MTMW14 Project 2: Using Shallow Water Equations to Model Ocean Gyres

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Introduction

In this project a model of a large scale ocean gyre is developed. The model is based on the Shallow Water Equations (SWEs), linearised about a resting state:

$$\frac{\partial \eta}{\partial t} = -H(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}), \tag{1}$$
$$\frac{\partial u}{\partial t} = +(f_0 + \beta y)v - g\frac{\partial \eta}{\partial x} - \gamma u + \frac{\tau_x}{\rho H}, \tag{2}$$
$$\frac{\partial v}{\partial t} = -(f_0 + \beta y)u - g\frac{\partial \eta}{\partial y} - \gamma v + \frac{\tau_y}{\rho H}. \tag{3}$$

Here η represents the height perturbation, u and v represent the column averaged zonal and meridional velocity perturbations. H is the average height, f_0 and β are the Coriolis and β parameters, g is the acceleration due to gravity, $gamma$ represents drag processes, and τ_x and τ_y represent the wind stress forcings. The equations were solved using a square domain, with the sides being given by L. The values used for the parameters were:

parameter	value	unit
L	1×10^6	m
H	1000	m
f_0	1×10^{-4}	s^{-1}
β	1×10^{-11}	m^{-1} s^{-1}
g	10	m s^{-2}
γ	1×10^{-6}	s^{-1}
ρ	1000	kg m^{-3}
τ_x	$-cos(\frac{\pi y}{L})$	XX
τ_y	0	XX

Following (XX Matsuno Beckers Deleersnijder) the SWEs are solved on an Arakawa-C grid using the forward-backward time scheme. The Arakawa-C grid was chosen so that e.g. spatial derivatives of u in the x direction would be available at the points where η is calculated. The domain is taken to be the size of the η grid points, as can be seen in Figure 1.

First η, u then v are calculated (in that order):

$$\eta^{n+1} = \eta^n - H\Delta t(\frac{\partial u^n}{\partial x} + \frac{\partial v^n}{\partial y}), \tag{4}$$
$$u^{n+1} = u^n + (f_0 + \beta y)\Delta t v^n - g\Delta t\frac{\partial \eta^{n+1}}{\partial x} - \gamma \Delta t u^n + \Delta t\frac{\tau_x}{\rho H}, \tag{5}$$
$$v^{n+1} = v^n - (f_0 + \beta y)\Delta t u^{n+1} - g\Delta t\frac{\partial \eta^{n+1}}{\partial y} - \gamma \Delta t v^n + \Delta t\frac{\tau_y}{\rho H}. \tag{6}$$

Second η, v then u are calculated (note, order of u and v calculations has been swapped):

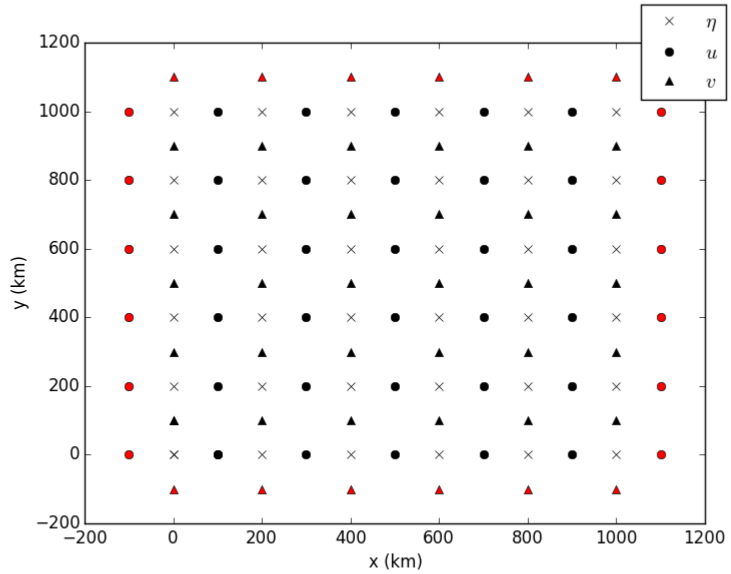

Figure 1: Shows where η, u and v are calculated on the Arakawa-C grid for $\Delta x = \Delta y = 250000\,m$ (lower than the lowest resolution used in this project, for illustration only). Note, u is one bigger in the x direction than η (and similar for v in y direction), and therefore the minimum and maximum u coordinates lie $\frac{\Delta x}{2}$ outside the domain (and similar for v). The red grid-point values show values which are held at 0 due to the boundary conditions.

$$\eta^{n+2} = \eta^{n+1} - H\Delta t(\frac{\partial u^{n+1}}{\partial x} + \frac{\partial v^{n+1}}{\partial y}), \tag{7}$$

$$v^{n+2} = v^{n+1} - (f_0 + \beta y)\Delta t u^{n+1} - g\Delta t\frac{\partial \eta^{n+2}}{\partial y} - \gamma\Delta t v^{n+1} + \Delta t\frac{\tau_y}{\rho H}, \tag{8}$$

$$u^{n+2} = u^{n+1} + (f_0 + \beta y)\Delta t v^{n+2} - g\Delta t\frac{\partial \eta^{n+1}}{\partial x} - \gamma\Delta t u^{n+1} + \Delta t\frac{\tau_x}{\rho H}. \tag{9}$$

To calculate u on the other two grids (see 1), spatial averging must be used. E.g. to calculate u on the η grid, the average of two u grid-points in the x direction must be used, and to calculate u on the v grid, four grid-points in the x and y directions must be used. Similar calculations apply to v and η on the other grids. Derivatives are calculated using the two values on either side of the grid-point where they are needed (using the midpoint method which is 2nd order accurate, indeed this is the strength of the Arakawa-C grid).

Task A

To model the Western Boundary Current (WBC), a sufficient number of grid-points must span this distance. If the size of the WBC is taken to be $\frac{1}{10}^{\text{th}}$ of the zonal extent of the domain, or 100 km, and four grid-points are required across this to capture its variation, this would mean a minimum spatial resolution of $\Delta x = 25\,km$ should be used. In this Task, a value of $\Delta x = \Delta y = 20\,km$ was used.

The fastest signals propagating in this system are gravity-inertia waves. These have a phase speed of $\sqrt{gH}$, and this can be treated as an upper bound for the CFL criterion, i.e. $\sqrt{gH}\frac{\Delta t}{\Delta x} <= 1$ will be a necessary condition for the scheme to be stable. In this task Δt was taken to be 139 s so as to just satisfy this criterion.

The model was run for one day, and various plots of u, v and η are shown in Figure 2. Overall, an anti-clockwise gyre is set up, as can be seen in plots (a) and (b) because u is negative along the southern boundary, and v is positive along the western boundary. Plot (d) shows that the height perturbation, η,

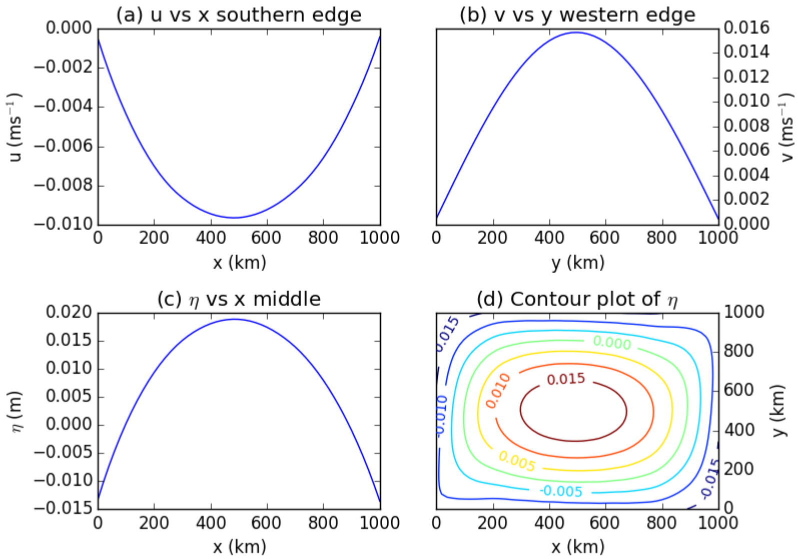

Figure 2: Four plots showing u, v and η along three different zonal/meridional extents after one day - (a), (b) and (c). Plot (d) shows η over the whole domain after one day.

is almost symmetrical after one day, although anomalously high values can be seen in the northeast and southwest corners.

Task B

The energy stored in the gyre can be calculated by summing the contributions from the kinetic energy (terms one and two in the integrand in Equation 10) and the gravitational potential energy (term three):

$$E(u, v, \eta) = \int_0^L \int_0^L \frac{1}{2}\rho(H(u^2 + v^2) + g\eta^2)dxdy \tag{10}$$

This can be approximated over the domain using Simpson's rule in 2D. This can then be calculated for every timestep, as is shown in Figure 3. For this figure, the model was run at two resolutions: once as before, and once with $\Delta x = \Delta y = 10\,km$ (i.e. twice the spatial resolution) and $\Delta t = 30\,s$ so that the (strict) CFL criterion is still satisfied. The model was run for 100 days in each case, and it can be seen that the energy reaches a peak at around 35 days before reducing by 2.05%(1.26%) for $\Delta x = 20$ km ($\Delta x = 10$ km) over the next 65 days.

Given that the energy in the model is approximately constant after 100 days, it can be compared to the steady state analytical solution as found in (XXX Musgrave). The only unknown in these equations is the value of η_0, for which the value of $\eta(0, \frac{L}{2})$ is used. One way of doing these comparisons is to work out the perturbations of u, v and η as defined by e.g. $u' = u - u_{st}$, then calculating the energy using Equation 10 - $E(u', v', \eta')$. When $\Delta x = 20$ km, the total energy is 18.4 TJ, and when $\Delta x = 10$ km, the total energy is 1.99 TJ. Therefore, increasing the resolution increases the accuracy of the modelled steady state solution.

Appendix A

All code can be downloaded from the following link:
https://github.com/markmuetz/mtmw14

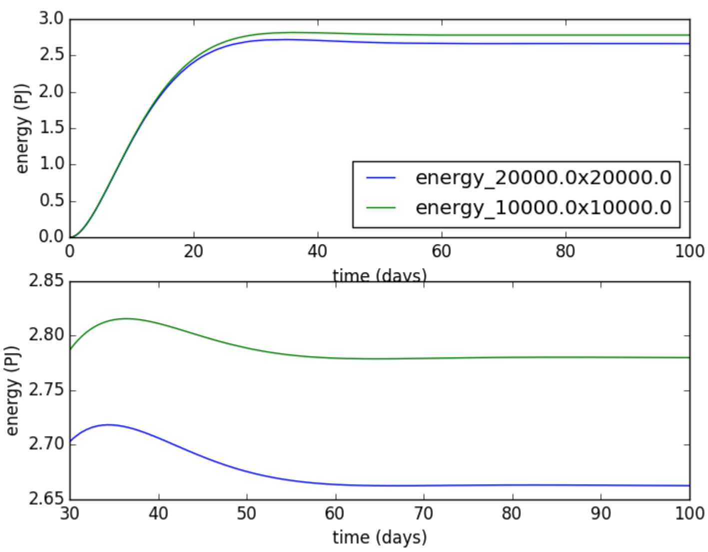

Figure 3: The effect of spatial resolution on the total energy of the system. Doubling the spatial resolution (blue curve) increases the overall energy of the system. It also reduces the energy difference between the analytical steady state solution and the steady state obtained by running the SWE model for 100 days.