Quiz 5 - Two-species Stability Analysis

My name is: ____________________________

1) Determine the elements of the Community Matrix $\mathbf{A}$ for the two-species model

$$\frac{dX}{dt} = (a - bY)X$$
$$\frac{dY}{dt} = (pbX - q)Y \, ,$$

remembering that, generally-speaking,

$$A_{ij} = \left. \frac{\partial \frac{dx_i}{dt}}{\partial x_j} \right|_{\vec{N}^*} .$$

We first expand and define

$$\frac{dX}{dt} = (a - bY)X = aX - bXY = f_X$$
$$\frac{dY}{dt} = (pbX - q)Y = pbXY - qY = f_Y \, .$$

We must then determine $\frac{\partial f_i}{\partial x_j}$ for all combinations of $i \in \{X, Y\}$ and $j \in \{X, Y\}$ and evaluate each at the equilibrium:

$$A_{11} = A_{XX} = \left. \frac{\partial f_X}{\partial X} \right|_{X^*, Y^*} = \left. \frac{\partial (aX - bXY)}{\partial X} \right|_{X^*, Y^*} = a - bY^*$$
$$A_{12} = A_{XY} = \left. \frac{\partial f_X}{\partial Y} \right|_{X^*, Y^*} = \left. \frac{\partial (aX - bXY)}{\partial Y} \right|_{X^*, Y^*} = 0 - bX^* = -bX^*$$
$$A_{21} = A_{YX} = \left. \frac{\partial f_Y}{\partial X} \right|_{X^*, Y^*} = \left. \frac{\partial (pbXY - qY)}{\partial X} \right|_{X^*, Y^*} = pbY^* - 0 = pbY^*$$
$$A_{22} = A_{YY} = \left. \frac{\partial f_Y}{\partial Y} \right|_{X^*, Y^*} = \left. \frac{\partial (pbXY - qY)}{\partial Y} \right|_{X^*, Y^*} = pbX^* - q \, .$$

These can be further simplified by substitution of the equilibrium population sizes obtained by solving for the isoclines. These are

$$\frac{dX}{dt} = aX - bXY = 0 \implies Y^* = \frac{a}{b}$$

for species X's isocline, and

$$\frac{dY}{dt} = pbXY - qY = 0 \implies X^* = \frac{q}{pb} \, .$$

Therefore we have that

$$\mathbf{A} = \begin{bmatrix} a - b\frac{a}{b} & -b\frac{q}{pb} \\ pb\frac{a}{b} & pb\frac{q}{pb} - q \end{bmatrix} = \begin{bmatrix} 0 & -\frac{q}{p} \\ ap & 0 \\ . & \end{bmatrix}$$

2) What is the biological interpretation of elements A_{12} and A_{21}?

By definition, $A_{ij} = \frac{\partial f_i}{\partial x_j} = \lim_{x \to 0} \frac{\Delta f_i}{\Delta x_j}$, thus, the ij^{th} element of $\mathbf{A}$ is to interpreted as the effect of (an infinitesimally small) change in the population size of species j on the population growth rate of species i. Infinitesimally small may be interpreted empirically as a single individual. Thus it is the per capita effect of j on the population growth rate of species i. A_{12} and A_{21} are thus the per capita effect of species 2 (Y) on the population growth rate of species 1 (X), and the per capita effect of species 1 (X) on the population growth rate of species 2 (Y), respectively.

3) How would you determine whether a given equilibrium point (not necessarily the coexistence equilibrium) will exhibit the following dynamics after a pulse perturbation?

a) locally stable,
$\lambda_i < 0 \; \forall \; i \in \{1, 2\}$

b) locally unstable (including attractor-repeller dynamics),
$\lambda_i > 0 \; \forall \; i \in \{1, 2\}$ (unstable)
$\lambda_1 < 0$ and $\lambda_2 > 0$ (saddle)
$\lambda_1 > 0$ and $\lambda_2 < 0$ (saddle)

c) or neutrally stable
$\lambda_i = 0 \; \forall \; i \in \{1, 2\}$