Mathematics 2, Part 4, Homework 2

I. Duals and duals and duals

1)

For the first task, as instructed we can rewrite the (D) problem in the standard form:

$$\begin{aligned} \max -b^\top y \\ -A^\top y \leq -c \\ y \geq 0 \end{aligned}$$

Then we can dualize according to the given pattern to get:

$$\begin{aligned} \max -c^\top x \\ -A^\top x \geq -b \\ x \geq 0 \end{aligned}$$

And finally by multiplying with -1, we get back to the (P) problem:

$$\begin{aligned} \max c^\top x \\ A^\top x \leq b \\ x \geq 0 \end{aligned}$$

2)

For this problem we took the (P) and (D) problem from [1]:

$$\begin{array}{ll} \text{(P):} & \text{(D):} \\ \min c^\top x & \max b^\top y \\ Ax = b & A^\top y + s = c \\ x \geq 0 & s \geq 0,\ y \in \mathbb{R}^m \end{array}$$

We show that they are dual by starting from the (D) problem. Firstly, we eliminate the slack s to get: $A^T y \leq c$. Then we can convert y into nonnegative variables: $y = y_+ - y_-$. By doing that we can rewrite the maximization to be:

$$\max \ \begin{bmatrix} b & -b \end{bmatrix} \begin{bmatrix} y_+ \\ y_- \end{bmatrix}$$

And the constraints:

$$\begin{bmatrix} A^T & -A^T \end{bmatrix} \begin{bmatrix} y_+ \\ y_- \end{bmatrix} \leq c, \begin{bmatrix} y_+ \\ y_- \end{bmatrix} \geq 0$$

By doing this we achieve the standard form and can now use the pattern formula to get:

$$\min \ c^T x$$

$$\begin{bmatrix} A^T & -A^T \end{bmatrix} x \ \geq \ \begin{bmatrix} b \\ -b \end{bmatrix}$$

$$x \geq 0$$

If we look closely at the second line we can see that it must hold that: $A^T x \geq b$ and $-A^T x \geq -b$, which translates to $A^T x \leq b$ if we multiply by -1. With that, the only option is that $A^T x = b$. With that we arrive at our (P) problem.

II. Interior point algorithm implementation

5)

Here we compute the dual of problem X. We can notice that this is the (P) problem from task 2), so we need to firstly extract the vectors and the matrix:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad c = \begin{bmatrix} -3 \\ -4 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 3 & 3 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 1 \end{bmatrix}$$

Than we can simply use the formula shown in task 2) to calculate the dual which now looks like the (D) problem from task 2). By evaluating the matrix multiplications we get the requested dual problem:

$$\max 4y_1 + 3y_2 + 4y_3$$

$$\begin{aligned} 3y_1 + 3y_2 + y_3 + s_1 &= -3 \\ 3y_1 + y_2 + 4y_3 + s_2 &= -4 \\ 3y_1 + s_3 &= 0 \\ y_2 + s_4 &= 0 \\ y_3 + s_5 &= 0 \\ s_1, s_2, s_3, s_4, s_5 &\geq 0 \\ y_1, y_2, y_3 &\in \mathbb{R} \end{aligned}$$

6)

Since the elements of x and s are strictly positive, we only need to check if $Ax = b$ and $A^T y + s = c$. We confirmed that using Pyhthon and with that we can claim that these vectors are strictly feasible solutions to both problem X and it's dual.

7)

To show that the vectors are a good starting solution, we need to check the invariants from [1] and the lectures. We already checked the primal and dual feasibility in the previous task, so we only need to satisfy the third one:

$$\sigma^2 = \sum_{i=1}^{m} \left(\frac{x_i s_i}{\mu} - 1 \right)^2 \leq \frac{1}{4}$$

By picking μ ad the average of $x_i s_i$ and calculating the left-hand size we get 0.0016, which easily satisfies the condition. Picking $\mu = 1$ is also appropriate since we get 0.0025.

References

[1] Kurt Mehlhorn and Sanjeev Saxena. A still simpler way of introducing interior-point method for linear programming. *Computer Science Review*, 22:1–11, 2016.