Lecture 11 – 2-D Stability Analysis – Competition Part 2

Today: Go over graphical analysis LV-competition
...then proceed to do formal stability analysis of equilibria
Next class move on to Consumer-Resource & nonlinear intxns.

2-spp. competition Main questions:
When can both spp. coexist?
When can sp 1 out-compete sp 2?
When can sp 1 invade sp 2?

Naive simulations

R-demonstration - (Vector field section.)

Walk through first set of parameter values. "Naive simulations"

Then show full table before showing results with R code for other parameters.

With $r_1 = r_2 = K_1 = K_2 = 1$ and $N_1(0) = N_2(0) = 0.01$

α_{21}	α_{12}	Outcome	
0.5	0.7	coexist	
1.5	0.5	sp1	
0.5	1.5	sp2	$N_1(0) = N_2(0)$
1.5	1.7	sp2	
1.5	1.7	sp1	$N_1(0) > N_2(0)$

⇒ Priority effect
⇒ Alternative stable states

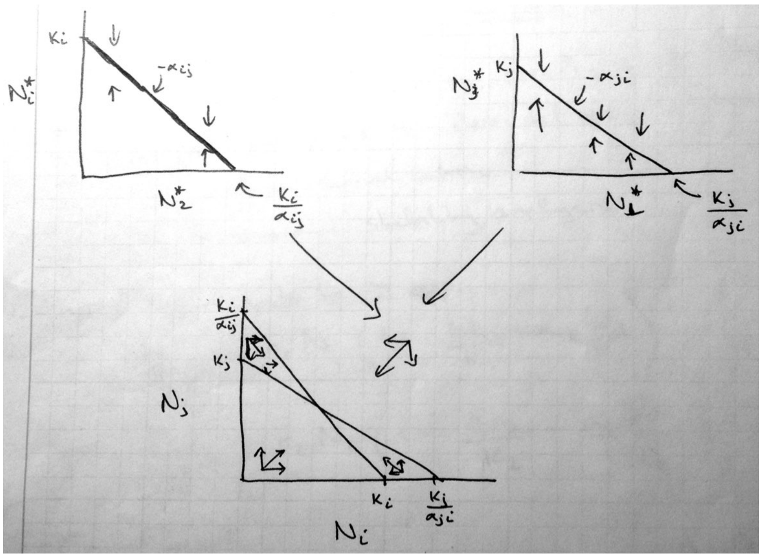
Graphical Analysis
Earlier we showed that from $\frac{dN_i}{dt} = 0$ that:

$$N_i^* = K_i - \alpha_{ij} N_j \quad \text{and} \quad N_j^* = K_j - \alpha_{ji} N_i$$
$$\Rightarrow N_j = \frac{K_j - N_i^*}{\alpha_{ij}}$$

Thus isocline interesects x-axis at

$$N_j = \frac{K_j - 0}{\alpha_{ij}} = \frac{K_j}{\alpha_{ij}}$$

Same goes for 2nd species.



Work through all other cases on board

R-code: Work through other parameter values

Inferences/Conclusions:

Coexistence:	intra > inter for both species
	(i.e. $\frac{K_j}{\alpha_{jj}} = \frac{K_j}{1} < \frac{K_j}{\alpha_{ji}} \Rightarrow \alpha_{jj} > \alpha_{ji}$)
Sp i dominance:	intra > inter for i & intra < inter for j
Priority effect:	intra < inter for both
	(i.e. $\frac{K_j}{\alpha_{jj}} = \frac{K_j}{1} > \frac{K_j}{\alpha_{ji}} \Rightarrow \alpha_{jj} < \alpha_{ji}$)

Evaluate stability of equilibria

Just as for 1 species model:

Step #1: Solve for N^* steady state equilibria

Step #2: Find f' and evaluate at N^*

Let's use non-dimensionalized version of the LV-competition model!

Let:

$$u_1 = \frac{N_1}{K_1} \qquad u_2 = \frac{N_2}{K_2} \qquad \tau = r_1 t$$
$$\rho = \frac{r_2}{r_1} \qquad a_{12} = \alpha_{12}\frac{K_2}{K_1} \qquad a_{21} = \alpha_{21}\frac{K_1}{K_2}$$

u_i - % of carrying capacity

τ - 'growth time'

ρ - relative growth rates

a_{ij} - relative resource requirements

New model becomes:

$$\frac{du_1}{d\tau} = u_1(1 - u_1 - a_{12}u_2) \qquad \frac{du_2}{d\tau} = \rho u_2(1 - u_2 - a_{21}u_1)$$

⇒ Mathematica *'Class10-LVcompetition-nonDim'*

Isoclines:

$$u_1^* = \frac{1 - u_2}{a_{21}} \qquad u_2^* = \frac{1 - u_1}{a_{12}}$$

Step #1: Solve for u^* steady state equilibria

$$(u_1^*, u_2^*) = \begin{cases} 0, & 0 \\ 0, & 1 \\ 1, & 0 \\ \frac{1-a_{12}}{1-a_{12}a_{21}}, & \frac{1-a_{21}}{1-a_{12}a_{21}} \end{cases}$$

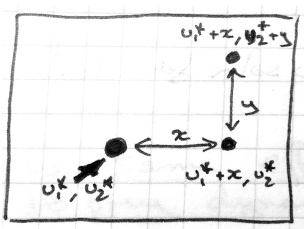
Step #2: Evaluate $f'(N^*)$

...but now in two dimensions: $f_1'(N_1^*, N_2^*)$ and $f_2'(N_1^*, N_2^*)$

⇒ Stability analysis in two dimensions ⇐

Perturbation to both u_1^* and u_2^* by amounts x and y.

$$u_1 = u_1^* + x \qquad u_2 = u_2^* + y$$
$$\Rightarrow \quad x = u_1^* - u_1 \qquad y = u_2^* - u_2$$



Thus:

$$\frac{du_1}{d\tau} = \frac{d(u_1^* + x)}{d\tau} = f_1(u_1^* + x, u_2^* + y)$$

and

$$\frac{du_2}{d\tau} = \frac{d(u_2^* + y)}{d\tau} = f_2(u_1^* + x, u_2^* + y)$$

Because u_1^* and u_2^* are constants w.r.t τ...

$$\frac{dx}{d\tau} = \frac{du_1}{d\tau}$$

and

$$\frac{dy}{d\tau} = \frac{du_2}{d\tau}$$

Recall that we approximated response of single-sp. perturbation using a Taylor expansion:

$$f(N^* + n_t) \approx f(N^*) + f'(N^*) \cdot n_t + h.o.t.$$

Do same thing in two dimensions (note use of *partial* derivatives):

To simplify syntax, let's denote $f_i(u_1^*, u_2^*) = f_i$
(i.e. f_i is implicitly evaluated at steady state abundances)

$$\begin{aligned} f_1(u_1^* + x, u_2^* + y) =& f_1(u_1^*, u_2^*) + \frac{\partial f_1}{1!\,\partial u_1} \cdot x + \frac{\partial^2 f_1}{2!\,\partial u_1^2} \cdot x^2 + ... \\ & ... + \frac{\partial f_1}{1!\,\partial u_2} \cdot y + \frac{\partial^2 f_1}{2!\,\partial u_2^2} \cdot y^2 + h.o.t. \\ \approx& f_1(u_1^*, u_2^*) + \left.\frac{\partial f_1}{\partial u_1}\right|_{u_1^*, u_2^*} \cdot x + \left.\frac{\partial f_1}{\partial u_2}\right|_{u_1^*, u_2^*} \cdot y \end{aligned}$$

Since at steady-state (by definition)..

$$f_1(u_1^*, u_2^*) = 0$$

...we can simplify further to:

$$f_1(u_1^* + x, u_2^* + y) \approx \underbrace{\left.\frac{\partial f_1}{\partial u_1}\right|_{u_1^*, u_2^*}}_{A_{11}} \cdot x + \underbrace{\left.\frac{\partial f_1}{\partial u_2}\right|_{u_1^*, u_2^*}}_{A_{12}} \cdot y$$

Similarly

$$f_2(u_1^* + x, u_2^* + y) \approx \underbrace{\left.\frac{\partial f_2}{\partial u_1}\right|_{u_1^*, u_2^*}}_{A_{21}} \cdot x + \underbrace{\left.\frac{\partial f_2}{\partial u_2}\right|_{u_1^*, u_2^*}}_{A_{22}} \cdot y$$

Organize these elements into a ***Jacobian matrix***:

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Jacobian contains the partial derivatives of each i^{th} variable's function f_i w.r.t j.

$\mathbf{A}$ is referred to as the ***Community matrix*** when f_i's reflect population growth rates (i.e. $f_i = \frac{dN_i}{dt}$).

Interpretation of elements:

How a small perturbation to j affects the population growth rate of i

with other species held constant

The Jacobian's eigenvalues provide insight into the stability of the steady state(s). We will return to this in more detail next time.

For now note that eigenvalues can have both 'Real' and 'Imaginary' parts...

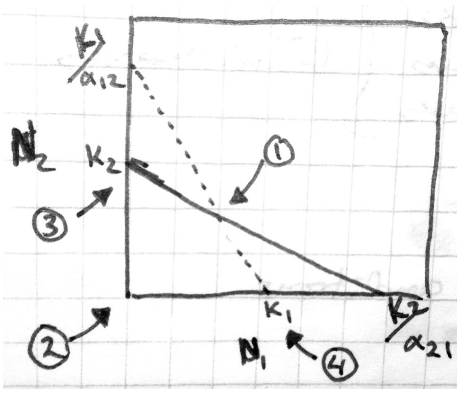
Eigenvalues (λ_i)	**Interpretation**
All parts < 0	Stable point attractor
Some real parts < 0	Saddle equilibrium (repellor-attractor)
No real parts < 0	Unstable point repellor
Real parts $= 0$	Neutral stability
No imaginary parts	No oscillations
With imaginary parts	Oscillations

Evaluate eigenvalues of LV-competition equilibria

$\Rightarrow$ Mathematica *'Class11-LVcompetition-nonDim'*

When $\alpha_{ii} > \alpha_{ij}$

(intra- is stronger than inter-specific competition)...



1 Both $\lambda's < 0 \Rightarrow$ Stable coexistence

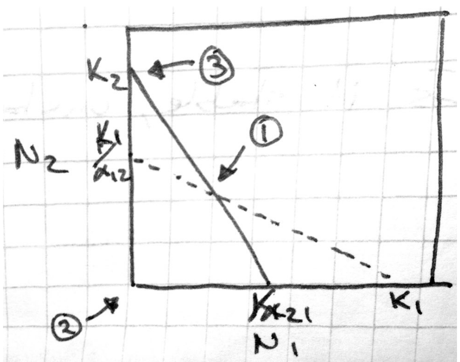
2 Both $\lambda's > 0 \Rightarrow$ Unstable

3 $\lambda_1 > 0$ & $\lambda_2 < 0 \Rightarrow$ unstable (attractor-repellor)

4 $\lambda_1 < 0$ & $\lambda_2 > 0 \Rightarrow$ unstable (attractor-repellor) 3 and 4 are mutually invasible

When $\alpha_{ii} > \alpha_{ij}$ & $\alpha_{jj} < \alpha_{ji}$

(inter- is stronger than intra-specific competition for one species but not the other)...



1 $\lambda_1 < 0$ & $\lambda_2 > 0 \Rightarrow$ unstable (attractor-repellor)

2 Both $\lambda's > 0 \Rightarrow$ Unstable

3 Both $\lambda's < 0 \Rightarrow$ Stable

...could do for third and fourth scenarios too.
