THEORETICAL PART

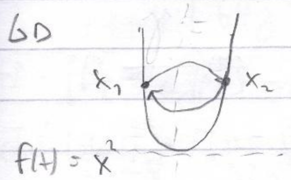
GD HW 2

1 Sequence $\{x_i\}_i$, p-periodic, $N, p \in \mathbb{N}$:

$x_{N+j} = x_{N+j+p}, \forall j \in \mathbb{N}$

strictly convex function, $\gamma, \mu > 0$, x_1 starting point

a) GD



$$\boxed{x_1 = -1}$$

$$x_2 = x_1 - \gamma \cdot 2 x_1 \quad (1)$$
$$x_1 = x_2 - \gamma 2 x_2 \quad (2)$$

Yes if we pick the starting point $x_1 = -1$ and $\gamma = 1$ we get

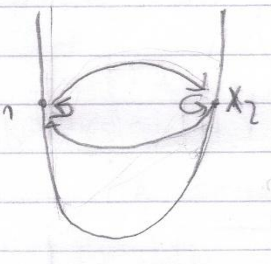
$$x_1 \rightleftarrows x_2$$

$$x_2 = -1 + 2\gamma \quad (1)$$
$$-1 = x_2 - 2\gamma x_2 \quad (2)$$

$$\cancel{-1} = \cancel{-1} + 2\gamma - 2\gamma(-1 + 2\gamma) =$$
$$2\gamma + 2\gamma - 4\gamma^2 = 0 \Rightarrow 2\gamma(1 - \gamma) = 0$$
$$\Rightarrow \gamma = 1 \quad (\gamma > 0)$$

POLYAK GD

b) $f(x) = x^2 \quad f'(x) = 2x$



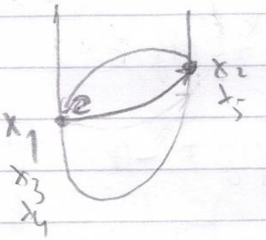
$$\boxed{x_1 = -1 \quad \gamma = 1}$$

$$x_2 = -1 + 2\gamma = 1$$
$$x_3 = 1 - 2\gamma + \mu \cdot (1+1) = 1 - 2\gamma + 2\mu = x_2$$
$$\cancel{1} - 2\gamma + 2\mu = \cancel{1}$$
$$\boxed{\mu = \tfrac{1}{2} 2\gamma = \gamma = 1}$$

$$x_2 = -1 + 2 = 1$$
$$x_3 = 1 - 2 + 2 = 1$$
$$x_4 = 1 - 2 + 1 \cdot (1 - 1) = -1$$
$$x_5 = -1 + 2 + 1 \cdot (-1 - 1) = -1$$
$$x_6 = -1 + 2 + 1 \cdot (-1 + 1) = 1$$
$$\vdots$$

However the previous is 4-periodic, Let's find 3-periodic

1B) POLKAKA



$f(x) = x^2 \qquad f' = 2x$

$\boxed{x_1 = -1}$

$x_2 = -1 + 2\gamma$

$x_3 = x_2 - 2\gamma x_2 + \mu(x_2 - x_1) = -1$

$x_2(1-2\gamma) + \mu(x_2 + 1) = -1$

$x_4 = x_3 - 2\gamma x_3 + \mu(x_3 - x_2) = -1$

$-1 + 2\gamma + \mu(-1 - x_2) = -1$

① $x_2(1-2\gamma) + \mu(x_2+1) = -1$

② $-1 + 2\gamma + \mu(-1 - x_2) = -1$

③ $x_2 = -1 + 2\gamma$

① $(-1+2\gamma)(1-2\gamma) + \mu(-1+2\gamma+1) = \boxed{-1 + 4\gamma - 4\gamma^2 + 2\mu\gamma = -1}$

② $\cancel{-1} + 2\gamma + \mu(-2\gamma) = \cancel{-1}\ 0 \implies \boxed{\mu = 1}$

$\implies \cancel{-1} + 4\gamma - 4\gamma^2 + 2\gamma = \cancel{-1} \qquad 6\gamma - 4\gamma^2 = 0$

$2\gamma(3 - 2\gamma) = 0$

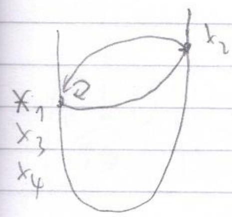
$\boxed{\gamma = \frac{3}{2}}$

Try:

$x_2 = -1 + 2 \cdot \frac{3}{2} = 2, \quad x_3 = 2 - \frac{3}{\cancel{2}}\cancel{2} \cdot 2 + 3 = -1$

$x_4 = -1 + 2 \cdot \frac{3}{2} + 1(-1-2) = -1 + 3 - 3 = -1$ ✓✓

1c) Nesterov



$\boxed{x_1 = -1}$ $\boxed{x_2 = -1+2\gamma}$

$x_3 = \cancel{-1}+2\gamma - 2\gamma\cdot(-1+2\gamma+\mu(2\gamma)) + 2\mu\gamma = \cancel{-1}$

$2\gamma + 2\gamma - 4\gamma^2 - 4\mu\gamma^2 + 2\mu\gamma = 0$ $\boxed{1}$

$x_4 = -1 - 2\gamma(-1+\mu(-2\gamma)) - 2\mu\gamma = -1$

$\cancel{-1} + 2\gamma + 4\mu\gamma^2 - 2\mu\gamma = \cancel{-1}$

$\boxed{1}$ $2\gamma - 2\gamma^2 - 2\mu\gamma^2 + \mu\gamma = 0 \ / :\gamma \Rightarrow 2 - 2\gamma - 2\mu\gamma + \mu = 0$

$\boxed{2}$ $\gamma + 2\mu\gamma - \mu = 0 \ / :\gamma \Rightarrow 1 + 2\mu\gamma - \mu = 0$ $)+$

$3 - 2\gamma = 0$ $\boxed{\gamma = \frac{3}{2}}$

$\frac{3}{2} + 2\mu\cdot\frac{3}{\cancel{2}} - \mu = 0 \Rightarrow 2\mu = -\frac{3}{2}$

$\boxed{\mu = -\frac{1}{2}}$ //

Completly different f $\Rightarrow$ $f = \begin{cases}|x|, & x\neq 0\\ 0, & x=0\end{cases}$ $f' = \begin{cases}1, & x>0\\ -1, & x<0\\ 0, & x=0\end{cases}$

$x_1 = -1$

$x_2 = -1 - \gamma f'(-1) = -1+\gamma = 1 \Rightarrow \boxed{\gamma=2}$

$x_3 = 1 - \gamma f'(1+\mu(2)) + \mu\cdot 2 =$

$= 1 - 2 + 2\mu = 1 \Rightarrow \boxed{\mu = 1}$

$\cancel{x_4}$

Let's check:

$x_1 = -1$

$x_2 = -1 - 2\cdot f'(-1) = -1+2 = 1$

$x_3 = 1 - 2\cdot f'(\underbrace{1+\mu\cdot 2}_{3}) + \underbrace{\mu\cdot 2}_{2} = 1-2+2 = 1$

$x_4 = 1 - 2\cdot f'(1) = 1-2 = -1$

⋮

However this is not strictly convex

And it is also 4-periodic

$\boxed{2}$ $f(x,y,z) = x^2 + 2y^2 - 2yz + 4z^2 + 3x - 4y + 5z =$

$Df = [2x+3,\ 4y - 2z - 4,\ -2y + 8z + 5]$

$$H_f = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 8 \end{bmatrix}$$ it's clearly PD

$\det(H) = 2\cdot(4\cdot 8 - 4) > 0$

eigenvalues:

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & -2 \\ 0 & -2 & 8-\lambda \end{vmatrix} = (2-\lambda)\cdot((4-\lambda)\cdot(8-\lambda) - 4) =$$

$= (2-\lambda)(32 - 12\lambda + \lambda^2 - 4) =$

$= (2-\lambda)(\lambda^2 - 12 + 28) = 0$

$\boxed{\beta = 6 + 2\sqrt{2}}$ $\boxed{\alpha = 2}$ $\lambda_1 = 2$ $\lambda_{2,3} = \frac{12 \pm \sqrt{144 - 112}}{2} = \frac{12 \pm \sqrt{32}}{2} =$

$\boxed{\sqrt{\mu} = \frac{\sqrt{\beta} - \sqrt{\alpha}}{\sqrt{\beta} + \sqrt{\alpha}} \quad \gamma = \frac{4}{(\sqrt{\alpha} + \sqrt{\beta})^2}}$ $= \frac{12 \pm 4\sqrt{2}}{2} = 6 \pm 2\sqrt{2}$