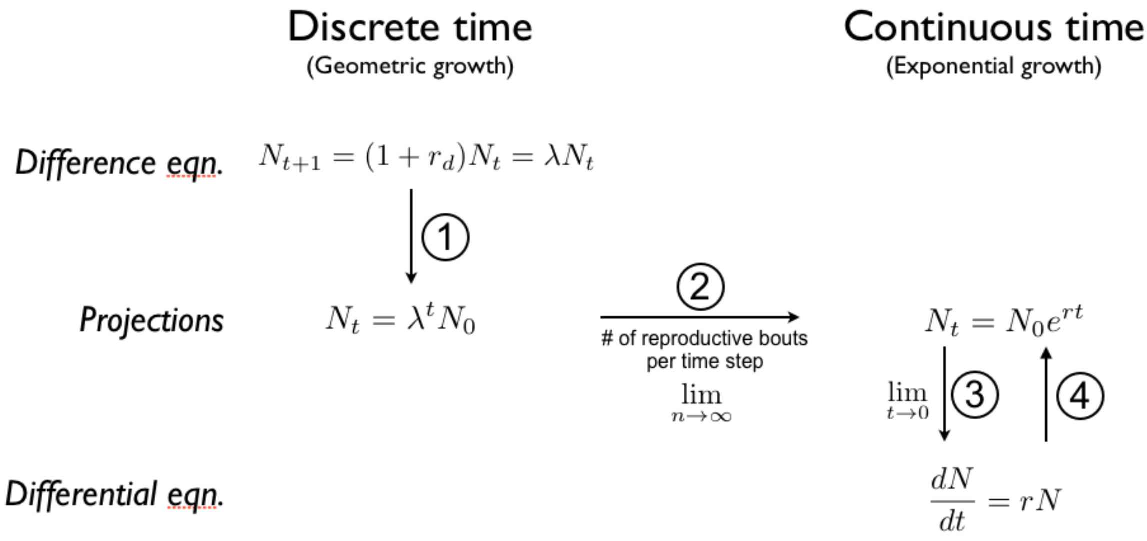
Lecture 2 - Summary

Purpose of last lecture was to show connections between model equations that most of you have probably seen before:



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1 e.g.,

$$N_{t+2} = \lambda N_{t+1} = \lambda(\underbrace{\lambda N_t}_{N_{t+1}}) = \lambda^2 N_t$$

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2 Demonstration of how Euler's number e emerges:

$$\boxed{\lim_{n\to\infty} \left(1 + \tfrac{1}{n}\right)^n = e^1 = e}$$

Let $r_d = 1$; $t = 1\ year$

1 event per time-step:

$$N_1 = \lambda N_0 = (1 + r_d)N_0$$

2 events per time-step:

$$N_1 = \lambda^2 N_0 = \left(1 + \tfrac{r_d}{2}\right)^2 N_0$$

n events per time-step:

$$N_1 = \left(1 + \tfrac{r_d}{n}\right)^n N_0$$

$$\lambda = \frac{N_1}{N_0} = \left(1 + \tfrac{r_d}{n}\right)^n$$

Limit as n goes to infinitity:

$$\lambda = \lim_{n\to\infty} \left(1 + \tfrac{r_d}{n}\right)^n = e^r$$

Therefore:

$$\lambda = (1 + r_d) = e^r$$

And we have:

$$N_t = N_0 \lambda^t = N_0 e^{rt}$$

Defining the natural-log as the 'anti-exponential' — $ln(e^x) = x$ — we also have:

$$ln(\lambda) = ln(1 + r_d) = r$$

All three reflect the *intrinsic per capita growth rate*:

$$\begin{aligned} r_d =& b_d - d_d && \text{discrete time} \\ r =& b - d && \text{continuous time} \end{aligned}$$

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(3) How to we get instantaneous population-level growth rate from projection equation, $N_0 e^{rt}$?
That is, how do we show that:

$$\lim_{\Delta t \to 0} \left(\frac{\Delta N_t}{\Delta t} \right) = \frac{dN}{dt}$$

Need to take the derivative of $N_0 e^{rt}$ with respect to time t.
Use Chain Rule:

$$\frac{d(XY)}{dt} = \frac{d(X)}{dt} \cdot Y + X \cdot \frac{d(Y)}{dt}$$

(The derivative of a product is the sum of the product of the derivative of each term times the other term.)
Thus:

$$\frac{d(N_0 \cdot e^{rt})}{dt} = \frac{d(N_0)}{dt} \cdot (e^r)^t + N_0 \cdot \frac{d((e^r)^t)}{dt}$$

Note:
Derivative of a constant = 0
Derivative of $a^x = ln(a) \cdot a^x$.
Thus:

$$\begin{aligned} \frac{d(N_0 e^{rt})}{dt} &= 0 \cdot (e^r)^t + ln(e^r) \cdot (e^r)^t \cdot N_0 \\ &= r \cdot (e^r)^t \cdot N_0 \\ &= r \cdot e^{rt} \cdot N_0 \\ &= r \cdot N_0 e^{rt} \end{aligned}$$

Since $N = N_0 e^{rt}$ for any time t...

$$= rN = \frac{dN}{dt}$$

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(4) Could also go in opposite direction from $\frac{dN}{dt} \to N_0 e^{rt}$:

$$\begin{aligned} &\frac{dN}{dt} = rN \\ &\frac{1}{N}\frac{dN}{dt} = r \\ &\int_0^T \frac{1}{N}\frac{dN}{dt}\, dt = \int_0^T r\, dt \quad \text{(Think of T as a constant, and t in dt as a variable)} \\ &\int_0^T \frac{1}{N}\frac{dN}{dt} = rt|_0^T = r \cdot T - r \cdot 0 \end{aligned}$$

Using $\int \frac{1}{x}\, dx = ln(x)$...

$$\begin{aligned} &ln(N(T)) - ln(N(0)) = rT \\ &ln\left(\frac{N(T)}{N(0)}\right) = rT \\ &\frac{N(T)}{N(0)} = e^{rT} \\ &N(T) = N(0)e^{rT} \end{aligned}$$
