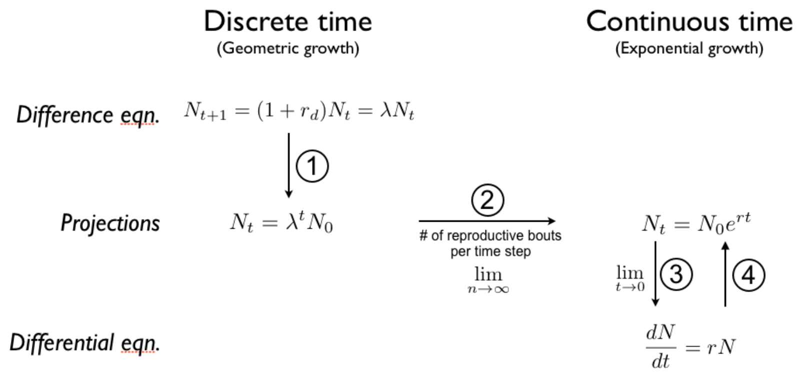
Lecture 2 - Summary

The purpose of last lecture was to show connections between alternative descriptions of population growth. So, to summarize:



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(1) We started with $N_{t+1} = N_t + B + D + I + E$, then substituted $E = I$, $B = bN_t$ and $D = dN_t$, where b and d were per capita rates. Using $r_d = b - d$ and $\lambda = 1 + r_d$, we then simplified down to $N_{t+1} = (1_+ r_d)N_t = \lambda N_t$. Thus, for example,

$$N_{t+2} = \lambda N_{t+1} = \lambda(\underbrace{\lambda N_t}_{N_{t+1}}) = \lambda^2 N_t,$$

such that more generally we can write $\boxed{N_t = \lambda^t N_0}$ for any arbitrary t time-steps into the future.

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(2) From the discrete to the continuous: Let's say that in t=1 year an annually reproducing (and dying) population grows by a factor of λ. From that one reproductive event we'd have:

1 event per time-step:

$$N_1 = \lambda N_0 = (1 + r_d)N_0$$

Now, if we had 2 reproductive events per year but achieved the same total amount of population growth over the year, we'd write

2 events per time-step:

$$N_1 = \left(1 + \tfrac{r_d}{2}\right)^2 N_0$$

By extension, if we had an n reproductive events per year but achieved the same total amount of population growth over the year, we'd write:

n events per time-step:

$$N_1 = \left(1 + \tfrac{r_d}{n}\right)^n N_0$$

Now, λ is nothing more than the discrete population growth factor (how much the population size changed over one year), so let's divide both sides by N_0 to isolate λ:

$$\lambda = \frac{N_1}{N_0} = \left(1 + \tfrac{r_d}{n}\right)^n$$

We won't prove it, but by demonstration (see next page):

$$\lambda = \lim_{n\to\infty} \left(1 + \tfrac{r_d}{n}\right)^n = e^r$$

Therefore, we have

$$\boxed{\lambda = (1 + r_d) = e^r}$$

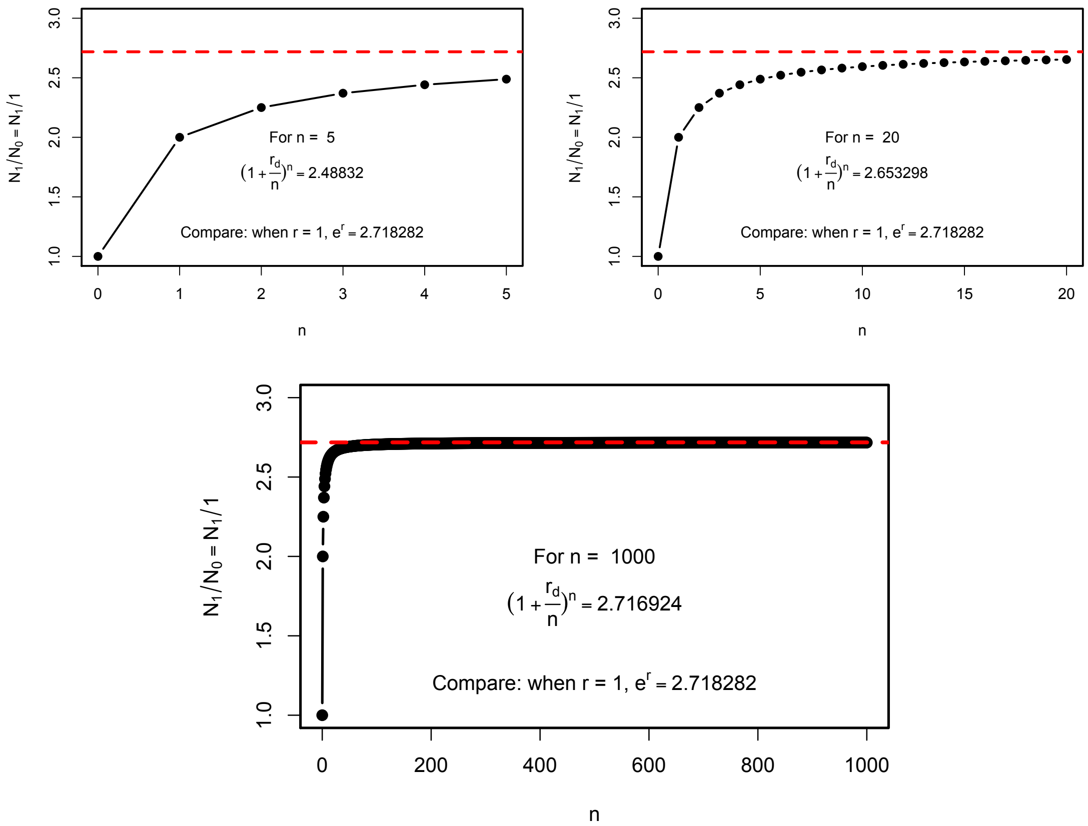
and we can relate (convert between) discrete-time and continuous-time population growth:

$$\boxed{N_t = N_0\lambda^t = N_0 e^{rt}}$$

How does Euler's number, e, emerge?
Let's let $n = 1,\ N_0 = 1,\ r_d = 1$
The true of $e = exp(1) = 2.71828...$, but we'll estimate it as $\lambda = \frac{N_1}{N_0}$ for ever increasing values of n:

$$
\begin{aligned}
n = 1 \Rightarrow \frac{N_1}{1} &= \left(1 + \tfrac{1}{1}\right)^1 = 2 \\
n = 2 \Rightarrow \frac{N_1}{1} &= \left(1 + \tfrac{1}{2}\right)^2 = 2.25 \\
n = 3 \Rightarrow \frac{N_1}{1} &= \left(1 + \tfrac{1}{3}\right)^3 = 2.30707... \\
n \to \infty \Rightarrow \frac{N_1}{1} &= \left(1 + \tfrac{r_d}{n}\right)^n = 2.71828... = e
\end{aligned}
$$

In graphical form:



Defining the natural-log as the 'anti-exponential' — $ln(e^x) = x$ — we also have:

$$ln(\lambda) = ln(1 + r_d) = r$$

All three (λ, r_d, and r) reflect a *per capita growth rate* for either geometric or exponential growth:

$$
\begin{aligned}
r_d &= b_d - d_d && \text{discrete time} \\
r &= b - d && \text{continuous time}
\end{aligned}
$$

(3) How to we get instantaneous population-level growth rate from projection equation, $N_0 e^{rt}$? That is, how do we show that:

$$\lim_{\Delta t \to 0} \left(\frac{\Delta N_t}{\Delta t} \right) = \frac{dN}{dt}$$

Need to take the derivative of $N_0 e^{rt}$ with respect to time t.
Use Chain Rule:

$$\frac{d(XY)}{dt} = \frac{d(X)}{dt} \cdot Y + X \cdot \frac{d(Y)}{dt}$$

(The derivative of a product is the sum of the product of the derivative of each term times the other term.)
Thus:

$$\frac{d(N_0 \cdot e^{rt})}{dt} = \frac{d(N_0)}{dt} \cdot (e^r)^t + N_0 \cdot \frac{d((e^r)^t)}{dt}$$

Note:
Derivative of a constant = 0
Derivative of $a^x = ln(a) \cdot a^x$.
Thus:

$$\begin{aligned} \frac{d(N_0 e^{rt})}{dt} &= 0 \cdot (e^r)^t + ln(e^r) \cdot (e^r)^t \cdot N_0 \\ &= r \cdot (e^r)^t \cdot N_0 \\ &= r \cdot e^{rt} \cdot N_0 \\ &= r \cdot N_0 e^{rt} \end{aligned}$$

Since $N = N_0 e^{rt}$ for any time t...

$$= rN = \frac{dN}{dt}$$

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(4) Could also go in opposite direction from $\frac{dN}{dt} \to N_0 e^{rt}$:

$$\begin{aligned} &\frac{dN}{dt} = rN \\ &\frac{1}{N} \frac{dN}{dt} = r \\ &\int_0^T \frac{1}{N} \frac{dN}{dt} \, dt = \int_0^T r \, dt \quad \text{(Think of T as a constant, and t in dt as a variable)} \\ &\int_0^T \frac{1}{N} \frac{dN}{dt} = rt|_0^T = r \cdot T - r \cdot 0 \end{aligned}$$

Using $\int \frac{1}{x} \, dx = ln(x)$...

$$\begin{aligned} &ln(N(T)) - ln(N(0)) = rT \\ &ln \left(\frac{N(T)}{N(0)} \right) = rT \\ &\frac{N(T)}{N(0)} = e^{rT} \\ &N(T) = N(0) e^{rT} \end{aligned}$$
