Lecture 17 – Tipping Points

Concepts:

- Alternative Stable States
- Hysteresis
- System potential
- Critical slowing down and early-warning signals

Alternative Stable States

⇒ ‘Two (or more) equilibria under *the same* conditions’

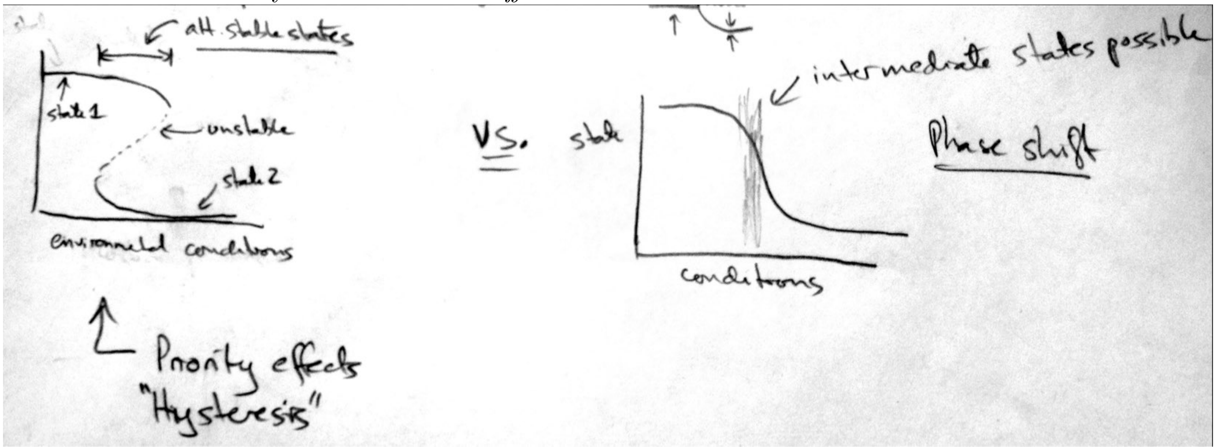
e.g., LV competition model

Saddle-node bifurcation when *inter-* > *intra*-specific competition

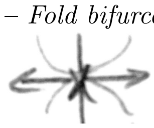
Either 1st of 2nd species persists, depending on initial conditions (*Priority effect*)

Contrast to *Phase shift*

⇒ ‘Different community structure under *different* conditions’

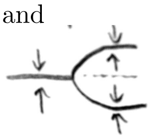


Today – *Fold bifurcation*: Alternative Stable States due to combination of:



Saddle-node bifurcation

and



Pitchfork bifurcation

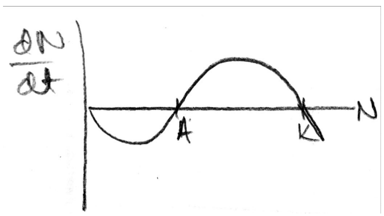
System Potential

Ball & cup diagram is not just an analogy!

Example: Logistic w/ Allee Effect

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)\left(\frac{N}{A} - 1\right)$$

(Note slightly different formulation that in Problem Set, but same effect.)



1. Solve for equilibria. **2.** Determine stability. **3.** Interpret.
⇒ Mathematica

$$N^* = \begin{cases} 0 \\ A \\ K \end{cases} \qquad \frac{d\frac{dN}{dt}}{dN} = \begin{cases} -r \\ r - \frac{A}{K}r > 0 \text{ when } A < K \\ r - \frac{K}{A}r < 0 \text{ when } A > K \end{cases} \qquad \Rightarrow \begin{cases} \text{stable} \\ \text{unstable} \\ \text{stable} \end{cases}$$

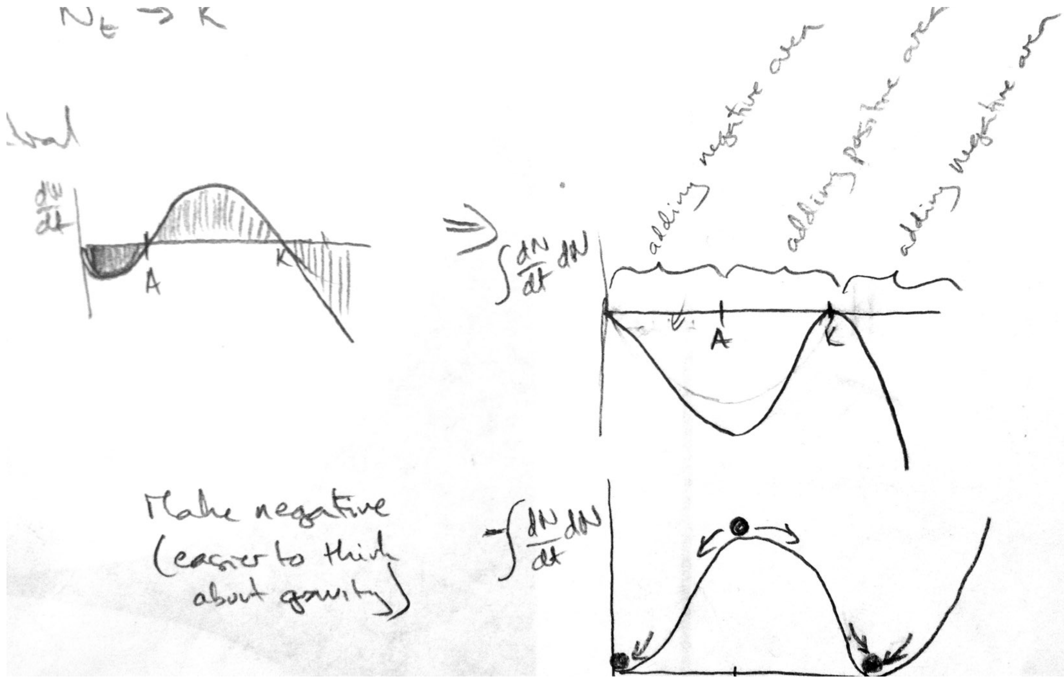
Therefore

$$N_0 < A \Rightarrow N_t \to 0$$
$$N_0 > A \Rightarrow N_t \to K$$

Def. System Potential (*Phi*):

$$\boxed{\Phi = -\int_{-\infty}^{\infty} \frac{dN}{dt} dN}$$

$\int \frac{dN}{dt} dN$ is nothing more than area under the curve in plot of $\frac{dN}{dt}$ vs. N



Exploitation model

(commonly used model to illustrate tipping points)

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \frac{cN^2 P}{b + N^2}$$

= Logistic growth with Type 3 functional response.

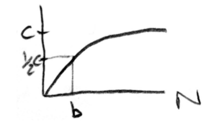
Side-note on functional response formulation:
Holling Type II:

$$f(N) = \frac{aN}{1 + ahN}$$

a - per capita attack rate $\qquad h$ - handling time

Michaelis-Menten:

$$f(N) = \frac{cN}{b+N} \qquad c \text{ - maximum feeding rate (capacity)} = \frac{1}{h} \qquad b \text{ - half-saturation constant} = \frac{1}{ah}$$



Holling Type III:

$$f(N) = \frac{aN^m}{1+ahN^m} \Rightarrow \frac{cN^2}{b+N^2} \qquad m - \text{ Hill exponent, typically set to } m = 2$$

Exploitation model

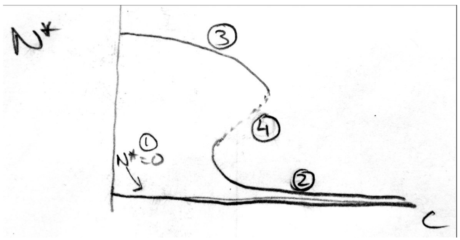
Let $P = 1$ (fixed, arbitrary. Limited by something else.)

1. Solve for equilibria: $\Rightarrow$ Mathematica

$$N^* = \begin{cases} 0 \\ > 0 \\ \text{crazy looking symbolically} \\ \text{crazy looking symbolically} \end{cases}$$

2. Plot equilibria as function of control parameter c

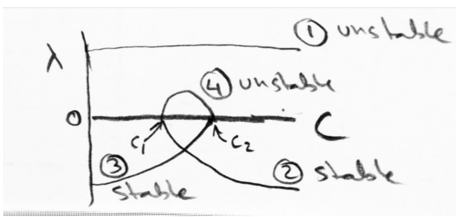
(c = feeding capacity, i.e.*maximum exploitation rate*)



3. Determine stability

$$\frac{d\frac{dN}{dt}}{dN} = \lambda = \frac{2cN^3P}{(b+N^2)^2} - \frac{2cNP}{b+N^2} - \frac{rN}{K} + r\left(1 - \frac{N}{K}\right)$$

4. Evaluate λ at N^* and plot as function of control parameter c



Unstable trivial equilibrium (1) $\Rightarrow$ Popn will always be present unless extinct.
Unstable non-trivial equilibrium (4) sandwiched between two stable equilibria (2) and (3)

Starting in (3), increasing c causes $\lambda \to 0$, then passing critical value c_2 causes $\lambda << 0$ jump.
Starting in (2), decreasing c causes $\lambda \to 0$, then passing critical values c_1 causes $\lambda << 0$ jump.

$\Rightarrow$ Two **'Critical transitions'**

Small changes in control parameter causes large 'catastrophic' change in system state
To get back to 'old' state requires larger reversal of control parameter than what caused shift.

Hysteresis

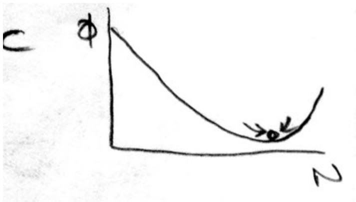
Fishery context - harvest rate

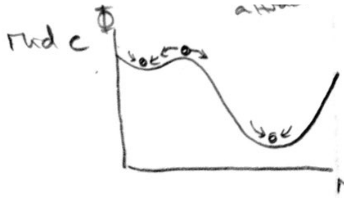
Slow increase causes only small decrease in stock size until, all of a sudden, stock crashes.
Pulling back a little makes little difference due to hysteresis.

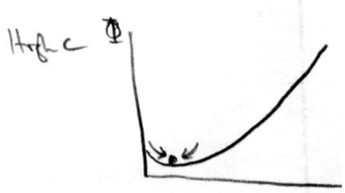
System Potential & Critical slowing down

$$\Phi = -\int \frac{dN}{dt} dN$$

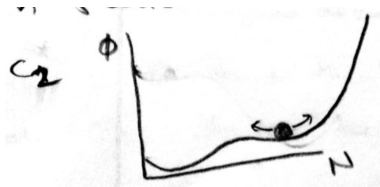
⇒ Mathematica (First Manipulate plot of c)

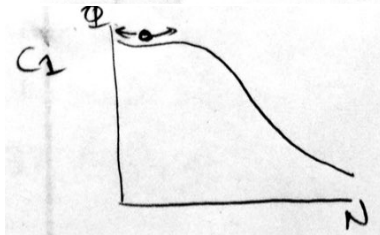






Observe that the landscape flattens out near critical values c_1 and c_2 where the 2nd basin emerges.





$$\text{As } \lambda \to 0 \quad \Rightarrow \quad \underbrace{\frac{d(-\int \frac{dN}{dt} dN)}{dN}}_{\text{slope}=-\frac{dN}{dt}} \to 0$$

Thus intuitively: If we perturb the system, it should respond slower.

Evaluate intuition formally:

1-sp.	1-sp. example	n-spp.
(1) Solve for equil $$\frac{dN}{dt} = f(N) = 0$$ $$\Rightarrow N^*$$	$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) = 0$$ $$\Rightarrow N^* = \begin{cases} 0 \\ K \end{cases}$$	$$\frac{dN_i}{dt} = f_i(\vec{N}) = 0 \; \forall i$$ $$\Rightarrow \vec{N}^*$$
(2) Determine stability of N^* Determine $$\frac{df(N)}{dN}$$	$$r - 2r\frac{N}{K}$$	Evaluate $$\mathbf{A} = \frac{\partial f_i(\vec{N})}{\partial N_j} \quad \forall i, j$$
Evaluate at N^* $$\left.\frac{df(N)}{dN}\right\|_{N^*} = \lambda$$	$$r - 2r\frac{K}{K}$$ $$= r - 2r$$ $$= -r$$	$$\mathbf{A}\|_{N^*}$$ (i) Routh-Hurwitz Criteria (ii) Eigenvalues λ_i (*scalar representation of Jacobian transformation matrix*)
(3) Interpret λ's $x =$ perturbation $$\frac{dx}{dt} = \lambda x$$ $$\Rightarrow x_t = x_0 e^{\lambda t}$$ Stable if $Re(\lambda) < 0$	$$\frac{dx}{dt} = -rx$$ $$\Rightarrow x_t = x_0 e^{-rt}$$ $\Rightarrow$ Stable	$$\vec{n} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \frac{d\vec{n}}{dt} = \vec{\lambda}\vec{n}$$ $$\Rightarrow \vec{n}_t = \vec{n}_0 e^{\vec{\lambda} t}$$ Stable if $Re(\lambda_i) < 0 \;\; \forall i$

Want to know how long it takes for a perturbation of size x_0 to decay ($x_t = 0$).
But since this is exponential decay, it never goes to zero! (Only at limit.)

Thus pick arbitrary value of x_t and determine the time to reach it.
Choose x_t such that:

$$x_t = \frac{x_0}{e} \quad \Rightarrow \text{ Perturbation decays to } \frac{1}{e}(\approx 63\%) \text{ of } x_0.$$

Thus:

$$\begin{aligned} \frac{x_0}{e} &= x_0 e^{\lambda t} \\ \frac{1}{e} &= e^{\lambda t} \\ \ln\left(\frac{1}{e}\right) &= \lambda t \\ -1 &= \lambda t \\ t &= -\frac{1}{\lambda} = T_R \qquad \textit{`Characteristic Return Time'} \end{aligned}$$

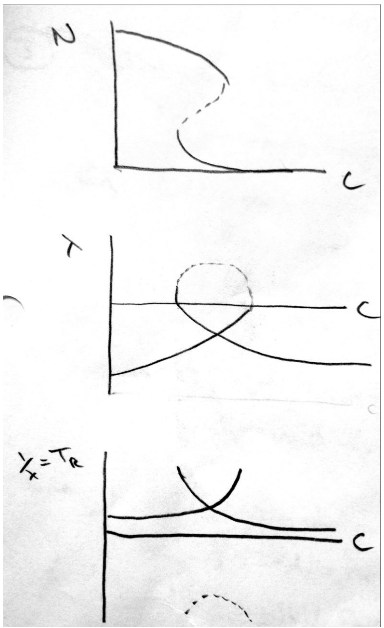
For 1-sp. Logistic:
$T_R = \frac{-1}{-r} = \frac{1}{r}$ $\quad\Rightarrow$ Return time depends on r. Larger $r \Rightarrow$ shorter Return time

For n-spp.

$$T_R = -\frac{1}{Re(\lambda_{\max})} = -\frac{1}{Re(\lambda_1)}$$

$\lambda_{\max} = \lambda_1 =$ 'Dominant' or 'Leading' eigenvalue (least negative; first of ordered eigenvalues)
Represents the slowest *combination* of species
(recall that we're in transformed coordinate system)

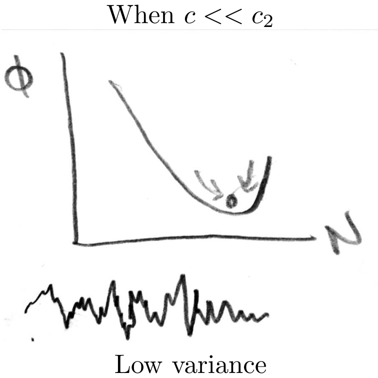
Leading indicators / Early warning signals
⇒ Mathematica



As $\lambda \to 0$, $T_R \to \infty$ ⇒ *Critical Slowing Down*

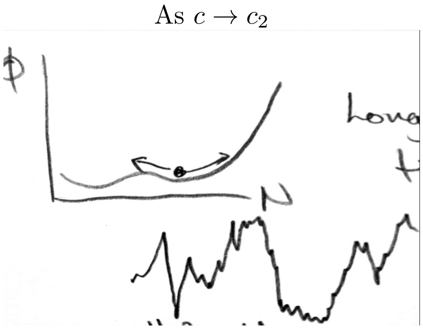
Above is all about deterministic systems. But think about real world, stochastic systems.
i.e. *Near* equilibrium, constantly perturbed by small amounts (small pulse perturbations)

When $c << c_2$



Low variance
Low autocorrelation
(independent and random perturbations)

As $c \to c_2$



Longer return times
Higher variance
Higher autocorrelation

Rise in variance and auto-correlation (spatial or temporal)
= Early warning signal of critical bifurcation

But, note that critical slowing down occurs for bifurcation types, not just 'tipping points'!
Including Hopf bifurcations! (False positives)
