

# Lecture 13 – Eigenvalues

## Concepts:

- n-spp. stability
- eigenvalues & complex numbers
- Matrix determinant

## Matrix Algebra & Conventions

**A** - matrix (bold)

$\vec{w}$  - vector (italicized)

$A_{ij}$  or  $w_i$  - scalars

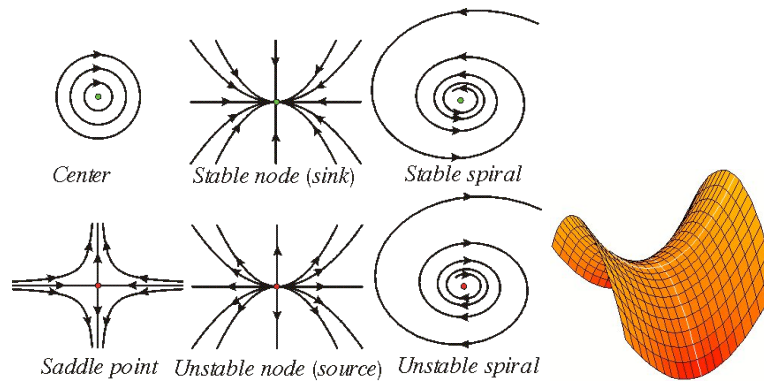
$\begin{bmatrix} a \\ b \end{bmatrix}$

- column vector

$\begin{bmatrix} a & b \end{bmatrix}$  - row vector

## Why do eigenvalues indicate stability properties?

Eigenvalues ( $\lambda_i$ )	Interpretation
$Re(\lambda_i) < 0$ for all $i$	Stable node
$Re(\lambda_i) < 0$ for some $i$	Saddle node
$Re(\lambda_i) > 0$ for all $i$	Unstable node
$Re(\lambda_i) = 0$ for all $i$	Neutrally stable
$Im(\lambda_i) = 0$ for all $i$	No oscillations
$Im(\lambda_i) \neq 0$ for some $i$	Oscillations



Recall, for perturbations  $x$  and  $y$  to species  $u_1$  and  $u_2$ :

$$\frac{dx}{dt} = f_1(u_1^* + x, u_2^* + y) \approx \underbrace{\frac{\partial f_1}{\partial u_1} \Big|_{u_1^*, u_2^*}}_{A_{11}} \cdot x + \underbrace{\frac{\partial f_1}{\partial u_2} \Big|_{u_1^*, u_2^*}}_{A_{12}} \cdot y$$

and

$$\frac{dy}{dt} = f_2(u_1^* + x, u_2^* + y) \approx \underbrace{\frac{\partial f_2}{\partial u_1} \Big|_{u_1^*, u_2^*}}_{A_{21}} \cdot x + \underbrace{\frac{\partial f_2}{\partial u_2} \Big|_{u_1^*, u_2^*}}_{A_{22}} \cdot y$$

Therefore, if  $x$  and  $y$  are small, can approximate dynamics as

$$\begin{aligned} \frac{dx}{dt} &= A_{11} \cdot x + A_{12} \cdot y \\ \frac{dy}{dt} &= A_{21} \cdot x + A_{22} \cdot y \end{aligned}$$

In matrix form:

$$\underbrace{\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}}_{\frac{d\vec{n}}{dt}} = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{n}}$$

or equivalently

$$\frac{d\vec{n}}{dt} = \mathbf{A} \cdot \vec{n}$$

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**Side note:**

Matrix addition:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \cdot e + b \cdot g & a \cdot f + b \cdot h \\ c \cdot e + d \cdot g & c \cdot f + d \cdot h \end{bmatrix}$$

In **R** use **A % \* % B** for matrix multiplication (i.e. dot product).

The default is element-wise multiplication (Hadamard product).

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Recall integration of single-spp exponential growth model:

$$\frac{dN}{dt} = rN \quad \Rightarrow \quad N(t) = \int_0^t \frac{dN}{dt} dt = N(0) e^{rt}$$

$r < 0 \Rightarrow$  decay.

$r > 0 \Rightarrow$  increase.

Represent dynamics of perturbation is same way.

For  $i^{th}$  perturbation:

$$\frac{dn_i}{dt} = A_{i \cdot} \cdot n_i \quad \Rightarrow \quad n_i(t) = n_i(0) e^{A_{i \cdot} t}$$

... or for both perturbations:

$$\vec{n}_t = \vec{n}_0 e^{\mathbf{A}t}$$

But  $e^{\mathbf{A}}$  is  $e^{matrix}$  which is not possible!

But what if we could replace matrix **A** with some constant – call it  $\lambda$ .

That would require that:

$$\boxed{\mathbf{A}\vec{w} = \lambda\vec{w}}$$

Vector  $\vec{w}$  the **right eigenvector** of **A**.

$\vec{w}$  is a column vector.

**Side-note** (we won't actually use this fact)

The **left eigenvector** comes from:

$$\vec{v}\mathbf{A} = \vec{v}\lambda$$

$\vec{v}$  is a row vector.

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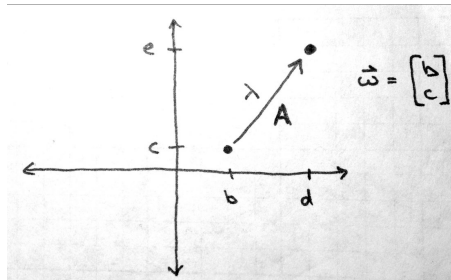
### What is an eigenvalue?

Can think of any square matrix  $A$  and its associated eigenvalues  $\lambda$  as equivalent transformation operations, ...like a multiplication by a 'slope' or 'direction'.

For example, if we let  $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} b \\ c \end{bmatrix}$ .

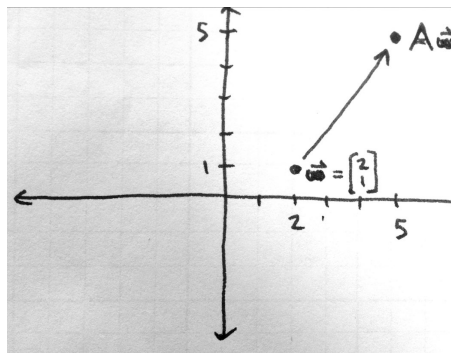
Then:

$$\begin{aligned} A\vec{w} &= \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \cdot \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} a_1b + a_2c \\ a_3b + a_4c \end{bmatrix} = \begin{bmatrix} d \\ e \end{bmatrix} \\ \lambda\vec{w} &= \lambda \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} \lambda b \\ \lambda c \end{bmatrix} = \begin{bmatrix} d \\ e \end{bmatrix} \end{aligned}$$

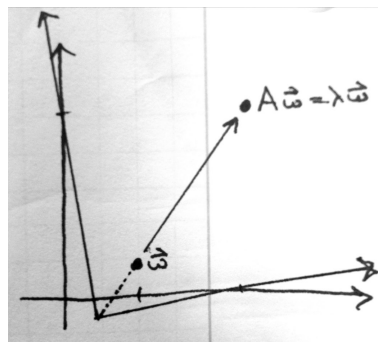


### Numerical example:

$$\text{Start at } \vec{w} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+1 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$



We can do the same thing with  $\lambda$  by defining a new coordinate system:



**How do we determine what  $\lambda$  is?**

Need to solve:

$$\mathbf{A}\vec{w} = \lambda\vec{w} \quad \Rightarrow \quad \mathbf{A}\vec{w} - \lambda\vec{w} = 0$$

First we have to write  $\lambda$  in matrix form using **Identity matrix**:

$$\lambda\mathbf{I} = \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

Thus:

$$\begin{aligned} \mathbf{A}\vec{w} - \lambda\vec{w} &= 0 \\ (\mathbf{A} - \lambda\mathbf{I})\vec{w} &= 0 \end{aligned}$$

Thus the solution is either

$$\vec{w} = 0 \quad \text{or} \quad (\mathbf{A} - \lambda\mathbf{I}) = 0$$

Not interested in  $\vec{w} = 0$ , but solving  $(\mathbf{A} - \lambda\mathbf{I}) = 0$  isn't any easier than what we started with!

However,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

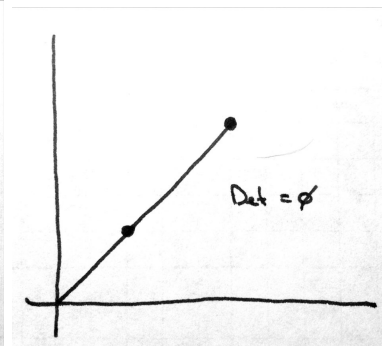
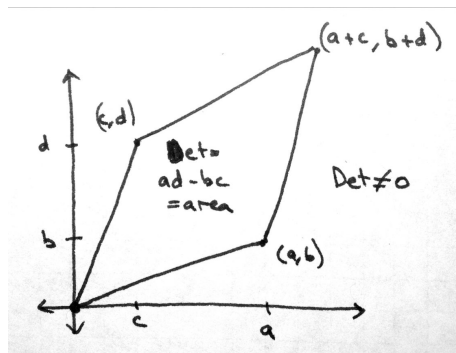
gives us a way to solve it.

**What is a matrix determinant?**

$$\det \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc$$

$$\det \left( \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = aei + bfg + cdh - ceg - fha - ibd$$

Graphical interpretation of the determinant is as a *volume* (or *area* for 2x2 matrices):



$$\det \left( \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right) = 2 \cdot 3 - 1 \cdot 1 = 5$$

$$\det \left( \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \right) = 2 \cdot 2 - 4 \cdot 1 = 0$$

**Back to solving for eigenvalues**

If  $\lambda$  is just a scaling factor, then the volume of  $\mathbf{A} - \lambda\mathbf{I}$  must equal 0.

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) \\ &= \det \left( \begin{bmatrix} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{bmatrix} \right) \\ &= (A_{11} - \lambda)(A_{22} - \lambda) - A_{12}A_{21} \\ &= \underbrace{\lambda^2 - (A_{11} + A_{22})\lambda + A_{11}A_{22} - A_{12}A_{21}}_{\text{Characteristic equation}} \end{aligned}$$

$\Rightarrow$  Solve for  $\lambda$  by solving the **characteristic equation**.

Solve for  $\lambda$  using quadratic formula:

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

By defining... (yes, this is confusing, but it's how it's done in the literature)

$$\lambda^2 - \underbrace{(A_{11} + A_{22})}_{A_1} \lambda + \underbrace{A_{11}A_{22} - A_{12}A_{21}}_{A_2}$$

That is:  $A_1 = A_{11} + A_{22}$  and  $A_2 = A_{11}A_{22} - A_{12}A_{21}$

...we have  $a = 1$ ,  $b = -A_1$  and  $c = A_2$ .

$$1 \cdot \lambda^2 + (-A_1)\lambda + A_2 = 0 \quad \Rightarrow \quad \begin{cases} \lambda_1 = \frac{A_1 + \sqrt{(-A_1)^2 - 4A_2}}{2} \\ \lambda_2 = \frac{A_1 - \sqrt{(-A_1)^2 - 4A_2}}{2} \end{cases}$$

The two solutions to  $\lambda$  are called a **complex conjugate pair**, or **complex conjugate roots** (=soltns.).

### Compare to single-sp. model

For 1-sp. model we used the sign of the slope  $\frac{df(N)}{dN}$  to infer stability:

$$\frac{df(N)}{dN} < 0 \quad \Rightarrow \quad \text{stable} \quad \text{vs.} \quad \frac{df(N)}{dN} > 0 \quad \Rightarrow \quad \text{unstable}$$

Now we're saying that it's the eigenvalues of  $\mathbf{A}$  that matter, not the slopes themselves.

But remember that the eigenvalues reflect the partial derivative slopes  $\frac{\partial f_i(\vec{N})}{\partial N_j}$ .

They reflect the same transformation of the perturbation.

In fact, for a 1-sp. model they are exactly the same:

$$\mathbf{A}\vec{w} = \lambda\vec{w} \quad \Leftrightarrow \quad \frac{df(N)}{dN} \cdot w = \lambda \cdot w \quad \Rightarrow \quad \frac{df(N)}{dN} = \lambda$$

### How do Complex numbers arise?

$$\lambda = \frac{A_1 \pm \sqrt{(-A_1)^2 - 4A_2}}{2}$$

When  $(-A_1)^2 < 4A_2 \Rightarrow \frac{1}{2}(A_1 \pm \sqrt{\text{negative}})$

But how do we take the  $\sqrt{\quad}$  of a negative number?

$\Rightarrow$  Imaginary unit  $i$ .

Define  $i^2 = -1$

$$\begin{aligned} \sqrt{-\#} &= \sqrt{-1 \cdot \#} = \sqrt{i^2 \#} = i\sqrt{\#} \\ \text{e.g., } \sqrt{-4} &= \sqrt{-1 \cdot 4} = \sqrt{i^2 4} = i\sqrt{4} = 2i \end{aligned}$$

$$\lambda = \frac{1}{2}A_1 \pm \frac{1}{2}\sqrt{(-A_1)^2 - 4A_2} i$$

There is nothing 'imaginary' about imaginary numbers! They're as 'real' as negative numbers!

*Real* numbers are just counts or fractional counts

(= easy to think about as an amount of something).

Can you imagine how much an amount of -10 is? No!

Negative counts don't exist. They're a mathematical convenience to subtract amounts.

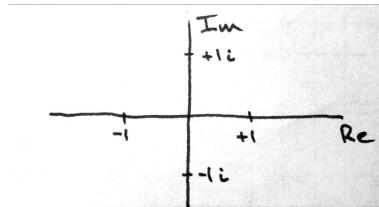
*Imaginary* numbers exist as much as negative numbers do!

They're a mathematical convenience to allow taking roots of negative numbers.

In order to solve... we invented...

$x - 5 = 0$	$\Rightarrow$	Integers (Real numbers)
$x + 5 = 0$	$\Rightarrow$	Negative integers (Real numbers)
$2x = 1$	$\Rightarrow$	Rational numbers (fractions, quotients of integers)
$x^2 = 2$	$\Rightarrow$	Irrational numbers (can't be expressed as fractions, e.g., $\sqrt{2}, \pi, e$ )
$x^2 = -1$	$\Rightarrow$	Imaginary numbers.

Think about imaginary numbers as representing a different number dimension:




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### Why do imaginary parts indicate oscillations?

Remember that  $\lambda$  represents dynamics of the perturbation:

$$\vec{n}_t = \vec{n}_0 e^{\lambda t}$$

With a complex part:

$$\vec{n}_t = \vec{n}_0 e^{(a \pm bi)t} = e^{at} \cdot e^{\pm ibt}$$

From the theory of complex numbers:

$$e^{i\Theta} = \cos \Theta + i \sin \Theta$$

$$e^{-i\Theta} = \cos \Theta - i \sin \Theta$$

Thinking of  $bt$  as  $\Theta$  gives:

$$e^{\pm ibt} = \cos(bt) \pm i \sin(bt)$$

Thus:

$$e^{(a \pm bi)t} = e^{at} \cdot [\cos(bt) \pm i \sin(bt)]$$

Therefore,  $a$  controls stability

- $a < 0 \Rightarrow$  oscillations will dampen
- $a > 0 \Rightarrow$  oscillations will amplify
- $a = 0 \Rightarrow$  neutral stability with cycles

And  $b$  controls the *frequency* of the oscillations.

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