Lecture 14 – Ruth-Hurwitz Criteria

Concepts:

- Determinant, Traces & Routh-Hurwitz criteria
- Touch on Complexity vs. Stability debate

Back to Biology: Classifying steady states

For 2x2 system:

$$\lambda^2 - \underbrace{(A_{11} + A_{22})}_{\text{Trace}} \lambda + \underbrace{A_{11}A_{22} - A_{12} + A_{21}}_{\text{Determinant}}$$

Routh-Hurwitz stabiliy criteria

Provide *biological insight* (not possible using just λ's)

Unfortunately the following applies only to 2 x 2 systems.

$\text{Tr}(\mathbf{A}) < 0 \quad \Rightarrow \quad A_{11} + A_{22} < 0$ is necessary for stability

$\Rightarrow$ *At least some species must be strongly self-limiting for stability* $\Leftarrow$

$\text{Det}(\mathbf{A}) > 0 \quad \Rightarrow \quad A_{11}A_{22} - A_{12}A_{21} > 0$ is necessary for stability

$\Rightarrow$ *Overall self-limitation must be stronger then interspecific effects for stability* $\Leftarrow$

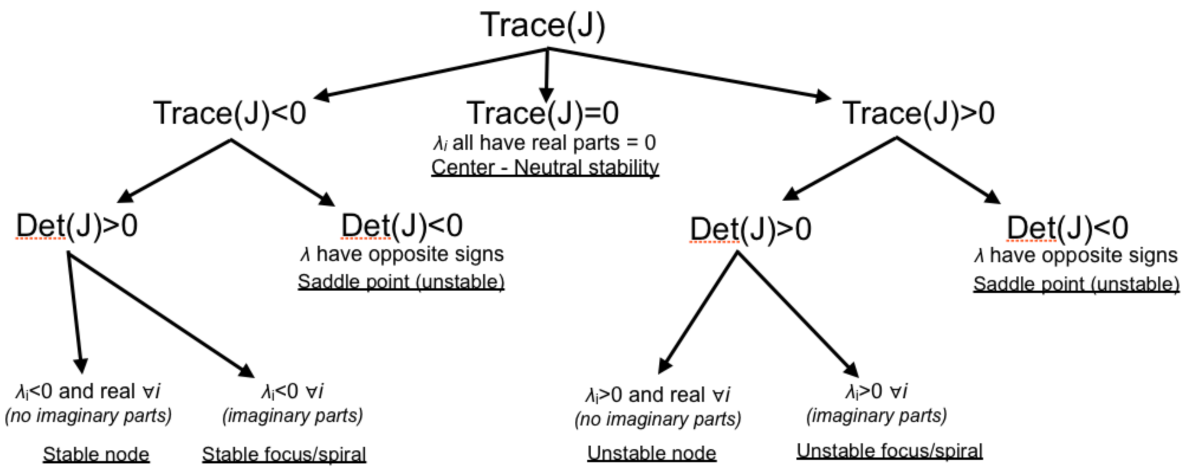
$\Rightarrow$ *Intra > inter-specific effects* $\Leftarrow$

Each condition by itself is *necessary, but not sufficient.*

$$\lambda = \tfrac{1}{2}\text{Tr}(\mathbf{A}) \pm \tfrac{1}{2}\sqrt{(-\text{Tr}(\mathbf{A}))^2 - 4 \cdot \text{Det}(\mathbf{A})}\, i$$

Whichever parts of $\sqrt{\ }$ is bigger determines with or without oscillations.

Bifurcation occurs at $(-\text{Tr}(\mathbf{A}))^2 = 4 \cdot \text{Det}(\mathbf{A})$.



MacArthur-Rosenzweig paradox of enrichment model - revisited

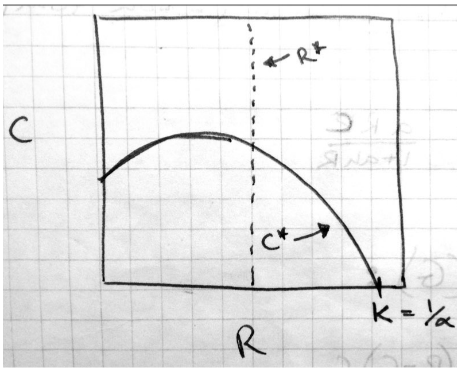
$$\frac{dR}{dt} = rR\left(1 - \frac{R}{K}\right) - \frac{aRC}{1+ahR} \qquad \frac{dC}{dt} = \frac{eaRC}{1+ahR} - dC$$

Prey isocline:

$$\frac{dR}{dt} = 0 \quad \Rightarrow \quad C^* = \frac{r(K-R)(1-ahR)}{aK}$$

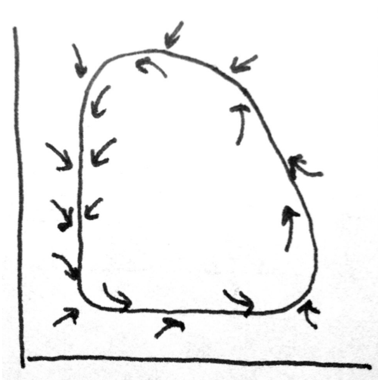
Predator isocline:

$$\frac{dC}{dt} = 0 \quad \Rightarrow \quad R^* = \frac{d}{a(e-dh)}$$



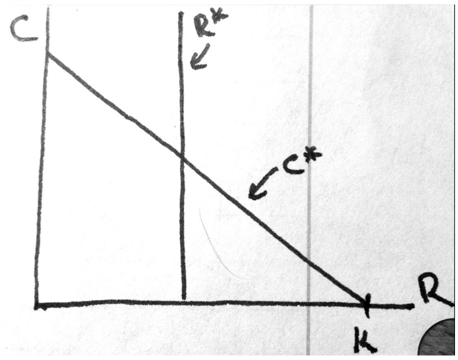
When $R* > K \Rightarrow$ predator extinct
Increasing K shifts C^* to right

When $R^* > \max C^* \Rightarrow$ stable fixed point
When $R^* < \max C^* \Rightarrow$ stable limit cycle
= *Hopf bifurcation*



Key control parameters:
As K increases, oscillation amplitude increases $\Rightarrow$ Risk of extinction
h influences position of R^* and how hump-shaped C^* is.

Note: If $h = 0 \Rightarrow C^* = \frac{r(K-R)}{aK}$, just like LV but with logistic prey:



Other parameters are less interesting for today.

Formal stability analysis

Step 1: Solve for steady state equilibria: *Mathematica*

Three solutions:

$$(R^*, C^*) = \begin{cases} 0 & 0 \\ K & 0 \\ \frac{d}{a(e-dh)} & \frac{er(aeK-d-adhK)}{Ka^2(e-dh)^2} \end{cases}$$

Step 2: Evaluate Jacobian at steady state(s)

Interested only in coexistence, so focus on 3rd

$$\mathbf{A}|_{R^*,C^*} = \begin{bmatrix} r - \frac{2rR^*}{K} - \frac{aC^*}{(1+ahR^*)^2} & \frac{-aR^*}{1+ahR^*} \\ \frac{eaC^*}{(1+ahR^*)^2} & \frac{eaR^*}{1+ahR} - d \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-dr(e+dh+ahK(dh-e))}{eaK(e-dh)} & \frac{-d}{e} \\ r\left(e - dh - \frac{d}{aK}\right) & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Note that for this model:

A_{21} will always be positive (look at first matrix) - *prey effect on pred*

A_{12} will always be negative - *pred effect on prey*

A_{11} can be positive or negative depending on R^* and C^*

Q: Why is $A_{22} = 0$? A: Consumer has no self-limitation (look back at model!)

Step 3: Assess stability using eigenvalues or Routh-Hurwitz Criteria

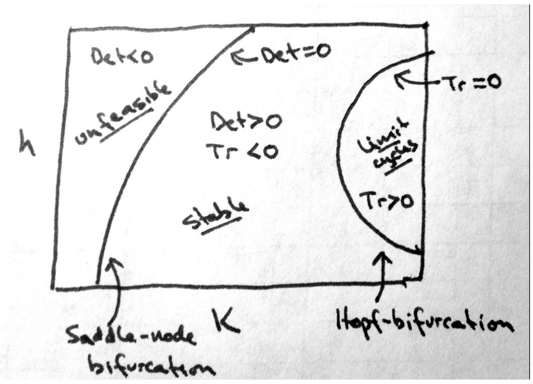
$\lambda_i < 0 \ \forall i \Rightarrow$ Stable fixed point

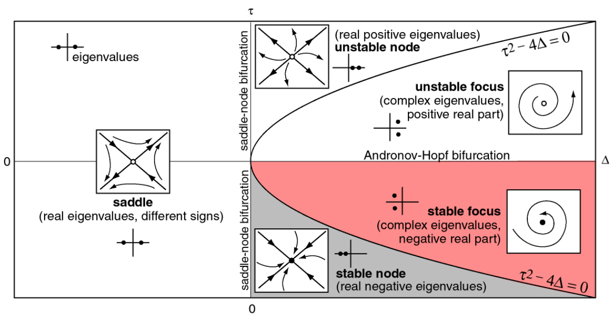
$Tr(\mathbf{A}) < 0$ & $Det(\mathbf{A}) > 0 \quad \Rightarrow \quad$ Stable fixed point

Graphical analysis:

Set $Tr(\mathbf{A})$ & $Det(\mathbf{A}) = 0$ and plot as functions of K and h:

Or plot $Tr(\mathbf{A}) = Det(\mathbf{A})$





Summary: Transitions from stability to local 'instability'

$$\lambda = \tfrac{1}{2}\mathrm{Tr}(\mathbf{A}) \pm \tfrac{1}{2}\sqrt{(-\mathrm{Tr}(\mathbf{A}))^2 - 4 \cdot \mathrm{Det}(\mathbf{A})}\ i$$

Zero real root:

When λ has only real part

i.e. $\mathrm{Tr}(\mathbf{A})^2 > 4 \cdot \mathrm{Det}(\mathbf{A})$

Transition at:

$\lambda = 0$

$Det(\mathbf{A}) = 0$

$Tr(\mathbf{A})$ can be $<$ or > 0

Result:

Qualitative change in type of steady-state.
Disappearance of fixed point equilibrium.
Appearance of new (boundary) equilibrium.

$\Rightarrow$ *Saddle-node bifurcation* $\Leftarrow$

Complex root with zero real part

When λ is complex

(i.e. $\mathrm{A}_1^2 < 4\mathrm{A}_2$)

Transition at:

$\lambda = 0 \pm i\sqrt{\mathrm{A}_2}$

$Det(\mathbf{A}) > 0$

$Tr(\mathbf{A}) = 0$

Result:

Transition from damped oscillations to sustained oscillations.
Equilibrium doesn't disappear.

$\Rightarrow$ *Hopf-bifurcation* $\Leftarrow$
