

Artificial Neural Networks

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I. IMPLEMENTATION OF FULLY CONNECTED NEURAL NETWORK FOR CLASSIFICATION

A. Implementation details

We implemented a fully connected neural network with an arbitrary number of layers for a classification task. We applied the sigmoid activation function at each layer, except for the input and output layers. Since the task involves multi-class classification, we used the softmax function on the output layer (applied to the logits).

To initialize the weights we used the Xavier(Glorot) normal initialization:

$$W \sim \mathcal{N}\left(0, c \cdot \frac{2}{n_{\text{in}} + n_{\text{out}}}\right)$$

To train the network, we used an approach where gradients are computed separately for each training example and then accumulated. We used the cross-entropy loss (log loss) as the criterion. After computing the loss, we applied backpropagation to calculate the gradients and then performed standard gradient descent to update the model parameters.

To correctly perform backpropagation, the following gradients were calculated. First, the gradient of the loss function with respect to the logits was computed. This was done by combining the gradient of the loss with respect to the probabilities and the gradient of the probabilities with respect to the logits (i.e., the gradient of the softmax):

$$\frac{\partial L}{\partial z_j} = \begin{cases} p_j - 1 & \text{if } j = y_i \\ p_j & \text{otherwise} \end{cases}$$

where y_i is the true class, z_j are the logits and p_j are the probabilities of each class.

Next, the gradients for the weights and biases in the linear layer were calculated:

$$\frac{\partial L}{\partial \mathbf{W}^{(k)}} = \delta^{(k)} \cdot (\mathbf{a}^{(k-1)})^T \quad \text{and} \quad \frac{\partial L}{\partial \mathbf{b}^{(k)}} = \delta^{(k)}$$

where $\delta^{(k)}$ is the gradient of the pre-activations in the current layer and $\mathbf{a}^{(k-1)}$ is the vector of activations in the previous layer.

Lastly, we needed to compute the gradient of the activations and consequently the pre-activations (note that in this case, k refers to the next layer, so it corresponds to $k-1$ in the previous equation):

$$\begin{aligned} \sigma'(z^{(k)}) &= \sigma(z^{(k)}) \cdot (1 - \sigma(z^{(k)})) \\ \delta^{(k)} &= (\delta^{(k+1)} \cdot \mathbf{W}^{(k+1)}) \circ \sigma'(z^{(k)}) \end{aligned}$$

where $\sigma(z^{(k)})$ represents the sigmoid function applied to the pre-activations.

B. Comparison of gradients to numerical gradients

To validate our implementation of backpropagation, we added the option to return both the numerical gradients and the gradients computed through backpropagation at a chosen layer. Our comparison procedure involved computing the gradients for all samples in the dataset (using the provided squares, doughnut, and example datasets), and then comparing

the results. To compute the numerical gradients we used the definition of the derivative:

$$\frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$$

and approximated it using $h = 0.0001$.

We used a network with six hidden layers, each with a different number of activations. We then checked the differences between the numerical and backpropagation-based gradients for both weights and biases. In all cases, the differences were within acceptable bounds—none of the gradient differences exceeded 1×10^{-7} .

C. Fitting the network to the squares and doughnut data

Next, we fitted our implementation to the squares and doughnut datasets to demonstrate that the network is capable of fully learning non-linear patterns.

In Figure 1, we observe that the network successfully fits the doughnut dataset. For this task, we used a network with a single hidden layer containing 3 activation units and a learning rate of 1.

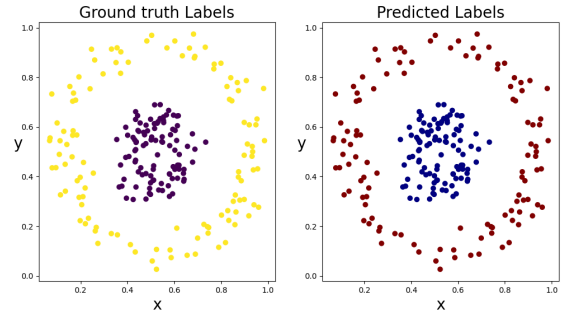


Figure 1. Network predictions fitted on the doughnut dataset.

Similarly, in Figure 2, the network also manages to perfectly fit the squares dataset. This time, we used one hidden layer with 5 activation units, keeping the learning rate at 1.

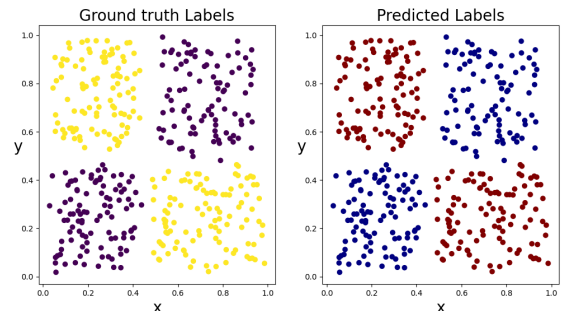


Figure 2. Network predictions fitted on the squares dataset.

II. EXTENDING THE ANN IMPLEMENTATION

A. Adding support for regression

We also implemented the regression counterpart of the same neural network. The key differences in implementation compared to the classification network are outlined below:

- The final layer contains a single output unit instead of one per class.
- No softmax function is applied in the output layer.
- The loss function used is the mean squared error, based on the assumption of a normally distributed target variable.
- The only change in the backpropagation algorithm occurs in the final layer, where the gradient is scaled by a constant factor of 2 (in comparison to the classification network).

B. Adding regularization

We also added the support for L2 regularization for both classification and regression networks. Note that the regularization was only applied to the weights, not to the biases. It was implemented by adding the correct term in gradient descent as shown by the following formula:

$$\mathbf{W}^{(k)} \leftarrow \mathbf{W}^{(k)} - \eta \left(\nabla_{\mathbf{W}^{(k)}} L + 2\lambda \mathbf{W}^{(k)} \right)$$

C. Adding reLU support

Next, we added support for the ReLU activation function. This required modifying the gradient computations, as the ReLU function and its derivative are defined as follows:

$$\begin{aligned} \text{ReLU}(x) &= \max(0, x) \\ \text{ReLU}'(x) &= \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Additionally, we allowed users to customize the activation function used at each layer of the network.

D. Comparing our implementation with PyTorch