MTMW14 Project 2: Using Shallow Water Equations to Model Ocean Gyres

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Introduction

In this project a model of a large scale ocean gyre is developed. The model is based on the Shallow Water Equations (SWEs), linearised about a resting state:

$$\frac{\partial \eta}{\partial t} = -H(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}), \tag{1}$$

$$\frac{\partial u}{\partial t} = +(f_0 + \beta y)v - g\frac{\partial \eta}{\partial x} - \gamma u + \frac{\tau_x}{\rho H}, \tag{2}$$

$$\frac{\partial v}{\partial t} = -(f_0 + \beta y)u - g\frac{\partial \eta}{\partial y} - \gamma v + \frac{\tau_y}{\rho H}. \tag{3}$$

Here η represents the height perturbation, u and v represent the column averaged zonal and meridional velocity perturbations. H is the average height, f_0 and β are the Coriolis and β parameters, g is the acceleration due to gravity, γ represents drag processes, and τ_x and τ_y represent the wind stress forcings. The equations were solved using a square domain, with the sides being given by L. The values used for the parameters were:

parameter	value	unit
L	1×10^6	m
H	1000	m
f_0	1×10^{-4}	s^{-1}
β	1×10^{-11}	m^{-1} s^{-1}
g	10	m s^{-2}
γ	1×10^{-6}	s^{-1}
ρ	1000	kg m^{-3}
τ_x	$-cos(\frac{\pi y}{L})$	XX
τ_y	0	XX

Following (XX Matsuno Beckers Deleersnijder) the SWEs are solved on an Arakawa-C grid using the forward-backward time scheme. The Arakawa-C grid was chosen so that e.g. spatial derivatives of u in the x direction would be available at the points where η is calculated. The domain is taken to be the size of the η grid points, as can be seen in Figure 1.

First η, u then v are calculated (in that order):

$$\eta^{n+1} = \eta^n - H\Delta t(\frac{\partial u^n}{\partial x} + \frac{\partial v^n}{\partial y}), \tag{4}$$

$$u^{n+1} = u^n + (f_0 + \beta y)\Delta t v^n - g\Delta t\frac{\partial \eta^{n+1}}{\partial x} - \gamma \Delta t u^n + \Delta t\frac{\tau_x}{\rho H}, \tag{5}$$

$$v^{n+1} = v^n - (f_0 + \beta y)\Delta t u^{n+1} - g\Delta t\frac{\partial \eta^{n+1}}{\partial y} - \gamma \Delta t v^n + \Delta t\frac{\tau_y}{\rho H}. \tag{6}$$

Second η, v then u are calculated (note, the order of u and v calculations has been swapped):

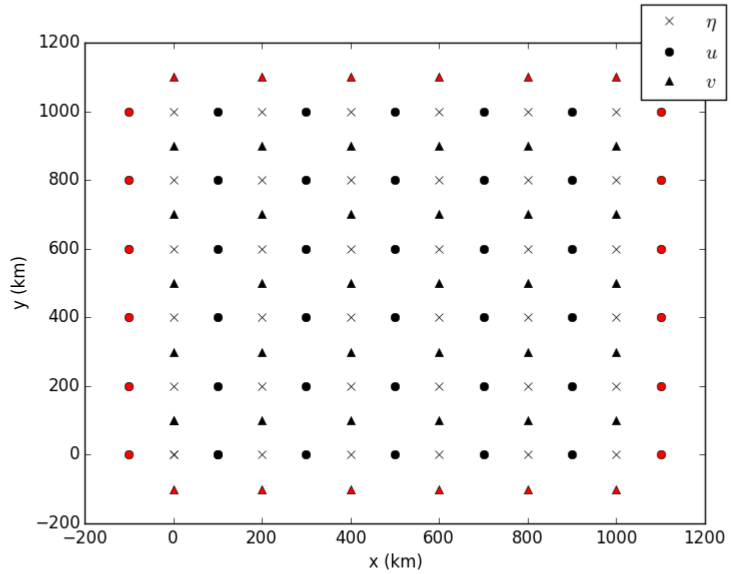

Figure 1: Shows where η, u and v are calculated on the Arakawa-C grid for $\Delta x = \Delta y = 250000\,m$ (lower than the lowest resolution used in this project, for illustration only). Note, u is one bigger in the x direction than η (and similar for v in y direction), and therefore the minimum and maximum u coordinates lie $\frac{\Delta x}{2}$ outside the domain (and similar for v). The red grid-point values show values which are held at 0 due to the boundary conditions.

$$\eta^{n+2} = \eta^{n+1} - H\Delta t(\frac{\partial u^{n+1}}{\partial x} + \frac{\partial v^{n+1}}{\partial y}), \tag{7}$$

$$v^{n+2} = v^{n+1} - (f_0 + \beta y)\Delta t u^{n+1} - g\Delta t\frac{\partial \eta^{n+2}}{\partial y} - \gamma\Delta t v^{n+1} + \Delta t\frac{\tau_y}{\rho H}, \tag{8}$$

$$u^{n+2} = u^{n+1} + (f_0 + \beta y)\Delta t v^{n+2} - g\Delta t\frac{\partial \eta^{n+1}}{\partial x} - \gamma\Delta t u^{n+1} + \Delta t\frac{\tau_x}{\rho H}. \tag{9}$$

To calculate u on the other two grids (see 1), spatial averging must be used. E.g. to calculate u on the η grid, the average of two u grid-points in the x direction must be used, and to calculate u on the v grid, four grid-points in the x and y directions must be used. Similar calculations apply to v and η on the other grids. Derivatives are calculated using the two values on either side of the grid-point where they are needed (using the midpoint method which is 2nd order accurate, indeed this is the strength of the Arakawa-C grid).

Task A

To model the Western Boundary Current (WBC), a sufficient number of grid-points must span this distance. If the size of the WBC is taken to be $\frac{1}{10}$ $^{\text{th}}$ of the zonal extent of the domain, or 100 km, and four grid-points are required across this to capture its variation, this would mean a minimum spatial resolution of $\Delta x = 25\,km$ should be used. In this Task, a value of $\Delta x = \Delta y = 20\,km$ was used. Kinematic boundary conditions are used throughout this study, i.e. u is held at 0 on the eastern and western boundaries, v is held at 0 on the northern and southern boundaries.

The fastest signals propagating in this system are gravity-inertia waves. These have a phase speed of $\sqrt{gH}$, and following XX Beckers…this can be used to calculate an upper bound for the CFL criterion, i.e. $\sqrt{gH}\frac{\Delta t}{\Delta x} <= \frac{1}{4}$ will be a necessary condition for the scheme to be stable. In practice, at the first resolution, it was found that the less stringent requirement of $\sqrt{gH}\frac{\Delta t}{\Delta x} <= 1$ was necessary to ensure numerical stability (from empirical experimentation). The first CFL criterion will be referred to as the strict criterion, and the second the lax criterion. In light of this, in this task Δt was taken to be 139 s so as to just satisfy this lax criterion. Satisfying the lax criterion allows for a larger timestep and therefore less time is needed to run the simulations.

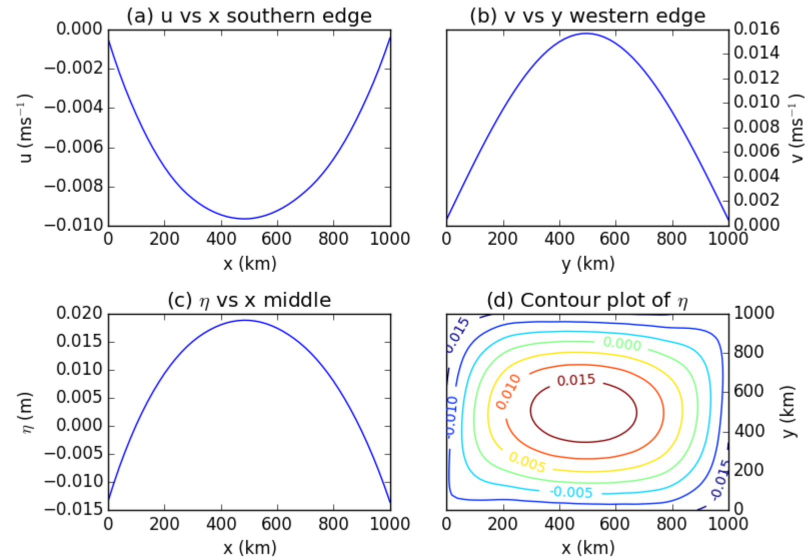

Figure 2: Four plots showing u, v and η along three different zonal/meridional extents after one day - (a), (b) and (c). Plot (d) shows η over the whole domain after one day.

The model was run for one day, and various plots of u, v and η are shown in Figure 2. Overall, an anti-clockwise gyre is set up, as can be seen in plots (a) and (b) because u is negative along the southern boundary, and v is positive along the western boundary. Plot (d) shows that the height perturbation, η, is almost symmetrical after one day, although anomalously high values can be seen in the northeast and southwest corners.

Task B

The energy stored in the gyre can be calculated by summing the contributions from the kinetic energy (terms one and two in the integrand in Equation 10) and the gravitational potential energy (term three):

$$E(u, v, \eta) = \int_0^L \int_0^L \frac{1}{2}\rho(H(u^2 + v^2) + g\eta^2)dxdy \tag{10}$$

This can be approximated over the domain using Simpson's rule in 2D. This can then be calculated for every timestep, as is shown in Figure 3. For this figure, the model was run at two resolutions: once as before with $\Delta x = \Delta y = 10\,km$, and once with $\Delta x = \Delta y = 10\,km$ (i.e. twice the spatial resolution). A time step of $\Delta t = 30\,s$ was use in both cases, so that the strict CFL criterion is still satisfied for the high resolution run and that only changes in the spatial resolution are being compared between the two runs. (It was found that running the model at the higher resolution with only the lax criterion being satisfied lead to numerical instability after around 50 days.) The model was run for 100 days in each case, and it can be seen that the energy reaches a peak at around 35 days before reducing by 2.05%(1.26%) for $\Delta x = 20$ km ($\Delta x = 10$ km) over the next 65 days.

Task C

Given that the energy in the model is approximately constant after 100 days, it can be compared to the steady state analytical solution as found in (XXX Musgrave). The only unknown in these equations is the value of η_0, for which the modelled value of $\eta(0, \frac{L}{2})$ is used. One way of doing these comparisons is to work out the perturbations of u, v and η as defined by e.g. $u' = u - u_{st}$, then calculating the energy difference using Equation 10 - $E(u', v', \eta')$. Using this equation, when $\Delta x = 20$ km, the total energy difference between the analytical and the modelled steady state solution is 18.4 TJ, and when $\Delta x = 10$ km, the total energy difference is 4.98 TJ. Therefore, increasing the resolution increases the accuracy of the modelled steady state solution. The 2$^{\text{nd}}$ order spatial accuracy of the 2D SWE numerical model on an Arakawa-C grid can be seen from the fact that when the resolution is doubled, the error, as measured by the total energy difference, is approximately four times smaller.

Task C

One way of solving a non-linear version of the SWE equations is to implement a semi-Lagrangian numerical scheme. Here the full Lagrangian rates of change ($\frac{D}{Dt}$ as opposed to $\frac{\partial}{\partial t}$) are used, and the departure point of each of each grid-point is calculated to work out the previous value of u, v and η to be used. It is therefore necessary to calculate the departure points for each of the u, v and η grids used in the Arakawa-C grid. To calculate the departure point, the $\boldsymbol{u}$ = (u,v) field at two previous timesteps is used. Following Durran pp366-368, first, a 2$^{\text{nd}}$ order estimate for $\boldsymbol{u}^{n+\frac{1}{2}}$ is calculated using $\boldsymbol{u}^{n+\frac{1}{2}} = \frac{3}{2}\boldsymbol{u}^n - \frac{1}{2}\boldsymbol{u}^{n-1}$. This is then used to calculate a value of $\tilde{\boldsymbol{u}}$, the departure point, through the use of an intermediate departure point using the following formulae:

$$\boldsymbol{x_*} = \boldsymbol{x}^{n+1} - \boldsymbol{u}(\boldsymbol{x}^{n+1}, t^n)\frac{\Delta t}{2}, \tag{11}$$

$$\tilde{\boldsymbol{x}}^n = \boldsymbol{x}^{n+1} - \boldsymbol{u}(\boldsymbol{x_*}, t^{n+\frac{1}{2}})\Delta t. \tag{12}$$

(13)

The departure points tell us where to take the values of u, v and η from in the previous timestep. However, it is unlikely that these departure points will lie on grid-points, therefore it is necessary to interpolate between the values of these three fields to calculate the values at the departure points. This is achieved using the `RectBivariateSpline` class from the `scipy.interpolate` namespace. By default this uses a 2D cubic spline interpolation, and this was used to implement the interpolation of the three fields in this Task.

Appendix A

All code can be downloaded from the following link:
https://github.com/markmuetz/mtmw14

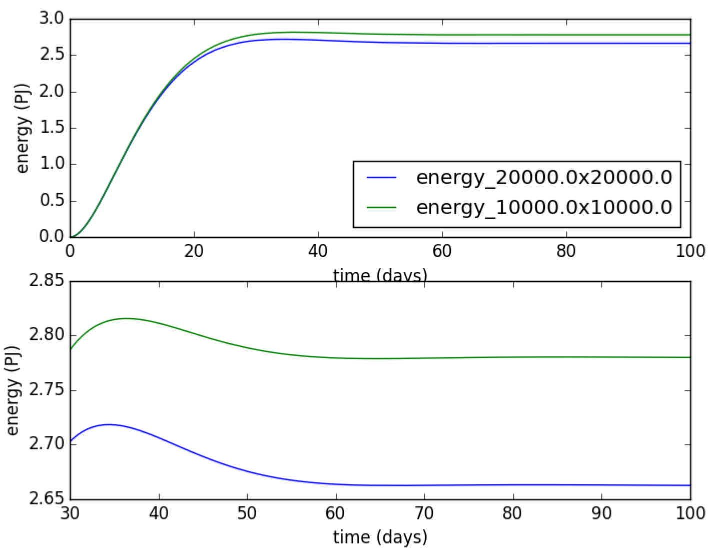

Figure 3: The effect of spatial resolution on the total energy of the system. Doubling the spatial resolution (blue curve) increases the overall energy of the system. It also reduces the energy difference between the analytical steady state solution and the steady state obtained by running the SWE model for 100 days.