NEW YORK UNIVERSITY ABU DHABI

On the evolution of the Diffusion equation and Maxwell-Cattaneo Theory of Thermal Waves

Marko Brnović[a,]

[a]*New York University Abu Dhabi*

Abstract

This paper elaborates on the evolution of the ubiquitous diffusion equation, particularly in the direction of Maxwell-Cattaneo formulation. I start by introducing the theory of diffusion and providing an alternative derivation of the diffusion equation. Then, the constraint concerning finite energy propagation speed is invoked and the according flux correction is deduced, thus modifying the diffusion equation from a parabolic to a hyperbolic model. The paper is especially focused on the theory of thermal waves in solid bodies in three dimensional space.

1. Evolution of the Diffusion equation

1.1. Considering concentration

Partial differential equations hardly ever correspond to a single model of a physical system. Rather, they encompass a range of phenomena which is evident when appropriate physical quantities are replaced[1]. *Diffusion* is a concept which can be used to describe various examples in invasion biology, propagation of the electric signal in the heart, 'random' motion of particles in a fluid or heat transport through a bounded region of space[2][3]. However, the underlying logic can be reduced to the movement of particles, under no *particular force*[1]. Consider an ensemble of particles, whose volume is infinitesimally small, in a fluid which can be discretised. It is important to note the discrete particle nature of the fluid since it is hard to mathematically model the interaction between a continuous fluid and a particle. If on the other hand, the fluid is composed of discrete entities, one can notice that each interaction between the fluid and any particle in the ensemble results in a 'jump' or an infinitesimal movement[4]. Here, note that the interactions between the particles within the ensemble is neglected, so that we can observe each individual particle.

Given a particle in an arbitrary region of 3D space, we shall assign to it a vector $\mathbf{r}$ such that it fully characterises its position. After each collision, the particle will have acquired an infinitesimal 'jump' ϵ such that $\mathbf{r} \to \mathbf{r} + \epsilon$. And here lies the key moment where we must assume that, on average, the overall position of the particles does not change (isotropic diffusion). This means that if we were to repeat this 'experiment' and look at the average infinitesimal displacement of each particle, it would be exactly 0. Note that since we are dealing with indistinguishable particles, this statement is equivalent to saying the average position of the *entire* ensemble does not change with time.

Loosely following Einstein's logic, suppose there exists a certain probability of collision between the fluid particles and our ensemble, and set the probability density $\psi(\mathbf{r}, t)$[4][5]. It follows that after an infinitesimally short time Δt, $\psi(\mathbf{r}, t) \to \psi(\mathbf{r} + \epsilon, t + \Delta t)$. For the set of all possible 'jumps', it is evident that

$$\int_{\Theta} \psi(\mathbf{r}, t)\, \mathrm{d}\mathbf{r} = 1 \tag{1}$$

for an arbitrary set of 'jumps'- Θ. Since the overall probability of jumps for each individual particle must equal 1 (each particle must jump somewhere, including $\epsilon = \mathbf{0}$). This is the *first* conserved quantity in this theory. And here it is also noticed that the macroscopic behaviour of these particles will be deduced from the microscopic considerations of ensemble interactions. The importance of it lies in the fact that deterministically, it is very information and time-consuming to determine how the system will evolve. So diffusion theory is a statistical theory, which is why it is often used in arbitrary particle motion. We shall start by observing the concentration of this ensemble as a physical quantity which evolves in space and time, depending on the probability of each collision ψ. We can investigate how a concentration ρ of the ensemble[2] at $\mathbf{r}$ changes from t to $t + \Delta t$. Namely, it holds that

$$\rho(\mathbf{r}, t + \Delta t) - \rho(\mathbf{r}, t) = \tag{2}$$

$$= \sum_{\Theta} \rho(\mathbf{r} - \epsilon, t)\, \psi(\mathbf{r} - \epsilon, t)\, \epsilon - \sum_{\Theta} \rho(\mathbf{r}, t)\, \psi(\mathbf{r}, t)\, \epsilon.$$

Email address: `mb7187@nyu.edu` (Marko Brnović)

[1]Here, we will first assume that there exists a certain scalar potential field extending over the space of the system we are considering, which can be later removed if we are to recover the diffusion equation.

[2]Here, note that the *concentration at a single point* is hardly a discrete concept (whereas this model started from assumptions involving discretisation) which will be alleviated by the fact that we let the magnitude of each jump tend to zero.

Which is simply saying that the change in concentration in an infinitesimal time interval is equal to the change in the expected values of the concentration, before and after the jump. Approximations like these are often used in literature including diffusion. Letting $\epsilon \to 0$, we obtain an integral by the Riemann-Stieltjes theorem:

$$\rho(\mathbf{r}, t+\Delta t) - \rho(\mathbf{r}, t) = \tag{3}$$
$$= \int_{\Theta} \rho(\mathbf{r}-\epsilon, t)\,\psi(\mathbf{r}-\epsilon, t)\,\mathrm{d}\epsilon - \int_{\Theta} \rho(\mathbf{r}, t)\,\psi(\mathbf{r}, t)\,\mathrm{d}\epsilon.$$

We now assume that the left integrand is at least twice differentiable[4]. Then, according to Taylor Theorem, we can find a Taylor polynomial of $\rho(\mathbf{r}-\epsilon, t)\,\psi(\mathbf{r}-\epsilon, t)$ centered at $\mathbf{r} = \epsilon$ of order 2. Since we are working with multispace-variable function, the terms in the polynomial will respectively be

$$\rho(\mathbf{r}, t)\,\psi(\mathbf{r}, t) \text{ for order } \epsilon^0 \tag{4}$$

$$-\vec{\nabla}\left(\rho(\mathbf{r}, t)\,\psi(\mathbf{r}, t)\,\epsilon\right) \text{ for order } \epsilon^1 \tag{5}$$

$$\nabla^2\left(\rho(\mathbf{r}, t)\,\psi(\mathbf{r}, t)\,|\epsilon|^2\right) \text{ for order } \epsilon^2. \tag{6}$$

Here, note that we have already made three assumptions. The first one is in equation(2), where we have tacitly assumed the relationship between the probabilities which vary with time and ones that vary with space. We have supported that argument by invoking $\Delta t \to 0$. In (3) we have assumed $\epsilon \to 0$, which allowed us to perform an integral (evolve to a continuous model as oppose to discrete jumps). And here (4-6), while taking the Taylor approximation of the integrand, we assumed that all terms corresponding to *epsilon*[3] or higher are negligible for the formulation of our theory. Integrals of equations (4-6) represent *moments* of the concentration function with respect to space. Moments are generally observed to second, eventually third order when developing a statistical theory. The most important part is the interpretation of these moments, as that is where the physical meaning of them decides to what extent they are negligible[6].

Since the integral of the Taylor polynomial of order 0 cancels out with the right integral in equation (3), we only need to find the physical relevance of (5) and (6). Notice that in the expansion, we have obtained that ρ is independent of the size of the jump ϵ. Although it may seem like it, this is not true for ψ, since the probability of collision at a certain point inherently depends on the size of the jump from earlier theoretical assumptions. Then it holds

$$-\int_{\Theta} \vec{\nabla}\left(\rho(\mathbf{r}, t)\,\psi(\mathbf{r}, t)\,\epsilon\right)\mathrm{d}\epsilon = -\vec{\nabla}\left(\rho(\mathbf{r}, t)\int_{\Theta}\psi(\mathbf{r}, t)\,\epsilon\mathrm{d}\epsilon\right) \tag{7}$$

And similarly

$$-\int_{\Theta} \nabla^2\left(\rho(\mathbf{r}, t)\,\psi(\mathbf{r}, t)\,|\epsilon|^2\right)\mathrm{d}\epsilon = \frac{1}{2}\nabla^2\left(\rho(\mathbf{r}, t)\int_{\Theta}\psi(\mathbf{r}, t)\,|\epsilon|^2\,\mathrm{d}\epsilon\right). \tag{8}$$

We then divide both sides of equation (3) by an infinitesimally short time interval and obtain

$$\frac{\rho(\mathbf{r}, t+\Delta t) - \rho(\mathbf{r}, t)}{\Delta t} = \tag{9}$$

$$= -\frac{1}{\Delta t}\vec{\nabla}\left(\rho(\mathbf{r}, t)\int_{\Theta}\psi(\mathbf{r}, t)\,\epsilon\mathrm{d}\epsilon\right) + \frac{1}{2\Delta t}\nabla^2\left(\rho(\mathbf{r}, t)\int_{\Theta}\psi(\mathbf{r}, t)\,|\epsilon|^2\,\mathrm{d}\epsilon\right).$$

Taking the limit as $\Delta t \to 0$ we obtain

$$\lim_{\Delta t \to 0}\frac{\rho(\mathbf{r}, t+\Delta t) - \rho(\mathbf{r}, t)}{\Delta t} = \frac{\partial\rho(\mathbf{r}, t)}{\partial t}. \tag{10}$$

But what are the terms on the right? It turns out it takes much more theoretical work with fluids to understand what they are. In particular, many models are concordant that

$$\lim_{\Delta t \to 0}\frac{\int_{\Theta}\psi(\mathbf{r}, t)\,\epsilon\mathrm{d}\epsilon}{\Delta t} = \mathbf{v}(\mathbf{r}, t) \tag{11}$$

where $\mathbf{v}$ is the velocity of the particles at some point $(\mathbf{r}, t)$. The second term in (10) is

$$\lim_{\Delta t \to 0}\frac{\int_{\Theta}\psi(\mathbf{r}, t)\,|\epsilon|^2\,\mathrm{d}\epsilon}{2\Delta t} = \mathbf{v}(\mathbf{r}, \mathbf{t}) = D(\mathbf{r}, t) \tag{12}$$

And yet again we ask from the fluid to be 'nice' in a way that both limits (11) and (12) exist and are finite[2]. But this is what a physical model should be like - it is hard to generalise when the underlying physical mechanisms are very elaborate. Moreover, the velocity we obtained in (11) tends to be proportional to the force in low Reynolds number (low turbulence) environments[7]. It follows that $\mathbf{v}$ is hardly considered constant at higher Reynolds numbers, and definitely not proportional to the force. So as we continue to *apply* our model, it is bound to have further approximations 'added' to it. As far as the least-approximated case of the Diffusion Equation is concerned, refer to equation 3.

But how outrageous is it to ask this of our model? Turns out: not that much. Conceptually, since the numerator in equation (11) can be assimilated to an *infinitesimal* expected jump[3], and it is divided by an infinitesimal time element - it is not that outrageous that the result is *some* kind of velocity. It just turns out (lucky for us) it is the velocity of the particles we were interested in. Furthermore, when calculating the second moment of the probability density in the numerator of (12), it becomes evident that it represents somewhat of a variance of the infinitesimal expected jump. Hence this is a 'correction' to the already obtained correlation between diffusion of concentration and velocity, since it involves some kind of variance. And this is the *Diffusion coefficient*. Notice that it is not constant, in general[8]. Had we kept more terms in our Taylor polynomial (to which end we would have to assume that our functions ρ and ψ are differentiable to higher orders) we would have gotten the correction to the diffusion coefficient, in terms of the third, fourth and higher order moments of the probability density. So the most outrageous thing about asking our model to behave nicely is really to not *need* any higher order corrections (presumably with turbulent flow).

[3] Infinitesimal since we integrate over all of the 'jumps' and the jumps themselves tend to zero, so the numerator tends to d$\mathbf{r}$

Now that we stopped and pondered the physical significance of the obtained quantities, we can combine (10), (11) and (12) to obtain the *relatively* simplified Diffusion equation

$$\boxed{\frac{\partial \rho(\mathbf{r},t)}{\partial t} = -\vec{\nabla}\cdot(\rho(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t)) + \vec{\nabla}\cdot\left(\rho(\mathbf{r},t)\,\vec{\nabla}\cdot D(\mathbf{r},t)\right)}. \tag{13}$$

And now we go back to the formulation of the initial problem. We chose ρ to be the variable which diffuses since it is an intuitive example to grasp diffusion with[4]. But in reality, this could be any physical quantity which can be manipulated with the assumptions we have taken, and behave 'nicely'. Temperature is one of those quantities[9], hence the alternative name to (13) - *Heat equation*. For temperature however, (which is a scalar field such as the concentration) it can be further simplified by setting $D(\mathbf{r},t)_{x,t} = D = \text{const}$ and $\mathbf{v}(\mathbf{r},t) = \mathbf{0}$, as approximated by Fourier. This then reduces (13) to

$$\frac{\partial}{\partial t}T(\mathbf{r},t) = D\nabla^2 T(\mathbf{r},t). \tag{14}$$

But why would this approximation make physical sense? We will revisit this question later on. Experimental evidence show that the Diffusion coefficient varies relatively weakly with space, especially in regions of space with uniform density and regular geometry[4]. Furthermore, if we go back the physical interpretation of $\mathbf{v}$, it is a product of some kind of a force, at low Reynolds numbers. Then, it makes sense to interpret this velocity as a consequence of a force, or somewhat of a potential gradient between the particles (in other words, particles would interact with each other - take for examples electrons which would exhibit electrodynamic repulsion). But what if we set this scalar potential field to zero (as one would do when considering diffusion of temperature, since there are no specific forces which would interfere with heat flow)?[2] It then clearly holds that $\mathbf{v} = 0$. Hence (14) has very sound physical sense, for diffusion of temperature for geometrically and materially simple systems. In particular, the solutions of the heat equation have served as good approximations to temperature distribution models and have thus been verified. Some of the solutions to the heat equation are[4]:

- A time-independent function linear in space (Figure 1a)
- A trigonometric function in space which is exponentially decreasing in time (Figure 1b)
- A Gaussian function (Figure 1c).

Since the Heat equation is a *linear* partial differential equation, it holds that any linear combination of its solutions will also be a solution[10] - which brings us to the general solution (example in Figure 1d). And this is where the heart and soul of this model are. We first look for idealised solutions where the function satisfying the boundary and initial conditions also satisfies the heat equation, and then we can combine all possible solutions so that the combination represents a more realistic case.

But looking at Figure 1, it does not immediately become clear as to why this would model a real physical system - in fact the solution which is linear in space in figure (a) and the combined function in figure (d) both tend to infinity, as we let $|\mathbf{r}| \to \infty$. This does not make physical sense, since we would need a source of infinite energy to even consider this solution. Thus, we need to impose further restrictions on our model. It turns out that if we want our solutions to represent physically attainable solutions, they need to obey physically allowable initial and boundary conditions. The initial condition should only depend on the space coordinates, in other words

$$T(\mathbf{r},0) = f(\mathbf{r}). \tag{15}$$

Moreover, if we are looking at a specific point in space, we may say that the initial condition is that the system has a temperature only at one or several points [5], that is that its *support* is:

$$supp\,(T(x,t)) = \begin{cases} f(x) \neq 0 & \text{for } x \in [a_1, a_2, \ldots, a_n] \\ 0 & \text{otherwise} \end{cases}. \tag{16}$$

Which is to say that in the beginning, T is only defined to be non-zero for a closed, finite set of points which is called a support[11]. This makes sense in the discrete model of particles, where the body is not considered a continuum. The support of a function is a valuable tool when analysing the physically allowable solutions of the diffusion equation.

Furthermore, it should hold (in this universe at least) that

$$\lim_{\mathbf{r}\to\infty} T(\mathbf{r},t) < \infty \quad \lim_{\mathbf{r}\to-\infty} T(\mathbf{r},t) < \infty \quad \lim_{t\to\infty} T(\mathbf{r},t) < \infty. \tag{17}$$

And in particular, it should most generally hold

$$\lim_{\mathbf{r}\to\infty} T(\mathbf{r},t) = \lim_{\mathbf{r}\to-\infty} T(\mathbf{r},t) = \lim_{t\to\infty} T(\mathbf{r},t) = 0. \tag{18}$$

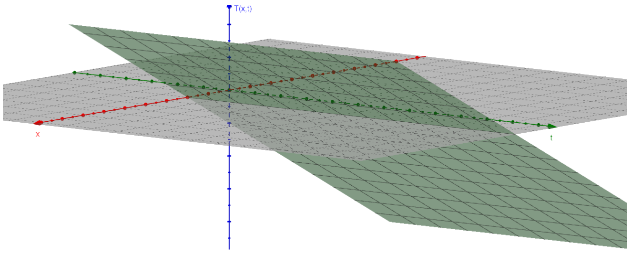
And this reduces the set of all possible solutions considerably. Why do these conditions make physical sense? And why do restrictions not arise in the Heat equation earlier in the derivation? To explain this, consider the original physical discrete system of particles which diffuse heat. Consider a discrete two-dimensional lattice of points (figure 2) which all have assigned a temperature which changes in time. Intuitively, if there are no heat sources around, and the system is left to reach thermal equilibrium, the temperature of points should become more levelled in the progression of time[3].

1.2. Considering Temperature

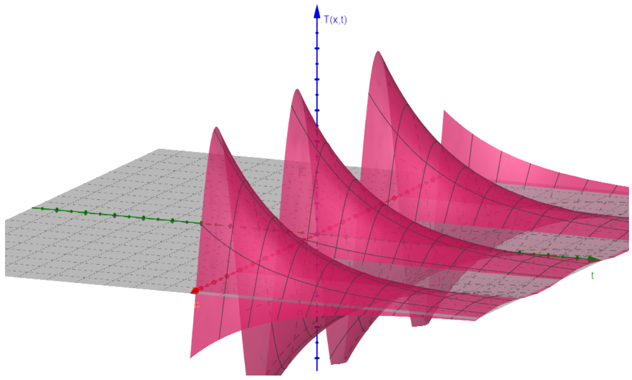
However, mathematical justification of the Heat equation still lacks. In our model it is evident that the particles which can be characterised as 'outliers' tend to 'accelerate' towards the equilibrium point faster than the particles which already have

[4]Note that in these figures the x-axis represents the simplification of the whole space or a coordinate basis for an arbitrary vector $\mathbf{r}$

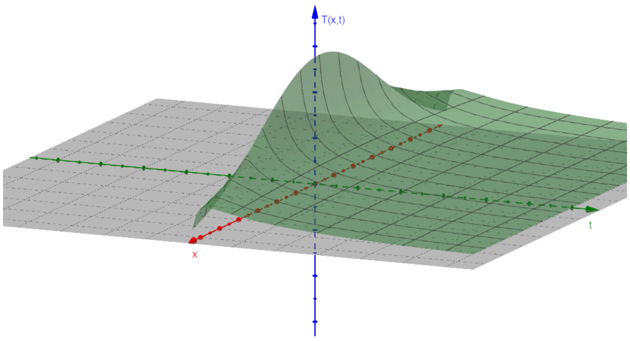
[5]This is a mathematical assumption to justify the fact that the point's temperature has not diffused yet.



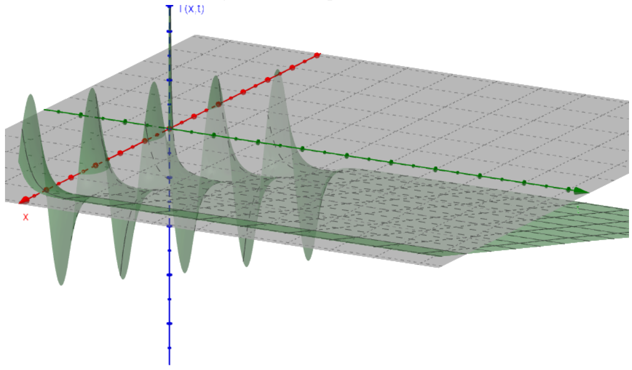
(a) Linear Solution



(b) Trigonometric/exponential solution



(c) Gaussian solution



(d) Linear combination

Figure 1: Solutions of the Heat equation. In each of the four graphs, Temperature is the vertical axis(blue), space axis is red and time axis is green.

a temperature which is bound to be equilibrium temperature in the limit of infinite time[6][9]. This means that somehow, the neighbourhood of each particle influences how quickly its temperature changes. If the particle is at a relatively low temperature and its neighbourhood (nearest neighbouring 4 points[7]). In this sense, the greater the difference of temperatures is between the surrounding points and a particular point in the middle, the faster it will accelerate towards an equilibrium temperature. But since we are specifying that only the neighbourhood has any effect on the particle, we can tell that the influence of the surrounding points on that point depends on the *distance* between them, which (in this model) is discrete. In particular, it is inversely proportional to the distance (the smaller the distance between the particle and its neighbourhood, the larger the influence on the speed of diffusion). And now, assume we extend our 2-dimensional model (figure 2) into three dimensions[8]. Here, a particle will have 6 points in its neighbourhood (Figure 3).

Suppose we take the average of those 6 and take away the temperature of our particle in the middle to have an approximation for $T(\mathbf{r},t) - T(\mathbf{r},t+\delta t)$ in our discrete model:

$$T(\mathbf{r},t) - T(\mathbf{r},t+\delta t) \propto \left(\frac{\sum_{i=1}^{6} T_i}{6} - T(\mathbf{r},t) \right) \tag{19}$$

Where T_i is the temperature of the ith particle. Suppose A and F correspond to $i = 1,2$ respectively. Similarly, E and C correspond to $i = 3,4$ and B and D correspond to $i = 5,6$. Consider all T_i constant for now, and later we will consider the whole system to be dynamic. Then, (18) becomes

$$T(\mathbf{r},t) - T(\mathbf{r},t+\delta t) \propto \frac{T_1 - T - (T - T_2)}{6} + \tag{20}$$

$$+ \frac{T_3 - T - (T - T_4)}{6} + \frac{T_5 - T - (T - T_6)}{6}$$

If we move away from the discrete metric and invoke the phenomenological fact that the further particles are away, the weaker the influence they have on diffusion, (19) becomes

$$T(\mathbf{r},t) - T(\mathbf{r},t+\delta t) \propto \frac{1}{\delta z}\frac{T_1 - T}{\delta z} - \frac{1}{\delta z}\frac{T - T_2}{\delta z} + \tag{21}$$

$$+ \frac{1}{\delta y}\frac{T_3 - T}{\delta y} - \frac{1}{\delta y}\frac{T - T_4}{\delta y} + \frac{1}{\delta x}\frac{T_5 - T}{\delta x} - \frac{1}{\delta x}\frac{T - T_6}{\delta x}$$

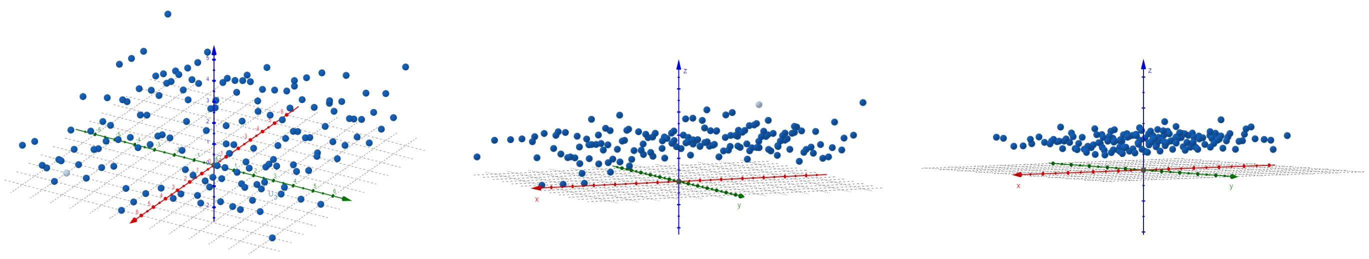
Which could be rewritten as

$$T(\mathbf{r},t) - T(\mathbf{r},t+\delta t) \propto \frac{1}{\delta z}\frac{\delta T_{z1}}{\delta z} - \frac{1}{\delta z}\frac{\delta T_{z2}}{\delta z} + \tag{22}$$

[6] This is also evident in empirical observations

[7] Notice here when we are talking about a neighbourhood, we are really moving towards a definition of infinitesimally close particles. It would be absolutely mathematically correct to count, as the neighbourhood the whole system of particles, but in the process of taking a limit, we deduce that the contribution of the other particles outside of the neighbourhood is negligible.

[8] Unfortunately, I am not aware yet as to how to draw in 4 dimensions.



(a) Distribution at $t = 0$ (b) Distribution at $t = \Delta t$ (c) Distribution at $t = 2\Delta t$

Figure 2: Discrete particle system reaching equilibrium. The vertical axis represents Temperature and the horizontal plane is the region of space where the discrete particles are defined

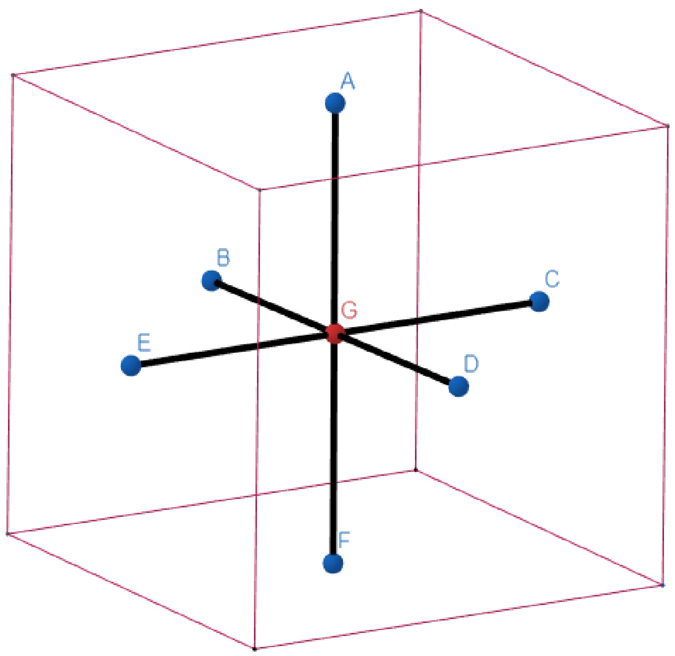


Figure 3: Point G represents the particle we are considering, A, B, C, D, E and F are in the neighbourhood

$$+\frac{1}{\delta y}\frac{\delta T_{y1}}{\delta y}-\frac{1}{\delta y}\frac{\delta T_{y2}}{\delta y}+\frac{1}{\delta x}\frac{\delta T_{x1}}{\delta x}-\frac{1}{\delta x}\frac{\delta T_{x2}}{\delta x}.$$

And noting that $\delta T_{i1} - \delta T_{i2} = \delta^2 T_i$ we get

$$T(\mathbf{r},t) - T(\mathbf{r},t+\delta t) \propto \frac{\delta^2 T_z}{\delta^2 z}+\frac{\delta^2 T_y}{\delta^2 y}+\frac{\delta^2 T_x}{\delta^2 x} \tag{23}$$

Letting our cube tend to an infinitesimally small one, we obtain

$$\delta T_t \propto \nabla^2 T. \tag{24}$$

Suppose the constant of proportionality is α. Then it would hold, for and arbitrary $\delta t > 0$ that

$$\frac{\delta T_t}{\delta t} = \frac{\alpha}{\delta t}\nabla^2 T. \tag{25}$$

Once again, we ask from our model to have

$$\lim_{\delta t\to 0}\frac{\alpha}{\delta t} < \infty. \tag{26}$$

And if so, then set

$$\lim_{\delta t\to 0}\frac{\alpha}{\delta t} = D \tag{27}$$

Hence by allowing $\delta t \to 0$ in (24), we recover the heat equation equivalent to (14):

$$\frac{\partial}{\partial t}T(\mathbf{r},t) = D\nabla^2 T(\mathbf{r},t). \tag{28}$$

Notice how it took close observation of two seemingly different physical concepts to arrive to the same model[12]. According to previous justification, one could accept that due to empirical observations, diffusion of temperature should follow the same law as that of concentration, but providing an approximation in both cases and arriving to the same equation (14) and (27) provides more substance to the theory of diffusion. Furthermore, had we included some kind of motion of our discrete particles (suppose molecular vibrations), we would have gotten another term in the equation proportional to the velocity of their movement **v** which would closer depict dynamic diffusion. For the purposes of this paper, for the sake of brevity and focusing on the main topic, we shall only observe static systems as they will suffice for the formulation of the problem.

And now, one may conclude that the diffusion model is applicable to temperature. Concordant with the boundary and initial conditions provided above - (15) and (17), and equation (28) - we can begin to postulate some solutions. Fourier cleverly noticed that any analytic function can be expressed as an infinite series of sines and cosines, which is a direct consequence of Taylor theorem and the orthogonality of those trigonometric functions[13]. He then showed that for an initial condition function $f(\mathbf{r},0)$ and appropriate boundary conditions, one may arrive to the general solution of the heat equation which reads

$$\boxed{T(\mathbf{r},t) = \sum_{n=0}^{\infty} T_n e^{-|\mathbf{k}_n|^2 Dt}\left(A_n\cos(\mathbf{k}_n\mathbf{r}) + B_n\sin(\mathbf{k}_n\mathbf{r})\right)} \tag{29}$$

$$\forall\, n \in \mathbb{N}_0.$$

Where

$$A_n = \frac{2}{L}\int_0^L f(\mathbf{r})\cos(\mathbf{k}_n\mathbf{r})\mathrm{d}x \tag{30}$$

And

$$B_n = \frac{2}{L}\int_0^L f(\mathbf{r})\sin(\mathbf{k}_n\mathbf{r})\mathrm{d}x. \qquad (31)$$

For $|\mathbf{k_n}| = \left|\frac{2n\pi}{L}\right|$ and the region is a box of sides L. It looks like we could stop here, since we found a three-dimensional solution of the heat equation, which definitely converges to 0 as $t \to 0$, as it should since it then makes physical sense. Also, we can write *a lot* (not all) initial condition functions using Fourier's trick, so it looks like we have reached a mathematically sound solution which perfectly obeys physical laws!
Well... Not exactly.

2. Formulation of the Problem

Let us consider some nuances we had overlooked in our model of a discrete two-dimensional system of particles (figure 2). And in fact, suppose an even yet simpler version, where *all* points but one in the middle are set at $T = 0$, at time $t = 0$ (no point has a fixed temperature). Suppose we set the point in the middle at some $T > 0$. The way we had described that system, due to the difference between the temperatures of that point and the neighbourhood, it will 'move' towards the neighbourhood. But since the neighbourhood does not have a fixed temperature, it will also 'move' towards that point. And furthermore, in the neighbourhoods of all the neighbourhood points, it is clear that they will move along, towards the equilibrium temperature $T > 0$. But at what 'speed' does this occur? That is, we know qualitatively and approximately quantitatively how the system's temperature should behave, but we do not know how 'quickly' the points around the centre point 'feel' the effect of it having a larger temperature. In fact, suppose they are some kind of thermometers. At what point in time do they register a higher temperature due to the presence of the middle point? And what about the points which are very far away[9] from the middle point? Do they feel this effect instantaneously? According to our model (14), it is certain that they should, since after an infinitesimally shirt time interval, the *whole system* undergoes a redistribution of temperatures according to the law! Namely, set this 'speed' of diffusion to be v_D. Then, from our assumptions it holds

$$v_D = \infty \qquad (32)$$

Is this concordant with physical observation?

To obtain a more holistic image, let us look back in time over the evolution of the heat equation. At the time Fourier first considered solutions to the Heat equation (18[th] century), a lot of physical phenomena were yet to be explained(as there are today). In particular, Einstein's theory of Special Relativity[15] had not yet been published and there was no theoretical explanation as to why propagation of information at an infinite speed would be physically unattainable. But in the beginning of the 20[th] century, Einstein said that no *matter* can travel as fast as or faster than light in vacuum[15]. In fact, this is a consequence of two separate postulates[10]:

- Same physical laws hold true in all inertial frames of reference
- It is a universal law that the speed of light in vacuum is constant and holds the value c.

Where an inertial referent frame is one that does not accelerate.[11] It is important to note that Einstein did not come up with any of these postulates. He merely put them together and realised that there had to be constraints which arise from the combination. The first postulate was first formulated by Galileo in the 17[th] century, and is central to Galilean relativity. Galileo observed idealised bodies moving at constant velocities ($\mathbf{v} = \mathbf{0}$ included). Coming up with thought experiments, he came to a conclusion that he could not find a system which moves at a constant velocity that does not behave like the one where $\mathbf{v} = \mathbf{0}$, in terms of obeying physical laws. So he postulated that this is the case always! In a mathematical system, a postulate like this would be called an *axiom*, since without it we would not be able ot formulate a theory. In physics however, there is no reason to assume anything is true, regardless of having countless evidence that it is true. In other words, since we did not *invent* physics, like we did mathematics as a tool to manipulate various concepts, physics is rather a consequence of universal laws which we cannot call to be *definitely* true. We can call them definitely false if they can be disproven, but unfortunately we have no physical tools to rigorously prove something is always true. However, the next best things we have are postulates on which the theory is based. So for our purposes, we will take them as true, until they can be contradicted.

The second postulate is a clear consequence of the fundamental theory of electromagnetism[16]. Namely, from Maxwell's equations we can deduce that when there is *some* matter to interfere with the propagation of electromagnetic waves, they travel at a constant speed

$$c = \frac{1}{\sqrt{\epsilon_0\epsilon_r\mu_0\mu_r}} \qquad (33)$$

where ϵ_0 and μ_0 are fundamental physical constants regarding the permitivity and permeability of free space, respectively. On the other hand, ϵ_r and μ_r depend on the properties of the medium through which electromagnetic waves travel (they are

[9] Here, we bring into question the notion of 'far away'. What exactly is 'far away'? In which units? It turns out, that far away could be in the neighbourhood or at an infinite distance from the point[14]. Nonetheless, the effect could not be registered instantaneously.

[10] Emphasis on the word 'postulate' since it represents a logical conclusion from many thought and physical experiments. A law is a postulate when it cannot be rigorously proven, but nonetheless has not yet been contradicted.

[11] With respect to what? Relativity deals exactly with these types of questions. It turns out that as long as the system does not accelerate with respect to an inertial reference frame, it is also inertial. But how do we know that the first one is inertial? It has to be inertial with respect to something. Two systems are inertial with respect to one another if an observer from the first system cannot identify a resultant force on the second one, and vise verse. Thus they are not accelerating with respect to each other.

both non-dimensional and are equal to 1 in vacuum)[16]. The former two can be experimentally obtained and are some of the most well-known constants in physics. Hence, the speed of light is too. And, if physical laws in all inertial frames of reference are the same, so is the constant speed of light. In other words, no matter how fast an observer moves, they will always register the same speed of light! This is really a fundamental result in physics which has reshaped many other physical theories. When one applies a constraint that the speed of light is the same for all observers travelling in constant velocities, interesting corollaries appear such as that no matter can travel as fast as or faster than light[17]. Light is just a part of the spectrum of all electromagnetic waves, which arise from Maxwell's equations when considering accelerating charged particles. *Charge* is a particle property we shall take to be inherent, since it has not been proven otherwise yet.

But here we are talking about temperature - temperature is not matter. It is in fact merely a scalar field where some value is assigned to each position in space - so why can it not change instantaneously[18]? To really understand this, we need to consider what temperature physically represents. Temperature is a convenient quantity to use to characterise how 'hot' or 'cold' something is. That 'something' must be matter. This is because temperature is a measure of the internal energy of the system - the higher the energy, the higher the temperature and vice versa. And although the internal energy of a matter-less system is a well-known quantity, we do not assign it a temperature since it does not *diffuse* in the sense of transferring heat through diffusion onto matter. But what is the 'internal energy' of the system? It is energy due to all of the attractive and repulsive potentials inside the system[12], along with the energy results from all types of motion such as vibration, translation and rotation[13]. Since all particles at non-zero temperatures (in Kelvin) exhibit some sort of internal motion, they will also have a temperature assigned at every point. So what the temperature really represents is how packed and disordered the system is[3]! In other words, if the system of particles is crammed into a small volume, where all particles tend to move relatively quickly and there are a lot of repulsive forces between them, it will have a high temperature and become more and more disordered through time[14].

So microscopically, diffusion is really a process where particles collide and interact with surrounding particles through fundamental forces, so that they transfer energy[4]. If this is an instantaneous process across the entire system, it would imply that the particles travel no distance and still manage to collide with its neighbourhood transferring it energy. But this cannot happen since we know that these particles can travel only slower than light. Furthermore, suppose the particles themselves represent atoms which are fully compact and immobile. One of the possible ways to transfer energy onto the surrounding system is by considering the accelerating charges within those atoms[16]. And since we know that accelerating charges (such as electrons) generate electromagnetic waves which themselves contain some energy, this is a beautiful model of how energy transfer can be described - due to the internal motion of charged particles within each one of those atoms. And this electromagnetic wave, when it 'hits' another particle in our system, transfers a discrete amount[15] of energy. So the *fastest* possible way that this transfer of energy (hence increase of temperature, since temperature is indicative of internal energy) can occur is through electromagnetic waves, which have a finite speed in all media! This implies that the speed of diffusion v_D must obey the constraint

$$v_D \leq c \implies v_D \leq \frac{1}{\sqrt{\epsilon_0 \epsilon_r \mu_0 \mu_r}} \tag{34}$$

Notice that (31) is in clear contradiction with (33), so either our model of diffusion is wrong, or Einstein's Special Relativity does not hold. It would be rather optimistic to attack one of the most frequently verified modern physical theories. So let us go with a not-so-outrageous assumption that our model is wrong in the first place. Perhaps we have somehow forgotten to impose the constraint that energy transfer (hence temperature diffusion) can travel only at a finite speed. If so, where?

In both derivations of the heat equation above we have not used a criterion which could impose a finite speed of propagation of diffusion. The constraints we had imposed only affected the particles which transmit heat, thereby reflecting the *discreteness* of the system. Now, let us consider a continuum and we will see that an opportunity for imposing a restriction will arise.

Consider an isolated three-dimensional object occupying a volume V in 3-D Euclidean space and denote temperature as a function in space $T(\mathbf{r}, t)$. Then, set the heat flux of that object ϕ to be

$$\phi(\mathbf{r}, t) = -\iint_{\mathscr{S}} \mathbf{E}(\mathbf{r}, t)\, \mathrm{d}\vec{A}. \tag{35}$$

Where $\mathbf{E}(\mathbf{r}, t)$ is the 'directed' transfer of energy inside the boundary of the body. It is defined as

$$\mathbf{E}(\mathbf{r}, t) = \vec{\nabla} Q(\mathbf{r}, t). \tag{36}$$

Where $Q(\mathbf{r}, t)$ is the internal energy of the system at a point in space $\mathbf{r}$ and time t. And $\mathscr{S}$ is the boundary of V, namely $\partial V = \mathscr{S}$. And $\mathrm{d}\vec{A}$ is the infinitesimal surface element of that boundary, whose direction is normal to the boundary, away from the interior int(V) (figure 4). So, the physical interpretation of flux is the energy that is either escaping or entering the

[12] Better known as Potential Energy.

[13] Better known as Kinetic Energy.

[14] I could go into explaining entropy here but I have decided not to since it would prove to be superfluous.

[15] Result is a consequence of Quantum Theory, since it has also been proven that it is impossible for a single electromagnetic wave to have a continuous spectrum of energy. see [19]

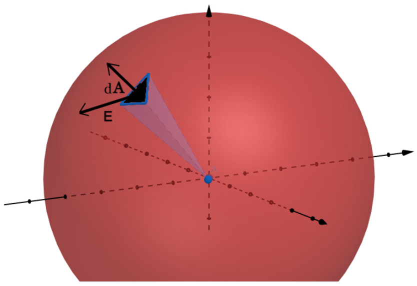


Figure 4: An example of how energy flux vs. boundary area could look like

body through its boundary. If we recall our previous discussion of the relationship between internal energy and temperature, it is safe to say they are somehow correlated. Consider the simplest (linear) case, where

$$Q(\mathbf{r}, t) = Q_0 T(\mathbf{r}, t). \tag{37}$$

For some $Q_0 \in \mathbb{R}$ whose units are that of energy. Combining (34)-(36) we obtain

$$\phi(\mathbf{r}, t) = -Q_0 \iint_{\mathscr{S}} \vec{\nabla} T(\mathbf{r}, t)\, \mathrm{d}\vec{A}. \tag{38}$$

Applying the nabla operator to (37) yields

$$\vec{\nabla}\phi(\mathbf{r}, t) = -Q_0 \vec{\nabla} \iint_{\mathscr{S}} \vec{\nabla} T(\mathbf{r}, t)\, \mathrm{d}\vec{A}. \tag{39}$$

When constructing a mathematical model, it is always important to take care of details, such as here. Tacitly assuming nabla operators commute in general with integrands can lead to mistakes in the model. There however, we can say that they commute only if both operations give us smooth functions (smooth and existent second derivatives). In other words, we should not have any discontinuities in our function's second derivatives. So now, we can write

$$\vec{\nabla}\phi(\mathbf{r}, t) = -Q_0 \iint_{\mathscr{S}} \nabla^2 T(\mathbf{r}, t)\, \mathrm{d}\vec{A}. \tag{40}$$

And this vector quantity tells us how much heat exits or enters our boundary, with a particular direction assigned to it. In fact, since it is the inflow or outflow of energy, it also represents how much energy has entered during some time Δt. But notice that, since none of the variables within it are vector quantities besides $\vec{\nabla}\phi$ and $\vec{A}$, this in fact gives us three equations, one for each spatial coordinate:

$$\frac{\partial}{\partial i}\phi(\mathbf{r}, t) = -Q_0 \iint_{\mathscr{S}} \frac{\partial^2}{\partial^2 i} T(\mathbf{r}, t)\, \mathrm{d}A_i. \tag{41}$$

For $i \in [x, y, z]$. Now, we can tell that the flux will flow in this direction, thus there will be an energy flow present with respect to time. To that end, define

$$\frac{1}{A} \iint_{\mathscr{S}} \frac{\partial Q(\mathbf{r}, t)}{\partial t} \mathrm{d}A = \frac{Q_0}{A} \iint_{\mathscr{S}} \frac{\partial T(\mathbf{r}, t)}{\partial t} \mathrm{d}A \tag{42}$$

To be the overall energy inflow or outflow per unit time. Notice the similarity between (40) and (41) - they both represent how much energy has input to some extent, albeit overall having a different integrand and hence different units. We postulate that they are in fact different by a factor of a certain constant. Thus, by equating (40) and (41) we obtain

$$\frac{\partial T(\mathbf{r}, t)}{\partial t} \propto \frac{\partial^2}{\partial^2 i} T(\mathbf{r}, t) \tag{43}$$

For $i \in [x, y, z]$. In a more compact manner

$$\frac{\partial}{\partial t} T(\mathbf{r}, t) = D \nabla^2 T(\mathbf{r}, t) \tag{44}$$

For some constant D. Thus we recover the heat equation. Notice a completely different mechanism here - we are not describing discrete particles interacting with the medium, we are observing the system as a continuum, and as such it has a particular physical quantity which is called flux. And flux is generally connected to quantities which fluctuate as a continuum. But this is the same as (14) and (24)? Have we gone wrong again? Well, due to the fact we have constructed the system through assumptions about its flux, it turns out we can impose some conditions on how quickly the temperature diffuses. This is due to the fact that the flux helps us to depict the 'diffusion velocity'. Hence we just have to find what the appropriate constraint is.

To make it a bit clearer, we shall rewrite the two equations that helped us in the derivation. This will thus constitute a system which will fully characterise the diffusion equation in terms of flux. In literature, they are most commonly written as:

$$\begin{cases} \rho c \frac{\partial}{\partial t} T(\mathbf{r}, t) = -\frac{\partial}{\partial i}\phi_1(\mathbf{r}, t) \\ \phi_i = -k \frac{\partial}{\partial i} T(\mathbf{r}, t) \end{cases} \tag{45}$$

Where $i \in [x, y, z]$. ρ and c are the density of the solid object and the specific heat capacity of the object. k is the thermal conductivity. In the previous derivations we had absorbed them into constants[16], namely $D = \frac{k}{\rho c}$. The first of the two equations in $(44)_1$ is better known as the First law of Thermodynamics[17] and the second equation is Fourier's law of flux, which is in fact an idealised approximation. Since they do not give us a constraint in energy transfer speed, let us consider making a change in either one of them. Going with the assumption that disobeying the first law of thermodynamics would make the universe implode, let us consider[18] the possibility of making a change to Fourier's approximation.

3. Maxwell-Cattaneo Contribution

Carlo Cattaneo was a mathematician who paid specific attention to theories which are incompatible with Special Relativity,

[16] And rightfully so, they can most of the time in solid diffusion be approximated as constants.

[17] Colloquially, conservation of energy.

[18] Ergo, deduce.

and this one in particular. He has been famously accredited to have come up with the first hyperbolic model of diffusion [20], which is what in fact solved the problem of infinite speed of propagation. But before we elaborate on that let us firstly take a short digression.

3.1. The second sound

With the birth of modern physics at the turn of the 20th century, a lot of physical theories were generated, disproved or modified (as testified above by Special Relativity). And the unparalleled theory which influenced and generated many is the theory of Quantum Mechanics. The interpretation of quantum phenomena is probabilistic in large, and that is reflected in its corollary - the wave particle duality. In the above section when we considered relativistic constraints we considered particles with certain internal energies as balls which collide with their surrounding and thus transfer heat. But quantum mechanics tells us that particles indeed behave like waves[19], and this phenomenon is especially seen for particles of very high speed and small size. In particular, the particles of the solid whose temperatures diffuse can be observed as waves in some specific physical limits such as very low temperatures. And when these limits are obtained, the wave-like propagation of heat through the system is observed, which is ceremonially called *The Second Sound*[14][19].

To be more specific, due to a temperature scalar field inside the object, particle-like excitations of this field are allowed and occur more frequently in the above mentioned condition[21]. When the density of these excitations changes (think of it as a continuum), a wave-like disturbance through the solid is emitted. Since waves carry energy, this wave too can transfer that energy onto other particles, thus providing an alternative for diffusion[20] So the conclusion we can obtain from the second sound is that there exist physical phenomena which support Einstein's proposition about finite energy transfer.

The second sound is a phenomenon which does not contribute visibly to energy transfer in solids at room temperature, but is notable at exceptionally low temperatures (between 0 and 10 Kelvin)[22]. Since the quantum mechanical proposition it has been experimentally verified so that its existence is not questioned, but its effects certainly are.

3.2. Attempt #1

Since equations in (44) give us back the heat equation (43), suppose we 'fix' the model by assuming that the diffusion coefficient is not constant. In particular, suppose it is a linear function of temperature[21] such as $D = kT$ Also to reduce calculations, suppose the model is 1-dimensional, namely temperature only diffuses in the x-direction. Then, if we recall (13), in a more general form (that is, before we took D to be constant) the diffusion equation is:

$$\frac{\partial}{\partial t}T(x,t) = \frac{\partial}{\partial x}\left(T(x,t)\frac{\partial}{\partial x}D(x,t)\right) = k\frac{\partial}{\partial x}\left(T(x,t)\frac{\partial}{\partial x}T(x,t)\right) \tag{46}$$

We then want to impose a constraint on how quickly the point $T = 0$ moves with respect to time after diffusion starts. Suppose some kind of a relation between x and t:

$$x = g(t). \tag{47}$$

Then (45) becomes

$$\dot{g}(t)\frac{\partial T}{\partial x} = k\left(\frac{\partial T}{\partial x}\right)^2 + k\frac{\partial^2 T}{\partial x^2}. \tag{48}$$

Then suppose we want to make a model such that the speed at which the line $T = 0$ moves is at the most c-the speed of light. Further suppose this function is an increasing, decaying exponential

$$\dot{g}(t) = c\left(1 - e^{-\lambda t}\right) \implies x = c\left(t - e^{-\lambda t}\right) + \frac{c}{\lambda}. \tag{49}$$

Regardless of the solution of equation (48) (which cannot be computed numerically for general $g(t)$) we have managed to make a theoretical change which now allows us to confirm that the point where $T = 0$, or the point $b + \delta$ for any $\delta > 0$, moves at a finite speed as T evolves in time - the system diffuses concordant with relativistic prediction.

We can furthermore suppose another dependence of x on t:

$$x = x_0 \sin(\omega t) \tag{50}$$

For some positive constants x_0 and ω, such that $\omega x_0 < c$. In this case the speed of propagation would be:

$$\dot{x}(t) = \omega x_0 \cos(\omega t). \tag{51}$$

And furthermore, since we can express $\dot{g}(t)$ in terms of $g(t)$, we can obtain a general solution T of this case:

$$T(x) = \int_1^x \frac{e^{1/2\omega x_0\left(\sqrt{1-\zeta^2}\zeta + \sin^{-1}(\zeta)\right)}}{c_1 + \int_1^\zeta e^{1/2\omega x_0\left(\sqrt{1-\xi^2}\xi + \sin^{-1}(\xi)\right)}\mathrm{d}\xi}\mathrm{d}\zeta + c_2 \tag{52}$$

But it may be argued with full right that the methodology of the computation was too restrictive. We assumed a linear dependence of D on T, and furthermore assumed two scenarios where the speed of the point $T = 0$ moves at either an increasing speed, or an oscillating one, and we could still just barely recover any sort of solutions.

[19]The first one being the basic disturbance of particles in a medium, such as sound propagation through air or water.

[20]Indeed, an alternative. Diffusion occurs due to the random interactions of particles, whether it is a change of concentration or temperature. As such it is a probabilistic phenomenon which depends on the randomness of motion within a continuum. On the other hand, the second sound represents a wave like transfer of energy which has a different physical origin.

[21]This is not that outrageous for finite regions in space. Sometimes the diffusion coefficient can be approximated using the exponential, and even further to a linear function. In particular, see [3].

3.3. Attempt #2: Cattaneo flux

When we consider the Fourier flux in (44) we can observe that it in fact takes in the *instantaneous* value of the temperature's spatial derivative. However, energy transfer at a finite speed would have to imply that a slight 'diffusion' of temperature would only affect the whole flux after some time τ[14]. In other words, this is our culprit! Since we have had a latent assumption that the flux is immediately influenced by the spatial variation of temperature, we kept getting infinite speeds of energy propagation. And now we know from (44) and the observation above that the overall flux should be

$$\phi_i(\mathbf{r}, t+\tau) = -k\frac{\partial}{\partial i}T(\mathbf{r}, t). \tag{53}$$

Disregarding the physical process that this describes, we could (carefully) assume that the function ϕ_i is smooth and take a Taylor polynomial approximation of the left hand side of (52). We thus obtain

$$\phi_i(\mathbf{r}, t) + \tau\frac{\partial}{\partial t}\phi_i(\mathbf{r}, t) \approx -k\frac{\partial}{\partial i}T(\mathbf{r}, t) \tag{54}$$

But why would this work physically? Cattaneo considered (53) and insightfully used a statistical correction to the Fourier flux[20]. He then made an analogy for the diffusion of a gaseous system. He deduced that the flux from (51) is approximately[22]

$$\phi_i = -k\frac{\partial T}{\partial i} + \sigma\frac{\partial^2 T}{\partial i \partial t}. \tag{55}$$

Cattaneo[20], in particular considered the *second sound* contribution of the system to the diffusion, and obtained a correction to the initial flux, which was faulty from our assumption about infinite speed. Here, σ is a proportionality parameter which characterises how quickly the temperature gradient changes. The second sound is a wave-like form of transfer of energy arising from the temperature density fluctuations in time, which is concordant with the term on the right in (54). Cattaneo then differentiates (54) with respect to time and obtains

$$\frac{\sigma}{k}\frac{\partial\phi_i}{\partial t} = -\sigma\frac{\partial^2 T}{\partial t \partial i} + \frac{\sigma^2}{k}\frac{\partial^3 T}{\partial i \partial^2 t} \tag{56}$$

By multiplying both sides by σ/k. Cattaneo[20] then adds (54) and (55) noting that both derivatives with respect to i and t exist and are continuous, thus obtaining

$$\phi_i + \frac{\sigma}{k}\frac{\partial\phi_i}{\partial t} = -k\frac{\partial T}{\partial i} + \frac{\sigma^2}{k}\frac{\partial^3 T}{\partial i \partial^2 t}. \tag{57}$$

If we set $\tau = \sigma/k$, we obtain (53) apart from the third-order derivative. And in fact, if we use the conservation of energy to write this equation in terms of temperature we obtain

$$\tau\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{k}{\rho c}\frac{\partial^2 T}{\partial i^2} - \frac{\sigma^2}{\rho c k}\frac{\partial^4 T}{\partial i^2 \partial t^2} \tag{58}$$

(57) is indeed a correction to the diffusion equation, but the meaning is yet unknown. Cattaneo observed motion of statistically describable particles and reached a conclusion of the shape of their flux due to their *second sound*[14]. The last term in (57) will tend to zero as $\sigma \to 0$ more quickly so than the first term on the left, so Cattaneo reached a conclusion that an approximation easier to work with is:

$$\tau\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{k}{\rho c}\frac{\partial^2 T}{\partial i^2} \tag{59}$$

For $i \in [x, y, z]$. In particular, if we add up all of the equations for all i's, we obtain

$$\boxed{\tau\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{k}{\rho c}\nabla^2 T} \tag{60}$$

As we can absorb the factor of three in the constants. So finally, (60) is the beginning of Maxwell-Cattaneo theory of diffusion[23]. Notice that if σ is really small (contribution of second sound is negligible), equation (60) reduces to the heat equation. However, the heat equations (13) and (60) represent two different classes of partial differential equations (PDE). (13) is a type of a parabolic PDE[24], whose characteristic solutions (with respect to the initial and boundary conditions) and linearity we had already elaborated on (see Figure 1). On the other hand, (60) is a hyperbolic PDE, from the same class the infamous wave equation comes from[23]. In fact, this equation is a combination of the wave equation and the heat equation, so its solutions will carry characteristics of the solutions of both.

4. Examples

Suppose we want to find the separable solutions of (60). Since it is a linear PDE, it satisfies the property of the linearity of solutions - any linear combination of the solutions is also a solution. Assume

$$T = f(\mathbf{r})\,g(t). \tag{61}$$

Thus, (60) becomes

$$\tau f(\mathbf{r})\frac{\mathrm{d}^2 g(t)}{\mathrm{d}t^2} + f(\mathbf{r})\frac{\mathrm{d}g(t)}{\mathrm{d}t} = g(t)\frac{k}{\rho c}\nabla^2 f(\mathbf{r}) \tag{62}$$

$$\implies \frac{\tau}{g(t)}\frac{\mathrm{d}^2 g(t)}{\mathrm{d}t^2} + \frac{1}{g(t)}\frac{\mathrm{d}g(t)}{\mathrm{d}t} = \frac{k}{\rho c}\frac{1}{f(\mathbf{r})}\nabla^2 f(\mathbf{r}) = \lambda. \tag{63}$$

For a constant θ. Then, for the spatial part we obtain

$$\nabla^2 f(\mathbf{r}) = \frac{\rho c \lambda}{k} f(\mathbf{r}). \tag{64}$$

We then search for the separable solutions to $f(\mathbf{r})$:

$$f(x, y, z) = A(x)B(y)C(z). \tag{65}$$

[22] Notice that I sometimes stop tagging along the $(\mathbf{r}, t)$ since a) it is evident which variables depend on space and time and b) to save column space.

[23] In fact, it is also an equation from the class of semilinear wave equations.

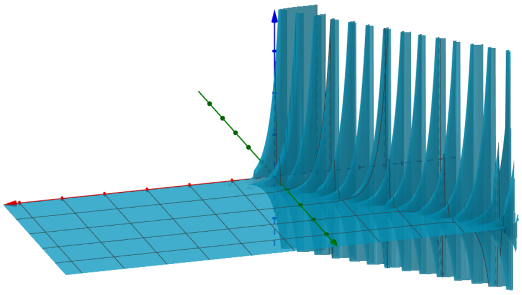


Figure 5: Damped solution from the Maxwell-Cattaneo Theory. Red axis is space, green axis is time and blue axis is temperature.

And we obtain accordingly

$$A(x) = a_x e^{\sqrt{\frac{\rho c \lambda}{k}} x} + b_x e^{-\sqrt{\frac{\rho c \lambda}{k}} x} \tag{66}$$

$$B(y) = a_y e^{\sqrt{\frac{\rho c \lambda}{k}} y} + b_y e^{-\sqrt{\frac{\rho c \lambda}{k}} y} \tag{67}$$

$$A(x) = a_z e^{\sqrt{\frac{\rho c \lambda}{k}} z} + b_z e^{-\sqrt{\frac{\rho c \lambda}{k}} z}. \tag{68}$$

On the other hand, the temporal part satisfies the equation

$$\frac{\mathrm{d}^2 g(t)}{\mathrm{d}t^2} + \frac{1}{\tau}\frac{\mathrm{d}g(t)}{\mathrm{d}t} - \frac{\lambda}{\tau} g(t) = 0. \tag{69}$$

Which is a well-known equation for damping. Its solution is (by considering solutions of the form Ae^{kt}:

$$g(t) = e^{-\frac{t}{\tau}}\left(a_t e^{\frac{\sqrt{1+4\lambda t}}{2\tau}} + b_t e^{-\frac{\sqrt{1+4\lambda t}}{2\tau}}\right). \tag{70}$$

Depending on the value of λ, the system may have various states (oscillatory in space ($\lambda < 0$), bound, decreasing exponential ($\lambda > 0$) etc.). Also, depending on the value of τ, there will be several levels of damping[24].

However, recall that not only the differential equation specifies the solution, but it should also be specified by its initial and boundary conditions. For example, suppose

$$T(\mathbf{r}, t) = f(\mathbf{r}) \tag{71}$$

Is its initial condition. A typical boundary condition is given by

$$f(a, y, z) = L_x < \infty \; f(x, a, z) = L_y < \infty \; f(x, y, a) = L_z < \infty. \tag{72}$$

We could then formulate the Fourier series of $f(\mathbf{r})$ as in (28) and find all of the corresponding terms of the progression of Temperature. Thus, an arbitrary solution such as the one in figure 5 can be obtained.

Comparing figure 5 and figure 1, we can immediately say that a large difference is that the solutions do not seem to 'travel in time' in figure 1, which reflects the fact that they represent infinitely fast energy transfer[25]. On the other hand, on Figure 5, we can clearly identify a wave-like structure emerging from $t = 0$ and propagating onwards. Thus this solution is representative of the fact that temperature is being diffused at a finite speed, and moreover it is going to be less than that of light.

Note that solutions are not covered in full generality, for there are many more which could be a topic of a paper or a book on their own. A more complete set of solutions can be considered from the particularly interesting point of view of breaking symmetry within Lie Algebra which is clearly more complex compared to finding separable solutions[25]. And in fact, such solutions are rarely left in functional form (mostly because they do not have one).

5. Applications and Conclusions

Diffusion is a ubiquitous process. It can be used to characterise the wavefunction of the Schrödinger equation, concentration of particles or in fact the transfer of energy and thus flow of temperature. By considering the discrepancies and shortcomings of the somewhat obsolete Fourier flux and fixed it due to the contribution of the second sound, we have obtained a hyperbolic form of the Heat Equation. From there we have constructed separable solutions for arbitrary initial conditions and through linearity confirmed that solutions could still be obtained in a similar fashion as with the parabolic equation.

Further research of the hyperbolic heat equation and further refinements are necessary to better understand many processes which this equation characterises. A future improvement in this topic may be constructing a simulation and observing how the spread of temperature varies in time for various supports and elaborating further how supports can affect the shape of the temperature scalar field.

5.1. Acknowledgments

- I acknowledge it was exceptionally hard putting the pieces of the original Cattaneo paper [20] since the original is unfortunately unavailable on the internet.
- I would like to thank Professor Francesco Paparella who introduced me to the topic and pushed me to make advancements throughout the project with intriguing literature.

[24] Underdamping, overdamping and critical damping to be exact.

[25] One way to look at this is to project T onto the x-t plane and look for a correlation between x and t. For example, consider drawing a straight line through all points (T_0, x, t) for some constant T_0. Evidently this line has a form $x = at + b$, hence the finite propagation speed. If on the other hand, we do the same for solutions in figure 1, we deduce that every line that contains the points (T_0, x, t) spikes up (if it exists in the first place), therefore we can say $x = \infty$-infinite propagation speed.

References

[1] Gerald B Folland. *Introduction to partial differential equations*, volume 102. Princeton university press, 1995.

[2] Merkel Henry Jacobs. Diffusion processes. In *Diffusion Processes*, pages 1–145. Springer, 1935.

[3] Erich Wimmer, Walter Wolf, Jürgen Sticht, Paul Saxe, Clint B Geller, Reza Najafabadi, and George A Young. Temperature-dependent diffusion coefficients from ab initio computations: Hydrogen, deuterium, and tritium in nickel. *Physical Review B*, 77(13):134305, 2008.

[4] Tristan S Ursell. The diffusion equation a multi-dimensional tutorial. *California Institute of Technology, Pasadena, Tech. Rep*, 2007.

[5] A Einstein. On the movement of small particles suspended in stationary liquids required by the molecular kinetic theory of heat. *Ann. d. Phys*, 17 (549-560):1, 1905.

[6] Persi Diaconis. Application of the method of moments in probability and statistics. *Moments in mathematics*, 37:125–142, 1987.

[7] Ramin Golestanian and Armand Ajdari. Analytic results for the three-sphere swimmer at low reynolds number. *Physical Review E*, 77(3): 036308, 2008.

[8] CY Lee and C Ro Wilke. Measurements of vapor diffusion coefficient. *Industrial & Engineering Chemistry*, 46(11):2381–2387, 1954.

[9] Geoffrey I Taylor. Diffusion by continuous movements. *Proceedings of the london mathematical society*, 2(1):196–212, 1922.

[10] Richard Bellman et al. A stability property of solutions of linear differential equations. *Duke Mathematical Journal*, 11(3):513–516, 1944.

[11] Thomas Little Heath. *A history of Greek mathematics*, volume 1. Clarendon, 1921.

[12] Bernard D Coleman and David R Owen. On the nonequilibrium behavior of solids that transport heat by second sound. *Computers & mathematics with applications*, 9(3):527–546, 1983.

[13] William Henry Young. On classes of summable functions and their fourier series. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 87(594):225–229, 1912.

[14] Brian Straughan. *Heat waves*, volume 177. Springer Science & Business Media, 2011.

[15] Albert Einstein. The theory of special relativity. In *The Meaning of Relativity*, pages 23–53. Springer, 1922.

[16] Richard Fitzpatrick. *Maxwells Equations and the Principles of Electromagnetism*. Laxmi Publications, Ltd., 2010.

[17] Luis Gonzalez-Mestres. Properties of a possible class of particles able to travel faster than light. *arXiv preprint astro-ph/9505117*, 1995.

[18] Jean-Paul Blaizot, Andreas Ipp, Ramon Mendez-Galain, and Nicolas Wschebor. Perturbation theory and non-perturbative renormalization flow in scalar field theory at finite temperature. *Nuclear Physics A*, 784(1-4): 376–406, 2007.

[19] David J Griffiths and Darrell F Schroeter. *Introduction to quantum mechanics*. Cambridge University Press, 2018.

[20] Carlo Cattaneo. Sulla conduzione del calore. *Atti Sem. Mat. Fis. Univ. Modena*, 3:83–101, 1948.

[21] R Srinivasan. Second sound. *Resonance*, 4(3):16–24, 1999.

[22] JC Ward and J Wilks. Iii. second sound and the thermo-mechanical effect at very low temperatures. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 43(336):48–50, 1952.

[23] D Graffi. Personal communication to prof. *F. Franchi. From the Accademia dei Lincei, Roma*, 1984.

[24] Avner Friedman. *Partial differential equations of parabolic type*. Courier Dover Publications, 2008.

[25] Célestin Wafo Soh. Symmetry breaking and exact solutions of the hyperbolic heat equation with variable medium properties. *Adv. Stud. Theor. Phys.*, 2:71–85, 2008.