Mathematics 2
Homework for Part 3

The solutions are to be submitted as one .zip file to the appropriate mailbox on ucilnica. The solutions should contain a .pdf file containing a clear and well described procedure and a code. The homework is worth 3 points. The written exam is worth additional 7 points. You should submit your solutions before taking the written exam.

1 Numerical integration vs Monte Carlo integration

Let X be a random variable following the (a, b, c)-PERT distribution with $b > a$, $c > a$, $a, b, c \in \mathbb{R}$, and the pdf

$$p(x) = \frac{(x-a)^{\alpha-1}(c-x)^{\beta-1}}{B(\alpha,\beta)(c-a)^{\alpha+\beta-1}}$$

supported on $[a, c]$, where

$$\alpha = 1 + 4\frac{b-a}{c-a}, \qquad \beta = 1 + 4\frac{c-b}{c-a}$$

and $B(x, y)$ is the Beta function. Let $a = 0$, $b = 10$ and $c = 100$. It turns out that

$$E[X] = \frac{a+4b+c}{6} \quad \text{and} \quad \mathrm{var}[X] = \frac{(E[X]-a)(c-E[X])}{7}.$$

1. Use the trapezoid rule to estimate $E[X]$ to 4 decimal places. How many function evaluations you need to do?
2. Use the Central limit theorem to estimate the number of points required to obtain the estimate of $E[X]$ to 2 decimal places using Monte Carlo integration when sampling from the uniform distribution on $[a, c]$.
3. Verify the estimates above on a few numerical samples.
4. Compare and comment on the results obtained by adaptive trapezoid rule and the Monte Carlo integration.

2 Importance sampling

We would like to compute the integral

$$I = \int_0^1 x^{-3/4} \cdot e^{-x}\, dx.$$

1. Plot the integrand $f(x) = x^{-3/4} \cdot e^{-x}$ on $[0, 1]$.
2. Sampling from the uniform distribution estimate I using the Monte Carlo integration with samples of size $n = 10^7$. Compute the average and the standard deviation for 10 samples.
3. Estimate I as $E_q(\frac{f}{q})$ for $q(x) = cx^{-3/4}$ by the following steps:
   (a) Determine c so that q is a density.
   (b) Sample from q using inversion sampling.
   (c) Repeat step (2) above.
4. Compare and comment the results obtained by both methods.

3 Metropolis–Hastings algorithm

Let X be a random variable following the (α, η)-Weibull distribution with the parameters $\alpha, \eta \in (0, \infty)$, support $x \in (0, \infty)$, and the pdf

$$p(x|\alpha, \eta) = \alpha\eta x^{\alpha-1} \exp(-x^{\alpha}\eta).$$

Let the prior distribution $\pi(\alpha, \eta)$ be proportional to $\exp(-\alpha - 2\eta) \cdot \eta$. We observe the data $\vec{x} = (x_1, x_2, x_3, x_4) = (0.3, 0.5, 0.75, 0.4)$. We would like to approximate the posterior distribution for α, η:

$$p(\alpha, \eta|\vec{x}) \propto \underbrace{p(\vec{x}|\alpha, \eta)\pi(\alpha, \eta)}_{f(\alpha,\eta)}.$$

To do this we need to sample from the distribution with density proportional to $f(\alpha, \eta)$.

Compute the mean and the variance of α and η using Metropolis–Hastings algorithm with:

1. a multivariate normal proposal. Use mean (α, η), while choose few different covariance matrices.
2. a proposal distribution

$$q(\alpha', \eta' \mid \alpha, \eta) = \frac{1}{\alpha\eta} \exp(-\frac{\alpha'}{\alpha} - \frac{\eta'}{\eta}).$$

Note that the density

$$q(\alpha', \eta' \mid \alpha, \eta) = \underbrace{\left(\frac{1}{\alpha}\exp(-\frac{\alpha'}{\alpha})\right)}_{\alpha' \sim \mathrm{Exp}(\frac{1}{\alpha})} \cdot \underbrace{\left(\frac{1}{\eta}\exp(-\frac{\eta'}{\eta})\right)}_{\eta' \sim \mathrm{Exp}(\frac{1}{\eta})}$$

is the product of the densities of two exponential distributions, which means the proposals α', η' are independent. Hence, to sample α' and η' you need to sample from $\mathrm{Exp}(\frac{1}{\alpha})$ and $\mathrm{Exp}(\frac{1}{\eta})$, respectively. The latter can be done using inversion sampling.

For each of the above scenarios:

1. Generate 5 independent chains of 1000 samples.
2. Apply standard MCMC diagnostics for each algorithm/run (traceplot for each parameter and all chains at the same time), autocovariance and the ESS.
3. Compare both algorithms in sampling from the target distribution. Include a discussion of how difficult it was to tune the MCMC parameters.
4. Estimate the probability $(\alpha, \eta) \in [1.3, \infty) \times [1.3, \infty)$ using a proposal distribution q.