WIOD - Sensitivity - short

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1 Theoretical model - Cobb Douglas

To understand the structure of the world economy we generalize the model introduced in **?**.

Model formulation: Consider a world with $J \geq 1$ countries (economies) $\{con_1, \dots, con_J\}$ and $S \geq 1$ sectors (industries) $\{sec_1, \dots, sec_S\}$ as a network $G = (V, E)$ of N nodes in which each node represents a country-industry pair. Country-industry pairs $(con_{\hat{i}}, sec_r)$ are mapped to the nodes in V with $(con_{\hat{i}}, sec_r) \rightarrow (\hat{i} - 1)S + r$, for $\hat{i} = 1, \dots, J$ and $r = 1, \dots, S$. Note that the nodes $1, \dots, S$ correspond to the country 1, con_1, the nodes $S + 1, \dots, 2S$ are related to the country 2, con_2, and so on.

Consider a static world-economy consisting of N competitive county-industry pairs, each producing a distinct product. Each product can be either consumed by the households or used as an intermediate input for production of other goods. Firms in each industry employ Cobb-Douglas production technologies with constant returns to scale to transform intermediate inputs and labor into final products. In particular, the output y_i of country-industry pair i is given by

$$y_i = \zeta_i \xi_i \left[\ell_i\right]^{\alpha_i} \prod_{j=1}^{N} \left[\chi_{ji}\right]^{a_{ji}}, \tag{1}$$

χ_{ji} is the amount (quantity) of good j produced by country-industry pair j used as input by country-industry pair i, ℓ_i is the amount of labor hired by firms in country-industry pair i, ζ_i is a Hicks-neutral productivity shock, and $\xi_i > 0$ is a normalization constant. The exponent $a_{ji} \geq 0$ in (1) represents the share of good j (produced by county-industry pair j) that are used in the production technology of good i (in the county-industry pair i), i.e., the ji-th entry of the matrix $\mathbf{A}$. A larger a_{ji} means that good j is more important in producing in, whereas $a_{ji} = 0$ implies that good j is not a required input for n's production technology. We assume that, for each i, $\alpha_i > 0$, and $a_{ji} \geq 0$ for all j, and

$$\alpha_i + \sum_{j=1}^{N} a_{ji} = 1, \tag{2}$$

so that the production function of each country-industry pair exhibits constant returns to scale.

As the output of each industry is used as input for other industries or consumed in the final good sector, the market-clearing condition for country-industry pair i can be written as

$$y_i = c_i + \sum_{j=1}^{N} \chi_{ij}, \tag{3}$$

where c_i is the (final) amount of goods produced in the country-industry pair i that are consumed (final consumption of the output of country-industry i). The preference side of this world economy is summarized by a representative household with a utility function

$$u(c_1, c_2, \dots, c_N, \ell) = \prod_{i=1}^{N} \left[c_i\right]^{\beta_i}, \tag{4}$$

where $\beta_i \in (0, 1)$ designates the weight of the good i (produced by country-industry pair i) in the representative household's preferences (with the normalization $\sum_i^N \beta_i = 1$). Denoting the price of the output of country-industry n by p_n, and assuming that income comes only from labor, $\omega\ell$, the representative household's budget constraint can be written as

$$\sum_{i=1}^{N} p_i c_i = \omega\ell.$$

Model properties: We focus on the competitive equilibrium of this static economy, so that all country-sector pairs (firms) maximize profits and the representative household maximizes its utility, in both cases taking all prices as given, and the market-clearing conditions for each good and labor are satisfied. The Cobb-Douglas production functions in (1), combined with profit maximization, imply

$$a_{ji} = \frac{p_j \chi_{ji}}{p_i y_i}, \tag{5}$$

$$\alpha_i = \frac{\omega \ell_i}{p_i y_i}. \tag{6}$$

Utility maximization in turn yields

$$\frac{p_i c_i}{\beta_i} = \frac{p_j c_j}{\beta_j}. \tag{7}$$

Since total household income is equal to labor income, we have

$$\sum_{i=1}^{N} p_i c_i = \omega \ell,$$

which yields

$$p_i c_i = \beta_i \omega \ell, \tag{8}$$

for all i. We assume that (i) labor is the only primary factor of production, (ii) all firms make zero profits, and (iii) total labor supply is normalized to 1, $\ell = 1$, gross domestic product in the world economy, or gross world product (GWP), is equal to the market wage, ω:

$$\mathrm{GWP} = \omega = \sum_i p_i c_i. \tag{9}$$

Using (3), (5), and (8) for equilibrium revenues per country-industry pair, we have

$$\begin{aligned} r_i \equiv p_i y_i &= p_i \sum_j \chi_{ij} + c_i p_i \\ &= p_i \sum_j \frac{a_{ij} p_j y_j}{p_i} + \beta_i \omega \ell \\ &= \sum_j a_{ij} r_j + \beta_i \omega. \end{aligned}$$

Writing $\mathbf{r} = [r_1, \ldots, r_N]^T$ and $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_N]^T$, the last equation can be written as

$$\mathbf{r} = \omega \left(\mathbf{I} - \mathbf{A}\right)^{-1} \boldsymbol{\beta}\ . \tag{10}$$

We now define the world influence index as

$$\mathbf{v} = \left(\mathbf{I} - \mathbf{A}\right)^{-1} \boldsymbol{\beta}. \tag{11}$$

Therefore, by combining (10) and (11) with (9) we have $\mathbf{v} = \mathbf{r}/\omega$ or

$$v_i = \frac{p_i y_i}{\sum_j p_j c_j} = \frac{p_i y_i}{\mathrm{GWP}}. \tag{12}$$

These quantities are also known as Domar weights (defined as world-sectoral sales divided by GWP **??**).

Remarks: Several remarks are in order.

First, in a special case, when $\alpha_i = \alpha$ for all i, since $\alpha + \sum_j a_{ji} = 1$ and defining Ω_{ji} such that $a_{ji} = (1-\alpha)\Omega_{ji}$ with $\sum_j \Omega_{ji} = 1$, Eq. (11) reduces to

$$\mathbf{v} = \left[\mathbf{I} - (1-\alpha)\boldsymbol{\Omega}\right]^{-1}\ \boldsymbol{\beta} \tag{13}$$

where $\boldsymbol{\Omega} = [\Omega_{ji}]$ is $N \times N$ stochastic matrix.

Second, if country-industry pairs source equally from all other pairs and also sell in equal proportions to final demand, (13) collapses to $\mathbf{v} = 1/N$. This is a restatement of the classical diversification argument by Lucas **?**: aggregate volatility is then proportional to $1/\sqrt{N}$ and shocks to individual sectors in a country wash out in the aggregate from a law of large numbers argument.

Third, with homogeneous input shares and heterogeneity in final demand, (13) collapses to

$$\mathbf{v} = \left[\mathbf{I} - \frac{(1-\alpha)}{N}\right]^{-1} \boldsymbol{\beta}.$$

For large N, this can be approximated as $\mathbf{v} \approx \boldsymbol{\beta}$. Hence the influence of individual firms is proportional to their sales to final demand. Moreover, if $\alpha = 1$, (13) becomes $\mathbf{v} = \boldsymbol{\beta} = \frac{\mathbf{r}}{\mathbf{r1}}$. This is a direct statement of the granular hypothesis presented by Gabaix **?**: in the presence of a sufficiently skewed sales Herfindahl index that is consistent with a power law with fat tails, shocks to large firms can contribute significantly to aggregate fluctuations.

Finally, with heterogeneous input shares and homogeneous sales to final demand, that is $\beta_i = 1/N$, (13) collapses to

$$\mathbf{v} = \frac{1}{N}[\mathbf{I} - (1-\alpha)\boldsymbol{\Omega}]^{-1}\mathbf{1}.$$

This is a modified influence vector introduced in **?** (as the factor α does not appear in the equation): if the distribution of $\mathbf{v}$ is consistent with a power law distribution with fat tails, idiosyncratic shocks to important suppliers can propagate through the network of production and show up in the aggregate of the economy.

Note on GWP: Plugging χ_{ji} and ℓ_i from (5) and (6) into i's production function (1), and setting (which simplifies our key expressions without any bearing on our results)

$$\xi_i \alpha_i^{\alpha_i} \prod_j [a_{ji}]^{a_{ji}} = 1,$$

we have

$$\begin{aligned} y_i &= \zeta_i \xi_i \left[\frac{\alpha_i p_i y_i}{\omega}\right]^{\alpha_i} \prod_j \left[\frac{a_{ji} p_i y_i}{p_j}\right]^{a_{ji}} \\ &= \zeta_i \xi_i \alpha_i^{\alpha_i} \frac{1}{\omega^{\alpha_i}} p_i y_i \prod_j [a_{ji}]^{a_{ji}} \prod_j [p_j]^{-a_{ji}} \\ &= \zeta_i \left[\frac{p_i y_i}{\omega}\right]^{\alpha_i} \prod_j \left[\frac{p_i y_i}{p_j}\right]^{a_{ji}}. \end{aligned}$$

Therefore,

$$\alpha_i \log \omega = \varepsilon_i + \log p_i - \sum_j a_{ji} \log p_j,$$

where $\varepsilon_i = \log \zeta_i$ are microeconomic shocks that are i.i.d. across country-industry pairs, are symmetrically distributed around the origin with full support over $\mathbb{R}$, and have a finite standard deviation, which we normalize to one. The previous equation implies that

$$\log \frac{p_i}{\omega} = \sum_j a_{ji} \log \frac{p_j}{\omega} - \varepsilon_i.$$

Rewriting this system of equations in matrix form we get that $\hat{\mathbf{p}} = \mathbf{A}^T \hat{\mathbf{p}} - \varepsilon$, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)^T$ and $\hat{\mathbf{p}} = (\log(p_1/\omega), \ldots, \log(p_N/\omega)$ denote the vectors of log relative prices and productivity shocks, respectively. Consequently, the equilibrium vector of (log) relative prices is given by

$$\hat{\mathbf{p}} = -(\mathbf{I} - \mathbf{A}^T)^{-1}\varepsilon = -\mathbf{L}\varepsilon, \tag{14}$$

where $\mathbf{L} = [l_{ij}]$ is the Leontief inverse. This can be also written as

$$\log \frac{p_i}{GWP} = -\sum_{j=1}^N l_{ji} \varepsilon_j. \tag{15}$$

Equation (11) implies $v_i = p_i y_i / GWP = \sum_{j=1}^N l_{ij} \beta_j$. Combining with equation (15), the following result holds:

$$\log(y_i) = \sum_{j=1}^N l_{ij} \varepsilon_j + \delta_i, \tag{16}$$

Table 1: Aggregated World input-output table for 2014: $\mathbf{v}$ vector.

Domar weights							
CHN		EU		USA		ROW	
Goods	Services	Goods	Services	Goods	Services	Goods	Services
0.23	0.19	0.12	0.34	0.30	0.54	0.09	0.32

Table 2: Aggregated World input-output table for 2014: α vector.

Labor elasticity							
CHN		EU		USA		ROW	
Goods	Services	Goods	Services	Goods	Services	Goods	Services
0.17	0.53	0.58	0.37	0.27	0.45	0.61	0.46

where δ_i is some constant that is independent of the shocks. Multiplying both sides of (15) by β_i and summing over all country-industry pairs i lead to

$$\log(GWP) = \sum_{j=1}^{N}\sum_{i=1}^{N}\beta_i l_{ji}\varepsilon_j + \sum_{i=1}^{N}\beta_i \log p_i.$$

By normalizing the price index $\prod_i p_i^{\beta_i} = 1$, we obtain

$$\log(\text{GWP}) = \sum_i \beta_i \sum_j l_{ji}\varepsilon_j \tag{17}$$

Since $\mathbf{L} = \widetilde{\mathbf{L}}^T$, from (17) and (11) it follows the aggregate output of the world economy is given by

$$y \equiv \log(\text{GWP}) = \sum_{i=1}^{N} v_i\varepsilon_i. \tag{18}$$

Equation (18), related to **???**, shows that in a competitive world economy with constant returns to scale technologies, world aggregate output is a linear combination of country-industry level productivity shocks, with coefficients v_n given by Domar weights. Moreover, the Domar weight of each country-industry pair n depends only on the preference shares, $\beta_1, \dots, \beta_N$, and the corresponding column of the world-economy's Leontief inverse.

Empirical example: Let us return to our aggregated economy example and calculate the corresponding Domar weights $\mathbf{v}$, the labor input share α and the preference shares β. The first quantity can be directly calculated using equation (11). Moreover, using equation (6) and the properties of the World input output table we can estimate the entries of the α vector as

$$\alpha_i = \frac{w_i}{x_i}.$$

In a similar fashion, due to the properties described with (7) we can estimate the elements of the β vector as

$$\beta_n = \frac{f_i}{\sum_j f_j}.$$

The results are depicted in Tables 1-2. **Discuss...**

Table 3: Aggregated World input-output table for 2014: β vector.

Preference shares							
CHN		EU		USA		ROW	
Goods	Services	Goods	Services	Goods	Services	Goods	Services
0.05	0.09	0.04	0.18	0.09	0.31	0.03	0.21

2 GWP equation - updated

First part - Cobb Douglas

Assumptions:

1. p is the multiplier of the GWP, $p = 1$, stable, $p < 1$: decrease, $p > 1$: increase
2. β and ε remain constant through the change of GWP.
3. p can be written as a product of the multipliers p_{ij} of each link: $p = \prod_i^N \prod_j^N p_{ij}$

$$\log(GWP) = \sum_{i=1}^{N} \varepsilon_i \sum_{j=1}^{N} l_{ij}\beta_j \tag{19}$$

$$\begin{aligned} GWP &= \prod_i^N \left[\prod_j^N \exp^{l_{ij}\beta_j} \right]^{\varepsilon_i} \\ &= \prod_i^N \prod_j^N \left[\exp^{l_{ij}\beta_j}\right]^{\varepsilon_i} \\ &= \prod_i^N \prod_j^N \exp^{l_{ij}\beta_j\varepsilon_i} \end{aligned}$$

$$\begin{aligned} pGWP &= p \prod_i^N \prod_j^N \exp^{l_{ij}\beta_j\varepsilon_i} \\ &= \prod_i^N \prod_j^N p_{ij} \exp^{l_{ij}\beta_j\varepsilon_i} \\ &= \prod_i^N \prod_j^N \exp^{log(p_{ij})} \exp^{l_{ij}\beta_j\varepsilon_i} \\ &= \prod_i^N \prod_j^N \exp^{l_{ij}\beta_j\varepsilon_i + log(p_{ij})} \\ &= \prod_i^N \prod_j^N \exp^{\beta_j\varepsilon_i(l_{ij} + \frac{log(p_{ij})}{\beta_j\varepsilon_i})} \end{aligned}$$

$$log(pGWP) = \sum_{i=1}^{N} \varepsilon_i \sum_{j}^{N} (l_{ij} + \frac{log(p_{ij})}{\beta_j\varepsilon_i})\beta_j \qquad = \sum_{i=1}^{N} \varepsilon_i \sum_{j}^{N} (l_{ij} + \Delta l_{ij})\beta_j$$

In other words, $\Delta l_{ij} = \frac{log(p_{ij})}{\beta_j\varepsilon_i}$ and $p_{ij} = \exp^{\Delta l_{ij}\beta_j\varepsilon_i}$.

Second part - Leontief derivative

The sensitivity of the fundamental matrix $\mathbf{L}$ with respect to $\boldsymbol{\theta}$ is given by

$$\frac{d\,\mathrm{vec}\,\mathbf{L}}{d\,\boldsymbol{\theta}^T} = \left(\mathbf{L}^T \otimes \mathbf{L}\right) \frac{d\,\mathrm{vec}\,\mathbf{A}}{d\,\boldsymbol{\theta}^T} \tag{20}$$

Assume now that we consider an (input/output) chain given by the matrix $\mathbf{P}$, Eq. **??**. Let the (i,j) entry of the matrix $\mathbf{A}$, for fixed values of i and j, is θ and we ask the question what is the rate of change of $\mathbf{L}$ in response to a change in the the parameter θ, holding the all other entries of matrix $\mathbf{A}$ constant. Then the vector $d\,\mathrm{vec}\,\mathbf{A}/d\,\theta$ has only one nonzero entry at $n(i-1)+j$ which is equal to 1. The $n(i-1)+j$ column of the matrix $\left(\mathbf{L}^T \otimes \mathbf{L}\right)$ reads

$$[\ell_{j1}\ell_{1i} \quad \ell_{j2}\ell_{1i} \quad \dots \ \ell_{jn}\ell_{1i} \ \dots \ \ell_{j1}\ell_{2i} \quad \ell_{j2}\ell_{2i} \quad \dots \ \ell_{jn}\ell_{2i} \ \dots \ \ell_{j1}\ell_{ni} \quad \ell_{j2}\ell_{ni} \quad \dots \ \ell_{jn}\ell_{ni}]^T$$

Rewriting Eq. 20 as a $n \times n$ matrix, we finally derive that the sensitivity matrix of $\mathbf{L}$ with respect to the (i,j) entry of the matrix $\mathbf{Q}$, $\mathbf{L}_{ij}^{sen}$, is

$$\mathbf{L}_{ij}^{sen} = \begin{bmatrix} \ell_{j1}\ell_{1i} & \ell_{j1}\ell_{2i} & \dots & \ell_{j1}\ell_{ni} \\ \ell_{j2}\ell_{1i} & \ell_{j2}\ell_{2i} & \dots & \ell_{j2}\ell_{ni} \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{jn}\ell_{1i} & \ell_{jn}\ell_{2i} & \dots & \ell_{jn}\ell_{ni} \end{bmatrix} \tag{21}$$

From this, we have that if we have a change ΔA, such that only the value of a_{ij} has changed: $\Delta a_{ij} = 1$, and $\Delta a_{pq} = 0$, for all $p \neq i$ and $q \neq j$, we have $\boldsymbol{\Delta}\mathbf{L} = \mathbf{L}_{ij}^{sen}$. Then $\boldsymbol{\Delta}\mathbf{L}$ is a $N \times N$ matrix with elements:

$$\boldsymbol{\Delta}\mathbf{L} = \begin{bmatrix} \Delta\ell_{11} & \Delta\ell_{12} & \dots & \Delta\ell_{1N} \\ \Delta\ell_{21} & \Delta\ell_{22} & \dots & \Delta\ell_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta\ell_{N1} & \Delta\ell_{N2} & \dots & \Delta\ell_{NN} \end{bmatrix} \tag{22}$$

Third part - Connection between parts one and two

Using the differentials $\Delta\ell_{ij}$ calculated in part two and inserting them in the formula $p_{ij} = \exp^{\Delta\ell_{ij}\beta_j\varepsilon_i}$ calculated in part one, we can calculate the multiplicators p_{ij} for each i,j pair. Once we have them, we can calculate $p = \prod_i^N \prod_j^N p_{ij}$ as a scalar metric, which represents the multiplicator of the GWP. Thus we can calculate p for each ΔA as scalar representing the sensitivity of the i,j link.

3 Country P results

The P matrix is a $N \times N$ matrix, where each element p^{uv} represents the multiplier of the GWP if there is a change $\boldsymbol{\Delta}\mathbf{L} = \mathbf{L}_{uv}^{sen}$ representing the derivative of the u,v pair in the A matrix, such that $p^{uv} = \prod_i^N \prod_j^N p_{ij}^{uv}$, and $p_{ij}^{uv} = \exp^{\Delta\ell_{ij}^{uv}\beta_j\varepsilon_i}$. For clarification: u,v represent the indices of the element a_{uv} for which the derivative is calculated, and i,j represent the elements of the $\mathbf{L}_{uv}^{sen}$ matrix.

As seen, the value of p_{ij} depends on the change $\Delta\ell_{ij}$, but also on the value ε_i and β_j. Therefore, β has a large impact on the final results in addition to $\Delta\ell_{ij}$.

4 Shock matrix

The celebrated theorem of **?** states that for efficient economies and under minimal assumptions, the first-order macroeconomic impact of microeconomic shocks is given by (see, for example, **?**):

$$\frac{d\log Y}{d\log A_i} = \lambda_i \tag{23}$$

where Y is the equilibrium aggregate output, A_i is Hicks-neutral technology, and λ_i is Domar weight. We consider the model suggested in **?** (see also Appendix A), for which it follows that:

$$\log(\mathrm{GWP}) = \sum_{i=1}^{n} \varepsilon_i \lambda_i \tag{24}$$

$$= \sum_{i=1}^{n} \varepsilon_i \sum_{j=1}^{n} \ell_{ij}\beta_j \tag{25}$$

where $\varepsilon_i = \log A_i$, ℓ_{ij} is the (i,j) entry of the Leontief-inverse matrix, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} \equiv [\ell_{ij}]$, and $\beta_i \in (0,1)$ designates the weight of the good i (produced by country-industry pair i) in the representative household's preferences (with the normalization $\sum_i \beta_i = 1$). Equation (24) shows that in a competitive world economy with constant returns to scale technologies, world aggregate output is a linear combination of country-industry level productivity shocks, with coefficients λ_i given by Domar weights. Moreover, Eq. (25), the Domar weight of each country-industry pair i depends only on the preference shares, $\beta_1, \ldots, \beta_n$, and the corresponding column of the world-economy's Leontief inverse.

For a fixed pair (u,v), let Δa^{uv} be a disturbance of the (u,v) entry of the matrix $\mathbf{A}$. This disturbance generates a change in the GWP which we assume can be written as

$$\mathrm{GWP}^{new} = p^{uv}\mathrm{GWP}$$

where p^{uv} is a multiplier of the GWP, which can be written as a product of the multipliers p_{ij}^{uv} of each link (i,j), that is

$$p^{uv} = \prod_{i,j} p_{ij}^{uv} \tag{26}$$

Combining equations (24), (25), and (26) it is easy to verify that

$$\log(\mathrm{GWP}^{new}) = \sum_{i=1}^{n} \varepsilon_i \sum_{j=1}^{n} \left(\ell_{ij} + \Delta\ell_{ij}^{uv}\right)\beta_j \tag{27}$$

with

$$\Delta\ell_{ij}^{uv} = \frac{\log p_{ij}^{uv}}{\beta_j \varepsilon_i} \tag{28}$$

Define a $n \times n$ matrix $\mathbf{P}$, which will be called *shock matrix*, as:

$$\mathbf{P} = [\log p^{uv}] = \left[\sum_{i,j} \log p_{ij}^{uv}\right] \tag{29}$$

Next theorem shows the effect on the GWP (or GDP) the disturbance Δa^{uv} has. The matrix $\mathbf{P}$ can be computed as

$$\mathbf{P} = \left[\sum_{i,j} \ell_{vi}\ell_{ju}\Delta a^{uv}\beta_j\varepsilon_i\right] \tag{30}$$

Moreover, assuming that $\Delta a^{uv} < 0$ (sudden drop of production in the node u for $u = v$, or sudden drop of the network linkage uv when $u \neq v$) and defining $u_{max} = \min_{uu} \log p^{uu}$ and $(u,v)_{max} = \min_{uv} \log p^{uv}$, $u \neq v$, we can compute the largest effect on the (world) economy a country-industry pair, a country, and an industry have, as well as the effect of network linkage has on the (world) economy for different networks.

Assume u and v are fixed and define a vector $\mathbf{c} = [c_1, c_2, \ldots, c_n]^T$ with

$$c_j = \frac{\Delta\ell_{uj}^{uv}\,\beta_j}{\Delta\ell_{uv}^{uv}\,\beta_v} = \frac{\ell_{ju}\beta_j}{\ell_{vu}\beta_v} \tag{31}$$

The equation (31) shows the ordering of the shock propagation Δa^{uv}: the maximum effect the shock will have for such j for which the product $\ell_{jv}\beta_j$ is maximal. This is unexpected result: if a perturbation occurs on the link uv, the largest effect will be on the link $uj^\star$ for which $j^\star = \max_j \ell_{jv}\beta_j$. Note that c_j (or the ordering) does not depend on the shock, ε_j and v. Also all quantities in (31) can be computed from WIOD.

Properties of the sensitive matrix – Sensitivity matrix is similar to the matrix

$$\begin{bmatrix} \mathbf{0}_{n-1} & \mathbf{u} \\ \mathbf{v}^T & a \end{bmatrix}$$

where $\mathbf{0}_{n-1}$ is $(n-1) \times (n-1)$ null matrix, $\mathbf{u}$ and $\mathbf{v}$ are $n-1$ dimensional vectors such that $\mathbf{v}^T\mathbf{u} = 0$, and $a > 0$. In this case, the eigenvalues of sensitivity matrix are $n-2$ zeros and

$$\frac{a \pm \sqrt{a^2 + 4\mathbf{v}^T\mathbf{u}}}{2}$$

Since $\mathbf{v}^T\mathbf{u} = 0$, the eigenvalues of the sensitivity matrix are $n-1$ zeros and a.