Appendix 2 Some Matrix Operations

Matrices are boxes that hold numbers in rows and columns. Generally in this book, we use the boldface type to denote matrices and vectors.

Matrix Multiplication

Matrix multiplication can be illustrated with an example from age-structured population growth. In Chapter 3, you learn how to form Leslie matrices: $n_1(0)$ is the number of individuals in age class 1 at time zero, s_1 is the survival rate of age class 1 to age class 2, and F_1 is the fecundity of females in age class 1.

Let's plug in some actual numbers: $n_1(0) = 30$, $n_2(0) = 20$, $n_3(0) = 10$, $s_1 = 0.6$, $s_2 = 0.9$, $F_1 = 0$, $F_2 = 1$, and $F_3 = 2$. One time step into the future we have

$$\begin{aligned} n_1(1) &= n_1(0)\,(\mathrm{F}_1) + n_2(0)(F_2) + n_3(0)(F_3) \\ &= 30(0) + 20(1) + 10(2) + 40; \\ n_2(1) &= n_1(0)(s_1) \\ &= 30(0.6) = 18; \\ n_3(1) &= n_2(0)(s_2) \\ &= 20(0.9) = 18. \end{aligned}$$

This can all be put in a much more concise format by using a Leslie matrix equation:

$$\begin{bmatrix} n_1(t+1) \\ n_2(t+1) \\ n_3(t+1) \end{bmatrix} = \begin{bmatrix} F_1 & F_2 & F_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{bmatrix} \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix}.$$

Using boldface to designate matrices and vectors, we can write the last expression as

$$\mathbf{n}(\mathrm{t}+1) = \mathbf{L}\,\mathbf{n}(t).$$

Note that this Leslie matrix is a square matrix: the number of rows equals the number of columns. To use the matrix to multiply the population vector, successively take each row of the matrix and multiply that row by the population vector. This vector by vector multiplication yields a scalar; these scalars stack up in **n** to become the new population vector at $t + 1$:

$$\begin{aligned} F_1n_1(t) + F_2n_2(t) + F_3n_3(t) &= n_{1(t} + 1); \\ s_1n_1(t) + 0n_2(t) + 0n_3(t) &= n_2(t+1); \\ 0n_1(t) + s_2n_2(t) + 0n_3(t) &= n_3(t+1). \end{aligned}$$

So for our example we have for $t = 1$:

$$\begin{bmatrix} 0 & 1 & 2 \\ 0.6 & 0 & 0 \\ 0 & 0.9 & 0 \end{bmatrix} \begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix} \begin{matrix} = 0(30)+1(20)+2(10) = 40 \\ = 0.6(30)+0(20)+0(10) = 18 \\ = 0(30)+0.9(20)+0(10) = 18. \end{matrix}$$

The next time step, $t = 2$, yields

$$\begin{bmatrix} 0 & 1 & 2 \\ 0.6 & 0 & 0 \\ 0 & 0.9 & 0 \end{bmatrix}\begin{bmatrix} 40 \\ 18 \\ 18 \end{bmatrix} \begin{matrix} = 0(40)+1(18)+2(18) = 54 \\ = 0.6(40)+0(18)+0(18) = 24 \\ = 0(40)+0.9(18)+0(18) = 16.2. \end{matrix}$$

Remember this answer because you are now going to learn how to get it again by using a slightly different approach. The double iteration we just did to reach $\mathbf{n}(t+2)$ can be described by

$$\begin{aligned} \mathbf{n}(t+2) &= \mathbf{L}\,\mathbf{n}(t+1) \\ &= \mathbf{L}\,\mathbf{L}\,\mathbf{n}(t). \end{aligned}$$

You just learned how to multiply a matrix times a vector, but how do you multiply two matrices (e.g., $\mathbf{L}$ times $\mathbf{L}$)? Let's be more general and consider the product

$$\mathbf{C} = \mathbf{A}\,\mathbf{B}.$$

One rule of matrix–matrix multiplication is that, for the multiplication to even be defined, the number of columns of $\mathbf{A}$ must equal the number of rows in $\mathbf{B}$. So if $\mathbf{A}$ is of size $n \times m$ (i.e., n rows and m columns) and $\mathbf{B}$ is $m \times s$, then the product $\mathbf{A}\,\mathbf{B}$ is possible. However, If $\mathbf{B}$, were, say, of size $n \times m$, then the product $\mathbf{A}\,\mathbf{B}$ would not be defined. As a simple device we have the following matrix multiplication rule:

	$\mathbf{A}$	$\mathbf{B}$	$=$	$\mathbf{C}$
Dimensions:	$(n \times m)$	$(m \times s)$		
Rule: These two numbers must be the same.				
The dimension of the product matrix **C** is				$(n \times s)$

Before, we multiplied $\mathbf{L}$ (size 3×3) times $\mathbf{n}$ (size 3×1), so this is a compatible product. The result we found was a 3×1 vector. Note that, by this rule, in general the multiplication $\mathbf{A}\,\mathbf{B}$ may be permitted while the multiplication $\mathbf{B}\,\mathbf{A}$ may not be. Unlike the situation for scalar multiplication, the product of two matrices is different, depending on the order of multiplication. In other words $\mathbf{A}\,\mathbf{B} \neq \mathbf{B}\,\mathbf{A}$ except under restricted circumstances (e.g., $\mathbf{A} = 0$).

The multiplication of two matrices is performed similarly to that of a matrix times a vector, which you have already learned. The element c_{ij} of the matrix $\mathbf{C}$ is produced by multiplying the ith row of $\mathbf{A}$ times the jth column of $\mathbf{B}$. An example will illustrate.

Find $\mathbf{C} = \mathbf{A}\,\mathbf{B}$.

$$\begin{array}{cc} \mathbf{A} & \mathbf{B} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 2 \end{bmatrix} \\ (2\times 2) & (2 \times 3) \end{array}$$

Therefore $\mathbf{C}$ will be of size (2×3). Then

$$\begin{aligned} c_{11} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 2 \end{bmatrix} \\ &= (1)(1)+(2)(3) = 7, \\ c_{12} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 2 \end{bmatrix} \\ &= (1)(2)+(2)(1) = 4, \\ c_{13} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 2 \end{bmatrix} \\ &= (1)(2)+(2)(2) = 6, \end{aligned}$$

and so on, to reach

$$\mathbf{C}=\begin{bmatrix} 7 & 4 & 6 \\ 15 & 10 & 14 \end{bmatrix}.$$

Returning to our age-structure example, since $\mathbf{L}$ is a square matrix (size 3×3), multiplying $\mathbf{L}$ by itself is defined and this product is a matrix also with dimension 3×3. Thus

$$\mathbf{L}^2=\mathbf{L}\,\mathbf{L}=\begin{bmatrix} 0 & 1 & 2 \\ 0.6 & 0 & 0 \\ 0 & 0.9 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0.6 & 0 & 0 \\ 0 & 0.9 & 0 \end{bmatrix}=\begin{bmatrix} 0.6 & 1.8 & 0 \\ 0 & 0.6 & 1.2 \\ 0.54 & 0 & 0 \end{bmatrix}.$$

Now we multiply the matrix $\mathbf{L}^2$, which we just found, times $\mathbf{n}(t)$ to see if we get the same answer for $\mathbf{n}(t+2)$ that we got previously by doing the double iteration:

$$\mathbf{L}^2\mathbf{n}(t)=\mathbf{n}(t+2)$$

or

$$\begin{bmatrix} 0.6 & 1.8 & 0 \\ 0 & 0.6 & 1.2 \\ 0.54 & 0 & 0 \end{bmatrix}\begin{bmatrix} 30 \\ 20 \\ 10 \end{bmatrix}=\begin{bmatrix} 54 \\ 24 \\ 16.2 \end{bmatrix}$$

As expected, we do.

TRANSPOSE OF A MATRIX

The transpose operation is accomplished by simply flipping a matrix on its side—by exchanging rows and columns. That is, the first row becomes the first column; the second row becomes the second column, and so on. We usually use the symbol T as a superscript to denote a transpose. Here are two examples. If

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & 2 \\ 0.6 & 0 & 0 \\ 0 & 0.9 & 0 \end{bmatrix},\quad \text{then}\quad \mathbf{A}^T=\begin{bmatrix} 0 & 0.6 & 0 \\ 1 & 0 & 0.9 \\ 2 & 0 & 0 \end{bmatrix},$$

and if

$$\mathbf{B}=\begin{bmatrix} 0 & 8 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}\quad \text{then}\quad \mathbf{B}^T=\begin{bmatrix} 0 & 1 & 2 \\ 8 & 3 & 4 \end{bmatrix}.$$

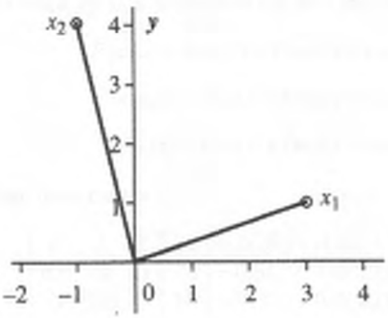
Note that the transpose is defined for matrices (or vectors) of any dimension, not just square ones).

DETERMINANT OF A MATRIX

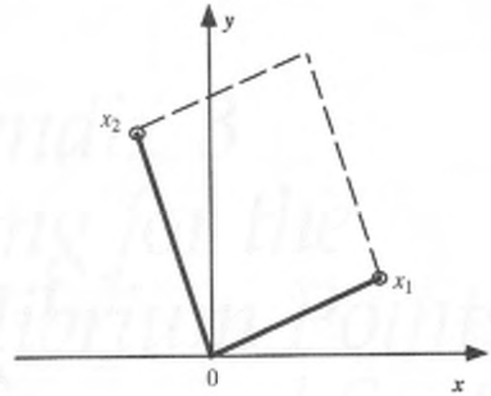
Think of a 2×2 matrix as two columns of numbers (i.e., two column vectors x_1 and x_2). Each vector expresses a point in two-dimensional space, as in

$$\mathbf{A}=[x_1\ x_2]=\begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}.$$

Think of this matrix as representing the two vectors in this picture.



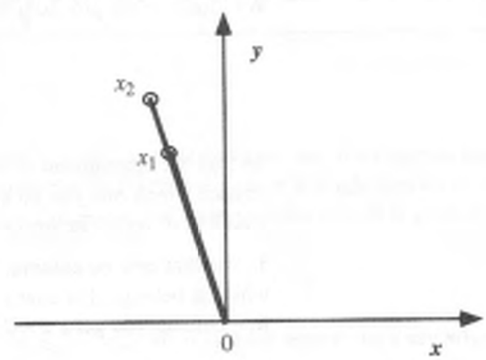
The determinant of a square matrix is a number (i.e., a scalar, not a matrix) whose absolute value is equal to the area of a parallelogram encompassed by these two vectors, as shown.



Taking the determinant of a square matrix for n = 2 Suppose that for **A** =

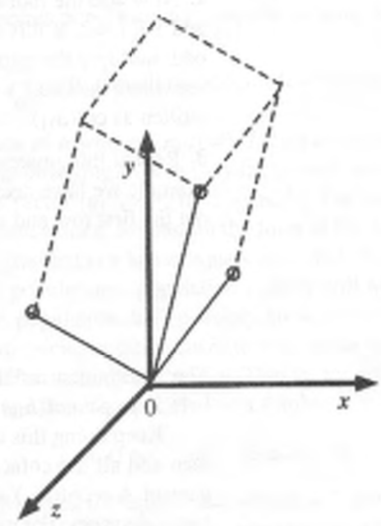
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The determinant equals the difference between the two cross products: $a_{11}a_{22} - a_{21}a_{12}$. If the two vectors were superimposed on each other, then the area would be zero, as depicted.

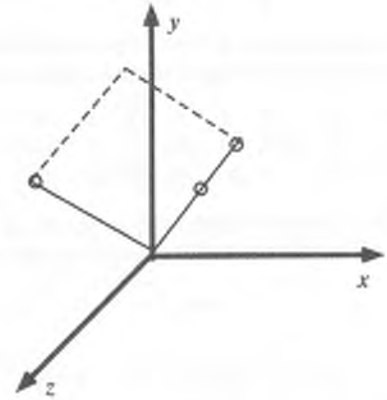


In this way a determinant tells you whether the vectors making up a matrix are all **linearly independent** or whether some duplicate others' directions.

When we extend this idea to three dimensions, the picture for a matrix with a nonzero determinant (the determinant is the volume of the dashed three-dimensional box) looks like this.



If two of the vectors were linearly dependent, then the picture would look like a flattened cardboard box in three-dimensional space.



This box is only two-dimensional, so the determinant is zero; in this case we we say that this matrix has **rank** 2. If all three vectors were linearly dependent, then we'd only have a line in three-dimensional space; and we'd say that the matrix has rank 1. But, if the determinant of the original matrix were nonzero, as in the diagram showing the three dimensional box, it would have rank 3.

The determinant of a matrix with a single element is simply that element; that is, the determinant of 2 is 2. Calculating the determinant of a square matrix for $n = 3$ or higher can be tedious. We illustrate the process for a 3×3 matrix. Suppose that

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

To find the determinant of matrix **A** choose any single row or column. It doesn't make any difference which one you pick, except that calculations will be easier if you pick a row or column with lots of zeros. In this case, there are no zeros, so let's just choose row 1.

1. For that row or column, begin with the first element and strike out the row and column to which it belongs. For row 1 you would begin with a_{11} (= 1). After striking out the first row and first column you get a 2×2 matrix:

$$\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}.$$

Then take the determinant of the resulting 2×2 matrix: $(5)(9) - (6)(8) = -3$. The determinant of this submatrix is called the **minor** of a_{11}, written as **minor**(a_{11}).

2. Now add the indexes (i.e., subscripts) of the original element chosen—in this case a_{11}—and add $1 + 1 = 2$. If this sum is an even number, as it is here, multiply the minor by $(+1)(a_{ij})$; if it is odd, multiply the minor by $(-1)(a_{ij})$. For this example you have in total $(1)(1)(-3) = -3$. (Do you see where the two 1's come from?) The result of this multiplication gives the **cofactor** of a_{11} often written as cof(a_{11}).

3. Repeat this process for the next element in the chosen row or column of matrix **A.** In this example we have decided to "expand" along the first row, so the next element is $a_{12} = 2$. Strike out the first row and second column to get

$$\begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & \cancel{5} & 6 \\ 7 & \cancel{8} & 9 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}.$$

The determinant of this submatrix is $(4)(9) - (6)(7) = -6 = \text{minor}(a_{12})$. Multiply this value by $(-1)(2)$ to get $\text{cof}(a_{12}) = 12$.

Keep doing this until you've finished all the calculations for the chosen row or column and then add all the cofactors. This sum gives you the determinant. For this example, the determinant of $\mathbf{A} = \text{cof}(a_{11}) + \text{cof}(a_{12}) + \text{cof}(a_{13}) = (-3) + (12) + (-9) = 0$. The process is the same for larger matrices. For example, for a 4×4 matrix, you would need to take the determinant of four matrices, each of size 3×3.

Appendix 3 Solving for the Equilibrium Points of Dynamical Systems and Finding the Inverse of a Square Matrix

In ecological dynamical systems the state variables are often species densities or resource levels, but they can be subcategories like different ages of a single species or males and females. Let n_i be the density of species i. Suppose that its population growth is given by the differential equation

$$\frac{dn_i}{dt} = n_i f_i(\mathbf{n}, t), \tag{A.8}$$

where $\mathbf{n}$ is the vector $(n_1, n_2, \cdots n_m)$. We may have several (let's say m) species with the per capita growth of each described by as yet unspecified equations called the f functions. (Remember: there's nothing sacred about choosing the symbol f for this; you could use g or σ, if you like. The Greek letters look more impressive, but they're harder to find on the keyboard.)

These m different per capita growth equations given by the f_i functions can potentially be complicated functions involving nonlinear terms; not all f_i's might have terms involving every species, but in general they could.

At equilibrium each species population has stopped growing by definition. Thus

$$\frac{dn_i}{dt} = 0 \quad \text{for all the species } (i = 1 \text{ to } m).$$

Our goal is to find the values of $\mathbf{n}$ where $f_i(\mathbf{n}, t) = 0$ for all m species. At what densities will all species population sizes stop changing? There may be several; each is an equilibrium point and is denoted $\mathbf{n}^*$, meaning the vector $(n_1^*, n_2^*, n_3^*, \ldots, n_m^*)$. The asterisk indicates the particular values of $\mathbf{n}$ that are equilibrium values. Because of the form of Eq. (A.8), there are two basic ways that dn_i/dt may equal zero. The first is when n_i equals zero, and the second is when $f_i(\mathbf{n}, t)$ equals zero. For closed biological populations, population growth will always be zero when $n_i = 0$; if there are no individuals, the population can't possibly grow.

One way to find a multispecies equilibrium is to send some or all of the species to zero and let the others reach an equilibrium in their absence. That is, for a four-species system, we might set species 1 and 2 to zero and then for species 3 and 4 solve

$$f_3(\mathbf{n}, t) = 0 \quad \text{and} \quad f_4(\mathbf{n}, t) = 0.$$

These two equations can be solved for equilibrium values n_3^* and n_4^* in the absence of species 1 and 2. We took this approach in Chapter 14 on competition when we tried to find boundary solutions for three Lotka–Volterra competitors. Extending that case to four species, we get two equations:

$$0 = (K_3 - n_3^* - \alpha_{34}n_4^*] \quad \text{and} \quad 0 = (K_4 - n_4^* - \alpha_{43}n_3^*).$$

After some algebra, this pair of equations can be solved to yield

$$\mathbf{n}^* = \begin{bmatrix} 0 \\ 0 \\ \dfrac{(K_3 - a_{34}K_4)}{(1 - a_{34}a_{43})} \\ \dfrac{(K_4 - a_{43}K_3)}{(1 - a_{34}a_{43})} \end{bmatrix}$$

for the boundary solution corresponding to $n_1 = n_2 = 0$.

Many subsets of species can be set to "zero"; the possibilities are numerous and include the *trivial* equilibrium where all species have zero density. For example for $m = 3$ species, you have the following potential equilibriums to evaluate: (0, 0, 0), (n_1, 0 ,0), (0, n_2, 0), (0, 0, n_3), (n_1, n_2, 0), (n_1, 0, n_2), and (0, n_2, n_3). Sometimes, of course, when you set one species to zero, the others may not even be able to reach a positive equilibrium. If the prey is at zero in a one-predator, one-prey system, the only possible equilibrium for the predator is also zero.

The *interior* equilibrium point is that particular equilibrium point $\mathbf{n}^*$ that satisfies

$$f_i(\mathbf{n}, t) = 0 \quad \text{for } all\ i \tag{A.9}$$

Note the emphasis on the word *all*. You can solve for the interior equilibrium if you have m independent equations (one for each i) in m unknowns (the m values of n_i^*).

If the solution to Eq. (A.9) is strictly positive (i.e., every n_i^* is greater than zero), the interior equilibrium point is then said to be a *feasible interior equilibrium*. A separate question is whether that equilibrium point is *stable*. Stability takes very different logic and math to evaluate, as you learned in Chapters 5 and 12. Usually, use of the term *stable* is restricted to the meaning that the system returns to the equilibrium following small perturbations away from it, although other definitions are possible.

Sometimes the algebraic solution of Eq. (A.9) yields an equilibrium point with the biologically nonsensical result that one or more of the species is at a negative density! Since, in reality, negative densities cannot be reached in a community, we say that the interior point is *unfeasible*. In this case the community will not reach the unfeasible equilibrium; instead, populations grow until a boundary equilibrium is reached (assuming that one of the boundaries is stable). The confusing part of multispecies dynamics, however, is that it's not necessarily true that if, say, species 2 and 6 have negative n_i^*, the boundary equilibrium involving all species but 2 and 6 will be a feasible or stable boundary equilibrium. We show this in Chapter 14 for two competitors (see Figure 14.26). It's also not necessarily true that the boundary equilibrium involving the absence of just species 2 (or just species 6) is itself necessarily unfeasible. Basically an unfeasible interior equilibrium point is not very helpful for determining which sets of species will be able to stably coexist and where the trajectory will end up starting from some initial conditions.

When the f_i are linear equations (e.g., the Lotka–Volterra competition equations), it is, of course, easier to solve for the interior equilibrium point than when these equations are nonlinear. However, in general, things can still be complicated since, even with linear per-capita systems, a number of possible equilibrium points exist and more than one of them may be locally stable. If this occurs, there are *alternative domains of attraction* or *alternative stable points*. For such a system, you might end up at one equilibrium point for some initial starting densities and another equilibrium point for other initial conditions. You can even see this in Lotka–Volterra competition involving just two species when $(\alpha_{12})(\alpha_{21}) > 1$ (see Chapter 14).

In the case of nonlinear f's, as in Chapter 12 on predator–prey dynamics, the equilibrium point can be feasible but unstable, and instead the equilibrium behavior of the system is a fixed cycle (i.e., not a point). In other words, the species densities keep oscillating up and down. You can even get limit cycles as the solution of these equations when the f's are linear if there are many interacting species. You will see this in Chapter 14 for three Lotka–Volterra competitors. It's still an active research problem in mathematics to determine these cyclic solutions when the number of interacting state variables (species) is large and the equations are nonlinear. Often researchers have to resort to simulation on a computer. Finally, the asymptotic behavior of some nonlinear systems may be neither a point or a cycle but rather a strange attractor of varying size and dimension; this is what we called chaos in Chapter 5.

Consider the more manageable case where the f's are linear. Now you can use some matrix tricks. Imagine a simple linear case of m Lotka–Volterra competitors and use linear algebra to solve for the *interior* equilibrium point. In this case after setting all the growth equations (i.e., the f_i) to zero, you're left with the linear matrix equation

$$\mathbf{K} - \mathbf{A}\,\mathbf{n}^* = \mathbf{0},$$

where $\mathbf{K}$ is the m by 1 vector of carrying capacities, $\mathbf{A}$ is the α matrix, $\mathbf{n}^*$ is the m by 1 vector of equilibrium densities and $\mathbf{0}$ is an m by 1 vector of zeros. (Remember, $\alpha_{ii} = 1$.) Rearranging yields

$$\mathbf{K} = \mathbf{A}\,\mathbf{n}^*. \tag{A.10}$$

To solve for $\mathbf{n}^*$ you now have two choices.

1. Apply **Cramer's rule** to Eq. (A.10). Cramer's rule works like this: To solve for the equilibrium level of species 1, n_1^*, take the ratio of the determinants of two different matrices: in the numerator form the matrix $\mathbf{A}$ (1), which is simply the matrix $\mathbf{A}$ with the first column replaced by the column of K's. In the denominator place the matrix of $\mathbf{A}$. Hence for a three-species system,

$$\mathbf{n}_1^* = \frac{|\mathbf{A}(1)|}{|\mathbf{A}|} = \frac{\begin{vmatrix} K_1 & a_{12} & a_{13} \\ K_2 & a_{22} & a_{23} \\ K_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

Similarly, the equilibrium density of species 2 is $n_2^* = \det(\mathbf{A}(2))/\det(\mathbf{A})$, where now, the matrix $\mathbf{A}(2)$ is formed by replacing the second column of matrix $\mathbf{A}$ with the $\mathbf{K}$ vector. In this manner, all n_i^* can be found.

2. Rearrange Eq. (A.10) to get $\mathbf{n}^* = \mathbf{A}^{-1}\,\mathbf{K}$. This involves finding the matrix $\mathbf{A}^{-1}$ called the *inverse of* $\mathbf{A}$. It's not at all like scalar arithmetic in the sense that the inverse of a matrix is not simply the inverse of each element in it. The concept, however, is similar: We want to find a matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is the **identity matrix** . This matrix has 1's down the diagonal and 0's everywhere else. Thus

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdot\cdot & 0 \\ 0 & 1 & 0 & \cdot\cdot & 0 \\ 0 & 0 & 1 & \cdot\cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdot\cdot & 1 \end{bmatrix}$$

How to Find the Inverse of a Matrix, A^{-1}

Only square matrices have inverses. There are basically four steps in the calculation of an inverse matrix for a 2×2 matrix.

1. Form a new matrix $\mathbf{B}$, where each element b_{ij} is found by taking the determinant of the $\mathbf{A}$ matrix that results from striking out the ith row and jth column. The determinants of these submatrices are the same minors that we developed earlier. Here is an example.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$ Striking out the first row and first column yields the single element 4; the determinant of a matrix with the single element 4 is simply 4; put 4 in the (1, 1) position. $$\longrightarrow \begin{bmatrix} 4 & \\ & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$ Striking out the first row and second column yields the single element 3; the determinant of 3 is 3; put 3 in the (1, 2) position. $$\longrightarrow \begin{bmatrix} 4 & 3 \\ & \end{bmatrix}$$

$$\begin{bmatrix} \cancel{1} & 2 \\ \cancel{3} & \cancel{4} \end{bmatrix}$$ Striking out the second row and first column yields the single element 2; the determinant of 2 is 2; put 2 in the (2, 1) position. $\longrightarrow \begin{bmatrix} 4 & 3 \\ 2 & \end{bmatrix}$

$$\begin{bmatrix} 1 & \cancel{2} \\ \cancel{3} & \cancel{4} \end{bmatrix}$$ Striking out the second row and second column yields the single element 1; put 1 in the (2, 2) position. $\longrightarrow \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

Thus $\mathbf{B} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$.

2. Change the signs of the elements of **B** in the following manner: for each minor associated with the a_{ij} element of **A**, if $i + j$ is an even number, then multiply the corresponding b_{ij} element of **B** by +1; if the sign of $i + j$ is an odd number, then multiply b_{ij} by −1. Matrix **B** becomes

$$\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}.$$

3. Next divide each element of matrix **B** by the determinant of **A**. For example, in this example the determinant of **A** is $4 - 6 = -2$ so we get

$$\frac{\mathbf{B}}{\det(\mathbf{A})} = \begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix}. \tag{A.11}$$

4. Finally, take the **transpose** of the matrix in Eq. (A.11); this now is the inverse matrix $\mathbf{A}^{-1}$. Recall that the transpose operation involves flipping the matrix on its side so that the *i*th row now becomes the *i*th column. Thus

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}.$$

To verify that this answer is correct, multiply $\mathbf{A}\,\mathbf{A}^{-1}$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since the product $\mathbf{A}\,\mathbf{A}^{-1}$ equals the identity matrix, you know that you did the inverse operation correctly. In practice, taking the inverse of a large matrix can be a tedious process and it's usually easier to let computers do the calculations.

Exercise: What is the inverse of matrix **A**:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ -1 & 3 & 4 \end{bmatrix}.$$

Partial Solution: Expanding along the first row of **A** gives

$$\begin{bmatrix} \cancel{1} & \cancel{0} & \cancel{-2} \\ \cancel{0} & 2 & 3 \\ \cancel{-1} & 3 & 4 \end{bmatrix}$$ The determinant of this 2×2 submatrix is $(2)(4) - (3)(3) = 1$. Therefore put 1 in the (1, 1) position of **B**. $\longrightarrow \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix}$

$$\begin{bmatrix} \cancel{1} & \cancel{0} & \cancel{-2} \\ 0 & \cancel{2} & 3 \\ -1 & \cancel{3} & 4 \end{bmatrix}$$ The determinant of this 2×2 submatrix is $(0)(4) - (3)(-1) = 3$. Therefore put 3 in the (1, 2) position of **B**. $\longrightarrow \begin{bmatrix} 1 & 3 & \\ & & \\ & & \end{bmatrix}$

Finish this process and verify that you have found the correct inverse by testing to see whether $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$.

Some Useful Facts from Linear Algebra

- For two square matrices **A** and **B**, the inverse of their product is

$$(\mathbf{A}\,\mathbf{B})^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}.$$

Note the reversal of multiplication order.

- The transpose of **A B** is

$$(\mathbf{A}\,\mathbf{B})^{T} = \mathbf{B}^{T}\,\mathbf{A}^{T}.$$

Note the reversal of multiplication order.

- If **D** is a diagonal matrix with elements d_{ii}, the result of multiplying **D A** is to multiply each element in the *i*th *row* of **A** by the *i*th element of **D**, d_{ii}. For example, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{11} & a_{22} \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix},$$

then

$$\mathbf{DA} = \begin{bmatrix} d_{11}a_{11} & d_{11}a_{12} \\ d_{22}a_{11} & d_{22}a_{22} \end{bmatrix}.$$

The result of multiplying **A D** is to multiply each element in the *i*th *column* of **A** by the *i*th element of **D**, d_{ii}, or

$$\mathbf{A\,D} = \begin{bmatrix} d_{11}a_{11} & d_{22}a_{12} \\ d_{11}a_{11} & d_{22}a_{22} \end{bmatrix}.$$

- A matrix **A** may be written in block form as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix},$$

such that blocks $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are square. A theorem in linear algebra gives the determinant of **A** as

$$|\mathbf{A}| = |\mathbf{A}_{22}|\,|\mathbf{A}_{11} - \mathbf{A}_{12}\,\mathbf{A}_{22}^{-1}\,\mathbf{A}_{21}|.$$

When either $\mathbf{A}_{12}$ or $\mathbf{A}_{21}$ contain all zeros, then this simplifies further to

$$|\mathbf{A}| = |\mathbf{A}_{22}|\,|\mathbf{A}_{11}|.$$

For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 1/4 & 1 \end{bmatrix},$$

then

$$|\mathbf{A}| = 1(1 - 0.5) = 0.5.$$

- The inverse of a diagonal matrix **D** has elements $1/d_{ii}$ on the diagonal and zeros elsewhere. For example,

$$\mathbf{D}^{-1} = \begin{bmatrix} \dfrac{1}{d_{11}} & 0 \\ 0 & \dfrac{1}{d_{22}} \end{bmatrix}.$$

- The eigenvalues of a triangular matrix are simply the elements on the diagonal. For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix},$$

then the eigenvalues of **A** are 1, 1, and 4.

Appendix 4 Some Useful Mathematical Identities and Approximations

1. e is the base of the natural logarithms = 2.7138. . . .
2. $\ln 1 = 0$
3. As $a \to 0$, $\ln a \to -\infty$.
4. $\ln ab = \ln a + \ln b$
5. $\ln(a + b) \neq \ln a + \ln b$
6. $e^{z \ln a} = a^z$
7. $\ln(1 + x) \approx x \quad \text{when } |x| << 1$
8. $e^x \approx 1 + x \quad \text{when } |x| << 1$
9. $\dfrac{a_1}{a_2} - \dfrac{a_3}{a_4} = \dfrac{a_1 a_4 - a_2 a_3}{a_2 a_4}$
10. The two roots of the quadratic equation

$$ax^2 + bx + c = 0$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

11. $e^{iz} = \cos z + i \sin z \quad \text{where } i = \sqrt{-1}$
12. $e^{-iz} = \cos z - i \sin z$
13. $e^{i\pi} = -1$ (Think about it!)
14. The **absolute value** (also called the magnitude or modulus) of a **complex** number, $z = a + bi$, is

$$|z| = \sqrt{a^2 + b^2}$$

and

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad \text{for any numbers } z_1 \text{ and } z_2.$$

15. $1° = (\pi/180)$ radians = 0.01745 radians.
16. 1 radian = $(180/\pi)°$ = 57.296°.

SOME SERIES, SUMS, AND APPROXIMATIONS

1. $\dfrac{a}{(1-x)}=\sum_{i=0}^{\infty} ax^i$ when $-1<x<1$

2. $\dfrac{a(1-x^n)}{(1-x)}=\sum_{i=0}^{n} ax^i$ when $-1<x<1$

3. $\dfrac{1}{(1-x)^2}=\sum_{i=0}^{\infty} ix^{(i-1)}$ when $-1<x<1$

4. $\dfrac{1}{(1-x)^2}=\sum_{i=0}^{\infty} (i+1)x^i$ when $-1<x<1$

5. $e^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots=\sum_{i=0}^{\infty}\dfrac{x^i}{1!}$ for all real x,

   where $n! = n$ factorial $= n(n-1)(n-2)\cdots 1$. For example $3! = (3)(2)(1) = 6$.

6. $(x+1)^n=\sum_{i=0}^{n}\binom{n}{i}x^i$ when $-1<x<1$,

   where $\binom{n}{i}=\dfrac{n!}{i!(n-i)!}$,

   e.g., $(0.5+1)^3=1+(3)(0.5)+(3)(0.5)^2+(1)(0.5)^3=3.375$.

7. $\dfrac{1}{1+x}=\sum_{i=0}^{\infty}(-1)^i x^i$ for $-1<x<1$

8. $\ln(1+x)=\sum_{i=1}^{\infty}(-1)^{i+1}\left(\dfrac{x^i}{i}\right)$ for $-1<x\le 1$

9. $\sum_i\sum_j a_i b_j \neq \sum_i a_i \sum_j b_j$

10. $\sum_i\sum_j a_{ij} b_j = \sum_i\left(\left(\sum_i a_{ij}\right)b_j\right)$

11. Using 5: $e^x \cong 1+x$ when $|x| \ll 1$.

12. Using 8: $\ln(1-x)\approx -x$ when $|x|\ll 1$.

13. Taylor's series approximation for a continuous function $F(x)$ at point x^* is

$$F(x)\approx F(x^*)+\frac{F'(x^*)(x-x^*)}{1!}+\frac{F''(x^*)(x-x^*)^2}{2!}+\frac{F'''(x^*)(x-x^*)^3}{3!}+\ldots,$$

   where F' is the first derivative, F'' is the second derivative, and so on.

14. Taylor's series approximation for a continuous function $F(x, y)$ at point (x^*, y^*) is

$$F(x,y)\approx F(x^*,y^*)+\frac{1}{1!}\left[(x-x^*)\frac{\partial F(x^*,y^*)}{\partial x}+(y-y^*)\frac{\partial F(x^*,y^*)}{\partial y}\right]+$$

$$\frac{1}{2!}\left[\left((x-x^*)\frac{\partial}{\partial x}+(y-y^*)\frac{\partial}{\partial y}\right)^2 F(x^*,y^*)\right]+\ldots.$$

Appendix 5 Calculus

Derivatives

Function	Derivative
1. $y = \text{constant}$	$\frac{dy}{dx} = 0$
2. $y = x^n$	$\frac{dy}{dx} = nx^{n-1}$
3. $y = \sin x$	$\frac{dy}{dx} = \cos x$
4. $y = \cos x$	$\frac{dy}{dx} = -\sin x$
5. $y = \ln x$	$\frac{dy}{dx} = \frac{1}{x}$
6. $y = e^x$	$\frac{dy}{dx} = e^x$

Reduction formulas for functions of x: in the following, y, u, and v are arbitrary functions of x; f is a function of u; and a is a constant.

Function	Derivative
7. au	$a\left(\frac{du}{dx}\right)$
8. y	$\frac{dy}{dx}$
9. $u + v$	$\frac{du}{dx} + \frac{dv}{dx}$
10. $\sum_i u_i$	$\sum_i \frac{du_i}{dx}$
11. uv	$u\frac{dv}{dx} + v\frac{du}{dx}$
12. $\frac{u}{v}$	$\frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$
13. $f(u(x))$	$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}.$

14. $y = y(x)$ $$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

Differential $$dy = \frac{dy}{dx}(dx).$$

Alternative notations for a derivative:

$$\frac{dy}{dx},\quad y',\quad f'(x),\quad \frac{df}{dx},\quad \frac{d}{dx}f(x),\quad D_x y$$

INTEGRATION

The indefinite integral is given without the constant of integration because the limits of integration are not specified. The inclusion of a multiplicative constant involves only a simple transformation (a and b are constants).

$$\int f(ax)dx = \frac{1}{a}\int f(x)dx$$

$$\int bf(x)dx = b\int f(x)dx$$

An integral that is the sum of several terms may be broken up, as in

$$\int [f_1(x) + f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx.$$

The following are general forms for some easily integrable functions.

1. $\int x^n dx = \frac{x^{n+1}}{n+1}$ (except for $n = -1$)
2. $\int \frac{1}{x} dx = \ln x$
3. $\int e^x dx = e^x$
4. $\int (\ln x)dx = x\ln x - x$
5. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
6. $\int \sin x\, dx = -\cos x$
7. $\int \cos x\, dx = \sin x$
8. $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin^2 x$
9. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$
10. $\int e^{-ax} dx = \frac{-1}{a}e^{-ax}$