Lecture 13 – Eigenvalues

Concepts:

- Eigenvalues
- Matrix determinant
- Complex numbers

Matrix Algebra & Conventions

$\mathbf{A}$ - matrix (bold) $\qquad$ $\vec{w}$ - vector (italicized) $\qquad$ A_{ij} or w_i - scalars

$\begin{bmatrix} a \\ b \end{bmatrix}$ - column vector $\qquad$ $\begin{bmatrix} a & b \end{bmatrix}$ - row vector

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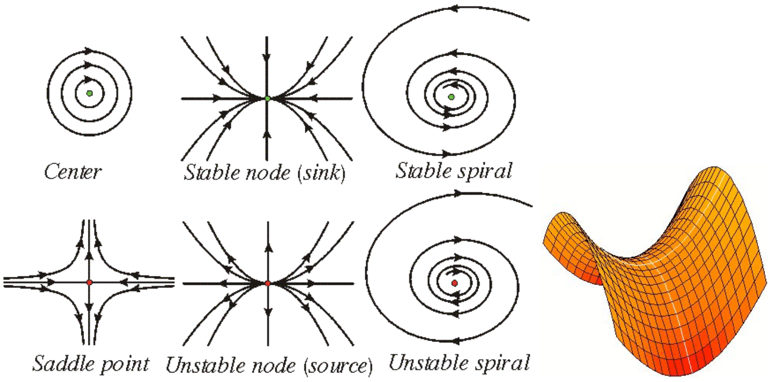
Why do eigenvalues indicate stability properties?

Eigenvalues (λ_i)	**Interpretation**
$Re(\lambda_i) < 0$ for all i	Stable node
$Re(\lambda_i) < 0$ for some i	Saddle node
$Re(\lambda_i) > 0$ for all i	Unstable node
$Re(\lambda_i) = 0$ for all i	Neutrally stable
$Im(\lambda_i) = 0$ for all i	No oscillations
$Im(\lambda_i) \neq 0$ for some i	Oscillations



Recall, for perturbations x and y to species u_1 and u_2:

$$\frac{dx}{dt} = f_1(u_1^* + x, u_2^* + y) \approx \underbrace{\left.\frac{\partial f_1}{\partial u_1}\right|_{u_1^*, u_2^*}}_{A_{11}} \cdot x + \underbrace{\left.\frac{\partial f_1}{\partial u_2}\right|_{u_1^*, u_2^*}}_{A_{12}} \cdot y$$

and

$$\frac{dy}{dt} = f_2(u_1^* + x, u_2^* + y) \approx \underbrace{\left.\frac{\partial f_2}{\partial u_1}\right|_{u_1^*, u_2^*}}_{A_{21}} \cdot x + \underbrace{\left.\frac{\partial f_2}{\partial u_2}\right|_{u_1^*, u_2^*}}_{A_{22}} \cdot y$$

Therefore, if x and y are small, can approximate dynamics as

$$\frac{dx}{dt} = A_{11} \cdot x + A_{12} \cdot y$$
$$\frac{dy}{dt} = A_{21} \cdot x + A_{22} \cdot y$$

In matrix form:

$$\underbrace{\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}}_{\frac{d\vec{n}}{dt}} = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{n}}$$

or equivalently

$$\frac{d\vec{n}}{dt} = \mathbf{A} \cdot \vec{n}$$

Side note:
Matrix addition:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \cdot e + b \cdot g & a \cdot f + b \cdot h \\ c \cdot e + d \cdot g & c \cdot f + d \cdot h \end{bmatrix}$$

In ***R*** use **A % * % B** for matrix multiplication (i.e. dot product).

The default is element-wise multiplication (Hadamard product).

Recall integration of single-spp exponential growth model:

$$\frac{dN}{dt} = rN \qquad \Rightarrow \qquad N(t) = \int_0^t \frac{dN}{dt} dt = N(0)\, e^{rt}$$

$r < 0 \Rightarrow$ decay.
$r > 0 \Rightarrow$ increase.

Represent dynamics of perturbation is same way.
For i^{th} perturbation:

$$\frac{dn_i}{dt} = A_{i\cdot} \cdot n_i \qquad \Rightarrow \qquad n_i(t) = n_i(0)\, e^{A_{i\cdot} t}$$

... or for both perturbations:

$$\vec{n}_t = \vec{n}_0\, e^{\mathbf{A}t}$$

But $e^{\mathbf{A}}$ is e^{matrix} for which usual definition of exponential function doesn't work!
How do we make it work?

What if we could replace matrix **A** with some constants – call it/them λ.
That would require that:

$$\boxed{\mathbf{A}\vec{w} = \lambda\vec{w}}$$

Vector $\vec{w}$ the **right eigenvector** of **A**.
$\vec{w}$ is a column vector.

Side-note (we won't actually use this fact)
The **left eigenvector** comes from:

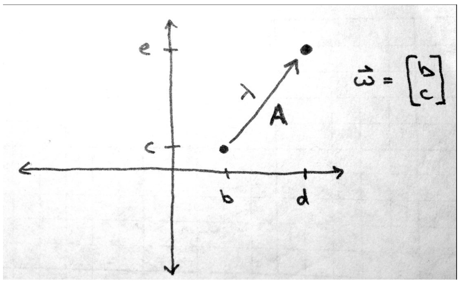
$$\vec{v}\mathbf{A} = \vec{v}\lambda$$

$\vec{v}$ is a row vector.

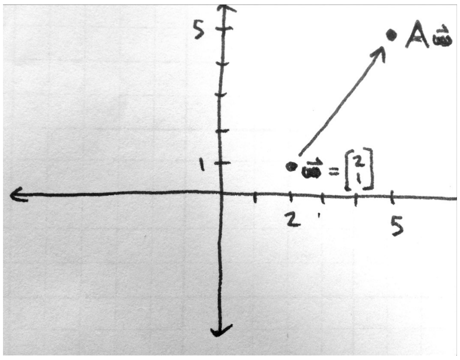
What is an eigenvalue?
Can think of any square matrix **A** and its associated eigenvalues λ as equivalent transformation operations,
...like a multiplication by a 'slope' or 'direction'.

For example, if we let $\mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} b \\ c \end{bmatrix}$.
Then:

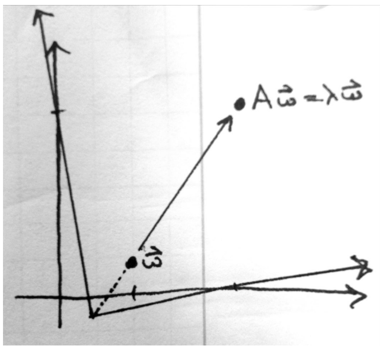
$$\mathbf{A}\vec{w} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \cdot \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} a_1 b + a_2 c \\ a_3 b + a_4 c \end{bmatrix} = \begin{bmatrix} d \\ e \end{bmatrix}$$
$$\lambda\vec{w} = \lambda \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} \lambda b \\ \lambda c \end{bmatrix} = \begin{bmatrix} d \\ e \end{bmatrix}$$



Numerical example:
Start at $\vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ $\Rightarrow$ $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+1 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$



We can do the same thing with λ by defining a new coordinate system:



How do we determine what λ is?
Need to solve:

$$\mathbf{A}\vec{w} = \lambda\vec{w} \qquad \Rightarrow \qquad \mathbf{A}\vec{w} - \lambda\vec{w} = 0$$

First we have to write λ in matrix form using **Identity matrix**:

$$\lambda\mathbf{I} = \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

Thus:

$$\mathbf{A}\vec{w} - \lambda\vec{w} = 0$$
$$(\mathbf{A} - \lambda\mathbf{I})\vec{w} = 0$$

Thus the solution is either

$$\vec{w} = 0 \qquad \text{or} \qquad (\mathbf{A} - \lambda\mathbf{I}) = 0$$

Not interested in $\vec{w} = 0$, but solving $(\mathbf{A} - \lambda\mathbf{I}) = 0$ isn't any easier than what we started with!
However,

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

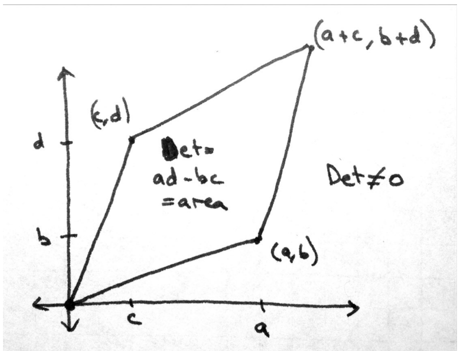
gives us a way to solve it.

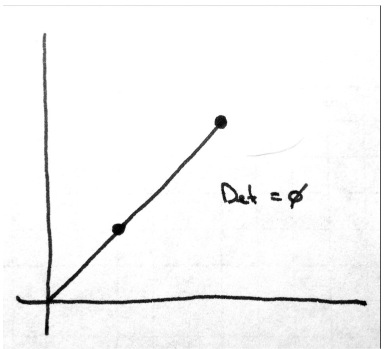
What is a matrix determinant?

$$det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

$$det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = aei + bfg + cdh - ceg - fha - ibd$$

Graphical interpretation of the determinant is as a *volume* (or *area* for 2x2 matrices):





$$det\left(\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 1 \cdot 1 = 5 \qquad\qquad det\left(\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\right) = 2 \cdot 2 - 4 \cdot 1 = 0$$

Back to solving for eigenvalues
If λ is just a scaling factor, then the volume of $\mathbf{A} - \lambda\mathbf{I}$ must equal 0.

$$\begin{aligned} 0 &= det(\mathbf{A} - \lambda\mathbf{I}) \\ &= det\left(\begin{bmatrix} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{bmatrix}\right) \\ &= (A_{11} - \lambda)(A_{22} - \lambda) - A_{12}A_{12} \\ &= \underbrace{\lambda^2 - (A_{11} + A_{22})\lambda + A_{11}A_{22} - A_{12}A_{21}}_{\text{Characteristic equation}} \end{aligned}$$

$\Rightarrow$ Solve for λ by solving the **characteristic equation**.

Solve for λ using quadratic formula:

$$ax^2 + bx + c = 0 \qquad \Rightarrow \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

By defining... (*yes, this is confusing, but it's how it's done in the literature*)

$$\lambda^2 - \underbrace{(A_{11} + A_{22})}_{\mathrm{A}_1}\lambda + \underbrace{A_{11}A_{22} - A_{12}A_{21}}_{\mathrm{A}_2}$$

That is: $\mathrm{A}_1 = A_{11} + A_{22}$ and $\mathrm{A}_2 = A_{11}A_{22} - A_{12}A_{21}$

...we have $a = 1$, $b = -\mathrm{A}_1$ and $c = \mathrm{A}_2$.

$$1 \cdot \lambda^2 + (-\mathrm{A}_1)\lambda + \mathrm{A}_2 = 0 \qquad \Rightarrow \qquad \begin{cases} \lambda_1 = \frac{\mathrm{A}_1 + \sqrt{(-\mathrm{A}_1)^2 - 4\mathrm{A}_2}}{2} \\ \lambda_2 = \frac{\mathrm{A}_1 - \sqrt{(-\mathrm{A}_1)^2 - 4\mathrm{A}_2}}{2} \end{cases}$$

The two solutions to λ are called a **complex conjugate pair**, or **complex conjugate roots** (=soltns.).

Compare to single-sp. model

For 1-sp. model we used the sign of the slope $\frac{d\,f(N)}{dN}$ to infer stability:

$$\frac{d\,f(N)}{dN} < 0 \quad \Rightarrow \quad \text{stable}$$

Now we're saying that it's the eigenvalues of $\mathbf{A}$ that matter, not the slopes themselves.
But remember that the eigenvalues reflect the partial derivative slopes $\frac{\partial\, f_i(\vec{N})}{\partial N_j}$.
They reflect the same transformation of the perturbation.

In fact, for a 1-sp. model they are exactly the same:

$$\mathbf{A}\vec{w} = \lambda\vec{w} \qquad \Leftrightarrow \qquad \frac{d\,f(N)}{dN} \cdot w = \lambda \cdot w \qquad \Rightarrow \qquad \frac{d\,f(N)}{dN} = \lambda$$

How do Complex numbers arise?

$$\lambda = \frac{\mathrm{A}_1 \pm \sqrt{(-\mathrm{A}_1)^2 - 4\mathrm{A}_2}}{2}$$

When $(-\mathrm{A}_1)^2 < 4\mathrm{A}_2 \quad \Rightarrow \quad \frac{1}{2}(\mathrm{A}_1 \pm \sqrt{\text{negative}})$
But how do we take the $\sqrt{\ }$ of a negative number?
$\Rightarrow$ Imaginary unit i.

$$\boxed{\text{Define } i^2 = -1}$$

$$\begin{array}{rcccccccc} \sqrt{-\#} & = & \sqrt{-1 \cdot \#} & = & \sqrt{i^2\#} & = & \sqrt{i^2}\sqrt{\#} = & i\sqrt{\#} \\ \text{e.g.,} \ \sqrt{-4} & = & \sqrt{-1 \cdot 4} & = & \sqrt{i^2 4} & = & i\sqrt{4} & = & i2 \end{array}$$

$$\lambda = \tfrac{1}{2}\mathrm{A}_1 \pm \tfrac{1}{2}\sqrt{(-\mathrm{A}_1)^2 - 4\mathrm{A}_2}\; i$$

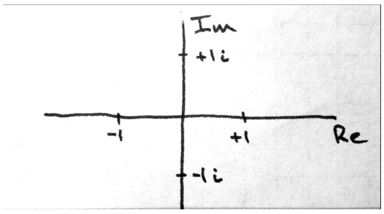
There is nothing 'imaginary' about imaginary numbers! They're as 'real' as negative numbers!
Real numbers are just counts or fractional counts
(= easy to think about as an amount of something).
Can you imagine how much an amount of -10 is? No!
Negative counts don't exist. They're a mathematical convenience to subtract amounts.
Imaginary numbers exist as much as negative numbers do!
They're a mathematical convenience to allow taking roots of negative numbers.

In order to solve... we invented...

$$\left.\begin{array}{lll} x-5=0 & \Rightarrow & \text{Integers} \\ x+5=0 & \Rightarrow & \text{Negative integers} \end{array}\right\} \text{(Real numbers)}$$

$$\begin{array}{lll} 2x=1 & \Rightarrow & \text{Rational numbers (fractions, quotients of integers)} \\ x^2=2 & \Rightarrow & \text{Irrational numbers (can't be expressed as fractions, e.g., } \sqrt{2}, \pi, e) \\ x^2=-1 & \Rightarrow & \text{Imaginary numbers.} \end{array}$$

Think about imaginary numbers as representing a different number dimension:



Why do imaginary parts indicate oscillations?
Remember that λ represents dynamics of the perturbation:

$$\vec{n}_t = \vec{n}_0 e^{\lambda t}$$

With a complex part:

$$\vec{n}_t = \vec{n}_0 e^{(a \pm bi)t} = e^{at} \cdot e^{ibt}$$

From the theory of complex numbers:

$$e^{i\Theta} = \cos\Theta + i\sin\Theta$$

Thinking of bt as Θ gives:

$$e^{ibt} = \cos(bt) + i\sin(bt)$$

Thus:

$$e^{(a \pm bi)t} = e^{at} \cdot [\cos(bt) \pm i\sin(bt)]$$

Therefore, a controls stability

$$\begin{array}{ll} a < 0 \Rightarrow & \text{oscillations will dampen} \\ a > 0 \Rightarrow & \text{oscillations will amplify} \\ a = 0 \Rightarrow & \text{neutral stability with cycles} \end{array}$$

And b controls the *frequency* of the oscillations.

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n-spp. stability
Note that we can determine the eigenvalues of an arbitrary-sized square matrix (not just 2-by-2).
We can therefore use eigenvalues to evaluate the local stability of any system of equations for any number of interacting species!
