Fluid Dynamics: Interesting Problems

Flow down a ramp and Poiuiseille flow

Marko Brnović

Problem 1

Consider a viscous fluid with kinematic viscosity $\mu = 1$ and density $\rho = 1$ flowing down a ramp with an angle θ of 10 degrees with the horizontal. For simplicity, pick units so that the gravity constant g also equals one.

a) For a uniform flow with height h, derive an expression for the shear $u(y)$ and the corresponding mass flow per unit width

$$Q(h) = \int_0^h \rho\ u(y)\ dy.$$

b) Now consider a variable flow, with $h = h(x,t)$ and $u = u(x,y,t)$. Under the assumption that the dependence on x and t is slow, we can assume that the relation you derived before for $Q(h)$ is still valid locally. Then use conservation of mass to write an equation for the evolution of $h(x,t)$.

c) Linearize the equation above by assuming that

$$h = 1 + \eta, \quad \eta \ll 1,$$

and write down the solution $\eta(x,t)$ to the initial value problem

$$\eta(x,0) = \eta_0(x).$$

d) Returning now the the fully nonlinear equation of part b, find a change of variables $r = r(h)$ so that the equation reduces to the Burgers equation for r

$$r_t + r\ r_x = 0.$$

– Along which lines in (x,t) space will r (and therefore also h) be constant?

– Consider a situation where the initial data $h_0(x)$ grows from $h = 1$ at $x = -\infty$ to a maximum value $h = 2$ at $x = 0$ and then decays again to $h = 1$ at $x = \infty$. When a shock first forms, will it form on the section where h grows or where it decays?

Solution

Part a

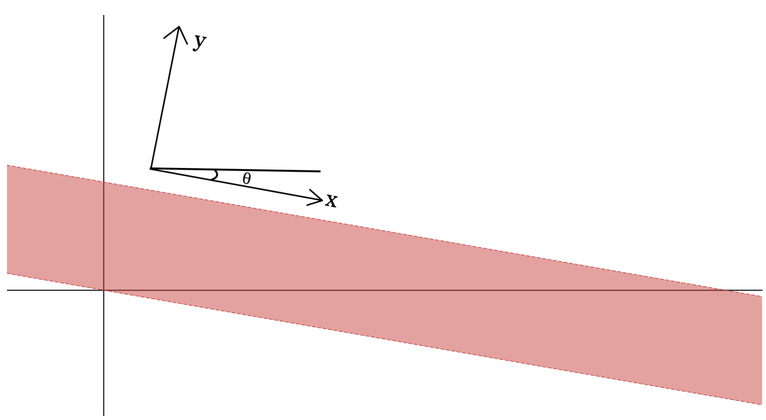

Figure 1: Visualisation of the problem

Problem 1 continued on next page...

For this part we need to recall the Navier-Stokes (N-S) equations in both x and y directions, as well as specify what those directions are (see Figure 1).

In the x direction, the N-S equation is written as

$$\frac{\partial u}{\partial t} + \left(\mathbf{u} \cdot \vec{\nabla}\right) u + \frac{1}{\rho}\frac{\partial p}{\partial x} = \mu \nabla^2 u + g \sin(\theta). \tag{1}$$

In the y direction, it is

$$\frac{\partial v}{\partial t} + \left(\mathbf{u} \cdot \vec{\nabla}\right) v + \frac{1}{\rho}\frac{\partial p}{\partial y} = \mu \nabla^2 v - g \cos(\theta). \tag{2}$$

In order to simplify the equations, we make some assumptions:

1. The flow is time-independent (steady).

2. u does not depend on x, since it extends to $\pm\infty$ in both directions.

3. The total velocity $\mathbf{u}$ is parallel to the x direction. A corollary from this is that $v = 0$.

4. The pressure on the moving boundary ($y = h$) is 0.

5. The flow is incompressible.

6. No-slip: at $y = 0$, $u = 0$.

7. No shear stress due to air: $\left.\frac{\partial u}{\partial y}\right|_{y=h} = 0$.

Applying these assumptions, equations (1) and (2) simplify to:

$$\frac{\partial p}{\partial x} = \eta \frac{\partial^2 u}{\partial y^2} + \rho g \sin(\theta) \tag{3}$$

and

$$\frac{\partial p}{\partial y} = -\rho g \cos(\theta) \tag{4}$$

respectively, where the identity $\eta = \mu\rho$ was used for simplification. From equation (4), we can deduce

$$p = -\rho g y \cos(\theta) + f(x). \tag{5}$$

If we input this expression for p into (3), we can see that on the left hand side, there will be a term with $f'(x)$, while on the right hand side there are no terms which contain an x dependence. We can thus deduce

$$f'(x) = 0 \implies f(x) = Ax + B \tag{6}$$

Furthermore, since we want a reasonable solution according to the assumption 4. (otherwise we would have an instance of infinite pressure), it is evident that $A = 0$, so that $f(x)$ is a constant. Moreover, it holds that

$$p(h) = -\rho g h \cos(\theta) + p_0 = 0 \implies p_0 = \rho g h \cos(\theta). \tag{7}$$

So finally,

$$p(y) = -\rho g \cos(\theta)(h - y). \tag{8}$$

Putting this in equation (3) yields

$$\eta \frac{\partial^2 u}{\partial y^2} = -\rho g \sin(\theta) \implies u = -\frac{\rho g y^2}{2\eta} \sin(\theta) + Ay + B. \tag{9}$$

Implementing assumptions 6. and 7. respectively, we obtain

$$B = 0 \quad \wedge \quad A = \frac{\rho g y}{\eta}\sin(\theta). \tag{10}$$

Putting this in (9) and using the identity $\mu = \frac{\eta}{\rho}$, we obtain

$$u = \frac{g\sin(\theta)}{2\mu} y\left(2h - y\right). \tag{11}$$

Substituting given values for μ, ρ and g yields

$$u = \frac{\sin(10°)}{2} y\left(2h - y\right). \tag{12}$$

Furthermore, since u is only a function of y we can calculate the mass flow rate per unit width as

$$Q(h) = \int_0^h \rho u(y)\mathrm{d}y = \frac{\sin(10°)}{2}\int_0^h \left(2hy - y^2\right)\mathrm{d}y \tag{13}$$

$$= \frac{h^3\sin(10°)}{3}. \tag{14}$$

Part b

Conservation of mass tells us

$$\frac{\partial}{\partial t}\iiint_V \mathrm{d}\tau + \oiint_{\mathcal{C}} \mathbf{u}\cdot\mathrm{d}A = 0. \tag{15}$$

Which, in our 1-dimensional case yields

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0 \implies \frac{\partial h}{\partial t} + Q'(h)\frac{\partial h}{\partial x} = 0. \tag{16}$$

Thus, the evolution for $h(x,t)$ can be modelled by the equation

$$\frac{\partial h}{\partial t} + h^2\sin(10°)\frac{\partial h}{\partial x} = 0. \tag{17}$$

Or equivalently,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0 \quad \text{along the line} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = h^2\sin(10°). \tag{18}$$

Part c

If we make the suggested substitution for h, (17) becomes

$$\frac{\partial \eta}{\partial t} + (1+\eta)^2\sin(10°)\frac{\partial \eta}{\partial x} = 0. \tag{19}$$

Using a small η approximation yields $(1+\eta)^2 \approx 1$, so (19) in linearised form becomes

$$\frac{\partial \eta}{\partial t} + \sin(10°)\frac{\partial \eta}{\partial x} = 0. \tag{20}$$

Which in turn implies

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = 0 \quad \text{along} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \sin(10°). \tag{21}$$

Lines corresponding to the second equation in (21) are linear in nature, this means that along lines depicted by Figure 2, η is constant. According to the graph, it is evident that

$$\eta(x,t) = \eta(x - \sin(10°)t, 0) \implies \boxed{\eta(x,t) = \eta_0\left(x - \sin(10°)t\right)} \tag{22}$$

Problem 1 continued on next page. . .

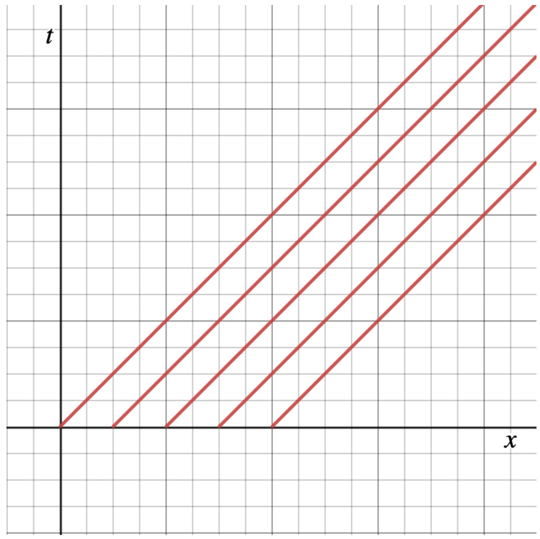

Figure 2: Riemann invariants in the x-t plane (not to scale!)

Part d

We have to transform (17) into Burgers' equation. To that end, we assume a relationship $h = f(r)$, which transforms (17) into

$$\frac{\partial f}{\partial r}\frac{\partial r}{\partial t} + f^2 \sin(10°)\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} = 0 \implies \frac{\partial r}{\partial t} + f^2 \sin(10°)\frac{\partial r}{\partial x} = 0 \tag{23}$$

which is equivalent to (17). From (23) it is evident that

$$r = h^2 \sin(10°). \tag{24}$$

And as a consequence, (17) gets transformed to

$$\frac{\partial r}{\partial t} + r\frac{\partial r}{\partial x} = 0. \tag{25}$$

From (24), it is evident that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 0 \quad \text{if} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = r. \tag{26}$$

Suppose the initial profile from the problem looks like Figure 3.

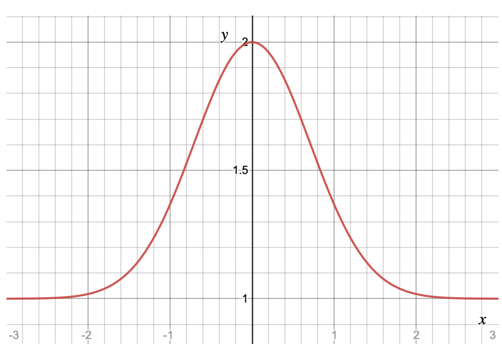

Figure 3: Example of an initial profile of h

Then, Riemann invariants will not all have a constant slope (like those in part c), but a slope which varies linearly with r (hence quadratically with h!). Hence, to the right of the central initial peak, the height decays and the consequent Riemann invariants have a smaller slope - they do not intersect. From the left of the central peak however, the height increases, so the slopes of the Riemann invariants increase - they intersect. A consequence of Riemann invariants intersecting is a shock creation, so a shock will form on **the section where h grows - to the left of the peak**.

Problem 2

This problem is similar in spirit to the first one, but for Poiseuille's flow down a pipe with slowly varying cross-section. Adopt again, for simplicity, a kinematic viscosity $\mu = 1$ and fluid density $\rho = 1$.

a) For a uniform pipe of radius R and pressure gradient P_x, derive Poiseuille's flow $u(r)$ and the corresponding mass flow rate $Q(R, P_x)$.

b) Consider now a pipe with varying radius $R(x)$, yet varying slowly enough that we can assume that your expression above for Q is still valid. Conservation of mass implies that Q is uniform through the pipe. Assuming that $Q = Q_0$ is also independent of time, derive an ODE for the pressure $P(x)$.

c) Consider a pipe with a contraction near $x = 0$, specifically described by

$$R(x) = 2 - e^{-\frac{x^2}{2}},$$

and $Q_0 = 1$. Solve your ODE for $P(x)$ from part c numerically using the trapezoidal rule:

$$P(x + \Delta x) \approx P(x) + \frac{P_x(x) + P_x(x + \Delta x)}{2}\Delta x$$

for a small enough Δx, and plot the resulting $P(x)$. You may start your integration at an x where $R(x)$ is nearly constant, say $x = -5$, and adopt any value for P there, as it is not the absolute value of the pressure but its gradient that matters for the flow.

Solution

Part a

Given the problem, we must make several assumptions before deriving the velocity of Poiseuille flow down a pipe.

1. The pressure gradient is locally constant - it has no dependence on x in particular.

2. The flow is steady - $\mathbf{u}$ is not a function of time.

3. The flow velocity does not have a component other than that in the x direction - in vectorial form:

$$\mathbf{u} = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}. \tag{27}$$

4. The flow velocity does not depend on the angle between the central axis of the tube and the line parallel to it, going through a particular point (the flow is cylindrically symmetric - only depends on the distance from the central axis, as shown by Figure 4). This implies $\mathbf{u} = u(r)\hat{x}$.

5. No-slip: $u(R) = 0$.

Then, we recall the Navier-Stokes equation in a single dimensional case:

$$\frac{\partial u}{\partial t} + \left(\mathbf{u} \cdot \vec{\nabla}\right) u + \frac{1}{\rho}\frac{\partial p}{\partial x} = \mu \nabla^2 u. \tag{28}$$

Problem 2 continued on next page...

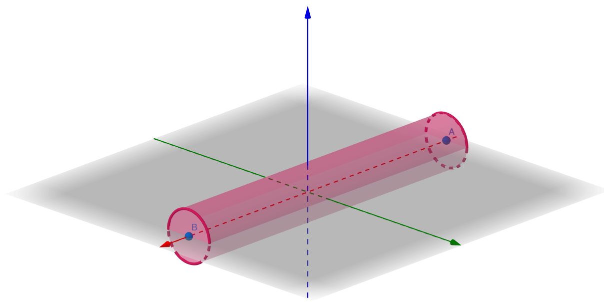

Figure 4: A pipe in the x direction facilitating Poiseuille flow

Applying our assumptions reduces (28) to

$$\frac{\partial p}{\partial x} = \eta \nabla^2 u \tag{29}$$

where we used the relationship between the constants $\eta = \mu\rho$. Now, recalling the cyclindrical form of the Laplace operator[1] and noting the fact that $\frac{\partial^2 u}{\partial \theta^2} = 0$, we obtain

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{P_x}{\eta}. \tag{30}$$

Inputting this PDE into a computing software we obtain a general solution

$$u(r) = \frac{P_x r^2}{4\eta} + a \ln r + b. \tag{31}$$

Applying the no-slip condition at $r = R$, and noting that $\eta = 1$ it must hold that

$$a = 0 \quad \wedge \quad b = -\frac{P_x R^2}{4} \implies \boxed{u(r) = \frac{P_x}{4}\left(r^2 - R^2\right)}. \tag{32}$$

We can use (32) to calculate the volume flow rate (hence mass, since $\rho = 1$):

$$Q = \oiint_{\mathcal{C}} \mathbf{u} \cdot \mathrm{d}\vec{A} = \frac{P_x}{4} \int_0^{2\pi} \int_0^R \left(r^3 - R^2 r\right) \mathrm{d}r\mathrm{d}\theta \tag{33}$$

where we used the fact that the velocity is parallel to the cross-sectional area vector, and that $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$ in polar coordinates. The integral is easily evaluated to be

$$\boxed{Q = -\frac{P_x \pi}{8} R^4} \tag{34}$$

Part b

Conservation of mass tells us that

$$\frac{\partial Q}{\partial x} = 0 \implies Q = Q_0 \tag{35}$$

[1]Remark: Here we have a change of coordinates from Cartesian (x, y, z) to cylindrical (x, r, θ) where $y = r\cos\theta$ and $z = r\sin\theta$.

where Q_0 is not a function of time, as given by the problem. Thus, the ODE for P_x is given by

$$\frac{\mathrm{d}P}{\mathrm{d}x} = -\frac{8Q_0}{(R(x))^4\,\pi} \tag{36}$$

Part c

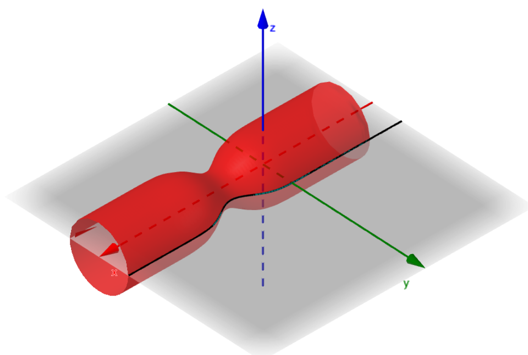

Figure 5: The modified tube around the origin

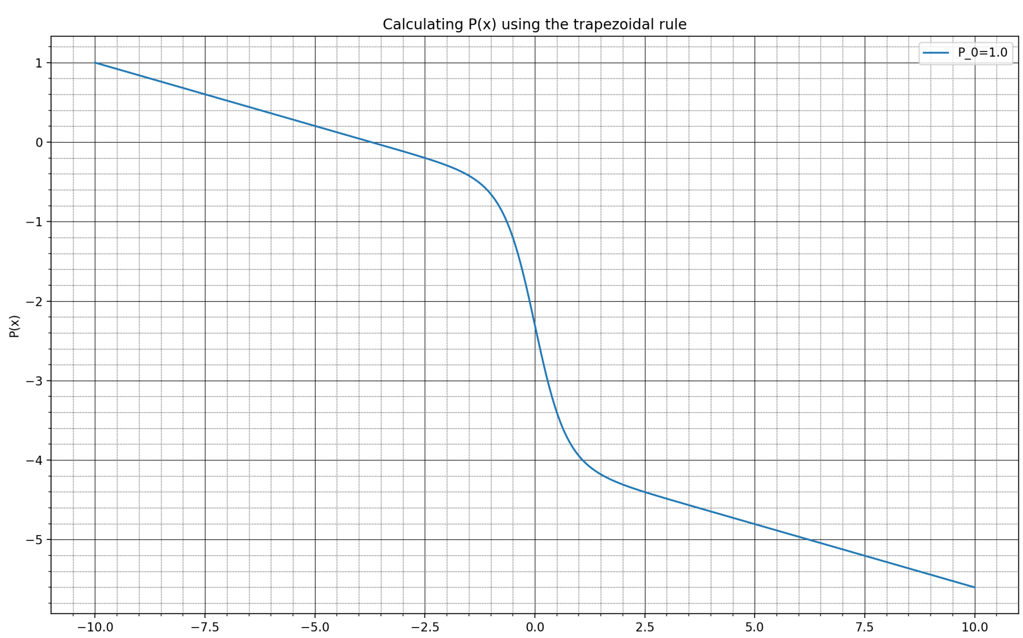

Figure 6: Dependence of P against x

This tube has a slowly varying radius, much like that of a solid of revolution of a Gaussian distribution (see Figure 5). The suggested numerical method was undertaken in a Python script and the results are reported in Figure 6.

In this script, the position interval was taken to be $x \in [-10, 10]$ with a step of $\mathrm{d}x = 10^{-4}$, and $P(-10) = P_0 = 1$. Away from the origin, the pressure decays linearly - which is expected for a constant volume flow Q, characteristic for flows influenced by a constant gravitational field[2].

It is worth noting that the initial condition at $x = -10$ affects $P(x)$ only to the extent of shifting the graph up by P_0, as shown by Figure 7.

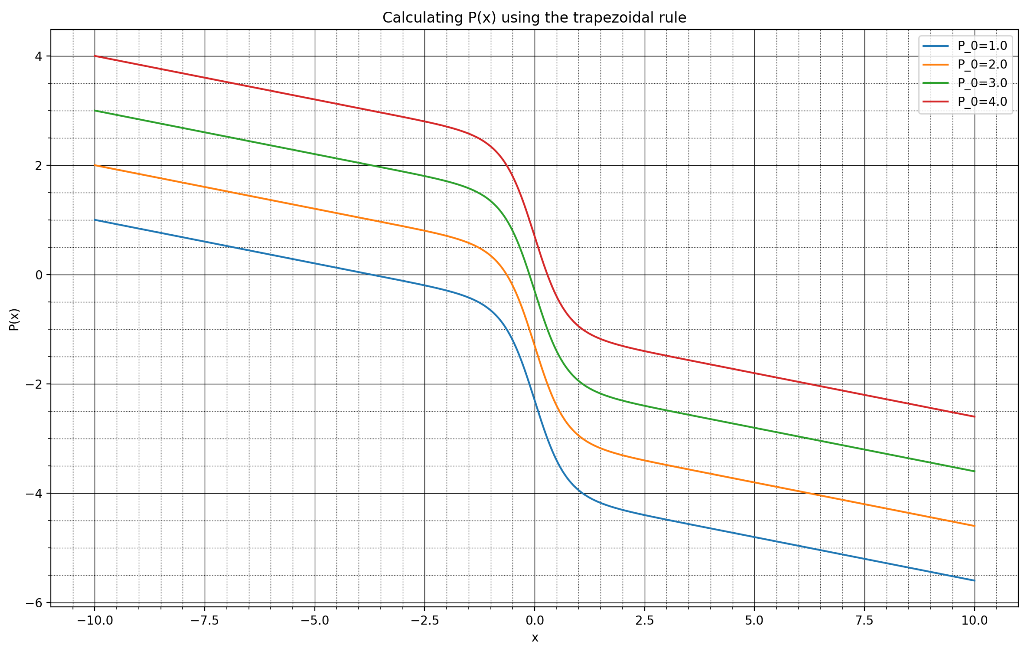

Figure 7: Dependence of P against x for various initial conditions

Furthermore, the code used to obtain Figures 6 and 7 is shown in Figure 8.

[2]This pressure gradient may not be influenced a constant gravitational field, the comparison is purely a curiosity.

trapezium.py

```python
import numpy as np
import matplotlib.pyplot as plt
dx=0.01
x=np.arange(-10,10,dx)
Q = 1.0
P_x = -(8*Q)/((2-np.exp(-(x**2)/2))**4*(np.pi))

P_0=[1.0,2.0,3.0,4.0]

P=[]
P.append(P_0[0])
for i in range (len(P_x)-1):
    P.append(P[-1]+(P_x[i]+P_x[i+1])*dx/2.0)

# P_1=[]
# P_1.append(P_0[1])
# for i in range (len(P_x)-1):
#   P_1.append(P_1[-1]+(P_x[i]+P_x[i+1])*dx/2.0)

# P_2=[]
# P_2.append(P_0[2])
# for i in range (len(P_x)-1):
#   P_2.append(P_2[-1]+(P_x[i]+P_x[i+1])*dx/2.0)

# P_3=[]
# P_3.append(P_0[3])
# for i in range (len(P_x)-1):
#   P_3.append(P_3[-1]+(P_x[i]+P_x[i+1])*dx/2.0)


plt.title("Calculating P(x) using the trapezoidal rule")
plt.minorticks_on()
# Customize the major grid
plt.grid(which='major', linestyle='-', linewidth='0.5', color='black')
# Customize the minor grid
plt.grid(which='minor', linestyle=':', linewidth='0.5', color='black')
plt.xlabel('x')
plt.ylabel('P(x)')
plt.plot(x,P)
# plt.plot(x,P_1)
# plt.plot(x,P_2)
# plt.plot(x,P_3)
plt.legend(['P_0=1.0'], loc='best')
plt.show()
```

Figure 8: Code for trapezoidal rule calculation