Fluid Dynamics

Interesting Problems on complex variables and inviscid flow

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Problem 1

Remark:

Fundamental proof in Complex Analysis!

In order for the function $w(z)$ (where $w = u + iv$ and $z = x + iy$) to have the same derivative in the $\mathrm{d}z = \mathrm{d}x$ and $\mathrm{d}z = i\mathrm{d}y$ directions, the Cauchy-Riemann conditions

$$u_x = v_y, \quad u_y = -v_x$$

need to hold. Then

$$w'(z) = u_x + iv_x = v_y - iu_y$$

Verify that, if the Cauchy-Riemann conditions hold, then the derivative in any direction is the same.

Solution

By definition,

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{w\left(z + \Delta z\right) - w\left(z\right)}{\Delta z} = \lim_{\Delta z \to 0} \frac{u\left(z + \Delta z\right) - u\left(z\right)}{\Delta z} + i \lim_{\Delta z \to 0} \frac{v\left(z + \Delta z\right) - v\left(z\right)}{\Delta z}. \tag{1}$$

Since w is a function of two real variables x and y, we ought to be able perform some kind of variable transform. As the hint says, if we adopt the notion that $x = t\cos\left(\theta\right)$ and $y = t\sin\left(\theta\right)$, by specifying what θ is, we can take a derivative in any direction on our two-dimensional plane. In particular, suppose we already choose that angle and apply the transformation of variables. Then, from (1) it holds

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{1}{\cos\left(\theta\right) + i\sin\left(\theta\right)} \lim_{\Delta t \to 0} \frac{u\left(\left[\cos\left(\theta\right) + i\sin\left(\theta\right)\right]\left[t + \Delta t\right]\right) - u\left(\left[\cos\left(\theta\right) + i\sin\left(\theta\right)\right]t\right)}{\Delta t} + \tag{2}$$

$$+ \frac{i}{\cos\left(\theta\right) + i\sin\left(\theta\right)} \lim_{\Delta t \to 0} \frac{v\left(\left[\cos\left(\theta\right) + i\sin\left(\theta\right)\right]\left[t + \Delta t\right]\right) - v\left(\left[\cos\left(\theta\right) + i\sin\left(\theta\right)\right]t\right)}{\Delta t}. \tag{3}$$

Since the only variation is in t, (2) becomes

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{1}{\cos\left(\theta\right) + i\sin\left(\theta\right)} \frac{\partial u}{\partial t} + i \frac{1}{\cos\left(\theta\right) + i\sin\left(\theta\right)} \frac{\partial v}{\partial t}. \tag{4}$$

Applying the rationalisation principle to (3) we obtain

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \cos\left(\theta\right)\left(\frac{\partial u}{\partial t} + i\frac{\partial v}{\partial t}\right) + \sin\left(\theta\right)\left(\frac{\partial v}{\partial t} - i\frac{\partial u}{\partial t}\right). \tag{5}$$

Applying the chain rule we get

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \cos\left(\theta\right)\frac{\partial x}{\partial t}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \sin\left(\theta\right)\frac{\partial x}{\partial y}\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right) \tag{6}$$

Which is equivalent to

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \cos^2\left(\theta\right)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \sin^2\left(\theta\right)\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right). \tag{7}$$

We can now use the fact that the Cauchy-Riemann conditions hold. Namely, we have that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \wedge \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{8}$$

Combining (7) and (6) and using the trigonometric identity $\cos^2\left(\theta\right) + \sin^2\left(\theta\right) = 1$ we obtain

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{9}$$

Or equivalently,

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}. \tag{10}$$

Problem 1 continued on next page...

Remark

I have obtained an expression for the complex derivative of a function w and the final expression (before considering Cauchy-Riemann conditions) involved partial derivatives with respect to x and y and an arbitrary angle θ such that $\tan(\theta) = \frac{y}{x}$. After considering Cauchy-Riemann conditions, it transpires that two of the possible ways to formulate this derivative only include variables x and y, and it does not depend on the angle θ. If the derivative does not depend on θ, it is evident that it is the same for any direction.

Problem 2

Consider a fluid flow in the whole plane with a fluid source at $(x, y) = (1, 0)$ and a sink at $(x, y) = (2, 0)$, both with unit strength. Moreover, assume that a positive unit circulation develops around the sink (as they often do in bath tubs) but none around any closed path enclosing both the source and the sink.
Using complex variables, write down the corresponding flow and plot a few representative streamlines.

Solution

Since there is a sink, it has a singularity at $(2, 0)$. Evidently there should be an influx of fluid into the sink, since it has unit strength, the flux F_{sink} will be equal to -1. Since there is a unit circulation, the potential corresponding to the sink will be

$$W_{sink} = -\frac{1}{2\pi}\ln(z-2) - \frac{i}{2\pi}\ln(z-2) = -\frac{1+i}{2\pi}\ln(z-2). \tag{11}$$

Since the source has a unit strength, it holds that $F_{source} = 1$. Since there is zero circulation around both the sink and the source, we can conclude that there must a be a negative unit circulation surrounding the source, as that of the source and the sink would add up to 1.
The potential of the point source will be

$$W_{source} = \frac{1}{2\pi}\ln(z-1) + \frac{i}{2\pi}\ln(z-1) = \frac{1+i}{2\pi}\ln(z-1). \tag{12}$$

Overall, the potential will be

$$W = \frac{1+i}{2\pi}\ln(z-1) - \frac{1+i}{2\pi}\ln(z-2) = \frac{1+i}{2\pi}\ln\frac{z-1}{z-2}. \tag{13}$$

We will use the identity

$$\frac{\mathrm{d}W}{\mathrm{d}z} = u - iv \tag{14}$$

And the result

$$\frac{\mathrm{d}W}{\mathrm{d}z} = \frac{1+i}{2\pi}\left[\frac{1}{z-1} - \frac{1}{z-2}\right] = \frac{1+i}{2\pi}\left[\frac{x-1-iy}{(x-1)^2+y^2} - \frac{x-2-iy}{(x-2)^2+y^2}\right] \tag{15}$$

To conclude

$$u = \frac{1}{2\pi}\left[\frac{x-1+y}{(x-1)^2+y^2} - \frac{x-2+y}{(x-2)^2+y^2}\right] \tag{16}$$

And

$$v = \frac{1}{2\pi}\left[\frac{y-x+1}{(x-1)^2+y^2} - \frac{y-x+2}{(x-2)^2+y^2}\right]. \tag{17}$$

Since we are looking for the streamlines, we must solve the equation:

$$\frac{u}{\mathrm{d}x} = \frac{v}{\mathrm{d}y} \Leftrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{y-x+1}{(x-1)^2+y^2} - \frac{y-x+2}{(x-2)^2+y^2}}{\frac{x-1+y}{(x-1)^2+y^2} - \frac{x-2+y}{(x-2)^2+y^2}}. \tag{18}$$

When inputting (17) into Wolfram—Alpha, the following field lines of the solutions appear (Figure 1).

Problem 2 continued on next page. . .

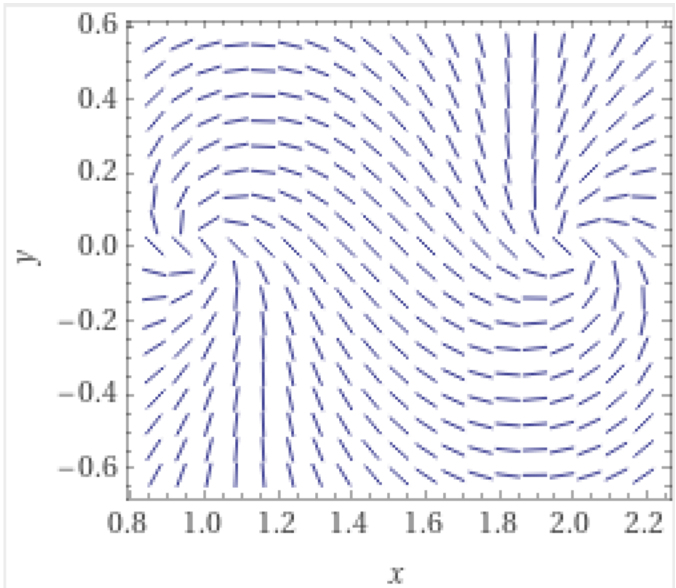

Figure 1: Field lines of the sink-source doublet in Wolfram—Alpha

Since the resolution is not very high, I decided to find the general solution of the differential equation (17) and plot several solutions in Desmos. The general solution to (17) is given by

$$\frac{1}{2}\ln\frac{(x-2)^2+y^2}{(x-1)^2+y^2}+\arctan\frac{x-1}{y}-\arctan\frac{x-2}{y}=C \tag{19}$$

Where C is a constant such that $C\in\mathbb{R}$. Several resulting streamlines are shown in Figure 2.

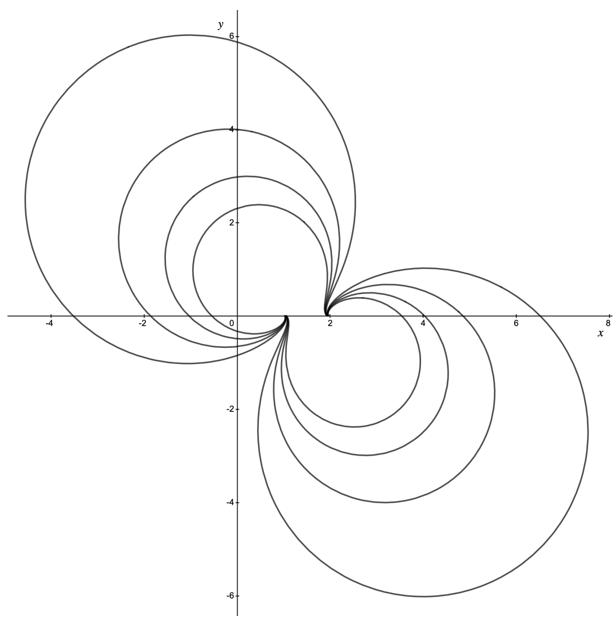

Figure 2: Streamlines of the sink-source doublet in Desmos

Problem 3

Using the method of images, transform the flow of problem 2 into one defined in the semi-infinite domain

$$x \geq 0,$$

with the same source, sink and corresponding circulations, but with no fluid flux through the boundary at $x = 0$. Again, plot a few representative streamlines.

Solution

The method of images tells us we should place a source and sink which are optically symmetric to the original setup, with respect to the line $x = 0$. In other words, the model of the new setup will take shape of Figure 3. Notice that the arrows on any circle surrounding the sink or the source correspond to the direction of circulation, whereas the arrows pointing towards or away from sinks and source correspond to the direction of the flux.

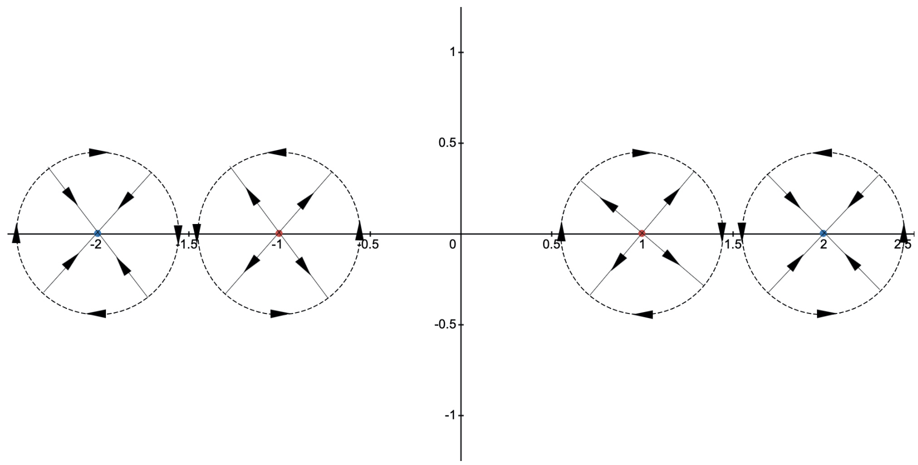

Figure 3: Superimposed mirror image of the sink-source doublet

We can now observe the flow on the left side to conclude that both fluxes of the sink and source are equivalent to that of their mirror images. On the other hand, the respective circulations have changed sign. The sink on the left now has a negative unit circulation, whereas the source has a positive unit circulation. To translate that into the complex potentials:

$$W^*_{sink} = -\frac{1}{2\pi}\ln(z+2) + \frac{i}{2\pi}\ln(z+2) = -\frac{1-i}{2\pi}\ln(z+2) \tag{20}$$

And

$$W^*_{source} = \frac{1}{2\pi}\ln(z+1) - \frac{i}{2\pi}\ln(z+1) = \frac{1-i}{2\pi}\ln(z+1). \tag{21}$$

Overall, the potential (combined with the previous potential from problem 2) will be

$$W^* = \frac{1-i}{2\pi}\ln(z+1) - \frac{1-i}{2\pi}\ln(z+2) + \frac{1+i}{2\pi}\ln(z-1) - \frac{1+i}{2\pi}\ln(z-2) \tag{22}$$

$$\implies W^* = \frac{1-i}{2\pi}\ln\frac{z+1}{z+2} + \frac{1+i}{2\pi}\ln\frac{z-1}{z-2}. \tag{23}$$

Similarly to the previous problem, we must find the derivative of the complex potential with respect to z to obtain information about the flow velocity.

$$\frac{\mathrm{d}W^*}{\mathrm{d}z} = \frac{1-i}{2\pi}\left[\frac{1}{z+1} - \frac{1}{z+2}\right] + \frac{1+i}{2\pi}\left[\frac{1}{z-1} - \frac{1}{z-2}\right] \tag{24}$$

Problem 3 continued on next page. . .

$$\Leftrightarrow \frac{\mathrm{d}W^*}{\mathrm{d}z} = \frac{1-i}{2\pi}\left[\frac{x+1-iy}{(x+1)^2+y^2} - \frac{x+2-iy}{(x+2)^2+y^2}\right] + \frac{1+i}{2\pi}\left[\frac{x-1-iy}{(x-1)^2+y^2} - \frac{x-2-iy}{(x-2)^2+y^2}\right]. \tag{25}$$

From (24) and (13) we can deduce

$$u^* = \frac{1}{2\pi}\left[\frac{x+1-y}{(x+1)^2+y^2} - \frac{x+2-y}{(x+2)^2+y^2} + \frac{x-1+y}{(x-1)^2+y^2} - \frac{x-2+y}{(x-2)^2+y^2}\right] \tag{26}$$

And

$$v^* = \frac{1}{2\pi}\left[\frac{y+x+1}{(x+1)^2+y^2} - \frac{y+x+2}{(x+2)^2+y^2} + \frac{y-x+1}{(x-1)^2+y^2} - \frac{y-x+2}{(x-2)^2+y^2}\right]. \tag{27}$$

To find the equations of the streamlines, we again solve the differential equation

$$\frac{u^*}{\mathrm{d}x} = \frac{v^*}{\mathrm{d}y} \tag{28}$$

This is deemed too complicated to solve by a powerful computing software Mathematica, but I have graphed several vector field lines and superimposed corresponding streamlines using several different softwares (Desmos mainly). The result is in Figure 4.

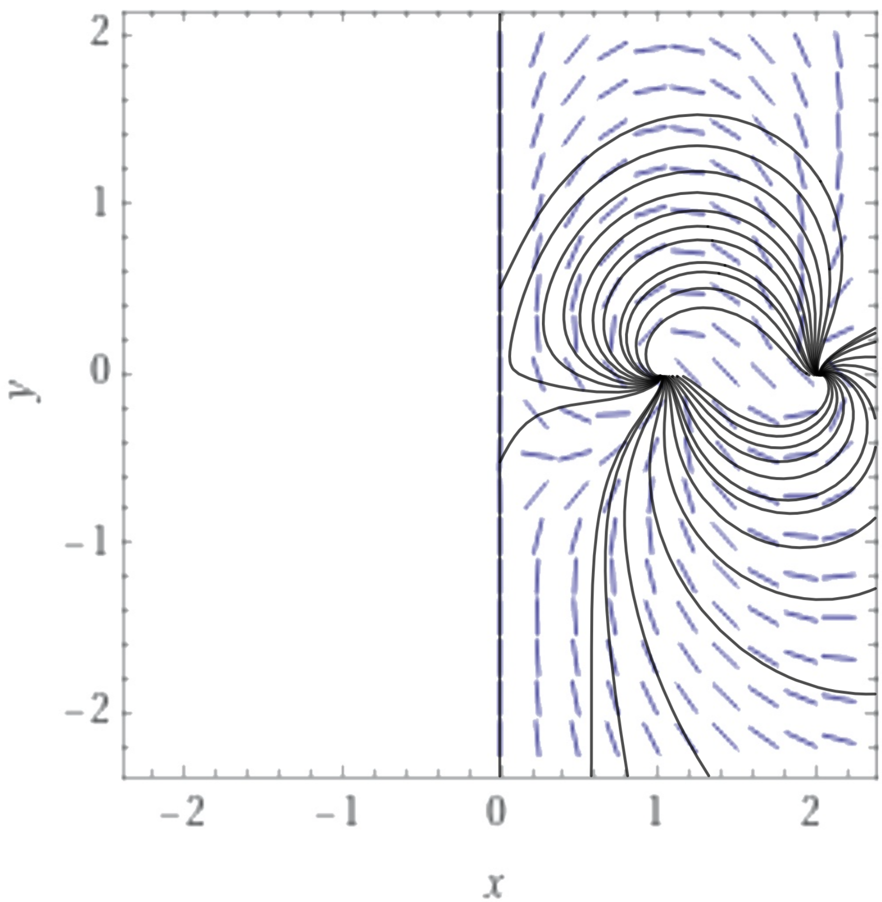

Figure 4: Visualisation of streamlines arising from a sink-source doublet in a semi-infinite domain

Notice how the line $x = 0$ is a streamline of the flow. This is expected since the flow has to 'obey' the domain in which it is situated, it cannot flow outside of it.

Problem 4

In problem 3, what's the total force that the fluid exerts on the boundary at $x = 0$, assuming that the pressure decays to $P = 0$ as $y \to \infty$? (You can leave your answer expressed as a real integral, you are not required to actually evaluate it.)

Solution

To start with, we recall Bernoulli's identity:

$$\frac{\rho \left|\mathbf{u}\right|^2}{2} + P + \rho g y = \text{const.} \tag{29}$$

Suppose we wish to determine the velocity at the boundary. Namely, if we observe equations (25) and (26) with a substitution $x = 0$, we obtain

$$u^* = 0 \tag{30}$$

And

$$v^* = \frac{1}{\pi}\left[\frac{y+1}{y^2+1} - \frac{y+2}{y^2+4}\right]. \tag{31}$$

So really, it holds that $|\mathbf{u}| = v^*$, as it should since we do not expect any horizontal flow!

Suppose we also assume constant density and we neglect the gravitational field[1] for now. Furthermore, since (as evident from equation 30) $\lim_{y\to} \left|\mathbf{u}\right|^2 = 0$ and if we assume that $\lim_{y\to\infty} P = 0$, the constant on the right hand side must be zero! Thus the Bernoulli principle reduces (28) to

$$P = -\frac{\rho v^{*2}}{2}. \tag{32}$$

The negative sign to the pressure only stands for the fact that the force which the fluid is applying to the boundary is in the negative x direction, as we will obtain soon.

Drawing analogies from the three-dimensional case of pressure and forces, the equation

$$\mathbf{F} = \iint_{\mathcal{S}} P \mathrm{d}S \hat{n} \tag{33}$$

Where $\mathbf{F}$ is the force, $\mathcal{S}$ is an area where the force is acting and $\hat{n}$ is the normal vector to that area. In the 2-D case (32) reduces to

$$F = \int_{\mathcal{C}} P \mathrm{d}l \tag{34}$$

Where $\mathcal{C}$ is some linear domain and $\mathrm{d}l$ is an infinitesimal length of that domain. In our case, it is evident that

$$F = \int_{-\infty}^{\infty} P \mathrm{d}y = -\frac{\rho}{2\pi^2}\int_{-\infty}^{\infty}\left[\frac{y+1}{y^2+1} - \frac{y+2}{y^2+4}\right]^2 \mathrm{d}y. \tag{35}$$

The integral was solved computationally through Wolfram—Alpha and the resulting force is

$$\boxed{F = -\frac{\rho}{12\pi}} \tag{36}$$

[1]Several problems if we do not. Some of them being varying gravitational field and infinite pressure.