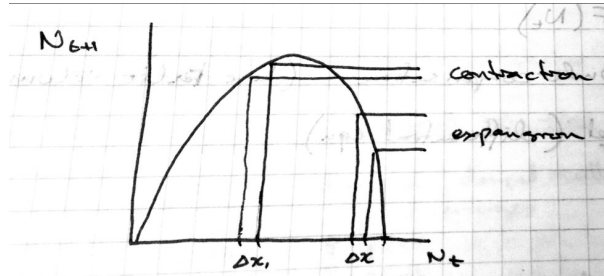# Lecture 9 − 1-D Stability Analysis - Part 2

Today: 1D stability cont., non-dimensionalization, and paper discussion
Next class: 2 sp. LV competition (Bring laptops – *might* need them depending on how far we get)
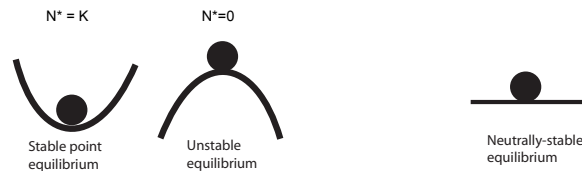

Ricker plots & Intuitive notion of Lyapunov exponent

$\rightarrow$ The larger r_d, the steeper the slopes, the more expansion.

Local stability analysis
Through simulation we found $r_d < 2$ for $N^* = K$ to be stable
(either monotonic dampening or damped oscillations).

Now let's determine this formally...
Discrete-time model:

$$N_{t+1} = F(N_t)$$

Step 1: Solve $F(N)$ for N^*:

$$F(N_t) = N_t + r_d N_t \left(1 - \frac{N_t}{K}\right)$$

By definition: $F(N^*) = N_t$

$$F(N^*) = N_t = N_t + r_d N_t - \frac{r_d N_t^2}{K}$$

Solve for N_t: $\quad 0 = r_d N_t - \frac{r_d N_t^2}{K}$

$$\frac{r_d N_t^2}{K} = r_d N_t$$

$$r_d N_t^2 = r_d N_t K$$

$$\rightarrow N^* = K \text{ and } N^* = 0$$
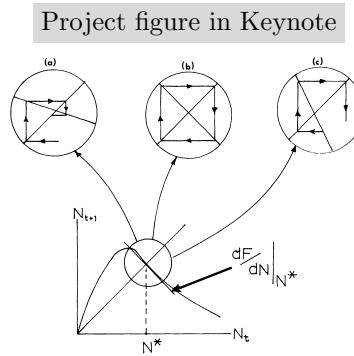
Step 2: Find slope of $F(N_t)$ evaluated at N^*:

$$F'(N^*) = \left. \frac{dF(N_t)}{dN} \right|_{N^*}$$

$$\boxed{\text{If } |F'(N^*)| < 1 \rightarrow \text{ stable.}}$$

For discrete logistic:

$$F'(N^*) = 1 + r_d - 2r_d \frac{N^*}{K}$$

$$= 1 + r_d - 2r_d \frac{K}{K}$$

$$= 1 - r_d$$

1

Thus, discrete logistic reaches stable point equilibrium when $0 < r_d < 2$.

Continuous-time model:

$$\frac{dN}{dt} = f(N) = rN\left(1 - \frac{N}{K}\right)$$

Step 1: Solve $f(N)$ for N^*:

By definition: At N^* when $f(N^*) = 0$

$$0 = rN - \frac{rN^2}{K}$$

$$\frac{rN^2}{K} = rN$$

$$rN^2 = rNK$$

$$\rightarrow N^* = K \text{ and } N^* = 0$$

Step 2: Find slope of $f(N)$ at N^*.

$$f'(N^*) = \left.\frac{df(N_t)}{dN}\right|_{N^*}$$

$$\boxed{\text{If } f'(N^*) < 0 \rightarrow \text{ stable.}}$$

For continuous-logistic:

$$f'(N^* = K) = \frac{d}{dN}\left(rN - \frac{rN^2}{K}\right)$$

$$= r - \frac{2rN}{K}$$

$$= r - \frac{2rK}{K}$$

$$= r - 2r$$

$$= -r$$

Thus, continuous logistic reaches stable point equilibrium when $r > 0$.

Summary
Step #1: Solve for N^*.
Step #2: Evaluate $f'(N^*)$
       For discrete:
          Stable if $|F'(N^*)| < 1$
              (for discrete logistic, stable if $0 < r_d < 2$)
       For continuous:
          Stable if $f'(N^*) < 0$
              (for continuous logistic, stable if $r > 0$)

1-D Stability analysis - Dig deeper (In discrete time, but same applies to continuous time models)

$$N_{t+1} = F(N_t) \quad \to \quad N^*$$

Now add small perturbation to N^* by adding n:

$$N_t = N^* + n_t$$

We therefore expect:

$$N_{t+1} = F(N^* + n_t)$$

We want to know if popn will return to N^*, but will ask a slightly different question to get the answer:

Since
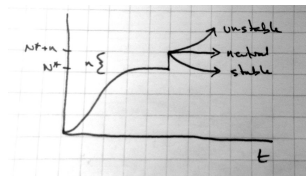
$$n_t = N_t - N^*$$

can write

$$n_{t+1} = N_{t+1} - N^*$$

Thus:

$$n_{t+1} = F(N^* + n_t) - N^*$$

Thus the question is: Does n_t grow or shrink with time?

Q: How do we find the solution of $F(N^* + n_t)$?
     (F could be a very complicated function!)
A: We approximate it:

$$\underbrace{F(N^* + n_t)}_{y} \approx \underbrace{F(N^*)}_{intercept} + \underbrace{F'(N^*)}_{slope} \cdot \underbrace{n_t}_{x}$$

Taylor Series
Recall polynomial series:

$$y = \sum_{i=0} \beta_i x^i$$

Taylor series is similar, but with derivatives...

$$F(N^* + n_t) = \sum_{i=0}^{\infty} \frac{F^{(i)}(N^*)}{i!} n_t^i = F(N^*) + \underbrace{\frac{F'(N^*)}{1!} n_t^1 + \frac{F''(N^*)}{2!} n_t^2 + \frac{F'''(N^*)}{3!} n_t^3 + \dots}_{\text{h.o.t.}}$$

Show figure from *'Class-Ex-TaylorExp.R'.*

Higher order terms get smaller and smaller.
First two terms are good approximation when n_t is small.
Thus:

$$
\begin{aligned}
n_{t+1} &= F(N^* + n_t) - N^* \\
&\approx F(N^*) + F'(N^*) \cdot n_t - N^* \\
&\approx N^* + F'(N^*) \cdot n_t - N^* \\
&\approx F'(N^*) \cdot n_t \\
&= \lambda n_t \quad (\lambda \text{ is the ``eigenvalue'' (for single-spp. model))!}
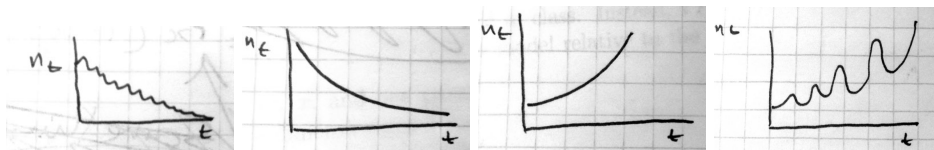\end{aligned}
$$

If $\lambda < 1 \rightarrow n_{t+1} < n_t \rightarrow$ Perturbation decays (i.e. stable system)
If $\lambda > 1 \rightarrow n_{t+1} < n_t \rightarrow$ Perturbation expands (i.e. unstable system)

Note that n_t could be a *removal* (i.e. $n_t < 0$) of individuals (rather than an *addition*).

Thus **for discrete-time model:**

$$\boxed{|F'(N^*)| < 1 \ \text{ for stability}}$$

$-1 < \lambda < 0 \rightarrow$ decay w/ damped oscillations
$0 < \lambda < 1 \rightarrow$ geometric decay
$\lambda > 1 \rightarrow$ geometric growth
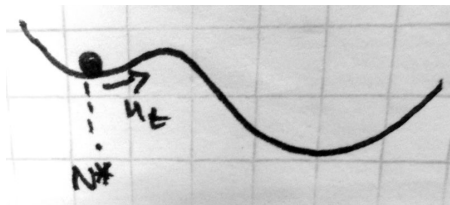$\lambda < -1 \rightarrow$ divergent oscillations

For continuous model:

$$\boxed{\left.\frac{d\,f(N)}{dN}\right|_{N^*} < 0 \ \text{for stability}}$$

We'll get much deeper into stability of continuous-time models from now on.
Will also talk about how oscillations occur and how their presence is analytically determined.

Remember that we're only dealing with **local** stability (i.e. small n_t) not global stability!

Intro to Mathematica
 Local stability analysis of continuous logistic

Walk through Mathematica code: *'Class9-Stability-cLogistic.nb'*

Functions use square brackets [ ].
Important functions:
 Solve[y=ax,x]
 D[f(x),x]
 Simplify
 func /. x → y
 var /. x

Non-dimensionalization

The equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) = \lim_{\Delta t \to 0}\left(\frac{\Delta N}{\Delta t}\right)$$

has 2 parameters (+1 that's implicit!) and 1 variable.

$$r : \frac{\#}{\#time} \qquad\qquad K : \# \qquad\qquad N : \#$$

Define x to be the population size relative to the carrying capacity (i.e. fraction of the carrying capacity).:

$$x := \frac{N}{K}$$

x is dimensionless!

(Note: The symbol := means *define*. Sometimes the symbol $\equiv$ meaning *equivalent* is also used.)

Rearrange to $N = xK$ and substitute:

$$\frac{dxK}{dt} = rxK\left(1 - \frac{xK}{K}\right)$$

$$\frac{dx}{dt} = rx\left(1 - x\right)$$

Only 1 variable (x)and 1 parameter (r), same dynamical properties.
Can reduce further...

$$\tau := rt$$

(τ = 'tau') - dimensionless - the units of r are #'s per # per time ($= 1/t$).
That is,

$$\frac{d}{dt} = \frac{d}{d\tau} \cdot \frac{d\tau}{dt}$$

Rewriting $r = \tau/t = d\tau/dt$, this simplifies to

$$\frac{d}{dt} = \frac{d}{d\tau}r$$

Therefore

$$\frac{dx}{dt} = rx\left(1 - x\right)$$

$$\frac{drx}{d\tau} = rx\left(1 - x\right)$$

$$\frac{dx}{d\tau} = x\left(1 - x\right)$$

Only 1 variable and 0 parameters!!!
Of course, 'time-scale' τ is tricky to conceptualize.
   $\to$ scaled realized rate of population dynamics (i.e. dN/dt or dx/dt)...
      ...relative to the maximum growth rate of the population (r).

Model is thus in terms of the realized growth relative to the maximum intrinsic growth.

Won't do all that much more non-dimensionalization (except in 2-spp. competition model, next), but can be extremely useful.
e.g., when absolute values of parameters are not known, but their relative values is (or can be approximated, or qualitatively known > 1 or < 1)