ for $n \in \mathbb{N}$ and $x \in E$. So Proposition 10.3 yields the desired result. □

We now give a criterion for checking uniform convergence of a series of functions without knowing its pointwise convergence.

Proposition 10.17 (Cauchy Criterion for Uniform Convergence of Series). *Let (f_k) be a sequence of real-valued functions on a set E. Then the series $\sum_{k\geq 1} f_k$ converges uniformly on E if and only if its sequence of partial sum functions is uniformly Cauchy on E, that is, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ satisfying*

$$m \geq n \geq n_0 \textit{ and } x \in E \Longrightarrow \left|\sum_{k=n+1}^{m} f_k(x)\right| < \epsilon.$$

Proof. The result follows by applying Proposition 10.4 to the sequence (S_n) of partial sum functions of the series $\sum_{k=1}^{\infty} f_k$. □

The next two propositions give tests, due to Weierstrass and Dirichlet, that are useful for checking the uniform convergence of a series of functions. The first is particularly useful when the series happens to be absolutely convergent for each point, while the second is particularly useful when the series is pointwise convergent, but the convergence may not be absolute.

Proposition 10.18 (Weierstrass M-Test for Uniform Convergence of Series). *Let (f_k) be a sequence of real-valued functions on a set E. Suppose there is a sequence (M_k) of real numbers such that $|f_k(x)| \leq M_k$ for all $k \in \mathbb{N}$ and all $x \in E$, and the series $\sum_{k\geq 1} M_k$ is convergent. Then the series $\sum_{k\geq 1} f_k(x)$ converges uniformly for $x \in E$ and absolutely for each $x \in E$.*

Proof. Note that

$$\left|\sum_{k=n+1}^{m} f_k(x)\right| \leq \sum_{k=n+1}^{m} |f_k(x)| \leq \sum_{k=n+1}^{m} M_k \quad \text{for all } m \geq n \text{ in } \mathbb{N} \text{ and } x \in E.$$

Let $\epsilon > 0$ be given. By the Cauchy Criterion for the convergent series $\sum_{k\geq 1} M_k$ given in Section 9.1, there exists $n_0 \in \mathbb{N}$ such that $\sum_{k=n+1}^{m} M_k < \epsilon$ for all $m \geq n \geq n_0$. Now Propositions 10.17 and 9.11 yield the desired result. □

Examples 10.19. (i) Let us consider the geometric series $\sum_{k\geq 0} x^k$, where $x \in (-1, 1)$. Let $r \in (0, 1)$ and $M_k := r^k$ for $k = 0, 1, \ldots$. Then by the Weierstrass M-Test, the geometric series converges uniformly on $[-r, r]$, since $|x^k| \leq M_k$ for all $k = 0, 1, \ldots$ and all $x \in [-r, r]$, and the series $\sum_{k\geq 0} M_k$ is convergent.

(ii) For $k \in \mathbb{N}$, let $f_k(x) := (-1)^k(x + k)/k^2$ and $x \in [0, 1]$. We show that the series $\sum_{k\geq 1} f_k$ converges uniformly on $[0, 1]$. For $k \in \mathbb{N}$, let $g_k(x) := (-1)^k x/k^2$ for $x \in [0, 1]$, and let $M_k := 1/k^2$. Observe that $|g_k(x)| \leq M_k$ for $k \in \mathbb{N}$ and $x \in [0, 1]$, and that the series $\sum_{k\geq 1} M_k$ is convergent. So by the Weierstrass M-Test, the series $\sum_{k\geq 1} g_k(x)$ converges uniformly for $x \in [0, 1]$ and absolutely for each $x \in [0, 1]$. Also, the series of real numbers $\sum_{k\geq 1}(-1)^k/k$ converges (uniformly on $[0, 1]$), as shown in Example 9.1 (iv). Since $f_k(x) = g_k(x) + (-1)^k/k$ for $k \in \mathbb{N}$ and $x \in [0, 1]$, the series $\sum_{k\geq 1} f_k(x)$ converges uniformly for $x \in [0, 1]$, but it does not converge absolutely for any $x \in [0, 1]$, since $\sum_{k\geq 1} 1/k$ is divergent. ◇

Proposition 10.20 (Dirichlet Test for Uniform Convergence of Series). *Let (f_k) and (g_k) be sequences of real-valued functions defined on a set E satisfying the following conditions:*

- *$(f_k(x))$ is a monotonic sequence in $\mathbb{R}$ for each $x \in E$.*
- *$f_k \to 0$ uniformly on E.*
- *f_1 is bounded on E.*
- *(G_n) defined by $G_n := \sum_{k=1}^{n} g_k$ for $n \in \mathbb{N}$ is uniformly bounded on E.*

Then the series $\sum_{k\geq 1} f_k g_k$ converges uniformly on E. Moreover, for every $n \in \mathbb{N}$, the function f_n is bounded on E and

$$\left|\sum_{k=n}^{\infty} f_k(x)g_k(x)\right| \leq 2\alpha_n\beta \quad \textit{for all } x \in E,$$

where $\alpha_n := \sup\{|f_n(x)| : x \in E\}$ and $\beta := \sup\{|G_m(x)| : m \in \mathbb{N}$ and $x \in E\}$.

Proof. By the Dirichlet Test (Proposition 9.21), the series $\sum_{k\geq 1} f_k g_k$ converges pointwise on E, and if $\beta_1(x) := \sup\{|G_m(x)| : m \in \mathbb{N}\}$ for $x \in E$, then

$$\left|\sum_{k=n}^{\infty} f_k(x)g_k(x)\right| \leq 2|f_n(x)|\beta_1(x) \leq 2|f_n(x)|\beta \quad \text{for all } n \in \mathbb{N} \text{ and } x \in E.$$

Now for each $x \in E$, since $(f_k(x))$ is monotonic and $f_k(x) \to 0$, we see that either $f_1(x) \leq f_2(x) \leq \cdots \leq 0$ or $f_1(x) \geq f_2(x) \geq \cdots \geq 0$. Thus the boundedness of f_1 on E implies the boundedness of each f_n on E. Consequently,

$$\left|\sum_{k=n}^{\infty} f_k(x)g_k(x)\right| \leq 2\alpha_n\beta \quad \text{for all } n \in \mathbb{N} \text{ and } x \in E.$$

Since $f_n \to 0$ uniformly on E, we obtain $\alpha_n \to 0$. Hence by Proposition 10.16, the series $\sum_{k\geq 1} f_k g_k$ converges uniformly on E. □

A similar result, known as the **Abel Test for Uniform Convergence of Series**, is given in Exercise 10.16.

Corollary 10.21 (Leibniz Test for Uniform Convergence of Series). *Let (f_k) be a monotonic sequence of real-valued functions defined on a set E such that $f_k \to 0$ uniformly on E and f_1 is bounded on E. Then the series $\sum_{k\geq 1}(-1)^{k-1} f_k$ converges uniformly on E. Moreover, for every $n \in \mathbb{N}$, the function f_n is bounded on E and*

$$\left| \sum_{k=n}^{\infty} (-1)^{k-1} f_k(x) \right| \leq |f_n(x)| \quad \textit{for all } x \in E.$$

Proof. For $k \in \mathbb{N}$ and $x \in E$, define $g_k(x) := (-1)^{k-1}$. For $n \in \mathbb{N}$, let $G_n := \sum_{k=1}^{n} g_k$ be the partial sum function defined on E. Then $|G_n(x)| \leq 1$ for all $n \in \mathbb{N}$ and all $x \in E$. By Proposition 10.20, the series $\sum_{k\geq 1}(-1)^{k-1} f_k$ converges uniformly on E. Also, the estimate for $\left| \sum_{k=n}^{\infty}(-1)^{k-1} f_k(x) \right|$ follows from Corollary 9.22. ☐

Corollary 10.22 (Uniform Convergence of Trigonometric Series). *Let (a_k) be a monotonic sequence of real numbers such that $a_k \to 0$. Let $\delta \in \mathbb{R}$ with $0 < \delta \leq \pi$ and $E := \{x \in \mathbb{R} : \delta \leq |x| \leq \pi\}$. Then the series $\sum_{k\geq 1} a_k \sin kx$ and $\sum_{k\geq 1} a_k \cos kx$ converge uniformly for $x \in E$.*

Proof. Let $x \in E$. As we have seen in the proof of Corollary 9.23, each partial sum function of the series $\sum_{k\geq 1} \sin kx$ and also of the series $\sum_{k\geq 1} \cos kx$ is at most $1/|\sin(x/2)|$. Moreover, $1/|\sin(x/2)| = 1/\sin(|x/2|) \leq 1/\sin(\delta/2)$ for all $x \in E$. By Proposition 10.20, it follows that the series $\sum_{k\geq 1} a_k \sin kx$ and $\sum_{k\geq 1} a_k \cos kx$ converge uniformly for $x \in E$. ☐

We remark that in Corollary 10.22, we can also obtain estimates for $\left| \sum_{k=n}^{\infty} a_k \sin kx \right|$ and for $\left| \sum_{k=n}^{\infty} a_k \cos kx \right|$ as an immediate consequence of Proposition 10.20 with $\beta = 1/\sin(\delta/2)$. The series in Corollary 10.22 are instances of a general **trigonometric series** given by

$$a_0 + \sum_{k\geq 1} (a_k \cos kx + b_k \sin kx), \quad \text{where } a_0, a_1, a_2, \ldots, b_1, b_2, \ldots \text{ are in } \mathbb{R}.$$

Examples 10.23. (i) Let $p \in (0, \infty)$, and let $f_k(x) := x^k/k^p$ for $x \in [0, 1]$ and $k \in \mathbb{N}$. Then (f_k) is a monotonic sequence of bounded functions on $[0, 1]$. Since $|f_k(x)| \leq 1/k^p$ for $k \in \mathbb{N}$ and $x \in [0, 1]$, Proposition 10.3 shows that $f_k(x) \to 0$ uniformly for $x \in [0, 1]$. By Corollary 10.21, it follows that the series $\sum_{k\geq 1}(-1)^{k-1} x^k/k^p$ converges uniformly for $x \in [0, 1]$, and also that $\left| \sum_{k=n}^{\infty}(-1)^{k-1} x^k/k^p \right| \leq 1/n^p$ for $n \in \mathbb{N}$ and $x \in [0, 1]$.

(ii) For $x \in \mathbb{R}$ and $p \in (0, \infty)$, consider the series

$$\sum_{k \geq 1} \frac{\sin kx}{k^p} \quad \text{and} \quad \sum_{k \geq 1} \frac{\cos kx}{k^p}.$$

If $p > 1$, then they converge uniformly for $x \in \mathbb{R}$ by the Weierstrass M-Test (Proposition 10.18), since $|\sin kx| \leq 1$ and $|\cos kx| \leq 1$ for all $x \in \mathbb{R}$, and the series $\sum_{k \geq 1} 1/k^p$ is convergent. Next, let $p \leq 1$. If $\delta \in \mathbb{R}$ with $0 < \delta \leq \pi$ and $E := \{x \in \mathbb{R} : \delta \leq |x| \leq \pi\}$, then the above series converge uniformly on E by Corollary 10.22. However, they do not converge uniformly on $(0, 1/2]$. To see this, consider the partial sum functions

$$S_n(x) := \sum_{k=1}^{n} \frac{\sin kx}{k^p} \quad \text{and} \quad T_n(x) := \sum_{k=1}^{n} \frac{\cos kx}{k^p}$$

for $n \in \mathbb{N}$ and $x \in (0, 1/2]$. If $x_n := 1/2n$ for $n \in \mathbb{N}$, then

$$S_{2n}(x_n) - S_n(x_n) = \sum_{k=n+1}^{2n} \frac{\sin kx_n}{k^p} \geq \sum_{k=n+1}^{2n} \frac{\sin(1/2)}{(2n)^p} \geq \frac{\sin(1/2)}{2^p},$$

where the last inequality follows since $p \leq 1$, and likewise

$$T_{2n}(x_n) - T_n(x_n) = \sum_{k=n+1}^{2n} \frac{\cos kx_n}{k^p} \geq \sum_{k=n+1}^{2n} \frac{\cos 1}{(2n)^p} \geq \frac{\cos 1}{2^p}.$$

Thus the Cauchy Criterion for uniform convergence is violated. ◇

Given sequences (a_k) and (λ_k) in $\mathbb{R}$ such that $0 \leq \lambda_k \leq \lambda_{k+1}$ for all $k \in \mathbb{N}$ and $\lambda_k \to \infty$, the infinite series of real-valued functions

$$\sum_{k \geq 1} a_k e^{-\lambda_k x}$$

is known as a **Dirichlet series**. The case $\lambda_k := \ln k$ for $k \in \mathbb{N}$ gives an important instance of Dirichlet series, namely $\sum_{k \geq 1} a_k / k^x$. Also, the case $\lambda_k := k$ for $k \in \mathbb{N}$ gives the power series $\sum_{k \geq 1} a_k y^k$, where $y := e^{-x}$.

Corollary 10.24 (Uniform Convergence of Dirichlet Series). *Let (a_k) and (λ_k) be sequences in $\mathbb{R}$ such that $0 \leq \lambda_k \leq \lambda_{k+1}$ for all $k \in \mathbb{N}$ and $\lambda_k \to \infty$, and consider the Dirichlet series $\sum_{k \geq 1} a_k e^{-\lambda_k x}$.*

(i) *Suppose there exist $\alpha, \delta \in (0, \infty)$ and $x_0 \in \mathbb{R}$ such that $|a_k| \leq \alpha e^{\lambda_k x_0}$ and $\lambda_k \geq \delta k$ for all $k \in \mathbb{N}$. Let $x_1 \in (x_0, \infty)$. Then the Dirichlet series converges uniformly for $x \in [x_1, \infty)$ and absolutely for each $x \in [x_1, \infty)$.*

(ii) *Suppose there exists $x_0 \in \mathbb{R}$ such that the set $\left\{\sum_{k=1}^{n} a_k e^{-\lambda_k x_0} : n \in \mathbb{N}\right\}$ of partial sums is bounded in $\mathbb{R}$. Let $x_1 \in (x_0, \infty)$. Then the Dirichlet series converges uniformly for $x \in [x_1, \infty)$.*

Proof. (i) We note that

$$|a_k e^{-\lambda_k x}| \le (\alpha e^{\lambda_k x_0})(e^{-\lambda_k x_1}) = \alpha e^{-\lambda_k (x_1 - x_0)} \quad \text{for all } k \in \mathbb{N} \text{ and } x \in [x_1, \infty).$$

Now the series $\sum_{k \ge 1} e^{-\lambda_k (x_1 - x_0)}$ is convergent by the Root Test (Proposition 9.17 (i)), since $x_1 - x_0 > 0$ and

$$\left(e^{-\lambda_k (x_1 - x_0)}\right)^{1/k} = e^{-\lambda_k (x_1 - x_0)/k} \le e^{-\delta (x_1 - x_0)} < 1 \quad \text{for all } k \in \mathbb{N}.$$

Hence by the Weierstrass M-Test (Proposition 10.18), $\sum_{k \ge 1} a_k e^{-\lambda_k x}$ converges uniformly for $x \in [x_1, \infty)$ and absolutely for each $x \in [x_1, \infty)$.

(ii) Let $E_1 := [x_1, \infty)$. For $k \in \mathbb{N}$, define $f_k, g_k : E_1 \to \mathbb{R}$ by $f_k(x) := e^{-\lambda_k (x - x_0)}$ and $g_k(x) := a_k e^{-\lambda_k x_0}$ for $x \in E_1$. For each $x \in E_1$, the sequence $(f_k(x))$ is monotonically decreasing, since $\lambda_k \le \lambda_{k+1}$ for all $k \in \mathbb{N}$. Also, since $|f_k(x)| \le e^{-\lambda_k (x_1 - x_0)}$ for all $x \in E_1$, where $\lambda_k \to \infty$, we see that $f_k(x) \to 0$ uniformly for $x \in E_1$. Further, f_1 is clearly bounded on E_1. By our hypothesis, the sequence (G_n) of functions on E_1 defined by $G_n(x) := \sum_{k=1}^{n} a_k e^{-\lambda_k x_0}$ is uniformly bounded on E_1. Hence by the Dirichlet Test (Proposition 10.20), the series $\sum_{k \ge 1} a_k e^{-\lambda_k x}$ converges uniformly for $x \in E_1$. □

Remarks 10.25. (i) If in part (ii) of Corollary 10.24, we assume that the series $\sum_{k \ge 1} a_k e^{-\lambda_k x_0}$ is convergent, then proceeding as in the proof of this corollary, but using the Abel Test (Exercise 10.16) in place of the Dirichlet Test (Proposition 10.20), we can conclude that the series $\sum_{k \ge 1} f_k(x) g_k(x)$, that is, the Dirichlet series $\sum_{k \ge 1} a_k e^{-\lambda_k x}$, converges uniformly for $x \in [x_0, \infty)$.

(ii) Suppose $x_0 \in \mathbb{R}$ is such that the series $\sum_{k \ge 1} a_k e^{-\lambda_k x_0}$ is convergent. Then the set $\left\{\sum_{k=1}^{n} a_k e^{-\lambda_k x_0} : n \in \mathbb{N}\right\}$ is a bounded subset of $\mathbb{R}$, and hence by part (ii) of Corollary 10.24, the Dirichlet series $\sum_{k \ge 1} a_k e^{-\lambda_k x}$ is convergent for each $x > x_0$. This also shows that if $x_0 \in \mathbb{R}$ is such that the series $\sum_{k \ge 1} a_k e^{-\lambda_k x_0}$ is divergent, then the series $\sum_{k \ge 1} a_k e^{-\lambda_k x}$ is divergent for every $x < x_0$. Let us consider the set $E := \big\{x \in \mathbb{R} : \sum_{k \ge 1} a_k e^{-\lambda_k x} \text{ is convergent}\big\}$. Suppose $E \ne \emptyset$ and $E \ne \mathbb{R}$. Then the set E is in fact bounded below, and we define $\xi := \inf E$. By convention, we let $\xi := \infty$ if $E = \emptyset$, and $\xi := -\infty$ if $E = \mathbb{R}$. Then ξ has the property that the Dirichlet series converges at every $x \in \mathbb{R}$ with $x > \xi$, and it diverges at every $x \in \mathbb{R}$ with $x < \xi$. We call ξ the **abscissa of convergence** of the given Dirichlet series. When $\xi \in \mathbb{R}$, the Dirichlet series may converge or it may diverge at ξ. For example, $\xi = 0$ for both the Dirichlet series $\sum_{k \ge 1} e^{-kx}/k$ and $\sum_{k \ge 1} e^{-kx}/k^2$, but the first series diverges at $x = \xi$, while the second series converges at $x = \xi$. Also, Example 9.1 (iii) shows that $\xi = 1$ for the Dirichlet series $\sum_{k \ge 1} 1/k^x$, and that the series diverges at $x = \xi$.

(iii) Under the hypotheses of part (ii) of Corollary 10.24, the sum function of a Dirichlet series can be shown to be infinitely differentiable on (x_0, ∞), and its nth derivative can be found for each $n \in \mathbb{N}$. See Exercise 10.45. ◇

Results regarding boundedness, continuity, integrability, and differentiability of the sum function of a convergent series of functions can be deduced from the corresponding results for the sequence of its partial sums. Thus we obtain the following results in respect of uniformly convergent series of functions.

Theorem 10.26.

(i) *The sum function of a uniformly convergent series of real-valued bounded functions defined on an arbitrary set is bounded, and the sequence of partial sum functions of the series is uniformly bounded.*

(ii) *The sum function of a uniformly convergent series of real-valued continuous functions defined on a subset E of $\mathbb{R}$ is continuous, and so if $\sum_{k\geq 1} f_k$ is uniformly convergent on E and (x_m) is a convergent sequence in E, then*

$$\lim_{m\to\infty} \sum_{k=1}^{\infty} f_k(x_m) = \sum_{k=1}^{\infty} \lim_{m\to\infty} f_k(x_m).$$

(iii) *The sum function of a uniformly convergent series of integrable functions defined on $[a,b]$ is integrable, and the series can be integrated term by term, that is, if $\sum_{k\geq 1} f_k$ is uniformly convergent on $[a,b]$ and if each f_k is integrable on $[a,b]$, then the function $\sum_{k=1}^{\infty} f_k$ is integrable on $[a,b]$ and*

$$\int_a^b \left(\sum_{k=1}^{\infty} f_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)dx.$$

(iv) *An analogue of part* (iii) *above holds for improper integrals if the partial sum functions of the series are dominated by a nonnegative function whose improper integral is convergent, that is, if $\sum_{k\geq 1} f_k$ is uniformly convergent on $[a,b]$ for every $b \in [a,\infty)$, and if there is $G : [a,\infty) \to \mathbb{R}$ such that $\left|\sum_{k=1}^{n} f_k\right| \leq G$ on $[a,\infty)$ for all $n \in \mathbb{N}$ and such that $\int_{x\geq a} G(x)dx$ is convergent, and further, if each improper integral $\int_{x\geq a} f_k(x)dx$ is convergent, then the improper integral $\int_{x\geq a} \left(\sum_{k=1}^{\infty} f_k(x)\right)dx$ is convergent, and*

$$\int_a^\infty \left(\sum_{k=1}^{\infty} f_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^\infty f_k(x)dx.$$

(v) *If a series of real-valued continuously differentiable functions defined on $[a,b]$ is convergent at one point of $[a,b]$ and if the "derived" series is uniformly convergent on $[a,b]$, then the given series converges uniformly on $[a,b]$, the sum function is continuously differentiable on $[a,b]$, and the series can be differentiated term by term, that is, if $\sum_{k\geq 1} f_k$ converges at one point of $[a,b]$, each f_k is continuously differentiable on $[a,b]$, and the series $\sum_{k\geq 1} f_k'$ is uniformly convergent on $[a,b]$, then the series $\sum_{k\geq 1} f_k$ converges uniformly to a continuously differentiable function on $[a,b]$, and*

$$\left(\sum_{k=1}^{\infty} f_k \right)'(x) = \sum_{k=1}^{\infty} f_k'(x) \quad \textit{for each } x \in [a,b].$$

Proof. We note that if a real-valued function f_k is bounded on a set E for each $k \in \mathbb{N}$, then so is the function $S_n := \sum_{k=1}^{n} f_k$ for each $n \in \mathbb{N}$. Similar results hold for continuity, integrability, and differentiability of functions. Also, the operations of Riemann integration and differentiation are "additive", that is, for each $n \in \mathbb{N}$,

$$\int_a^b \left(\sum_{k=1}^{n} f_k(x) \right) dx = \sum_{k=1}^{n} \int_a^b f_k(x) dx \quad \text{and} \quad \left(\sum_{k=1}^{n} f_k \right)' = \sum_{k=1}^{n} f_k'.$$

As a consequence, we may apply the results in Remark 10.15 to the sequence (S_n) of partial sum functions of the given series to obtain (i)–(v) above. □

Examples 10.27. (i) Let $p \in (0, \infty)$, and let $S(x) := \sum_{k=1}^{\infty} (-1)^{k-1} x^k / k^p$ for $x \in [0, 1]$. From Example 10.23 (i) and part (ii) of Theorem 10.26, we see that S is a continuous function on $[0, 1]$.

(ii) For $k = 0, 1, 2, \ldots$ and $t \in (-1, 1)$, let $f_k(t) := t^{2k}$ and $g_k(t) := (-1)^k$. Then $f_{k-1} \geq f_k$ for all $k \in \mathbb{N}$, and thus (f_k) is a monotonic sequence of bounded functions on $(-1, 1)$. Let $0 < r < 1$. As seen in Example 10.19 (i), $f_k \to 0$ uniformly on $[-r, r]$. By the Leibniz Test (Corollary 10.21), the series $\sum_{k \geq 0} (-1)^k t^{2k}$ converges uniformly for $t \in [-r, r]$. Also, the sum of this geometric series is $1/(1 + t^2)$ for all $t \in (-1, 1)$. Let $x \in (-1, 1)$, and let $r := |x|$. Integrating term by term, we obtain

$$\begin{aligned} \int_0^x \frac{1}{1+t^2} dt &= \int_0^x \left(\sum_{k=0}^{\infty} (-1)^k t^{2k} \right) dt \\ &= \sum_{k=0}^{\infty} \int_0^x (-1)^k t^{2k} dt \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}. \end{aligned}$$

This gives us the power series expansion of the arctan function on $(-1, 1)$.

(iii) For $k = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$, let $f_k(x) := x^k / k!$. Since $f_0(0) = 1$ and $f_k(0) = 0$ for all $k \in \mathbb{N}$, the series $\sum_{k=0}^{\infty} f_k(0)$ is clearly convergent. The derived series $\sum_{k \geq 0} f_k'$ is given by

$$\sum_{k \geq 0} f_k'(x) = \sum_{k \geq 1} \frac{x^{k-1}}{(k-1)!} = \sum_{k \geq 0} \frac{x^k}{k!} \quad \text{for } x \in \mathbb{R},$$

which is the same as the series $\sum_{k \geq 0} f_k$. Let $r \in (0, \infty)$. Now $|f_k(x)| \leq r^k / k!$ for $k = 0, 1, 2, \ldots$ and $x \in [-r, r]$. We have seen in Example 9.25 (ii) that the series $\sum_{k \geq 0} r^k / k!$ is convergent. So by the Weierstrass M-Test (Proposition 10.18), the derived series $\sum_{k \geq 0} f_k'$ converges uniformly on

$[-r, r]$. Hence by part (v) of Theorem 10.26, the series $\sum_{k\geq 0} f_k$ converges uniformly on $[-r, r]$. Let $S(x)$ denote the sum of this series for $x \in [-r, r]$. Differentiating term by term, we obtain

$$S'(x) = \sum_{k=1}^{\infty} f_k'(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = S(x) \quad \text{for } x \in [-r, r].$$

Let us define $T(x) := S(x)e^{-x}$ for $x \in \mathbb{R}$. Then $T'(x) = S'(x)e^{-x} + S(x)(-e^{-x}) = 0$ for all $x \in [-r, r]$. Also, $T(0) = S(0)e^0 = 1$. Hence $T(x) = 1$, that is, $S(x) = e^x$ for all $x \in [-r, r]$. Since $r > 0$ is arbitrary, we obtain a power series expansion of the exponential function on $\mathbb{R}$. This expansion was also obtained in Example 9.34 (ii) as the Taylor series of the exponential function around 0. $\diamond$

Remark 10.28. It follows from part (ii) of Theorem 10.26 and from part (a) of Example 10.1 (v) that the uniform sum of a series of continuous functions is continuous, but the uniform sum of a series of differentiable functions need not be differentiable. This can be used to construct an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ that is nowhere differentiable, that is, $f'(c)$ does not exist for each $c \in \mathbb{R}$. We refer to the book of Rudin [71, Theorem 7.18] or of Gelbaum and Olmsted [32, Ex. 8, Ch. 3] for such examples. $\diamond$

Uniform Convergence of Power Series

The series of functions in Examples 10.27 are particular cases of power series $\sum_{k\geq 0} c_k x^k$, where $c_k \in \mathbb{R}$ for $k = 0, 1, 2, \ldots$, considered in Section 9.3. We now prove an important result regarding the uniform convergence of a power series. Recall that the radius of convergence of a power series is ∞ if the power series is absolutely convergent for all $x \in \mathbb{R}$; otherwise, its radius of convergence is defined to be the unique nonnegative real number r such that the power series is absolutely convergent for each $x \in \mathbb{R}$ satisfying $|x| < r$ and divergent for each $x \in \mathbb{R}$ satisfying $|x| > r$. (See Proposition 9.27.)

Proposition 10.29 (Uniform Convergence of Power Series). *Let r be the radius of convergence of a power series $\sum_{k\geq 0} c_k x^k$. If $s \in \mathbb{R}$ is such that $0 < s < r$, then the power series converges uniformly on $[-s, s]$. Let $S : (-r, r) \to \mathbb{R}$ denote the sum function of the power series given by $S(x) := \sum_{k=0}^{\infty} c_k x^k$ for $x \in (-r, r)$. Then*

(i) *S is continuous on $(-r, r)$.*
(ii) *For every $x \in (-r, r)$,*

$$\int_0^x S(t)dt = \sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}.$$

(iii) *S is in fact differentiable on $(-r, r)$ and*

$$S'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1} \quad \text{for } x \in (-r, r).$$

Further, S is infinitely differentiable on $(-r, r)$, and $c_k = S^{(k)}(0)/k!$ for $k \in \mathbb{N}$. Consequently, $\sum_{k \geq 0} c_k x^k$ is the Taylor series of S around 0.

Proof. Let $s \in (0, r)$. Then the series $\sum_{k \geq 0} |c_k| s^k$ is convergent. Moreover, $|c_k x^k| \leq |c_k| s^k$ for all $x \in [-s, s]$. Thus the Weierstrass M-Test (Proposition 10.18) shows that the power series converges uniformly on $[-s, s]$.

(i) Since each term of the power series is a continuous function on $[-s, s]$, part (ii) of Theorem 10.26 shows that S is continuous on $[-s, s]$. Since this holds for every $s \in (0, r)$, it follows that S is continuous on $(-r, r)$.

(ii) Let $x \in (-r, r)$. Since each term of the power series is an integrable function on the closed interval between 0 and x, part (iii) of Theorem 10.26 shows that S is integrable on this interval, and

$$\int_0^x S(t)dt = \sum_{k=0}^{\infty} \int_0^x c_k t^k dt = \sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}.$$

(iii) Let $x \in (-r, r)$, and choose $s \in \mathbb{R}$ such that $|x| < s < r$. We claim that the series $\sum_{k \geq 1} k c_k s^{k-1}$ is absolutely convergent. Let $t \in \mathbb{R}$ be such that $s < t < r$. Define $a_k := k c_k s^{k-1}$ and $b_k := |c_k| t^k$ for $k \in \mathbb{N}$. Then

$$|a_k| = b_k \frac{k}{s} \left(\frac{s}{t}\right)^k \quad \text{for all } k \in \mathbb{N}.$$

The series $\sum_{k \geq 1} b_k$ is convergent, since $0 < t < r$. Also, $k(s/t)^k \to 0$, since $0 < (s/t) < 1$. (See Example 2.7 (i).) Hence $(k/s)(s/t)^k \leq 1$, and so $|a_k| \leq b_k$ for all large $k \in \mathbb{N}$. By the Comparison Test (Proposition 9.11), we see that the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, as claimed. Now the Weierstrass M-Test shows that the derived series $\sum_{k \geq 1} k c_k x^{k-1}$ converges uniformly for $x \in [-s, s]$. Hence by part (v) of Theorem 10.26, S is differentiable on $[-s, s]$, and $S'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}$ for $x \in [-s, s]$.

The above argument successively shows that $S', S'', \ldots$ exist, and for $j \in \mathbb{N}$,

$$S^{(j)}(x) = \sum_{k=j}^{\infty} k(k-1)\cdots(k-j+1) c_k x^{k-j} \quad \text{for } j \in \mathbb{N} \text{ and } x \in (-r, r).$$

In particular, $S^{(j)}(0) = j(j-1)\cdots 2 \cdot 1 \cdot c_j = j! c_j$, that is, $c_j = S^{(j)}(0)/j!$ for $j \in \mathbb{N}$. Since $c_0 = S(0)$ as well, we see that $\sum_{k \geq 0} c_k x^k = \sum_{k \geq 0} S^{(k)}(0) x^k / k!$, which is the Taylor series of the function S around 0. $\square$

We have already illustrated the above results in Examples 10.27. Also, see Exercises 10.20, 10.21, and 10.22. Similar results hold for a power series around an arbitrary point $a \in \mathbb{R}$, namely for the series $\sum_{k\geq 0} c_k(x-a)^k$.

We proved the following result in Chapter 9 (Proposition 9.31). We now give an alternative proof.

Corollary 10.30. *Let r be the radius of convergence of the power series $\sum_{k\geq 0} c_k x^k$. Then r is also the radius of convergence of the "integrated" power series $\sum_{k\geq 0} c_k x^{k+1}/(k+1)$ and of the "derived" power series $\sum_{k\geq 1} kc_k x^{k-1}$.*

Proof. Let r_0 and r_1 be the radii of convergence of $\sum_{k\geq 0} c_k x^{k+1}/(k+1)$ and of $\sum_{k\geq 1} kc_k x^{k-1}$ respectively. Parts (ii) and (iii) of Proposition 10.29 show that $r_0 \geq r$ and $r_1 \geq r$. On the other hand, applying part (ii) of Proposition 10.29 to $S_1(x) := \sum_{k=1}^{\infty} kc_k x^{k-1}$, we see that $r \geq r_1$, while applying part (iii) of Proposition 10.29 to $S_0(x) := \sum_{k=0}^{\infty} c_k x^{k+1}/(k+1)$, we see that $r \geq r_0$. It follows that $r_0 = r = r_1$. □

Remark 10.31. Let $\sum_{k\geq 0} c_k x^k$ be a power series with a positive radius of convergence r, and let $S : (-r, r) \to \mathbb{R}$ denote its sum function. As we have seen in Section 9.3, the power series may or may not converge at $x = r$. If it does converge, then the sum function S can be defined at $x = r$ as well, and it is in fact continuous at $x = r$. (See Exercise 10.44 (i).) Even if a power series does not converge at $x = r$ but the sequence of its partial sum functions is bounded at $x = r$ and we let $S(r) := 0$, then S is integrable on $[0, r]$, and $\int_0^r S(x)dx$ is obtained on integrating term by term. (See Exercise 10.44 (ii).) An example of such a power series is $\sum_{k\geq 0}(-1)^k x^k$, for which $r = 1$. ◇

10.4 Weierstrass Approximation Theorems

In this section, we shall prove two celebrated results of Weierstrass about (real-valued) continuous functions defined on a closed and bounded interval in $\mathbb{R}$. In Proposition 10.6, we saw that the uniform limit of a sequence of continuous functions on a subset E of $\mathbb{R}$ is continuous, and part (ii) of Theorem 10.26 shows that the sum function of a uniformly convergent series of continuous functions is continuous. Let us now reverse the procedure and inquire whether a given continuous function is the uniform limit of a sequence of some special continuous functions, and whether it is the sum function of a uniformly convergent series of some special continuous functions. These special functions should be conceptually easy to deal with and should also be computationally simple to handle. If E is a closed and bounded interval in $\mathbb{R}$, two natural choices for such special continuous functions are the polynomial functions, and the so-called trigonometric polynomial functions.

Uniform Approximation by Polynomials

Weierstrass showed that every real-valued continuous function on a closed and bounded interval in $\mathbb{R}$ is the uniform limit of a sequence of polynomial functions. We shall give a constructive proof of this result involving a particular kind of polynomial function introduced by Bernstein.

Given a function $f : [0,1] \to \mathbb{R}$ and $n \in \mathbb{N}$, we define the nth **Bernstein polynomial function** $B_n(f) : [0,1] \to \mathbb{R}$ associated with f by

$$B_n(f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

Note that $B_n(f)(0) = f(0)$, $B_n(f)(1) = f(1)$, and given any $g : [0,1] \to \mathbb{R}$, $B_n(\alpha f + \beta g) = \alpha B_n(f) + \beta B_n(g)$ for all $\alpha, \beta \in \mathbb{R}$ and for each $n \in \mathbb{N}$. Also, if $f \geq 0$ on $[0,1]$, then $B_n(f) \geq 0$ on $[0,1]$ for each $n \in \mathbb{N}$.

Define ϕ_0, ϕ_1, $\phi_2 : [0,1] \to \mathbb{R}$ by

$$\phi_0(x) := 1, \quad \phi_1(x) := x, \quad \text{and} \quad \phi_2(x) := x^2 \quad \text{for } x \in [0,1].$$

By the well-known Binomial Theorem,

$$\sum_{k=0}^{n} \binom{n}{k} s^k t^{n-k} = (s+t)^n \quad \text{for all } n \in \mathbb{N} \text{ and } s, t \in \mathbb{R}.$$

We shall refer to the above equality as the **binomial equality**, and we shall use it to find $B_n(\phi_j)$ for $n \in \mathbb{N}$ and $j = 1, 2, 3$.

Lemma 10.32. *For all $n \in \mathbb{N}$,*

$$B_n(\phi_0) = \phi_0, \quad B_n(\phi_1) = \phi_1, \quad \textit{and} \quad B_n(\phi_2) = \frac{n-1}{n}\phi_2 + \frac{1}{n}\phi_1.$$

Consequently, $B_n(\phi_j) \to \phi_j$ for each $j = 0, 1, 2$. Further,

$$\sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \frac{x(1-x)}{n} \quad \textit{for all } n \in \mathbb{N} \textit{ and } x \in [0,1].$$

Proof. Let $n \in \mathbb{N}$ and $x \in [0,1]$. By the binomial equality with $s = x = 1 - t$,

$$B_n(\phi_0)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = \big(x + (1-x)\big)^n = 1 = \phi_0(x).$$

In the binomial equality, keeping t fixed and differentiating both sides with respect to s, and then multiplying both sides by s/n, we obtain

$$\sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} s^k t^{n-k} = (s+t)^{n-1} s \quad \text{for all } n \in \mathbb{N} \text{ and } s, t \in \mathbb{R}.$$

Letting $s := x$ and $t := 1 - x$, we see that

$$B_n(\phi_1)(x) = \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} = x = \phi_1(x).$$

Again, in the penultimate equality, keeping t fixed and differentiating with respect to s, and then multiplying both sides by s/n, we obtain

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n}\right)^2 s^k t^{n-k} = \frac{n-1}{n}(s+t)^{n-2}s^2 + \frac{1}{n}(s+t)^{n-1}s \quad \text{for all } s, t \in \mathbb{R}.$$

Letting $s := x$ and $t := 1 - x$, we see that

$$B_n(\phi_2)(x) \ = \ \frac{n-1}{n}x^2 + \frac{1}{n}x \ = \ \frac{n-1}{n}\phi_2(x) + \frac{1}{n}\phi_1(x).$$

Thus $B_n(\phi_0) = \phi_0$, $B_n(\phi_1) = \phi_1$, and $B_n(\phi_2) = (n-1)\phi_2/n + \phi_1/n$ for all $n \in \mathbb{N}$. Consequently, $B_n(\phi_j) \to \phi_j$ for each $j = 0, 1, 2$. Also,

$$\begin{aligned}\sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} &= \sum_{k=0}^{n} \left(x^2 - \frac{2k}{n}x + \frac{k^2}{n^2}\right) \binom{n}{k} x^k (1-x)^{n-k} \\ &= x^2 - 2x^2 + \frac{n-1}{n}x^2 + \frac{x}{n} = \frac{x(1-x)}{n}. \quad \square\end{aligned}$$

Proposition 10.33 (Bernstein Theorem). *Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then the sequence $(B_n(f))$ of Bernstein polynomial functions associated with f converges uniformly to f on $[0,1]$.*

Proof. By part (i) of Proposition 3.10, the function f is bounded. Let $\alpha \in \mathbb{R}$ be such that $|f(x)| \le \alpha$ for all $x \in [0,1]$. Also, by Proposition 3.20, f is uniformly continuous on $[0,1]$. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon/2$ for all x, $y \in [0,1]$ satisfying $|x - y| < \delta$. Now let $n \in \mathbb{N}$ and $x \in [0,1]$. Since $B_n(\phi_0) = 1$,

$$\begin{aligned}B_n(f)(x) - f(x) &= B_n(f)(x) - f(x)B_n(\phi_0)(x) \\ &= \sum_{k=0}^{n} \left(f\left(\frac{k}{n}\right) - f(x)\right)\binom{n}{k} x^k (1-x)^{n-k}.\end{aligned}$$

Since $x^k(1-x)^{n-k} \ge 0$ for $k = 0, 1, \ldots, n$, we obtain

$$|B_n(f)(x) - f(x)| \le \sum_{k=0}^{n} \left|f\left(\frac{k}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k}.$$

Let us break up the sum over $k = 0, 1, \ldots, n$ on the right-hand side into two parts. Let $\sum'$ denote the sum over k for which $|(k/n) - x| < \delta$, and let $\sum''$ denote the sum over k for which $|(k/n) - x| \ge \delta$. Then

$$\sum{}' \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} < \frac{\epsilon}{2} \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = \frac{\epsilon}{2}.$$

Also, by Lemma 10.32,

$$\begin{aligned} &\sum{}'' \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} \\ &\leq 2\alpha \sum{}'' \binom{n}{k} x^k (1-x)^{n-k} \\ &\leq \frac{2\alpha}{\delta^2} \sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} \\ &= \frac{2\alpha}{\delta^2} \frac{x(1-x)}{n}, \end{aligned}$$

which is less than or equal to $\alpha/2n\delta^2$, since $x(1-x) \leq 1/4$ by the A.M-G.M. inequality (Proposition 1.11). Let $n_0 \in \mathbb{N}$ satisfy $n_0 > \alpha/\delta^2\epsilon$. Then

$$\sum{}'' \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} < \frac{\epsilon}{2} \quad \text{for all } n \geq n_0.$$

Thus $|B_n(f)(x) - f(x)| < (\epsilon/2) + (\epsilon/2) = \epsilon$ for all $n \geq n_0$. Since n_0 is independent of $x \in [0,1]$, we see that $B_n(f) \to f$ uniformly on $[0,1]$. □

It is evident from the above proof that if $f : [0,1] \to \mathbb{R}$ is a bounded function and if f is continuous at some $x \in [0,1]$, then $B_n(f)(x) \to f(x)$. The proof of the Bernstein Theorem given above is adapted from the article [45] of Kadison and Liu. They also show that $B_n(f)'(x) \to f'(x)$ if f is differentiable at $x \in [0,1]$, and $B_n(f)' \to f'$ uniformly on $[0,1]$ if f is continuously differentiable on $[0,1]$. (Compare Exercise 10.25.)

Theorem 10.34 (Weierstrass Polynomial Approximation Theorem). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then there is a sequence of polynomial functions on $[a,b]$ that converges uniformly to f on $[a,b]$.*

Proof. If $a = b$, then f itself is a polynomial function. Now let $a < b$. Define $\psi : [0,1] \to [a,b]$ by $\psi(x) := (b-a)x + a$ for $x \in [0,1]$. Then ψ is a one-one continuous function from $[0,1]$ onto $[a,b]$, and its inverse $\psi^{-1} : [a,b] \to \mathbb{R}$ is given by $\psi^{-1}(t) := (t-a)/(b-a)$ for $t \in [a,b]$. It is also one-one, continuous, and maps $[a,b]$ onto $[0,1]$. Define $g : [0,1] \to \mathbb{R}$ by $g := f \circ \psi$. Then g is a continuous function on $[0,1]$. By Proposition 10.33, there is a sequence (Q_n) of polynomial functions on $[0,1]$ such that $Q_n \to g$ uniformly on $[0,1]$, namely $Q_n := B_n(g)$ for $n \in \mathbb{N}$. Define $P_n := Q_n \circ \psi^{-1}$ for $n \in \mathbb{N}$. Then each P_n is a polynomial function on $[a,b]$. In fact,

$$P_n(t) = \frac{1}{(b-a)^n} \sum_{k=0}^{n} f\left(\frac{bk + (n-k)a}{n}\right) \binom{n}{k} (t-a)^k (b-t)^{n-k}$$

for $n \in \mathbb{N}$ and $t \in [a, b]$. Also, $P_n \to f$ uniformly on $[a, b]$, since for $t \in [a, b]$,

$$|P_n(t) - f(t)| = |Q_n(\psi^{-1}(t)) - g(\psi^{-1}(t))| = |Q_n(x) - g(x)|,$$

where $x := (t - a)/(b - a)$. $\square$

As a consequence of the above result, we see that every continuous function on $[a, b]$ (even if it is not differentiable at any point) can be uniformly approximated by a sequence of infinitely differentiable functions!

Uniform Approximation by Trigonometric Polynomials

Weierstrass showed that every real-valued continuous function on a closed and bounded interval having the same values at the endpoints is the uniform limit of a sequence of trigonometric polynomial functions. We shall give a constructive proof of this result using a particular kind of trigonometric polynomial function as suggested by Fejér.

Given a nonnegative integer n and real numbers $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$, the function $T_n : [-\pi, \pi] \to \mathbb{R}$ defined by

$$T_n(x) := a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

is called a **trigonometric polynomial function**. Since the functions cos and sin are continuous on $[-\pi, \pi]$ (Proposition 7.17), we see that T_n is a continuous function on $[-\pi, \pi]$. Also, we note that $T_n(\pi) = T_n(-\pi)$.

Suppose f is an integrable function on $[-\pi, \pi]$, and suppose there are real numbers $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots,$ such that the **trigonometric series**

$$a_0 + \sum_{k \geq 1} (a_k \cos kx + b_k \sin kx)$$

converges to $f(x)$ uniformly for $x \in [-\pi, \pi]$. Then f must be continuous on $[-\pi, \pi]$ by part (ii) of Theorem 10.26, and $f(\pi)$ must equal $f(-\pi)$. Next, by part (iii) of Theorem 10.26, we can integrate term by term so as to obtain

$$\int_{-\pi}^{\pi} f(x)dx = 2\pi a_0 + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos kx \, dx + b_k \int_{-\pi}^{\pi} \sin kx \, dx \right) = 2\pi a_0.$$

Further, the following relations can be easily verified. For all nonnegative integers k, j with $k \neq j$,

$$\int_{-\pi}^{\pi} (\sin kx)(\cos jx)dx = \int_{-\pi}^{\pi} (\cos kx)(\cos jx)dx = \int_{-\pi}^{\pi} (\sin kx)(\sin jx)dx = 0,$$

whereas for every positive integer k,

$$\int_{-\pi}^{\pi}(\sin kx)(\cos kx)dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi}\sin^2 kx\,dx = \pi = \int_{-\pi}^{\pi}\cos^2 kx\,dx.$$

(Compare Exercise 7.30.) Using these relations and integrating term by term, we obtain

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \pi a_k \quad \text{and} \quad \int_{-\pi}^{\pi} f(x)\sin kx\,dx = \pi b_k \quad \text{for each } k \in \mathbb{N}.$$

This motivates the following definitions. For an integrable real-valued function f on $[-\pi, \pi]$, the **Fourier coefficients** of f are defined by

$$a_0(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt,$$

$$a_k(f) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt, \quad \text{and} \quad b_k(f) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt \quad \text{for } k \in \mathbb{N}.$$

The series $a_0(f)+\sum_{k\geq 1}\big(a_k(f)\cos kx + b_k(f)\sin kx\big)$, whose terms are trigonometric polynomial functions on $[-\pi, \pi]$, is called the **Fourier series** of the function f. The orthogonality relations show that the Fourier series of a trigonometric polynomial function coincides with the function.

For a nonnegative integer n, let $S_n(f)$ denote the nth partial sum function of the Fourier series of f, that is, let

$$S_n(f)(x) := a_0(f) + \sum_{k=1}^{n}\big(a_k(f)\cos kx + b_k(f)\sin kx\big) \quad \text{for } x \in [-\pi, \pi].$$

The Fourier series of an integrable function $f : [-\pi, \pi] \to \mathbb{R}$ may not converge pointwise to f. In 1911, Kolmogorov gave an example of an *integrable* real-valued function f on $[-\pi, \pi]$ such that $S_n(f)(x) \not\to f(x)$ for every $x \in [-\pi, \pi]$. Earlier, in 1876, du Bois Raymond constructed a *continuous* real-valued function f on $[-\pi, \pi]$ satisfying $f(\pi) = f(-\pi)$ whose Fourier series does not converge pointwise. Methods of functional analysis can be used to establish that there is a large number of functions of this kind. (See, for example, [59, Theorem 9.4].) To improve our chances of approximating a function by trigonometric polynomial functions, we proceed as follows.

Let n be a nonnegative integer. Consider the arithmetic mean $\sigma_n(f)$ of the partial sums $S_0(f), S_1(f), \ldots, S_n(f)$ of the Fourier series of f, that is,

$$\sigma_n(f) := \frac{S_0(f) + S_1(f) + \cdots + S_n(f)}{n+1}.$$

In other words, $(\sigma_n(f))$ is the sequence of the Cesàro means of the Fourier series of f.

We note that $\sigma_n(f)(\pi) = \sigma_n(f)(-\pi)$, and if $g : [-\pi, \pi] \to \mathbb{R}$ is integrable, then $\sigma_n(\alpha f + \beta g) = \alpha\sigma_n(f) + \beta\sigma_n(g)$ for all $\alpha, \beta \in \mathbb{R}$ and each $n \geq 0$. We shall show that if $f \geq 0$ on $[-\pi, \pi]$, then $\sigma_n(f) \geq 0$ on $[-\pi, \pi]$ for $n \geq 0$.

Define $\psi_0, \psi_1, \psi_2 : [0,1] \to \mathbb{R}$ by

$$\psi_0(x) := 1, \quad \psi_1(x) := \cos x, \quad \text{and} \quad \psi_2(x) := \sin x \quad \text{for } x \in [-\pi, \pi].$$

Since $S_0(\psi_0)(x) = 1$, while $S_0(\psi_1)(x) = S_0(\psi_2)(x) = 0$ for all $x \in [-\pi, \pi]$, and since $S_k(\psi_j) = \psi_j$ for $k \in \mathbb{N}$ and $j = 0, 1, 2$, it follows that

$$\sigma_n(\psi_0) = \psi_0, \ \ \sigma_n(\psi_1) = \frac{n}{n+1}\psi_1, \ \ \text{and} \ \ \sigma_n(\psi_2) = \frac{n}{n+1}\psi_2 \quad \text{for all } n \in \mathbb{N}.$$

Consequently, $\sigma_n(\psi_j) \to \psi_j$ for each $j = 0, 1, 2$.

We now extend this result to every real-valued continuous function f on $[-\pi, \pi]$ satisfying $f(\pi) = f(-\pi)$. As a preparation, we take a closer look at the partial sums of the Fourier series of an integrable function on $[-\pi, \pi]$, and also at their arithmetic means.

Let $f : [-\pi, \pi] \to \mathbb{R}$ be integrable and satisfy $f(\pi) = f(-\pi)$. We extend the function f to $\mathbb{R}$ by requiring $f(x + 2\pi) = f(x)$ for $x \in \mathbb{R}$. Let n be a nonnegative integer. The definition of the Fourier coefficients of f yields

$$\begin{aligned} S_n(f)(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left(1 + 2\sum_{k=1}^{n}(\cos kt \cos kx + \sin kt \sin kx)\right)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left(1 + 2\sum_{k=1}^{n}\cos k(x-t)\right)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-s)\left(1 + 2\sum_{k=1}^{n}\cos ks\right)ds \end{aligned}$$

for $x \in [-\pi, \pi]$ by part (ii) of Proposition 7.18 and the substitution $t = x - s$.

We define the nth **Dirichlet kernel** D_n and the nth **Fejér kernel** K_n by

$$D_n(t) := 1 + 2\sum_{k=1}^{n}\cos kt \quad \text{and} \quad K_n(t) := \frac{1}{n+1}\sum_{k=0}^{n} D_k(t) \quad \text{for } t \in \mathbb{R}.$$

As we have shown above,

$$S_n(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)dt,$$

and hence we obtain the following integral representation for $\sigma_n(f)$:

$$\sigma_n(f)(x) = \frac{1}{2\pi(n+1)}\int_{-\pi}^{\pi} f(x-t)\left(\sum_{k=0}^{n} D_k(t)\right)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_n(t)dt.$$

We shall now derive closed-form expressions for the nth Dirichlet kernel D_n and the nth Fejér kernel K_n. Let $t \in \mathbb{R}$. Using the elementary identity

$$2\cos t_1 \sin t_2 = \sin(t_1 + t_2) - \sin(t_1 - t_2) \quad \text{for all } t_1, t_2 \in \mathbb{R},$$

and evaluating a finite telescoping sum, we obtain

$$D_n(t)\sin\frac{t}{2} = \left(1 + 2\sum_{k=1}^{n}\cos kt\right)\sin\frac{t}{2} = \sin\frac{(2n+1)t}{2}.$$

(See Exercise 7.30.) Hence, if t is not an integral multiple of 2π, then

$$D_n(t) = \sin\frac{(2n+1)t}{2}\bigg/\sin\frac{t}{2},$$

while if t is an integral multiple of 2π, then $D_n(t) = 1 + 2\sum_{k=0}^{n} 1 = 2n+1$.

Next, if t is not an integral multiple of 2π, then

$$K_n(t)\sin^2\frac{t}{2} = \frac{1}{n+1}\left(\sum_{k=0}^{n} D_k(t)\right)\sin^2\frac{t}{2} = \frac{1}{n+1}\left(\sum_{k=0}^{n}\sin\frac{(2k+1)t}{2}\right)\sin\frac{t}{2}.$$

Using $2\sin t_1 \sin t_2 = \cos(t_1 - t_2) - \cos(t_1 + t_2)$ for all $t_1, t_2 \in \mathbb{R}$, we obtain

$$\left(\sum_{k=0}^{n}\sin\frac{(2k+1)t}{2}\right)\sin\frac{t}{2} = \frac{1}{2}\sum_{k=0}^{n}\big(\cos kt - \cos(k+1)t\big) = \frac{1}{2}\big(1 - \cos(n+1)t\big),$$

which equals $\sin^2(n+1)t/2$. Hence, if t is not an integral multiple of 2π, then

$$K_n(t) = \sin^2\frac{(n+1)t}{2}\bigg/(n+1)\sin^2\frac{t}{2},$$

while if t is an integral multiple of 2π, then

$$K_n(t) = \frac{1}{n+1}\sum_{k=0}^{n}(2k+1) = \frac{1}{n+1}\big(n(n+1) + n + 1\big) = n+1.$$

Lemma 10.35 (Properties of the Fejér Kernel). *Let n be a nonnegative integer. Then*

(i) *K_n is a continuous function on $\mathbb{R}$.*

(ii) $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1.$

(iii) *$K_n(t) \geq 0$ for all $t \in \mathbb{R}$.*

(iv) *If $\delta > 0$, then* $\displaystyle\int_{\delta\leq|t|\leq\pi} K_n(t)dt \to 0$ *as $n \to \infty$.*

Proof. (i) Since each Dirichlet kernel D_k is clearly a continuous function on $\mathbb{R}$, so is $K_n = \left(\sum_{k=0}^{n} D_k\right)/(n+1)$.

(ii) Since $\int_{-\pi}^{\pi} D_k(t)dt = \int_{-\pi}^{\pi}\left(1 + 2\sum_{j=1}^{k}\cos jt\right)dt = 2\pi$ for each $k \geq 0$,

$$\int_{-\pi}^{\pi} K_n(t)dt = \frac{1}{n+1}\int_{-\pi}^{\pi}\left(\sum_{k=0}^{n} D_k(t)\right)dt = \frac{1}{n+1}\sum_{k=0}^{n}\int_{-\pi}^{\pi} D_k(t)dt = 2\pi.$$

(iii) Let $t \in \mathbb{R}$. If t is an integral multiple of 2π, then $K_n(t) = n+1 \geq 0$, and otherwise, $K_n(t) = (\sin^2(n+1)t/2)/(n+1)\sin^2(t/2) \geq 0$.

(iv) Let $\delta > 0$ be given. If $\delta \leq |t| \leq \pi$, then $\sin^2(\delta/2) \leq \sin^2(t/2)$, and so

$$0 \leq K_n(t) \leq \frac{1}{(n+1)\sin^2(\delta/2)} \to 0 \quad \text{as } n \to \infty.$$

Hence by Proposition 10.3, we see that $K_n(t) \to 0$ uniformly for $t \in E$, where $E := \{t \in [-\pi, \pi] : \delta \leq |t|\}$, and so by Proposition 10.9, $\int_E K_n(t)dt \to 0$. □

By part (iii) of the above proposition, if $f \geq 0$ on $[-\pi, \pi]$, then $\sigma_n(f) \geq 0$ on $[-\pi, \pi]$ for each $n = 0, 1, 2, \ldots$, as we had promised to show.

Proposition 10.36 (Fejér Theorem). *Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuous function such that $f(\pi) = f(-\pi)$. Then the sequence $(\sigma_n(f))$ of arithmetic means of the partial sum functions of the Fourier series of f converges uniformly to f on $[-\pi, \pi]$, that is, the Fourier series of f is* **uniformly Cesàro convergent** *on $[-\pi, \pi]$, and its Cesàro sum is equal to f.*

Proof. By part (i) of Proposition 3.10, the function f is bounded on $[-\pi, \pi]$. Let $\alpha \in \mathbb{R}$ be such that $|f(x)| \leq \alpha$ for all $x \in [-\pi, \pi]$. Since f is continuous on $[-\pi, \pi]$ and satisfies $f(-\pi) = f(\pi)$, the extended 2π-periodic function f is continuous on $\mathbb{R}$. In particular, by Proposition 3.20, f is uniformly continuous on $[-2\pi, 2\pi]$. Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that $\delta < \pi$ and

$$|f(u) - f(v)| < \frac{\epsilon}{2} \quad \text{for all } u,\, v \in [-2\pi, 2\pi] \text{ satisfying } |u - v| \leq \delta.$$

Let $n \in \mathbb{N}$ and $x \in [-\pi, \pi]$. By part (ii) of Lemma 10.35, $\int_{-\pi}^{\pi} K_n(t)dt = 2\pi$, and hence

$$\begin{aligned}\sigma_n(f)(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_n(t)dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)K_n(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \big(f(x-t) - f(x)\big)K_n(t)dt.\end{aligned}$$

Now, by part (iii) of Lemma 10.35,

$$|\sigma_n(f)(x) - f(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|K_n(t)dt.$$

Let us break up the integral over $[-\pi, \pi]$ on the right-hand side into two parts, one over the interval $[-\delta, \delta]$, and the other over the union of the intervals $[-\pi, -\delta]$ and $[\delta, \pi]$. First, note that if $|t| \leq \delta$, then $x - t$ is in $[-2\pi, 2\pi]$, since $x \in [-\pi, \pi]$ and $\delta < \pi$. Thus by part (ii) of Lemma 10.35,

$$\frac{1}{2\pi}\int_{|t|\le\delta}|f(x-t)-f(x)|K_n(t)dt \le \frac{1}{2\pi}\frac{\epsilon}{2}\int_{-\pi}^{\pi}K_n(t)dt = \frac{\epsilon}{2}.$$

Further,

$$\frac{1}{2\pi}\int_{\delta\le|t|\le\pi}|f(x-t)-f(x)|K_n(t)dt \le \frac{\alpha}{\pi}\int_{\delta\le|t|\le\pi}K_n(t)dt.$$

Let $n_0\in\mathbb{N}$ be such that $\int_{\delta\le|t|\le\pi}K_n(t)dt < \epsilon\pi/2\alpha$ for all $n\ge n_0$ by part (iv) of Lemma 10.35. Hence

$$\frac{1}{2\pi}\int_{\delta\le|t|\le\pi}|f(x-t)-f(x)|K_n(t)dt < \frac{\epsilon}{2} \quad \text{for all } n\ge n_0.$$

Thus $|\sigma_n(f)(x)-f(x)| < (\epsilon/2)+(\epsilon/2)=\epsilon$ for all $n\ge n_0$. Since n_0 is independent of $x\in[-\pi,\pi]$, we conclude that $\sigma_n(f)\to f$ uniformly on $[-\pi,\pi]$. □

Let $f:[-\pi,\pi]\to\mathbb{R}$ be an integrable function. Suppose $x\in(-\pi,\pi)$ and f is continuous at x. It is evident from the proof of Proposition 10.36 that $\sigma_n(f)(x)\to f(x)$, that is, the Fourier series of f is Cesàro convergent at x and its Cesàro sum is equal to $f(x)$. Also, if $x=\pm\pi$, then these results hold if $f(\pi)=\lim_{x\to\pi^-}f(x)=\lim_{x\to-\pi^+}f(x)=f(-\pi)$.

If, in addition to the hypothesis of the Fejér theorem, $ka_k(f)\to 0$ and $kb_k(f)\to 0$, then by an analogue of Corollary 9.7 for uniform convergence of sequences of functions given in Exercise 10.46, it can be shown that the Fourier series of f converges uniformly to f on $[-\pi,\pi]$. (See Exercises 10.47 and 10.48.)

Let f be continuously differentiable on $[-\pi,\pi]$ such that $f(\pi)=f(-\pi)$ and $f'(\pi)=f'(-\pi)$. Then $\sigma_n(f)'\to f'$ uniformly on $[-\pi,\pi]$, and the Fourier series of f converges uniformly to f on $[-\pi,\pi]$. (See Exercises 10.29 and 10.49.)

Theorem 10.37 (Weierstrass Trigonometric Polynomial Approximation Theorem). *Let $f:[-\pi,\pi]\to\mathbb{R}$ be a continuous function such that $f(\pi)=f(-\pi)$. Then there is a sequence of trigonometric polynomial functions on $[-\pi,\pi]$ that converges uniformly to f on $[-\pi,\pi]$.*

Proof. Let n be a nonnegative integer and $x\in[-\pi,\pi]$. Then

$$\begin{aligned}\sigma_n(f)(x) &= a_0(f)+\frac{1}{n+1}\sum_{k=1}^{n}\left(\sum_{j=1}^{k}a_j(f)\cos jx+b_j(f)\sin jx\right)\\ &= a_0(f)+\frac{1}{n+1}\sum_{j=1}^{n}\left(\sum_{k=j}^{n}a_j(f)\cos jx+b_j(f)\sin jx\right)\\ &= a_0(f)+\frac{1}{n+1}\sum_{j=1}^{n}(n-j+1)\big(a_j(f)\cos jx+b_j(f)\sin jx\big).\end{aligned}$$

Thus $\sigma_n(f)$ is a trigonometric polynomial function, and so the desired result follows from Proposition 10.36. □

An analogue of Theorem 10.37 can be proved for a continuous function f defined on an interval $[a, b]$ that satisfies $f(a) = f(b)$ in the same way as Theorem 10.34 was deduced from Proposition 10.33.

10.5 Bounded Convergence

In this section, we consider a mode of convergence that is stronger than pointwise convergence but is often weaker than uniform convergence.

Let E be a set, and let (f_n) be a sequence of real-valued functions defined on E. We say that (f_n) **converges boundedly** on E if (f_n) converges pointwise on E and is uniformly bounded on E. In this case, we may also say that (f_n) **converges boundedly** for $x \in E$. If (f_n) converges boundedly on E and if f is the pointwise limit of (f_n), we write $f_n \to f$ boundedly on E.

Pointwise convergent sequences that are not boundedly convergent are given in part (a) of Example 10.1 (i) and in parts (b) and (c) of Example 10.1 (iii), whereas boundedly convergent sequences that are not uniformly convergent are given in parts (b) and (c) of Example 10.1 (i), in Example 10.1 (ii), and in part (a) of Example 10.1 (iii). On the other hand, Proposition 10.5 shows that if (f_n) is a sequence of bounded functions defined on a set E and if $f_n \to f$ uniformly on E, then $f_n \to f$ boundedly on E. For an example of a sequence that converges uniformly but not boundedly, we may consider $E := \mathbb{R}$ and $f_n(x) := x + (1/n)$ for $n \in \mathbb{N}$ and $x \in E$.

Suppose a sequence (f_n) of functions defined on a set E converges boundedly to a function f on E. Then there exists $\alpha \in \mathbb{R}$ such that $|f_n(x)| \le \alpha$ for all $n \in \mathbb{N}$ and $x \in E$. Since $|f(x)| = \lim_{n\to\infty} |f_n(x)| \le \alpha$ for all $x \in E$, we see that f is bounded on E. However, Example 10.1 (ii) shows that a sequence of continuous functions can converge boundedly to a discontinuous function.

Further, part (a) of Example 10.1 (iii) shows that a sequence of real-valued integrable functions on $[a, b]$ can converge boundedly to a nonintegrable function on $[a, b]$. We shall prove an important theorem of Arzelà that says that if a sequence (f_n) of integrable functions on $[a, b]$ converges boundedly to a function f that is in fact integrable on $[a, b]$, then $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$. In view of Proposition 10.5, Arzelà's result improves upon Proposition 10.9, provided the limit function f is integrable. We shall prove a modification of this result for improper integrals, and also a companion result on differentiation. This will yield versions of the results in parts (iii), (iv), and (v) of Remark 10.15 for bounded convergence of a sequence of functions.

We begin with a preliminary result on Riemann integration that will be crucial in our proof of the theorem of Arzelà.

Lemma 10.38. *Let $f : [a, b] \to \mathbb{R}$ be a nonnegative bounded function, and let $\epsilon > 0$ be given. Then there is a nonnegative continuous piecewise linear function g on $[a, b]$ such that $g \le f$ on $[a, b]$ and*

$$L(f) < \int_a^b g(x)dx + \epsilon.$$

Proof. By the definition of the lower integral $L(f)$ of f, there is a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that $L(P, f) > L(f) - (\epsilon/2)$, where $L(P, f) = \sum_{i=1}^n m_i(x_i - x_{i-1})$ with $m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\} \geq 0$ for $i = 1, \ldots, n$. Choose $\delta > 0$ such that $2\delta < \min\{x_i - x_{i-1} : i = 1, \ldots, n\}$ and $2\delta \sum_{i=1}^n m_i < \epsilon$, and define $g : [a, b] \to \mathbb{R}$ as follows. (See Figure 10.1.) For $i = 1, \ldots, n$ and $x \in [x_{i-1}, x_i]$, let

$$g(x) := \begin{cases} m_i(x - x_{i-1})/\delta & \text{if } x_{i-1} \leq x < x_{i-1} + \delta, \\ m_i & \text{if } x_{i-1} + \delta \leq x \leq x_i - \delta, \\ m_i(x_i - x)/\delta & \text{if } x_i - \delta < x \leq x_i. \end{cases}$$

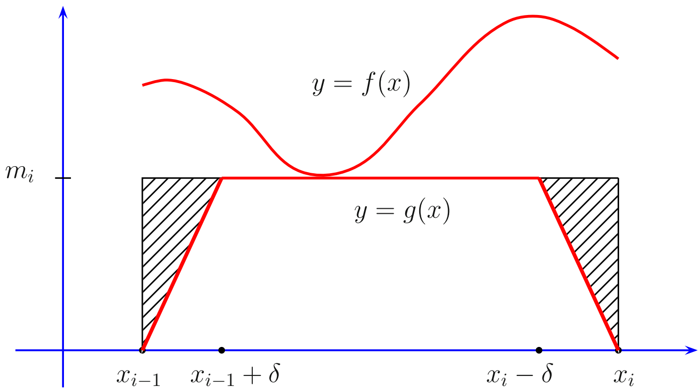

Fig. 10.1. Trapezoidal graph of the function g on $[x_{i-1}, x_i]$ in the proof of Lemma 10.38. The area of each shaded part is equal to $m_i\delta/2$.

Then $g(x_i) = 0$ for $i = 0, 1, \ldots, n$, and g is a nonnegative continuous piecewise linear function on $[a, b]$. Since $g(x) \leq m_i \leq f(x)$ for all $x \in [x_{i-1}, x_i]$ and $i = 1, \ldots, n$, we see that $g \leq f$ on $[a, b]$. Also,

$$\begin{aligned} \int_a^b g(x)dx &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} g(x)dx \\ &= \sum_{i=1}^n \left(\frac{1}{2}m_i\delta + m_i(x_i - \delta - x_{i-1} - \delta) + \frac{1}{2}m_i\delta\right) \\ &= \sum_{i=1}^n m_i(x_i - x_{i-1}) - \delta \sum_{i=1}^n m_i > L(P, f) - \frac{\epsilon}{2}. \end{aligned}$$

Hence $L(f) < L(P, f) + (\epsilon/2) < \int_a^b g(x)dx + \epsilon$, as desired. $\square$

The following result improves upon the Monotone Convergence Theorem for continuous functions (Corollary 10.11).

Proposition 10.39 (Monotone Convergence Theorem). *Let (f_n) be a monotonic sequence of real-valued integrable functions on $[a, b]$ such that $f_n \to f$ on $[a, b]$, where f is integrable on $[a, b]$. Then $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$.*

Proof. First suppose $f_n \geq f_{n+1}$ for all $n \in \mathbb{N}$. Then $f_n - f \geq f_{n+1} - f$, and $f_n - f$ is integrable on $[a, b]$ for each $n \in \mathbb{N}$. Replacing the sequence (f_n) by the sequence $(f_n - f)$ if necessary, we assume without loss of generality that $f = 0$ and $f_n \geq 0$ on $[a, b]$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$ be given. We shall construct a sequence (h_n) of continuous functions defined on $[a, b]$ such that for each $n \in \mathbb{N}$,

$$f \leq h_{n+1} \leq h_n \leq f_n \quad \text{and} \quad \int_a^b f_n(x)dx < \int_a^b h_n(x)dx + \epsilon,$$

and then use the Dini theorem (Proposition 10.7) for (h_n).

First note that for each $n \in \mathbb{N}$, by Lemma 10.38, there is a continuous function $g_n : [a, b] \to \mathbb{R}$ such that $0 \leq g_n \leq f_n$ and

$$\int_a^b f_n(x)dx = L(f_n) \leq \int_a^b g_n(x)dx + \frac{\epsilon}{2^n}.$$

For $n \in \mathbb{N}$, define $h_n := \min(g_1, \ldots, g_n)$. Fix $n \in \mathbb{N}$. Now by Corollary 3.4, $h_n : [a, b] \to \mathbb{R}$ is a continuous function. Also, $0 \leq h_n \leq g_n \leq f_n$. Consider $x \in [a, b]$, and let $i \in \{1, \ldots, n\}$ be such that $g_i(x) = \min\{g_1(x), \ldots, g_n(x)\}$. Since $g_n(x) \leq f_n(x) \leq f_i(x)$, we obtain

$$f_n(x) - h_n(x) = f_n(x) - g_i(x) \leq f_n(x) - g_n(x) + f_i(x) - g_i(x),$$

and so $0 \leq f_n(x) - h_n(x) \leq \sum_{k=1}^n \big(f_k(x) - g_k(x)\big)$ for all $x \in [a, b]$. Hence

$$0 \leq \int_a^b \big(f_n(x) - h_n(x)\big)dx \leq \sum_{k=1}^n \int_a^b \big(f_k(x) - g_k(x)\big)dx \leq \sum_{k=1}^n \frac{\epsilon}{2^k} < \epsilon.$$

Thus $0 \leq \int_a^b f_n(x)dx < \int_a^b h_n(x)dx + \epsilon$ for each $n \in \mathbb{N}$. Clearly, $h_n \geq h_{n+1}$ for all $n \in \mathbb{N}$, and $h_n \to 0$ on $[a, b]$, since $f_n \to 0$ on $[a, b]$. Hence by Corollary 10.11, $\int_a^b h_n(x)dx \to 0$, and so there exists $n_0 \in \mathbb{N}$ such that $\int_a^b h_n(x)dx < \epsilon$ for all $n \geq n_0$. It follows that $0 \leq \int_a^b f_n(x)dx < 2\epsilon$ for all $n \geq n_0$. Consequently, $\int_a^b f_n(x)dx \to 0$, that is, $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$.

Next, suppose $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$. Then $-f_n \geq -f_{n+1}$ for all $n \in \mathbb{N}$. By the argument given above, $\int_a^b -f_n(x)dx \to \int_a^b -f(x)dx$, which implies that $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$. $\square$

We are now in a position to prove the theorem of Arzelà mentioned earlier. Our proof uses the Monotone Convergence Theorem established above.

Proposition 10.40 (Arzelà Bounded Convergence Theorem). *Let (f_n) be a sequence of real-valued integrable functions on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ an integrable function such that $f_n \to f$ boundedly on $[a,b]$. Then*

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

Further, if we let $F_n(x) := \int_a^x f_n(t)dt$ and $F(x) := \int_a^x f(t)dt$ for $x \in [a,b]$, then $F_n \to F$ boundedly on $[a,b]$.

Proof. We note that $|f_n - f|$ is integrable on $[a,b]$ and $|f_n - f| \le |f_n| + |f|$, while $\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| \le \int_a^b |f_n - f|(x)dx$ for each $n\in\mathbb{N}$. Replacing (f_n) by $(|f_n - f|)$ if necessary, we assume without loss of generality that $f = 0$ and $f_n \ge 0$ on $[a,b]$ for all $n\in\mathbb{N}$. Since (f_n) is uniformly bounded, there exists $\alpha\in\mathbb{R}$ such that $0 \le f_n(x) \le \alpha$ for all $n\in\mathbb{N}$ and $x\in[a,b]$.

Let $\epsilon > 0$ be given. In order to show that $\int_a^b f_n(x)dx \to 0$, we shall construct a sequence (h_n) of integrable functions defined on $[a,b]$ such that

$$0 \le h_{n+1} \le h_n \quad \text{and} \quad \int_a^b f_n(x)dx < \int_a^b h_n(x)dx + \epsilon \text{ for each } n \in \mathbb{N},$$

and then appeal to the Monotone Convergence Theorem (Proposition 10.39).

For $n,m \in \mathbb{N}$ with $m \ge n$, define $\varphi_{n,m} := \max(f_n, f_{n+1}, \ldots, f_m)$. By Corollary 6.20, each $\varphi_{n,m}$ is integrable on $[a,b]$. Also, $0 \le \varphi_{n,m} \le \varphi_{n,m+1} \le \alpha$ for all $n,m\in\mathbb{N}$ with $m \ge n$. Hence for each fixed $n\in\mathbb{N}$, the sequence $\left(\int_a^b \varphi_{n,m}(x)dx\right)$ in $\mathbb{R}$ is monotonically increasing and is bounded above by $\alpha(b-a)$. So by Proposition 2.8, it is convergent, and by Proposition 2.22, it is Cauchy. Hence we can recursively obtain $m_1 < m_2 < \cdots$ in $\mathbb{N}$ such that

$$0 \le \int_a^b \varphi_{n,j}(x)dx - \int_a^b \varphi_{n,m_n}(x)dx < \frac{\epsilon}{2^n} \quad \text{for each } n\in\mathbb{N} \text{ and for all } j \ge m_n.$$

Note that $m_n \ge n$ for all $n\in\mathbb{N}$, since $m_1 \ge 1$ and $m_i < m_{i+1}$ for all $i\in\mathbb{N}$. For $n\in\mathbb{N}$, let us write $g_n := \varphi_{n,m_n}$. Then $0 \le f_n \le g_n$ for $n\in\mathbb{N}$. Also, since $f_n \to 0$ on $[a,b]$, it is easy to see that $g_n \to 0$ on $[a,b]$. Further, for $n\in\mathbb{N}$ and $i = 1,\ldots,n$,

$$g_n = g_i + (g_n - g_i) \le g_i + \big(\max(g_i,\ldots,g_n) - g_i\big) = g_i + \varphi_{i,m_n} - \varphi_{i,m_i},$$

since $m_i \le m_n$ and $\max(g_i,\ldots,g_n) = \max(f_i,\ldots,f_{m_n}) = \varphi_{i,m_n}$. Thus

$$g_n \le g_i + (\varphi_{i,m_n} - \varphi_{i,m_i}) \le g_i + \sum_{k=1}^{n} (\varphi_{k,m_n} - \varphi_{k,m_k}) \quad \text{for } i = 1,\ldots,n.$$

Considering the minimum over $i \in \{1, \ldots, n\}$, we obtain

$$g_n \leq \min(g_1, \ldots, g_n) + \sum_{k=1}^{n} (\varphi_{k,m_n} - \varphi_{k,m_k}).$$

Define $h_n := \min(g_1, \ldots, g_n)$. Since $m_n \geq m_k$ for $k = 1, \ldots, n$, we see that

$$\int_a^b \big(g_n(x) - h_n(x)\big)dx \leq \sum_{k=1}^{n} \int_a^b \big(\varphi_{k,m_n} - \varphi_{k,m_k}\big)dx \leq \sum_{k=1}^{n} \frac{\epsilon}{2^k} < \epsilon.$$

Thus $0 \leq \int_a^b f_n(x)dx \leq \int_a^b g_n(x)dx < \int_a^b h_n(x)dx + \epsilon$ for each $n \in \mathbb{N}$. Now h_n is an integrable function on $[a, b]$ and $h_n \geq h_{n+1}$ for each $n \in \mathbb{N}$. Also, $h_n \to 0$ on $[a, b]$, since $0 \leq h_n \leq g_n$ for all $n \in \mathbb{N}$ and $g_n \to 0$ on $[a, b]$. By the Monotone Convergence Theorem (Proposition 10.39), $\int_a^b h_n(x)dx \to 0$, and so there exists $n_0 \in \mathbb{N}$ such that $0 \leq \int_a^b h_n(x)dx < \epsilon$ for all $n \geq n_0$. It follows that $0 \leq \int_a^b f_n(x)dx < 2\epsilon$ for all $n \geq n_0$. Hence $\int_a^b f_n(x)dx \to 0 = \int_a^b f(x)dx$.

Finally, let $F_n(x) := \int_a^x f_n(t)dt$ and $F(x) := \int_a^x f(t)dt$ for $x \in [a, b]$. By what we have proved above, $F_n \to F$ on $[a, b]$. Also, since $0 \leq F_n(x) \leq \alpha(b-a)$ for all $x \in [a, b]$, we see that $F_n \to F$ boundedly on $[a, b]$. □

We now give a modification of Proposition 10.40 for improper integrals.

Proposition 10.41 (Arzelà Dominated Convergence Theorem). *Let $a \in \mathbb{R}$, and let (f_n) be a sequence of real-valued functions on $[a, \infty)$ such that $f_n \to f$ on $[a, \infty)$, where $f : [a, \infty) \to \mathbb{R}$ and each f_n is integrable on $[a, b]$ for every $b \in [a, \infty)$.*

(i) *Suppose there is a function $g : [a, \infty) \to \mathbb{R}$ such that $|f_n| \leq g$ on $[a, \infty)$ for all $n \in \mathbb{N}$ and $\int_{t \geq a} g(t)dt$ is convergent. Then each $\int_{t \geq a} f_n(t)dt$ and the $\int_{t \geq a} f(t)dt$ are convergent, and $\int_a^\infty f_n(t)dt \to \int_a^\infty f(t)dt$.*

(ii) *Suppose $0 \leq f_n \leq f$ on $[a, \infty)$ for each $n \in \mathbb{N}$. If $\int_{t \geq a} f(t)dt$ is convergent, then each $\int_{t \geq a} f_n(t)dt$ is convergent and $\int_a^\infty f_n(t)dt \to \int_a^\infty f(t)dt$, and further, if $\int_{t \geq a} f(t)dt$ diverges to ∞ and each $\int_{t \geq a} f_n(t)dt$ is convergent, then $\int_a^\infty f_n(t)dt \to \infty$.*

Proof. Let $b \in [a, \infty)$.

(i) Since $|f_n| \leq g$ for all $n \in \mathbb{N}$ and g is bounded on $[a, b]$, we see that $f_n \to f$ boundedly on $[a, b]$. Also, $|f(t)| = \lim_{n \to \infty} |f_n(t)| \leq g(t)$ for all $t \in [a, \infty)$. Hence by the Comparison Test for Improper Integrals (Proposition 9.46), each $\int_{t \geq a} f_n(t)dt$ and $\int_{t \geq a} f(t)dt$ are convergent. The proof of $\int_a^\infty f_n(t)dt \to \int_a^\infty f(t)dt$ proceeds exactly on the same lines as the proof of Proposition 10.12, except that we invoke the Arzelà Bounded Convergence Theorem (Proposition 10.40) in place of Proposition 10.9.

(ii) Since $0 \le f_n \le f$ for all $n \in \mathbb{N}$ and f is bounded on $[a,b]$, we see that $f_n \to f$ boundedly on $[a,b]$. If $\int_{t\ge a} f(t)dt$ is convergent, then let $g := f$ in part (i) above to obtain $\int_a^\infty f_n(t)dt \to \int_a^\infty f(t)dt$. Next, suppose $\int_{t\ge a} f(t)dt$ diverges to ∞ and each $\int_{t\ge a} f_n(t)dt$ is convergent. Given any $\alpha \in \mathbb{R}$, there exists $b_0 \in [a,\infty)$ such that $\int_a^{b_0} f(t)dt > \alpha$. Now by Proposition 10.40, $\int_a^{b_0} f_n(t)dt \to \int_a^{b_0} f(t)dt$. Hence there exists $n_0 \in \mathbb{N}$ such that $\int_a^{b_0} f_n(t)dt > \alpha$ for all $n \ge n_0$. Thus $\int_a^\infty f_n(t)dt > \alpha$ for all $n \ge n_0$. Since $\alpha \in \mathbb{R}$ is arbitrary, we see that $\int_a^\infty f_n(t)dt \to \infty$. □

Examples 10.42. (i) Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) := nx/(nx+1)$ for each $n \in \mathbb{N}$. Then (f_n) converges boundedly (but not uniformly) to the integrable function f on $[0,1]$ defined by $f(0) := 0$ and $f(x) := 1$ for $x \in (0,1]$. Hence by Proposition 10.40, $\int_0^1 f_n(x)dx \to \int_0^1 f(x)dx$. This can also be seen directly as follows:

$$\int_0^1 f_n(x)dx = \int_0^1 \frac{nx}{nx+1}dx = 1 - \frac{\ln(n+1)}{n} \to 1 = \int_0^1 f(x)dx.$$

(ii) Let $u > 0$. For $n \in \mathbb{N}$, define $f_n : [0,\infty) \to \mathbb{R}$ by

$$f_n(t) := \begin{cases} \left(1 - \dfrac{t}{n}\right)^n t^{u-1} & \text{if } t \in \big[1/n, n\big], \\ 0 & \text{otherwise.} \end{cases}$$

Define $g : [0,\infty) \to \mathbb{R}$ by $g(0) := 0$ and $g(t) := e^{-t}t^{u-1}$ for $t \in (0,\infty)$. Let $n \in \mathbb{N}$. Now it is easy to see that $es \le e^s$, and so $s^n e^{n(1-s)} \le 1$ for all $s \in [0,1]$. It follows that $0 \le \big(1-(t/n)\big)^n \le e^{-t}$ for all $t \in [0,n]$. Thus $|f_n| \le g$ on $[0,\infty)$. Also, $f_n \to g$ on $[0,\infty)$ by Corollary 7.7. Just before defining the gamma function in Section 9.6, we saw that the improper integral $\int_{t>0} g(t)dt$ converges to $\Gamma(u)$. By a slight modification of part (i) of Proposition 10.41 to accommodate a combination of improper integrals of the first kind and of the second kind, we obtain

$$\int_{0+}^\infty \left(1 - \frac{t}{n}\right)^n t^{u-1}dt = \int_{0+}^\infty f_n(t)dt \to \int_{0+}^\infty g(t)dt = \Gamma(u). \qquad \diamond$$

Here is a companion result of Proposition 10.40 on differentiation.

Proposition 10.43. *Let (f_n) be a sequence of real-valued differentiable functions on $[a,b]$ such that each f_n' is integrable on $[a,b]$, (f_n) converges at a point $c \in [a,b]$, and (f_n') converges boundedly to a continuous function on $[a,b]$. Then there is a continuously differentiable function f on $[a,b]$ such that $f_n \to f$ boundedly and $f_n' \to f'$ boundedly on $[a,b]$.*

Proof. We argue exactly on the same lines as the argument in the proof of Proposition 10.13, except that we invoke the Arzelà Bounded Convergence Theorem (Proposition 10.40) in place of Proposition 10.9. □

Bounded Convergence of Series

We say that a series $\sum_{k\geq 1} f_k$ of real-valued functions defined on a set E **converges boundedly** on E if the sequence (S_n) of its partial sum functions converges boundedly on E, that is, if the sequence (S_n) is pointwise convergent and uniformly bounded on E. In this case, we may also say that the series $\sum_{k\geq 1} f_k(x)$ **converges boundedly** for $x \in E$.

Let (f_k) be a sequence of real-valued bounded functions defined on a set E. Then for each $n \in \mathbb{N}$, the nth partial sum function $S_n := \sum_{k=1}^{n} f_k$ is bounded on E. If the series $\sum_{k\geq 1} f_k$ converges uniformly on E, then it follows from Proposition 10.5 that it converges boundedly on E. However, the converse does not hold, as the following example shows. Let $E := (-1, 0]$, and for $k := 0, 1, \ldots$ and $x \in E$, let $f_k(x) := x^k$. Then

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{and} \quad |S_n(x)| = \left|\sum_{k=0}^{n} x^k\right| = \frac{1-x^{n+1}}{1-x} \leq 1 \quad \text{for all } x \in E.$$

Thus the series $\sum_{k\geq 0} f_k$ converges boundedly on E. Next, consider $n \in \mathbb{N}$, and find $x_n \in E$ such that $(n+1)\ln|x_n| \geq -\ln 2$. (This is possible, since $\ln|x| \to 0$ as $|x| \to 1$.) Then $|S_{n+1}(x_n) - S_n(x_n)| = |x_n|^{n+1} \geq 1/2$. Hence the series $\sum_{k\geq 0} f_k$ does not converge uniformly on E.

Let us give some tests for bounded convergence of a series of functions.

Proposition 10.44. *Let (f_k) and (g_k) be sequences of real-valued functions on a set E.*

(i) **(Comparison Test for Bounded Convergence of Series).** *Suppose $|f_k| \leq g_k$ on E for each $k \in \mathbb{N}$ and the series $\sum_{k\geq 1} g_k$ converges boundedly on E. Then the series $\sum_{k\geq 1} f_k(x)$ converges boundedly for $x \in E$ and absolutely for each $x \in E$.*

(ii) **(Dirichlet Test for Bounded Convergence of Series).**[1] *Suppose (f_k) is monotonic and $f_k(x) \to 0$ for every $x \in E$, and moreover, the function f_1 is bounded on E. For $n \in \mathbb{N}$, let $G_n := \sum_{k=1}^{n} g_k$, and suppose the sequence (G_n) is uniformly bounded on E. Then the series $\sum_{k\geq 1} f_k g_k$ converges boundedly on E. Moreover,*

$$\left|\sum_{k=n}^{\infty} f_k(x) g_k(x)\right| \leq 2|f_n(x)|\beta \quad \textit{for all } n \in \mathbb{N} \textit{ and } x \in E,$$

where $\beta := \sup\{|G_m(x)| : m \in \mathbb{N} \text{ and } x \in E\}$.

Proof. (i) The result follows from the Comparison Test (Proposition 9.11) and the inequalities

[1] This test was communicated by Jonathan Lewin in a personal email.

$$\left|\sum_{k=1}^{n} f_k(x)\right| \leq \sum_{k=1}^{n} |f_k(x)| \leq \sum_{k=1}^{n} g_k(x) \quad \text{for } n \in \mathbb{N} \text{ and } x \in E.$$

(ii) By the Dirichlet Test (Proposition 9.21), the series $\sum_{k\geq 1} f_k g_k$ converges pointwise on E. Consider $x \in E$ and define $\beta_1(x) := \sup\{|G_m(x)| : m \in \mathbb{N}\}$. Letting $a_k := f_k(x)$ and $b_k := g_k(x)$ for $k \in \mathbb{N}$ in Proposition 9.21, we obtain

$$\left|\sum_{k=1}^{n} f_k(x)g_k(x)\right| \leq |f_1(x)|\beta_1(x) \quad \text{and} \quad \left|\sum_{k=n}^{\infty} f_k(x)g_k(x)\right| \leq 2|f_n(x)|\beta_1(x)$$

for all $n \in \mathbb{N}$. Since f_1 is bounded on E, and $\beta_1(x) \leq \beta$ for all $x \in E$, we obtain the desired results. □

Corollary 10.45 (Leibniz Test for Bounded Convergence of Series). *Let (f_k) be a monotonic sequence of real-valued functions defined on a set E such that f_1 is bounded on E and $f_k(x) \to 0$ for every $x \in E$. Then the series $\sum_{k\geq 1}(-1)^{k-1} f_k$ converges boundedly on E. Moreover, for every $n \in \mathbb{N}$, the function f_n is bounded on E and*

$$\left|\sum_{k=n}^{\infty}(-1)^{k-1} f_k(x)\right| \leq |f_n(x)| \quad \textit{for all } x \in E.$$

Proof. The desired results follow from part (ii) of Proposition 10.44 and Corollary 9.22 just as Corollary 10.21 follows from Proposition 10.20. □

We remark that if in part (ii) of Proposition 10.44 and in Corollary 10.45, the set E is a closed and bounded subset of $\mathbb{R}$ and if each f_k is a continuous function on E, then $f_k \to 0$ uniformly on E by the Dini Theorem (Proposition 10.7), and so Proposition 10.20 and Corollary 10.21 yield uniform convergence of the series $\sum_{k\geq 1} f_k g_k$ and $\sum_{k\geq 1}(-1)^{k-1} f_k$ on E, respectively.

In the following result, we consider term-by-term integration of a boundedly convergent series.

Proposition 10.46. *Suppose (f_k) is a sequence of real-valued integrable functions on $[a, b]$. If the series $\sum_{k\geq 1} f_k$ is boundedly convergent on $[a, b]$ and its sum function is integrable on $[a, b]$, then*

$$\int_a^b \left(\sum_{k=1}^{\infty} f_k(x)\right) dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)dx.$$

Proof. Apply the Arzelà Bounded Convergence Theorem (Proposition 10.40) to the sequence of partial sum functions of the series $\sum_{k\geq 1} f_k$. □

The above proposition should be compared with part (iii) of Theorem 10.26, in which term-by-term integration of a uniformly convergent series is considered. The following example illustrates how the above proposition can be useful in the absence of uniform convergence.

Example 10.47. For $k \in \mathbb{N}$, define $f_k : [0,1] \to \mathbb{R}$ by $f_k(0) := 0$ and $f_k(x) := 1/(kx+1)$ for $x \in (0,1]$. Then the sequence (f_k) is monotonic, $f_k(x) \to 0$ for all $x \in [0,1]$, and moreover, f_1 is bounded on $[0,1]$. In fact, $0 \le f_1(x) \le 1$ for all $x \in [0,1]$. Hence by Corollary 10.45, the series $\sum_{k\ge1}(-1)^{k-1}f_k$ converges boundedly on $[0,1]$, and so its sum function S is bounded on $[0,1]$.

Let $\delta > 0$ be given. We claim that the series $\sum_{k\ge1}(-1)^{k-1}f_k$ converges uniformly on $[\delta,1]$. Since $0 \le f_k(x) \le 1/(k\delta+1)$ for all $x \in [\delta,1]$, it follows from Proposition 10.3 that $f_k(x) \to 0$ uniformly for $x \in [\delta,1]$. Hence the Leibniz Test for uniform convergence of a series (Corollary 10.21) justifies our claim. Now since each summand $(-1)^{k-1}f_k$ is continuous on $[\delta,1]$, the sum function S of the series is continuous on $[\delta,1]$ by part (ii) of Theorem 10.26. It follows that S is continuous on $(0,1]$. By Corollary 6.12, S is an integrable function on $[0,1]$, and so by Proposition 10.46, term by term integration gives

$$\int_0^1 \left(\sum_{k=1}^{\infty}(-1)^{k-1}f_k(x)\right)dx = \sum_{k=1}^{\infty}\int_0^1 \frac{(-1)^{k-1}}{kx+1}dx = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{\ln(k+1)}{k}.$$

We note that the series $\sum_{k\ge1}(-1)^{k-1}f_k$ does not converge uniformly on $[0,1]$, since $f_k(1/k) = 1/2$ for each $k \in \mathbb{N}$, and so the kth term of the series does not tend to 0 uniformly on $[0,1]$. (Compare Exercise 10.11.) $\diamond$

In a similar manner, the Arzelà Dominated Convergence Theorem (Proposition 10.41) shows that we can replace the assumption of uniform convergence of a series in part (iv) of Theorem 10.26 by its pointwise convergence on $[a,\infty)$ if the sum function is assumed to be integrable on $[a,b]$ for every $b \in [a,\infty)$.

Further, in view of Proposition 10.43, we can replace the assumption of uniform convergence of the "derived" series in part (v) of Theorem 10.26 by its bounded convergence if the sum function is assumed to be continuous on $[a,b]$. Thus, if $\sum_{k\ge1} f_k$ converges at one point of $[a,b]$ and each f_k is continuously differentiable on $[a,b]$, and moreover, the series $\sum_{k\ge1} f_k'$ is boundedly convergent on $[a,b]$ to a continuous function, then the series $\sum_{k\ge1} f_k$ converges boundedly to a continuously differentiable function, and for each $x \in [a,b]$,

$$\left(\sum_{k=1}^{\infty} f_k\right)'(x) = \sum_{k=1}^{\infty} f_k'(x).$$

10.6 Riemann Integrals Depending on a Parameter

Our aim is to discuss a continuous analogue of infinite series of functions, called improper integrals depending on a parameter. In the continuous analogue, the "discrete variable" k in an infinite series of the form $\sum_k f_k$, where each f_k is a real-valued function defined on a subset E of $\mathbb{R}$, is replaced by a "continuous variable" t. In the case of series, k varies over $\mathbb{N}$ or more generally over the set of integers $\geq m$, where m is a fixed integer. In the case of improper integrals, we shall let t vary over the set of real numbers $\geq a$, where a is a fixed real number. A partial sum function such as $\sum_{k=1}^{n} f_k(u)$, where $n \in \mathbb{N}$ and u varies over E, will correspond to a partial integral function given by $\int_a^x f(t,u)dt$, where $x \in [a, \infty)$ and the parameter u varies over the set E.

Finite sums of functions preserve properties such as boundedness, continuity, integrability and differentiability of the summands, and the order of summations can be interchanged. However, Riemann integrals depending on a parameter do not in general preserve such properties of the integrands, and iterated Riemann integrals may not be equal. (See, for example, Exercises 10.35 and 10.36.) Thus, prior to discussing improper integrals depending on a parameter, it would be useful to first discuss when such properties are preserved and when the order of iterated Riemann integrals can be interchanged in the case of Riemann integrals. We therefore take up the study of Riemann integrals depending on a parameter in this section.

Let $a, b \in \mathbb{R}$ with $a \leq b$, and let E be a (parameter) set. Consider a real-valued function f defined on $[a,b] \times E$. Given (a parameter value) $u \in E$, define the function $f(\cdot, u) : [a,b] \to \mathbb{R}$ by $f(\cdot,u)(t) := f(t,u)$ for $t \in [a,b]$. Likewise, given $t \in [a,b]$, define the function $f(t,\cdot) : E \to \mathbb{R}$ by $f(t,\cdot)(u) := f(t,u)$ for $u \in E$. Suppose for each fixed (parameter value) $u \in E$, the function $f(\cdot,u)$ is integrable on $[a,b]$. Then $F : E \to \mathbb{R}$ defined by

$$F(u) := \int_a^b f(t,u)dt \quad \text{for } u \in E$$

is called the **integral function** corresponding to f. We denote it by the symbol $\int_a^b f(t,\cdot)dt$. We shall now study boundedness, continuity, integrability, and differentiability of this integral function.

Proposition 10.48 (Boundedness and Continuity of an Integral Function). *Let $a, b \in \mathbb{R}$ with $a \leq b$, and let E be a set. Suppose $f : [a,b] \times E \to \mathbb{R}$ is a bounded function such that for each fixed $u \in E$, the function $f(\cdot,u)$ is integrable on $[a,b]$. Then*

(i) *The integral function F corresponding to f is bounded on E.*

(ii) *Assume that E is a subset of $\mathbb{R}$ and that for each $t \in [a,b]$, the function $f(t,\cdot)$ is continuous on E. Then the integral function F corresponding to f is continuous on E, and therefore if $u_n \to u$ in E, then*

$$\lim_{n\to\infty} \int_a^b f(t,u_n)dt = \int_a^b \lim_{n\to\infty} f(t,u_n)dt.$$

Proof. Let $\alpha \in \mathbb{R}$ be such that $|f(t,u)| \leq \alpha$ for all $(t,u) \in [a,b] \times E$.

(i) Since $|F(u)| \leq \alpha(b-a)$ for all $u \in E$, the function F is bounded on E.

(ii) Let $u \in E$. Consider a sequence (u_n) in E such that $u_n \to u$. For $n \in \mathbb{N}$, define $g_n := f(\cdot, u_n)$, and let $g := f(\cdot, u)$. Then g_n and g are integrable functions on $[a, b]$. Since the function $f(t, \cdot)$ is continuous at u, we see that $g_n(t) := f(t, u_n) \to f(t, u) = g(t)$ for each $t \in [a, b]$, that is, $g_n \to g$ on $[a, b]$. Also, $|g_n(t)| \le \alpha$ for all $n \in \mathbb{N}$ and all $t \in [a, b]$. Thus $g_n \to g$ boundedly on $[a, b]$. So by the Arzelà Bounded Convergence Theorem (Proposition 10.40),

$$F(u_n) = \int_a^b f(t, u_n)dt = \int_a^b g_n(t)dt \to \int_a^b g(t)dt = \int_a^b f(t, u)dt = F(u).$$

This shows that the function F is continuous at $u \in E$. Also,

$$\lim_{n\to\infty} \int_a^b f(t, u_n)dt = \int_a^b f(t, u)dt = \int_a^b \lim_{n\to\infty} f(t, u_n)dt,$$

since for each $t \in [a, b]$, the function $f(t, \cdot)$ is continuous at u . □

Remark 10.49. In the above proposition, let $E := [c, \infty)$, where $c \in \mathbb{R}$. Suppose $f : [a, b] \times [c, \infty)$ is a real-valued bounded function, and the function $f(t, \cdot)$ is continuous on E for each $t \in [a, b]$. The proof given above shows that if $\lim_{u\to\infty} f(t, u)$ exists for each $t \in [a, b]$, and if the function g defined by $g(t) := \lim_{u\to\infty} f(t, u)$ is integrable on $[a, b]$, then

$$\lim_{u\to\infty} \int_a^b f(t, u)dt = \int_a^b g(t)dt = \int_a^b \lim_{u\to\infty} f(t, u)dt.$$

A similar comment holds if $E := (-\infty, d]$, where $d \in \mathbb{R}$. ◇

We now turn to iterated Riemann integrals.

Proposition 10.50 (Fichtenholz–Lichtenstein Theorem). *Let a, b, c, d be real numbers with $a \le b$ and $c \le d$, and let f be a real-valued function on $[a, b]\times[c, d]$. Assume that*

- *f is bounded on $[a, b]\times[c, d]$.*
- *For each $u \in [c, d]$, the function $f(\cdot, u)$ is integrable on $[a, b]$. Also, the function $F : [c, d] \to \mathbb{R}$ defined by $F(u) := \int_a^b f(t, u)dt$ is integrable.*
- *For each $t \in [a, b]$, the function $f(t, \cdot)$ is integrable on $[c, d]$. Also, the function $G : [a, b] \to \mathbb{R}$ defined by $G(t) := \int_c^d f(t, u)du$ is integrable.*

Then $\int_c^d F(u)du = \int_a^b G(t)dt$, that is,

$$\int_c^d \left(\int_a^b f(t, u)dt \right) du = \int_a^b \left(\int_c^d f(t, u)du \right) dt.$$

Proof. Let $\alpha \in \mathbb{R}$ be such that $|f(t, u)| \le \alpha$ for all $(t, u) \in [a, b] \times E$. For $n \in \mathbb{N}$, let $\mathsf{P}_n := \{u_{n,0}, u_{n,1}, \ldots, u_{n,n}\}$ denote the partition of $[c, d]$ into n equal parts, and let $\mathcal{T}_n := \{u_{n,1}, \ldots, u_{n,n}\}$. Consider the Riemann sum

$$S(\mathsf{P}_n, \mathcal{T}_n, F) := \frac{d-c}{n}\sum_{j=1}^{n} F(u_{n,j}) = \frac{d-c}{n}\sum_{j=1}^{n}\int_a^b f(t, u_{n,j})dt$$

for the function F corresponding to P_n and the tag set $\mathcal{T}_n$. Since the integral function F is integrable on $[c, d]$, and since the mesh $\mu(\mathsf{P}_n) = (d-c)/n \to 0$, Corollary 6.33 shows that $S(\mathsf{P}_n, \mathcal{T}_n, F) \to \int_c^d F(u)du$.

On the other hand, fix $t \in [a, b]$, and for $n \in \mathbb{N}$, define

$$G_n(t) := \frac{d-c}{n}\sum_{j=1}^{n} f(t, u_{n,j}),$$

the Riemann sum for the function $f(t, \cdot)$ corresponding to P_n and the tag set $\mathcal{T}_n$. Since the function $f(t, \cdot)$ is integrable on $[c, d]$, and since $\mu(\mathsf{P}_n) = (d-c)/n \to 0$, Corollary 6.33 shows that $G_n(t) \to \int_c^d f(t, u)du = G(t)$. Also,

$$|G_n(t)| \le (d-c)\max\{|f(t, u_{n,j})| : j = 1, \ldots, n\} \le (d-c)\alpha \quad \text{for all } n \in \mathbb{N}.$$

Thus for $n \in \mathbb{N}$, we obtain a function $G_n : [a, b] \to \mathbb{R}$ such that $G_n \to G$ boundedly on $[a, b]$. Also, G is assumed to be integrable on $[a, b]$. So by the Arzelà Bounded Convergence Theorem (Proposition 10.40), we obtain

$$S(\mathsf{P}_n, \mathcal{T}_n, F) := \frac{d-c}{n}\sum_{j=1}^{n}\int_a^b f(t, u_{n,j})dt = \int_a^b G_n(t)dt \to \int_a^b G(t)dt.$$

Hence $\int_c^d F(u)du = \int_a^b G(t)dt$, as desired. □

Remarks 10.51. (i) The hypotheses on the integrability of F and G in the above theorem can be dropped. In fact, one can deduce these statements as a part of the proof. See Notes and Comments on this chapter.

(ii) The Fichtenholz–Lichtenstein Theorem can be partially extended to improper integrals by replacing the Riemann integral on $[a, b]$ by the improper integral over $[a, \infty)$. (See Proposition 10.65.) A similar result will hold if we replace the Riemann integral on $[c, d]$ by the improper integral over $[c, \infty)$. Both the replacements can be made simultaneously if the integrand f is nonnegative on $[a, \infty) \times [c, \infty)$. (See Exercise 10.41.) But the conclusion of the Fichtenholz–Lichtenstein Theorem does not hold for improper integrals in full generality. (See Exercise 10.42.) ◇

Let $a, b \in \mathbb{R}$ with $a < b$, and let E be an interval in $\mathbb{R}$. If for a fixed $u \in E$, the function $f(\cdot, u)$ is differentiable on $[a, b]$, then we denote its derivative at $t \in [a, b]$ by $D_1 f(t, u)$. This gives a function $D_1 f(\cdot, u) : [a, b] \to \mathbb{R}$. Similarly, if for a fixed $t \in [a, b]$, the function $f(t, \cdot)$ is differentiable on E, then we denote its derivative at $u \in E$ by $D_2 f(t, u)$. This gives a function $D_2 f(t, \cdot) : [c, d] \to \mathbb{R}$.

Proposition 10.52 (Derivative of an Integral Function). *Let $a, b \in \mathbb{R}$ with $a < b$, and let E be an interval in $\mathbb{R}$. Suppose $f : [a, b] \times E \to \mathbb{R}$ is such that for each $u \in E$, the function $f(\cdot, u)$ is integrable on $[a, b]$, and for each fixed $t \in [a, b]$, the function $f(t, \cdot)$ is differentiable on E. Further, suppose that the function $D_2 f : [a, b] \times E \to \mathbb{R}$ is bounded, and the Riemann integral $\int_a^b D_2 f(t, u)dt$ exists for each $u \in E$. Then the integral function F corresponding to f is differentiable on E, and for each $u \in E$,*

$$F'(u) = \int_a^b D_2 f(t, u)dt.$$

If in addition, for each $t \in [a, b]$, the function $D_2 f(t, \cdot)$ is continuous on E, then the integral function F is continuously differentiable on E.

Proof. Let $\beta \in \mathbb{R}$ be such that $|D_2 f(t, u)| \leq \beta$ for all $(t, u) \in [a, b] \times E$. Consider $u \in E$, and a sequence (u_n) in $E \setminus \{u\}$ such that $u_n \to u$. Fix $n \in \mathbb{N}$ and define $H_n : [a, b] \to \mathbb{R}$ by

$$H_n(t) := \frac{f(t, u_n) - f(t, u)}{u_n - u} \quad \text{for } t \in [a, b].$$

Clearly, H_n is integrable on $[a, b]$ and $\int_a^b H_n(t)dt = \big(F(u_n) - F(u)\big)/(u_n - u)$. Let $t \in [a, b]$. By the Mean Value Theorem (Proposition 4.20), there is v_n in the interval between u_n and u such that $f(t, u_n) - f(t, u) = (u_n - u)D_2 f(t, v_n)$, that is, $H_n(t) = D_2 f(t, v_n)$. It follows that $|H_n(t)| \leq \beta$. Further, by the definition of derivative, $H_n(t) \to D_2 f(t, u)$ as $n \to \infty$. Thus $H_n \to D_2(\cdot, u)$ boundedly on $[a, b]$, where the function $D_2(\cdot, u)$ is assumed to be integrable on $[a, b]$. Now by the Arzelà Bounded Convergence Theorem (Proposition 10.40),

$$\frac{F(u_n) - F(u)}{u_n - u} = \int_a^b H_n(t)dt \to \int_a^b D_2 f(t, u)dt.$$

Thus the integral function F is differentiable at u, and $F'(u) = \int_a^b D_2 f(t, u)dt$.

Next, suppose that for each $t \in [a, b]$, the function $D_2 f(t, \cdot)$ is continuous on E. Applying part (ii) of Proposition 10.48 to the function $D_2 f$, we see that F' is continuous on E, that is, F is continuously differentiable on E. □

Examples 10.53. (i) By part (ii) of Proposition 10.48,

$$\lim_{u \to 0} \int_0^\pi [t] \sin(t + u^2)dt = \int_0^\pi [t] \lim_{u \to 0} \sin(t + u^2)dt = \int_0^\pi [t] \sin t \, dt,$$

which is equal to $\cos 1 + \cos 2 + \cos 3 + 3$.

(ii) By Proposition 10.50,

$$\int_0^1 \left(\int_0^{\ln 2} 2t\, u\, e^{t^2 u} du \right) dt = \int_0^{\ln 2} \left(\int_0^1 2t\, u\, e^{t^2 u} dt \right) du = \int_0^{\ln 2} (e^u - 1)du,$$

which is equal to $1 - \ln 2$.

(iii) Let $f(t,u) := e^{-u^2(1+t^2)}/(1+t^2)$ for $(t,u) \in [0,1] \times [0,\infty)$, and let

$$F(u) := \int_0^1 f(t,u)dt = \int_0^1 \frac{e^{-u^2(1+t^2)}}{(1+t^2)}dt \quad \text{for } u \in [0,\infty).$$

By Proposition 10.52, F is differentiable, and for each $u \in [0,\infty)$,

$$F'(u) = \int_0^1 D_2 f(t,u)dt = \int_0^1 -2u\, e^{-u^2} e^{-u^2t^2} dt = -2e^{-u^2} \int_0^u e^{-s^2} ds,$$

where the last equality is obtained by the substitution $s = ut$. This result can be used to evaluate the improper integral $\int_0^\infty e^{-s^2} ds$ as follows. Let

$$G(u) := \int_0^u e^{-s^2} ds \quad \text{for } u \in [0,\infty).$$

By part (i) of the Fundamental Theorem of Calculus (Proposition 6.24), $G'(u) = e^{-u^2}$, and so $F'(u) = -2G'(u)G(u)$ for $u \in [0,\infty)$. Consider $H : [0,\infty) \to \mathbb{R}$ defined by $H(u) := F(u) + G(u)^2$ for $u \in [0,\infty)$. Then $H'(u) = 0$ for all $u \in [0,\infty)$. By Corollary 4.23, H is a constant function on $[0,\infty)$. Now $F(0) = \int_0^1 1/(1+t^2)dt = \arctan 1 = \pi/4$, as we have seen in Remark 7.15, and $G(0) = 0$. Hence $H(0) = F(0) + G(0)^2 = \pi/4$. Thus

$$H(u) = \int_0^1 \frac{e^{-u^2(1+t^2)}}{1+t^2}dt + \left(\int_0^u e^{-s^2} ds\right)^2 = \frac{\pi}{4} \quad \text{for all } u \in [0,\infty).$$

Since $e^{-u^2} \to 0$ as $u \to \infty$, Remark 10.49 shows that

$$0 + \left(\int_0^\infty e^{-s^2} ds\right)^2 = \frac{\pi}{4}, \quad \text{and so} \quad \int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2}. \qquad \diamond$$

10.7 Improper Integrals Depending on a Parameter

As indicated in the beginning of Section 10.6, we shall now discuss continuous analogues of infinite series of functions, namely, improper integrals of the form $\int_{t \ge a} f(t,u)dt$, where $a \in \mathbb{R}$ and the variable t of integration ranges over $[a,\infty)$, while the parameter u ranges over a set E. In order to study them collectively, we shall denote them by $\int_{t \ge a} f(t,\cdot)dt$.

Let $a \in \mathbb{R}$ and let E be a set. Consider a real-valued function f defined on $[a,\infty) \times E$. As in Section 10.6, for a fixed $u \in E$, we define the function $f(\cdot,u) : [a,\infty) \to \mathbb{R}$ by $f(\cdot,u)(t) := f(t,u)$ for $t \in [a,\infty)$. Likewise, for a fixed $t \in [a,\infty)$, we define the function $f(t,\cdot) : E \to \mathbb{R}$ by $f(t,\cdot)(u) := f(t,u)$ for $u \in E$. Suppose for each fixed $u \in E$, the function $f(\cdot,u)$ is integrable on $[a,x]$ for every $x \in [a,\infty)$. Then we let

$$F(x,u) := \int_a^x f(t,u)dt \qquad \text{for } x \in [a,\infty) \text{ and } u \in E.$$

For $x \in [a,\infty)$, the function $F(x,\cdot) : E \to \mathbb{R}$ given by $(x,u) \longmapsto F(x,u)$ is called a **partial integral function** of the improper integral $\int_{t\geq a} f(t,\cdot)dt$ depending on a parameter belonging to a set E. We say that the improper integral $\int_{t\geq a} f(t,\cdot)dt$ **converges pointwise** on E if $\lim_{x\to\infty} F(x,u)$ exists for each $u \in E$. In this case, we define

$$I(u) := \lim_{x\to\infty} F(x,u) = \int_a^\infty f(t,u)dt \quad \text{for } u \in E.$$

The resulting function $I : E \to \mathbb{R}$ is called the **improper integral function** corresponding to f.

In Section 10.6, we denoted the partial integral function $F(b,\cdot)$ simply by F, and deduced its properties such as boundedness, continuity, integrability, and differentiability from the corresponding properties of the functions $f(t,\cdot)$, where $t \in [a,b]$. Under pointwise convergence, these properties of the partial integral functions may not carry over to the improper integral function I, as the following examples show.

Examples 10.54. (i) Let $f(t,u) := 1/t^u$ for $(t,u) \in [1,\infty) \times (1,2]$. Clearly, $0 \leq f(t,u) \leq 1$ for all $(t,u) \in [1,\infty) \times (1,2]$. By part (i) of Proposition 10.48, we see that the partial integral function $F(x,\cdot)$ is bounded on $(1,2]$ for every $x \in [1,\infty)$. But since $I(u) = \int_1^\infty (1/t^u)dt = 1/(u-1)$ for $u \in (1,2]$, the improper integral function I is not bounded on $(1,2]$.

(ii) Let $f(t,u) := ue^{-t|u|}$ for $(t,u) \in [0,\infty) \times [-1,1]$. By part (ii) of Proposition 10.48, we see that the partial integral function $F(x,\cdot)$ is continuous on $[-1,1]$ for every fixed $x \in [0,\infty)$. Let $I(u) := \int_0^\infty ue^{-t|u|}dt$ for $u \in [-1,1]$. It is easy to see that $I(u) = -1$ if $u \in [-1,0)$, $I(0) = 0$, and $I(u) = 1$ if $u \in (0,1]$. Hence the improper integral function I is not continuous at 0.

(iii) Let $f(t,u) := (2tu - t^2u^2)e^{-tu}$ for $(t,u) \in [0,\infty) \times [0,1]$. Then

$$F(x,u) = \int_0^x (2tu - t^2u^2)e^{-tu}dt = t^2ue^{-tu}\Big|_{t=0}^{t=x} = x^2ue^{-xu}$$

for every $x \in [0,\infty)$ and $u \in [0,1]$. Also, for $t \in [0,\infty)$,

$$\int_0^1 f(t,u)du = \int_0^1 (2tu - t^2u^2)e^{-tu}du = tu^2e^{-tu}\Big|_{u=0}^{u=1} = te^{-t}.$$

By Proposition 10.50, we obtain $\int_0^1 \left(\int_0^x f(t,u)dt\right)du = \int_0^x \left(\int_0^1 f(t,u)du\right)dt$ for every $x \in [0,\infty)$. On the other hand, $I(u) = \lim_{x\to\infty} F(x,u) = 0$ for every $u \in [0,1]$, while $\int_0^\infty te^{-t}dt = 1$. Hence

$$\int_0^1 \left(\int_0^\infty f(t,u)dt\right)du = \int_0^1 I(u)du = 0, \text{ but } \int_0^\infty \left(\int_0^1 f(t,u)du\right)dt = 1.$$

Thus $\int_0^1 \left(\int_0^\infty f(t,u)dt\right)du \neq \int_0^\infty \left(\int_0^1 f(t,u)du\right)dt$.

(iv) Let $f(t,u) := u^3 e^{-tu^2}$ for $(t,u) \in [0,\infty) \times [-1,1]$. Then $D_2 f(t,u) = u^2(3-2tu^2)e^{-tu^2}$, and so $|D_2 f(t,u)| \le u^2(3e^{-tu^2} + 2tu^2 e^{-tu^2}) \le 3+2 = 5$ for all $(t,u) \in [0,\infty) \times [-1,1]$. Hence by Proposition 10.52, we see that $F'(x,u) := \int_0^x D_2 f(t,u)dt$ for every $x \in [0,\infty)$ and $u \in [-1,1]$. On the other hand, since $I(u) := \int_0^\infty u^3 e^{-tu^2} dt$ for $u \in [-1,1]$, it is easy to see that $I(u) = u$ for all $u \in [-1,1]$. Thus the improper integral function I is differentiable at 0 and $I'(0) = 1$. But since $D_2 f(t,0) = 0$ for all $t \in [0,\infty)$, we obtain $\int_0^\infty D_2 f(t,0)dt = 0$. Hence $I'(0) \ne \int_0^\infty D_2 f(t,0)dt$. ◇

The above examples motivate us to consider a stronger concept of convergence of an improper integral depending on a parameter. This is done in the next subsection.

Uniform Convergence of Improper Integrals

Henceforth, whenever we consider an improper integral $\int_{t\ge a} f(t,\cdot)dt$ depending on a parameter in a set E, it will be tacitly assumed that $a \in \mathbb{R}$ and $f : [a,\infty) \times E \to \mathbb{R}$ is a function such that for each fixed $u \in E$, the function $f(\cdot,u)$ is integrable on $[a,x]$ for every $x \in [a,\infty)$.

We say that an improper integral $\int_{t\ge a} f(t,\cdot)dt$ depending on a parameter in E **converges uniformly** on E if there is a function $I : E \to \mathbb{R}$ such that

$$F(x,u) \to I(u) \text{ as } x \to \infty \quad \text{uniformly for } u \in E,$$

that is, for every $\epsilon > 0$, there exists $x_0 \in [a,\infty)$ satisfying

$$x \in [x_0,\infty) \text{ and } u \in E \Longrightarrow |F(x,u) - I(u)| < \epsilon.$$

In this case, we may also say that the improper integral $\int_{t\ge a} f(t,u)dt$ **converges uniformly** for $u \in E$. Note that the real number $x_0 \in [a,\infty)$ mentioned above is independent of $u \in E$, although it may depend on the functions $f(t,\cdot)$, where $t \in [a,\infty)$, and on ϵ.

Clearly, an improper integral $\int_{t\ge a} f(t,u)dt$ converges uniformly for $u \in E$ if and only if there exist a function $I : E \to \mathbb{R}$ and $x_0 \in [a,\infty)$ such that the function $|F(x,\cdot) - I|$ is bounded on E for each $x \in [x_0,\infty)$, and moreover, $\alpha(x) \to 0$ as $x \to \infty$, where $\alpha(x) := \sup\{|F(x,u) - I(u)| : u \in E\}$ for $x \ge x_0$.

Results involving uniform convergence of series of functions have analogues for uniform convergence of improper integrals. The following test is useful for checking uniform convergence of a pointwise convergent improper integral. In effect, it says that an improper integral depending on a parameter converges uniformly if its tail is uniformly small.

Proposition 10.55 (Test for Uniform Convergence of Improper Integrals). *Suppose an improper integral $\int_{t\ge a} f(t,\cdot)dt$ depending on a parameter in a set E converges pointwise on E, and there exist $x_0 \in [a,\infty)$ and a function $\beta : [x_0,\infty) \to \mathbb{R}$ such that $\left|\int_x^\infty f(t,u)dt\right| \le \beta(x)$ for all $x \in [x_0,\infty)$ and $u \in E$. If $\beta(x) \to 0$ as $x \to \infty$, then $\int_{t\ge a} f(t,\cdot)dt$ converges uniformly on E.*

Proof. Let $\alpha(x) := \sup\{|F(x,u) - I(u)| : u \in E\}$ for $x \in [x_0, \infty)$. Then $\alpha(x) \le \beta(x)$ for $x \in [x_0, \infty)$, and so if $\beta(x) \to 0$, then $\alpha(x) \to 0$ as $x \to \infty$. This implies that $\int_{t \ge a} f(t, \cdot)dt$ converges uniformly on E. □

We now give a criterion for checking uniform convergence of an improper integral without knowing its pointwise convergence.

Proposition 10.56 (Cauchy Criterion for Uniform Convergence of Improper Integrals). *An improper integral $\int_{t \ge a} f(t, \cdot)dt$ depending on a parameter in a set E converges uniformly on E if and only if for every $\epsilon > 0$, there exists $x_0 \in [a, \infty)$ such that*

$$y \ge x \ge x_0 \textit{ and } u \in E \Longrightarrow \left| \int_x^y f(t,u)dt \right| < \epsilon.$$

Proof. Suppose $\int_{t \ge a} f(t, \cdot)dt$ converges uniformly on E. Let $\epsilon > 0$ be given. Then there exists $x_0 \in [a, \infty)$ such that $\left| \int_x^\infty f(t,u)dt \right| < \epsilon/2$ for all $x \ge x_0$ and all $u \in E$. Hence

$$\left| \int_x^y f(t,u)dt \right| = \left| \int_x^\infty f(t,u)dt - \int_y^\infty f(t,u)dt \right| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $y \ge x \ge x_0$ and all $u \in E$.

Conversely, suppose the stated condition holds. The Cauchy Criterion for improper integrals given in Section 9.4 shows that the improper integral $\int_{t \ge a} f(t,u)dt$ is convergent for each $u \in E$. Define $I(u) := \int_a^\infty f(t,u)dt$ for $u \in E$. Let $\epsilon > 0$ be given. The stated condition implies that there exists $x_0 \in [a, \infty)$ such that

$$y \ge x \ge x_0 \text{ and } u \in E \Longrightarrow \left| \int_a^y f(t,u)dt - \int_a^x f(t,u)dt \right| < \epsilon.$$

Fix $x \ge x_0$ and let $y \to \infty$. Thus we obtain $\left| I(u) - \int_a^x f(t,u)dt \right| \le \epsilon$ for all $u \in E$. This implies that $\int_{t \ge a} f(t, \cdot)dt$ converges uniformly on E. □

The next two propositions give tests, due to Weierstrass and Dirichlet, that are extremely useful for checking the uniform convergence of improper integrals depending on a parameter. The first is particularly useful when the improper integeral happens to be absolutely convergent for each value of the parameter, whereas the second is particularly useful when the improper integral is pointwise convergent, but the convergence may not be absolute.

Proposition 10.57 (Weierstrass M-Test for Uniform Convergence of Improper Integrals). *Consider an improper integral $\int_{t \ge a} f(t, \cdot)dt$ depending on a parameter in a set E. Suppose there is a function $M : [a, \infty) \to \mathbb{R}$ such that $|f(t,u)| \le M(t)$ for all $t \in [a, \infty)$ and all $u \in E$. If the improper integral $\int_{t \ge a} M(t)dt$ is convergent, then the improper integral $\int_{t \ge a} f(t,u)dt$ converges uniformly for $u \in E$ and absolutely for each $u \in E$.*

Proof. Assume that the improper integral $\int_{t \geq a} M(t)dt$ is convergent. Now

$$\left| \int_x^y f(t,u)dt \right| \leq \int_x^y |f(t,u)|dt \leq \int_x^y M(t)dt \quad \text{for all } y \geq x \geq a \text{ and } u \in E.$$

Let $\epsilon > 0$ be given. By the Cauchy criterion for the convergence of improper integrals given in Section 9.4, there exists $x_0 \in [a, \infty)$ such that $\int_x^y M(t)dt < \epsilon$ for all $y \geq x \geq x_0$. Hence Proposition 10.56 implies the uniform convergence of $\int_{t \geq a} f(t, \cdot)dt$ on E, while the Comparison Test for Improper Integrals (Proposition 9.46) implies the absolute convergence of $\int_{t \geq a} f(t,u)dt$ for $u \in E$. □

Proposition 10.58 (Dirichlet Test for Uniform Convergence of Improper Integrals). *Suppose $a \in \mathbb{R}$, E is a set, and $f, g : [a, \infty) \times E \to \mathbb{R}$ are functions satisfying the following conditions:*

- *For each $u \in E$, the function $f(\cdot, u)$ is monotonic and differentiable on $[a, \infty)$, and $D_1 f(\cdot, u)$ is integrable on $[a, x]$ for every $x \in [a, \infty)$.*
- *$f(t,u) \to 0$ uniformly for $u \in E$ as $t \to \infty$.*
- *The function $f(a, \cdot)$ is bounded on E.*
- *For each $u \in E$, the function $g(\cdot, u)$ is continuous on $[a, \infty)$ and the function $G : [a, \infty) \times E \to \mathbb{R}$ defined by $G(x,u) := \int_a^x g(t,u)dt$ is bounded.*

Then the improper integral $\int_{t \geq a} f(t,u)g(t,u)dt$ converges uniformly for $u \in E$. Moreover, for every $x \in [a, \infty)$, the function $f(x, \cdot)$ is bounded on E and

$$\left| \int_x^\infty f(t,u)g(t,u)dt \right| \leq 2\alpha(x)\beta \quad \textit{for all } u \in E,$$

where $\alpha(x) := \sup\{|f(x,u)| : u \in E\}$, $\beta := \sup\{|G(y,u)| : (y,u) \in [a, \infty) \times E\}$.

Proof. By the Dirichlet Test for Improper Integrals (Proposition 9.53), the improper integral $\int_{t \geq a} f(t, \cdot)g(t, \cdot)dt$ converges pointwise on E, and if we let $\beta_a(u) := \sup\{|G(y,u)| : y \in [a, \infty)\}$ for $u \in E$, then

$$\left| \int_x^\infty f(t,u)g(t,u)dt \right| \leq 2|f(x,u)|\beta_a(u) \quad \text{for all } x \in [a, \infty) \text{ and } u \in E.$$

Now for each $u \in E$, the function $f(\cdot, u)$ is monotonic on $[a, \infty)$, and $f(t,u) \to 0$ as $t \to \infty$, and so we see that either $f(a,u) \leq f(x,u) \leq 0$ or $f(a,u) \geq f(x,u) \geq 0$ for every $x \in [a, \infty)$. Thus the boundedness of $f(a, \cdot)$ on E implies the boundeness of $f(x, \cdot)$ on E for every $x \in [a, \infty)$. Consequently,

$$\left| \int_x^\infty f(t,u)g(t,u)dt \right| \leq 2\alpha(x)\beta \quad \text{for every } x \in [a, \infty) \text{ and } u \in E.$$

Since $f(x,u) \to 0$ uniformly for $u \in E$ as $x \to \infty$, we clearly obtain $\alpha(x) \to 0$ as $x \to \infty$. Hence by Proposition 10.55, the improper integral $\int_{t \geq a} f(t,u)g(t,u)dt$ converges uniformly for $u \in E$. □

A similar result, known as the **Abel Test for Uniform Convergence of Improper Integrals**, is given in Exercise 10.37.

Given $a \in \mathbb{R}$ and a function $f : [a, \infty) \to \mathbb{R}$ that is integrable on $[a, x]$ for every $x \geq a$, the improper integrals

$$\mathcal{F}_s(f)(u) := \frac{2}{\pi} \int_{t \geq a} f(t) \sin ut \, dt \quad \text{and} \quad \mathcal{F}_c(f)(u) := \frac{2}{\pi} \int_{t \geq a} f(t) \cos ut \, dt$$

are called the **Fourier sine integral** of f and the **Fourier cosine integral** of f. These can be viewed as analogues of the trigonometric series.

Corollary 10.59 (Uniform Convergence of Fourier Integrals). *Let $a \in \mathbb{R}$, and let $f : [a, \infty) \to \mathbb{R}$ be a monotonic function such that $f(t) \to 0$ as $t \to \infty$. Suppose f is differentiable and f' is integrable on $[a, x]$ for every $x \in [a, \infty)$. Let $\delta > 0$ be given, and let $E := \{u \in \mathbb{R} : |u| \geq \delta\}$. Then $\mathcal{F}_s(f)(u)$ and $\mathcal{F}_c(f)(u)$ converge uniformly for $u \in E$.*

Proof. Since f is monotonic and $f(t) \to 0$ as $t \to \infty$, we see that f is bounded on $[a, \infty)$. Define $g : [a, \infty) \times E \to \mathbb{R}$ by $g(t, u) := \sin ut$ for $t \in [a, \infty)$ and $u \in E$. For each $u \in E$, the function $g(\cdot, u)$ is continuous on $[a, \infty)$. If we let $G(x, u) := \int_a^x g(t, u) dt$ for $x \in [a, \infty)$ and $u \in E$, then

$$|G(x, u)| = \frac{|\cos ua - \cos ux|}{|u|} \leq \frac{2}{|u|} \leq \frac{2}{\delta} \quad \text{for all } x \in [a, \infty) \text{ and } u \in E.$$

Thus the function G is bounded on $[a, \infty) \times E$. Hence by Proposition 10.58, $\mathcal{F}_s(f)(u)$ converges uniformly for $u \in E$. Similarly, we can see that $\mathcal{F}_c(f)(u)$ converges uniformly for $u \in E$. □

Examples 10.60. (i) Let $\delta > 0$ be given. We show that the improper integral

$$\int_{t \geq 0} \frac{\sin(t + u)}{1 + t^2 u^2} dt$$

converges uniformly for $u \in [\delta, \infty)$ and absolutely for each $u \in [\delta, \infty)$. Clearly, $|\sin(t+u)/(1+t^2u^2)| \leq 1/(1+t^2\delta^2)$ for $t \in [0, \infty)$ and $u \in [\delta, \infty)$. Also, since $\int_1^\infty (1/t^2) dt = 1$ (Example 9.36 (iii)), we see that

$$\int_0^\infty \frac{1}{1 + t^2 \delta^2} dt = \int_0^1 \frac{1}{1 + t^2 \delta^2} dt + \int_1^\infty \frac{1}{1 + t^2 \delta^2} dt \leq 1 + \frac{1}{\delta^2}.$$

Hence the desired result follows from Proposition 10.57.

(ii) Let $\delta > 0$ be given. We show that the improper integral

$$\int_{t \geq 0} \frac{\cos tu}{t + u} dt$$

converges uniformly for $u \in [\delta, \infty)$. Define $f,\, g : [0,\infty)\times(0,\infty) \to \mathbb{R}$ by $f(t,u) := 1/(t+u)$ and $g(t,u) := \cos tu$. Fix $u \in [\delta,\infty)$. Then the function $f(\cdot,u)$ is monotonically decreasing and differentiable on $[0,\infty)$. Since $D_1 f(t,u) = -1/(t+u)^2$ for $t \in [0,\infty)$, the function $D_1 f(\cdot,u)$ is integrable on $[0,x]$ for every $x \in [0,\infty)$. Also, the function $g(\cdot,u)$ is continuous on $[0,\infty)$. Further, since $0 \le f(t,u) \le 1/(t+\delta)$ for $t \in [0,\infty)$ and $u \in [\delta,\infty)$, we see that $f(t,u) \to 0$ uniformly for $u \in [\delta,\infty)$ as $t \to \infty$, and moreover,

$$|G(x,u)| = \left| \int_0^x \cos tu\, dt \right| = \left| \frac{\sin xu}{u} \right| \le \frac{1}{\delta} \quad \text{for } (x,u) \in [0,\infty)\times[\delta,\infty).$$

Hence the desired result follows from Proposition 10.58.

(iii) Let $\delta > 0$ be given, and let $E := \{u \in \mathbb{R} : |u| \ge \delta\}$. Then by Corollary 10.59, the Fourier sine and cosine integrals

$$\frac{2}{\pi}\int_{t\ge 1} \frac{\sin ut}{t^p} dt \quad \text{and} \quad \frac{2}{\pi}\int_{t\ge 1} \frac{\cos ut}{t^p} dt$$

converge uniformly for $u \in E$ if $p > 0$. But they do not converge uniformly for $u \in (0, 1/2]$ if $0 < p \le 1$. To see this, consider the partial integral functions

$$F_s(x,u) := \frac{2}{\pi}\int_1^x \frac{\sin ut}{t^p} dt \quad \text{and} \quad F_c(x,u) := \frac{2}{\pi}\int_1^x \frac{\cos ku}{t^p} dt$$

for $x \in [1,\infty)$ and $u \in (0,1/2]$. If $u_n := 1/2n$ for $n \in \mathbb{N}$, then

$$\frac{\pi}{2}\big(F_s(2n,u_n) - F_s(n,u_n)\big) = \int_n^{2n} \frac{\sin u_n t}{t^p} dt \ge \int_n^{2n} \frac{\sin(1/2)}{(2n)^p} dt \ge \frac{\sin(1/2)}{2^p},$$

where the last inequality follows since $p \le 1$, and likewise

$$\frac{\pi}{2}\big(F_c(2n,u_n) - F_c(n,u_n)\big) = \int_n^{2n} \frac{\cos u_n t}{t^p} dt \ge \int_n^{2n} \frac{\cos 1}{(2n)^p} dt \ge \frac{\cos 1}{2^p}.$$

Thus the Cauchy Criterion for uniform convergence is violated. ◇

Given a function $f : [0,\infty) \to \mathbb{R}$ that is integrable on $[0,x]$ for every $x \ge 0$, the improper integral

$$\mathcal{L}(f)(u) := \int_{t\ge 0} f(t)e^{-ut} dt$$

is called the **Laplace integral** of f. It can be viewed as a continuous analogue of a Dirichlet series $\sum_{k\ge 1} a_k e^{-kx}$.

Corollary 10.61 (Uniform Convergence of Laplace Integrals). *Let $f : [0,\infty) \to \mathbb{R}$ be a function that is integrable on $[0,x]$ for every $x \ge 0$.*

(i) *Suppose there exist* $\alpha \in (0, \infty)$ *and* $u_0 \in \mathbb{R}$ *such that* $|f(t)| \le \alpha e^{u_0 t}$ *for all* $t \in [0, \infty)$. *Let* $u_1 \in (u_0, \infty)$. *Then* $\mathcal{L}(f)(u)$ *converges uniformly for* $u \in [u_1, \infty)$ *and absolutely for each* $u \in [u_1, \infty)$.

(ii) *Suppose* f *is continuous on* $[0, \infty)$ *and there exists* $u_0 \in \mathbb{R}$ *such that the set* $\left\{\int_0^x f(t)e^{-u_0 t}dt : x \in [0, \infty)\right\}$ *is bounded. Let* $u_1 \in (u_0, \infty)$. *Then* $\mathcal{L}(f)(u)$ *converges uniformly for* $u \in [u_1, \infty)$.

Proof. (i) Clearly, $|f(t)e^{-ut}| \le (\alpha e^{u_0 t})(e^{-u_1 t}) = \alpha e^{-(u_1 - u_0)t}$ for all $t \in [0, \infty)$ and $u \in [u_1, \infty)$. Also, since $u_1 - u_0 > 0$, the improper integral $\int_{t \ge 0} e^{-(u_1 - u_0)t}dt$ is convergent. Hence by the Weierstrass M-Test (Proposition 10.57) $\mathcal{L}(f)(u)$ converges uniformly for $u \in [u_1, \infty)$ and absolutely for each $u \in [u_1, \infty)$.

(ii) Let $E_1 := [u_1, \infty)$. Define $h,\, g : [0, \infty) \times E_1 \to \mathbb{R}$ by

$$h(t, u) := e^{-(u-u_0)t} \quad \text{and} \quad g(t, u) := f(t)e^{-u_0 t} \text{ for } t \in [0, \infty) \text{ and } u \in E_1.$$

For each $u \in E_1$, the function $h(\cdot, u)$ is monotonically decreasing and differentiable on $[0, \infty)$, and $D_1 h(\cdot, u)$ is continuous on $[0, \infty)$. Also, since $|h(t, u)| \le e^{-(u_1 - u_0)t}$ for $u \in E_1$, we see that $h(t, u) \to 0$ uniformly for $u \in E_1$ as $t \to \infty$. Further, $h(0, u) = 1$ for all $u \in E_1$. By the hypothesis in (ii), the function $g(\cdot, u)$ is continuous on $[0, \infty)$ for each $u \in E_1$, and the function $G : [a, \infty) \times E_1 \to \mathbb{R}$ given by $G(x, u) := \int_0^x g(t, u)dt = \int_0^x f(t)e^{-u_0 t}dt$ is bounded. Hence by the Dirichlet Test (Proposition 10.58), $\mathcal{L}(f)(u)$ converges uniformly for $u \in E_1$. □

Remarks 10.62. (i) If in part (ii) of Corollary 10.61, we assume that $\mathcal{L}(f)(u_0)$ is convergent, then proceeding as in the proof of this corollary, but using the Abel Test (Exercise 10.37) in place of the Dirichlet Test (Proposition 10.58), we can conclude that $\mathcal{L}(f)(u)$ converges uniformly for $u \in [u_0, \infty)$.

(ii) Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous, and $u_0 \in \mathbb{R}$ is such that $\mathcal{L}(f)(u_0)$ is convergent. Then the set $\left\{\int_0^x f(t)e^{-u_0 t}dt : x \in [0, \infty)\right\}$ is bounded, and so by part (ii) of Corollary 10.61, $\mathcal{L}(f)(u)$ is convergent for $u > u_0$. This also shows that if $u_0 \in \mathbb{R}$ is such that $\mathcal{L}(f)(u_0)$ is divergent, then $\mathcal{L}(f)(u)$ is divergent for every $u < u_0$. Let us consider the set $E := \{u \in \mathbb{R} : \mathcal{L}(f)(u) \text{ is convergent}\}$. Suppose $E \ne \emptyset$ and $E \ne \mathbb{R}$. Then the set E is in fact bounded below, and we define $\xi := \inf E$. By convention, we let $\xi := \infty$ if $E = \emptyset$, and $\xi := -\infty$ if $E = \mathbb{R}$. Then ξ has the property that $\mathcal{L}(f)(u)$ converges at every $u > \xi$, and it diverges at every $u < \xi$. We call ξ the **abscissa of convergence** of the Laplace integral of f. When $\xi \in \mathbb{R}$, the Laplace integral of f may converge or it may diverge at $u = \xi$. For example, consider

$$f(t) := \begin{cases} 0 & \text{if } 0 \le t < 1, \\ 1/t & \text{if } 1 \le t < \infty, \end{cases} \quad \text{and} \quad g(t) := \begin{cases} 0 & \text{if } 0 \le t < 1, \\ 1/t^2 & \text{if } 1 \le t < \infty. \end{cases}$$

Then $\xi = 0$ for both the Laplace integrals $\mathcal{L}(f)$ and $\mathcal{L}(g)$, but $\mathcal{L}(f)$ diverges at $u = \xi$, while $\mathcal{L}(g)$ converges at $u = \xi$.

(iii) Under the hypotheses of part (ii) of Corollary 10.61, it can be shown that the function $\mathcal{L}(f) : (u_0, \infty) \to \mathbb{R}$ is infinitely differentiable, and its nth derivative can be found for each $n \in \mathbb{N}$. See Exercise 10.55. ◇

Properties of Improper Integral Functions

Let $a \in \mathbb{R}$ and let E be a set. Also, let f be a real-valued function defined on $[a, \infty) \times E$. We shall show how properties such as boundedness, continuity, and integrability of the partial integral function corresponding to f, namely of the function $F(x, \cdot) : E \to \mathbb{R}$ given by

$$F(x, u) := \int_a^x f(t, u)dt \quad \text{for } u \in E,$$

carry over to the properties of the improper integral function corresponding to f, namely of the function $I : E \to \mathbb{R}$ given by

$$I(u) := \int_a^\infty f(t, u)dt \quad \text{for } u \in E,$$

provided $\int_{t \geq a} f(t, \cdot)dt$ converges uniformly to I on E, that is, $F(x, u) \to I(u)$ uniformly for $u \in E$ as $x \to \infty$. Moreover, we shall see that the differentiability of f carries over to that of I, provided that the improper integral $\int_{t \geq a} D_2 f(t, \cdot)dt$ is uniformly convergent on E.

Statements and proofs of Proposition 10.63, 10.65, and 10.67 that follow resemble the proofs of Propositions 10.5, 10.6, 10.9, and 10.13 regarding boundedness, continuity, integrability, and differentiability of uniform limits of sequences of functions.

Proposition 10.63 (Boundedness and Continuity of an Improper Integral Function). *Let $a \in \mathbb{R}$ and let E be a set. Also, let f be a real-valued function defined on $[a, \infty) \times E$. Suppose $\int_{t \geq a} f(t, \cdot)dt$ converges uniformly to a function I on E. For $x \in [a, \infty)$, let $F(x, \cdot) : E \to \mathbb{R}$ denote the partial integral function corresponding to f.*

(i) *Suppose that for each $x \in [a, \infty)$, the function $F(x, \cdot)$ is bounded on E. Then the improper integral function I is bounded on E. Also, there exist $\alpha \in \mathbb{R}$ and $x_0 \in [a, \infty)$ such that $|F(x, u)| \leq \alpha$ for all $x \in [x_0, \infty)$ and $u \in E$.*

(ii) *Suppose $E \subseteq \mathbb{R}$ and for each $x \in [a, \infty)$, the function $F(x, \cdot)$ is continuous on E. Then the improper integral function I is continuous on E.*

Proof. (i) Since the improper integral $\int_{t \geq a} f(t, u)dt$ converges to $I(u)$ uniformly for $u \in E$, there exists $x_0 \in [a, \infty)$ such that $|F(x, u) - I(u)| < 1$ for all $x \in [x_0, \infty)$ and all $u \in E$. Also, since $F(x_0, \cdot)$ is bounded, there exists $\beta \in \mathbb{R}$ such that $|F(x_0, u)| \leq \beta$ for all $u \in E$. Then

$$|I(u)| \le |I(u) - F(x_0, u)| + |F(x_0, u)| < 1 + \beta \quad \text{for all } u \in E.$$

Hence the function I is bounded on E. Moreover, if we let $\alpha := 2 + \beta$, then $|F(x, u)| \le |F(x, u) - I(u)| + |I(u)| \le \alpha$ for all $x \in [x_0, \infty)$ and all $u \in E$.

(ii) Let $\epsilon > 0$ be given. Since the improper integral $\int_{t \ge a} f(t, u)dt$ converges to $I(u)$ uniformly for $u \in E$, there is $x_0 \in [a, \infty)$ such that $|F(x_0, u) - I(u)| < \epsilon$ for all $u \in E$. Consider $u_0 \in E$. Then by the continuity of $F(x_0, \cdot)$ at u_0, there exists $\delta > 0$ such that if $u \in E$ and $|u - u_0| < \delta$, then $|F(x_0, u) - F(x_0, u_0)| < \epsilon$, and consequently,

$$|I(u) - I(u_0)| \le |I(u) - F(x_0, u)| + |F(x_0, u) - F(x_0, u_0)| + |F(x_0, u_0) - I(u_0)|,$$

which is less than 3ϵ. Thus the function I is continuous at each u_0. $\square$

Proposition 10.63 together with Examples 10.54 (i) and (ii) shows that the improper integrals $\int_{t \ge 1} (1/t^u)dt$ and $\int_{t \ge 0} ue^{-t|u|}dt$ do not converge uniformly for $u \in (1, 2]$ and for $u \in [-1, 1]$, respectively.

Corollary 10.64. *Let $a \in \mathbb{R}$, and let E be a set. Suppose $f : [a, \infty) \times E \to \mathbb{R}$ is such that the improper integral $\int_{t \ge a} f(t, \cdot)dt$ converges uniformly to a function I on E, and such that f is bounded on $[a, x] \times E$ for each $x \in [a, \infty)$. Then*

(i) *The improper integral function I corresponding to f is bounded on E.*

(ii) *Suppose $E \subseteq \mathbb{R}$, and for each $t \in [a, \infty)$, the function $f(t, \cdot)$ is continuous on E. Then the improper integral function I corresponding to f is continuous on E, and so if $u_n \to u$ in E, then*

$$\lim_{n \to \infty} \int_a^\infty f(t, u_n)dt = \int_a^\infty \lim_{n \to \infty} f(t, u_n)dt.$$

Proof. By Proposition 10.48, the partial integral function $F(x, \cdot)$ is bounded on E for each $x \in [a, \infty)$, and if the hypothesis in (ii) is satisfied, then $F(x, \cdot)$ is continuous on E for each $x \in [a, \infty)$. Hence the desired results in (i) and (ii) above follow from parts (i) and (ii) of Proposition 10.63, respectively. Now let $u_n \to u$ in E. Then $\int_a^\infty f(t, u_n)dt = I(u_n) \to I(u) = \int_a^\infty f(t, u)dt$. Moreover, $f(t, u) = \lim_{n \to \infty} f(t, u_n)$, since $f(t, \cdot)$ is a continuous function on E for each $t \in [a, \infty)$. This establishes the equality stated in (ii). $\square$

Proposition 10.65 (Riemann Integral of an Improper Integral Function). *Let $a, c, d \in \mathbb{R}$ with $c \le d$. Suppose $f : [a, \infty) \times [c, d] \to \mathbb{R}$ satisfies the following conditions (in addition to our tacit assumption stated in the beginning of the previous subsection).*

- *$\int_{t \ge a} f(t, \cdot)dt$ converges uniformly to a real-valued function I on $[c, d]$.*
- *The partial integral function $F(x, \cdot)$ is integrable on $[c, d]$ for each fixed $x \in [a, \infty)$.*

- *For each fixed* $t \in [a, \infty)$, *the function* $f(t, \cdot)$ *is integrable on* $[c, d]$. *Moreover, the function* $G : [a, \infty) \to \mathbb{R}$ *defined by* $G(t) := \int_c^d f(t, u)du$ *is integrable on* $[a, x]$ *for every* $x \in [a, \infty)$.
- $\int_c^d F(x, u)du = \int_a^x G(t)dt$, *that is,* $\int_c^d \big(\int_a^x f(t, u)dt \big) du = \int_a^x \big(\int_c^d f(t, u)du \big) dt$ *for each* $x \in [a, \infty)$.

Then the improper integral function I *is integrable on* $[c, d]$, *and the improper integral* $\int_{t \geq a} G(t)dt$ *is convergent, and also,* $\int_c^d I(u)du = \int_a^\infty G(t)dt$, *that is,*

$$\int_c^d \left(\int_a^\infty f(t, u)dt \right) du = \int_a^\infty \left(\int_c^d f(t, u)du \right) dt.$$

Proof. According to our hypotheses, for each $x \in [a, \infty)$, the partial integral function $F(x, \cdot)$ corresponding to f is integrable on $[c, d]$, and as $x \to \infty$, $F(x, u) \to I(u)$ uniformly for $u \in [c, d]$. To show that the improper integral function I is integrable on $[c, d]$, we first note that by part (i) of Proposition 10.63, the function I is bounded on $[c, d]$. Let $x \in [a, \infty)$, and define

$$\alpha_x := \sup\{|F(x, u) - I(u)| : u \in [c, d]\}.$$

Then $F(x, u) - \alpha_x \leq I(u) \leq F(x, u) + \alpha_x$ for all $u \in [c, d]$. Thus in view of the definitions of the lower and the upper Riemann integrals, we obtain

$$L(F(x, \cdot)) - \alpha_x(d - c) \leq L(I) \leq U(I) \leq U(F(x, \cdot)) + \alpha_x(d - c).$$

Since $F(x, \cdot)$ is integrable on $[c, d]$, we see that $L(F(x, \cdot)) = U(F(x, \cdot))$, and so $0 \leq U(I) - L(I) \leq 2\alpha_x(d - c)$. Now $\alpha_x \to 0$ as $x \to \infty$ because of the uniform convergence of $F(x, \cdot)$ to I on $[c, d]$. Thus $L(I) = U(I)$, that is, the function I is integrable on $[c, d]$. Also, as $x \to \infty$,

$$\left| \int_c^d F(x, u)du - \int_c^d I(u)du \right| \leq \int_c^d |F(x, u) - I(u)|du \leq \alpha_x(d - c) \to 0.$$

Thus $\int_c^d F(x, u)du \to \int_c^d I(u)du$ as $x \to \infty$. But according to our hypotheses, $\int_c^d F(x, u)du = \int_a^x G(t)dt$ for each $x \in [a, \infty)$. Hence $\int_a^x G(t)dt \to \int_c^d I(u)du$ as $x \to \infty$, that is, $\int_{t \geq a} G(t)dt$ is convergent and $\int_a^\infty G(t)dt = \int_c^d I(u)du$. □

Proposition 10.65 together with Example 10.54 (iii) shows that the improper integral $\int_{t \geq 0} (2tu - t^2u^2)e^{-tu}dt$ does not converge uniformly for $u \in [0, 1]$.

Corollary 10.66. *Let* $a, c, d \in \mathbb{R}$ *with* $c \leq d$ *and let* $f : [a, \infty) \times [c, d] \to \mathbb{R}$ *be a function. Suppose the improper integral* $\int_{t \geq a} f(t, \cdot)dt$ *converges uniformly to a function* I *on* $[c, d]$. *If the hypotheses stated in the Fichtenholz–Lichtenstein Theorem (Proposition 10.50) hold with the point* b *replaced by an arbitrary point* $x \in [a, \infty)$, *and with the integral function* F *on* $[c, d]$ *replaced by the partial integral function* $F(x, \cdot)$ *on* $[c, d]$, *then the improper integral function* I *is integrable on* $[c, d]$ *and* $\int_c^d \big(\int_a^\infty f(t, u)dt \big) du = \int_a^\infty \big(\int_c^d f(t, u)du \big) dt$.

Proof. By Proposition 10.50, $\int_c^d \big(\int_a^x f(t,u)dt\big)du = \int_a^x \big(\int_c^d f(t,u)du\big)dt$ for all $x \in [a,\infty)$. Hence the desired results follow from Proposition 10.65. $\square$

Proposition 10.67 (Derivative of an Improper Integral Function). *Let $a, c, d \in \mathbb{R}$, with $c < d$, and let f be a real-valued function on $[a,\infty)\times[c,d]$. Suppose the following conditions hold.*

- *For each $t \in [a,\infty)$, the function $f(t,\cdot)$ is differentiable on $[c,d]$.*
- *For each $u \in [c,d]$, the functions $f(\cdot,u)$ and $D_2 f(\cdot,u)$ are integrable on $[a,x]$ for each $x \in [a,\infty)$.*
- *For each $x \in [a,\infty)$, the partial integral function $F(x,\cdot) := \int_a^x f(t,\cdot)dt$ is differentiable on $[c,d]$ and its derivative $D_2F(x,\cdot)$ is continuous on $[c,d]$, and further, $D_2F(x,u) = \int_a^x D_2 f(t,u)dt$ for all $u \in [c,d]$.*
- *There exists $u_0 \in [c,d]$ such that the improper integral $\int_{t\geq a} f(t,u_0)dt$ is convergent, and further, the improper integral $\int_{t\geq a} D_2 f(t,u)dt$ converges uniformly for $u \in [c,d]$.*

Then the improper integral $\int_{t\geq a} f(t,\cdot)dt$ converges uniformly to a continuously differentiable function I on $[c,d]$, and

$$I'(u) = \int_a^\infty D_2 f(t,u)dt \quad \textit{for } u \in [c,d].$$

Proof. Consider the improper integral $J(u) := \int_{t\geq a} D_2 f(t,u)dt$ and the corresponding partial integral $G(x,u) := \int_a^x D_2 f(t,u)dt$ for $u \in [c,d]$, where $x \in [a,\infty)$. According to our hypotheses, the improper integral $J(u)$ converges uniformly for $u \in [c,d]$, and for $x \in [a,\infty)$, the functions $G(x,\cdot)$ and $D_2F(x,\cdot)$ are equal and are continuous on $[c,d]$. Hence by part (ii) of Proposition 10.63 (ii), the function J is continuous on $[c,d]$.

Let $x \in [a,\infty)$. By the Fundamental Theorem of Calculus (part (ii) of Proposition 6.24),

$$F(x,u) = F(x,u_0) + \int_{u_0}^u D_2F(x,v)dv = F(x,u_0) + \int_{u_0}^u G(x,v)dv \quad \text{for } u \in [c,d].$$

Let $\int_{t\geq a} f(t,u_0)dt$ converge to $I(u_0)$. Then $\int_{t\geq x} f(t,u_0)dt$ is convergent and

$$F(x,u_0) = I(u_0) - \int_x^\infty f(t,u_0)dt, \quad \text{while} \quad G(x,v) = J(v) - \int_x^\infty D_2 f(t,v)dt$$

for $v \in [c,d]$. Consequently, for $u \in [c,d]$,

$$\left| F(x,u) - I(u_0) - \int_{u_0}^u J(v)dv \right| \leq \left| \int_x^\infty f(t,u_0)dt \right| + \left| \int_{u_0}^u \left(\int_x^\infty D_2 f(t,v)dt \right) dv \right|.$$

Let $\epsilon > 0$ be given. Since the improper integral $\int_{t\geq a} f(t,u_0)dt$ is convergent, there exists $x_0 \in [a,\infty)$ such that $\left|\int_x^\infty f(t,u_0)dt\right| < \epsilon$ for all $x \in [x_0,\infty)$, and since the improper integral $\int_{t\geq a} D_2 f(t,u)dt$ converges uniformly for $u \in [c,d]$, there exists $x_1 \in [a,\infty)$ such that $\left|\int_x^\infty D_2 f(t,u)dt\right| < \epsilon$ for all $x \in [x_1,\infty)$ and all $u \in [c,d]$. Hence

$$\left|F(x,u) - I(u_0) - \int_{u_0}^{u} J(v)dv\right| < \epsilon + |u-u_0|\epsilon \leq (1+d-c)\epsilon$$

for all $x \in \mathbb{R}$ with $x \geq \max\{x_0, x_1\}$ and all $u \in [c,d]$. It follows that

$$F(x,u) \to I(u_0) + \int_{u_0}^{u} J(v)dv \quad \text{uniformly for } u \in [c,d] \quad \text{as } x \to \infty,$$

that is, the improper integral $\int_{t\geq a} f(t,u)dt$ converges uniformly to $I(u)$, where

$$I(u) := I(u_0) + \int_{u_0}^{u} J(v)dv \quad \text{for } u \in [c,d].$$

Since the function J is continuous on $[c,d]$, the Fundamental Theorem of Calculus (part (i) of Proposition 6.24) shows that the function I is differentiable and $I' = J$ on $[c,d]$. Thus I is continuously differentiable on $[c,d]$, and $I'(u) = \int_a^\infty D_2 f(t,u)dt$ for $u \in [c,d]$, as desired. □

Proposition 10.67 together with Example 10.54 (iv) shows that the improper integral $\int_{t\geq 0} u^2(3-2tu^2)e^{-tu^2}dt$ does not converge uniformly for $u \in [-1,1]$.

Corollary 10.68. *Let $a, c, d \in \mathbb{R}$ with $c < d$ and let $f : [a,\infty) \times [c,d] \to \mathbb{R}$ be a function. Suppose the hypotheses stated in Proposition 10.52 hold with the point b replaced by an arbitrary point $x \in [a,\infty)$. If there exists $u_0 \in [c,d]$ such that the improper integral $\int_{t\geq a} f(t,u_0)dt$ is convergent, and if the improper integral $\int_{t\geq a} D_2 f(t,\cdot)dt$ is uniformly convergent on $[c,d]$, then the improper integral $\int_{t\geq a} f(t,\cdot)dt$ converges uniformly to a continuously differentiable function I on $[c,d]$, and $I'(u) = \int_a^\infty D_2 f(t,u)dt$ for $u \in [c,d]$.*

Proof. By Proposition 10.52, for each $x \in [a,\infty)$, the partial integral function $F(x,\cdot) := \int_a^x f(t,\cdot)dt$ is differentiable on $[c,d]$ and its derivative $D_2F(x,\cdot)$ is continuous on $[c,d]$, and further, $D_2F(x,u) = \int_a^x D_2 f(t,u)dt$ all $u \in [c,d]$. Hence the desired results follow from Proposition 10.67. □

Examples 10.69. (i) Let $f(t,u) := u^2/(1+t^2u^2)$ for $t \in [1,\infty)$ and $u \in \mathbb{R}$. Since $|f(t,u)| \leq 1/t^2$ for all $t \in [1,\infty)$ and $u \in \mathbb{R}$, and since $\int_{t\geq 1}(1/t^2)dt$ is convergent, the improper integral $\int_{t\geq 1} f(t,u)dt$ converges uniformly for $u \in \mathbb{R}$ by the Weierstrass M-Test (Proposition 10.57). Fix $x \in [1,\infty)$, and consider the partial integral function $F(x,\cdot) : \mathbb{R} \to \mathbb{R}$ given by $F(x,u) := \int_1^x f(t,u)dt$ for $u \in \mathbb{R}$. Now $|f(t,u)| \leq 1$ for all $(t,u) \in [1,x] \times \mathbb{R}$, and for each $t \in [1,x]$, the function $f(t,\cdot)$ is continuous on $\mathbb{R}$. Hence by part (ii) of

Proposition 10.48, the integral function $F(x,\cdot)$ is continuous on $\mathbb{R}$. Next, part (ii) of Proposition 10.63 shows that the improper integral function $I : \mathbb{R} \to \mathbb{R}$ given by $I(u) := \int_1^\infty f(t,u)dt$ is continuous on $\mathbb{R}$. In particular,

$$\lim_{u\to 1}\int_1^\infty \frac{u^2}{1+t^2u^2}dt = \int_1^\infty \lim_{u\to 1}\left(\frac{u^2}{1+t^2u^2}\right)dt = \int_1^\infty \frac{1}{1+t^2}dt = \frac{\pi}{4},$$

as we have seen in Remark 7.15. Also, in view of Remark 10.49,

$$\lim_{u\to \infty}\int_1^\infty \frac{u^2}{1+t^2u^2}dt = \int_1^\infty \lim_{u\to \infty}\left(\frac{u^2}{1+t^2u^2}\right)dt = \int_1^\infty \frac{1}{t^2}dt = 1.$$

(ii) Let $c,\, d \in \mathbb{R}$ with $0 < c < d$. Define $G : [0,\infty) \to \mathbb{R}$ by $G(0) := d - c$ and $G(t) := (e^{-c\,t} - e^{-d\,t})t^{-1}$ for $t \in (0,\infty)$. By L'Hôpital's Rule (Proposition 4.39), $\lim_{t\to 0^+} G(t) = d-c$. Hence the function G is continuous on $[0,\infty)$. To find $\int_0^\infty G(t)dt$, define $f(t,u) := e^{-tu}$ for $(t,u) \in [0,\infty)\times[c,d]$, and note that $\int_c^d f(t,u)du = G(t)$ for $t \in [0,\infty)$. Also, since $|f(t,u)| \le e^{-ct}$ for all $(t,u) \in [0,\infty)\times[c,d]$, and since the improper integral $\int_{t\ge 0} e^{-ct}dt$ is convergent, we see that the improper integral $\int_{t\ge 0} f(t,u)dt$ converges uniformly for $u \in [c,d]$ by the Weierstrass M-Test (Proposition 10.57). Let $x \in [0,\infty)$. Since

$$F(x,u) := \int_0^x f(t,u)dt = \frac{1-e^{-xu}}{u} \quad \text{for } u \in [c,d],$$

the function $F(x,\cdot)$ is integrable on $[c,d]$. Also, $0 \le f(t,u) \le 1$ for all (t,u) in $[0,b]\times[c,d]$. By the Fichtenholz–Lichtenstein Theorem (Proposition 10.50),

$$\int_0^b\left(\int_c^d e^{-tu}du\right)dt = \int_c^d\left(\int_0^b e^{-tu}dt\right)du.$$

Now Proposition 10.65 shows that

$$\begin{aligned}\int_0^\infty G(t)dt &= \int_0^\infty\left(\int_c^d e^{-tu}du\right)dt\\ &= \int_c^d\left(\int_0^\infty e^{-tu}dt\right)du = \int_c^d \frac{1}{u}du = \ln d - \ln c = \ln(d/c).\end{aligned}$$

(iii) To evaluate the improper integral function $I : \mathbb{R} \to \mathbb{R}$ defined by $I(u) := \int_0^\infty e^{-t^2}\cos 2tu\, dt$ for $u \in \mathbb{R}$, we first find its derivative I'. Let $c \in (0,\infty)$, and let $E_c := [-c,c]$. Define $f : [0,\infty)\times E_c \to \mathbb{R}$ by $f(t,u) := e^{-t^2}\cos 2tu$. Then $D_2 f(t,u) = -2te^{-t^2}\sin 2ut$ for $(t,u) \in [0,\infty)\times E_c$. Fix $x \in [0,\infty)$, and let $F(x,u) := \int_0^x f(t,u)dt$ for $u \in E_c$. Clearly, the Riemann integral $\int_0^x D_2 f(t,u)dt$ exists for every $u \in E_c$. Also, the function $D_2 f$ is bounded on $[0, x] \times E_c$. By Proposition 10.52, $D_2F(x, u) = \int_0^x D_2 f(t, u)dt$ for all $u \in E_c$, and further, the derivative $D_2F(x, \cdot)$ is continuous on E_c, since the function $D_2 f(t, \cdot)$ is continuous on E_c for each $t \in [0, x]$.

Next, Example 9.56 (ii) shows that $\int_{t\ge 0} f(t, 0)dt$ is convergent, and since $|D_2 f(t, u)| \le 2te^{-t^2}$ for all $u \in E_c$, Example 9.36 (ii) and the Weierstrass M-Test (Proposition 10.57) show that the improper integral $\int_{t\ge 0} D_2 f(t, u)dt$ converges uniformly for $u \in E_c$. Now by Proposition 10.67, the improper integral function I is continuously differentiable on E_c, and $I'(u) = \int_0^\infty D_2 f(t, u)dt$ for $u \in E_c$. Integrating by parts (Proposition 6.28), we obtain

$$\begin{aligned}\int_0^x D_2 f(t, u)dt &= \int_0^x (\sin 2tu)(-2te^{-t^2})dt \\ &= (\sin 2tu)e^{-t^2}\Big|_{t=0}^{t=x} - 2u\int_0^x (\cos 2tu)e^{-t^2}dt \\ &= (\sin 2xu)e^{-x^2} - 2u\int_0^x e^{-t^2}\cos 2tu\,dt\end{aligned}$$

for $u \in E_c$. Letting $x \to \infty$, we obtain

$$I'(u) = -2u\int_0^\infty e^{-t^2}\cos 2tu\,dt = -2uI(u) \quad \text{for } u \in E_c.$$

Define $J(u) := I(u)e^{u^2}$ for $u \in E_c$. Then J is differentiable and $J'(u) = e^{u^2}\big(I'(u) + 2uI(u)\big) = 0$ for all $u \in E_c$. By Corollary 4.23, we see that J is a constant function on E_c. Also, $J(0) = I(0) = \int_0^\infty e^{-t^2}dt = \sqrt{\pi}/2$, as we have seen in Example 10.53 (iii). Hence $I(u) = (\sqrt{\pi}/2)e^{-u^2}$ for all $u \in E_c$. Since c is an arbitrary positive real number, we obtain

$$I(u) = \int_0^\infty e^{-t^2}\cos 2tu\,dt = \frac{\sqrt{\pi}}{2}e^{-u^2} \quad \text{for } u \in \mathbb{R}. \qquad \diamond$$

Remarks 10.70. (i) The Arzelà Dominated Convergence Theorem can be used to obtain continuity, integrability, or differentiability of an improper integral function just as the Arzelà Bounded Convergence Theorem was used to obtain these properties of an integral function in Propositions 10.48, 10.50, and 10.52. We note that if an improper integral $\int_{t\ge a} f(t, \cdot)dt$ converges pointwise on E, and if there is a function $g : [a, \infty) \to \mathbb{R}$ such that $|f(t, u)| \le g(t)$ for all $(t, u) \in [a, \infty) \times E$, and further, if the improper integral $\int_{t\ge a} g(t)dt$ is convergent, then by the Weierstrass M-Test (Proposition 10.57), the improper integral $\int_{t\ge a} f(t, \cdot)dt$ converges uniformly on E.

(ii) We say that an improper integral $\int_{t\ge a} f(t, \cdot)dt$ depending on a parameter in a set E **converges boundedly** on E if it converges pointwise on E and if there are $\alpha \in \mathbb{R}$ and $x_0 \in [a, \infty)$ such that $|F(x, u)| \le \alpha$ for all $x \in [x_0, \infty)$

and all $u \in E$, where $F(x,u) := \int_a^x f(t,u)dt$. Uniform convergence of an improper integral does not imply, and is not implied by, its bounded convergence. See Exercise 10.38. On the other hand, part (i) of Proposition 10.63 shows that if the improper integral $\int_{t \geq a} f(t,\cdot)dt$ converges uniformly on E, and if each partial integral function $F(x,\cdot)$ is bounded on E, then the improper integral $\int_{t \geq a} f(t,\cdot)dt$ converges boundedly on E. Note that if for each $x \in [a,\infty)$, the function $f : [a,x] \times E \to \mathbb{R}$ is bounded, then the partial integral function $F(x,\cdot)$ is bounded on E by part (i) of Proposition 10.48. ◇

Uniform Convergence of Related Integrals

In Section 9.6, we saw how the treatment of improper integrals of the type $\int_{t \geq a} f(t)dt$ can be used to discuss the convergence of other types of improper integrals, namely $\int_{t \leq b} f(t)dt$, $\int_{t \in \mathbb{R}} f(t)dt$, $\int_{a<t\leq b} f(t)dt$, $\int_{a\leq t<b} f(t)dt$, $\int_{a<t<b} f(t)dt$, $\int_{a<t} f(t)dt$, and $\int_{t<b} f(t)dt$. The same holds for uniform convergence of such integrals depending on a parameter. We shall not treat them in detail, except for stating a couple of illustrative results below.

Proposition 10.71 (Weierstrass M-Test for Uniform Convergence of Improper Integrals of the Second Kind). *Let $a, b \in \mathbb{R}$ with $a < b$. Consider an improper integral $\int_{a<t\leq b} f(t,\cdot)dt$ depending on a parameter in a set E, where f is a real-valued function on $(a,b] \times E$. If there is a function $M : (a,b] \to \mathbb{R}$ such that $|f(t,u)| \leq M(t)$ for all $(t,u) \in (a,b] \times E$, and if the improper integral $\int_{a<t\leq b} M(t)dt$ is convergent, then the improper integral $\int_{a<t\leq b} f(t,u)dt$ converges uniformly for $u \in E$ and absolutely for each $u \in E$.*

Proof. Similar to Proposition 10.57. □

See Exercise 10.57 for the corresponding Dirichlet Test.

Proposition 10.72. *Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let f be a real-valued function on $(a,b] \times [c,d]$. Suppose the following conditions hold:*

- *For each $t \in (a,b]$, the function $f(t,\cdot)$ is differentiable on $[c,d]$.*
- *For each $u \in [c,d]$, the functions $f(\cdot,u)$ and $D_2 f(\cdot,u)$ are integrable on $[x,b]$ for each $x \in (a,b]$.*
- *For each $x \in (a,b]$, the partial integral function $F(x,\cdot) := \int_x^b f(t,\cdot)dt$ is differentiable on $[c,d]$ and its derivative $D_2 F(x,\cdot)$ is continuous on $[c,d]$, and further, $D_2 F(x,u) = \int_x^b D_2 f(t,u)dt$ for all $u \in [c,d]$.*
- *There exists $u_0 \in [c,d]$ such that the improper integral $\int_{a<t\leq b} f(t,u_0)dt$ is convergent, and further, the improper integral $\int_{a<t\leq b} D_2 f(t,u)dt$ converges uniformly for $u \in [c,d]$.*

Then the improper integral $\int_{a<t\le b} f(t,\cdot)dt$ *is uniformly convergent to a continuously differentiable function* I *on* $[c,d]$*, and*

$$I'(u) = \int_{a^+}^{b} D_2 f(t,u)dt \quad \textit{for } u \in [c,d].$$

Proof. Similar to Proposition 10.67. □

More about the Gamma Function

In Section 9.6, we defined the gamma function $\varGamma : (0,\infty) \to \mathbb{R}$ by

$$\varGamma(u) := \int_{0^+}^{\infty} e^{-t}t^{u-1}dt \quad \text{for } u \in (0,\infty).$$

We also described some of its properties in Proposition 9.60. We observe that the gamma function is the improper integral function $\int_{0^+}^{\infty} g(t,\cdot)dt$ defined on $(0,\infty)$, where $g : (0,\infty)\times(0,\infty) \to \mathbb{R}$ is given by $g(t,u) := e^{-t}t^{u-1}$. We shall now use the results proved in this chapter for integrals depending on a parameter to establish additional properties of the gamma function. First, we record an auxiliary result that will be useful in establishing these properties.

Lemma 10.73. (i) $\ln x < x^{1/j}$ *for all* $j \in \mathbb{N}$ *and all* $x \in \mathbb{R}$ *with* $x \ge j^{2j}$*.*
(ii) *Given a nonnegative integer* k *and* $c, d \in \mathbb{R}$ *with* $0 < c < d$*, the improper integral* $\int_{t\ge 1} e^{-t}t^{u-1}(\ln t)^k\, dt$ *converges uniformly for* $u \in (0,d]$ *and the improper integral* $\int_{0<t\le 1} e^{-t}t^{u-1}(\ln t)^k\, dt$ *converges uniformly for* $u \in [c,\infty)$*.*

Proof. (i) First we show that $x > 2\ln x$ for every positive real number x. Consider $g : (0,\infty) \to \mathbb{R}$ defined by $g(x) := x - 2\ln x$. Then g is differentiable and $g'(x) = (x-2)/x \ge 0$ for $x \in [2,\infty)$, while $g'(x) \le 0$ for $x \in (0,2]$. Thus g is decreasing on $(0,2]$ and increasing on $[2,\infty)$. Hence $g(x) \ge g(2) = 2(1-\ln 2) > 0$ for all $x \in (0,\infty)$.

Next, let $j \in \mathbb{N}$. Consider $h : (0,\infty) \to \mathbb{R}$ defined by $h(x) := x^{1/j} - \ln x$. Then h is differentiable and $h'(x) = (x^{1/j} - j)/jx \ge 0$ for $x \in [j^j,\infty)$. Thus h is increasing on $[j^j,\infty)$. Hence $h(x) \ge h(j^{2j}) = j^2 - 2j\ln j = j(j - 2\ln j)$ for all $x \in [j^{2j},\infty)$. But $j > 2\ln j$ as shown above, and so the desired inequality follows.

(ii) Let k be a nonnegative integer, and let c, $d \in \mathbb{R}$ with $0 < c < d$. Choose $j \in \mathbb{N}$ such that $c > k/j$. Since the improper integral defining $\varGamma(u)$ is convergent whenever $u > 0$, it follows that the improper integrals

$$\int_{t\ge 1} e^{-t}t^{d+k-1}dt \quad \text{and} \quad \int_{0<t\le 1} e^{-t}t^{c-(k/j)-1}dt$$

are convergent. Further, if $u \in (0,d]$, then for all $t \in [1,\infty)$,

$$0 \le e^{-t}t^{u-1}(\ln t)^k < e^{-t}t^{d-1}t^k = e^{-t}t^{d+k-1},$$

since $\ln t < t$, whereas if $u \in [c, \infty)$, then for all $t \in (0, j^{-2j}]$,

$$0 \le |e^{-t}t^{u-1}(\ln t)^k| = e^{-t}t^{u-1}\left(\ln \frac{1}{t}\right)^k \le e^{-t}t^{c-1}t^{-k/j} = e^{-t}t^{c-(k/j)-1}$$

by (i) above. Hence by the Weierstrass M-Test (Propositions 10.57 and 10.71), we conclude that the improper integral $\int_{t\ge 1} e^{-t}t^{u-1}(\ln t)^k\,dt$ converges uniformly for $u \in (0, d]$ and the improper integral $\int_{0<t\le 1} e^{-t}t^{u-1}(\ln t)^k\,dt$ converges uniformly for $u \in [c, \infty)$. □

Proposition 10.74. (i) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ *and for* $n \in \mathbb{N}$,

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdots (2n-3)(2n-1)}{2^n}\sqrt{\pi}.$$

(ii) *The gamma function is continuous on* $(0, \infty)$. *In fact, it is infinitely differentiable on* $(0, \infty)$ *and*

$$\Gamma^{(k)}(u) = \int_{0+}^{\infty} e^{-t}t^{u-1}(\ln t)^k dt \quad \textit{for } k \in \mathbb{N} \textit{ and } u \in (0, \infty).$$

(iii) *The gamma function is strictly convex on* $(0, \infty)$. *Also, there is a unique* $a \in (1, 2)$ *such that* Γ *is strictly decreasing on* $(0, a)$ *and strictly increasing on* (a, ∞). *Further,* $\Gamma(u) \to \infty$ *as* $u \to \infty$.

Proof. (i) We note that $\Gamma\left(\frac{1}{2}\right) = \int_{0+}^{\infty} e^{-t}t^{-1/2}dt = 2\int_0^{\infty} e^{-s^2}ds$, using the substitution $t = s^2$. But $\int_0^{\infty} e^{-s^2}ds = \sqrt{\pi}/2$, as we saw in Example 10.53 (iii). Hence $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. This together with the relation $\Gamma(u+1) = u\,\Gamma(u)$ for all $u \in (0, \infty)$ established in part (ii) of Proposition 9.60 implies the desired formula for $\Gamma\left(n + \frac{1}{2}\right)$ using an easy induction on $n \in \mathbb{N}$.

(ii) Let $c, d \in \mathbb{R}$ with $0 < c < d$. Define

$$\Gamma_1(u) := \int_1^{\infty} e^{-t}t^{u-1}dt \quad \text{and} \quad \Gamma_2(u) := \int_{0+}^{1} e^{-t}t^{u-1}dt \quad \text{for } u \in (0, \infty).$$

By part (ii) of Lemma 10.73 (with $k = 0$), the improper integral $\int_{t\ge 1} e^{-t}t^{u-1}dt$ converges uniformly for $u \in (0, d]$. Hence part (ii) of Proposition 10.48 shows that the function Γ_1 is continuous on $(0, d]$. Also, an analogue of this result for improper integrals of the second kind together with part (ii) of Lemma 10.73 shows that the function Γ_2 is continuous on $[c, \infty)$. Consequently, the gamma function $\Gamma := \Gamma_1 + \Gamma_2$ is continuous on $[c, d]$. Since $c > 0$ and $d > c$ are arbitrary, it follows that Γ is continuous on $(0, \infty)$.

Let $g(t, u) := e^{-t}t^{u-1}$ for $(t, u) \in (0, \infty) \times (0, \infty)$. Then $D_2 g(t, u) = e^{-t}t^{u-1}(\ln t)$ for $(t, u) \in (0, \infty) \times (0, \infty)$. We note that the function $D_2 g$ is bounded on $[1, x] \times (0, d]$ for every $x \in [1, \infty)$ and also on $[x, 1] \times [c, \infty)$ for every $x \in (0, 1]$.

For $x \in [1, \infty)$, define $G_1(x, u) := \int_1^x g(t, u)dt$ for $u \in (0, d]$. By Proposition 10.52, we obtain

$$D_2 G_1(x, u) = \int_1^x D_2 g(t, u)dt = \int_1^x e^{-t} t^{u-1} (\ln t)dt \quad \text{for } u \in (0, d].$$

Now using part (ii) of Lemma 10.73 (with $k = 1$) and Proposition 10.67, we see that the function Γ_1 is differentiable on $(0, d]$, and

$$\Gamma_1'(u) = \int_1^\infty D_2 g(t, u)dt = \int_1^\infty e^{-t} t^{u-1} (\ln t)dt \quad \text{for } u \in (0, d].$$

Similarly, for $x \in (0, 1]$,

$$D_2 G_1(x, u) = \int_x^1 D_2 g(t, u)dt = \int_x^1 e^{-t} t^{u-1} (\ln t)dt \quad \text{for } u \in [c, \infty).$$

Using part (ii) of Lemma 10.73 (with $k = 1$) and Proposition 10.72, we see that the function Γ_2 is differentiable on $[c, \infty)$, and

$$\Gamma_2'(u) = \int_{0+}^1 D_2 g(t, u)dt = \int_{0+}^1 e^{-t} t^{u-1} (\ln t)dt \quad \text{for } u \in [c, \infty).$$

Hence $\Gamma = \Gamma_1 + \Gamma_2$ is differentiable on $[c, d]$, and

$$\Gamma'(u) = \Gamma_1'(u) + \Gamma_2'(u) = \int_{0+}^\infty e^{-t} t^{u-1} (\ln t)dt \quad \text{for } u \in [c, d].$$

Since $c > 0$ and $d > c$ are arbitrary, we see that Γ is differentiable on $(0, \infty)$, and $\Gamma'(u) = \int_{0+}^\infty e^{-t} t^{u-1} (\ln t)dt$ for all $u \in (0, \infty)$.

Applying the argument given above to Γ' in place of Γ (and using part (ii) of Lemma 10.73 with $k = 2$), we see that Γ' is differentiable on $(0, \infty)$, and $\Gamma''(u) = \int_{0+}^\infty e^{-t} t^{u-1} (\ln t)^2 dt$ for all $u \in (0, \infty)$. Continuing in this way, we see that $\Gamma^{(k)}$ exists and is given by the desired expression for each $k \in \mathbb{N}$.

(iii) As observed in (ii) above,

$$\Gamma''(u) = \int_{0+}^\infty e^{-t} t^{u-1} (\ln t)^2 dt \quad \text{for all } u \in (0, \infty).$$

We note that for each $u \in (0, \infty)$, the integrand above is positive and continuous on $(0, \infty)$. Hence $\Gamma'' > 0$ on $(0, \infty)$, and so by part (iii) of Proposition 4.34, Γ is strictly convex on $(0, \infty)$. Next, $\Gamma(2) = (2-1)! = 1 = (1-0)! = \Gamma(1)$, as we have seen in part (iii) of Proposition 9.60. Hence by the Rolle Theorem (Proposition 4.17), there exists $a \in (1, 2)$ such that $\Gamma'(a) = 0$. Such a number a is unique, since the function Γ' is strictly increasing on $(0, \infty)$ (part (iii) of Proposition 4.33). It follows that $\Gamma' < 0$ on $(0, a)$, and $\Gamma' > 0$ on (a, ∞). So the function Γ is strictly decreasing on $(0, a)$ and strictly increasing on (a, ∞). Further, since $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$ (part (iii) of Proposition 9.60), we see that $\Gamma(u) \to \infty$ as $u \to \infty$. (See Figure 10.2.) □

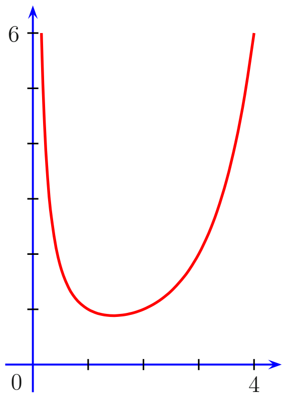

Fig. 10.2. The graph of the gamma function on $[0.1555, 4]$.

Notes and Comments

Unlike our treatment of sequences of real numbers in Chapter 2, and of series of real numbers in Chapter 9, we have not given formal definitions of sequences and series of real-valued functions. However, at this stage an interested reader can easily formulate them as follows. If E is a set and if $\mathbb{R}^E$ denotes the set of all functions from E to $\mathbb{R}$, then a **sequence of real-valued functions** *on E is a map from $\mathbb{N}$ to $\mathbb{R}^E$. Thus a sequence (f_n) of real-valued functions on E is formally the map that associates to $n \in \mathbb{N}$ the function $f_n : E \to \mathbb{R}$. Likewise, a* **series of real-valued functions** *on E is an ordered pair $\big((f_k), (S_n)\big)$ of sequences (f_k) and (S_n) of real-valued functions on E such that $S_n(x) = \sum_{k=1}^{n} f_k(x)$ for all $n \in \mathbb{N}$ and $x \in E$. Such a series is denoted by $\sum_{k\geq 1} f_k$, and in case this series converges pointwise to a function $S : E \to \mathbb{R}$, then we denote S by $\sum_{k=1}^{\infty} f_k$. A formal definition of an improper integral depending on a parameter can be given in a similar manner. Given $a \in \mathbb{R}$, a set E, and a function $f : [a, \infty) \times E \to \mathbb{R}$, the improper integral of f on $[a, \infty)$ depending on a parameter in E is denoted by $\int_{t\geq a} f(t, \cdot)dt$, and in case this improper integral converges to a function $I : E \to \mathbb{R}$, then we denote I by $\int_a^{\infty} f(t, \cdot)dt$.*

The two celebrated results given in Section 10.4 (Propositions 10.34 and 10.37) were proved by K. Weierstrass in a remarkable paper of 1885. (See [86].) His proofs used approximate identities, convolutions, and complex function theory. For a detailed and readable account of the developments related to the contributions of Weierstrass to approximation theory, we refer to [68]. In 1912, S. Bernstein gave an elegant proof of the polynomial approximation theorem using special polynomials that are now called Bernstein polynomials. (See [11].) In 1900, L. Fejér proved the trigonometric polynomial approximation theorem using the arithmetic means of the partial sums of Fourier series. (See [28].) We have followed these approaches in Propositions 10.33 and 10.36. There is a notable similarity in the proofs of Bernstein and Fejér. Both proofs *involve "positive linear maps". In 1953, P. P. Korovkin proved surprisingly strong results about such maps from which Propositions 10.33 and 10.36 can be deduced as special cases. (See [52, Theorems 4 and 5 of Chapter 1] and [59, Theorems 3.11 and 3.13].) M. H. Stone substantially generalized the results of Weierstrass in 1937, and later simplified his proof in 1948. He considered the "algebra" of continuous functions on an arbitrary "compact Hausdorff topological space" (rather than a closed and bounded interval), and more general "subalgebras" of continuous functions (rather than the subalgebra of polynomial functions). His result is now known as the Stone–Weierstrass theorem. See, for example, [71, Theorem 7.32].*

Section 10.5 is based on the concept of bounded convergence of a sequence of functions. Let (f_n) be a sequence of integrable functions defined on an interval $[a, b]$. It is customary to require uniform convergence of this sequence to justify an interchange of the operation of taking the limit as $n \to \infty$ with the operation of Riemann integration. In 1885, C. Arzelà proved an interesting result that says that instead of uniform convergence, we may require only pointwise convergence and uniform boundedness of the sequence (f_n), provided the pointwise limit is integrable on $[a, b]$. (See [7].) However, this result remained inaccessible to students of elementary analysis because its initial proof was based on the theory of Lebesgue measure. Elementary proofs not involving the Lebesgue theory were offered by several mathematicians including F. Riesz, L. Bieberbach, E. Landau, F. Hausdorff, H. S. Carslaw, H. A. Lauwerier, J. D. Weston, W. F. Eberlein, W. A. Luxemburg, and J. W. Lewin. We have based our proof of the Arzelà theorem (Proposition 10.40) on the elementary proof given by Luxemburg in [62], mainly because it neatly utilizes the Dini theorem for continuous functions (which we have proved in Proposition 10.7) to obtain a monotone convergence theorem for integrable functions. This proof is admittedly more involved than our proofs of other results in this chapter. However, it is simple minded and straightforward.

A result stronger than the Arzelà theorem stated above is true: If a sequence (f_n) of integrable functions on $[a, b]$ converges pointwise and is uniformly bounded on $[a, b]$, then the sequence $\left(\int_a^b f_n(x)dx\right)$ converges in $\mathbb{R}$. (In this version, the pointwise limit function f plays no role.) While this is an immediate consequence of the Lebesgue Bounded Convergence Theorem, it can be proved without using the Lebesgue theory. (See [55] and [57, Section 14.9].)

One of the important consequences of the Arzelà Theorem is a result on the validity of the interchange of the order of integration of a real-valued bounded function f on $[a, b] \times [c, d]$ (Proposition 10.50) that was first presented by G. Fichtenholz to the Faculty of Physical and Mathematical Sciences of the University of Odessa in 1910, and was independently published by L. Lichtenstein in the same year. (See [29] and [58].) Using the stronger version of the Arzelà Theorem mentioned above, one can drop the hypotheses of the integrability of the functions F and G made in Proposition 10.50. Thus if $f : [a, b] \times [c, d] \to \mathbb{R}$ is a bounded function such that for every $u \in [c, d]$, the function $f(\cdot, u)$ is in-

tegrable on $[a,b]$ *and for every* $t \in [a,b]$*, the function* $f(t,\cdot)$ *is integrable on* $[c,d]$*, then* $\int_c^d \big(\int_a^b f(t,u)dt\big)du = \int_a^b \big(\int_c^d f(t,u)du\big)dt$*. (See [56] and [57, Theorem 16.6.2].) This result does not involve the double integral of* f*, and so it is different from the well-known Fubini's theorem given, for example, in [33, Proposition 5.28]. Also, it is essentially a result about Riemann integrals in the sense that its analogue for Lebesgue integrals is not true even under the so-called continuum hypothesis. (See the last paragraph of [56].)*

In Section 10.6, we deal with properties of the Riemann integral of a function that depends on a parameter. They lay the foundation for the results of Section 10.7 concerning uniform convergence of an improper integral depending on a parameter. As pointed out in the beginning of Section 10.6, a similar foundation was not necessary for the results of Section 10.3 concerning uniform convergence of an infinite series of functions. This is because a finite sum of functions clearly inherits properties of summands such as continuity, integrability, and differentiability, while the same is not obvious for the Riemann integral of a function depending on a parameter. Other than the boundedness of a real-valued function defined on an arbitrary set E *considered in Section 1.3, in this book we have treated properties of functions of only one real variable. In particular, we have refrained from alluding to properties such as bivariate continuity and bivariate differentiability (since this is a book on one-variable calculus), and so it was a significant challenge to present the results of Sections 10.6 and 10.7. The Arzelà Theorem came to our rescue in this situation, and it enabled us to prove Propositions 10.48, 10.50, and 10.52 within the scope of this book. An alternative way of proving the continuity and the differentiability of an integral function would have been to use the concept of "pointwise equicontinuity" of a family of real-valued functions of a real variable. This approach is indicated in Exercises 10.50 and 10.51.*

Exercises

Part A

10.1. (i) For $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(k/n!) := 1$ if $k = 0, 1, 2, \ldots, n!$ and $f_n(x) := 0$ otherwise. Show that each f_n is integrable on $[0,1]$, but $f_n \to f$ on $[0,1]$, where the Dirichlet function f is not integrable.

(ii) For $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) := n^3 x e^{-nx}$. Show that each f_n is integrable on $[0,1]$, and $f_n \to 0$ on $[0,1]$, but $\int_0^1 f_n(x)dx \to \infty$.

(iii) For $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) := n^2 x e^{-nx}$. Show that each f_n is integrable on $[0,1]$ and $f_n \to 0$ on $[0,1]$, but $\int_0^1 f_n(x)dx \to 1$.

10.2. Let E be a set, and let (f_n) be a sequence of real-valued functions defined on E. We say that (f_n) is **pointwise bounded** on E if for each $x \in E$, there exists $\alpha_x \in \mathbb{R}$ such that $|f_n(x)| \le \alpha_x$ for all $n \in \mathbb{N}$. Show that a pointwise convergent sequence is pointwise bounded, but the converse does not hold. However, if a monotonic sequence is pointwise bounded on E, then show that it is pointwise convergent on E. If (f_n) is monotonic and uniformly bounded on E, must (f_n) be uniformly convergent on E?

10.3. For $n \in \mathbb{N}$, define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) := x^n e^{-nx}$. Show that $f_n \to 0$ uniformly on $[0, \infty)$. (Hint: For $n \in \mathbb{N}$, f_n attains its maximum at 1.)

10.4. For $n \in \mathbb{N}$, define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) := \big(1 + (x/n)\big)^n$. Show that $f_n \le f_{n+1}$ on $[0, \infty)$ for each $n \in \mathbb{N}$. Also, show that for every $b \in [0, \infty)$, the sequence (f_n) converges uniformly to the exponential function on $[0, b]$. (Compare Exercise 2.8. Hint: Example 2.10 (ii) and Corollary 7.7.)

10.5. Let E be an interval in $\mathbb{R}$, and let a sequence (f_n) of real-valued functions on E converge pointwise to a function f on E. If each f_n is monotonic on E, must f be monotonic on E? If each f_n is convex on E, must f be convex on E? If each f_n has the IVP on E, must f have the IVP on E?

10.6. Let E be a subset of $\mathbb{R}$, and let a sequence (f_n) of real-valued functions on E converge uniformly to a function f on E. If each f_n is uniformly continuous on E, show that f is uniformly continuous on E.

10.7. Let $g : [-1, 1] \to \mathbb{R}$ be a continuous function such that $g(0) \ne 0$. Let $n \in \mathbb{N}$, and define $f_n(x) := e^{-nx^2} g(x)$ for $x \in [-1, 1]$. Show that (f_n) converges pointwise but not uniformly on $[-1, 1]$. Find $\lim_{n\to\infty} \int_{-1}^{1} f_n(x)dx$.

10.8. Let $k \in \mathbb{N}$ and define $f_k(x) := x/k(x + k)$ for $x \in [0, 1]$. Show that $\int_0^1 \big(\sum_{k=1}^{\infty} f_k(x)\big)dx = \gamma$, where γ is the Euler constant. (See the note on Exercise 7.2.)

10.9. Let (f_n) and (g_n) be sequences of functions that are uniformly convergent on a set E. Show that the sequence $(f_n + g_n)$ converges uniformly on E, but the sequence $(f_n g_n)$ may not converge uniformly on E. What can you say if f_n and g_n are bounded functions on E for all $n \in \mathbb{N}$?

10.10. Let $n \in \mathbb{N}$ and define $f_n(x) := \ln(nx + 1)/n$ for $x \in [0, 1]$. Show that (f_n) converges uniformly on $[0, 1]$, and (f_n') converges pointwise but not uniformly on $[0, 1]$. (Compare Example 10.2 (i).)

10.11. **(kth Term Test for Uniform Convergence)** Let (f_k) be a sequence of real-valued functions defined on a set E such that the series $\sum_{k\ge1} f_k$ converges uniformly on E. Show that $f_k \to 0$ uniformly on E. Deduce that the series $\sum_{k\ge0} x^k/k!$ does not converge uniformly for $x \in \mathbb{R}$.

10.12. Let (f_k) be a sequence of nonnegative continuous functions defined on a closed and bounded subset E of $\mathbb{R}$. Suppose the series $\sum_{k\ge1} f_k(x)$ converges pointwise on E and the sum function is continuous on E. Show that the series converges uniformly on E. (Hint: Proposition 10.7)

10.13. For $k \in \mathbb{N}$ and $x \in (-1, \infty)$, let $f_k(x) := x^k/(1 + x)^k$. If a and b are real numbers such that $b > a > -1/2$, then show that the series $\sum_{k\ge0} f_k$ converges uniformly on $\in [a, b]$, but it does not converge uniformly either on $(-1/2, b]$ or on $[a, \infty)$. (Hint: If $r \in (0, 1)$ satisfies $-a \le (a + 1)r$ and $b \le (b + 1)r$, then $|x| \le |x + 1|r$ for all $x \in [a, b]$.)

10.14. Show that the series $\sum_{k\ge1}(-1)^k/(x^2 + k)$ converges uniformly for $x \in \mathbb{R}$, and $\big|\sum_{k=n}^{\infty}(-1)^k/(x^2 + k)\big| \le 1/n$. (Hint: Corollary 10.21)

10.15. Let $\delta \in (0,\infty)$. If $E := \{x \in \mathbb{R} : \delta \le |x| \le \pi\}$, then show that the series

$$\sum_{k\ge 2} \frac{\sin kx}{\ln k} \quad \text{and} \quad \sum_{k\ge 2} \frac{\cos kx}{\ln k}$$

converge uniformly for $x \in E$. (Hint: Corollary 10.22)

10.16. **(Abel Test for Uniform Convergence of Series)** Let (f_k) and (g_k) be sequences of real-valued functions defined on a set E such that (f_k) is uniformly bounded on E, the sequence $(f_k(x))$ is monotonic for each $x \in E$, and the series $\sum_{k\ge1} g_k$ converges uniformly on E. Show that the series $\sum_{k\ge1} f_k g_k$ converges uniformly on E. Further, show that

$$\left|\sum_{k=n}^{\infty} f_k(x) g_k(x)\right| \le 3\alpha\,\beta_n \quad \text{for all large } n \in \mathbb{N} \text{ and all } x \in E,$$

where $\alpha := \sup\{|f_k(x)| : k \in \mathbb{N} \text{ and } x \in E\}$ and $\beta_n := \sup\left\{\left|\sum_{k=n}^{m} g_k(x)\right| : m \in \mathbb{N} \text{ with } m \ge n \text{ and } x \in E\right\}$. (Hint: Exercise 9.15)

10.17. For $k \in \mathbb{N}$, let $f_k : [a,b] \to \mathbb{R}$ be integrable, and suppose the series $\sum_{k\ge1} f_k$ is uniformly convergent on $[a,b]$. Show that the series $\sum_{k\ge1} \int_a^x f_k(t)dt$ converges uniformly for $x \in [a,b]$. (Hint: Proposition 10.9)

10.18. Let $a \in (0,\infty)$. Show successively that the three series

$$\sum_{k\ge1} \frac{(-1)^k}{k^2} \sin\left(\frac{x}{k}\right), \quad \sum_{k\ge1} \frac{(-1)^k}{k} \cos\left(\frac{x}{k}\right) \quad \text{and} \quad \sum_{k\ge1} (-1)^k \sin\left(\frac{x}{k}\right)$$

converge uniformly for $x \in [-a,a]$. (Hint: Proposition 10.18, part (v) of Theorem 10.26, and Exercise 10.17.)

10.19. Let $a > 0$ be given. Show that the series $\sum_{k\ge0} \cos(x/k)$ does not converge for any $x \in [-a,a]$, but the "derived" series $-\sum_{k\ge1} (1/k)\sin(x/k)$ converges uniformly for $x \in [-a,a]$. (Compare part (v) of Theorem 10.26.)

10.20. (i) Show that

$$\ln x = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(x-1)^k}{k} \quad \text{for } x \in (0,2).$$

(Hint: Part (ii) of Proposition 10.29. Compare Example 9.34 (iii).)

(ii) Show that

$$\sin x = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} \quad \text{and} \quad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad \text{for } x \in \mathbb{R}.$$

(Hint: Exercise 7.24 and Part (ii) of Proposition 10.29. Compare Example 9.34 (i) and Exercise 9.21.)

10.21. Show that

$$\sum_{k=1}^{\infty} x^k/k = -\ln(1-x) \quad \text{and} \quad \sum_{k=1}^{\infty} kx^k = x/(1-x)^2$$

for $x \in (-1, 1)$. (Hint: Parts (i) and (ii) of Proposition 10.29)

10.22. Prove the following.

(i) The power series $\sum_{k\geq 1} x^k/k^2$ converges uniformly for $x \in [-1, 1]$.

(ii) The power series $\sum_{k\geq 1} x^k/k$ converges uniformly for $x \in [-1, 0]$, but it does not converge uniformly for $x \in [0, 1)$. (Hint: Corollary 10.21, Proposition 10.5, and Exercise 10.21)

(iii) The power series $\sum_{k\geq 0} x^k$ does not converge uniformly for $x \in (-1, 0]$ or for $x \in [0, 1)$ even though it converges pointwise for $x \in (-1, 1)$.

10.23. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) := x$ if $x \in [0, 1/2]$, and $f(x) := 1-x$ if $x \in (1/2, 1]$. For $n \in \mathbb{N}$ and $x \in [0, 1]$, define

$$f_n(x) := \sum_{k=0}^{[n/2]} \binom{n-1}{k-1} x^k(1-x)^{n-k} + \sum_{[n/2]+1}^{n} \binom{n-1}{k} x^k(1-x)^{n-k}.$$

Show that $f_n \to f$ uniformly on $[0, 1]$. (Hint: Find $B_n(f)$ for $n \in \mathbb{N}$.)

10.24. Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) := |x|$ for $x \in [-1, 1]$. Find a sequence (P_n) of polynomial functions on $[-1, 1]$ such that $P_n \to f$ uniformly on $[-1, 1]$, and $P_n(0) = 0$ for all $n \in \mathbb{N}$.

10.25. Let f be a continuously differentiable function on $[a, b]$. Show that there is a sequence (P_n) of polynomial functions on $[a, b]$ such that $P_n \to f$ uniformly on $[a, b]$, and $P_n' \to f'$ uniformly on $[a, b]$.

10.26. Let f be a real-valued differentiable function on $[-\pi, \pi]$ satisfying $f(\pi) = f(-\pi)$. If f' is integrable on $[-\pi, \pi]$, then show that the Fourier coefficients of f' are given by $a_0(f') = 0$, and $a_k(f') = kb_k(f)$, while $b_k(f') = -ka_k(f)$ for $k \in \mathbb{N}$. (Hint: Proposition 6.28)

10.27. Let f be a real-valued continuous function on $[-\pi, \pi]$ satisfying $f(\pi) = f(-\pi)$. If the Fourier series of f converges pointwise to a function g on $[-\pi, \pi]$, then show that g is continuous, and in fact, $g = f$. Deduce that if the series $\sum_{k\geq 1}(|a_k(f)| + |b_k(f)|)$ is convergent, then the Fourier series of f converges uniformly to f on $[-\pi, \pi]$ and absolutely for each $x \in [-\pi, \pi]$.

10.28. Let $f, g : [-\pi, \pi] \to \mathbb{R}$ be continuous functions satisfying $f(\pi) = f(-\pi)$ and $g(\pi) = g(-\pi)$. If each Fourier coefficient of f is equal to the corresponding Fourier coefficient of g, then show that $f = g$.

10.29. Let f be a continuously differentiable function on $[-\pi, \pi]$ satisfying $f(\pi) = f(-\pi)$ and $f'(\pi) = f'(-\pi)$. Show that $\sigma_n(f) \to f$ as well as $\sigma_n(f)' \to f'$ uniformly on $[-\pi, \pi]$. (Hint: For $n = 0, 1, 2, \ldots$, apply Proposition 10.52 to the integral representation of $\sigma_n(f)$ to obtain $\sigma_n(f)' = \sigma_n(f')$.)

10.30. Let $g : [0, 1] \to \mathbb{R}$ be an integrable function such that $g(0) = 0$ and $\int_0^1 g(x)dx \neq 0$. (For example, $g(x) := x$ or $g(x) := \sin \pi x$ for $x \in [0, 1]$.)

Let (c_n) be a sequence in $\mathbb{R}$, and for $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) := c_n g(nx)$ if $0 \le x \le (1/n)$ and $f_n(x) := 0$ if $(1/n) < x \le 1$. Show that $f_n \to 0$ on $[0,1]$. Further, show that $f_n \to 0$ boundedly on $[0,1]$ if and only if the sequence (c_n) is bounded, and $f_n \to 0$ uniformly on $[0,1]$ if and only if $c_n \to 0$, and in both cases, $\int_0^1 f_n(x)dx \to 0$, but if $c_n := n$ for $n \in \mathbb{N}$, then $\left(\int_0^1 f_n(x)dx\right)$ may not converge to 0.

10.31. Let $f : [a,b] \to \mathbb{R}$ be nonnegative and bounded. Show that for every $\epsilon > 0$, there is a continuous piecewise linear function $h : [a,b] \to \mathbb{R}$ such that $f \le h$ on $[a,b]$ and $\int_a^b h(x)dx - \epsilon < U(f)$. (Compare Lemma 10.38.)

10.32. **(kth Term Test for Bounded Convergence)** Let (f_k) be a sequence of real-valued functions on a set E such that the series $\sum_{k\ge1} f_k$ converges boundedly on E. Show that $f_k \to 0$ boundedly on E. In particular, let $E := [0,1]$, and for $k \in \mathbb{N}$, define $f_k : E \to \mathbb{R}$ by $f_k(x) := k$ if $0 \le x \le 1/k$ and $f_k(x) := 0$ if $1/k < x \le 1$. Deduce that the series $\sum_{k\ge1} f_k(x)$ does not converge boundedly for $x \in E$.

10.33. Let $f : [-\pi,\pi] \to \mathbb{R}$ be an integrable function. If (f_n) is a sequence of real-valued integrable functions on $[-\pi,\pi]$ such that $f_n \to f$ boundedly on $[-\pi,\pi]$, then show that $a_0(f_n) \to a_0(f)$, and $a_k(f_n) \to a_k(f)$ as well as $b_k(f_n) \to b_k(f)$ for each $k \in \mathbb{N}$.

10.34. Suppose a trigonometric series $a_0 + \sum_{k\ge1}(a_k \cos kx + b_k \sin kx)$ converges boundedly for $x \in [-\pi,\pi]$, and its sum function f is integrable on $[-\pi,\pi]$. Show that the series is the Fourier series of the function f, that is, $a_0 = a_0(f)$, and $a_k = a_k(f)$, $b_k = b_k(f)$ for all $k \in \mathbb{N}$. (Hint: Proposition 10.46)

10.35. Consider $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by $f(t,u) := 1/t$ if $0 < u \le t \le 1$, while $f(t,u) := 1/u$ if $0 < t \le u \le 1$, and $f(t,u) := 0$ otherwise. Show that for each fixed $t \in [0,1]$, the function $f(t,\cdot)$ is bounded on $[0,1]$, but the integral function F corresponding to f is not bounded on $[0,1]$. Why does this not contradict part (i) of Proposition 10.48?

10.36. Consider $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by $f(t,u) := 1/t^2$ if $0 < u \le t \le 1$, while $f(t,u) := -1/u^2$ if $0 < t \le u \le 1$, and $f(t,u) := 0$ otherwise. Show that for each fixed $u \in [0,1]$, the function $f(\cdot,u)$ is integrable on $[0,1]$, and if $F(u) := \int_0^1 f(t,u)dt$ for $u \in [0,1]$, then $\int_0^1 F(u)du$ exists, while for each fixed $t \in [0,1]$, the function $f(t,\cdot)$ is integrable on $[0,1]$, and if $G(t) := \int_0^1 f(t,u)du$ for $t \in [0,1]$, then $\int_0^1 G(t)dt$ exists. Show that $\int_0^1 F(u)du \ne \int_0^1 G(t)dt$. Why does this not contradict Proposition 10.50?

10.37. **(Abel Test for Uniform Convergence of Improper Integrals)** Let $a \in \mathbb{R}$ and let E be a set. Let $f, g : [a,\infty) \times E \to \mathbb{R}$ be such that for each $u \in E$, the function $f(\cdot,u)$ is monotonic and differentiable on $[a,\infty)$, and $D_1 f(\cdot,u)$ is integrable on $[a,x]$ for every $x \in [a,\infty)$, while the function $g(\cdot,u)$ is continuous on $[a,\infty)$. Also, suppose f is bounded on $[a,\infty) \times E$ and the improper integral $\int_{t\ge a} g(t,u)dt$ converges uniformly for $u \in E$. Show that the improper integral $\int_{t\ge a} f(t,u)g(t,u)dt$ converges uniformly for $u \in E$. Further, show that

$$\left|\int_x^\infty f(t,u)g(t,u)dt\right| \le 3\alpha\,\beta_x \quad \text{ for all large } x \in [a,\infty) \text{ and all } u \in E,$$

where $\alpha := \sup\{|f(t,u)| : t \in [a,\infty),\ u \in E\}$ and $\beta_x := \sup\left\{\left|\int_x^y g(t,u)dt\right| : (y,u) \in [x,\infty)\times E\right\}$. (Hint: Exercise 9.32. Compare Proposition 10.58.)

10.38. (i) Define $f : [0,\infty) \times [0,1] \to \mathbb{R}$ by $f(t,u) := 1/t^2$ if $0 < u < t \le 1$, $f(t,u) := -1/u^2$ if $0 < t < u \le 1$, and $f(t,u) := 0$ otherwise. Let $I(u) := -1$ for $u \in [0,\infty)$. Show that the improper integral $\int_{t\ge 0} f(t,\cdot)dt$ converges uniformly, but not boundedly, to I on $[0,1]$.

(ii) Define $f : [0,\infty) \times [-1,1] \to \mathbb{R}$ by $f(t,u) := ue^{-t|u|}$. Also, define $I : [-1,1] \to \mathbb{R}$ by $I(u) := -1$ for $u \in [-1,0)$, $I(0) := 0$, and $I(u) := 1$ for $u \in (0,1]$. Show that the improper integral $\int_{t\ge 0} f(t,\cdot)dt$ converges boundedly, but not uniformly, to I on $[-1,1]$.

10.39. In Propositions 10.12 and 10.41, if (b_n) is a sequence in $[a,\infty)$ such that $b_n \to \infty$, then show that $\int_a^{b_n} f_n(x)dx \to \int_a^\infty f(x)dx$.

10.40. Let $E := [a,\infty) \times [c,\infty)$, and let $f : E \to \mathbb{R}$ be a nonnegative function. Suppose that for every $(b,d) \in E$, the iterated integral $\int_a^b \left(\int_c^d f(t,u)du\right)dt$ exists. Let (b_n) be a sequence in (a,∞) such that $b_n \to \infty$, and let (d_n) be a sequence in $[c,\infty)$ such that $d_n \to \infty$. Show that

$$\int_a^{b_n}\left(\int_c^{d_n} f(t,u)du\right)dt \to \int_a^\infty\left(\int_c^\infty f(t,u)du\right)dt.$$

(Hint: Apply part (ii) of Proposition 10.41 and Exercise 10.39 to the sequence (G_n), where $G_n : [a,\infty) \to \mathbb{R}$ is defined by $G_n(t) := \int_c^{d_n} f(t,u)du$ if $t \in [a,b_n]$ and $G_n(t) := 0$ if $t > b_n$ for $n \in \mathbb{N}$.)

10.41. Let f be a nonnegative function defined on $[a,\infty) \times [c,\infty)$. Suppose the iterated integrals $\int_a^b \left(\int_c^d f(t,u)du\right)dt$ and $\int_c^d \left(\int_a^b f(t,u)dt\right)du$ exist for every $(b,d) \in [a,\infty) \times [c,\infty)$. Show that

$$\int_c^\infty\left(\int_a^\infty f(t,u)dt\right)du = \int_a^\infty\left(\int_c^\infty f(t,u)du\right)dt.$$

(Hint: Proposition 10.50 and Exercise 10.40)

10.42. Let $f(t,u) := (t^2-u^2)/(t^2+u^2)^2$ for $(t,u) \in [1,\infty) \times [1,\infty)$. Show that

$$\int_1^\infty\left(\int_1^\infty f(t,u)dt\right)du = \pi/4 = -\int_1^\infty\left(\int_1^\infty f(t,u)du\right)dt.$$

(Hint: Use the substitution $t = u\tan\theta$.)

Part B

10.43. **(Pólya Theorem)** If a sequence of monotonic functions defined on $[a,b]$ converges pointwise to a continuous function on $[a,b]$, then show that the convergence is uniform on $[a,b]$. (Hint: It suffices to assume that each term of the sequence is an increasing function. Use Proposition 3.20.)

10.44. Let $r \in (0, \infty)$ be the radius of convergence of a power series $\sum_{k\geq 0} c_k x^k$, and let $S : (-r, r) \to \mathbb{R}$ denote its sum function.

(i) **(Abel Theorem for Power Series: Continuity at r)** Suppose the series $\sum_{k\geq 0} c_k r^k$ is convergent. Show that

$$\lim_{x\to r^-} \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k r^k.$$

(Hint: Assume $r = 1$ with no loss of generality. For $m \geq 0$, write $\sum_{k=0}^{m} c_k x^k = (1 - x)\sum_{k=0}^{m-1} C_k x^k + C_m x^m$, where $C_m := \sum_{k=0}^{m} c_k$. Then $S(x) = (1-x)\sum_{k=0}^{\infty} C_k x^k$ and $(1-x)\sum_{k=0}^{\infty} x^k = 1$ for $|x| < 1$.)

(ii) **(Integration on $[0, r]$)** Suppose the sequence of the partial sums of the series $\sum_{k\geq 0} c_k r^k$ is bounded. Define $S(r) := 0$. Show that the function S is integrable on $[0, r]$, and

$$\int_0^r S(x)dx = \sum_{k=0}^{\infty} c_k \frac{r^{k+1}}{k+1}.$$

Deduce that $\ln 2 = \sum_{k=0}^{\infty}(-1)^k/(k+1)$ and $\pi/4 = \sum_{k=0}^{\infty}(-1)^k/(2k+1)$. (Hint: For $x \in [0, r)$, write $c_k x^k = a_k b_k$, where $a_k := (x/r)^k$ and $b_k := c_k r^k$ for $k \geq 0$. Use Propositions 9.21 and 10.40.)

10.45. Under the hypotheses of part (ii) of Corollary 10.24, show that the sum function S of a Dirichlet series is infinitely differentiable on (x_0, ∞). For a fixed $n \in \mathbb{N}$, prove that the nth derivative of S is given by

$$S^{(n)}(x) = (-1)^n \sum_{k=0}^{\infty} a_k \lambda_k^n e^{-\lambda_k x} \quad \text{for } x \in (x_0, \infty).$$

(Hint: Part (v) if Theorem 10.26. For fixed $n \in \mathbb{N}$ and $x \in (x_0, \infty)$, the sequence $\left(\lambda_k^n e^{-\lambda_k(x-x_0)}\right)$ is decreasing when $\lambda_k \geq n/(x - x_0)$.)

10.46. Let (f_n) be a sequence of real-valued bounded functions defined on a set E, and let (g_n) be the sequence of arithmetic means of (f_n). If $f : E \to \mathbb{R}$ and $f_n \to f$ uniformly on E, then show that $g_n \to f$ uniformly on E. Conversely, if (g_n) is uniformly convergent on E, then show that (f_n) is uniformly convergent on E if and only if $\frac{1}{n}\sum_{k=1}^{n}(f_n - f_k) \to 0$ uniformly on E. (Compare Proposition 2.15 and use Proposition 10.5)

10.47. Let $(f_k)_{k\geq 0}$ be a sequence of real-valued bounded functions on a set E. If the series $\sum_{k\geq 0} f_k$ is uniformly convergent on E, then show that it is uniformly Cesàro convergent on E, and its Cesàro sum is equal to its sum. Conversely, if $\sum_{k\geq 0} f_k$ is uniformly Cesàro convergent on E, then show that $\sum_{k\geq 0} f_k$ is uniformly convergent on $E \iff \left(\sum_{k=1}^{n} k f_k\right)/(n+1) \to 0$ uniformly on E. Deduce that if $\sum_{k\geq 0} f_k$ is uniformly Cesàro convergent on E and if $k f_k \to 0$ uniformly on E, then $\sum_{k\geq 0} f_k$ is uniformly convergent on E. (Hint: Exercise 10.46. Compare Proposition 9.6 and Corollary 9.7.)

10.48. Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuous function such that $f(\pi) = f(-\pi)$. Show that $a_k(f) \to 0$ and $b_k(f) \to 0$. If in fact $ka_k(f) \to 0$ and $kb_k(f) \to 0$, then show that the Fourier series of f converges uniformly to f on $[-\pi, \pi]$. (Hint: Proposition 10.36 and Exercise 10.47)

10.49. Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuously differentiable function such that $f(\pi) = f(-\pi)$ and $f'(\pi) = f'(-\pi)$. Show that the Fourier series of f converges uniformly to f on $[-\pi, \pi]$. (Hint: Exercises 10.26 and 10.48)

10.50. Let E be a subset of $\mathbb{R}$ and $\mathcal{F}$ a family of real-valued functions defined on E. We say that $\mathcal{F}$ is **pointwise equicontinuous** on E if given $u_0 \in E$ and $\epsilon > 0$, there exists $\delta > 0$ such that $|f(u) - f(u_0)| < \epsilon$ for all $f \in \mathcal{F}$ and $u \in E$ with $|u - u_0| < \delta$, while $\mathcal{F}$ is **uniformly equicontinuous** on E if given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(u) - f(v)| < \epsilon$ for all $f \in \mathcal{F}$ and $u, v \in E$ with $|u - v| < \delta$. If (f_n) is a uniformly convergent sequence of real-valued continuous functions defined on E, then show that the family $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$ is pointwise equicontinuous on E, and if in addition, the set E is closed and bounded, then $\mathcal{F}$ is uniformly equicontinuous.

10.51. Let E be a set, and let $f : [a, b] \times E \to \mathbb{R}$ be a function such that for every $u \in E$, the Riemann integral $\int_a^b f(t, u)dt$ exists.
   (i) Suppose $E \subset \mathbb{R}$ and the family $\mathcal{F} := \{f(t, \cdot) : t \in [a, b]\}$ is pointwise equicontinuous on E. Show that the integral function F corresponding to f is continuous on E.
   (ii) Suppose E is an interval in $\mathbb{R}$, and for each $t \in [a, b]$, the function $f(t, \cdot)$ is differentiable on E and for every $u \in E$, the Riemann integral $\int_a^b D_2 f(t, u)dt$ exists. If the family $\mathcal{G} := \{D_2 f(t, \cdot) : t \in [a, b]\}$ is pointwise equicontinuous on E, then show that the integral function F corresponding to f is continuously differentiable on E and $F'(u) = \int_a^b D_2 f(t, u)dt$ for every $u \in E$.

10.52. **(Abel Integral Theorem)** Let $g : [a, \infty) \to \mathbb{R}$ be a continuous function, and let $u_0 \in \mathbb{R}$ be such that the improper integral $\int_{t \geq a} e^{-u_0 t} g(t)dt$ is convergent. Show that

$$\lim_{u \to u_0^+} \int_a^\infty e^{-ut} g(t)dt = \int_a^\infty e^{-u_0 t} g(t)dt.$$

(Hint: Let $u_0 := 0$. For $t \in [a, \infty)$ and $u \in (0, \infty)$, let $f(t, u) := e^{-ut}$ in Proposition 10.58. Use the continuity of the function $u \longmapsto e^{-au}$ at 0.)

10.53. Let $a, u \in (0, \infty)$. Prove the following equalities successively.
   (i) $\displaystyle\int_0^\infty e^{-ut} \cos at \, dt = \frac{u}{(u^2 + a^2)}$,
   (ii) $\displaystyle\int_0^\infty e^{-ut} \frac{\sin at}{t} dt = \frac{\pi}{2} - \arctan \frac{u}{a}$,
   (iii) $\displaystyle\int_0^\infty \frac{\sin t}{t} dt = \lim_{u \to 0^+} \int_0^\infty e^{-ut} \frac{\sin t}{t} dt = \frac{\pi}{2}$.
   (Hint: Propositions 6.28, 6.29, 6.24, and Exercise 10.52)

10.54. (i) Let $f : [0, \infty) \to \mathbb{R}$ be a bounded function that is integrable on $[0, x]$ for every $x \in [0, \infty)$. If f is continuous at 0, then show that

$$\lim_{u \to 0^+} \int_0^\infty \frac{uf(t)}{u^2 + t^2} dt = \frac{\pi}{2} f(0).$$

(Compare Example 10.69 (i))

(ii) Let c, $d \in \mathbb{R}$ be such that $0 < c < d$. Show that

$$\int_0^\infty \frac{\cos ct - \cos dt}{t^2} dt = \frac{\pi}{2}(d - c).$$

(Hint: Corollary 10.59, Exercise 10.53 (iii). Compare Example 10.69 (ii).)

(iii) For $u \in \mathbb{R}$, evaluate $\int_0^\infty e^{-t^2} \sin 2tu \, dt$. (Compare Example 10.69 (iii))

10.55. Under the hypotheses of part (ii) of Corollary 10.61, show that the function $u \longmapsto \mathcal{L}(f)(u)$ is infinitely differentiable on (u_0, ∞). For $n \in \mathbb{N}$, prove that

$$\mathcal{L}(f)^{(n)}(u) = (-1)^n \int_0^\infty e^{-ut} t^n f(t) dt = (-1)^n \mathcal{L}(f_n)(u) \quad \text{if } u \in (u_0, \infty),$$

where $f_n(t) := t^n f(t)$ for $t \in [0, \infty)$. In particular, if $n \in \mathbb{N}$ and $f(t) := 1$ for all $t \in [0, \infty)$, then show that $\mathcal{L}(f_n)(u) = n!/u^{n+1}$ for all $u \in (0, \infty)$. (Hint: Proposition 10.67. For fixed $n \in \mathbb{N}$ and $u \in (u_0, \infty)$, the function $t \longmapsto e^{-(u-u_0)t} t^n$ is decreasing on the interval $(n/(u - u_0), \infty)$.)

10.56. Under the hypotheses of Proposition 10.65, let $G(y, t) := \int_c^y f(t, u) du$ for $y \in [c, d]$ and $t \in [a, \infty)$. Show that $\int_a^x G(y, t) dt \to \int_c^y I(u) du$ as $x \to \infty$ uniformly for $y \in [c, d]$.

10.57. **(Dirichlet Test for Uniform Convergence of Improper Integrals of the Second Kind)** Let $a, b \in \mathbb{R}$ with $a < b$, and let E be a set. Let $f, g : (a, b] \times E \to \mathbb{R}$ be such that for each $u \in E$, the function $f(\cdot, u)$ is monotonic and differentiable on $(a, b]$, and the function $D_1 f(\cdot, u)$ is integrable on $[x, b]$ for every $x \in (a, b]$, while the function $g(\cdot, u)$ is continuous on $(a, b]$. Further, suppose $f(t, u) \to 0$ uniformly for $u \in E$ as $t \to a^+$, and the function $G : (a, b] \times E \to \mathbb{R}$ defined by $G(x, u) := \int_x^b g(t, u) dt$ is bounded on $(a, b] \times E$. Show that the improper integral $\int_{a < t \le b} f(t, u) g(t, u) dt$ converges uniformly for $u \in E$. Further, show that there exists $x_0 \in (a, b]$ such that for each $x \in (a, x_0]$,

$$\left| \int_{a^+}^x f(t, u) g(t, u) dt \right| \le 2\alpha(x)\beta \quad \text{for all } u \in E,$$

where $\alpha(x) := \sup\{|f(x, u)| : u \in E\}$ and

$$\beta := \sup\{|G(y, u)| : y \in (a, b] \text{ and } u \in E\}.$$

(Compare Proposition 10.58. Hint: Exercise 9.42.)

10.58. (i) Suppose $f : (0, b] \to \mathbb{R}$ is a monotonic and differentiable function such that $f(t) \to 0$ as $t \to 0$, and f' is integrable on $[x, b]$ for every $x \in (a, b]$. Let $\delta > 0$ be given. Show that the improper integrals $\int_{0<t\le b} f(t)\sin(u/t)t^{-2}dt$ and $\int_{0<t\le b} f(t)\cos(u/t)t^{-2}dt$ converge uniformly for $u \in (-\infty, -\delta] \cup [\delta, \infty)$. Further, show that for $x \in (0, b]$,

$$\left|\int_{0^+}^{x} f(t)\sin(u/t)t^{-2}dt\right| \quad \text{and} \quad \left|\int_{0^+}^{x} f(t)\cos(u/t)t^{-2}dt\right|$$

are less than or equal to $4|f(x)|/\delta$.

(ii) Suppose $f : (0, b] \to \mathbb{R}$ is a continuous function, and there exists $u_0 \in \mathbb{R}$ such that $\left\{ \int_x^b e^{-u_0/t} f(t)dt : x \in (0, b] \right\}$ is a bounded subset of $\mathbb{R}$. If $u_1 \in (u_0, \infty)$, then show that the improper integral $\int_{0<t\le b} e^{-u/t} f(t)dt$ converges uniformly for $u \in [u_1, \infty)$. Also, for $x \in (0, b]$, show that

$$\left|\int_{0^+}^{x} e^{u/t} f(t)dt\right| \le 2\beta e^{(u_1-u_0)/x},$$

where $\beta := \sup\left\{ \int_x^b e^{-u_0/t} f(t)dt : x \in (0, b] \right\}$. (Hint: Exercise 10.57)

10.59. Let $b \in (0, \infty)$, and suppose $f : (0, b] \to \mathbb{R}$ is integrable on $[x, b]$ for every $x \in (0, b]$. If there exist $\beta \in (0, \infty)$ and $p \in (0, 1)$ such that $|f(t)| \le \beta t^{-p}$ for all $t \in (0, b]$, then show that the improper integral $\int_{0<t\le b} f(t)e^{-ut}dt$ converges uniformly for $u \in [0, \infty)$, and absolutely for each $u \in [0, \infty)$.

10.60. (i) Let $a \in (0, 1)$. Show that the improper integral $\int_{0<t\le 1}(\ln ut)dt$ converges uniformly for $u \in [a, 1/a]$. Evaluate $\int_{0^+}^{1}(\ln ut)dt$.

(ii) Let $p \in (0, 1)$. Show that the improper integrals

$$\int_{0<t\le 1} e^{-ut}(\sin t)t^{-p}dt \quad \text{and} \quad \int_{0<t\le 1} e^{-ut}(\cos t)t^{-p}dt$$

converge uniformly for $u \in [0, \infty)$.

(iii) Let $q \in (0, 2)$ and let $\delta > 0$ be given. Show that the improper integrals

$$\int_{0<t\le 1} \sin(u/t)t^{-q}dt \quad \text{and} \quad \int_{0<t\le 1} \cos(u/t)t^{-q}dt$$

converge uniformly for $u \in (-\infty, \delta] \cup [\delta, \infty)$.

(iv) Let $q \in [1, 2)$ and $\delta > 0$ be given. Show that the improper integrals

$$\int_{0<t\le 1} e^{-u/t}\sin(1/t)t^{-q}dt \quad \text{and} \int_{0<t\le 1} e^{-u/t}\cos(1/t)t^{-q}dt$$

converge uniformly for $u \in [\delta, \infty)$.

(Hint: Proposition 10.57, Exercises 10.59, and 10.58)

10.61. Show that the improper integral $\int_{t\geq 0} e^{-t}\cos(ut^2)dt$ converges uniformly for $u \in \mathbb{R}$. Further, show that the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(u) := \int_0^\infty e^{-t}\cos(ut^2)dt \quad \text{for } u \in \mathbb{R}$$

is infinitely differentiable at $u = 0$ and moreover, $\phi^{(k)}(0) = 0$ if k is odd, while $\phi^{(k)}(0) = (-1)^{k/2}(2k)!$ if k is even. Deduce that the Taylor series of ϕ diverges at each $u \neq 0$. (Hint: Propositions 10.67, 9.60 and 9.8).

10.62. Define the **digamma function** $\psi : (0,\infty) \to \mathbb{R}$ by $\psi(u) := \Gamma'(u)/\Gamma(u)$. Show that $\psi(u+1) = (1/u) + \psi(u)$ for all $u \in (0,\infty)$. Also, show that

$$\psi(n+1) = \Gamma'(1)+\sum_{k=1}^{n}\frac{1}{k} \quad \text{and} \quad \Gamma'(n+1) = n!\left(\Gamma'(1)+\sum_{k=1}^{n}\frac{1}{k}\right) \quad \text{for } n \in \mathbb{N}.$$

10.63. Show that the gamma function attains its minimum value on $(0,\infty)$ at a unique point between 1 and 2, and the minimum value is between 0 and 1. Also, show that $\Gamma'(u) \to -\infty$ as $u \to 0$ and $\Gamma'(u) \to \infty$ as $u \to \infty$.

10.64. Let $c, d \in \mathbb{R}$ with $0 < c < d$. Show that

$$\int_c^d \Gamma(u)du = \int_{0^+}^{1^-}\frac{t^{d-1}-t^{c-1}}{e^t(\ln t)}dt + \int_{1^+}^{\infty}\frac{t^{d-1}-t^{c-1}}{e^t(\ln t)}dt.$$

10.65. **(Bivariate Continuity of the Beta Function)** If $E := (0,\infty)\times(0,\infty)$ and $f : (0,1)\times E \to \mathbb{R}$ is defined by $f(t,p,q) := t^{p-1}(1-t)^{q-1}$, then show that for each $c \in (0,\infty)$, the improper integral $\int_{0<t<1} f(t,p,q)dt$ converges uniformly for $(p,q) \in [c,\infty)\times[c,\infty)$. Deduce that if $p, q \in (0,\infty)$, and if (p_n) and (q_n) are sequences in $(0,\infty)$ such that $p_n \to p$ and $q_n \to q$, then $\beta(p_n,q_n) \to \beta(p,q)$.

10.66. Let $p, q \in (0,\infty)$. Show that

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Deduce that $\beta(p+1,q) = p\,\beta(p,q)/(p+q)$ for all $p,q \in (0,\infty)$. Prove that

$$D_1\beta(p,q) = \beta(p,q)\left(\frac{\Gamma'(p)}{\Gamma(p)} - \frac{\Gamma'(p+q)}{\Gamma(p+q)}\right).$$

(Hint: Exercise 10.41 gives $\Gamma(p)\Gamma(q) = \int_{0^+}^{\infty}(e^{-s}\int_{0^+}^{s^-}(s-u)^{p-1}u^{q-1}du)ds$.)

A

Construction of the Real Numbers

In Chapter 1 we have described the sets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ of natural numbers, integers, and rational numbers, respectively, as follows:

$$\mathbb{N} = \{1, 2, 3, \dots\}, \quad \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\},$$

and

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, \ n \neq 0 \right\}.$$

The set $\mathbb{R}$ of "real numbers" was then introduced as a set containing $\mathbb{Q}$ and the "irrational numbers" in such a way that the elements of $\mathbb{R}$ are in one-to-one correspondence with the points on the "number line". But $\mathbb{R}$ defies a simplistic description such as that given above for $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$. Thus, while we can conceive easily what a rational number is, it is a little harder to say precisely what a real number is. For this reason, we made an assumption that there exists a set $\mathbb{R}$ containing $\mathbb{Q}$ that satisfies the three sets of properties given in Section 1.1, namely the Algebraic Properties A1-A5, the Order Properties O1-O2, and the Completeness Property. The main aim of this appendix is to show that such a set $\mathbb{R}$ does indeed exist and is essentially unique. The approach that we shall take is due to Cantor, and uses Cauchy sequences of rational numbers. In what follows, we shall assume familiarity with the set $\mathbb{Q}$ and the usual algebraic operations on $\mathbb{Q}$ as well as the usual order relation that permits us to talk of the subset $\mathbb{Q}^+$ of positive rational numbers in such a way that the properties A1-A5 and O1-O2 in Section 1.1 are satisfied if we replace $\mathbb{R}$ by $\mathbb{Q}$ throughout. This appendix is divided into three sections, which are organized as follows. In the first section below, we discuss some preliminaries about equivalence relations and equivalence classes. Then in the next section, we outline a construction of $\mathbb{R}$ using Cauchy sequences of rational numbers. The "uniqueness" of $\mathbb{R}$ is formally established in the last section.

A.1 Equivalence Relations

The notion of an equivalence relation is basic to much of mathematics, and it will be useful in our formal construction of $\mathbb{R}$ from $\mathbb{Q}$. The most basic equivalence relation on any set is that of equality denoted by $=$. Fundamental properties of this relation motivate the following definition.

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Let S be a set. A **relation** on S is a subset of $S \times S$. If $\sim$ is a relation on S and $a, b \in S$, then we usually write $a \sim b$ to indicate that the ordered pair (a, b) is an element of the subset $\sim$ of $S \times S$. A relation $\sim$ on S is called an **equivalence relation** if (i) $\sim$ is **reflexive**, that is, $a \sim a$ for all $a \in S$, (ii) $\sim$ is **symmetric**, that is, $b \sim a$ whenever $a, b \in S$ satisfy $a \sim b$, and (iii) $\sim$ is **transitive**, that is, $a \sim c$ whenever $a, b, c \in S$ satisfy $a \sim b$ and $b \sim c$.

If $\sim$ is an equivalence relation on a set S and if $a \in S$, then the set $\{x \in S : x \sim a\}$ is called the **equivalence class** of a and is denoted[1] by $[a]$; in general, a subset E of S is called an **equivalence class** (with respect to $\sim$) if $E = [a]$ for some $a \in S$. A key fact about equivalence relations is the following result, which basically says that an equivalence relation on a set partitions the set into disjoint equivalence classes.

Proposition A.1. *Let S be a set and let $\sim$ be an equivalence relation on S. Then any two equivalence classes (with respect to $\sim$) are either disjoint or identical. Consequently, if $\mathcal{E}$ denotes the collection of distinct equivalence classes with respect to $\sim$, then*

$$S = \bigcup_{E \in \mathcal{E}} E,$$

where the union is disjoint.

Proof. Let $a, b \in S$ and suppose the equivalence classes $[a]$ and $[b]$ are not disjoint, that is, there exists $c \in [a] \cap [b]$. Then $c \sim a$ and $c \sim b$. Hence using the fact that $\sim$ is an equivalence relation, we see that for every $x \in S$,

$$x \in [a] \iff x \sim a \iff x \sim c \iff x \sim b \iff x \in [b].$$

This shows that $[a] = [b]$. Thus any two equivalence classes are either disjoint or identical. Finally, since $a \in [a]$ for each $a \in S$, we obtain $S = \bigcup_{a \in S} [a]$. $\square$

We give several examples of equivalence relations and corresponding equivalence classes below. The detailed verification of the assertions made in these examples is left to the reader.

Examples A.2. (i) On the set $\mathbb{N}$, define a relation $\sim$ by

$$m \sim n \iff m \text{ and } n \text{ have the same } \textbf{parity}, \text{ that is, } (-1)^m = (-1)^n.$$

Then $\sim$ is an equivalence relation. There are exactly two equivalence classes with respect to $\sim$, namely the set of odd positive integers and the set of even positive integers.

[1] When $S \subseteq \mathbb{Q}$, the notation $[a]$ for the equivalence class of an element a of S conflicts with the notation used in the text for the integer part of a. To avoid any possible confusion, we shall always use in this appendix the notation $\lfloor a \rfloor$ for the integer part of a.

(ii) On the set $\mathbb{N} \times \mathbb{N}$, define a relation $\sim$ by

$$(a, b) \sim (c, d) \iff a + d = b + c \quad \text{for } (a, b), (c, d) \in \mathbb{N} \times \mathbb{N}.$$

Then $\sim$ is an equivalence relation, and the equivalence classes with respect to $\sim$ are in one-to-one correspondence with the set $\mathbb{Z}$ of all integers.

(iii) Let $S = \{(m, n) : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$. The relation $\sim$ on S defined by

$$(a, b) \sim (c, d) \iff ad = bc \quad \text{for } (a, b), (c, d) \in S$$

is an equivalence relation on S, and the equivalence classes with respect to $\sim$ are in one-to-one correspondence with the set $\mathbb{Q}$ of all rational numbers.

(iv) Let $n \in \mathbb{N}$. Consider the relation $\sim$ on $\mathbb{Z}$ defined by

$$a \sim b \iff a - b \text{ is divisible by } n \qquad \text{for } a, b \in \mathbb{Z}.$$

Then $\sim$ is an equivalence relation, called **congruence modulo** n. There are exactly n distinct equivalence classes with respect to $\sim$ given by $C_0, C_1, \ldots, C_{n-1}$, where for $0 \leq i < n$, the set C_i consists of integers that leave remainder i when divided by n. These equivalence classes are known as **residue classes modulo** n, and the set $\{C_0, C_1, \ldots, C_{n-1}\}$ of all residue classes modulo n is sometimes denoted by $\mathbb{Z}/n\mathbb{Z}$.

We remark that examples (ii) and (iii) above can be used to formally construct $\mathbb{Z}$ from $\mathbb{N}$, and to construct $\mathbb{Q}$ from $\mathbb{Z}$. For an axiomatic treatment of $\mathbb{N}$, we refer to the book of Landau [54].

A.2 Cauchy Sequences of Rational Numbers

We shall now define the notion of a Cauchy sequence in $\mathbb{Q}$. This is completely analogous to the notion discussed in Chapter 2, except that we will refrain from using real numbers anywhere. In particular, ϵ will denote a positive rational number, that is, $\epsilon \in \mathbb{Q}^+$. Note that since it is well understood what positive rational numbers are, the notion of the absolute value of a rational number is well-defined and satisfies the basic properties given in Proposition 1.8.

A **sequence** in $\mathbb{Q}$ is a function from $\mathbb{N}$ to $\mathbb{Q}$. We usually write (a_n) to denote the sequence $\mathbf{a} : \mathbb{N} \to \mathbb{Q}$ defined by $\mathbf{a}(n) := a_n$ for $n \in \mathbb{N}$. The rational number a_n is called the n**th term** of the sequence (a_n). A sequence (a_n) of rational numbers is said to be

1. **bounded above** if there exists $\alpha \in \mathbb{Q}$ such that $a_n \leq \alpha$ for all $n \in \mathbb{N}$,
2. **bounded below** if there exists $\beta \in \mathbb{Q}$ such that $a_n \geq \beta$ for all $n \in \mathbb{N}$,
3. **bounded** if it is bounded above as well as bounded below,
4. **Cauchy** if for every $\epsilon \in \mathbb{Q}^+$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $m, n \in \mathbb{N}$ with $m, n \geq n_0$.

We shall also say that a sequence (c_n) of rational numbers is **null**, and write $c_n \to 0$, if for every $\epsilon \in \mathbb{Q}^+$, there is $n_0 \in \mathbb{N}$ such that $|c_n| < \epsilon$ for all $n \geq n_0$.

Examples A.3. (i) Let (a_n) be the sequence in $\mathbb{Q}$ defined by $a_n := 1/n$ for $n \in \mathbb{N}$. Then (a_n) is a null sequence. Indeed, given any $\epsilon \in \mathbb{Q}^+$, say $\epsilon = p/q$, where $p, q \in \mathbb{N}$, the positive integer $n_0 := q+1$ satisfies $n_0 > 1/\epsilon$. Hence $|a_n| < \epsilon$ for all $n \geq n_0$.

(ii) Let (a_n) be the sequence in $\mathbb{Q}$ defined by $a_n := (n-1)/n$ for $n \in \mathbb{N}$. Then (a_n) is a Cauchy sequence. To see this, let $\epsilon = p/q \in \mathbb{Q}^+$ be given, where $p, q \in \mathbb{N}$. Now the positive integer $n_0 := 2(q+1)$ satisfies $n_0 > 2/\epsilon$. Hence

$$|a_n - a_m| = \left| \frac{n-1}{n} - \frac{m-1}{m} \right| = \left| \frac{1}{m} - \frac{1}{n} \right| \leq \frac{2}{n_0} < \epsilon \quad \text{for all } m, n \geq n_0.$$

Note that although (a_n) is a Cauchy sequence, it is not a null sequence. In fact, $|a_n| \geq 1/2$ for all $n \geq 2$.

Proposition A.4. (i) *Every Cauchy sequence of rational numbers is bounded.*
(ii) *Every null sequence of rational numbers is Cauchy.*
(iii) *Let (a_n) be a Cauchy sequence of rational numbers that is not a null sequence. Then there exist $\epsilon_0 \in \mathbb{Q}^+$ and $n_0 \in \mathbb{N}$ such that $|a_n - a_{n_0}| < \epsilon_0$ and $|a_n| \geq \epsilon_0$ for all $n \geq n_0$.*

Proof. (i) Let (a_n) be a Cauchy sequence. Then there exists $k \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $m, n \geq k$. Consequently,

$$|a_n| \leq \alpha \text{ for all } n \in \mathbb{N}, \quad \text{where} \quad \alpha := \max\{|a_1|, \ldots, |a_{k-1}|, |a_k| + 1\}.$$

Hence (a_n) is bounded.

(ii) Let (c_n) be a null sequence. Given any $\epsilon \in \mathbb{Q}^+$, there exists $n_0 \in \mathbb{N}$ such that $|c_n| < \epsilon/2$ for all $n \geq n_0$. Then

$$|c_n - c_m| \leq |c_n| + |c_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } n \geq n_0.$$

Hence (c_n) is Cauchy.

(iii) Since (a_n) is not a null sequence, there exists $\epsilon \in \mathbb{Q}^+$ such that for every $k \in \mathbb{N}$, there exists $n_1 \geq k$ satisfying $|a_{n_1}| \geq \epsilon$. Also, since (a_n) is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that $|a_m - a_n| < \epsilon/2$ for all $m, n \geq n_0$. Let $k := n_0$, and find $n_1 \geq n_0$ such that $|a_{n_1}| \geq \epsilon$. Then

$$\epsilon \leq |a_{n_1}| \leq |a_{n_1} - a_n| + |a_n| \leq \frac{\epsilon}{2} + |a_n|, \text{ and hence } |a_n| \geq \frac{\epsilon}{2} \text{ for all } n \geq n_0.$$

Thus $\epsilon_0 := \epsilon/2 \in \mathbb{Q}^+$ and $n_0 \in \mathbb{N}$ have the desired property. □

Now let us define

$$\mathscr{C} := \text{the set of all Cauchy sequences of rational numbers.}$$

Further, consider the relation $\sim$ on $\mathscr{C}$ defined by

$$(a_n) \sim (b_n) \iff (a_n - b_n) \text{ is a null sequence}, \qquad \text{where } (a_n), (b_n) \in \mathscr{C}.$$

Proposition A.5. *The relation $\sim$ is an equivalence relation on $\mathscr{C}$.*

Proof. Clearly, $\sim$ is reflexive and symmetric. Suppose $(a_n), (b_n), (c_n) \in \mathscr{C}$ are such that $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$. Given any $\epsilon \in \mathbb{Q}^+$, the number $\epsilon/2$ is also in $\mathbb{Q}^+$. Hence there exist $n_1, n_2 \in \mathbb{N}$ such that

$$|a_n - b_n| < \frac{\epsilon}{2} \text{ for all } n \geq n_1 \quad \text{and} \quad |b_n - c_n| < \frac{\epsilon}{2} \text{ for all } n \geq n_2.$$

Now if $n_0 = \max\{n_1, n_2\}$, then for each $n \geq n_0$,

$$|a_n - c_n| = |(a_n - b_n) + (b_n - c_n)| \leq |a_n - b_n| + |b_n - c_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $(a_n) \sim (c_n)$. Thus $\sim$ is transitive as well. $\square$

We are now ready to define a model for $\mathbb{R}$ that we seek to construct. Let

$$\mathcal{R} := \text{the set of all equivalence classes of elements of } \mathscr{C} \text{ with respect to } \sim.$$

As in the previous section, the equivalence class of a Cauchy sequence (a_n) in $\mathscr{C}$ with respect to $\sim$ will be denoted by $[(a_n)]$. Given any $r \in \mathbb{Q}$, the constant sequence (r), that is, the sequence (r_n) with $r_n = r$ for all $n \in \mathbb{N}$, is clearly Cauchy. We will denote by $\mathcal{Q}$ the subset of $\mathcal{R}$ consisting of the equivalence classes of constant sequences of rational numbers. It is clear that the map from $\mathbb{Q}$ to $\mathcal{Q}$ given by $r \longmapsto [(r)]$ is one-one and onto. Thus we can, and will, identify $\mathcal{Q}$ with $\mathbb{Q}$. In particular, the equivalence classes of the constant sequences (0) and (1) will be denoted simply by 0 and 1, respectively.

We now define addition and multiplication on the set $\mathcal{R}$ as follows.

$$[(a_n)] + [(b_n)] = [(a_n + b_n)] \quad \text{and} \quad [(a_n)] \cdot [(b_n)] = [(a_n b_n)] \quad \text{for } (a_n), (b_n) \in \mathscr{C}.$$

Proposition A.6. *The operations of addition and multiplication on $\mathcal{R}$ are well-defined and satisfy the following algebraic properties:*

A1. $a + (b + c) = (a + b) + c$ *and* $a(bc) = (ab)c$ *for all* $a, b, c \in \mathcal{R}$.
A2. $a + b = b + a$ *and* $ab = ba$ *for all* $a, b \in \mathcal{R}$.
A3. $a + 0 = a$ *and* $a \cdot 1 = a$ *for all* $a \in \mathcal{R}$.
A4. *Let* $a \in \mathcal{R}$. *Then there exists* $a' \in \mathcal{R}$ *such that* $a + a' = 0$. *Further, if* $a \neq 0$, *then there exists* $a^* \in \mathcal{R}$ *such that* $aa^* = 1$.
A5. $a(b + c) = ab + ac$ *for all* $a, b, c \in \mathcal{R}$.

Proof. To show that the operations of addition and multiplication on $\mathcal{R}$ are well-defined, it suffices to show that for all (a_n), (a'_n), (b_n), $(b'_n) \in \mathscr{C}$,

$$(a_n) \sim (a'_n) \text{ and } (b_n) \sim (b'_n) \Longrightarrow (a_n + b_n) \sim (a'_n + b'_n) \text{ and } (a_n b_n) \sim (a'_n b'_n).$$

The assertion $(a_n + b_n) \sim (a'_n + b'_n)$ follows from the definition, since $|(a_n + b_n) - (a'_n + b'_n)| \leq |a_n - a'_n| + |b_n - b'_n|$. To see that $(a_n b_n) \sim (a'_n b'_n)$, we use part (i) of Proposition A.4 and obtain $\alpha', \beta \in \mathbb{Q}^+$ such that $|a'_n| \leq \alpha'$ and $|b_n| \leq \beta$ for all $n \in \mathbb{N}$, so that

$$|a_n b_n - a'_n b'_n| = |(a_n - a'_n)b_n + a'_n(b_n - b'_n)| \leq \beta|a_n - a'_n| + \alpha'|b_n - b'_n|.$$

Now since $a_n - a'_n \to 0$ and $b_n - b'_n \to 0$, it is readily seen that $a_n b_n - a'_n b'_n \to 0$.

Having established that addition and multiplication on $\mathcal{R}$ are well-defined, we see that properties A1, A2, A3, and A5 are immediate consequences of the definition and the corresponding properties of rational numbers.

Moreover, for every $a = [(a_n)] \in \mathcal{R}$, the element $a' := [(-a_n)]$ is clearly in $\mathcal{R}$ and it satisfies $a + a' = 0$. Finally, suppose $a = [(a_n)] \in \mathcal{R}$ is such that $a \neq 0$. Then by part (iii) of Proposition A.4, there exist $\epsilon_0 \in \mathbb{Q}^+$ and $n_0 \in \mathbb{N}$ such that $|a_n| \geq \epsilon_0$ for all $n \geq n_0$. In particular, $a_n \neq 0$ for all $n \geq n_0$. Define the sequence (a_n^*) in $\mathbb{Q}$ by $a_n^* := 1$ for $1 \leq n < n_0$ and $a_n^* = 1/a_n$ for $n \geq n_0$. Then $|a_n^* - a_m^*| \leq (1/\epsilon_0^2)|a_n - a_m|$ for all $n, m \geq n_0$. Since (a_n) is a Cauchy sequence, this implies that (a_n^*) is also a Cauchy sequence. Moreover, $a_n a_n^* - 1 = 0$ for all $n \geq n_0$, and so $(a_n a_n^*) \sim (1)$. This proves A4. $\square$

As noted in Section 1.1, several simple properties (such as, $a \cdot 0 = 0$ for all $a \in \mathcal{R}$) are formal consequences of properties A1–A5 proved in Proposition A.6, and these will now be tacitly assumed; also uniqueness of the additive inverse $a' \in \mathcal{R}$ for $a \in \mathcal{R}$, and of the multiplicative inverse $a^* \in \mathcal{R}$ for $a \in \mathcal{R}$ with $a \neq 0$ as in A4, is a formal consequence of Proposition A.6, and we will adopt the usual notation $-a$ for a', and $1/a$ or a^{-1} for a^*. We remark also that as a consequence of Proposition A.6, the addition and multiplication on $\mathcal{R}$ are compatible with the usual addition and multiplication on $\mathbb{Q}$ when $\mathbb{Q}$ is identified with the subset $\mathcal{Q}$ of $\mathcal{R}$ as before.

Now let us turn to order properties. We shall say that a sequence $(a_n) \in \mathscr{C}$ is **positive** if it satisfies the following property:

$$\text{There exist } r \in \mathbb{Q}^+ \text{ and } n_0 \in \mathbb{N} \text{ such that } a_n \geq r \text{ for all } n \geq n_0.$$

Note that if $(a'_n) \in \mathscr{C}$ is such that $(a'_n) \sim (a_n)$ and (a_n) satisfies the above property, then so does (a'_n). Indeed, since $(a'_n) \sim (a_n)$, there exists $n_1 \in \mathbb{N}$ such that $|a'_n - a_n| < r/2$ for all $n \geq n_1$. Now if we let $r' := r/2$ and $n_2 := \max\{n_0, n_1\}$, then we obtain

$$a'_n > a_n - \frac{r}{2} \geq r - \frac{r}{2} = r' \text{ for all } n \geq n_2.$$

With this in view, we define $\mathcal{R}^+$ to be the set of all equivalence classes of positive sequences in $\mathscr{C}$. It is clear that $\mathcal{R}^+$ is a well-defined subset of $\mathcal{R}$.

Proposition A.7. *The set $\mathcal{R}^+$ satisfies the following order properties:*

O1. *Given any $a \in \mathcal{R}$, exactly one of the following statements is true:*

$$a \in \mathcal{R}^+; \qquad a = 0; \qquad -a \in \mathcal{R}^+.$$

O2. *If* $a, b \in \mathcal{R}^+$, *then* $a + b \in \mathcal{R}^+$ *and* $ab \in \mathcal{R}^+$.

Proof. Let $(a_n) \in \mathscr{C}$ be such that $[(a_n)] \neq 0$. By part (iii) of Proposition A.4, there exist $\epsilon_0 \in \mathbb{Q}^+$ and $n_0 \in \mathbb{N}$ such that $|a_n - a_{n_0}| < \epsilon_0$ and $|a_n| \geq \epsilon_0$ for all $n \geq n_0$. In particular, $a_{n_0} \neq 0$. Now if $a_{n_0} > 0$, then the above inequalities imply $a_n = a_{n_0} + (a_n - a_{n_0}) > \epsilon_0 - \epsilon_0 = 0$ for all $n \geq n_0$, and consequently, $a_n \geq \epsilon_0$ for all $n \geq n_0$. Likewise, if $a_{n_0} < 0$, then $-a_n \geq \epsilon_0$ for all $n \geq n_0$. Thus $[(a_n)] \in \mathcal{R}^+$ or $-[(a_n)] = [(-a_n)] \in \mathcal{R}^+$. This proves O1.

Next, if $(a_n), (b_n) \in \mathscr{C}$ are such that $[(a_n)], [(b_n)] \in \mathcal{R}^+$, then there exist $r_1, r_2 \in \mathbb{Q}^+$ and $n_1, n_2 \in \mathbb{N}$ such that

$$a_n > r_1 \text{ for all } n \geq n_1 \quad \text{and} \quad b_n > r_2 \text{ for all } n \geq n_2.$$

Now if we let $n_0 = \max\{n_1, n_2\}$, then we clearly have

$$a_n + b_n > r_1 + r_2 \text{ for all } n \geq n_0 \quad \text{and} \quad a_n b_n > r_1 r_2 \text{ for all } n \geq n_0.$$

Since $r_1 + r_2,\ r_1 r_2 \in \mathbb{Q}^+$, we obtain $[(a_n)] + [(b_n)] \in \mathcal{R}^+$ and $[(a_n)][(b_n)] \in \mathcal{R}^+$. This proves O2. □

Using the set $\mathcal{R}^+$, we can define an order relation on $\mathcal{R}$ exactly as in Section 1.1, namely, for all $a, b \in \mathcal{R}$, we write $a < b$ or $b > a$ if $b - a \in \mathcal{R}^+$. Moreover, we shall write $a \leq b$ or $b \geq a$ to mean that either $a < b$ or $a = b$. The usual properties of this order relation, as listed in (i), (ii), and (iii) on page 3 (with $\mathbb{R}$ replaced by $\mathcal{R}$), and also the fact that $1 > 0$ are formal consequences of O1 and O2, and will thus be tacitly assumed. Moreover, the notions of a subset of $\mathcal{R}$ being bounded above, bounded below, or bounded as well as the notions of upper bound, lower bound, supremum, and infimum for subsets of $\mathcal{R}$ can now be defined exactly as they were defined for subsets of $\mathbb{R}$ in Chapter 1. Note also that the order relation on $\mathcal{R}$ that we have just defined is compatible with the known order relation on the set $\mathbb{Q}$, that is, if $r, s \in \mathbb{Q}$ and if $[(r)], [(s)]$ are the corresponding elements of $\mathcal{Q}$, then $r < s$ if and only if $[(r)] < [(s)]$.

We shall now proceed to prove that the set $\mathcal{R}$, which we have constructed from $\mathbb{Q}$, has the completeness property. As a preliminary step, we will first show that $\mathcal{R}$ has the archimedean property. It may be recalled that in Proposition 1.3, the archimedean property of $\mathbb{R}$ was deduced from the assumption that $\mathbb{R}$ has the completeness property. Here we will give a direct proof to show that $\mathcal{R}$ has the archimedean property, and later, use it to derive the completeness property of $\mathcal{R}$.

Proposition A.8. *Given any* $a \in \mathcal{R}$, *there is some* $k \in \mathbb{N}$ *such that* $k > a$.

Proof. First note that if $a \in \mathcal{Q}$, then a corresponds to a unique rational number p/q, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Hence $k := |p| + 1$ clearly satisfies $k > a$. Now suppose $a = [(a_n)]$ is an arbitrary element of $\mathcal{R}$, where (a_n) is a Cauchy sequence of rational numbers. Then for $\epsilon = 1/2$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < 1/2$ for all $n, m \geq n_0$. Since the archimedean property holds for rational numbers, there exists $\ell \in \mathbb{N}$ such that $|a_{n_0}| < \ell$. Hence for each $n \geq n_0$, we obtain

$$|a_n| \leq |a_n - a_{n_0}| + |a_{n_0}| < \frac{1}{2} + \ell \quad \text{and hence} \quad |(1+\ell) - a_n| \geq (1+\ell) - |a_n| > \frac{1}{2}.$$

It follows that $[(a_n)] < [(1+\ell)]$, that is, $k := 1 + \ell \in \mathbb{N}$ satisfies $k > a$. □

The archimedean property proved in Proposition A.8 enables us to define the integer part of every $x \in \mathcal{R}$ exactly as in the paragraph following Proposition 1.3 of Chapter 1, and this, in turn, permits us to deduce that between any two elements of $\mathcal{R}$, there is a rational number. It is important to note that this result uses only the algebraic and order properties together with the archimedean property (Propositions A.6, A.7, and A.8).

Proposition A.9. *Given any $a, b \in \mathcal{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. The proof is identical to that of Proposition 1.6 and hence omitted. A more general result (Proposition A.12) is proved in the next section. □

Corollary A.10. *Let $(r_n) \in \mathscr{C}$ and let $a = [(r_n)]$ be the corresponding element of $\mathcal{R}$. If there exists $n_0 \in \mathbb{N}$ such that $r_n \geq 0$ for all $n \geq n_0$, then $a \geq 0$. More generally, if there exist $\alpha, \beta \in \mathcal{R}$ and $n_0 \in \mathbb{N}$ such that $\beta \leq r_n \leq \alpha$ for all $n \geq n_0$, then $\beta \leq a \leq \alpha$.*

Proof. Suppose there exists $n_0 \in \mathbb{N}$ such that $r_n \geq 0$ for all $n \geq n_0$. Let, if possible, $a < 0$. Then by Proposition A.9, there exists $s \in \mathbb{Q}$ such that $a < s < 0$. Now $-s > 0$ and $r_n - s \geq -s$ for all $n \geq n_0$. So it follows from the definition of $\mathcal{R}^+$ that $a - s > 0$, which is a contradiction. This proves that $a \geq 0$.

Next suppose $\alpha, \beta \in \mathcal{R}$ and $n_0 \in \mathbb{N}$ are such that $\beta \leq r_n \leq \alpha$ for all $n \geq n_0$. By Proposition A.9, there exist $\alpha_n, \beta_n \in \mathbb{Q}$ such that $\beta - \frac{1}{n} < \beta_n < \beta$ and $\alpha < \alpha_n < \alpha + \frac{1}{n}$ for each $n \in \mathbb{N}$. This implies that $(\alpha_n), (\beta_n) \in \mathscr{C}$. Moreover, $\alpha = [(\alpha_n)]$ and $\beta = [(\beta_n)]$. (Verify!) Now applying the first assertion in the corollary to $(r_n - \beta_n)$ and $(\alpha_n - r_n)$, we obtain $\beta \leq a \leq \alpha$. □

We are now ready to prove that the set $\mathcal{R}$ has the completeness property.

Proposition A.11. *Every nonempty subset of $\mathcal{R}$ that is bounded above has a supremum.*

Proof. Let $\mathcal{S}$ be a nonempty subset of $\mathcal{R}$ that is bounded above. Since $\mathcal{S}$ is nonempty, there is some $a_0 \in \mathcal{S}$, and since $\mathcal{S}$ is bounded above, there is some $\alpha_0 \in \mathcal{R}$ such that α_0 is an upper bound of $\mathcal{S}$, that is, $a \leq \alpha_0$ for all $a \in \mathcal{S}$. Now let $\beta_1 := (a_0 + \alpha_0)/2$. If β_1 is an upper bound of $\mathcal{S}$, we let $a_1 := a_0$ and $\alpha_1 := \beta_1$, whereas if β_1 is not an upper bound of $\mathcal{S}$, then there exists $b \in \mathcal{S}$

such that $\beta_1 < b$, and in this case, we let $a_1 := b$ and $\alpha_1 := \alpha_0$. In any case, $a_0 \le a_1$ and $\alpha_0 \ge \alpha_1$, and moreover,

$$a_1 \in \mathcal{S}, \quad \alpha_1 \text{ is an upper bound of } \mathcal{S}, \quad \text{and} \quad 0 \le \alpha_1 - a_1 \le \frac{\alpha_0 - a_0}{2}.$$

Next, we replace (a_0, α_0) by (a_1, α_1) and proceed as before. In general, given $n \in \mathbb{N}$ and $a_i \in \mathcal{S}$ and upper bounds α_i of $\mathcal{S}$ with $0 \le (\alpha_i - a_i) \le (\alpha_0 - a_0)/2^i$ for $0 \le i \le n-1$ and with $a_0 \le a_1 \le \cdots \le a_{n-1}$ and $\alpha_0 \ge \alpha_1 \ge \cdots \ge \alpha_{n-1}$, we choose $a_n \in \mathcal{S}$ and an upper bound α_n of $\mathcal{S}$ as follows. Let $\beta_n := (a_{n-1} + \alpha_{n-1})/2$. If β_n is an upper bound of $\mathcal{S}$, we let $a_n := a_{n-1}$ and $\alpha_n := \beta_n$, whereas if β_n is not an upper bound of $\mathcal{S}$, then there exists $b \in \mathcal{S}$ such that $\beta_n < b$, and in this case, we let $a_n := b$ and $\alpha_n := \alpha_{n-1}$. In any case, $a_{n-1} \le a_n$ and $\alpha_{n-1} \ge \alpha_n$, and moreover,

$$a_n \in \mathcal{S}, \quad \alpha_n \text{ is an upper bound of } \mathcal{S}, \quad \text{and} \quad 0 \le \alpha_n - a_n \le \frac{\alpha_0 - a_0}{2^n}.$$

Note that if $a_n = \alpha_n$ for some $n \ge 0$, then clearly α_n is the supremum of $\mathcal{S}$.

Now suppose $a_n < \alpha_n$ for all $n \ge 0$. By Proposition A.9, for each $n \in \mathbb{N}$, there exists $r_n \in \mathbb{Q}$ such that $a_n < r_n < \alpha_n$. We claim that (r_n) is a Cauchy sequence. To see this, let $\epsilon \in \mathbb{Q}^+$ be given. Applying Proposition A.8 to $a = (\alpha_0 - a_0)/\epsilon$, we see that there exists $k \in \mathbb{N}$ such that

$$\frac{\alpha_0 - a_0}{k} < \epsilon \quad \text{and hence} \quad \frac{\alpha_0 - a_0}{2^k} < \epsilon,$$

where the last inequality follows by noting that $2^j \ge j$ for all $j \in \mathbb{N}$, as can be seen easily by induction on j. Now given any $m, n \in \mathbb{N}$ with $m \ge n \ge k$, since $a_m < r_m < \alpha_m$, $a_n < r_n < \alpha_n$, and $a_m \ge a_n$, we obtain

$$r_n - r_m < \alpha_n - r_m < \alpha_n - a_m \le \alpha_n - a_n.$$

In a similar way, since $\alpha_m \le \alpha_n$, we obtain

$$r_n - r_m > a_n - r_m > a_n - \alpha_m \ge a_n - \alpha_n.$$

It follows that

$$|r_n - r_m| < \alpha_n - a_n \le \frac{\alpha_0 - a_0}{2^n} \le \frac{\alpha_0 - a_0}{2^k} < \epsilon \quad \text{for all } m, n \ge k.$$

Thus $(r_n) \in \mathscr{C}$, and so $\alpha := [(r_n)] \in \mathcal{R}$. We shall now show that α is the supremum of $\mathcal{S}$. To this end, let us first observe that for every fixed $m \in \mathbb{N}$, the inequalities $a_m \le a_n < r_n < \alpha_n \le \alpha_m$ hold for all $n \ge m$, and so by Corollary A.10, we see that $a_m \le \alpha \le \alpha_m$.

Now suppose, if possible, α is not an upper bound of $\mathcal{S}$. Then there is $a \in \mathcal{S}$ such that $a > \alpha$. By Proposition A.9, there exists $\delta \in \mathbb{Q}^+$ such that $\delta < a - \alpha$. Further, by Proposition A.8, there is $m \in \mathbb{N}$ such that $m > (\alpha_0 - a_0)/\delta$, and so

$$0 \le \alpha_m - a_m \le \frac{\alpha_0 - a_0}{2^m} \le \frac{\alpha_0 - a_0}{m} < \delta.$$

Hence $\alpha_m < a_m + \delta \le \alpha + \delta < a$. But this contradicts the fact that α_m is an upper bound of $\mathcal{S}$. Hence α is an upper bound of $\mathcal{S}$.

Next, suppose α is not the least upper bound of $\mathcal{S}$. Then there exists $\beta \in \mathcal{R}$ with $\beta < \alpha$ such that β is an upper bound of $\mathcal{S}$. Again, choose $\delta \in \mathbb{Q}^+$ such that $0 < \delta < \alpha - \beta$ and $m \in \mathbb{N}$ such that $0 \le \alpha_m - a_m < \delta$. Then $a_m > \alpha_m - \delta \ge \alpha - \delta > \beta$, which is a contradiction, since $a_m \in \mathcal{S}$ and β is an upper bound of $\mathcal{S}$. It follows that α is the supremum of $\mathcal{S}$. □

A.3 Uniqueness of a Complete Ordered Field

In the previous section, we have shown that the set $\mathcal{R}$ possesses all the properties that were postulated for $\mathbb{R}$ in Chapter 1. In other words, we have established the existence of the set of all real numbers. We will now prove its "uniqueness". First, we introduce some useful terminology.

By a **field** we shall mean a set F that has operations of addition and multiplication (that is, maps from $F \times F$ to F that associate elements $a+b$ and ab of F to $(a,b) \in F \times F$) and has distinct elements 0_F and 1_F in it such that the five algebraic properties A1–A5 in Proposition A.6 are satisfied with $\mathcal{R}$ replaced throughout by F, and with 0 and 1 replaced by 0_F and 1_F. It is easy to see that in a field F, elements 0_F and 1_F satisfying A3 are unique, and these are sometimes called the additive identity and the multiplicative identity of F, respectively. Note that $\mathbb{Q}$ and $\mathcal{R}$ are examples of fields. A special case of Example A.2 (iv), namely the set $\mathbb{Z}/p\mathbb{Z}$ of residue classes modulo a prime number p, is a field having only finitely many elements.

If a field F contains a subset F^+ satisfying the two order properties O1–O2 with $\mathcal{R}^+$ replaced throughout by F^+, then F is called an **ordered field**; in this case, for every $a, b \in F$, we write $a < b$ or $b > a$ if $b - a \in F^+$; also, we write $a \le b$ or $b \ge a$ if either $a < b$ or $a = b$. The notions of boundedness, supremum, etc. are defined for subsets of an ordered field in exactly the same way as in the case of $\mathbb{R}$. Note that $\mathbb{Q}$ and $\mathcal{R}$ are ordered fields, but $\mathbb{Z}/p\mathbb{Z}$ is not. In fact, an ordered field F cannot be finite. Indeed, $1_F > 0_F$ (because otherwise $-1_F > 0_F$, and so $1_F = (-1_F)(-1_F) > 0_F$, which would be a contradiction). Hence for every $n \in \mathbb{N}$, if we let $n_F := 1_F + \cdots + 1_F$ (n times), then $n_F > 0_F$; moreover, $0_F < 1_F < 2_F < \cdots$, and so F contains infinitely many elements. Furthermore, in an ordered field F, for every $m \in \mathbb{Z}$ with $m < 0$, we let m_F denote the additive inverse of $(-m)_F$, that is, the unique element of F satisfying $(-m)_F + m_F = 0_F$. For every $r = m/n$ in $\mathbb{Q}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, we let $r_F := (m_F)(n_F)^{-1}$. It is easily seen that $r \longmapsto r_F$ gives a well-defined, one-one map of $\mathbb{Q} \to F$, which preserves algebraic and order operations, that is, for all $r, s \in \mathbb{Q}$,

$$(r+s)_F = r_F + s_F, \quad (rs)_F = r_F s_F, \quad \text{and} \quad r < s \Longrightarrow r_F < s_F.$$

Thus F contains a copy of $\mathbb{Q}$, namely $\mathbb{Q}_F := \{r_F : r \in \mathbb{Q}\}$. In an ordered field F, the **absolute value** of an element can be defined as in the case of $\mathbb{Q}$ or $\mathbb{R}$. Thus for every $a \in F$, we let $|a| := a$ if $a \geq 0_F$ and $|a| := -a$ if $a < 0_F$. It is easily seen that $|ab| = |a||b|$ and $|a + b| \leq |a| + |b|$ for all $a, b \in F$.

Let F be an ordered field. We say that F is **archimedean** if for every $a \in F$, there exists $n \in \mathbb{N}$ such that $n_F > a$. and we say that F is **complete** if every nonempty subset of F that is bounded above has a supremum in F. For example, both $\mathbb{Q}$ and $\mathcal{R}$ are archimedean ordered fields, and $\mathcal{R}$ is complete (thanks to Proposition A.11), but $\mathbb{Q}$ is not. In general, by arguing as in the proof of Proposition 1.3, we readily see that a complete ordered field is archimedean.

Let F be an archimedean ordered field and let $a \in F$. Then there exist $m, n \in \mathbb{N}$ such that $m_F > -a$ and $n_F > a$, that is, $-m_F < a < n_F$. Thus if k is the largest among the finitely many integers satisfying $-m \leq k \leq n$ and $k_F \leq a$, then k_F is called the **integer part** of a in F, and is denoted by $\lfloor a \rfloor$. Note that $\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1_F$. The following result is similar to Proposition A.9. The proof is similar to that of Proposition 1.6, but this time we include it.

Proposition A.12. *Let F be an archimedean ordered field and let $a_1, a_2 \in F$ satisfy $a_1 < a_2$. Then there is $r \in \mathbb{Q}$ such that $a_1 < r_F < a_2$.*

Proof. Since F is archimedean, there exists $n \in \mathbb{N}$ such that $n_F > (a_2 - a_1)^{-1}$, that is, $(n_F)^{-1} < (a_2 - a_1)$. Let $m \in \mathbb{N}$ be such that $m_F = \lfloor n_F a_1 \rfloor + 1_F$. Then $m_F - 1_F \leq n_F a_1 < m_F$. Hence

$$a_1 < m_F (n_F)^{-1} \leq (n_F a_1 + 1_F)(n_F)^{-1} = a_1 + (n_F)^{-1} < a_1 + (a_2 - a_1) = a_2.$$

Thus $r = m/n \in \mathbb{Q}$ satisfies $a_1 < r_F < a_2$. $\square$

Corollary A.13. *Let F be an archimedean ordered field. Suppose $a \in F$ satisfies $|a| < \epsilon_F$, that is, $-\epsilon_F < a < \epsilon_F$, for all $\epsilon \in \mathbb{Q}^+$. Then $a = 0_F$.*

Proof. In case $a > 0_F$, by Proposition A.12, there exists $r \in \mathbb{Q}$ such that $0_F < r_F < a$. Thus the hypothesis is contradicted if we take $\epsilon = r$. Likewise, we arrive at a contradiction if $a < 0_F$. Hence $a = 0_F$. $\square$

Let K and F be ordered fields. A map $f : K \to F$ is called an **order isomorphism** if f is both one-one and onto, and f preserves algebraic and order operations, that is, for all $x, y \in K$,

$$f(x + y) = f(x) + f(y), \quad f(xy) = f(x)f(y), \quad \text{and} \quad x < y \Longrightarrow f(x) < f(y).$$

If such a map exists, then we say that F is **order isomorphic** to K.

Proposition A.14. *Every complete ordered field is order isomorphic to $\mathcal{R}$.*

Proof. Let F be a complete ordered field. Define $f : \mathcal{R} \to F$ by

$$f(x) := \sup F_x, \quad \text{where} \quad F_x := \{r_F : r \in \mathbb{Q} \text{ and } r \le x\} \quad \text{for } x \in \mathcal{R}.$$

Note that f is well-defined. Indeed, given any $x \in \mathcal{R}$, by Proposition A.9, there exist $s, t \in \mathbb{Q}$ such that $x - 1 < s < x < t < x + 1$. It follows that $s_F \in F_x$ and t_F is an upper bound of F_x. Thus $\sup F_x$ exists, since F is complete. Note also that $f(r) = r_F$ for all $r \in \mathbb{Q}$. Indeed, $f(r) < r_F$ as well as $f(r) > r_F$ will both lead to a contradiction using Proposition A.12.

Let $x, y \in \mathcal{R}$ be such that $x < y$. Then $x < (x + y)/2 < y$, and using Corollary A.9, we can find $u, v \in \mathbb{Q}$ such that $x < u < (x + y)/2 < v < y$. Now u_F is an upper bound of F_x and v_F is an element of F_y. Hence we obtain $f(x) \le u_F < v_F \le f(y)$. Thus f is order-preserving, and therefore one-one.

To show that f is onto, suppose $a \in F$. In case $a = r_F \in \mathbb{Q}_F$ for some $r \in \mathbb{Q}$, then $a = f(r)$. Now suppose $a \notin \mathbb{Q}_F$. Let $\mathbb{Q}_a := \{r \in \mathbb{Q} : r_F \le a\}$. Since F is complete, it is archimedean, and therefore by Proposition A.12, there exist $r, s \in \mathbb{Q}$ such that $a - 1 < r_F < a$ and $a < s_F < a + 1$. This implies that the set $\mathbb{Q}_a$ is nonempty and bounded above. Hence $x := \sup \mathbb{Q}_a$ is a well-defined element of $\mathcal{R}$. We shall now show that $f(x) = a$.

First, suppose $x \in \mathbb{Q}$. Then $f(x) = x_F$. Now if $x_F < a$, then by Proposition A.12, there exists $r \in \mathbb{Q}$ such that $x_F < r_F < a$, and this leads to a contradiction, because on the one hand $x < r$, since $x, r \in \mathbb{Q}$ and $x_F < r_F$, but on the other hand, $r \le x$, since $r_F < a$ implies $r \in \mathbb{Q}_a$ and $x = \sup \mathbb{Q}_a$. Likewise, if $x_F > a$, then by Proposition A.12, there exists $s \in \mathbb{Q}$ such that $a < s_F < x_F$, but then s is an upper bound of $\mathbb{Q}_a$ (because $r \in \mathbb{Q}$ and $r_F \le a$ implies $r_F < s_F$ and hence $r < s$), and therefore $x = \sup \mathbb{Q}_a \le s$, which implies $x_F \le s_F$, and this contradicts $s_F < x_F$. Thus $f(x) = a$ when $x \in \mathbb{Q}$.

Next, suppose $x \notin \mathbb{Q}$. Let $r \in \mathbb{Q}$ with $r \le x$. Since $x \notin \mathbb{Q}$, we obtain $r < x$, and since $x = \sup \mathbb{Q}_a$, there exists $s \in \mathbb{Q}_a$ such that $r < s \le x$. Consequently, $r_F < s_F \le a$. It follows that a is an upper bound of F_x. Hence $f(x) \le a$. Furthermore, if $f(x) < a$, then by Proposition A.12, there exists $t \in \mathbb{Q}$ such that $f(x) < t_F < a$. But then $t \in \mathbb{Q}_a$, and so $t \le x$, which implies $t_F \le f(x)$, and so we obtain a contradiction. It follows that $f(x) = a$. Thus f is onto.

It remains to show that f preserves the algebraic operations. Let $x, y \in \mathcal{R}$ and let $\epsilon \in \mathbb{Q}^+$ be given. By Proposition A.9, there exist $r, s, u, v \in \mathbb{Q}$ such that

$$x - \frac{\epsilon}{4} < r < x < s < x + \frac{\epsilon}{4} \quad \text{and} \quad y - \frac{\epsilon}{4} < u < y < v < y + \frac{\epsilon}{4}.$$

Then $0 < s - r < \epsilon/2$ and $0 < v - u < \epsilon/2$. Since f is order-preserving, $r_F < f(x) < s_F$ and $u_F < f(y) < v_F$. Hence $r_F + u_F < f(x) + f(y) < s_F + v_F$. Moreover, $r + u < x + y < s + v$, and again since f is order-preserving, $r_F + u_F < f(x + y) < s_F + v_F$. Consequently,

$$f(x + y) - f(x) - f(y) < (s_F - r_F) + (v_F - u_F) < \frac{\epsilon_F}{2_F} + \frac{\epsilon_F}{2_F} = \epsilon_F.$$

By a similar argument, $f(x + y) - f(x) - f(y) > -\epsilon_F$. Now by Corollary A.13, we obtain $f(x + y) = f(x) + f(y)$. Thus, f preserves addition.

To show that f preserves multiplication, first consider $x, y \in \mathcal{R}$ with $x > 0$ and $y > 0$. Let $\epsilon \in \mathbb{Q}^+$ be given. By Proposition A.8, there exists $n \in \mathbb{N}$ such that $n > x$, $n > y$, and $n > \epsilon/6$. Now using Proposition A.9 and the assumption that $x > 0$ and $y > 0$, we obtain $r, s, u, v \in \mathbb{Q}^+$ such that

$$x - \frac{\epsilon}{6n} < r < x < s < x + \frac{\epsilon}{6n} \quad \text{and} \quad y - \frac{\epsilon}{6n} < u < y < v < y + \frac{\epsilon}{6n}.$$

Next, we use the usual trick of adding and subtracting suitable terms to write

$$xy - ru = (x-r)y + r(y-u) \quad \text{and} \quad sv - xy = (s-x)(v-y) + x(v-y) + y(s-x).$$

Consequently, $xy - ru < (\epsilon/6n)y + r(\epsilon/6n) < (\epsilon/6) + (\epsilon/6) < \epsilon/2$ and

$$sv - xy < \left(\frac{\epsilon}{6n}\right)^2 + x\frac{\epsilon}{6n} + y\frac{\epsilon}{6n} < \frac{\epsilon}{6n} + \frac{\epsilon}{6} + \frac{\epsilon}{6} \le \frac{\epsilon}{6} + \frac{\epsilon}{3} = \frac{\epsilon}{2}.$$

Thus $sv - (\epsilon/2) < xy < ru + (\epsilon/2)$, and since f is order-preserving, we obtain

$$s_F v_F - \frac{\epsilon_F}{2_F} < f(xy) < r_F u_F + \frac{\epsilon_F}{2_F}.$$

Again, since f is order-preserving, by arguing as before, we obtain

$$s_F v_F - \frac{\epsilon_F}{2_F} < f(x)f(y) < r_F u_F + \frac{\epsilon_F}{2_F}.$$

It follows that $-\epsilon_F < f(xy) - f(x)f(y) < \epsilon_F$. By Corollary A.13, we obtain $f(xy) = f(x)f(y)$. Finally, since f preserves addition, it is easily seen that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in \mathcal{R}$. Hence the result just proved, namely $f(xy) = f(x)f(y)$ for all positive $x, y \in \mathcal{R}$, implies that $f(xy) = f(x)f(y)$ for all $x, y \in \mathcal{R}$. So f preserves multiplication as well. $\square$

Remark A.15. In view of the results of the previous section and the uniqueness result in Proposition A.14, it makes sense to refer to any set satisfying the algebraic, order, and completeness properties as **the** set of all real numbers. The construction of $\mathcal{R}$ given in the previous section is one of the ways of constructing $\mathbb{R}$. Several other constructions are possible. The most prominent among these is a construction due to Dedekind, where the basic idea is to determine a real number x by means of the pair (L_x, R_x) of subsets of $\mathbb{Q}$, where $L_x := \{r \in \mathbb{Q} : r < x\}$ and $R_x := \{r \in \mathbb{Q} : r \ge x\}$. Such a pair is called a **Dedekind cut**, or simply a **cut**. More formally, a **cut** is a pair (L, R) of nonempty disjoint subsets of $\mathbb{Q}$ such that (i) $L \cup R = \mathbb{Q}$, (ii) L is downwards closed, that is, $s \in L$ whenever $s < t$ for some $t \in L$, (iii) R is upwards closed, that is, $s \in R$ whenever $s > t$ for some $t \in R$, and (iv) L has no maximum element. One defines addition, multiplication, and an order on the set of all cuts, and shows that this set is a complete ordered field. For more details about this approach, one can refer to the essays of Dedekind [24] or the appendix to Chapter 1 of Rudin [71]. For a host of other constructions for $\mathbb{R}$, we refer to the article of Weiss [87]. At any rate, by Proposition A.14, all these constructions yield essentially the same ordered field. $\diamond$

Remark A.16. In this book, we have used the word completeness (of $\mathbb{R}$, or more generally, of any ordered field F) to mean that every nonempty subset that is bounded above has a supremum. This is sometimes referred to as **order completeness**, especially when contrasted with other notions such as **monotone completeness** and **Cauchy completeness** that are defined as follows. An ordered field F is said to be **monotone complete** (resp. **Cauchy complete**) if every monotonic bounded sequence (resp. Cauchy sequence) in F is convergent. Note that in an ordered field, the notions of absolute value and of a sequence being monotonic, bounded, convergent, or Cauchy are defined in the same way as in the case of $\mathbb{R}$. Arguing as in the proofs of Propositions 2.8 and 2.22, we readily see that for every ordered field F,

$$F \text{ is complete} \Longrightarrow F \text{ is monotone complete} \Longrightarrow F \text{ is Cauchy complete.}$$

It can be shown that Cauchy completeness implies (order) completeness, provided that the ordered field is archimedean. (Compare Exercise 2.42 of Chapter 2.) Thus for an archimedean ordered field, the three notions of completeness are equivalent. Moreover, an ordered field that is monotone complete is necessarily archimedean (because otherwise, the sequence (n_F) would be convergent, and if n_F converges to a, then there exists $n_0 \in \mathbb{N}$ such that $a-1_F < n_F < a+1_F$ for $n \geq n_0$, but then $(n+2)_F > a+1_F$, which would be a contradiction!) and consequently, order complete. However, there do exist ordered fields that are Cauchy complete, but not archimedean and therefore neither order complete nor monotone complete; see, for instance, Ex. 4 and 7 in Chapter 1 of Gelbaum and Olmsted [32]. For more on various notions of completeness and related matters, see the article of Hall and Todorov [36]. Finally, we remark that the notion of Cauchy completeness can be readily defined for any field F that (is not necessarily ordered, but) has an "absolute value function", that is, a map from F to $\mathbb{R}^+$ given by $a \longmapsto |a|$ satisfying for all $a, b \in F$, the following: (i) $|a| = 0 \iff a = 0_F$, (ii) $|ab| = |a||b|$, and (iii) $|a+b| \leq |a| + |b|$. In the next section, we will formally introduce the field $\mathbb{C}$ of complex numbers and show that $\mathbb{C}$ is not an ordered field, but $\mathbb{C}$ has an absolute value function. Moreover, it is easy to show (using Proposition 2.22) that $\mathbb{C}$ is Cauchy complete. $\diamond$

B

Fundamental Theorem of Algebra

Although the set $\mathbb{R}$ of all real numbers is "complete", it has a lacuna from an algebraic point of view. Namely, there are polynomials with real coefficients that have no root in $\mathbb{R}$. The simplest among these is the polynomial x^2+1. By adjoining to $\mathbb{R}$ an "imaginary" root i of x^2+1, we obtain the complex numbers, which are sometimes "defined" as the numbers of the form $x+iy$, where $x, y \in \mathbb{R}$. It is not difficult to give a formal and precise definition of complex numbers and in particular, of the number i. We do this in Section B.1, and then outline how several of the notions about real-valued functions can be extended to complex-valued functions. Next, in Section B.2, we prove a remarkable result known as the Fundamental Theorem of Algebra, which basically says that if one can solve $x^2+1=0$, then one can solve every polynomial equation in one variable with real or complex coefficients!

B.1 Complex Numbers and Complex Functions

A **complex number** is defined as an ordered pair of real numbers. The set of all complex numbers is denoted by $\mathbb{C}$. Addition and multiplication in $\mathbb{C}$ are defined as follows. For all $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$, let

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &:= (x_1 + x_2,\, y_1 + y_2), \text{ and}\\ (x_1, y_1)(x_2, y_2) &:= (x_1x_2 - y_1y_2,\, x_1y_2 + y_1x_2).\end{aligned}$$

It is easily seen that with respect to these algebraic operations, $\mathbb{C}$ is a field. Moreover, the map given by $x \longmapsto (x, 0)$ gives a one-one map from $\mathbb{R}$ to $\mathbb{C}$, which preserves the algebraic operations. With this in view, we regard $\mathbb{R}$ as a subset of $\mathbb{C}$ by identifiyng a real number x with the ordered pair $(x, 0)$ in $\mathbb{C}$. We define $i := (0, 1)$. With the identification of $\mathbb{R}$ with a subset of $\mathbb{C}$ as above, we can write any $(x, y) \in \mathbb{C}$ as $x + iy$. Note that $i^2 = -1$, where we have again identified -1 with $(-1, 0)$. Let $z \in \mathbb{C}$. As noted above, $z = x + iy$ for unique $x, y \in \mathbb{R}$. We call x the **real part** of z and denote it by $\Re(z)$, and we call y the **imaginary part** of z and denote it by $\Im(z)$. The complex number $x - iy$ is called the **conjugate** of $z = x + iy$ and is denoted by $\overline{z}$. We also define the **absolute value** or the **modulus** of z to be the nonnegative real number $|z| := \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$. Note that this definition is consistent with

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that of the absolute value of a real number, and that the **Triangle Inequality** $|z_1 + z_2| \leq |z_1| + |z_2|$ holds for all $z_1, z_2 \in \mathbb{C}$. Also, note that

$$\max\{|\Re(z)|, |\Im(z)|\} \leq |z| \leq |\Re(z)| + |\Im(z)| \quad \text{for all } z \in \mathbb{C}.$$

Observe that $\mathbb{C}$ is not an ordered field. Indeed, if $\mathbb{C}$ had a subset $\mathbb{C}^+$ satisfying the two order properties O1 and O2 (as in Proposition A.7 with $\mathcal{R}^+$ replaced by $\mathbb{C}^+$ and $\mathcal{R}$ replaced by $\mathbb{C}$), then by O1, either $i \in \mathbb{C}^+$ or $-i \in \mathbb{C}^+$. Now O2 implies that $-1 = (\pm i)^2 \in \mathbb{C}^+$ and $1 = (-1)^2 \in \mathbb{C}^+$. This contradicts O1.

The **complex exponential** of a complex number z is defined by

$$e^z := e^x(\cos y + i \sin y), \quad \text{where } x = \Re(z) \text{ and } y = \Im(z).$$

Note that $|e^z| = e^{\Re(z)}$ and therefore $e^z \neq 0$ for all $z \in \mathbb{C}$. Note also that $e^{i\theta} = \cos\theta + i\sin\theta$ for all $\theta \in \mathbb{R}$. In particular, $e^{i\pi} + 1 = 0$.

We now consider **complex-valued** functions of a real variable, that is, functions whose codomain is $\mathbb{C}$ and whose domain is a subset of $\mathbb{R}$. For example, $h : \mathbb{R} \to \mathbb{C}$ defined by $h(t) := e^{it}$ is a complex-valued function of a real variable. In general, if $D \subseteq \mathbb{R}$ and if $h : D \to \mathbb{C}$ is a complex-valued function of a real variable, then there are unique functions $f, g : D \to \mathbb{R}$ such that $h(t) = f(t) + ig(t)$ for $t \in D$. We call f the **real part** and g the **imaginary part** of h, and write $h = f + ig$. The notions of boundedness, continuity, differentiability and Riemann integration extend to complex-valued functions of a real variable in a straightforward manner. In fact, we define $h : D \to \mathbb{C}$ to be

- **bounded** on D if both f and g are bounded on D,
- **continuous** at a point c of D if both f and g are continuous at c,
- **differentiable** at an interior point c of D if both f and g are differentiable at c; in this case, the complex number $h'(c) := f'(c) + ig'(c)$ is called the **derivative** of h at c,

In case $D := [a, b]$ for some $a, b \in \mathbb{R}$ with $a \leq b$, and $h : D \to \mathbb{C}$ is bounded, then we define h to be **integrable** on $[a, b]$ if both f and g are integrable $[a, b]$, and in this case, the complex number $\int_a^b f(t)dt + i\int_a^b g(t)dt$ is called the **integral** of h over $[a, b]$ and denoted by $\int_a^b h(t)dt$.

For example, if $D = [0, \pi]$ and $h : D \to \mathbb{R}$ is defined by $h(t) := e^{it}$, then h is differentiable at each $t \in D$ and $h'(t) = ie^{it}$. Also, h is integrable and

$$\int_a^b h(t)dt = \int_0^\pi \cos t dt + i\int_0^\pi \sin t dt = 0 - i((-1) - 1) = 2i = \frac{1}{i}\left(e^{i\pi} - e^0\right).$$

In a similar way, we can consider complex-valued functions of two or more real variables. Thus, for example, if $D \subseteq \mathbb{R}^2$ and $h : D \to \mathbb{C}$ is a complex-valued function of two real variables, then there are unique real-valued functions $f, g : D \to \mathbb{R}$ such that $h(t, u) = f(t, u) + ig(t, u)$ for $(t, u) \in D$. As in Section 10.6, we can fix one variable and differentiate or integrate with respect to another. Thus we let

$$D_1h(t,u) = D_1f(t,u) + iD_1g(t,u) \text{ and } D_2h(t,u) = D_2f(t,u) + iD_2g(t,u),$$

where D_1 indicates differentiation with respect to the first variable t (treating u as a constant) and D_2 indicates differentiation with respect to the second variable u (treating t as a constant). Likewise, if $D \subseteq \mathbb{R}^2$ is of the form $D = [a,b] \times E$ for some $a, b \in \mathbb{R}$ with $a \le b$ and $E \subseteq \mathbb{R}$, and if $h : D \to \mathbb{C}$ is continuous in t (which means both f and g are continuous in t), then we can consider the integral function $H : E \to \mathbb{C}$ given by

$$H(u) = \int_a^b h(t,u)dt := \int_a^b f(t,u)dt + i\int_a^b g(t,u)dt \quad \text{for } u \in E.$$

Moreover, the following analogue of Proposition 10.52 holds.

Proposition B.1. *Let E be an interval in $\mathbb{R}$, and let $a, b \in \mathbb{R}$ with $a \le b$. Suppose $h : [a,b] \times E \to \mathbb{C}$ is a complex-valued function of two real variables such that D_2h exists and is bounded on $[a,b] \times E$, and for each $u \in E$, the integral $\int_a^b D_2h(t,u)dt$ exists. Then the integral function $H : E \to \mathbb{C}$ given by $H(u) := \int_a^b h(t,u)dt$ is differentiable and $H'(u) = \int_a^b D_2h(t,u)dt$ for $u \in E$.*

Proof. Use Proposition 10.52 for the real and imaginary parts of h. $\square$

B.2 Polynomials and Their Roots

In this section, we shall give a proof due to Paul Loya [60] of the Fundamental Theorem of Algebra. The statement involves polynomials (in one variable) with complex coefficients, which are defined in the same way as polynomials with real coefficients. (See Section 1.3.) Notions of the degree and of the leading coefficient of a nonzero polynomial are also defined in the same way. In particular, by a **nonconstant polynomial** we mean a polynomial of positive degree. Also, by a **monic polynomial** we mean a polynomial whose leading coefficient is 1. We will begin by proving an elementary property of polynomials that will be useful later.

Lemma B.2. *Let $p(z)$ be a polynomial with coefficients in $\mathbb{C}$. Then for every $z_0 \in \mathbb{C}$, there exists $\delta_0 > 0$ such that $|p(z)| \ge |p(z_0)|/2$ for all $z \in \mathbb{C}$ with $|z - z_0| < \delta_0$.*

Proof. Write $p(z) = c_dz^d + c_{d-1}z^{d-1} + \cdots + c_1z + c_0$, where $c_0, c_1, \ldots, c_d \in \mathbb{C}$. Let us first consider $z_0 := 0$, so that $p(z_0) = c_0$. Note that if $c_0 = 0$, then there is nothing to prove. Let us now assume that $c_0 \ne 0$. By writing $c_0 = p(z) - z\left(c_dz^{d-1} + c_{d-1}z^{d-2} + \cdots + c_1\right)$, we see that the triangle inequality gives $|p(z)| \ge |c_0| - |z|\left(|c_d||z|^{d-1} + \cdots + |c_1|\right)$. Consider $\delta_0 \in \mathbb{R}$ defined by $\delta_0 := |c_0|/2\left(|c_d| + \cdots + |c_1| + |c_0|\right)$. Then $0 < \delta_0 \le 1/2 < 1$ and

$$|z| < \delta_0 \Longrightarrow |z| \left(|c_d||z|^{d-1} + \cdots + |c_1|\right) \le \delta_0 \left(|c_d| + \cdots + |c_1|\right) < \frac{|c_0|}{2}.$$

Consequently, $|p(z)| \ge |c_0|/2 = |p(0)|/2$ whenever $|z| < \delta_0$.

Next, consider any $z_0 \in \mathbb{C}$. Substituting $z = (z - z_0) + z_0$ in $p(z)$, we see that $p(z) = p^*(z - z_0)$, where $p^*(w) = c_d^* w^d + \cdots + c_1^* w + c_0^*$ is a polynomial in w whose coefficients $c_0^*, \ldots, c_d^* \in \mathbb{C}$ are determined by $c_0, \ldots, c_d$ and z_0, and moreover, $c_0^* = p(z_0)$. From the first part of the proof, we find $\delta_0 > 0$ such that $|p^*(w)| \ge |c_0^*|/2$ for all $w \in \mathbb{C}$ with $|w| < \delta_0$. Puting $w = z - z_0$, we obtain the desired result. $\square$

We remark that the above lemma can also be deduced from the continuity of polynomial functions of a complex variable. However, since we have not discussed continuity of functions of a complex variable, we have chosen to give a direct proof.

Proposition B.3 (Fundamental Theorem of Algebra). *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof. Let $p(z)$ be a nonconstant polynomial of degree d with coefficients in $\mathbb{C}$. Dividing $p(z)$ by its leading coefficient, we may assume that $p(z)$ is monic. Thus we can write $p(z) = z^d + c_{d-1}z^{d-1} + \cdots + c_1 z + c_0$, where $c_0, \ldots, c_{d-1} \in \mathbb{C}$. Suppose $p(z)$ has no root in $\mathbb{C}$, that is, $p(z_0) \ne 0$ for all $z_0 \in \mathbb{C}$. In particular, $c_0 = p(0) \ne 0$. Consider the function $h : [-\pi, \pi] \times \mathbb{R} \to \mathbb{C}$ defined by

$$h(t, u) := \frac{1}{p\left(ue^{it}\right)} = \frac{1}{u^d e^{idt} + \cdots + c_1 u e^{it} + c_0} \quad \text{for } t \in [-\pi, \pi] \text{ and } u \in \mathbb{R}.$$

Clearly, h is differentiable in each of the two variables t and u. Moreover, a direct computation shows that

$$D_2 h(t, u) = \frac{-\left(du^{d-1}e^{idt} + \cdots + c_1 e^{it}\right)}{\left(u^d e^{idt} + \cdots + c_1 u e^{it} + c_0\right)^2} \quad \text{for } t \in [-\pi, \pi] \text{ and } u \in \mathbb{R}.$$

It is clear that for each fixed $u \in \mathbb{R}$, the function from $[-\pi, \pi]$ to $\mathbb{C}$ given by $t \longmapsto D_2 h(t, u)$ is continuous. Now let $\alpha \in \mathbb{R}$ with $\alpha > 0$. Then

$$\left|du^{d-1}e^{idt} + \cdots + c_1 e^{it}\right| \le d\alpha^{d-1} + \cdots + |c_1| \quad \text{for } t \in [-\pi, \pi] \text{ and } u \in [-\alpha, \alpha].$$

Next, we show that there exists $\delta > 0$ such that $\left|p\left(ue^{it}\right)\right| \ge \delta$ for all $t \in [-\pi, \pi]$ and $u \in [-\alpha, \alpha]$. Suppose this is not the case. Then there are sequences (t_n) in $[-\pi, \pi]$ and (u_n) in $[-\alpha, \alpha]$ such that $\left|p\left(u_n e^{it_n}\right)\right| < 1/n$ for all $n \in \mathbb{N}$. By the Bolzano–Weierstrass Theorem (Proposition 2.17), there exist a subsequence (t_{n_k}) of (t_n) and $t_0 \in [-\pi, \pi]$ such that $t_{n_k} \to t_0$. In turn, there exist a subsequence $(u_{n_{k_j}})$ of (u_{n_k}) and $u_0 \in [-\alpha, \alpha]$ such that $u_{n_{k_j}} \to u_0$. Let $z_j := u_{n_{k_j}} e^{it_{n_{k_j}}}$ for $j \in \mathbb{N}$ and let $z_0 := u_0 e^{it_0}$. Then $p(z_0) \ne 0$, and by Lemma B.2, there exists $\delta_0 > 0$ such that $|p(z)| \ge |p(z_0)|/2$ for all $z \in \mathbb{C}$ with

$|z - z_0| < \delta_0$. Since $z_j \to z_0$, there exists $j_0 \in \mathbb{N}$ such that $|z_j - z_0| < \delta_0$ for $j \geq j_0$, and hence

$$0 < \frac{|p(z_0)|}{2} \leq |p(z_j)| < \frac{1}{n_{k_j}} \quad \text{for all } j \geq j_0.$$

But this is impossible, since $n_{k_j} \to \infty$ as $j \to \infty$. Hence there exists $\delta > 0$ such that $\left|p\left(ue^{it}\right)\right| \geq \delta$ for all $t \in [-\pi, \pi]$ and $u \in [-\alpha, \alpha]$. It follows that

$$|D_2 h(t,u)| \leq \frac{d\alpha^{d-1} + \cdots + |c_1|}{\delta^2} \quad \text{for all } t \in [-\pi, \pi] \text{ and } u \in [-\alpha, \alpha].$$

Thus the hypothesis of Proposition B.1 is satisfied with $E = [-\alpha, \alpha]$. Since $\alpha > 0$ is arbitrary, we see that the integral function $H : \mathbb{R} \to \mathbb{C}$ given by $H(u) = \int_{-\pi}^{\pi} h(t,u)dt$ is differentiable and

$$H'(u) = \int_{-\pi}^{\pi} D_2 h(t,u)dt \quad \text{for all } u \in \mathbb{R}.$$

If $u = 0$, then $D_2 h(t,u) = D_2 h(t,0) = -c_1 e^{it}/c_0^2$ for $t \in [-\pi, \pi]$, and so $H'(0) = -\left(c_1/c_0^2\right) \int_{-\pi}^{\pi} e^{it} dt = -\left(c_1/ic_0^2\right)\left(e^{i\pi} - e^{-i\pi}\right) = 0$, whereas

$$D_2 h(t,u) = \frac{-\left(u^d ide^{idt} + \cdots + c_1 uie^{it}\right)}{iu\left(u^d e^{idt} + \cdots + c_1 ue^{it} + c_0\right)^2} = \frac{1}{iu} D_1 h(t,u), \quad \text{if } u \neq 0.$$

Consequently, for all $u \in \mathbb{R}$ with $u \neq 0$,

$$H'(u) = \frac{1}{iu} \int_{-\pi}^{\pi} D_1 h(t,u)dt = \frac{1}{iu}\left[h(\pi,u) - h(-\pi,u)\right] = 0,$$

where the second equality follows from the Fundamental Theorem of Calculus (part (ii) of Proposition 6.24) and the last equality follows since $e^{\pi i} = e^{-\pi i}$. Thus $H'(u) = 0$ for all $u \in \mathbb{R}$. Hence by Corollary 4.23, H is a constant function, and so

$$H(u) = H(0) = \int_{-\pi}^{\pi} h(t,0)dt = \frac{2\pi}{c_0} \neq 0 \quad \text{for all } u \in \mathbb{R}.$$

On the other hand, we show that $H(n) \to 0$ as $n \to \infty$. To this end, let

$$g_n(t) := h(t,n) = \frac{1}{p(ne^{it})} \quad \text{for } n \in \mathbb{N} \text{ and } t \in [-\pi, \pi].$$

Now for all $n \in \mathbb{N}$ and $t \in [-\pi, \pi]$,

$$\begin{aligned}
|p(ne^{it})| &= |n^d e^{idt} + \cdots + c_1 ne^{it} + c_0| \\
&\geq \left|n^d e^{idt}\right| - \left|c_{d-1} n^{d-1} e^{i(d-1)t} + \cdots + c_1 ne^{it} + c_0\right| \\
&= n^d \left(1 - \left|\frac{c_{d-1}}{n} e^{i(d-1)t} + \cdots + \frac{c_1}{n^{d-1}} e^{it} + \frac{c_0}{n^d}\right|\right) \\
&\geq n^d \left(1 - \frac{|c_{d-1}|}{n} - \cdots - \frac{|c_1|}{n^{d-1}} - \frac{|c_0|}{n^d}\right) \\
&\geq \frac{n^d}{2}, \quad \text{provided } n \geq 2\left(|c_0| + |c_1| + \cdots + |c_{d-1}|\right).
\end{aligned}$$

This implies that $g_n(t) \to 0$ as $n \to \infty$ for each $t \in [-\pi, \pi]$, and also that the sequence $(|g_n|)$ is uniformly bounded on $[-\pi, \pi]$. So by applying the Arzelà Bounded Convergence Theorem (Proposition 10.40) to the real and imaginary parts of g_n, we obtain

$$H(n) = \int_{-\pi}^{\pi} h(t,n)dt = \int_{-\pi}^{\pi} g_n(t)dt \longrightarrow \int_{-\pi}^{\pi} 0\,dt = 0.$$

But since $H(u) = 2\pi/c_0 \neq 0$, this is a contradiction. It follows that $p(z)$ must have a root in $\mathbb{C}$. □

Corollary B.4. (i) *Every nonzero polynomial $p(z)$ of degree d with coefficients in $\mathbb{C}$ can be factored into d linear factors as*

$$p(z) = c(z - \alpha_1) \cdots (z - \alpha_d),$$

where $c \in \mathbb{C}$ with $c \neq 0$ and $\alpha_1, \ldots, \alpha_d$ are complex numbers.

(ii) *Every nonzero polynomial $p(z)$ of degree d with coefficients in $\mathbb{C}$ is a product of powers of distinct linear factors as*

$$p(z) = c(z - \lambda_1)^{m_1} \cdots (z - \lambda_k)^{m_k},$$

where $c \in \mathbb{C}$ with $c \neq 0$ and $\lambda_1, \ldots, \lambda_k$ are distinct complex numbers and $m_1, \ldots, m_k$ are positive integers satisfying $m_1 + \cdots + m_k = d$.

(iii) **(Real Fundamental Theorem of Algebra)** *Every nonzero polynomial $p(x)$ with coefficients in $\mathbb{R}$ can be factored as a finite product of linear polynomials and quadratic polynomials with negative discriminants.*

Proof. (i) The result is obvious when $d = 0$, since an empty product equals 1. For $d \geq 1$, it follows from Proposition B.3 by induction on d.

(ii) This follows from (i) by collating equal linear factors.

(iii) Note that if $\alpha \in \mathbb{C}$ is a root of a polynomial $p(x)$ with real coefficients, that is, if $p(\alpha) = 0$, then $p(\overline{\alpha}) = \overline{p(\alpha)} = 0$, and hence the conjugate $\overline{\alpha}$ of α is also a root of $p(x)$. Thus linear factors of $p(x)$ of the form $(x - \alpha)$, where α is a nonreal complex number, occur in conjugate pairs. Since $(x - \alpha)(x - \overline{\alpha})$ is the polynomial $x^2 - 2\Re(\alpha)x + |\alpha|^2$ with real coefficients whose discriminant is negative, we see that (i) implies (iii). □

Recall that part (iii) of Corollary B.4 was stated earlier in Chapter 1.

Remark B.5. Numerous proofs of the Fundamental of Algebra are known, and in fact, the theorem can be proved using techniques developed in many of the advanced courses in mathematics such as complex analysis, topology, and Galois theory. The proof that we have chosen is closer to the spirit of this book and uses some of the ideas developed in Chapter 10. For a variety of other proofs, see the nice book of Fine and Rosenberger [30]. ◇

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List of Symbols and Abbreviations

	Definition/Description	Page
$\mathbb{N}$	set of all positive integers	1
$\mathbb{Z}$	set of all integers	1
$\mathbb{Q}$	set of all rational numbers	1
$\mathbb{R}$	set of all real numbers	2
$\sum$	sum	3
$\prod$	product	3
$A := B$	A is defined to be equal to B	3
$\mathbb{R}^+$	set of all positive real numbers	3
$\sup S$	supremum of a subset S of $\mathbb{R}$	4
$\inf S$	infimum of a subset S of $\mathbb{R}$	4
$\max S$	maximum of a subset S of $\mathbb{R}$	5
$\min S$	minimum of a subset S of $\mathbb{R}$	5
$[x]$	integer part of a real number x	6
$\lfloor x \rfloor$	integer part or the floor of a real number x	6
$\lceil x \rceil$	ceiling of a real number x	6
$\sqrt[n]{a}$	nth root of a nonnegative real number a	7
$\sqrt{a}$	square root of a nonnegative real number a	7
$m \mid n$	m divides n	8
$m \nmid n$	m does not divide n	8
(a, b)	open interval $\{x \in \mathbb{R} : a < x < b\}$	9
$[a, b]$	closed interval $\{x \in \mathbb{R} : a \le x \le b\}$	9
$[a, b)$	semiopen interval $\{x \in \mathbb{R} : a \le x < b\}$	9
$(a, b]$	semiopen interval $\{x \in \mathbb{R} : a < x \le b\}$	9
(a, ∞)	semi-infinite interval $\{x \in \mathbb{R} : x > a\}$	9
$[a, \infty)$	semi-infinite interval $\{x \in \mathbb{R} : x \ge a\}$	9
$(-\infty, a)$	semi-infinite interval $\{x \in \mathbb{R} : x < a\}$	9
$(-\infty, a]$	semi-infinite interval $\{x \in \mathbb{R} : x \le a\}$	9
$\lvert a \rvert$	absolute value of a real number a	10

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	Definition/Description	**Page**
A.M.	arithmetic mean	12
G.M.	geometric mean	12
$D \setminus C$	$\{x \in D : x \notin C\}$	13
$D \times E$	$\{(x, y) : x \in D \text{ and } y \in E\}$	14
$f_{\vert C}$	restriction of $f : D \to E$ to a subset C of D	16
$g \circ f$	composite of g with f	16
f^{-1}	inverse of an injective function f	16
$f = 0$	f is the zero function (on its domain)	17
$f \leq g$	$f(x) \leq g(x)$ for all x	17
$f \geq 0$	$f(x) \geq 0$ for all x	17
$\mathbb{R}[x]$	set of all polynomials in x with coefficients in $\mathbb{R}$	18
$\deg p(x)$	degree of a nonzero polynomial $p(x)$	18
IVP	intermediate value property	29
H.M.	harmonic mean	34
GCD	greatest common divisor	37
LCM	least common multiple	37
(a_n)	sequence whose nth term is the real number a_n	41
$a_n \to a$	sequence (a_n) tends to a real number a	42
$\lim\limits_{n\to\infty} a_n$	limit of the sequence (a_n)	43
$a_n = O(b_n)$	(a_n) is big-oh of (b_n)	51
$a_n = o(b_n)$	(a_n) is little-oh of (b_n)	52
$a_n \sim b_n$	(a_n) is asymptotically equivalent to (b_n)	52
$a_n \to \infty$	sequence (a_n) tends to ∞	52
$a_n \to -\infty$	sequence (a_n) tends to $-\infty$	52
$\not\to$	does not tend to	53
$\limsup\limits_{n\to\infty} a_n$	limit superior of (a_n)	60
$\liminf\limits_{n\to\infty} a_n$	limit inferior of (a_n)	60
$\lim\limits_{x\to c} f(x)$	limit of $f(x)$ as x tends to c	84
$\lim\limits_{x\to c^-} f(x)$	left limit of $f(x)$ as x tends to c	89
$\lim\limits_{x\to c^+} f(x)$	right limit of $f(x)$ as x tends to c	90
$f(x) = O(g(x))$	$f(x)$ is big-oh of $g(x)$ as $x \to \infty$	92
$f(x) = o(g(x))$	$f(x)$ is little-oh of $g(x)$ as $x \to \infty$	92
$f(x) \sim g(x)$	$f(x)$ is asymptotically equivalent to $g(x)$	93
$f'(c),\ \left.\dfrac{df}{dx}\right\vert_{x=c}$	derivative of f at c	106
$f''(c),\ \left.\dfrac{d^2 f}{dx^2}\right\vert_{x=c}$	second derivative of f at c	114
$f^{(n)}(c),\ \left.\dfrac{d^n f}{dx^n}\right\vert_{x=c}$	nth derivative of f at c	114

	Definition/Description	Page
$f'_-(c)$	left derivative of f at c	115
$f'_+(c)$	right derivative of f at c	115
MVT	Mean Value Theorem	122
$\approx$	approximately equal	126
L'HR	L'Hôpital's Rule	135, 136
P_n	partition of $[a, b]$ into n equal parts	182
$m(f)$	infimum of $\{f(x) : x \in [a, b]\}$	182
$M(f)$	supremum of $\{f(x) : x \in [a, b]\}$	182
$m_i(f)$	infimum of $\{f(x) : x \in [x_{i-1}, x_i]\}$	182
$M_i(f)$	supremum of $\{f(x) : x \in [x_{i-1}, x_i]\}$	182
$L(P, f)$	lower sum for f with respect to P	183
$U(P, f)$	upper sum for f with respect to P	183
$L(f)$	lower Riemann integral of f	183
$U(f)$	upper Riemann integral of f	183
$\int_a^b f(x)dx$	Riemann integral of f on $[a, b]$	185
f^+	positive part of f	202
f^-	negative part of f	202
FTC	Fundamental Theorem of Calculus	204
$\int f(x)dx$	an indefinite integral of f	206
$\big[F(x)\big]_a^b$, $F(x)\big\vert_a^b$	$F(b) - F(a)$	206
$S(P, \mathcal{T}, f)$	Riemann sum for f corresponding to a partition P and a tag set $\mathcal{T}$	211
$\mu(P)$	mesh of a partition P	211
$\ell(D)$	length of a bounded subset D of $\mathbb{R}$	221
ln	logarithmic function	234
e	unique real number such that $\ln e = 1$	234
exp	exponential function	236
arctan	arctangent function	246
π	$2 \sup\{\arctan x : x \in (0, \infty)\}$	247
$\angle(OP_1, OP_2)$	angle between OP_1 and OP_2	269
$L_1 \parallel L_2$	lines L_1 and L_2 are parallel	271
$L_1 \nparallel L_2$	lines L_1 and L_2 are not parallel	271
$\measuredangle(L_1, L_2)$	(acute) angle between L_1 and L_2	271
$L_1 \perp L_2$	lines L_1 and L_2 are perpendicular	272
$L_1 \not\perp L_2$	lines L_1 and L_2 are not perpendicular	272
$\measuredangle(C_1, C_2; P)$	angle at P between C_1 and C_2	273
Area (R)	area of a region R	296
Vol (D)	volume of a solid body D	303, 307
$\ell(C)$	length of a curve C	316
Area (S)	area of a surface S	325
Av(f)	average of f	329
Av$(f; w)$	weighted average of f with respect to w	329

	Definition/Description	**Page**
$(\overline{x}, \overline{y})$	centroid of a curve or a planar region	330, 333
$(\overline{x}, \overline{y}, \overline{z})$	centroid of a surface or a solid body	331, 334
$Q(f)$	Quadrature Rule for f	341
$R(f)$	Rectangular Rule for f	341
$M(f)$	Midpoint Rule for f	341
$T(f)$	Trapezoidal Rule for f	342
$S(f)$	Simpson Rule for f	342
$R_n(f)$	Compound Rectangular Rule for f	343
$M_n(f)$	Compound Midpoint Rule for f	343
$T_n(f)$	Compound Trapezoidal Rule for f	344
$S_n(f)$	Compound Simpson Rule for f	345
$\sum_{k\geq 1} a_k$	series whose sequence of terms is (a_k)	366
$\sum_{k=1}^{\infty} a_k$	sum of $\sum_{k\geq 1} a_k$, when convergent	366
$\int_{t\geq a} f(t)dt$	improper integral of f on $[a, \infty)$	391
$\int_a^{\infty} f(t)dt$	value of $\int_{t\geq a} f(t)dt$, when convergent	392
$\int_{t\geq b} f(t)dt$	improper integral of f on $(-\infty, b]$	406
$\int_{-\infty}^{b} f(t)dt$	value of $\int_{t\geq b} f(t)dt$, when convergent	406
$\int_{\mathbb{R}} f(t)dt$	improper integral of f on $(-\infty, \infty)$	406
$\int_{-\infty}^{\infty} f(t)dt$	value of $\int_{\mathbb{R}} f(t)dt$, when convergent	406
$\int_{a<t\leq b} f(t)dt$	improper integral of f on $(a, b]$	407
$\int_{a^+}^{b} f(t)dt$	value of $\int_{a<t\leq b} f(t)dt$, when convergent	407
$\int_{a\leq t<b} f(t)dt$	improper integral of f on $[a, b)$	408
$\int_{a}^{b^-} f(t)dt$	value of $\int_{a\leq t<b} f(t)dt$, when convergent	408
$\int_{a<t<b} f(t)dt$	improper integral of f on (a, b)	408
$\int_{a^+}^{b^-} f(t)dt$	value of $\int_{a<t<b} f(t)dt$, when convergent	408
$\beta(p, q)$	beta function at $(p, q) \in (0, \infty) \times (0, \infty)$	413
$\Gamma(u)$	gamma function at $u \in (0, \infty)$	413
(f_n)	sequence whose nth term is the function f_n	426
$f_n \to f$	(f_n) converges pointwise to f	426
$f_n \to f$ uniformly	(f_n) converges uniformly to f	429
$\sum_{k\geq 1} f_k$	series of functions whose sequence of terms is (f_k)	438
$\sum_{k=1}^{\infty} f_k$	sum function of $\sum_{k\geq 1} f_k$, when convergent	438

	Definition/Description	**Page**
$B_n(f)$	nth Bernstein polynomial function associated with f	449
$a_k(f),\ b_k(f)$	Fourier coefficients of f	453
$S_n(f)$	nth partial sum of the Fourier series of f	453
$\sigma_n(f)$	arithmetic mean of $S_0(f), S_1(f), \ldots, S_n(f)$	453
D_n	nth Dirichlet kernel	454
K_n	nth Fejér kernel	454
$f_n \to f$ boundedly	(f_n) converges boundedly to f	458
$f(\cdot, u)$	function given by $t \longmapsto f(t, u)$ for a fixed u	467
$f(t, \cdot)$	function given by $u \longmapsto f(t, u)$ for a fixed t	467
$\int_a^b f(t, \cdot)dt$	integral function given by $u \longmapsto \int_a^b f(t, u)dt$	467
$D_1 f(\cdot, u)$	derivative of the function $f(\cdot, u)$ for a fixed u	469
$D_2 f(t, \cdot)$	derivative of the function $f(t, \cdot)$ for a fixed t	469
$\int_{t \geq a} f(t, \cdot)dt$	improper integral of f on $[a, \infty)$ depending on a parameter	472
$\int_a^\infty f(t, \cdot)dt$	improper integral function given by $u \longmapsto \int_a^\infty f(t, u)dt$	472
$\mathcal{F}_s(f)$	Fourier sine integral of f	476
$\mathcal{F}_c(f)$	Fourier cosine integral of f	476
$\mathcal{L}(f)$	Laplace integral of f	477
$\int_{a<t\leq b} f(t, \cdot)dt$	improper integral of f on $(a, b]$ depending on a parameter	486
$\int_{a^+}^b f(t, \cdot)dt$	improper integral function given by $u \longmapsto \int_{a^+}^b f(t, u)dt$	486
$\mathbb{Q}^+$	set of positive rational numbers	503
$[a]$	equivalence class of a	504
$\mathscr{C}$	set of all Cauchy sequences of rational numbers	506
$\mathcal{R}$	set of all equivalence classes of elements of $\mathscr{C}$	507
$[(a_n)]$	equivalence class of a sequence (a_n) in $\mathscr{C}$	507
(r)	constant sequence having each term equal to r	507
$\mathcal{Q}$	set of all equivalence classes of constant sequences in $\mathscr{C}$	507
$\mathcal{R}^+$	set of all equivalence classes of positive sequences in $\mathscr{C}$	508
0_F	additive identity in a field F	512
1_F	multiplicative identity in a field F	512
$\mathbb{C}$	set of all complex numbers	517
$\Re(z)$	real part of a complex number z	517
$\Im(z)$	imaginary part of a complex number z	517
$\overline{z}$	conjugate of a complex number z	517
$\|z\|$	absolute value of a complex number z	517

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