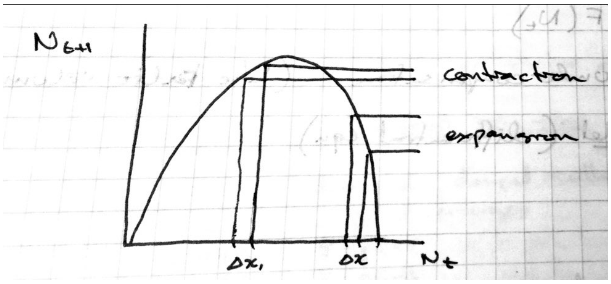
# Lecture 9 – 1-D Stability Analysis - Part 2

**Ricker plots & Intuitive notion of Lyapunov exponent**




$\rightarrow$ The larger $r_d$, the steeper the slopes, the more expansion.

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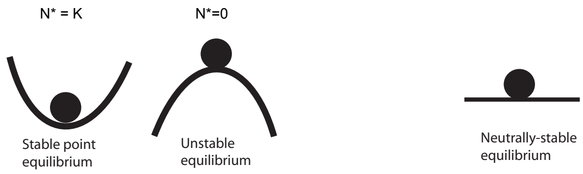
**Local stability analysis**
Through simulation we found $r_d < 2$ for $N^* = K$ to be stable
(either monotonic dampening or damped oscillations).




Now let's determine this formally...
**Discrete-time model:**

$$N_{t+1} = F(N_t)$$

**Step 1**: Solve $F(N)$ for $N^*$:

$$F(N_t) = N_t + r_d N_t \left(1 - \frac{N_t}{K}\right)$$

By definition: $F(N^*) = N_t$

$$F(N^*) = N_t = N_t + r_d N_t - \frac{r_d N_t^2}{K}$$

$$\text{Solve for } N_t: \quad 0 = r_d N_t - \frac{r_d N_t^2}{K}$$

$$\frac{r_d N_t^2}{K} = r_d N_t$$

$$r_d N_t^2 = r_d N_t K$$

$$\rightarrow N^* = K \text{ and } N^* = 0$$

**Step 2**: Find slope of $F(N_t)$ evaluated at $N^*$:

$$F'(N^*) = \left.\frac{dF(N_t)}{dN}\right|_{N^*}$$

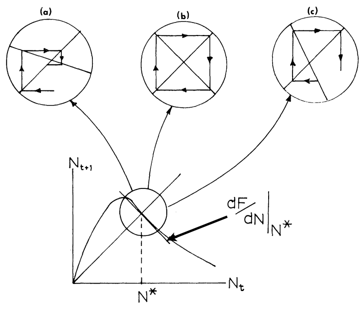
If $|F'(N^*)| < 1 \rightarrow$ stable.

For discrete logistic:

$$F'(N^*) = 1 + r_d - 2r_d\frac{N^*}{K}$$
$$= 1 + r_d - 2r_d\frac{K}{K}$$
$$= 1 - r_d$$

**Thus, discrete logistic reaches stable point equilibrium when** $0 < r_d < 2$.

Project figure in Keynote

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**Continuous-time model:**

$$\frac{dN}{dt} = f(N) = rN\left(1 - \frac{N}{K}\right)$$

**Step 1**: Solve $f(N)$ for $N^*$:

By definition: At $N^*$ when $f(N^*) = 0$

$$\begin{aligned} 0 &= rN - \frac{rN^2}{K} \\ \frac{rN^2}{K} &= rN \\ rN^2 &= rNK \\ \rightarrow N^* &= K \text{ and } N^* = 0 \end{aligned}$$

**Step 2**: Find slope of $f(N)$ at $N^*$.

$$f'(N^*) = \left.\frac{df(N_t)}{dN}\right|_{N^*}$$

If $f'(N^*) < 0 \rightarrow$ stable.

For continuous-logistic:

$$\begin{aligned} f'(N^* = K) &= \frac{d}{dN}\left(rN - \frac{rN^2}{K}\right) \\ &= r - \frac{2rN}{K} \\ &= r - \frac{2rK}{K} \\ &= r - 2r \\ &= -r \end{aligned}$$

**Thus, continuous logistic reaches stable point equilibrium when** $r > 0$.

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**Summary**

Step #1: Solve for $N^*$.

Step #2: Evaluate $f'(N^*)$

- For discrete:
  - Stable if $|F'(N^*)| < 1$
    - (for discrete logistic, stable if $0 < r_d < 2$)
- For continuous:
  - Stable if $f'(N^*) < 0$
    - (for continuous logistic, stable if $r > 0$)

## 1-D Stability analysis - Dig deeper

(Will do it in discrete time, but same applies to continuous time models)

$$N_{t+1} = F(N_t) \quad \rightarrow \quad N^*$$

Now add small perturbation to $N^*$ by adding $n$:

$$N_t = N^* + n_t$$

We therefore expect:

$$N_{t+1} = F(N^* + n_t)$$

We want to know if popn will return to $N^*$, but will ask a slightly different question to get the answer:

Since

$$n_t = N_t - N^*$$

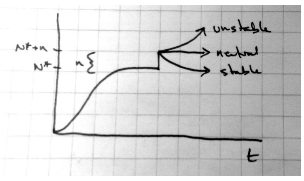
can write

$$n_{t+1} = N_{t+1} - N^*$$

Thus:

$$n_{t+1} = F(N^* + n_t) - N^*$$

**Thus the question is: Does $n_t$ grow or shrink with time?**




Q: How do we find the solution of $F(N^* + n_t)$?

($F$ could be a very complicated function!)

A: We approximate it:

$$\underbrace{F(N^* + n_t)}_{y} \approx \underbrace{F(N^*)}_{intercept} + \underbrace{F'(N^*)}_{slope} \cdot \underbrace{n_t}_{x}$$

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## Taylor Series

Recall polynomial series:

$$y = \sum_{i=0} \beta_i x^i$$

Taylor series is similar, but with derivatives...

$$F(N^* + n_t) = \sum_{i=0}^{\infty} \frac{f^{(i)}(N^*)}{i!} n_t^i = F(N^*) + \underbrace{\frac{F'(N^*)}{1!} n_t^1 + \frac{F''(N^*)}{2!} n_t^2 + \frac{F'''(N^*)}{3!} n_t^3 + ...}_{\text{h.o.t.}}$$

Show figure from *'Class-Ex-TaylorExp.R'*.

*Higher order terms* get smaller and smaller.
First to terms are good approximation when $n_t$ is small.
Thus:

$$\begin{aligned} n_{t+1} &= F(N^* + n_t) - N^* \\ &\approx F(N^*) + F'(N^*) \cdot n_t - N^* \\ &\approx N^* + F'(N^*) \cdot n_t - N^* \\ &\approx F'(N^*) \cdot n_t \\ &= \lambda n_t \qquad (\lambda \text{ is the "eigenvalue" (for single-spp. model))!} \end{aligned}$$

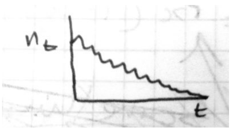
If $\lambda < 1 \rightarrow n_{t+1} < n_t \rightarrow$ Perturbation decays (i.e. stable system)
If $\lambda > 1 \rightarrow n_{t+1} < n_t \rightarrow$ Perturbation expands (i.e. unstable system)

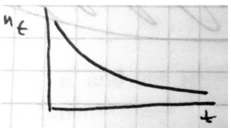
Note that $n_t$ could be a *removal* (i.e. $n_t > 0$) of individuals (rather than an *addition.*

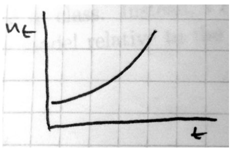
Thus **for discrete-time model:**

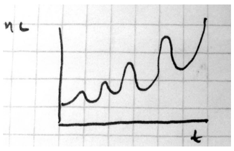
$$\boxed{|F'(N^*)| < 1 \ \text{ for stability}}$$

$$\begin{aligned} -1 < \lambda < 0 &\rightarrow \text{decay w/ damped oscillations} \\ 0 < \lambda < 1 &\rightarrow \text{geometric decay} \\ \lambda > 1 &\rightarrow \text{geometric growth} \\ \lambda < -1 &\rightarrow \text{divergent oscillations} \end{aligned}$$







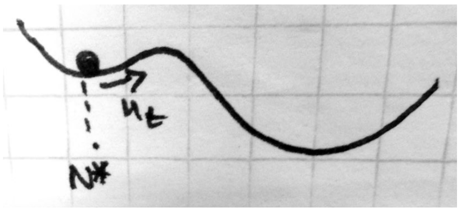



**For continuous model:**

$$\boxed{\left.\frac{d\,f(N)}{dN}\right|_{N^*} < 0 \text{ for stability}}$$

We'll get much deeper into stability of continuous-time models from now on.
Will also talk about how oscillations occur and their presence is analytically determined.

**Remember** that we're only dealing with **local** stability (i.e. small $n_t$) not global stability!

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**Intro to Mathematica**
**Local stability analysis of continuous logistic**

Walk through Mathematica code: *'Class9-Stability-cLogistic.nb'*

Functions use square brackets [ ].
Important functions:
*Solve[y=ax,x]*
*D[f(x),x]*
*Simplify*
*func /. x → y*
*var /. x*

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**Non-dimensionalization**

The equation

$$\frac{dN}{dt} = rn\left(1 - \frac{N}{K}\right) = \lim_{\Delta t \to 0}\left(\frac{\Delta N}{\Delta t}\right)$$

has 2 parameters (+1 that's implicit!) and 1 variable.

$r : \frac{\#}{\# time}$ $\qquad K : \#$ $\qquad N : \#$

Define $x$ to be the population size relative to the carrying capacity (i.e. fraction of the carrying capacity).:

$$x := \frac{N}{K}$$

$x$ is dimensionless!

(Note: The symbol := means *define*. Sometimes the symbol ≡ meaning *equivalent* is also used.)

Rearrange to $N = xK$ and substitute:

$$\frac{dxK}{dt} = rxK\left(1 - \frac{xK}{K}\right)$$
$$\frac{dx}{dt} = rx\,(1 - x)$$

Only 1 variable ($x$ )and 1 parameter ($r$), same dynamical properties.
Can reduce further...

$$\tau := rt$$

($\tau$ = 'tau') - dimensionless - the units of $r$ are #'s per # per time (= $1/t$).
That is,

$$\frac{d}{dt} = \frac{d}{d\tau} \cdot \frac{d\tau}{dt}$$

Rewriting $r = \tau/t = d\tau/dt$, this simplifies to

$$\frac{d}{dt} = \frac{d}{d\tau}r$$

Therefore

$$\frac{dx}{dt} = rx\,(1 - x)$$
$$\frac{drx}{d\tau} = rx\,(1 - x)$$
$$\frac{dx}{d\tau} = x\,(1 - x)$$

Only 1 variable and 0 parameters!!!
Of course, 'time-scale' $\tau$ is tricky to conceptualize.
→ scaled realized rate of population dynamics (i.e. $dN/dt$ or $dx/dt$)...
...relative to the maximum growth rate of the population ($r$).

Model is thus in terms of the realized growth relative to the maximum intrinsic growth.

Won't do all that much more non-dimensionalization (except in 2-spp. competition model, next), but can be extremely useful.
e.g., when absolute values of parameters are not known, but their relative values is (or can be approximated, or qualitatively known $> 1$ or $< 1$)