Article

The structure of the world economy: absorbing Markov chain approach

Olivera Kostoska [1,†,‡], Viktor Stojkoski [1,‡] and Ljupco Kocarev [2,*]

[1] Affiliation 1; e-mail@e-mail.com
[2] Affiliation 2; e-mail@e-mail.com
* Correspondence: e-mail@e-mail.com; Tel.: (optional; include country code; if there are multiple corresponding authors, add author initials) +xx-xxxx-xxx-xxxx (F.L.)
† Current address: Affiliation 3
‡ These authors contributed equally to this work.

Version February 27, 2020 submitted to Journal Not Specified

Abstract: The expansion of global production networks has raised many important questions about future changes in the world economy and interdependence among countries, including the propagation and effects of economic shocks. Here, we are approaching the structure and lengths of value chains and issues about 'upstreamness' and 'downstreamness' from a completely different perspective. By assigning a random endogenous variable to a network linkage representing the number of intermediate sales/purchases before absorption (final use or value added), the discrete-time absorbing Markov chains proposed here throw a new light on the world input/output networks. The variance of this variable can help assess the risk when shaping the chain length and optimize the level of production. Contrary to what might be expected simply on the basis of comparative advantage, the results reveal that both the input and output chains exhibit the same quasi-stationary product distribution. Finally, the metric we propose about the probability distribution of the global value added provides the policy makers with a useful tool for estimating the resilience of the world trading system and forecasting the macroeconomic developments.

Keywords: world economy; global production networks; discrete-time absorbing Markov chain; quasi-stationary product distribution; global value added matrix

1. Introduction

The ideas of networked economy, pervasive transmission channels, systemic risk and complexity have become increasingly important after the 2008 financial crisis, but nowadays they are major concern on the impact of global trade tensions. Over the last decades, international fragmentation of production has made a huge transformation in geography and dynamics of international trade. Such fragmentation of production activities has given rise to the Global Value Chains (GVCs) and greatly contributed to reinforce the structural interdependence worldwide. The global production networks are very complex, with flows of value-added representing a final outcome of the complex linkages that exist between firms in different industries and countries over time. The evaluation of these linkages calls for developing new tools that go well beyond the appraisal of bilateral gross trade flows. The network analysis and related metrics are extremely important in assessing the complexity of the whole structure of interactions (direct and indirect linkages) in the world economy, whilst the current research is still in infancy.

So far, various macroeconomic models have been developed, ranging from dynamic stochastic general equilibrium modeling to agent-based macroeconomic models. The former assumes, in a standard setting, an economy that is populated by both an infinitely-lived representative household and a representative firm, with homogenous production technology that is hit by exogenous shocks. The latter, on the other hand, considers an economy populated by heterogenous agents, whose far from equilibrium interactions continuously change the structure of the system. Most of the models are, to some extent, descended from the Leontief's work on input-output tables [1], in which firm/agent interactions are mainly characterized by the global production networks. The latter are typically described with the so-called multi-regional input-output (MRIO) models that combine, in a coherent framework, national input-output and trade flow tables. In addition to tracking GVCs, together with other methodologies, such as, for example, life cycle assessment, material flow analysis, MRIO models have been used for sustainability analysis addressing a wide range of policy and research questions regarding the impacts of global production networks [2]. There are several independently constructed global MRIO databases. In this paper, all theoretical results are accompanied with numerical computations of the World Input-Output Database (WIOD) [3].

Basically, our approach is motivated by the recent observations in two strands of theoretical and empirical literature: shock propagation in economic networks and value chain positioning. The production network can work as a channel for propagating shocks throughout the economy. The possibility that substantial aggregate fluctuations may originate from microeconomic shocks has long been abandoned in the literature (see for example [4]). This is mainly due to the 'diversification argument', which states that, in an economy consisting of n industries hit by independent shocks, aggregate fluctuations would have a magnitude that is roughly proportional to $1/\sqrt{n}$, a minor effect at high levels of disaggregation. This argument, however, disregard the input-output linkages between different firms and industries operating as a propagation channel of idiosyncratic microeconomic shocks throughout the economy [5–10]. The other main question of interest is the representation of network-originated macro fluctuations in terms of the economy's structural parameters. In line with the key observations of [5], different roles various sectors play as input suppliers to others may generate sizable aggregate volatility when compared against the standard diversification argument rate. Microeconomic shocks may propagate over the network, but if propagated symmetrically, they would average out, and thus, would have minimal aggregate effects (hence the diversification argument remains applicable). The diversification argument would not hold, however, if intersectoral input-output linkages do not display such symmetries. Put differently, when sectors are highly asymmetric as input suppliers, even with a large number of industries, shocks to sectors that are more important suppliers propagate strongly to the rest of the economy producing significant aggregate fluctuations. Similarly, an industry will take a more 'central' position in the network if it plays a more important role as an input supplier to other central industries, and thus, it will be more influential in determining the aggregate output (see for example the Bonacich centrality). This statement goes in line with the intuition that productivity shocks to an industry with more direct or indirect downstream customers should have sizable aggregate effects.

The second strand of the literature focuses on quantifying the relative production line position of industries/countries (or country-industry pairs) in (global) value chains. In point of fact, if shocks propagate downstream or upstream, the economic condition of a certain industry/country/country-industry pair is largely dependent on its relative position along the global output supply chain and the global input demand chain. The "upstreamness from final consumption" and "downstreamness from primary factors" are two numerical estimates (based on the length of output supply/input demand chains, respectively) that measure the country's/industry's/country-industry pair's position in global production processes [11–14]. Given the importance for policy makers, our first theorem contributes to existing literature by providing mathematical framework for ordering (ranking) the country-industry pairs according to values of these measures. However, two limitations, as argued by [15] may possibly reduce the reliability of 'upstreamness' and 'downstreamness': first, they all begin with an industry's gross output, whilst the production chain should begin with the industry's primary inputs (or the sources of value added).

Second, these measures do not imply each other and might suggest inconsistent positions for the same country-industry pairs.

This paper, by proposing two (input and output) discrete-time absorbing Markov chains, opts to investigate the positions, length of chains and structural interdependence of the world economy from a completely different perspective. The first theoretical linkage between Markov chain and Leontief input-output models was derived by Solow in 1952 [16]. In spite of the great variety of mathematical properties, so far, however, world economic networks have not been fully examined with Markov chain formalism. Here, we introduce a random endogenous variable pointing to the number of intermediate sales/purchases before absorption (final demand/primary inputs or sources of value added). The variance of this variable can serve as a useful guide for policy makers when determining the "optimal level of fragmentation". Further, our results reveal the same quasi-stationary product distribution of both chains. This, rather puzzling finding, indicates that the expected proportion of time spent in a state before absorption does not depend on the network features. Finally, our measures about the probability distribution of value added across countries allows for a more precise determination of the global production chain lengths than any presently available.

2. Materials and Methods

2.1. World Input-Output Database

All of the analyses performed in the main text were done with data gathered from the 2016 release of the World Input-Output Database (WIOD). Central in the WIOD is a time-series of world input-output tables. A world input-output table (WIOT) can be regarded as a set of national input-output tables that are connected with each other by bilateral international trade flows. A WIOT provides a comprehensive summary of all transactions in the global economy between industries and final users across countries. The columns in the WIOT contain information on production processes. When expressed as ratios to gross output, the cells in a column provide information on the shares of inputs in total costs. Such a vector of cost shares is often referred to as a production technology. Products can be used as intermediates by other industries, or as final products by households and governments (consumption) or firms (stocks and gross fixed capital formation). The distribution of the output of industries over user categories is indicated in the rows of the table. An important accounting identity in the WIOT is that gross output of each industry (given in the last element of each column) is equal to the sum of all uses of the output from that industry (given in the last element of each row).

This 2016 release provides an annual time-series of WIOTs from 2000 to 2014. It covers forty-three countries, including all twenty-eight members of the European Union (as of July 1, 2013) and fifteen other major economies. These countries have been chosen by considering both the requirement of data availability of sufficient quality and the desire to cover a major part of the world economy. Together, the countries cover more than 85 per cent of world GDP (at current exchange rates). In addition, a model for the remaining non-covered part of the world economy is estimated, called the "rest of the world" region. The WIOD is structured according to the recent industry and product classification ISIC Rev. 4 (or equivalently NACE Rev. 2). The underlying WIOTs cover 56 industries. More about the included countries and industries can be read in references [3,17]. The dataset itself is available at http://www.wiod.org/home.

As the data has more than 2 dimensions, it should be arranged as multidimensional arrays. Such arrays are usually called tensors. The order of a tensor is the number of dimensions. Vectors (tensors of order one) are denoted by boldface lowercase letters, e.g., $\mathbf{a}$. Matrices (tensors of order two) are denoted by boldface capital letters, e.g., $\mathbf{A}$. Higher-order tensors (order three or higher) are denoted by math calligraphy letters, e.g., $\mathcal{X}$. Scalars are denoted by lowercase or uppercase letters, e.g., a or A. The i-th entry of a vector $\mathbf{a}$ is denoted by a_i, element (i, j) of a matrix $\mathbf{A}$ is denoted by a_{ir} or a_i^r, element (i, j, r) of a third-order tensor $\mathcal{X}$ is denoted by x_{ijr} or x_{ij}^r, and element (i, j, r, s) of a forth-order tensor $\mathcal{Z}$ is denoted by z_{ijrs} (Z_{ijrs}) or z_{ij}^{rs} (Z_{ij}^{rs}).

The WIOD includes detailed data for J countries (indexed by $\hat{\imath}$ or $\hat{\jmath}$) and S sectors (indexed by r or s) organized as two tensors: 4-order tensor $\mathcal{Z} \in \mathbb{R}^{J\times J\times S\times S}$ with entries $z^{rs}_{\hat{\imath}\hat{\jmath}}$ describing the intermediate purchases (input flows) by industry s in country $\hat{\jmath}$ from sector r in country $\hat{\imath}$; and 3-order tensor $\mathcal{F} \in \mathbb{R}^{J\times J\times S}$ with entries $f^{r}_{\hat{\imath}\hat{\jmath}}$ denoting the final use in each country $\hat{\jmath}$ of output originating from sector r in country $\hat{\imath}$. In addition, the WIOD includes matrices $\mathbf{F}$, $\mathbf{X}$, and $\mathbf{W}$, uniquely determined by the tensors $\mathcal{Z}$ and $\mathcal{F}$. The entries of these matrices are

$$f^r_{\hat{\imath}} = \sum_{\hat{\jmath}=1}^{J} f^r_{\hat{\imath}\hat{\jmath}}, \tag{1}$$

$$x^r_{\hat{\imath}} = \sum_{s=1}^{S}\sum_{\hat{\jmath}=1}^{J} z^{rs}_{\hat{\imath}\hat{\jmath}} + f^r_{\hat{\imath}}, \tag{2}$$

$$w^s_{\hat{\jmath}} = x^s_{\hat{\jmath}} - \sum_{r=1}^{S}\sum_{\hat{\imath}=1}^{J} z^{rs}_{\hat{\imath}\hat{\jmath}}. \tag{3}$$

where $x^r_{\hat{\imath}}$ represents the value of gross output originating from sector r in country $\hat{\imath}$; the element $f^r_{\hat{\imath}}$ stands for the value of output from sector r in country $\hat{\imath}$ intended for final consumers worldwide; and $w^s_{\hat{\jmath}}$ indicates the country's $\hat{\jmath}$ value-added employed in the production of an industry s.

An N th-order tensor is an element of the tensor product of N vector spaces, each of which has its own coordinate system. Slices are two-dimensional sections of a tensor, defined by fixing all but two indices. Matricization, also known as unfolding or flattening, is the process of reordering the elements of an N-th order array into a matrix. For instance, a $2\times 2\times 3\times 3$ tensor can be arranged as a 6×6 matrix or a 2×18 matrix, and so on. It is also possible to vectorize a tensor; for example, $2\times 2\times 3\times 3$ tensor can be arranged as a 36 dimensional vector.

The world input-output table is obtained by unfolding the tensors $\mathcal{Z}$ and $\mathcal{F}$ into $JS\times JS$ and $JS\times S$ matrices respectively, and by unfolding the matrices $\mathbf{F}$, $\mathbf{X}$, and $\mathbf{W}$ into JS vectors. Therefore, WIOT consists of

- $n\times n$ matrix $\mathbf{Z} = [z_{ij}]$ with $i = (\hat{\imath}, r)$ and $j = (\hat{\jmath}, s)$ and $n = JS$, so that the elements of the matrix describe the intermediate inputs from country-industry pair i to country-industry pair j,

$$\mathbf{Z} = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{bmatrix}. \tag{4}$$

- n dimensional Final-use vector $\mathbf{f} = [f_1, \dots, f_n]^T$ with the i-th entry describing the final consumption in country-industry pair i,
- n dimensional Gross-output vector $\mathbf{x} = [x_1, \dots, x_n]^T$, where $x_i = \sum_j z_{ij} + f_i$.
- n dimensional Value-added vector $\mathbf{w} = [w_1, \dots, w_n]^T$, where $w_i = x_i - \sum_j z_{ji}$.

2.2. World-input and world-output networks

Consider a world with $J \geq 1$ countries (economies): country-1, …, country-J and $S \geq 1$ sectors (industries): sector-1, …, sector-S as a network $G = (V, E)$ of $n = JS$ nodes in which each node represents a country-industry pair, where $V = \{1, \dots, n\}$ is the set of nodes and E is the set of edges to be defined shortly. Country-industry pairs $(\hat{\imath}, r)$ are mapped to the nodes in V with $(\hat{\imath}, r) \to (\hat{\imath}-1)S + r$, for $\hat{\imath} = 1, \dots, J$ and $r = 1, \dots, S$. Note that the nodes $1, \dots, S$ correspond to the country-1, the nodes $S+1, \dots, 2S$ are related to the country-2, and so on.

We write $i = (\hat{\imath}, r)$ and $j = (\hat{\jmath}, s)$ so that country-industry pairs are indexed by i and j, $i, j = 1, \dots, n$. Two networks are associated with the vertex set $V = \{1, 2, \dots, n\}$. *World-input network* is represented by the adjacency input matrix $\mathbf{A} = [a_{ij}]$ for which $a_{ij} \equiv z_{ij}/x_j$. This normalization will be called "world-input" network, since, along the output supply chain, the country-industry pair i sells intermediate inputs to other country-industry pairs j's in the world economy (the corresponding links are denoted by the input coefficients a_{ij}).

For the *world-output network*, let $\mathbf{B} = [b_{ji}]$ represent the adjacency output matrix of the second normalization with a typical element $b_{ji} \equiv z_{ji}/x_j$. This specification will be called (is known as) "world-output" network, since, along the input demand chain, the country-industry i buys intermediate inputs worldwide (the corresponding links from all country-industry pairs j's to i are denoted by the output coefficients b_{ji}). Note that the matrices $\mathbf{A}$ and $\mathbf{B}$ are similar, $\mathbf{B} = \mathbf{X}_{dg}^{-1}\mathbf{A}\mathbf{X}_{dg}$, where $\mathbf{X}_{dg}$ is the diagonal matrix with elements of $\mathbf{x} = [x_1, \ldots, x_n]^T$ along its diagonal and zeros otherwise, share same eigenvalues, and their largest eigenvalue, λ, is real with $\lambda < 1$.

Here, the world economy is modelled as a linear economy $(\mathbf{A}, \mathbf{f})$, or equivalently as $(\mathbf{B}, \mathbf{w})$, where $\mathbf{f} = [f_1, \ldots, f_n]^T$ and $\mathbf{w} = [w_1, \ldots w_n]^T$. Let $\mathcal{L} = (\mathbf{I} - \mathbf{A})^{-1}$ and $\mathcal{G} = (\mathbf{I} - \mathbf{B})^{-1}$ be Leontief-inverse matrix and Ghosh-inverse matrix, respectively. Then Eq. (2) and Eq. (3) can be rewritten in a compact form as [11,12],

$$\mathbf{x} = \mathcal{L}\,\mathbf{f} \tag{5}$$
$$\mathbf{x} = \mathcal{G}^T\mathbf{w} \tag{6}$$

Two well-established metrics in input-output economics indicating the country-industry's (weighted) average position in global value chains [11,12], the output upstreamness (or upstreamness) (OU), $\mathbf{u} = [u_1, \ldots, u_n]^T$, and the input downstreamness (or downstreamness) (ID), $\mathbf{d} = [d_1, \ldots, d_n]^T$, are defined as $\mathbf{u} = \mathcal{G}\,\mathbf{1}$ and $\mathbf{d} = \mathcal{L}^T\mathbf{1}$, where $\mathbf{1}$ is a length-n column vector whose entries are all 1. In input-output analysis, u_i and d_i are used as measures of the importance of a certain country-industry pair i. That is, other things being equal, country-industry pairs with large u_i or d_i values are being more proper targets for economic stimulation because they will bring much greater benefits to the entire world economy (by extending more of its resources to other country-industry pars in the former case, or by triggering other country-industry pairs to increase their outputs in the latter).

2.3. Absorbing Markov chains

We next turn to alternative measures of the country-industry's average position in global value chains that can be estimated for the output supply chain (hence relative to final consumption) and for the input demand chain (thus relative to primary inputs). In order to understand the structure and organization of the world production, we associate with the input and output networks, two homogeneous discrete-time absorbing Markov chains, which will be called *input-chain* and *output-chain*. The state space of both chains is $V \cup \{0\} = \{0, 1, 2, \ldots, n\}$. The vertex set V is considered as a set of transient states. For the world-input network, the absorbing state 0 represents final use of output. On the other hand, for world-output network, the absorbing state 0 represents primary factors of production (or sources of value added). If we define

$$\gamma_i = \frac{f_i}{x_i} \tag{7}$$
$$\delta_i = \frac{w_i}{x_i} \tag{8}$$

the transition matrix $\mathbf{P}_{in}$ of the absorbing Markov input-chain reads

$$\mathbf{P}_{in} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \delta & \mathbf{A}^T \end{bmatrix}, \tag{9}$$

while the transition matrix $\mathbf{P}_{out}$ of the absorbing Markov output-chain is

$$\mathbf{P}_{out} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \gamma & \mathbf{B} \end{bmatrix}, \tag{10}$$

where T denotes transpose operator, $\gamma = [\gamma_1, \ldots, \gamma_n]^T$, $\delta = [\delta_1, \ldots, \delta_n]^T$, and $\mathbf{0}$ is a length-n column vector whose entries are all 0. Both matrices $\mathbf{P}_{in}$ and $\mathbf{P}_{out}$ are row stochastic. The column stochastic matrix $\mathbf{A}$ and row stochastic matrix $\mathbf{B}$ are the one-step transition probabilities of the (sub)Markov chain on V and δ_i and γ_i are the one-step transition probabilities of absorption into the state 0.

We analyse the world input/output network with a Markov process (input-chain or output-chain). Assume that the process starts in state $i \in V$ at time 0, and let $Y_{ij}^{(t)} = 1$ (or 0) if the process is (or is not) in the state j at time t. Further, let $X_{ij} = \sum_{t=0}^{\infty} Y_{ij}^{(t)}$ be a random variable representing the number of visits to state j before absorption and $X_i = \sum_{j \in V} X_{ij}$ be another random variable – the time to absorption. To simplify notation, let us write $\mathbf{P}$ for both matrices $\mathbf{P}_{in}$ and $\mathbf{P}_{out}$ and let $\mathbf{Q}$ be either $\mathbf{A}^T$ or $\mathbf{B}$. From the standard theory of absorbing Markov chains, the following equations can be derived [18]:

$$\mathbf{L} \equiv \left[\mathbb{E}[X_{ij}]\right] = (\mathbf{I} - \mathbf{Q})^{-1} \tag{11}$$

$$\mathbf{L}_2 \equiv \left[\mathrm{Var}[X_{ij}]\right] = \mathbf{L}(2\mathbf{L}_{dg} - \mathbf{I}) - \mathbf{L}_{sq} \tag{12}$$

$$\mathbf{g} \equiv \mathbf{g}(\mathbf{Q}) \equiv [\mathbb{E}[X_1], \ldots, \mathbb{E}[X_n]]^T = \mathbf{L1} \tag{13}$$

$$\mathbf{h} \equiv \mathbf{h}(\mathbf{Q}) \equiv [\mathrm{Var}[X_1], \ldots, \mathrm{Var}[X_n]]^T = (2\mathbf{L} - \mathbf{I})\mathbf{g} - \mathbf{g}_{sq} \tag{14}$$

where $\mathbb{E}[X]$ and $\mathrm{Var}[X]$ are expectation and variance of the random variable X, respectively, $\mathbf{L}_{dg} = [\ell_{ii}]$ is a diagonal matrix, $\mathbf{L}_{sq} = [\ell_{ij}^2]$, $\mathbf{1}$ is a length-n column vector whose entries are all 1, and $\mathbf{g}_{sq} = [g_1^2, \ldots, g_n^2]^T$.

When $\mathbf{Q} = \mathbf{A}^T$, the transpose of the matrix defined with Eq. (11) coincides with Leontief-inverse matrix $\mathcal{L}$. For $\mathbf{Q} = \mathbf{B}$, Eq, (11) reduces to the Ghosh-inverse matrix $\mathcal{G}$. Moreover, from Eq. (13), it follows that the expected number of steps before absorption is characterized with vectors (for the output and input networks) $\mathbf{u}$ and $\mathbf{d}$. The metric $\mathbf{u}$ is a measure of distance of a country-industry pair from the final demand. Therefore, u_i describes "how far" (in expected number of steps) the production of a county-industry pair is from the final use (or "average production line position"). The second quantity $\mathbf{d}$ is a measure of distance of a country-industry pair from the primary factors of production (or sources of value-added). In another words, d_i measures "average distance from primary inputs suppliers". Not all products need to have their production split into multiple stages. Services, for example, are less inclined to vertical specialization when supplier is required to have a close contact with the consumer. The variance (of the number of steps before absorption) can help measure the volatility (or the risk) a country-industry pair assumes when determining the production chain lengths. It could therefore permit the country-industry pairs to develop better models of production by optimizing the level of fragmentation (that is a trade-off between higher transaction/coordination costs and lower costs of production).

For both absorbing Markov chains, the input-chain and the output-chain, the set of transient state V is irreducible. In terms of the world-input network, it means that for arbitrary two country-industry pairs i and j, even for those directly not connected ($a_{ij} = 0$), the country-industry pair i, after finite number of jumps (hops/steps), sells intermediate inputs to the country-industry pair j. Similarly, irreducibility of the world-output network implies that an arbitrary country-industry pair, after finite number of purchases, buys intermediate inputs from all other country-industry pairs worldwide. However, since the Markov chains are absorbing, eventually, intermediate output sales and input purchases reach the absorbing state — state from which further jumps are impossible (e.g. final consumers in the output supply chain). An absorbing Markov chain is characterized with a quasi-stationary distribution which represents the proportion of time the process spends in the

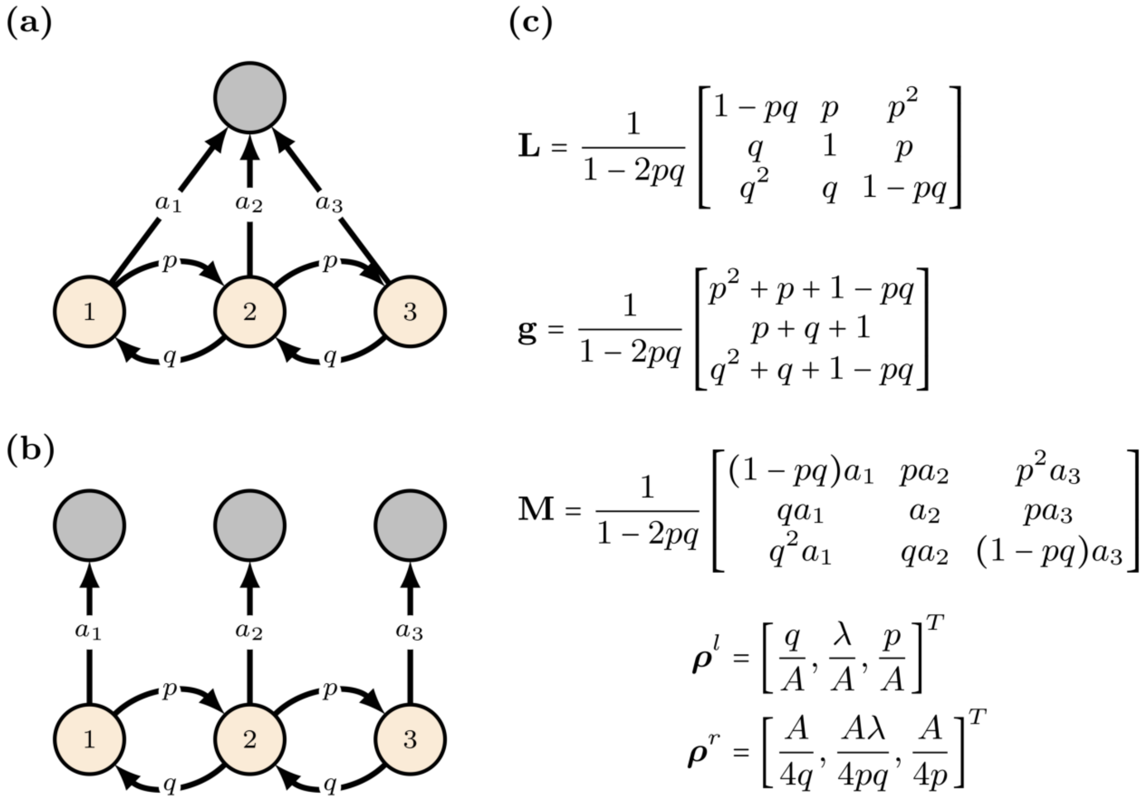

Figure 1. A simple example with three countries C_1, C_2, C_3 and one industry I. The world economy consists of three pairs: $1 \equiv (C_1, I)$, $2 \equiv (C_2, I)$, and $3 \equiv (C_3, I)$. The first pair produces intermediate goods in amount of p percentage of the total output for the second pair, and final goods in amount of a_1 percentage of the total output for the country C_1. The second pair produces intermediate goods in amount of q percentage of the total output for the first pair, and intermediate goods in amount of p percentage of the total output for the third pair, and final goods in amount of a_2 percentage of the total output for the country C_2. The third pair produces intermediate goods in amount of q percentage of the total output for the second pair, and final goods in amount of a_3 percentage of the total output for the country C_3. (a) World economy is represented as an absorbing Markov chain with one absorbing state. In this case $\boldsymbol{\gamma} = [a_1, a_2, a_3]^T$. (b) World economy is represented as an absorbing Markov chain with three absorbing states. In this case, $\mathbf{D}_\eta$, Eq. (18), is a 3×3 diagonal matrix with elements a_1, a_2, and a_3. (c) The Ghosh-inverse matrix $\mathbf{L} = (\mathbf{I} - \mathbf{B})^{-1} = \mathcal{G}$, Eq. (11), the output upstreamness, $\mathbf{g}$, Eq. (13), and the matrix $\mathbf{M}$. The largest eigenvalue of the matrix $\mathbf{B}$ is $\lambda = \sqrt{2pq}$. The left and right eigenvectors are also shown with $A = p + q + \lambda$. For this example, the product distribution does not depend on p and q and is equal to $\{1/4, 1/2, 1/4\}$.

transient state (e.g. for the output supply (resp. input demand) chain, the number of times a particular country-industry pair contributes along its individual production steps to total output (resp. total input) of a country-industry pair i) before absorption (given that the time to absorption is long).

We consider two quasi-stationary distributions that are derived, roughly speaking, by considering only those realisations in which the time to absorption is long. Assume that the process starts in state i with probability π_i. Let $\mathbf{P}^t = [p_{ij}^{(t)}]$. The probability that the process has been absorbed by time t is

$\sum_{i\in V}\pi_i p_{i0}^{(t)}$. For the conditional probability that the process is in state j at time t, given that the process is not absorb at the time t, one computes [19]:

$$\begin{aligned}\Pr\left[\text{in state } j \text{ at time } t \mid \text{not absorbed by time } t\right] &= \\ \frac{\sum_{i\in V}\pi_i p_{ij}^{(t)}}{\sum_{i\in V}\pi_i\left[1-p_{i0}^{(t)}\right]} = \rho_j^l + O\left(\left(\frac{|\lambda_2|}{\lambda}\right)^t\right)\end{aligned} \tag{15}$$

where $\boldsymbol{\rho}^l = [\rho_1^l,\ldots,\rho_n^l]^T$ is the left dominant eigenvector of the matrix $\mathbf{Q}$, where λ_2 is the second largest eigenvalue of the matrix $\mathbf{Q}$, and we have assumed, for simplicity only, that its multiplicity is 1. This vector is normalized, $\sum_i \rho_i^l = 1$, so it represents a quasi-stationary distribution of the absorbing Markov chain. Therefore, this quasi-stationary distribution represents for the output supply (resp. input demand) chain, the number of times a particular country-industry pair contributes along its individual production steps to total output (resp. total input) of a country-industry pair i before absorption (given that the time to absorption is long). Note that the limit as $t\to\infty$ is ρ_j^l which is independent of the probability distribution $\boldsymbol{\pi} = [\pi_1,\ldots,\pi_n]^T$. Equation (15) can be further generalized as [19]:

$$\begin{aligned}\Pr\left[\text{in state } j \text{ at time } \tau \mid \text{not absorbed by time } t\right] &= \\ \rho_j^l\rho_j^r + O\left(\left(\frac{|\lambda_2|}{\lambda}\right)^\tau\right) + O\left(\left(\frac{|\lambda_2|}{\lambda}\right)^{(t-\tau)}\right)\end{aligned} \tag{16}$$

where $\boldsymbol{\rho}^r = [\rho_1^r,\ldots,\rho_n^r]^T$ is the right dominant eigenvector of the matrix $\mathbf{Q}$. This vector is normalized, $\sum_i \rho_i^l\rho_i^r = 1$, and therefore,

$$\boldsymbol{\rho}_{prod} = \left[\rho_1^l\rho_1^r,\ldots,\rho_n^l\rho_n^r\right]^T \tag{17}$$

represents a quasi-stationary distribution of the absorbing Markov chain, which will be referred as *product distribution*. The left hand side of Eq. (16) converges to $\rho_j^l\rho_j^r$ as $\tau\to\infty$ and $t-\tau\to\infty$. Note that $\{\rho_i^l\rho_i^r\}$ may be described as the distribution of the random variable at time τ (τ large), given that absorption has not yet taken place and will not take place for a long time. The product distribution is more relevant than $\{\rho_i^l\}$ in the sense that the Eq, (15) is a "degenerate" case of the Eq. (16).

Next we consider the structure of the world network from the "final use perspective" and the "final value added perspective". Let $m_{i\hat{j}}$, where $i = 1,\ldots,n$ and $\hat{j} = 1,\ldots,J$ be the probability that production from country-industry pair i ends up as output purchased by the final users of country $\hat{j}$. We arrange the elements $m_{i\hat{j}}$ as $n\times J$ matrix $\mathbf{M}$, which is called a *global final demand matrix*. Let $\eta_{i\hat{j}} = \frac{f_{i,\hat{j}}}{x_i}$ and define $n\times J$ matrix $\mathbf{D}_\eta$ with elements $\eta_{i\hat{j}}$. In this case, Markov absorbing chain has J absorbing states. The transition matrix, Eq. (10), now reads

$$\mathbf{P}_{out} = \begin{bmatrix}\mathbf{I}_{J\times J} & \mathbf{0}_{J\times n}\\ \mathbf{D}_\eta & \mathbf{B}\end{bmatrix}, \tag{18}$$

where $\mathbf{I}_{J\times J}$ is identity matrix and $\mathbf{0}_{J\times n}$ zero matrix. A simple example, for illustrating all quantities introduced in this subsection is provided on Fig. 1.

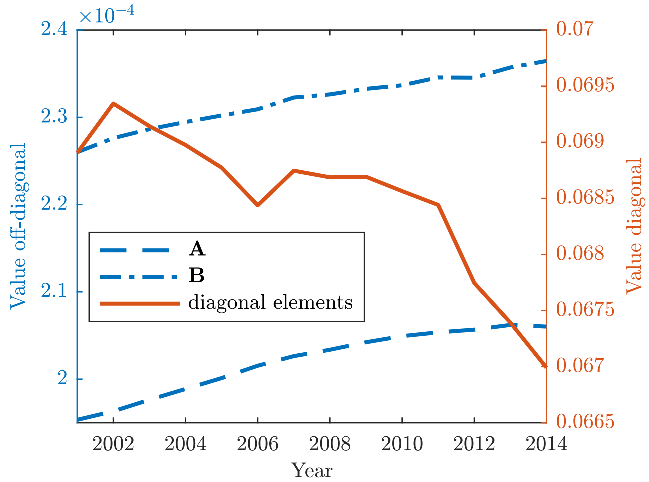

Figure 2. Mean values for the off-diagonal blocks of **A** and **B** (in blue). Mean values for the on-diagonal blocks of **A** and **B** (in orange).

3. Results and Discussions

The world economy is characterized by the structure of the input/output networks. Fig. 2 shows a growing fragmentation of production across countries worldwide. The **Z** matrix, Eq. (4), is rewritten as

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_{11} & \dots & \mathbf{Z}_{1J} \\ \vdots & \ddots & \vdots \\ \mathbf{Z}_{J1} & \dots & \mathbf{Z}_{JJ} \end{bmatrix}. \tag{19}$$

where each $\mathbf{Z}_{ij}$ is $S \times S$ matrix. Diagonal block matrices in Eq. (19) represent (domestic) input-output table of each country, while off-diagonal blocks show trade activities between countries. The adjacency matrices **A** and **B** are also rearranged in this way and their mean values are shown in Fig. 2. From 2000 onwards, the global production networks became more complex and increasingly interconnected as a result of massively increased trade in intermediates (goods or services) among countries, both as users of foreign inputs (off-diagonal blocks of matrix **B**) or suppliers of intermediate products to third countries for further processing and export (off-diagonal blocks of matrix **A**). It is noteworthy, however, that international fragmentation of production seems to have lost momentum in recent years, at least compared to 2000s, especially when it comes to forward GVC participation (many companies are probably less agile in responding to changing consumer demands). The consolidation of some value chains has been observed even during the 2008-2009 financial crisis, with some country-industry pairs switching back to domestic suppliers (slight increase in the mean values of the on-diagonal block in Fig. 2) in the context of difficult access to trade finance and risks connected with international suppliers. Nevertheless, the recent slowdown may also point to a number of potential structural shifts facing the world economy (with many companies deciding to reexamine their outsourcing and production strategies), which could dramatically change the configuration of global production landscape and determine the future of globalization in a systemic way. Figure 3 shows the dominant eigenvalue of the world input/output network as well as the country dominant eigenvalues of the input/output networks of all countries versus time. The mean value of dominant eigenvalues of the input/output networks of all countries, $\langle \lambda_{1i} \rangle$, is almost constant in time, while the dominant eigenvalue of the world input/output network, λ_1, increases in time. Since, when $\lambda \to 1$, the ordering (ranking) of upstreamness and downstreamness depend on the right (left) dominant eigenvector of the matrices **B** and **A**, so that, in principle, network structure is crucial for the world economy: it could happen that a

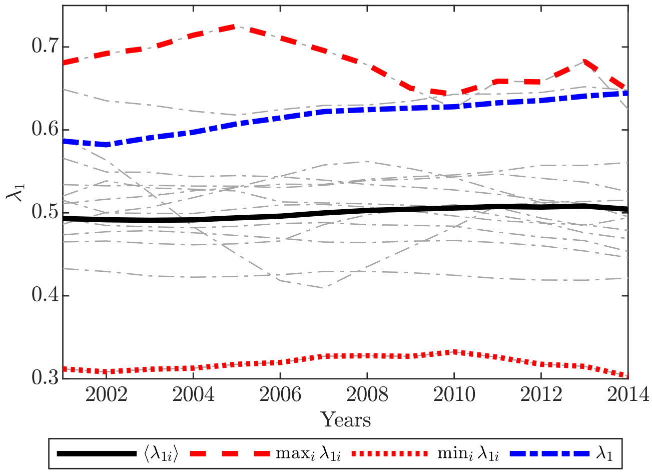

Figure 3. The dominant eigenvalue of the world input/output network versus time.

low-degree country-industry pair has greater influence (i.e. it is more relevant player in the global production networks) than high-degree hub.

Upstreamness and downstreamness are two well-established metrics in input-output economics. Larger u_i values are related to relatively higher level of upstreamness of the particular country-industry pair i. The latter thus provides little to final consumers worldwide and instead sells disproportionately large share of its output (as intermediate inputs) to other producing industries in the world economy. Conversely, larger d_i values are related to relatively higher level of downstreamness of a certain country-industry pair i. Clearly, the production process here relies disproportionately on intermediate inputs relative to the value-added from primary factors of production, and especially if purchases are made from those country-industry pairs which themselves use intermediate inputs intensively. It should also be noted that a country-industry with higher levels of downstreamness exhibit greater productivity fluctuations, because "upstream supply-side shocks accumulate while propagating downstream" [20]. In world-output network, the country-industry pairs with large values of d_i will produce complex and strong intermediate input demand edges with similar pairs (and vice versa for small values). In world-input network, on the other hand, the country-industry pairs with large u_i values will produce complex and strong intermediate output supply edges with similar pairs (and vice versa for small values). Next theorem describes how u_i and d_i are ordered (ranked) (see Appendix A for the proof of the theorem).

Theorem 1. *The following approximations hold:*

$$\mathbf{u} \approx \begin{cases} \mathbf{1}+\mathbf{B}\mathbf{1} & \text{for} \quad \lambda \to 0^+ \\ \frac{\sum_i \rho_i^l(\mathbf{B})}{1-\lambda}\boldsymbol{\rho}^r(\mathbf{B}) & \text{for} \quad \lambda \to 1^- \end{cases} \tag{20}$$

$$\mathbf{d} \approx \begin{cases} \mathbf{1}+\mathbf{A}^T\mathbf{1} & \text{for} \quad \lambda \to 0^+ \\ \frac{\sum_i \rho_i^r(\mathbf{A})}{1-\lambda}\boldsymbol{\rho}^l(\mathbf{A}) & \text{for} \quad \lambda \to 1^- \end{cases} \tag{21}$$

where $\boldsymbol{\rho}^l(\cdot)$ and $\boldsymbol{\rho}^r(\cdot)$ are the left and the right dominant eigenvectors of the matrices $\mathbf{A}$ *and* $\mathbf{B}$.

Note that in Eq. (20) and Eq. (21), $\mathbf{B1}$ and $\mathbf{A}^T\mathbf{1}$ are vectors of out-degree and in-degree of the nodes of the output-network and input-network, respectively. Therefore, when $\lambda \to 0$, it follows that the ordering (or the ranking) of the country-industry pairs depends only on the out-degree (in-degree) centrality, and thus, high-degree nodes are important, that is the country-industry pairs that import intermediates from many sources (or the most important recipients of foreign value added in global production networks) and the most important suppliers of value added in world GVCs

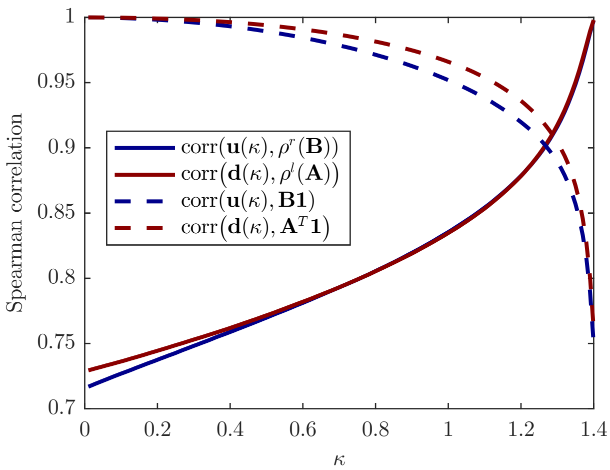

Figure 4. Spearman's rank correlation for the WIOD, year 2014. Upstreamness $\mathbf{u}(\kappa)$ and downstreamness $\mathbf{d}(\kappa)$ are computed as $\mathbf{u}(\kappa) = (\mathbf{I} - \kappa\mathbf{B})^{-1}\mathbf{1}$ and $\mathbf{d}(\kappa) = (\mathbf{I} - \kappa\mathbf{A}^T)^{-1}\mathbf{1}$, respectively, where κ is a parameter. $\kappa = 1$ corresponds to the values of the $\mathbf{u}$ and $\mathbf{d}$ of the world networks and the year 2014. The limits $\kappa \to 0^+$ and $\kappa \to (1/\lambda)^-$ correspond to $\lambda \to 0^+$ and $\lambda \to 1^-$, respectively (see Appendix A).

(or the general-purpose country-industry pairs whose value added contained in intermediate inputs is sent to a wide range of country-industry pairs for further processing. On the other hand, when $\lambda \to 1$, the ordering (ranking) depends only on the right (left) dominant eigenvector of the matrix $\mathbf{B}$ ($\mathbf{A}$). Therefore, in this case, it could happen that a low-degree country-industry pair has greater influence (i.e. it is more relevant player in the global production networks) than high-degree hub. Put differently, the centrality of a country-industry pair here is recurrently related to the pairs to which it is connected, that is a node's position depends on the importance of its neighbors (a node eigenvector centrality). Figure 4 depicts Spearman's rank correlation between upstreamness (downstreamness) and the out-degree (in-degree) centrality and right dominant eigenvector of the adjacency matrix $\mathbf{B}$ ($\mathbf{A}$) for the WIOD. In the limits $\kappa \to 0^+$ and $\kappa \to (1/\lambda)^-$ (that correspond to the limits $\lambda \to 0^+$ and $\lambda \to 1^-$, respectively, see the Appendix A), the orderings of $\mathbf{u}$ and $\mathbf{d}$ are perfectly matched with the ordering of the vectors from the right-hand side of the Eq. (20) and Eq. (21), respectively. When $\kappa = 1$, the ordering of $\mathbf{u}$ and $\mathbf{d}$ is 95% correlated with the ordering of degree centrality and 80% with eigenvectors' ordering.

Previous section introduces two random variables. Assume that the process starts in state i. The first random variable, X_{ij}, represents the number of visits to state j before absorption; the second, X_i is the time to absorption. Expectation and variance of these random variables are provided by Eq. (11), Eq. (12), Eq. (13), and Eq. (14), respectively. Figure 5 depicts the expectation and variance of the time to absorption for both input and output networks versus time.

The next result unveils that both the input chain and the output chain have exactly the same quasi-stationary product distribution (for the proof of the theorem, see Appendix B).

Theorem 2. *The World input/output network is characterized with*

$$\rho_i^l(\mathbf{A})\rho_i^r(\mathbf{A}) = \rho_i^l(\mathbf{B})\rho_i^r(\mathbf{B}) \text{ for all } i \in V. \tag{22}$$

Note that although $\rho_i^l(\mathbf{A}) \neq \rho_i^l(\mathbf{B})$ and $\rho_i^r(\mathbf{A}) \neq \rho_i^r(\mathbf{B})$, their products are equal to each other. Therefore, the product distribution does not depend on the type of network (input/output). This, rather surprising, fact runs counter to what might be expected at a time of rising importance for the global value chains.

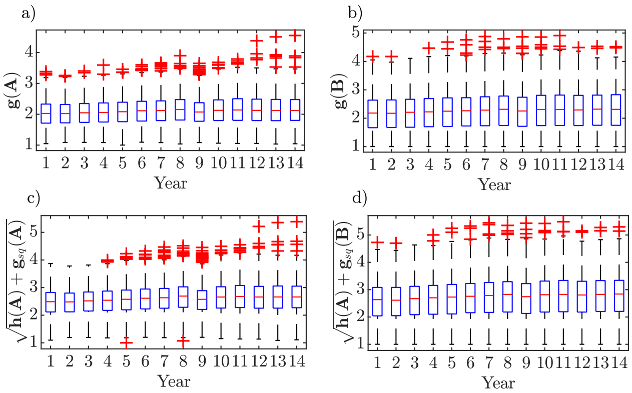

Figure 5. Box-plot for **g** and **h**.

In order to explain this surprising (unexpected) result, we interpret the probability $\rho_j^l \rho_j^r$ as limiting conditional mean ratios. It can be shown that:

$$\sum_{i \in V} \pi_i \mathbb{E}\left[\frac{X_{ij}}{X_i} \middle| X_i = t \right] = \rho_j^l \rho_j^r + O\left(\frac{1}{t}\right) \tag{23}$$

$$\sum_{i \in V} \pi_i \mathbb{E}\left[\frac{\sum_{k=0}^{t} Y_{ij}^{(k)}}{t} \middle| X_i > t \right] = \rho_j^l \rho_j^r + O\left(\frac{1}{t}\right) \tag{24}$$

The left hand side of Eq. (23) is the expected proportion of time spent in the state j before absorption. Since the input flows originated from the pair j are, at the same time, equal to output flows coming from the pair j, the expected proportion of time spent in the state j before absorption for the input network is equal to the expected proportion of time spent in the state j before absorption for the output network. Moreover, note that $\sum_{k=0}^{t} Y_{ij}^{(k)}$, see Eq. (24), equals the number of visits to state j up to time t. Therefore, the expected proportion of time spent in the state j before absorption is equal to $\rho_j^l \rho_j^r$ and does not depend on the network character. Figure 6 depicts the histograms of the product distribution $\rho_j^l \rho_j^r$, the output upstreamness $\mathbf{u}$, and the input downstreamness $\mathbf{d}$ for the WIOD and the year 2014.

The following result is a combination of classical results regarding absorbing Markov chains [18] with some new results (for the proof, see Appendix C).

Theorem 3. *(i) The global final demand matrix* $\mathbf{M}$ *can be computed as*

$$\mathbf{M} = (\mathbf{I} - \mathbf{B})^{-1} \mathbf{D}_\eta \tag{25}$$

such that $\mathbf{M}$ *is a row stochastic matrix.*

(ii)

$$(\mathbf{I} - \mathbf{A}^T)^{-1} \delta = \mathbf{1} \tag{26}$$

(iii) For an economy for which the row sums of $\mathbf{A}^T$ *are all equal to* c*, the input downstreamness is constant* $\mathbf{d} = \frac{1}{1-c}\mathbf{1}$*. If the row sums of* $\mathbf{B}$ *are all equal to* c*, then the the output upstreamness is constant* $\mathbf{u} = \frac{1}{1-c}\mathbf{1}$*.*

Note that the matrix $\mathbf{D}_\eta$ is the one-step transition matrix, while the vector (for a fixed i)

$$\mathbf{m}_i = [m_{i1}, \ldots, m_{iJ}]^T \tag{27}$$

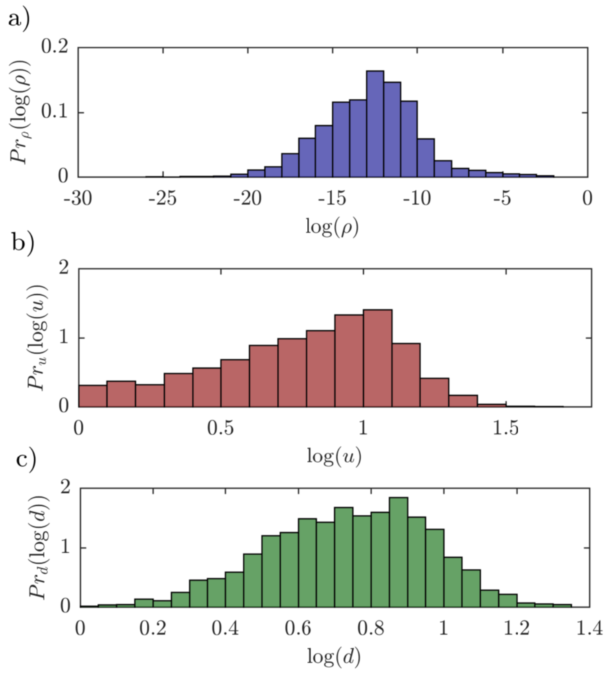

Figure 6. Histograms (WIOD, year 2014) of (a) the product distribution, (b) the output upstreamness, and (c) the input downstreamness. The product distribution does not depend on the type of network (input/output).

is a probability distribution showing how the final output of country-industry pair i is distributed among different countries. For a fixed industry r we arrange the elements $m_{(\hat{i},r),\hat{j}}$ in a $J \times J$ matrix,

$$\mathbf{PP} = \left[m_{(\hat{i},r),\hat{j}} \right], \tag{28}$$

which we call a *global final demand matrix of the industry r*, showing the global patterns of final demand for any given industry. Similarly, if we write $\mathbf{L} = [\ell_{ij}] \equiv (\mathbf{I} - \mathbf{A}^T)^{-1}$ and define $\zeta_{i\hat{j}} = \sum_{r=1}^{S} \ell_{i(\hat{j}r)} \delta_{\hat{j}r}$, it follows from Eq. 26 that the vector (for a fixed i)

$$\zeta_i \quad = \quad [\zeta_{i1}, \ldots, \zeta_{iJ}]^T \tag{29}$$

is the probability distribution showing how the value added of country-industry pair i is distributed among different countries. Again, for a fixed industry r we arrange the elements $\zeta_{(\hat{i},r),\hat{j}}$ in a $J \times J$ matrix,

$$\mathbf{WP} = \left[\zeta_{(\hat{i},r),\hat{j}} \right], \tag{30}$$

which we call a *global value added matrix of the industry r*, capturing the final impact after all stages of production have circulated throughout the world economy and showing the global patterns of value added for any given industry. This metric, which breaks down the distribution of gross trade flows along the sources and destinations of value added, provides a coherent answer to many important questions about the interconnections among countries, especially with regard to the aggregate impact and propagation of shocks. For example, the important role that particular countries play in international flows of value added raises the questions about the resilience of the world trading system if they suffer a large-scale economic shock. All of this has a major impact on forecasting the macroeconomic developments and on monetary policy decisions. Figure 7 shows global demand matrix for the warehousing and support activities for transportation in 2014. Each row (a horizontal line) is a probability distribution: the j element of the row i shows the probability that a good/service produced in the country i has been delivered to final consumers in country j. On

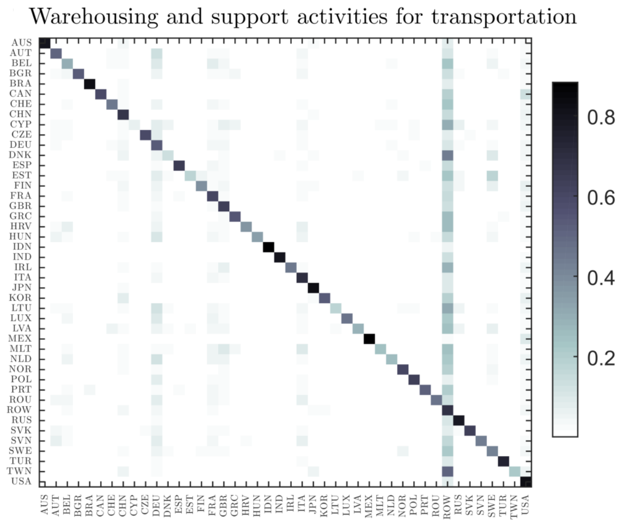

Figure 7. Final demand matrix patterns of the world economy for the warehousing and support activities for transportation in 2014.

the other, the column j (a vertical line) represents the final consumer buying patterns for a particular product (e.g. warehousing and support activities for transportation) in country j.

Finally, we introduce several global metrics:

$$\text{GDVA} = \sum_{\hat{i}=1}^{J} \sum_{r=1}^{S} \zeta_{\hat{i},r;\hat{i}} \tag{31}$$

$$\text{GIEVA} = \sum_{\hat{i}=1}^{J} \sum_{\hat{j}=1,\hat{j}\neq\hat{i}}^{J} \sum_{r=1}^{S} \zeta_{\hat{i},r;\hat{j}} \tag{32}$$

$$\text{GDFU} = \sum_{\hat{i}=1}^{J} \sum_{r=1}^{S} m_{\hat{i},r;\hat{i}} \tag{33}$$

$$\text{GIEFU} = \sum_{\hat{i}=1}^{J} \sum_{\hat{j}=1,\hat{j}\neq\hat{i}}^{J} \sum_{r=1}^{S} m_{\hat{i},r;\hat{j}} \tag{34}$$

The on-diagonal block of the global value-added matrix, which we call Global Domestic Value-Added (GDVA), refers to all domestic value-added flows within a given country-industry pair, irrespective of the number of steps along the chain (see Eq. (31)). Hence, the measure includes both, direct and indirect domestic value-added contained in exports. The former denotes the income of primary inputs directly involved in production (one stage), while the latter encompasses both the domestically produced intermediates embodied in exports (hence, the magnitude only depends on the density of domestic intersectoral linkages) and domestic value-added that is re-imported in the economy of origin (hence, the magnitude of value added depends on the density of intersectoral linkages among countries in the world economy). The off-diagonal blocks of the global value-added matrix, which we call Global Import-Export Value-Added (GIEVA), indicates the extent to which a country-industry pair is connected to global production networks for its foreign trade (see Eq. (32)). Hence, this measure refers to both, the import of intermediates contained in foreign value-added to produce exports (demand side in global production networks) and export of domestic value-added (embodied in intermediate inputs) to other countries that re-export them (embodied in other goods and services) to third countries in the world economy (supply side in global production networks). The on-diagonal block of the global final demand matrix, which we call Global Domestic Final Use (GDFU), refers to all output sales intended for final consumers in the country of origin (see Eq. (33)). The off-diagonal blocks of the global final

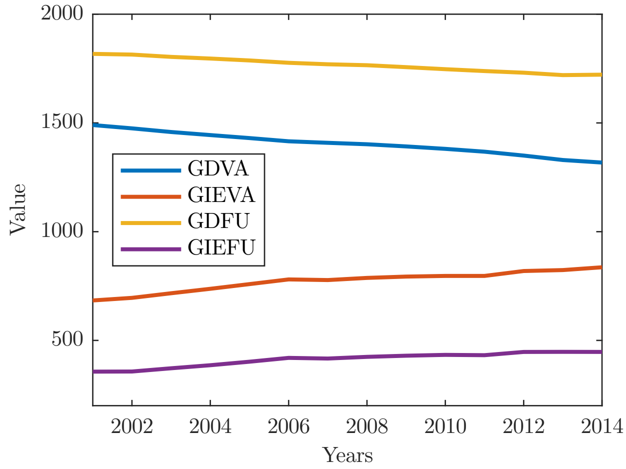

Figure 8. Global quantities over years.

demand matrix, which we call Global Import-Export Final Use (GIEFU), refers to all imports and exports destined for final consumers worldwide (see Eq. (34)). Figure 8 shows **to be written**

4. Conclusions

In this paper, there random variables have been analysed: (1) The time to absorption from the seller perspective (output supply chain) or supply side in global production networks, that is the number of times a particular country-industry pair contributes (domestic value added contained in intermediates sent to a partner economy for further processing and export) along the route of each unit of value added in the global economy before reaching the final consumers. (2) The time to absorption from the buyer perspective (input demand chain) or sourcing side in global production networks, that is the number of times an intermediate country-industry contributes in total input of a particular country-industry pair that incorporates foreign value added (or imports intermediate inputs to produce goods/services that are subsequently exported, along the output supply chain, for final consumption or intermediate use). (3) The time spent in a state before absorption or the number of times a country-industry contributes to its own production stages (it becomes an intermediate in its own supply or demand chains) before reaching the final consumers or the total input of a particular country-industry pair that incorporates foreign value. Based on these variables, several quantities have been introduced, listed in the Table 1. In addition to upstreamness and downstreamness), we introduce product distribution, country distribution of the final output and value added of a country-industry pair, and so on **to be finished**

Appendix A: Proof of the Theorem 1

Let $\mathbf{u}(\kappa) = (\mathbf{I} - \kappa\mathbf{B})^{-1}\mathbf{1}$ and $\mathbf{d}(\kappa) = (\mathbf{I} - \kappa\mathbf{A}^T)^{-1}\mathbf{1}$, where $\kappa > 0$ is a parameter such that $\kappa < 1/\lambda$. The first main result states that the vectors $\mathbf{u}(\kappa)$ and $\mathbf{d}(\kappa)$ can be approximated as

$$\mathbf{u}(\kappa) \approx \begin{cases} \mathbf{1} + \mathbf{B}\mathbf{1}, & \text{for} \quad \kappa \to 0^+, \\ \frac{\rho^l(\mathbf{B})^T\mathbf{1}}{1-\kappa\lambda}\rho^r(\mathbf{B}), & \text{for} \quad \kappa \to \frac{1}{\lambda}^-, \end{cases} \tag{35}$$

$$\mathbf{d}(\kappa) \approx \begin{cases} \mathbf{1} + \mathbf{A}^T\mathbf{1}, & \text{for} \quad \kappa \to 0^+, \\ \frac{\mathbf{1}^T\rho^r(\mathbf{A})}{1-\kappa\lambda}\rho^l(\mathbf{A}) & \text{for,} \quad \kappa \to \frac{1}{\lambda}^-, \end{cases} \tag{36}$$

where $\rho^l(\mathbf{M})$ and $\rho^r(\mathbf{M})$ are the left and right eigenvectors associated with the largest eigenvalue of $\mathbf{M}$, respectively. In what follows, we first provide a detailed background on relevant properties of positive matrices and functions of matrices. Afterwards, we give a full proof of the theorem.

	quantity	computed with	equation
$\mathbf{g}(\mathbf{B})$	n-dim vector	$\mathbf{B}$	(13)
$\mathbf{g}(\mathbf{A})$	n-dim vector	$\mathbf{A}$	(13)
$\mathbf{h}(\mathbf{B})$	n-dim vector	$\mathbf{B}$	(14)
$\mathbf{h}(\mathbf{A})$	n-dim vector	$\mathbf{A}$	(14)
$\boldsymbol{\rho}_{prod}$	n-dim vector	$\mathbf{A}$ or $\mathbf{B}$	(17)
$\mathbf{m}_i$	J-dim vector	$\mathbf{B}$ and $\mathbf{D}_\eta$	(27)
ζ_i	J-dim vector	$\mathbf{A}$ and δ	(29)
$\mathbf{PP}$	$J \times J$ matrix	$\mathbf{B}$ and $\mathbf{D}_\eta$	(28)
$\mathbf{WP}$	$J \times J$ matrix	$\mathbf{A}$ and δ	(30)
GDVA	scalar	$\mathbf{A}$ and δ	(31)
GIEVA	scalar	$\mathbf{A}$ and δ	(32)
GDFU	scalar	$\mathbf{B}$ and $\mathbf{D}_\eta$	(33)
GIEFU	scalar	$\mathbf{B}$ and $\mathbf{D}_\eta$	(34)

Table 1. Table to test captions and labels

Perron-Frobenius theorem for positive matrices

Recall some known properties of positive matrices. Let $\mathbf{M} = [M_{ij}]$ be an $N \times N$ positive matrix: $a_{ij} > 0$ for $1 \leq i, j \leq N$. Then the following statements hold.

- There is a positive real number λ_1, (called the Perron root, the Perron-Frobenius eigenvalue, the leading eigenvalue, or the dominant eigenvalue), such that λ_1 is an eigenvalue of $\mathbf{M}$ and any other eigenvalue (possibly, complex) in absolute value is strictly smaller than λ_1,

$$| \lambda_i | < \lambda_1, \tag{37}$$

  for $i = 2, \ldots, N$.
- λ_1 is a simple root of the characteristic polynomial of $\mathbf{M}$.
- There exists a right eigenvector $\boldsymbol{\rho}^r = [\rho_1^r, \ldots, \rho_N^r]^T$ of $\mathbf{M}$ with eigenvalue λ_1 such that $\mathbf{M}\boldsymbol{\rho}^r = \lambda_1 \boldsymbol{\rho}^r$, $\rho_i^r > 0$ for $i = 1, \ldots, N$. Respectively, there exists a positive left eigenvector $\boldsymbol{\rho}^l = \left[\rho_1^l, \ldots, \rho_N^l\right]^T$ such that $\left[\boldsymbol{\rho}^l\right]^T \mathbf{M} = \lambda_1 \left[\boldsymbol{\rho}^l\right]^T$ and $\rho_i^l > 0$ for all i.

Spectral Theorem for Diagonalizable Matrices

An $N \times N$ matrix $\mathbf{M}$ with spectrum $\sigma(\mathbf{M}) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ is said to diagonalizable if and only if there exist matrices $\{\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_N\}$ such that

$$\mathbf{M} = \lambda_1 \mathbf{G}_1 + \lambda_2 \mathbf{G}_2 + \ldots + \lambda_N \mathbf{G}_N, \tag{38}$$

where the $\mathbf{G}_i$'s have the following properties: (i) $\mathbf{G}_i$ is the projector onto $K(\mathbf{M} - \lambda_i \mathbf{I})$ along $R(\mathbf{M} - \lambda_i \mathbf{I})$; (ii) $\mathbf{G}_i \mathbf{G}_j = 0$ whenever $i \neq j$; and (iii) $\mathbf{G}_1 + \mathbf{G}_2 + \ldots + \mathbf{G}_N = \mathbf{I}$. The expansion (38) is known as the spectral decomposition of $\mathbf{M}$, and the $\mathbf{G}_i$'s are called the spectral projectors associated with $\mathbf{M}$. Moreover, if $\boldsymbol{\rho}^r$ and $\boldsymbol{\rho}^l$ are the respective right-hand and left-hand eigenvectors associated with a simple eigenvalue λ, then spectral projector associated with λ is:

$$\mathbf{G} = \frac{\boldsymbol{\rho}^r \left[\boldsymbol{\rho}^l\right]^T}{\left[\boldsymbol{\rho}^l\right]^T \boldsymbol{\rho}^r}. \tag{39}$$

If $\mathbf{M}$ is a diagonalizable matrix, then it is also similar to a diagonal matrix $\mathbf{D}$. Note that two $N \times N$ matrices $\mathbf{M}$ and $\mathbf{D}$ are said to be similar whenever there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{M}\mathbf{P} = \mathbf{D}$.

Functions of Matrices

Let $\mathbf{M} = \mathbf{PDP}^{-1}$ be a diagonalizable matrix where the eigenvalues in $\mathbf{D} = \text{diag}\,(\lambda_1 \mathbf{I}, \lambda_2 \mathbf{I}, \ldots, \lambda_N \mathbf{I})$ are grouped by repetition. For a function $f(z)$ that is defined at each λ_i, define

$$\begin{aligned} f(\mathbf{M}) &= \mathbf{P} f(\mathbf{D}) \mathbf{P}^{-1} \\ &= \mathbf{P} \begin{bmatrix} f(\lambda_1)\mathbf{I} & 0 & 0 & \ldots & 0 \\ 0 & f(\lambda_2)\mathbf{I} & 0 & \ldots & 0 \\ & \ldots\ldots\ldots\ldots\ldots\ldots & & & \\ 0 & 0 & 0 & \ldots & f(\lambda_N)\mathbf{I} \end{bmatrix} \mathbf{P}^{-1} \\ &= f(\lambda_1)\mathbf{G}_1 + f(\lambda_2)\mathbf{G}_2 + \ldots + f(\lambda_N)\mathbf{G}_N. \end{aligned} \tag{40}$$

We now briefly discuss functions of nondiagonalizable matrices following Ref. [21]. For an arbitrary matrix $\mathbf{M} \in \mathbb{C}^{N \times N}$ with $\sigma(\mathbf{M}) = \{\lambda_1, \ldots, \lambda_s\}$ where s is the number of distinct eigenvalues of $\mathbf{M}$, let k_i be the index of the eigenvalue λ_i (that is, the order of the largest Jordan block associated with λ_i in the Jordan canonical form of $\mathbf{M}$). A function $f : \mathbb{C} \to \mathbb{C}$ is said to be defined (or to exist) at $\mathbf{M}$ when $f(\lambda_i), f'(\lambda_i), \ldots, f^{(k_i - 1)}(\lambda_i)$ exist for each λ_i. If f exists at $\mathbf{M}$, then the value of f at $\mathbf{M}$ is defined to be

$$f(\mathbf{M}) = \sum_{i=1}^{s} \sum_{j=0}^{k_i - 1} \frac{f^{(j)}(\lambda_i)}{j!} (\mathbf{M} - \lambda_i \mathbf{I})^j \, \mathbf{G}_i. \tag{41}$$

For an arbitrary square matrix $\mathbf{M}$, as a particular example of f we consider the geometric series $f(z) = 1 + z + z^2 + \ldots$ also known as the Neumann series

$$f(\mathbf{M}) = \sum_{k=0}^{\infty} \mathbf{M}^k.$$

Let $\rho(\mathbf{M})$ be the spectral radius of $\mathbf{M}$ defined as $\rho(\mathbf{M}) = \max_{\lambda \in \sigma(\mathbf{M})} \mid \lambda \mid$. The following statements are equivalent: (i) The Neumann series converges. (ii) $\rho(\mathbf{M}) < 1$, and (iii) $\lim_{k \to \infty} \mathbf{M}^k = 0$. In which case, $(\mathbf{I} - \mathbf{M})^{-1}$ exists and

$$\sum_{k=0}^{\infty} \mathbf{M}^k = (\mathbf{I} - \mathbf{M})^{-1}.$$

Proof of the theorem

Let us now consider an arbitrary positive matrix $\mathbf{M}$. Let $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ be its eigenvalues. From the Perron-Frobenius theorem it follows that $\lambda_1 > \mid \lambda_2 \mid \geq \ldots \geq \mid \lambda_N \mid$, we consider the parametrized vector $\mathbf{g}$ defined as

$$\mathbf{g}(\kappa) = f(\kappa \mathbf{M})\mathbf{1} = (\mathbf{I} - \kappa \mathbf{M})^{-1} \mathbf{1}. \tag{42}$$

where $\kappa > 0$ is a real parameter such that $\kappa < 1/\lambda_1$. Let $\boldsymbol{\rho}^r(\mathbf{M})$ be the dominant right eigenvector of $\mathbf{M}$ and let $\boldsymbol{\rho}^l(\mathbf{M})$ be the dominant left eigenvector of $\mathbf{M}$. Assume that $\mathbf{M}$ is diagonalizable matrix. Then by combining (40) and (39) we have

$$\begin{aligned} \mathbf{g}(\kappa) &= [f(\kappa\lambda_1)\mathbf{G}_1 + f(\kappa\lambda_2)\mathbf{G}_2 + \ldots + f(\kappa\lambda_N)\mathbf{G}_N]\,\mathbf{1} \\ &= \left[f(\kappa\lambda_1) \frac{\boldsymbol{\rho}^r \left[\boldsymbol{\rho}^l\right]^T}{\left[\boldsymbol{\rho}^l\right]^T \boldsymbol{\rho}^r} + f(\kappa\lambda_2)\mathbf{G}_2 + \ldots + f(\kappa\lambda_N)\mathbf{G}_N \right] \mathbf{1}. \end{aligned}$$

Assuming $\left[\boldsymbol{\rho}^l\right]^T \boldsymbol{\rho}^r = 1$ (normalization) and since $f(\kappa\lambda_1) > 0$, $\boldsymbol{\rho}^l > 0$ and $\boldsymbol{\beta} > 0$ from the last equation it follows that

$$\frac{\mathbf{g}(\kappa)}{f(\kappa\lambda_1)\left[\boldsymbol{\rho}^l\right]^T \boldsymbol{\beta}} = \boldsymbol{\rho}^r + \frac{f(\kappa\lambda_2)}{f(\kappa\lambda_1)\left[\boldsymbol{\rho}^l\right]^T \mathbf{1}}\mathbf{G}_2\mathbf{1} + \ldots + \frac{f(\kappa\lambda_N)}{f(\kappa\lambda_1)\left[\boldsymbol{\rho}^l\right]^T \mathbf{1}}\mathbf{G}_N\mathbf{1}.$$

Let us examine the two limiting cases as $\kappa \to \frac{1}{\lambda_1}$ and $\kappa \to 0$.

First, since for $z = 1$, the series $f(z)$ diverges, as $\kappa \to \frac{1}{\lambda_1}$, the denominator of the right-hand side of the last equation approaches infinity. On the other hand, each derivative $f^{(j)}(z)$ of $f(z)$ can be expressed by a power series having the same radius of convergence as the power series expressing $f(z)$. From Eq. (37), it follows that $\left|f^{(j)}\left(\frac{\lambda_i}{\lambda_1}\right)\right| < \infty$, and hence,

$$\lim_{\kappa \to \frac{1}{\lambda_1}^-} \frac{\mathbf{g}(\kappa)}{f(\kappa\lambda_1)\left[\boldsymbol{\rho}^l\right]^T \mathbf{1}} = \boldsymbol{\rho}^r.$$

More generally, for nondiagonalizable matrices, from (41) it follows

$$\begin{aligned}
\mathbf{g}(\kappa) &= f(\kappa\lambda_1)\boldsymbol{\rho}^r \left[\boldsymbol{\rho}^l\right]^T \mathbf{1} + \sum_{i=2}^{s}\sum_{j=0}^{k_i-1} \frac{f^{(j)}(\kappa\lambda_i)}{j!}\left(\mathbf{M}-\lambda_i\mathbf{I}\right)^j \mathbf{G}_i\mathbf{1} \\
\frac{\mathbf{g}(\kappa)}{f(\kappa\lambda_1)\left[\boldsymbol{\rho}^l\right]^T \mathbf{1}} &= \boldsymbol{\rho}^r + \sum_{i=2}^{s}\sum_{j=0}^{k_i-1} \frac{1}{\left[\boldsymbol{\rho}^l\right]^T \mathbf{1}} \frac{f^{(j)}(\kappa\lambda_i)}{j! f(\kappa\lambda_1)}\left(\mathbf{M}-\lambda_i\mathbf{I}\right)^j \mathbf{G}_i\mathbf{1} && (43) \\
\lim_{\kappa \to \frac{1}{\lambda_1}^-} \frac{\mathbf{g}(\kappa)}{f(\kappa\lambda_1)\left[\boldsymbol{\rho}^l\right]^T \mathbf{1}} &= \boldsymbol{\rho}^r. && (44)
\end{aligned}$$

Since $\left[\boldsymbol{\rho}^l\right]^T \mathbf{1} > 0$, as $\kappa \to \frac{1}{\lambda_1}^-$ the rankings produced by $\mathbf{g}(\kappa)$ converge to those produced by the entries of $\boldsymbol{\rho}^r$.

For the behavior as $\kappa \to 0$ we have

$$\begin{aligned}
\mathbf{g}(\kappa) &= \mathbf{I1} + \kappa\mathbf{M1} + \kappa^2\mathbf{M}^2\mathbf{1} + \ldots \\
\frac{\mathbf{g}(\kappa) - \mathbf{1}}{\kappa} &= \mathbf{M1} + \kappa\mathbf{M}^2\mathbf{1} + \kappa^2\mathbf{M}^3\mathbf{1}\ldots \\
\lim_{\kappa\to 0^+}\left[\frac{\mathbf{g}(\kappa) - \mathbf{1}}{\kappa}\right] &= \mathbf{M1}. && (45)
\end{aligned}$$

Therefore, it follows that as $\kappa \to 0^+$ the rankings produced by $\mathbf{g}(\kappa)$ converge to those produced by the vector $\mathbf{M1}$.

Finally, by substituting $\mathbf{M}$ with $\mathbf{B}$ and $\mathbf{A}^T$, we get the results in Eqs. (35) and (36). In the special case, when we transform $\kappa\lambda_1 \to \lambda_1$ we are looking at the non-parametrized versions of the OU and ID. Then, Eqs. (35) and (36) reduce to

$$\begin{aligned}
\mathbf{u} &\approx \begin{cases} \mathbf{1} + \mathbf{B1} & \text{for} \quad \lambda \to 0^+, \\ \frac{\left[\rho^l(\mathbf{B})\right]^T \mathbf{1}}{1-\lambda}\rho^r(\mathbf{B}) & \text{for} \quad \frac{1}{\lambda}^- \to 1, \end{cases} \\
\mathbf{d} &\approx \begin{cases} \mathbf{1} + \mathbf{A}^T\mathbf{1} & \text{for} \quad \lambda \to 0^+, \\ \frac{\left[\rho^r(\mathbf{A})\right]^T \mathbf{1}}{1-\lambda}\rho^l(\mathbf{A}) & \text{for} \quad \frac{1}{\lambda}^- \to 1. \end{cases}
\end{aligned}$$

where $\lambda = \lambda_1(\mathbf{A}) = \lambda_1(\mathbf{B})$. Summarizing, we have proved that (i) for an economy with $\lambda \to 0$, ranking (ordering) of country-industry pairs depends on out-degree centrality; (ii) for an economy with $\lambda \to 1$, the ranking of sectors depends solely on the network structure.

Appendix B: Proof of Theorem 2

Let us study the quasistationary product distribution of the Markov input-chain and the Markov output-chain. This distribution describes the evolution of the state-space of the Markov chain in the regime before the random walker becomes absorbed. Formally for an arbitrary absorbing Markov chain described with a transition matrix $\mathbf{\Omega} = [\Omega_{ij}]$, the quasistationary product distribution $\boldsymbol{\pi}$ is defined as

$$\boldsymbol{\pi} = \hat{\mathbf{x}} \odot \hat{\mathbf{y}},$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are correspondingly the right and left eigenvector associated with the largest eigenvalue λ of $\mathbf{\Omega}$ and $\odot$ is the Hadamard (element-wise) product. The eigenvectors are normalized in a way such that $\sum_i \hat{y}_i = 1$ and $\hat{\mathbf{y}}^T\hat{\mathbf{x}} = 1$.

We claim that the quasistationary product distributions $\boldsymbol{\pi}(\mathbf{A})$ and $\boldsymbol{\pi}(\mathbf{B})$ of the input and output chains are the same. This can be proven in the following way. First, notice that the matrices $\mathbf{A}$ and $\mathbf{B}$ can be written as

$$\mathbf{A} = \mathbf{Z}\mathbf{X}^{-1}, \\ \mathbf{B} = \mathbf{X}^{-1}\mathbf{Z},$$

where $\mathbf{X}$ is a diagonal matrix with entries $x_{ij} = x_i$ if $i = j$ and 0 otherwise. It is widely known that such matrices are similar since we can write $\mathbf{A} = \mathbf{X}\mathbf{B}\mathbf{X}^{-1}$.

Let $\hat{\mathbf{x}}(\mathbf{A}) = [\hat{x}_1(\mathbf{A}), \dots, \hat{x}_N(\mathbf{A})]^T$ and $\mathbf{y}(\mathbf{A} = [\hat{y}_1(\mathbf{A}), \dots, \hat{y}_N(\mathbf{A})]^T$ be the right and the left dominant eigenvector of the matrix $\mathbf{A}$, respectively. Then.

$$\begin{aligned}
\mathbf{A}\hat{\mathbf{x}}(\mathbf{A}) &= \lambda\hat{\mathbf{x}}(\mathbf{A}) \\
\mathbf{Z}\mathbf{X}^{-1}\hat{\mathbf{x}}(\mathbf{A}) &= \lambda\hat{\mathbf{x}}(\mathbf{A}) \\
\mathbf{X}^{-1}\mathbf{Z}\mathbf{X}^{-1}\hat{\mathbf{x}}(\mathbf{A}) &= \lambda\mathbf{X}^{-1}\hat{\mathbf{x}}(\mathbf{A}) \\
\mathbf{B}\mathbf{X}^{-1}\hat{\mathbf{x}}(\mathbf{A}) &= \lambda\mathbf{X}^{-1}\hat{\mathbf{x}}(\mathbf{A}) \\
\mathbf{B}\hat{\mathbf{x}}(\mathbf{B}) &= \lambda\hat{\mathbf{x}}(\mathbf{B}),
\end{aligned}$$

where,

$$\hat{\mathbf{x}}(\mathbf{B}) = \mathbf{X}^{-1}\hat{\mathbf{x}}(\mathbf{A}), \tag{46}$$

is the right dominant eigenvector of the matrix $\mathbf{B}$. In a similar way,

$$\begin{aligned}
\hat{\mathbf{y}}^T(\mathbf{A})\mathbf{A} &= \lambda\hat{\mathbf{y}}^T(\mathbf{A}) \\
\hat{\mathbf{y}}^T(\mathbf{A})\mathbf{X}\mathbf{B}\mathbf{X}^{-1} &= \lambda\hat{\mathbf{y}}^T(\mathbf{A}) \\
\hat{\mathbf{y}}^T(\mathbf{A})\mathbf{X}\mathbf{B}\mathbf{X}^{-1}\mathbf{X} &= \lambda\hat{\mathbf{y}}^T(\mathbf{A})\mathbf{X} \\
\hat{\mathbf{y}}^T(\mathbf{A})\mathbf{X}\mathbf{B} &= \lambda\hat{\mathbf{y}}^T(\mathbf{A})\mathbf{X} \\
\hat{\mathbf{y}}^T(\mathbf{B})\mathbf{B} &= \lambda\hat{\mathbf{y}}^T(\mathbf{B}),
\end{aligned}$$

where,

$$\hat{\mathbf{y}}^T(\mathbf{B}) = \hat{\mathbf{y}}(\mathbf{A})^T\mathbf{X}, \tag{47}$$

is the left dominant eigenvector of the matrix $\mathbf{B}$. Combining equations (46) and (47) we obtain

$$\hat{\mathbf{x}}(\mathbf{A}) \odot \hat{\mathbf{y}}(\mathbf{A}) = \hat{\mathbf{x}}(\mathbf{B}) \odot \hat{\mathbf{y}}(\mathbf{B}),$$

thus concluding the proof.

Appendix C: Proof of the Theorem 3

Consider Markov absorbing chain with J absorbing states and a transition matrix:

$$\mathbf{P}_{out} = \begin{bmatrix} \mathbf{I}_{J\times J} & \mathbf{0}_{J\times n} \\ \mathbf{D}_\eta & \mathbf{B} \end{bmatrix},$$

where $\mathbf{I}_{J\times J}$ is identity matrix, $\mathbf{0}_{J\times n}$ zero matrix, and $\mathbf{B} = [b_{ij}]$ is the $n \times n$ adjacency output matrix. Let $V = \{1, 2, \ldots, n\}$ be the set of transition states – this is the set of all country-industry pairs. Let $S = \{s_1, s_2, \ldots, s_J\}$ be the set of all absorbing states. Starting in i, the process may be absorbed in $s_{\hat{j}} \in S$ in one or more steps. The probability of absorption in a single step is $\eta_{i\hat{j}}$. If this does not happen, the process may move either to another absorbing state (in which case it is impossible to reach $s_{\hat{j}}$), or to a transient state k. In the latter case there is probability $m_{k\hat{j}}$ of being absorbed in the state $s_{\hat{j}}$). Therefore, we have

$$m_{i\hat{j}} = \eta_{i\hat{j}} + \sum_{k\in V} b_{ik} m_{k\hat{j}}$$

which can be written in matrix form as $\mathbf{M} = \mathbf{D}_\eta + \mathbf{B}\mathbf{M}$. Thus,

$$\mathbf{M} = (\mathbf{I} - \mathbf{B})^{-1}\mathbf{D}_\eta \tag{48}$$

Note that

$$\lim_{t\to\infty} [\mathbf{P}_{out}]^t = \begin{bmatrix} \mathbf{I}_{J\times J} & \mathbf{0}_{J\times n} \\ \mathbf{M} & \mathbf{0}_{n\times n} \end{bmatrix},$$

hence the matrix $\mathbf{M}$ is row stochastic. Consider now an absorbing Markov chain with one absorbing state and a transition matrix $\mathbf{P}$ given by:

$$\mathbf{P} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \boldsymbol{\alpha} & \mathbf{Q} \end{bmatrix},$$

where $\boldsymbol{\alpha}$ and $\mathbf{Q}$ are δ and $\mathbf{A}^T$, respectively, for the input chain, and γ and $\mathbf{B}$, respectively, for the output chain. In this case, it follows from Eq. (48) that

$$\begin{aligned} (\mathbf{I} - \mathbf{A}^T)^{-1}\delta &= \mathbf{1} \\ (\mathbf{I} - \mathbf{B})^{-1}\gamma &= \mathbf{1} \end{aligned}$$

Assume now that the row sums of $\mathbf{Q}$ are all equal to a common value $c < 1$. In this case $\boldsymbol{\alpha} = (1 - c)\mathbf{1}$. The last two equations can be rewritten as

$$\begin{aligned} (\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{1} &= \frac{1}{1-c}\mathbf{1} \\ (\mathbf{I} - \mathbf{B})^{-1}\mathbf{1} &= \frac{1}{1-c}\mathbf{1}, \end{aligned}$$

thus concluding the proof of the theorem.

1. Leontief, W. *The Structure of American Economy, 1919–1939: An Empirical Application of Equilibrium Analysis*; Oxford University Press, 1951.
2. Peters, G.P.; Minx, J.C.; Weber, C.L.; Edenhofer, O. Growth in emission transfers via international trade from 1990 to 2008. *Proceedings of the National Academy of Sciences* **2011**, *108*, 8903–8908, [https://www.pnas.org/content/108/21/8903.full.pdf]. doi:10.1073/pnas.1006388108.

3. Timmer, M.P.; Dietzenbacher, E.; Los, B.; Stehrer, R.; De Vries, G.J. An illustrated user guide to the world input–output database: the case of global automotive production. *Review of International Economics* **2015**, *23*, 575–605.
4. Lucas Jr, R.E. Understanding business cycles. Carnegie-Rochester conference series on public policy. North-Holland, 1977, Vol. 5, pp. 7–29.
5. Acemoglu, D.; Carvalho, V.; Ozdaglar, A.; Tahbaz-Salehi, A. The network origins of aggregate fluctuations. *Econometrica* **2012**, *80*, 1977–2016.
6. Carvalho, V.M.; Tahbaz-Salehi, A. Production networks: A primer. *Annual Review of Economics* **2019**, *1*, 635–663.
7. Baqaee, D.R.; Farhi, E. The macroeconomic impact of microeconomic shocks: beyond Hulten's Theorem. *Econometrica* **2019**, *87*, 1155–1203.
8. Carvalho, V.M. From micro to macro via production networks. *Journal of Economic Perspectives* **2014**, *28*, 23–48.
9. Barrot, J.N.; Sauvagnat, J. Input specificity and the propagation of idiosyncratic shocks in production networks. *The Quarterly Journal of Economics* **2016**, *131*, 1543–1592.
10. Acemoglu, D.; Akcigit, U.; Kerr, W. Networks and the macroeconomy: An empirical exploration. *NBER Macroeconomics Annual* **2016**, *30*, 273–335.
11. Antràs, P.; Chor, D. Organizing the global value chain. *Econometrica* **2013**, *81*, 2127–2204.
12. Miller, R.E.; Temurshoev, U. Output upstreamness and input downstreamness of industries/countries in world production. *International Regional Science Review* **2017**, *40*, 443–475.
13. Fally, T. Production staging: measurement and facts. *Boulder, Colorado, University of Colorado Boulder, May* **2012**, pp. 155–168.
14. Antràs, P.; Chor, D.; Fally, T.; Hillberry, R. Measuring the upstreamness of production and trade flows. *American Economic Review* **2012**, *102*, 412–16.
15. Wang, Z.; Wei, S.J.; Yu, X.; Zhu, K. Characterizing global and regional manufacturing value chains: stable and evolving features. Technical report, Centro Studi Luca d'Agliano Development Studies, Working Paper 419, 2017.
16. Solow, R. On the Structure of Linear Models. *Econometrica* **1952**, *20*, 29–46.
17. Timmer, M.; Los, B.; Stehrer, R.; de Vries, G. An anatomy of the global trade slowdown based on the WIOD 2016 release. Technical report, Groningen Growth and Development Centre, University of Groningen, 2016.
18. Kemeny, J.; Snell, J. *Finite Markov Chains*; Springer, 1983.
19. Darroch, J.N.; Seneta, E. On quasi-stationary distributions in absorbing discrete-time finite Markov chains. *Journal of Applied Probability* **1965**, *2*, 88–100.
20. McNerney, J.; Savoie, C.; Caravelli, F.; Farmer, J.D. How production networks amplify economic growth. *arXiv preprint arXiv:1810.07774* **2018**.
21. Meyer, C.D. *Matrix analysis and applied linear algebra*; Vol. 71, Siam, 2000.