Quiz 4 - Lotka-Volterra Competition

My name is: ______________________________

Two species (N_1 and N_2) are locked in a deadly battle over a set of shared resources. They thereby compete with one another such that an individual of species N_2 utilizes α_{12} amount of the resources utilized by an individual of species N_1, and an individual of species N_1 utilizes α_{21} amount of the resources utilized by an individual of species N_2. By a simple extension of the single-species logistic model, we can include such competition between two species by writing

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - \frac{\alpha_{12} N_2}{K_1}\right)$$

to describe the population growth rate of species N_1, and

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{\alpha_{21} N_1}{K_2} - \frac{N_2}{K_2}\right)$$

to describe the population growth rate of species N_2.

(a) How many equilibria does this model have? Describe each of them qualitatively in terms of the population sizes of N_1 and N_2.

$$\text{4 equilibria: } (N_1^*, N_2^*) = \begin{cases} N_1 = 0, N_2 = 0 \\ N_1 > 0, N_2 = 0 \\ N_1 = 0, N_2 > 0 \\ N_1 > 0, N_2 > 0 \end{cases}$$

(b) Use the above equations to show that species N_1 will reach its equilibrium carrying capacity K_1 in the absence of species N_2.

Since $N_2 = 0$ we have $\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right)$. Solving this for the equilibria gives:

$$\begin{aligned} \frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) &= 0 \\ r_1 N_1 - r_1 N_1 \frac{N_1}{K_1} &= 0 \\ r_1 N_1 &= r_1 N_1 \frac{N_1}{K_1} \\ N_1^* &= K_1 \end{aligned}$$

(c) What is the equilibrium population size of N_2 in the absence of N_1?

By the same logic as in (b), $N_2^* = K_2$

(d) Use the equations to solve for the equilibrium population size of N_1 in the presence of N_2.

$$\begin{aligned} r_1 N_1 &= r_1 N_1 N_1 - r_1 \alpha_{12} N_1 \frac{N_2}{K_1} \\ r_1 N_1 K_1 &= r_1 N_1 N_1 + r_1 \alpha_{12} N_1 N_2 \\ K_1 &= N_1 - \alpha_{12} N_2 \\ N_1^* &= K_1 - \alpha_{12} N_2 \end{aligned}$$

By the same logic, $N_2^* = K_2 - \alpha_{21} N_1$, or by rearranging,

$$\begin{aligned} K_1 - N_1^* &= \alpha_{12} N_2^* \\ N_2^* &= \frac{K_1 - N_1^*}{\alpha_{12}} \end{aligned}$$

and similarly

$$N_1^* = \frac{K_2 - N_2^*}{\alpha_{21}}$$