Variance of Population Size with Stochastic Geometric Growth

Expected population size at time t

Suppose we have a population growing geometrically with intrinsic growth factor at time $t \in T$ λ_t, where $\lambda_1, \lambda_2, ..., \lambda_t$ are i.i.d. random variables with mean $\bar{\lambda}$ and variance σ^2. The population size at time t is $N_0 \prod_{i=1}^{t} \lambda_i$. The expected population size at time t is therefore

$$\begin{aligned}
\mathbb{E}[N_t] &= \mathbb{E}\left[N_0 \prod_{i=1}^{t} \lambda_i\right] \\
&= N_0 \prod_{i=1}^{t} \mathbb{E}[\lambda_i] \\
&= N_0(\bar{\lambda})^t.
\end{aligned}$$

Because we are assuming the λ_i's are independent, the expectation can be moved through the product.

Variance of N_t

Using the definition $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$,

$$\begin{aligned}
\mathrm{Var}(N_t) &= \mathbb{E}\left[\left(N_0 \prod_{i=1}^{t} \lambda_i\right)^2\right] - \mathbb{E}\left[N_0 \prod_{i=1}^{t} \lambda_i\right]^2 \\
&= \mathbb{E}\left[N_0^2 \prod_{i=1}^{t} \lambda_i^2\right] - N_0^2(\bar{\lambda})^{2t} \\
&= \left(N_0^2 \prod_{i=1}^{t} \mathbb{E}[\lambda_i^2]\right) - N_0^2(\bar{\lambda})^{2t}.
\end{aligned}$$

Rearrange the definition of variance to see that $\mathbb{E}[X^2] = \mathrm{Var}(X) + \mathbb{E}[X]^2$, we can substitute $(\sigma^2 + \bar{\lambda}^2)$ for $\mathbb{E}[\lambda_i^2]$ above.

$$\begin{aligned}
\mathrm{Var}(N_t) &= \left(N_0^2 \prod_{i=1}^{t} \mathbb{E}[\lambda_i^2]\right) - N_0^2(\bar{\lambda})^{2t} \\
&= \left(N_0^2 \prod_{i=1}^{t} (\sigma^2 + (\bar{\lambda})^2)_i\right) - N_0^2(\bar{\lambda})^{2t} \\
&= N_0^2(\sigma^2 + (\bar{\lambda})^2)^t - N_0^2(\bar{\lambda})^{2t} \\
&= N_0^2\left((\sigma^2 + (\bar{\lambda})^2)^t - (\bar{\lambda})^{2t}\right).
\end{aligned}$$

I'm not one-hundred percent sure this is correct, but it at least makes some sense. Variance increases with the number of reproductive events and with the square of the starting population, but goes to zero as the process-level variation σ^2 goes to zero.