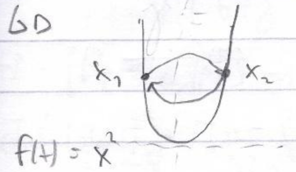
THEORETICAL PART

GD HW 2

1 Sequence $\{x_i\}_i$, p-periodic, $N, p \in \mathbb{N}$:
$x_{N+j} = x_{N+j+p}, \forall j \in \mathbb{N}$
- strictly convex function, $\gamma, \mu > 0$, x_1 starting point

a) GD



$\boxed{x_1 = -1}$

$x_2 = x_1 - \gamma \cdot 2x_1$ ①
$x_1 = x_2 - \gamma 2 x_2$ ②

Yes if we pick the starting point $x_1 = -1$ and $\gamma = 1$ we get

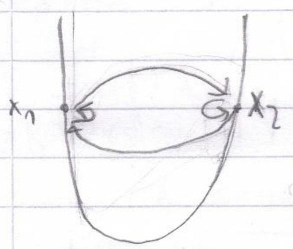
$X_1 \rightleftarrows X_2$

$x_2 = -1 + 2\gamma$ ①
$-1 = x_2 - 2\gamma x_2$ ②

$\cancel{-1} = \cancel{-1} + 2\gamma - 2\gamma(-1 + 2\gamma) =$
$2\gamma + 2\gamma - 4\gamma^2 = 0 \Rightarrow 2\gamma(1 - \gamma) = 0$
$\Rightarrow \gamma = 1 \quad (\gamma > 0)$

POLYAK GD

b) $f(x) = x^2 \quad f'(x) = 2x$



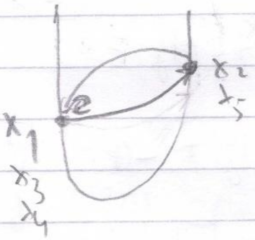
$\boxed{x_1 = -1 \quad \gamma = 1}$

$x_2 = -1 + 2\gamma = 1$
$x_3 = 1 - 2\gamma + \mu \cdot (1 + 1) = 1 - 2\gamma + 2\mu = x_2$
$\cancel{1} - 2\gamma + 2\mu = \cancel{1}$
$\boxed{\mu = \frac{1}{2} 2\gamma = \gamma = 1}$

$x_2 = -1 + 2 = 1$
$x_3 = 1 - 2 + 2 = 1$
$x_4 = 1 - 2 + 1 \cdot (1 - 1) = -1$
$x_5 = -1 + 2 + 1 \cdot (-1 - 1) = -1$
$x_6 = -1 + 2 + 1 \cdot (-1 + 1) = 1$
$\vdots$

However the previous is 4-periodic, Let's find 3-periodic

1B) POLYAKa



$f(x)=x^2$ $f'=2x$

$\boxed{x_1=-1}$

$x_2=-1+2\gamma$

$x_3 = x_2 - 2\gamma x_2 + \mu(x_2-x_1) = -1$

$x_2(1-2\gamma)+\mu(x_2+1) = -1$

$x_4 = x_3 - 2\gamma x_3 + \mu(x_3-x_2) = x_2$

$-1+2\gamma+\mu(-1-x_2) = x_2$

① $x_2(1-2\gamma)+\mu(x_2+1)=-1$

② $-1+2\gamma+\mu(-1-x_2)=x_2$

③ $x_2 = -1+2\gamma$

① $(-1+2\gamma)(1-2\gamma)+\mu(-1+2\gamma+1) = \boxed{-1+4\gamma-4\gamma^2+2\mu\gamma=-1}$

② $-\cancel{1}+\cancel{2\gamma}+\mu(-2\gamma) = -\cancel{1}\ 0 \Rightarrow \boxed{\mu=1}$

$\Rightarrow -\cancel{1}+4\gamma-4\gamma^2+2\gamma=-\cancel{1}$ $6\gamma-4\gamma^2=0$

$2\gamma(3-2\gamma)=0$

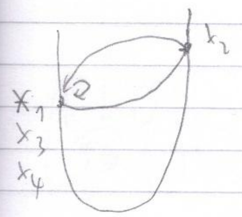
$\boxed{\gamma=\frac{3}{2}}$

Try:

$x_2=-1+2\cdot\frac{3}{2}=2$, $x_3 = 2-\frac{3}{2}\cdot 2\cdot 2+3=-1$

$x_4 = -1+2\cdot\frac{3}{2}+1(-1-2) = -1+3-3=-1$ ✓✓

1c) Nesterov



$\boxed{x_1 = -1}$ $\boxed{x_2 = -1 + 2\gamma}$

$x_3 = \cancel{-1} + 2\gamma - 2\gamma \cdot (-1 + 2\gamma + \mu(2\gamma)) + 2\mu\gamma = \cancel{-1}$

$2\gamma + 2\gamma - 4\gamma^2 - 4\mu\gamma^2 + 2\mu\gamma = 0$ $\boxed{1}$

$x_4 = -1 - 2\gamma(-1 + \mu(-2\gamma)) - 2\mu\gamma = -1$

$\cancel{-1} + 2\gamma + 4\mu\gamma^2 - 2\mu\gamma = \cancel{-1}$

$\boxed{1}$ $2\gamma - 2\gamma^2 - 2\mu\gamma^2 + \mu\gamma = 0 \;/ : \gamma \Rightarrow 2 - 2\gamma - \cancel{2\mu\gamma} + \mu = 0$

$\boxed{2}$ $\gamma + 2\mu\gamma - \mu = 0 \;/ : \gamma \Rightarrow 1 + \cancel{2\mu\gamma} - \mu = 0$)+

$3 - 2\gamma = 0$ $\boxed{\gamma = \frac{3}{2}}$

$\frac{3}{2} + \cancel{2}\mu \cdot \frac{3}{\cancel{2}} - \mu = 0 \Rightarrow 2\mu = -\frac{3}{2}$

$\boxed{\mu = -\frac{1}{2}}$ //

Completly different $f \Rightarrow f = \begin{cases} |x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$ $f' = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$

$x_1 = -1$

$x_2 = -1 - \gamma(-1) = -1 + \gamma = 1 \Rightarrow \boxed{\gamma = 2}$

$x_3 = 1 - \gamma f'(1 + \mu(2)) + \mu \cdot 2 =$

$= 1 - 2 + 2\mu = 1 \Rightarrow \boxed{\mu = 1}$

~~x_4~~

Let's check:

$x_1 = -1$

$x_2 = -1 - 2 \cdot f'(-1) = -1 + 2 = 1$

$x_3 = 1 - 2 \cdot f'(\underbrace{1 + \mu \cdot 2}_{3}) + \underbrace{\mu \cdot 2}_{2} = 1 - 2 + 2 = 1$

$x_4 = 1 - 2 \cdot f'(1) = 1 - 2 = -1$

⋮

However this is not strictly convex
And it is also 4-periodic

$\boxed{2}$ $f(x,y,z) = x^2 + 2y^2 - 2yz + 4z^2 + 3x - 4y + 5z =$

$$Df = [2x + 3,\ 4y - 2z - 4,\ -2y + 8z + 5]$$

$$H_f = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 8 \end{bmatrix}$$ it's clearly PD

$\det(H) = 2 \cdot (4 \cdot 8 - 4) > 0$

eigenvalues:

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & -2 \\ 0 & -2 & 8-\lambda \end{vmatrix} = (2-\lambda) \cdot ((4-\lambda) \cdot (8-\lambda) - 4) =$$

$$= (2-\lambda)(32 - 12\lambda + \lambda^2 - 4) =$$

$$= (2-\lambda)(\lambda^2 - 12\lambda + 28) = 0$$

$\lambda_1 = 2$ $\quad \lambda_{2,3} = \dfrac{12 \pm \sqrt{144 - 112}}{2} = \dfrac{12 \pm \sqrt{32}}{2} = \dfrac{12 \pm 4\sqrt{2}}{2} = 6 \pm 2\sqrt{2}$

$\boxed{\beta = 6 + 2\sqrt{2}}$ $\boxed{\alpha = 2}$

$$\boxed{\sqrt{\mu} = \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\beta} + \sqrt{\alpha}} \quad \gamma = \frac{4}{(\sqrt{\alpha} + \sqrt{\beta})^2}}$$

Programming part

I. Problems 3,4, and 5

In this section we implemented gradient descent, Polyak gradient descent, Nesterov Gradient descent, AdaGrad gradient descent, the Newton method, the BFGS method and the L_BFGS method. We tested all of the listed methods on three different functions and 2 different starting points(for each function). For each function we tested which method performs best in 2,5,10 and 100 steps. We also tested which performs best in 0.1, 1 and 2 seconds.

A. Function 1

The first funcion was given as:

$$f_1(x) = (x_0 - x_2)^2 + (2x_1 + x_2)^2 + (4x_0 - 2x_1 + x_2)^2 + x_0 + x_1$$

with the gradient of:

$$\nabla f_1(x) = \begin{bmatrix} 2(x_0 - x_2) + 8(4x_0 - 2x_1 + x_2) + 1 \\ 4(2x_1 + x_2) - 4(4x_0 - 2x_1 + x_2) + 1 \\ 6x_0 + 6x_2 \end{bmatrix}$$

and the Hessian matrix:

$$\nabla^2 f_1(x) = \begin{bmatrix} 34 & -16 & 6 \\ -16 & 16 & 0 \\ 6 & 0 & 6 \end{bmatrix}$$

This function has a nice constant positive definite Hessian so none of the functions really have problems to converge to the minimum. Note that for all the functions we used a learning rate (for gradient descent methods) of 0.001, except for the AdaGrad where it was set to 1. We set the momentum coefficient to 0.5. The m in the L-BFGS method was set to 10. We set our convergence condition to be where the difference in our positions is small enough. These conditions hold throughout all three functions.

In Table I we can see the results when setting the methods to different amounts of maximum iterations. Note that S1 and S2 denote the different starting points. Also note that Max S denotes the maximum number of steps. We can see that for this function the Newton and quasi-Newton methods converge very fast in comparison to the gradient descent methods, which is not suprising since this function is quadratic. Comparing just the gradient descent methods, we can see that AdaGrad is by far the best, followed by Polyak and Nesterov methods, and lastly the normal gradient descent which shows the slowest convergence.

We also tested the methods when limited to the specified time frames, however for this function all of the methods converged already in the smallest time frame. The results of the final convergence of methods is presented in Table II. We can see that all the methods converged to the same minimum however the Newton and quasi-Newton methods used considerabely less iterations to do so. Among the gradient descent methods, we can conclude the same as for the previous table.

Method	**Max S**	**f1 Value**		**Steps**	
		S1	**S2**	**S1**	**S2**
GD	2	-0.0039	10.2427	2	2
	5	-0.0096	9.2779	5	5
	10	-0.0185	8.0179	10	10
	50	-0.0713	3.7972	50	50
	100	-0.1114	1.9365	100	100
Polyak	2	-0.0049	10.0584	2	2
	5	-0.0153	8.3858	5	5
	10	-0.0317	6.4749	10	10
	50	-0.1109	1.9411	50	50
	100	-0.1532	0.6071	100	100
Nesterov	2	-0.0049	9.8163	2	2
	5	-0.0152	8.1940	5	5
	10	-0.0315	6.3939	10	10
	50	-0.1106	1.9378	50	50
	100	-0.1530	0.6091	100	100
AdaGrad	2	24.5474	7.3378	2	2
	5	0.7125	0.5004	5	5
	10	0.0095	-0.1905	10	10
	50	-0.1979	-0.1979	50	50
	100	-0.1979	-0.1979	100	100
Newton	2	-0.1979	-0.1979	1	1
	5	-0.1979	-0.1979	1	1
	10	-0.1979	-0.1979	1	1
	50	-0.1979	-0.1979	1	1
	100	-0.1979	-0.1979	1	1
BFGS	2	-0.1114	5.0201	2	2
	5	-0.1566	0.6331	5	5
	10	-0.1597	0.5838	10	10
	50	-0.1979	-0.1979	20	22
	100	-0.1979	-0.1979	20	22
L-BFGS	2	7.0000	6246.0000	2	2
	5	-0.1806	0.1495	5	5
	10	-0.1979	-0.1979	8	10
	50	-0.1979	-0.1979	8	11
	100	-0.1979	-0.1979	8	11

Table I
Performance of various optimization methods with different numbers of maximum steps and starting points for the first function

Method	**S1: f1**	**S1: Steps**	**S2: f1**	**S2: Steps**
GD	-0.1979	3142	-0.1979	3611
Polyak	-0.1979	1561	-0.1979	1794
Nesterov	-0.1979	1565	-0.1979	1798
AdaGrad	-0.1979	101	-0.1979	79
Newton	-0.1979	1	-0.1979	1
Bfgs	-0.1979	20	-0.1979	22
L-Bfgs	-0.1979	8	-0.1979	11

Table II
Performance of various optimization methods with different starting points for the first function

B. Function 2

The second function was given as:

$$f_2(x) = (x_1 - 1)^2 + (x_2 - 1)^2 + 100 \cdot (x_2 - x_1^2)^2 + 100 \cdot (x_3 - x_2^2)^2$$

with the gradient:

$$\nabla f_2(x) = \begin{bmatrix} 2(x_1 - 1) - 400x_1(x_2 - x_1^2) \\ 2(x_2 - 1) + 200(x_2 - x_1^2) - 400x_2(x_3 - x_2^2) \\ 100(x_3 - x_2^2) \end{bmatrix}$$

and the hessian matrix:

$$H_f(x) = \begin{bmatrix} 1200x_1^2 - 400x_2 + 2 & -400x_1 & 0 \\ -400x_1 & 1200x_2^2 - 400x_3 + 202 & -400x_2 \\ 0 & -400x_2 & 200 \end{bmatrix}$$

This function is not as simple as the previous one. In Table III we can see that all the methods, including the newton and the quasi-newton mtehods cannot converge even in 100 steps. In this case the performance of the gradient descent methods are closer to the Newton and quasi-Newton methods.

Method	**Max S**	**f2 Value**		**Steps**	
		S1	**S2**	**S1**	**S2**
GD	2	0.0743	4.9321	2	2
	5	0.0151	4.2101	5	5
	10	0.0149	4.1956	10	10
	100	0.0143	4.1571	100	100
Polyak	2	0.9840	6.6213	2	2
	5	0.0199	4.3692	5	5
	10	0.0262	4.1973	10	10
	100	0.0131	4.1104	100	100
Nesterov	2	160.9209	172.8538	2	2
	5	58.8873	73.0538	5	5
	10	5.6640	34.8767	10	10
	100	2.8777	1.2016	100	100
AdaGrad	2	100.2114	11.5469	2	2
	5	66.6059	4.8448	5	5
	10	0.2681	4.1950	10	10
	100	0.1661	4.0784	100	100
Newton	2	8.2461	17.5259	2	2
	5	1.1529	2.5068	5	5
	10	0.3387	0.6023	10	10
	100	0.0867	0.2468	100	100
BFGS	2	0.2435	4.2265	2	2
	5	0.0166	4.2030	5	5
	10	0.0149	4.2002	10	10
	100	0.0557	1.7524	100	100
L-BFGS	2	1.922×10^{10}	6.042×10^{10}	2	2
	5	0.3672	4.5203	5	5
	10	0.0178	4.2191	10	10
	100	0.0024	0.4304	100	100

Table III
PERFORMANCE OF VARIOUS OPTIMIZATION METHODS WITH DIFFERENT NUMBERS OF MAXIMUM STEPS AND STARTING POINTS FOR THE SECOND FUNCTION

Next, we can take a look at Table IV, where we see the performance of the methods for different maximum times. Here a change in performance is quite visible when we limit the time, especially for the Newton method, where if we observe the number of iterations, we can see that it increases with each time-span, meaning that it might not have converged even in the longest time span. The BFGS and L-BFGS methods are the only ones that manage to converge in the shortest time span.

Method	**Max Time (s)**	**f2 Value**		**Steps**	
		S1	**S2**	**S1**	**S2**
GD	0.1	7.9768×10^{-5}	2.1700×10^{-3}	10446	10428
	1	3.2708×10^{-8}	3.2679×10^{-8}	24966	30730
	2	3.2708×10^{-8}	3.2679×10^{-8}	24966	30730
Polyak	0.1	1.2445×10^{-6}	6.8638×10^{-6}	9055	10403
	1	3.2682×10^{-8}	3.2678×10^{-8}	12430	15350
	2	3.2682×10^{-8}	3.2678×10^{-8}	12430	15350
Nesterov	0.1	2.6412×10^{-5}	2.2838×10^{-6}	7943	9851
	1	3.2664×10^{-8}	3.2669×10^{-8}	14134	13784
	2	3.2664×10^{-8}	3.2669×10^{-8}	14134	13784
AdaGrad	0.1	2.0800×10^{-2}	2.9082×10^{0}	1517	1481
	1	3.6009×10^{-9}	1.1398×10^{-8}	13498	15287
	2	3.6010×10^{-9}	4.6223×10^{-9}	13498	15915
Newton	0.1	9.3400×10^{-2}	2.4997×10^{-1}	3487	3506
	1	3.0340×10^{-1}	2.2597×10^{-1}	31966	32128
	2	1.8400×10^{-1}	3.8305×10^{-1}	67541	66239
BFGS	0.1	1.9391×10^{-15}	4.7094×10^{-16}	319	188
	1	1.9391×10^{-15}	4.7094×10^{-16}	319	188
	2	1.9391×10^{-15}	4.7094×10^{-16}	319	188
L-BFGS	0.1	5.1612×10^{-11}	2.2550×10^{-8}	369	509
	1	5.1612×10^{-11}	2.2550×10^{-8}	369	509
	2	5.1612×10^{-11}	2.2550×10^{-8}	369	509

Table IV
OPTIMIZATION RESULTS FOR THE SECOND FUNCTION FOR DIFFERENT MAX TIMES AND STARTING POINTS.

C. Function 3

The last funcion is given as:

$$f_3(x) = (1.5 - x_1 + x_1 x_2)^2 + (2.25 - x_1 + x_1 x_2^2)^2 + (2.625 - x_1 + x_1 x_2^3)^2$$

The gradient and Hessian will be skipped because they cannot fit on this page.
For this function there is a big difference between the starting points. For the first one the all the methods, except for the Newton method actually manage some progress (the quasi-Newton methods even converge nicely even without many steps), while for the second starting point none of the functions (except for the AdaGrad) actually manage to decrease much. This is the the consequence of the gradient being really large in that area. Note that even we would have to lower the learning rate by a lot for the gradient descent methods to be able to achive some kind of descent. We can observe the results in Table V. For the gradient descent methods we might be able to solve this by setting a really low learning rate, however, then they would converge really slowly when they would leave the area with the large gradients. For the Polyak and Nesterov methods, maybe taking a larger momentum with the smaller learning rate might also help, however the momentum also can't be too large.

Method	**Max S**	**f3 Value**		**Steps**	
		S1	**S2**	**S1**	**S2**
GD	2	1.2808×10^{1}	1.5874×10^{106}	2	2
	5	1.1252×10^{1}	nan	5	5
	10	9.4597	nan	10	10
	100	2.0613	nan	100	100
Polyak	2	1.2476×10^{1}	1.5874×10^{106}	2	2
	5	9.8833	nan	5	5
	10	7.3880	nan	10	10
	100	8.9470×10^{-1}	nan	100	100
Nesterov	2	9.0595	2.5933×10^{185}	2	2
	5	7.4694	nan	5	5
	10	6.2562	nan	10	10
	100	9.4610×10^{-1}	nan	100	100
AdaGrad	2	7.8155×10^{-1}	7.9649×10^{3}	2	2
	5	2.9340×10^{-1}	1.6211×10^{4}	5	5
	10	8.5844×10^{-2}	4.1076×10^{3}	10	10
	100	1.5678×10^{-5}	3.0342×10^{2}	100	100
Newton	2	1.4203×10^{1}	1.4223×10^{4}	1	2
	5	1.4203×10^{1}	3.4089×10^{2}	1	5
	10	1.4203×10^{1}	1.4208×10^{1}	1	10
	100	1.4203×10^{1}	1.4203×10^{1}	1	12
BFGS	2	7.1161	8.4329×10^{10}	2	2
	5	4.3680	inf	5	5
	10	4.2374	nan	10	10
	100	4.6611×10^{-18}	nan	31	100
L-BFGS	2	3.6684×10^{8}	8.9245×10^{41}	2	2
	5	2.0989×10^{7}	1.7481×10^{5}	5	3
	10	3.1547	1.7481×10^{5}	10	3
	100	1.1897×10^{-14}	1.7481×10^{5}	29	3

Table V
PERFORMANCE OF VARIOUS OPTIMIZATION METHODS WITH DIFFERENT NUMBERS OF MAXIMUM STEPS AND STARTING POINTS FOR THE THIRD FUNCTION

Lastly we can take a look at Table VI. Here we again observe the performance of methods with different time limits. As expected from the previous results, for the second starting point only the AdaGrad method manages to get to some kind of minimum while the other methods fail. The only one that acutally doesn't get really high is the Newton method, which gets stuck in some local extreme.
For the first starting point all the methods except for the Newton method converge nicely. BFGS and L-BFGS converge especially quickly.

Method	**Max Time (s)**	**f3 Value**		**Steps**	
		S1	**S2**	**S1**	**S2**
GD	0.1	2.42548×10^{-4}	nan	7593	9214
	1	1.65662×10^{-8}	nan	23234	83842
	2	1.65662×10^{-8}	nan	23234	100000
Polyak	0.1	5.51462×10^{-6}	nan	6815	8012
	1	1.65567×10^{-8}	nan	11607	77093
	2	1.65567×10^{-8}	nan	11607	100000
Nesterov	0.1	1.22775×10^{-5}	nan	6170	6220
	1	1.65714×10^{-8}	nan	11617	64429
	2	1.65714×10^{-8}	nan	11617	100000
AdaGrad	0.1	1.08487×10^{-12}	22.33527	318	1168
	1	1.08487×10^{-12}	1.08831	318	13160
	2	1.08487×10^{-12}	0.227015	318	27017
Newton	0.1	14.203125	14.203125	1	12
	1	14.203125	14.203125	1	12
	2	14.203125	14.203125	1	12
BFGS	0.1	4.66106×10^{-18}	nan	31	3121
	1	4.66106×10^{-18}	nan	31	33981
	2	4.66106×10^{-18}	nan	31	71794
L-BFGS	0.1	1.18971×10^{-14}	1.74813×10^{5}	29	3
	1	1.18971×10^{-14}	1.74813×10^{5}	29	3
	2	1.18971×10^{-14}	1.74813×10^{5}	29	3

Table VI
OPTIMIZATION RESULTS FOR THE THIRD FUNCTION FOR DIFFERENT MAX TIMES AND STARTING POINTS

II. Problem 6

For this problem we fitted a linear regression to the data described in the instructions using the least squares method. So the function we minimized is given by:

$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}(kx_i + n - y_i)^2$$

and has the gradient:

$$\nabla L(k,n) = \begin{bmatrix} \frac{\partial L}{\partial k} \\ \frac{\partial L}{\partial n} \end{bmatrix} = \frac{2}{n}\sum_{i=1}^{n}\begin{bmatrix} (kx_i + n - y_i)x_i \\ (kx_i + n - y_i) \end{bmatrix}$$

We compared different optimizer methods (GD, SGD, Newton, BFGS, L-BFGS) and different number of data points. On Figure 1 we can see the fitted linear regression for 50, 100 and 1000 data points.

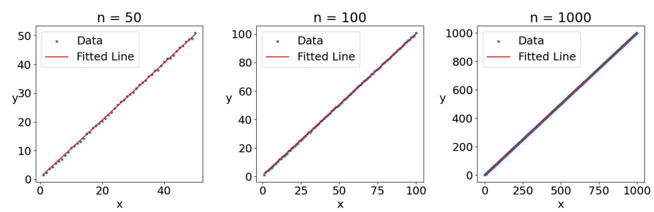


Figure 1. PGD for different learning rates and domains

Lastly, we can see the results for different optimizers and numbers of data points in Table VII. We can see that for smaller datasets, the Newton and quasi-Newton methods dominate, while for larger datasets the SGD is by far the quickest, with the normal gradient descent being the slowest. Note that here we set the learning rate to be: $\frac{1}{n^2}$, since the gradients are a summation, and are therefore larger with more points, and this formula proved to work nicely.

n	**Method**	**Iterations**	**Time (s)**
n = 50	GD	6769	0.2575
	SGD	351	0.0027
	Newton	1	0.0002
	BFGS	5	0.0002
	LBFGS	7	0.0003
n = 100	GD	11	0.0004
	SGD	223	0.0018
	Newton	1	0.0002
	BFGS	3	0.0001
	LBFGS	7	0.0004
n = 1000	GD	10	0.0005
	SGD	25	0.0002
	Newton	1	0.0002
	BFGS	5	0.0003
	LBFGS	3	0.0001
n = 10000	GD	10	0.0013
	SGD	12	0.0001
	Newton	1	0.0005
	BFGS	3	0.0003
	LBFGS	3	0.0002
n = 100000	GD	10	0.0084
	SGD	6	0.0001
	Newton	1	0.0026
	BFGS	6	0.0026
	LBFGS	3	0.0012
n = 1000000	GD	10	0.4266
	SGD	5	0.0001
	Newton	1	0.0945
	BFGS	4	0.0892
	LBFGS	4	0.0885

Table VII
Performance of Different Optimization Methods for the linear regression task