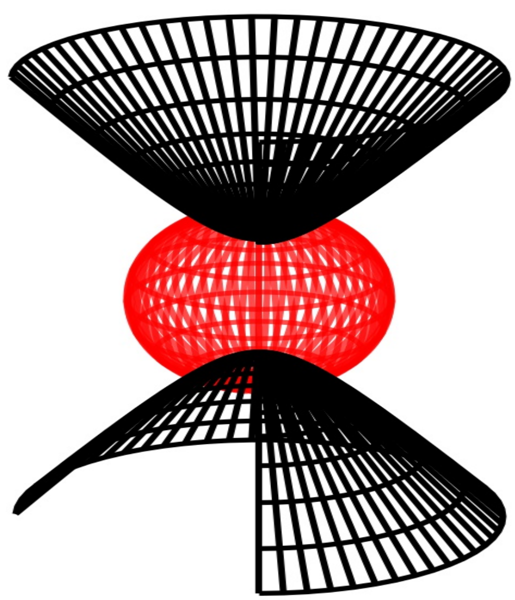
Elektrodinamika

Marko Imbrišak



PMF - Fizički odsjek

2020

Sadržaj

1. Osnove vektorske analize

1.1 Skalari, vektori i tenzori

Skalari su veličine koje se ne mijenjaju prilikom promjena koordinatnog sustava. Fizikalni primjeri su gustoća, masa i volumen. S druge strane, vektori se prilikom promjene koordinatnog sustava mijenjaju. Primjeri vektora su položaj, brzina i sila.

Matematiku vektora objasnit ćemo na primjeru koordinata položaja tijela. Koordinate tijela opisujemo vektorom položaja,

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z},$$

za $x, y, z \in \mathbb{R}$. Ovaj zapis možemo skratiti uvodeći oznaku $\mathbf{e}_i$, gdje je

$$\mathbf{e}_1 = \hat{x} \qquad \mathbf{e}_2 = \hat{y} \qquad \mathbf{e}_3 = \hat{z}, \tag{1.1}$$

dok koordinate analogno označavamo

$$r^1 = x \qquad r^2 = y \qquad r^3 = z. \tag{1.2}$$

U ovom zapisu, vektor položaja možemo zapisati pomoću sume

$$\mathbf{r} = \sum_{i=1}^{3} r^i \mathbf{e}_i. \tag{1.3}$$

Kako bismo pojednostavili ovaj zapis, uvodimo *Einsteinovu konvenciju* - ako se indeksi ponavljaju, sumacija se podrazumijeva.

$$\mathbf{r} = \sum_{i=1}^{3} r^i \mathbf{e}_i \to \mathbf{r} = r^i \mathbf{e}_i. \tag{1.4}$$

Skalarni produkt dvaju vektora **a** i **b** možemo prikazati kao produkt vektora i linearnog funkcionala (dualnog vektora), pri čemu vektor prikazujemo kao vektor stupac te linearni funkcional kao vektor redak.

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix} = a_1 b^1 + a_2 b^2 + a_3 b^3 \tag{1.5}$$

Vektorski produkt dvaju vektora je vektor koji računamo pomoću determinante

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{vmatrix}. \tag{1.6}$$

1.2 Koordinatni sustavi

Najčešći koordinatni sustavi su Kartezijev, cilindrični i sferni koordinatni sustav.

Primjer 1.2.1. Pronađite jedinične vektore za sferni koordinatni sustav te transformacije u Kartezijev. Sferni sustav zadan je koordinatama

$$\begin{aligned} x(r,\theta,\phi) &= r \sin\theta \cos\phi \\ y(r,\theta,\phi) &= r \sin\theta \sin\phi \\ z(r,\theta,\phi) &= r \cos\theta. \end{aligned}$$

Rješenje. Sferni sustav definiramo s (r, θ, ϕ). Primijetimo da vrijedi

$$\begin{aligned} x^2 + y^2 &= r^2 \sin^2\theta \left(\cos^2\phi + \sin^2\phi\right) && (1.7) \\ &= r^2 \sin^2\theta && (1.8) \\ z^2 &= r^2 \cos^2\theta && (1.9) \\ x^2 + y^2 + z^2 &= r^2 \left(\sin^2\theta \left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta\right) && (1.10) \\ &= r^2 \left(\sin^2\theta + \cos^2\theta\right) && (1.11) \\ &= r^2, && (1.12) \end{aligned}$$

iz čega dijeljenjem dobivamo poveznice s Kartezijevim koordinatama

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} && (1.13) \\ \sin\theta &= \sqrt{\frac{x^2 + y^2}{x^2 + y^2 + z^2}} && (1.14) \\ \cos\theta &= \frac{z}{\sqrt{x^2 + y^2 + z^2}} && (1.15) \\ \cos\phi &= \frac{x}{\sqrt{x^2 + y^2}} && (1.16) \\ \sin\phi &= \frac{y}{\sqrt{x^2 + y^2}}. && (1.17) \end{aligned}$$

Pronađimo nenormirane tangentne vektore

$$\mathbf{r} = r(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}) \tag{1.18}$$
$$\partial_r\mathbf{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{1.19}$$
$$\partial_\theta\mathbf{r} = r(\cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - sin\theta\hat{z}) \tag{1.20}$$
$$\partial_\phi\mathbf{r} = r(-\sin\theta\sin\phi\hat{x} + \sin\theta\cos\phi\hat{y}). \tag{1.21}$$

Normiranjem dobivamo jedinične vektore, izražene miješano

$$\hat{r} = \frac{\partial_r\mathbf{r}}{|\partial_r\mathbf{r}|} \tag{1.22}$$
$$= \frac{\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}}{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}} \tag{1.23}$$
$$= \frac{\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}}{\sqrt{\sin^2\theta + \cos^2\theta}} \tag{1.24}$$
$$= \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{1.25}$$
$$\hat{\theta} = \frac{\partial_\theta\mathbf{r}}{|\partial_\theta\mathbf{r}|} \tag{1.26}$$
$$= \frac{r(\cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - sin\theta\hat{z})}{r\sqrt{\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta}} \tag{1.27}$$
$$= \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - sin\theta\hat{z} \tag{1.28}$$
$$\hat{\phi} = \frac{\partial_\phi\mathbf{r}}{|\partial_\phi\mathbf{r}|} \tag{1.29}$$
$$= \frac{r(-\sin\theta\cos\phi\hat{x} + \sin\theta\cos\phi\hat{y})}{r\sqrt{\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi}} \tag{1.30}$$
$$= \frac{r(-\sin\theta\cos\phi\hat{x} + \sin\theta\cos\phi\hat{y})}{r\sin\theta} \tag{1.31}$$
$$= -\sin\phi\hat{x} + \cos\phi\hat{y}, \tag{1.32}$$

što je ujedno i najkorisniji prikaz te dobivamo jedinične vektore izražene u potpunosti Kartezijevim koordinatama

$$\hat{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2+y^2+z^2}} \tag{1.33}$$
$$\hat{\theta} = \frac{zx\hat{x} + zy\hat{y}}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}} - \hat{z}\sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}} \tag{1.34}$$
$$= \frac{z(x\hat{x} + y\hat{y}) + (x^2+y^2)\hat{z}}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}} \tag{1.35}$$
$$\hat{\phi} = -\frac{y}{\sqrt{x^2+y^2}}\hat{x} + \frac{x}{\sqrt{x^2+y^2}}\hat{y} \tag{1.36}$$
$$= \frac{-y\hat{x} + x\hat{y}}{\sqrt{x^2+y^2}}. \tag{1.37}$$

Također, možemo izraziti Kartezijeve jedinične vektore preko sfernih

$$
\begin{aligned}
\hat{x} &= (\hat{x}\cdot\hat{r})\hat{r} + (\hat{x}\cdot\hat{\theta})\hat{\theta} + (\hat{x}\cdot\hat{\phi})\hat{\phi} && (1.38)\\
&= \sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi} && (1.39)\\
\hat{y} &= (\hat{y}\cdot\hat{r})\hat{r} + (\hat{y}\cdot\hat{\theta})\hat{\theta} + (\hat{y}\cdot\hat{\phi})\hat{\phi} && (1.40)\\
&= \sin\theta\sin\phi\hat{r} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi} && (1.41)\\
\hat{z} &= (\hat{z}\cdot\hat{r})\hat{r} + (\hat{z}\cdot\hat{\theta})\hat{\theta} + (\hat{z}\cdot\hat{\phi})\hat{\phi} && (1.42)\\
&= \cos\theta\hat{r} - \sin\theta\hat{\theta}. && (1.43)
\end{aligned}
$$

Primjer 1.2.2. Pronađite jedinične vektore za cilindrični koordinatni sustav te transformacije u Kartezijev. Cilindrični sustav zadan je koordinatama

$$
\begin{aligned}
x(\rho,\phi,z) &= \rho\cos\phi\\
y(\rho,\phi,z) &= \rho\sin\phi\\
z(\rho,\phi,z) &= z.
\end{aligned}
$$

Rješenje. Cilindrični sustav definiramo s (ρ, ϕ, z). Primijetimo da vrijedi

$$
\begin{aligned}
x^2 + y^2 &= \rho^2\left(\sin^2\phi + \cos^2\phi\right) && (1.44)\\
&= \rho^2, && (1.45)
\end{aligned}
$$

iz čega možemo izraziti

$$
\begin{aligned}
\rho &= \sqrt{x^2+y^2} && (1.46)\\
\cos\phi &= \frac{x}{\sqrt{x^2+y^2}} && (1.47)\\
\sin\phi &= \frac{y}{\sqrt{x^2+y^2}}, && (1.48)
\end{aligned}
$$

Pronađimo nenormirane tangentne vektore

$$
\begin{aligned}
\mathbf{r} &= \rho(\cos\phi\hat{x} + \sin\phi\hat{y}) + z\hat{z} && (1.49)\\
\partial_\rho\mathbf{r} &= \cos\phi\hat{x} + \sin\phi\hat{y} && (1.50)\\
\partial_\phi\mathbf{r} &= \rho(-\sin\phi\hat{x} + \cos\phi\hat{y}) && (1.51)\\
\partial_z\mathbf{r} &= \hat{z}. && (1.52)
\end{aligned}
$$

Normiranjem dobivamo jedinične vektore, izražene miješano

$$\hat{\rho} = \frac{\partial_\rho \mathbf{r}}{|\partial_\rho \mathbf{r}|} \tag{1.53}$$
$$= \frac{\cos\phi\hat{x} + \sin\phi\hat{y}}{\sqrt{\theta\cos^2\phi + \sin^2\phi}} \tag{1.54}$$
$$= \cos\phi\hat{x} + \sin\phi\hat{y} \tag{1.55}$$
$$\hat{\phi} = \frac{\partial_\phi \mathbf{r}}{|\partial_\phi \mathbf{r}|} \tag{1.56}$$
$$= \frac{\rho(-\sin\phi\hat{x} + \cos\phi\hat{y})}{\rho\sqrt{\sin^2\phi + \cos^2\phi}} \tag{1.57}$$
$$= -\sin\phi\hat{x} + \cos\phi\hat{y} \tag{1.58}$$
$$\hat{z} = \frac{\partial_z \mathbf{r}}{|\partial_z \mathbf{r}|} \tag{1.59}$$
$$= \hat{z}. \tag{1.60}$$

što je ujedno i najkorisniji prikaz. Dobivamo jedinične vektore izražene u potpunosti Kartezijevim koordinatama

$$\hat{\rho} = \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2+y^2}} \tag{1.61}$$
$$\hat{\phi} = -\frac{y}{\sqrt{x^2+y^2}}\hat{x} + \frac{x}{\sqrt{x^2+y^2}}\hat{y} \tag{1.62}$$
$$= \frac{-y\hat{x} + x\hat{y}}{\sqrt{x^2+y^2}} \tag{1.63}$$
$$\hat{z} = \hat{z}. \tag{1.64}$$

Također, možemo izraziti Kartezijeve jedinične vektore preko cilindričnih

$$\hat{x} = (\hat{x}\cdot\hat{\rho})\hat{\rho} + (\hat{x}\cdot\hat{\phi})\hat{\phi} + (\hat{x}\cdot\hat{z})\hat{z} \tag{1.65}$$
$$= \cos\phi\hat{\rho} - \sin\phi\hat{\phi} \tag{1.66}$$
$$\hat{y} = (\hat{y}\cdot\hat{\rho})\hat{\rho} + (\hat{y}\cdot\hat{\phi})\hat{\phi} + (\hat{y}\cdot\hat{z})\hat{z} \tag{1.67}$$
$$= \sin\phi\hat{\rho} + \cos\phi\hat{\phi} \tag{1.68}$$
$$\hat{z} = (\hat{z}\cdot\hat{\rho})\hat{\rho} + (\hat{z}\cdot\hat{\phi})\hat{\phi} + (\hat{z}\cdot\hat{z})\hat{z} \tag{1.69}$$
$$= \hat{z} \tag{1.70}$$

Primjer 1.2.3. Povežite cilindrični i sferni koordinatni sustav.

Rješenje. Budući da opisujemo isti vektor, mora vrijediti $\mathbf{r}(r,\theta,\phi) = \mathbf{r}(\rho,\phi,z)$. Usporedbom dobivamo

$$r\sin\theta\cos\phi = \rho\cos\phi \tag{1.71}$$
$$r\sin\theta\sin\phi = \rho\sin\phi \tag{1.72}$$
$$r\cos\theta = z. \tag{1.73}$$

Budući da ϕ ima isti izraz u Kartezijevom sustavu u oba sustava, vrijedi

$$\rho = r \sin \phi \tag{1.74}$$
$$z = r \cos \theta \tag{1.75}$$
$$r = \sqrt{\rho^2 + z^2} \tag{1.76}$$
$$\sin \theta = \frac{\rho}{\sqrt{\rho^2 + z^2}} \tag{1.77}$$
$$\cos \theta = \frac{z}{\sqrt{\rho^2 + z^2}}. \tag{1.78}$$

Za izražavanje jediničnih vektora, koristimo prethodna dva zadatka

$$\hat{\rho} = \cos \phi \hat{x} + \sin \phi \hat{y} \tag{1.79}$$
$$= \cos \phi (\sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi}) \tag{1.80}$$
$$= + \sin \phi (\sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi}) \tag{1.81}$$
$$= \sin \theta \hat{r} + \cos \theta \hat{\theta} \tag{1.82}$$
$$\hat{\phi} = - \sin \phi \hat{x} + \cos \phi \hat{y} \tag{1.83}$$
$$= - \sin \phi (\sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi}) \tag{1.84}$$
$$+ \cos \phi (\sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi}) \tag{1.85}$$
$$= \hat{\phi} \tag{1.86}$$
$$\hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta}. \tag{1.87}$$

Obrat računamo uvrštavanjem

$$\hat{r} = (\hat{r} \cdot \hat{\rho}) \hat{\rho} + (\hat{r} \cdot \hat{\phi}) \hat{\phi} + (\hat{r} \cdot \hat{z}) \hat{z} \tag{1.88}$$
$$= \sin \theta \hat{\rho} + \cos \theta \hat{z} \tag{1.89}$$
$$= \frac{\rho \hat{\rho} + z \hat{z}}{\sqrt{\rho^2 + z^2}} \tag{1.90}$$
$$\hat{\theta} = (\hat{\theta} \cdot \hat{\rho}) \hat{\rho} + (\hat{\theta} \cdot \hat{\phi}) \hat{\phi} + (\hat{\theta} \cdot \hat{z}) \hat{z} \tag{1.91}$$
$$= \cos \theta \hat{\rho} - \sin \theta \hat{z} \tag{1.92}$$
$$= \frac{z \hat{\rho} - \rho \hat{z}}{\sqrt{\rho^2 + z^2}} \tag{1.93}$$
$$\hat{\phi} = \hat{\phi}. \tag{1.94}$$

Primjer 1.2.4. Izvedite jedinične vektore za tangentni sferni koordinatni sustav. Tangentni sferni sustav zadan je koordinatama

$$x(\mu, \nu, \psi) = \frac{\mu \cos \psi}{\mu^2 + \nu^2}$$
$$y(\mu, \nu, \psi) = \frac{\mu \sin \psi}{\mu^2 + \nu^2}$$
$$z(\mu, \nu, \psi) = \frac{\nu}{\mu^2 + \nu^2}.$$

Rješenje. Izrazimo poveznicu koordinatnih sustava

$$\sqrt{x^2+y^2} = \sqrt{\frac{\mu^2\left(\cos^2\psi+\sin^2\psi\right)}{\left(\mu^2+\nu^2\right)^2}} \tag{1.95}$$

$$= \frac{\mu}{\mu^2+\nu^2} \tag{1.96}$$

$$\sqrt{x^2+y^2+z^2} = \sqrt{\frac{\mu^2\left(\cos^2\psi+\sin^2\psi\right)+\nu^2}{\left(\mu^2+\nu^2\right)^2}} \tag{1.97}$$

$$= \sqrt{\frac{\mu^2+\nu^2}{(\mu^2+\nu^2)^2}} \tag{1.98}$$

$$= \frac{1}{\sqrt{\mu^2+\nu^2}} \tag{1.99}$$

$$\cos\psi = x\frac{\mu^2+\nu^2}{\mu} \tag{1.100}$$

$$= \frac{x}{\sqrt{x^2+y^2}} \tag{1.101}$$

$$\sin\psi = y\frac{\mu^2+\nu^2}{\mu} \tag{1.102}$$

$$= \frac{y}{\sqrt{x^2+y^2}} \tag{1.103}$$

$$\mu = \sqrt{x^2+y^2}(\mu^2+\nu^2) \tag{1.104}$$

$$= \frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} \tag{1.105}$$

$$\nu = z(\mu^2+\nu^2) \tag{1.106}$$

$$= \frac{z}{x^2+y^2+z^2}. \tag{1.107}$$

Pronađimo nenormirane vektore

$$\mathbf{r} = \frac{\mu\cos\psi}{\mu^2+\nu^2}\hat{x} + \frac{\mu\sin\psi}{\mu^2+\nu^2}\hat{y} + \frac{\nu}{\mu^2+\nu^2}\hat{z} \tag{1.108}$$

$$\partial_\mu\mathbf{r} = \frac{(\mu^2+\nu^2)-2\mu^2}{(\mu^2+\nu^2)}\cos\psi\hat{x} + \frac{(\mu^2+\nu^2)-2\mu^2}{(\mu^2+\nu^2)^2}\sin\psi\hat{y} + \frac{-2\mu\nu}{(\mu^2+\nu^2)}\hat{z} \tag{1.109}$$

$$= \frac{1}{(\mu^2+\nu)^2}((\nu^2-\mu^2)(\cos\psi\hat{x}+\sin\psi\hat{y})-2\mu\nu\hat{z}) \tag{1.110}$$

$$\partial_\nu\mathbf{r} = \frac{-2\mu\nu\cos\psi}{(\mu^2+\nu^2)^2}\hat{x} + \frac{-2\mu\nu\sin\psi}{(\mu^2+\nu^2)}\hat{y} + \frac{\mu^2+\nu^2-2\nu^2}{(\mu^2+\nu^2)^2}\hat{z} \tag{1.111}$$

$$= \frac{1}{(\mu^2+\nu^2)}\left(-2\mu\nu(\cos\psi\hat{x}+\sin\psi\hat{y})+(\mu^2-\nu^2)\hat{z}\right) \tag{1.112}$$

$$\partial_\psi\mathbf{r} = -\frac{\mu\sin\psi}{\mu^2+\nu^2}\hat{x} + \frac{\mu\cos\psi}{\mu^2+\nu^2}\hat{y}. \tag{1.113}$$

Normiramo

$$\hat{\mu} = \frac{(\nu^2 - \mu^2)(\cos\psi\hat{x} + \sin\psi\hat{y}) - 2\mu\nu\hat{z}}{\sqrt{(\nu^2 - \mu^2)^2 + 4\mu^2\nu^2}} \tag{1.114}$$

$$= \frac{(\nu^2 - \mu^2)(\cos\psi\hat{x} + \sin\psi\hat{y}) - 2\mu\nu\hat{z}}{\sqrt{(\nu^2 + \mu^2)^2}} \tag{1.115}$$

$$= \frac{(\nu^2 - \mu^2)(\cos\psi\hat{x} + \sin\psi\hat{y}) - 2\mu\nu\hat{z}}{\nu^2 + \mu^2} \tag{1.116}$$

$$\hat{\nu} = \frac{-2\mu\nu(\cos\psi\hat{x} + \sin\psi\hat{y}) + (\mu^2 - \nu^2)\hat{z}}{\sqrt{(\nu^2 - \mu^2)^2 + 4\mu^2\nu^2}} \tag{1.117}$$

$$= \frac{-2\mu\nu(\cos\psi\hat{x} + \sin\psi\hat{y}) + (\mu^2 - \nu^2)\hat{z}}{\sqrt{(\nu^2 + \mu^2)^2}} \tag{1.118}$$

$$= \frac{-2\mu\nu(\cos\psi\hat{x} + \sin\psi\hat{y}) + (\mu^2 - \nu^2)\hat{z}}{\nu^2 + \mu^2} \tag{1.119}$$

$$\hat{\psi} = \frac{-\frac{\mu\sin\psi}{\mu^2+\nu^2}\hat{x} + \frac{\mu\cos\psi}{\mu^2+\nu^2}\hat{y}}{\frac{\mu\sin\psi}{\mu^2+\nu^2}\sqrt{\sin^2\psi + \cos^2\psi}} \tag{1.120}$$

$$= -\sin\psi\hat{x} + \cos\psi\hat{y}. \tag{1.121}$$

Koristeći poveznice dobivamo

$$\hat{\mu} = \frac{(z^2 - x^2 - y^2)(x\hat{x} + y\hat{y}) - 2(x^2 + y^2)z\hat{z}}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2}} \tag{1.122}$$

$$\hat{\nu} = \frac{-2(x^2 + y^2)(x\hat{x} + y\hat{y}) - 2(z^2 - x^2 - y^2)z\hat{z}}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2}} \tag{1.123}$$

$$\hat{\psi} = \frac{-y\hat{x} + x\hat{y}}{\sqrt{x^2 + y^2}}. \tag{1.124}$$

Također možemo izračunati

$$\hat{x} = (\hat{x} \cdot \mu)\hat{\mu} + (\hat{x} \cdot \nu)\hat{\nu} + (\hat{x} \cdot \psi)\hat{\psi} \tag{1.125}$$

$$= \frac{(\nu^2 - \mu^2)\cos\psi\hat{\mu}}{\nu^2 + \mu^2} + \frac{-2\mu\nu\cos\psi}{\nu^2 + \mu^2}\hat{\nu} - \sin\psi\hat{\psi} \tag{1.126}$$

$$= \frac{(\nu^2 - \mu^2)\hat{\mu} - 2\mu\nu\hat{\nu}}{\nu^2 + \mu^2}\cos\psi - \sin\psi\hat{\psi} \tag{1.127}$$

$$\hat{y} = (\hat{y} \cdot \mu)\hat{\mu} + (\hat{y} \cdot \nu)\hat{\nu} + (\hat{y} \cdot \psi)\hat{\psi} \tag{1.128}$$

$$= \frac{(\nu^2 - \mu^2)\sin\psi}{\nu^2 + \mu^2}\hat{\mu} + \frac{-2\mu\nu\sin\psi}{\nu^2 + \mu^2}\hat{\nu} + \cos\psi\hat{\psi} \tag{1.129}$$

$$= \frac{(\nu^2 - \mu^2)\hat{\mu} - 2\mu\nu\sin\psi}{\nu^2 + \mu^2}\sin\psi + \cos\psi\hat{\psi} \tag{1.130}$$

$$\hat{z} = \frac{-2\mu\nu}{\nu^2 + \mu^2}\mu + \frac{(\mu^2 - \nu^2)}{\nu^2 + \mu^2}\hat{\nu} \tag{1.131}$$

$$= \frac{-2\mu\nu\hat{\mu} + (\mu^2 - \nu^2)\hat{\nu}}{\nu^2 + \mu^2}. \tag{1.132}$$

Primjer 1.2.5. Izračunati jedinične vektore za paraboličke koordinate definirane s

$$x(\sigma, \tau) = \sigma\tau \tag{1.133}$$

$$y(\sigma, \tau) = \frac{\tau^2 - \sigma^2}{2}. \tag{1.134}$$

Rješenje. Nenormirani tangentni vektori su

$$\mathbf{r} = \sigma\tau\hat{x} + \frac{\tau^2 - \sigma^2}{2}\hat{y} \tag{1.135}$$

$$\partial_\tau \mathbf{r} = \sigma\hat{x} + \tau\hat{y} \tag{1.136}$$

$$\partial_\sigma \mathbf{r} = \tau\hat{x} - \sigma\hat{y}. \tag{1.137}$$

Normiranjem dobivamo

$$\hat{\tau} = \frac{\partial_\tau \mathbf{r}}{|\partial_\tau \mathbf{r}|} \tag{1.138}$$

$$= \frac{\sigma\hat{x} + \tau\hat{y}}{\sqrt{\tau^2 + \sigma^2}} \tag{1.139}$$

$$\hat{\sigma} = \frac{\partial_\sigma \mathbf{r}}{|\partial_\sigma \mathbf{r}|} \tag{1.140}$$

$$= \frac{\tau\hat{x} - \sigma\hat{y}}{\sqrt{\tau^2 + \sigma^2}}. \tag{1.141}$$

Primjer 1.2.6. Pronaći gradijent i laplasijan funkcije u Kartezijevim koordinatama $f(x, y, z) = x^2yz$.

Rješenje. Gradijent računamo po formuli

$$\nabla f = (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z)(x^2yz) \tag{1.142}$$

$$= 2xyz\hat{x} + x^2z\hat{y} + x^2y\hat{z} \tag{1.143}$$

$$= x^2yz\left(\frac{2\hat{x}}{x} + \frac{\hat{y}}{y} + \frac{\hat{z}}{z}\right). \tag{1.144}$$

Laplasijan je

$$\nabla^2 f = (\partial_x^2 + \partial_y^2 + \partial_z^2)(x^2yz) \tag{1.145}$$

$$= 2yz + 0 + 0 \tag{1.146}$$

$$= 2yz. \tag{1.147}$$

Primjer 1.2.7. Pronaći gradijent i laplasijan funkcije zadane u cilindričnim koordinatama $f(\rho, \phi, z) = \rho^2 \ln z$.

Rješenje. Računamo gradijent

$$\nabla f = \left(\hat{\rho}\partial_\rho + \frac{\hat{\phi}}{\rho}\partial_\phi + \hat{z}\partial_z\right)\rho^2 \ln z \tag{1.148}$$

$$= \hat{\rho}2\rho \ln z + 0\hat{\phi} + \frac{\rho^2}{z}\hat{z} \tag{1.149}$$

$$= 2\rho \ln z\hat{\rho} + \frac{\rho^2}{z}\hat{z}. \tag{1.150}$$

Laplasijan

$$\nabla^2 f = \left(\frac{1}{\rho}\partial_\rho(\rho\partial_\rho) + \frac{\partial_\phi^2}{\rho^2} + \partial_z^2 \right) \rho^2 \ln z \tag{1.151}$$

$$= \frac{\ln z}{\rho}\partial_\rho(2\rho^2) + 0 + \frac{-\rho^2}{z^2} \tag{1.152}$$

$$= 4\ln z - \frac{\rho^2}{z^2}. \tag{1.153}$$

Primjer 1.2.8. Pronaći gradijent i laplasijan funkcije zadanu sfernim koordinatama $f(r, \theta, \phi) = r \sin\theta \cos\phi$ za $r \neq 0$.

Rješenje. Gradijent je

$$\nabla f = \left(\hat{r}\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \frac{\hat{\phi}}{r\sin\theta}\partial_\phi \right) f \tag{1.154}$$

$$= \hat{r}\sin\theta\cos\phi + \frac{r\cos\theta\cos\phi\hat{\theta}}{r} + \frac{r\sin\theta(-\sin\phi)\hat{\phi}}{r\sin\theta} \tag{1.155}$$

$$= \hat{r}\sin\theta\cos\phi + \cos\theta\cos\phi\hat{\theta} - \sin\theta\phi\phi. \tag{1.156}$$

Računamo i laplasijan

$$\nabla^2 f = \left(\frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 \right) f \tag{1.157}$$

$$= \frac{\sin\theta\cos\phi}{r^2}\partial_r(r^2) + \frac{r\cos\phi}{r^2\sin\theta}\partial_\theta(\sin\theta\cos\theta) + \frac{-r\sin\theta\cos\phi}{r^2\sin^2\theta} \tag{1.158}$$

$$= \frac{2\sin\theta\cos\phi}{r} + \frac{\cos\phi(\cos^2\theta - \sin^2\theta)}{r\sin\theta} - \frac{\cos\phi}{r\sin\theta} \tag{1.159}$$

$$= \frac{2\sin\theta\cos\phi}{r} + \frac{\cos\phi(1 - 2\sin^2\theta - 1)}{r\sin\theta} \tag{1.160}$$

$$= 0. \tag{1.161}$$

Primjer 1.2.9. Izračunati divergenciju i rotaciju vektorske funkcije $\mathbf{A} = y\hat{x} + z\hat{y} + x\hat{z}$.

Rješenje. Računamo divergenciju

$$\nabla \cdot \mathbf{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z \tag{1.162}$$

$$= \partial_x y + \partial_y z + \partial_z x \tag{1.163}$$

$$= 0, \tag{1.164}$$

i rotaciju

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} \tag{1.165}$$

$$= \hat{x}(0-1) - \hat{y}(1-0) + \hat{z}(0-1) \tag{1.166}$$

$$= -(\hat{x} + \hat{y} + \hat{z}). \tag{1.167}$$

Primjer 1.2.10. Izračunati divergenciju i rotaciju vektorske funkcije $\mathbf{A} = \rho\hat{\rho} + z\hat{z}$.

Rješenje. Divergenciju računamo po formuli

$$\begin{aligned}
\nabla \cdot \mathbf{A} &= \frac{1}{\rho}\partial_\rho(\rho A_\rho) + \frac{1}{\rho}\partial_\phi A_\phi + \partial_z A_z && (1.168)\\
&= \frac{1}{\rho}\partial_\rho(\rho^2) + 0 + \partial_z(z) && (1.169)\\
&= 2 + 1 && (1.170)\\
&= 3. && (1.171)
\end{aligned}$$

Rotaciju možemo najlakše predstaviti determinantom

$$\begin{aligned}
\nabla \times \mathbf{A} &= \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \partial_\rho & \partial_\phi & \partial_z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} && (1.172)\\
&= \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \partial_\rho & \partial_\phi & \partial_z \\ \rho & 0 & z \end{vmatrix} && (1.173)\\
&= 0\hat{\rho} + 0\hat{\phi} + 0\hat{z} && (1.174)\\
&= \vec{0}. && (1.175)
\end{aligned}$$

Primjer 1.2.11. Izračunati divergenciju i rotaciju vektorske funkcije $\mathbf{A} = C\sin\theta\sin\phi\hat{r} + r\hat{\theta}$.

Rješenje. Divergencija je

$$\begin{aligned}
\nabla \cdot \mathbf{A} &= \frac{1}{r^2}\partial_r(r^2 A_r) + \frac{\partial_\theta(\sin\theta A_\theta)}{r\sin\theta} + \frac{\partial_\phi A_\phi}{r\sin\theta} && (1.176)\\
&= \frac{C\sin\theta\sin\phi}{r^2}\partial_r(r^2) + \frac{r\partial_\theta(\sin\theta)}{r\sin\theta} + 0 && (1.177)\\
&= \frac{2C\sin\theta\sin\phi}{r} + \cot\theta, && (1.178)
\end{aligned}$$

a rotacija je

$$\begin{aligned}
\nabla \times \mathbf{A} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} && (1.179)\\
&= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ C\sin\theta\sin\phi & r^2 & 0 \end{vmatrix} && (1.180)\\
&= 0\hat{r} + \frac{-r\hat{\theta}}{r^2\sin\theta}(0 - C\sin\theta\cos\phi) + \frac{r\sin\theta\hat{\phi}}{r^2\sin\theta}(2r - C\cos\theta\sin\phi) && (1.181)\\
&= \frac{C}{r}\cos\phi\hat{\theta} + \frac{\hat{\phi}}{r}(2r - C\cos\theta\sin\phi). && (1.182)
\end{aligned}$$

1.3 Diracova δ funkcija

Diracova δ funkcija je distribucija koja za neku funkciju f ima svojstvo

$$\int \delta(x-a)f(x)dx = f(a). \tag{1.183}$$

U trodimenzionalnom sustavu također možemo uvesti Diracovu δ funkciju

$$\int \delta(\mathbf{r}-\mathbf{a})f(\mathbf{r})d^3\mathbf{r} = f(\mathbf{a}). \tag{1.184}$$

Ovakav nam izraz omogućuje izračunati gustoću točkastog naboja. Primjerice, naboj q na poziciji (x_0, y_0, z_0) možemo opisati gustoćom

$$\int d^3\mathbf{r}\rho(\mathbf{r}) = \int\int\int \rho_0\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)dxdydz = q \tag{1.185}$$

budući da je ρ_0 konstanta, vrijedi $\rho_0 = q$ te je gustoća

$$\rho(\mathbf{r}) = q\delta(x-x_0)\delta(y-y_0)\delta(z-z_0). \tag{1.186}$$

Možemo također računati kompliciranije izraze koji uključuju δ funkcije koristeći sljedeći izraz

$$\delta(x-f(x)) = \sum_{x_0,\, f(x_0)=0} \frac{\delta(x-x_0)}{|f'(x_0)|}. \tag{1.187}$$

Ako uvedemo Lameove koeficijente za transformaciju $x^i = x^i(\xi^1, \xi^2, \xi^3)$

$$h_i = \sqrt{\sum_j \left(\frac{\partial x^j}{\partial \xi^i}\right)^2}, \tag{1.188}$$

Možemo izračunati δ funkciju u proizvoljnom koordinatnom sustavu

$$\delta(\mathbf{r}-\mathbf{r}_0) = \frac{\delta(\xi^1-\xi^1_0)\delta(\xi^2-\xi^2_0)\delta(\xi^3-\xi^3_0)}{h_1h_2h_3}. \tag{1.189}$$

Primjer 1.3.1. Izračunati izraze

1. $\delta(x^2-1)$
2. $\delta(ax+b)$
3. $\delta(\cos\theta)$
4. $\delta(\sin\theta)$

Rješenje. 1. Izraz pod δ funkcijom ima dvije nultočke $x = \pm 1$. Primjenjujući formulu dobivamo

$$\delta(x^2-1) = \frac{\delta(x-1)}{2|1|} + \frac{\delta(x+1)}{2|-1|} = \frac{\delta(x-1)+\delta(x+1)}{2}. \tag{1.190}$$

2.

$$\delta(ax+b) = \frac{\delta(x-b/a)}{|(ax+b)'|} = \frac{\delta(x-b/a)}{|a|} \tag{1.191}$$

3. Nultočka funkcije je $\theta = \pi/2 + n\pi$, $n \in \mathbb{Z}$

$$\delta(\cos\theta) = \sum_n \frac{\delta(\theta - \pi/2 - n\pi)}{|\sin(\theta)|} = \sum_n \frac{\delta(\theta - \pi/2 - n\pi)}{|\sin(\pi/2 + n\pi)|} = \sum_n \delta(\theta - \pi/2 - n\pi). \tag{1.192}$$

4. Nultočka funkcije je $\theta = n\pi$, $n \in \mathbb{Z}$

$$\delta(\sin\theta) = \sum_n \frac{\delta(\theta - n\pi)}{|\cos(\theta)|} = \sum_n \frac{\delta(\theta - n\pi)}{|\cos(n\pi)|} = \sum_n \delta(\theta - n\pi). \tag{1.193}$$

Primjer 1.3.2. Izračunati δ funkciju u cilindričnom i sfernom koordinatnom sustavu

Rješenje. Lameovi koeficijenti za sferni sustav dani su izrazima

$$h_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} \tag{1.194}$$

$$= \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} = 1 \tag{1.195}$$

$$h_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} \tag{1.196}$$

$$= \sqrt{r^2(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta)} = r \tag{1.197}$$

$$h_\phi = \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2} \tag{1.198}$$

$$= \sqrt{r^2\sin^2\theta(\cos^2\phi + \sin^2\phi)} = r\sin\theta. \tag{1.199}$$

Diracova δ funkcija je tada

$$\delta(\mathbf{r} - \mathbf{r}_0) = \frac{\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)}{r^2\sin\theta} \tag{1.200}$$

Lameovi koeficijenti za cilindrični sustav dani su izrazima

$$h_\rho = \sqrt{\left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \rho}\right)^2} \tag{1.201}$$

$$= \sqrt{\cos^2\phi + \sin^2\phi} = 1 \tag{1.202}$$

$$h_\phi = \sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2} \tag{1.203}$$

$$= \sqrt{\rho^2(\cos^2\phi + \sin^2\phi)} = \rho \tag{1.204}$$

$$h_z = \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} \tag{1.205}$$

$$= 1. \tag{1.206}$$

Diracova δ funkcija je tada

$$\delta(\mathbf{r} - \mathbf{r}_0) = \frac{\delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(z - z_0)}{\rho} \tag{1.207}$$

Primjer 1.3.3. Izračunati gustoću sfere i prstena radijusa R.

Rješenje. Sfera je opisana samo jednom δ funkcijom $\delta(r-R)$. Djelić naboja sfere dq se nalazi na koordinatama θ_0 i ϕ_0. Ukupna δ funkcija je tada:

$$\delta(\mathbf{r}-\mathbf{r}_0)=\int \sin\theta d\theta d\phi \frac{\delta(r-R)\delta(\theta-\theta_0)\delta(\phi-\phi_0)}{r^2\sin\theta}=\frac{\delta(r-R)}{r^2}. \tag{1.208}$$

Gustoća naboja je proporcionalna delta funkciji:

$$\rho=N\delta(\mathbf{r}-\mathbf{r}_0). \tag{1.209}$$

Da bi našli konstantu N, uvršavamo u izraz za ukupni naboj:

$$q=\int_V \rho dV=\int N\frac{\delta(r-R)}{r^2}r^2\sin\theta dr d\theta d\phi=4\pi N\to N=\frac{q}{4\pi} \tag{1.210}$$

Gustoća sfere je tada

$$\rho=q\frac{\delta(r-R)}{4\pi r^2}. \tag{1.211}$$

Analogno za prsten:

$$\delta(\mathbf{r}-\mathbf{r}_0)=\int d\phi\frac{\delta(\rho-R)\delta(z-z_0)\delta(\phi-\phi_0)}{r}=\frac{\delta(\rho-R)\delta(z-z_0)}{\rho}. \tag{1.212}$$

Gustoća naboja je proporcionalna delta funkciji:

$$\rho_{nabojna}=N\delta(\mathbf{r}-\mathbf{r}_0). \tag{1.213}$$

Da bi našli konstantu N, uvršavamo u izraz za ukupni naboj:

$$q=\int N\frac{\delta(\rho-R)\delta(z-z_0)}{\rho}d\rho d\phi dz=2\pi N, \tag{1.214}$$

te je gustoća:

$$\rho_{nabojna}=q\frac{\delta(\rho-R)\delta(z-z_0)}{2\pi\rho}. \tag{1.215}$$

Primjer 1.3.4. Izračunati

1. $\int\limits_0^2 \delta'(x-1)x^3dx$
2. $\int \nabla^2\delta(\mathbf{r}-\mathbf{r}')f(\mathbf{r})d^3\mathbf{r}$
3. $\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|}=-4\pi\delta(\mathbf{r}-\mathbf{r}')$

Rješenje. Rješenja dobivamo parcijalnom integracijom:

$$\int\limits_0^2 \delta'(x-1)x^3dx=\delta(x-1)x^3\Big|_0^2-\int\limits_0^2 3x^2\delta(x-1)dx=-3 \tag{1.216}$$

$$\int\nabla^2\delta(\mathbf{r}-\mathbf{r}')f(\mathbf{r})d^3\mathbf{r}=\oint f(\mathbf{r})\nabla\delta(\mathbf{r}-\mathbf{r_0})\cdot d^2\mathbf{r}-\int\nabla\delta(\mathbf{r}-\mathbf{r_0})\cdot\nabla f(\mathbf{r})d^3\mathbf{r} \tag{1.217}$$

$$=\int\nabla^2 f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')d^3\mathbf{r}=\nabla^2 f(\mathbf{r}'). \tag{1.218}$$

Za zadnji zadatak koristimo Gaussov zakon. Primijetimo da je izraz koji tražimo

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{4\pi\epsilon_0}{q} \nabla \cdot \mathbf{E}. \tag{1.219}$$

Koristeći Gausov zakon

$$\int \nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r} = \int \frac{-4\pi\epsilon_0}{q} \nabla \cdot \mathbf{E} d^3\mathbf{r} \tag{1.220}$$

$$= \oint \frac{-4\pi\epsilon_0}{q} \mathbf{E} \cdot d^2\mathbf{r} \tag{1.221}$$

$$= \frac{-4\pi\epsilon_0}{q} \frac{q}{\epsilon_0} \tag{1.222}$$

$$= -4\pi \tag{1.223}$$

$$= \int -4\pi\delta(\mathbf{r}-\mathbf{r}') d^3\mathbf{r}. \tag{1.224}$$

1.4 Parametrizacije

1.4.1 Parametrizacija krivulja

Krivulju u prostoru parametriziramo jednim parametrom, u oznaci t kao $\mathbf{s}(t)$. Diferencijal duž krivulje je tada $d\mathbf{s} = \frac{d\mathbf{s}}{dt} dt$. Ako tražimo duljinu luka, moramo uzeti normu ovog vektora te imamo

$$l = \int \sqrt{(d\mathbf{s})^2} = \int \sqrt{\left(\frac{d\mathbf{s}}{dt}\right)^2} dt. \tag{1.225}$$

Ako želimo računati skalarni produkt polja $\mathbf{F}(t)$ duž krivulje, koristimo vektorski diferencijal $d\mathbf{s}$

$$\int \mathbf{F}(t) \cdot d\mathbf{s} = \int \mathbf{F}(t) \cdot \frac{d\mathbf{s}}{dt} dt. \tag{1.226}$$

Primjer 1.4.1. Neka su zadane točke $A = (1,1)$ i $B = (3,2)$ kroz koje prolazi pravac. Naći duljinu segmenta pravca te rad za silu $\mathbf{F} = 3x\hat{x} + \hat{y}$.

Rješenje. Zapišimo parametrizaciju pravca preko $\mathbf{s}$:

$$\mathbf{s} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1-t) + t \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1-t+3t \\ 1-t+2t \end{pmatrix} = \begin{pmatrix} x = 1+2t \\ y = 1+t \end{pmatrix}, \tag{1.227}$$

takav da $t \in [0,1]$, dok je diferencijal $d\mathbf{s}$:

$$d\mathbf{s} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} dt. \tag{1.228}$$

Duljina segmenta je onda

$$l = \int \sqrt{d\mathbf{s}^2} = \int_0^1 \sqrt{1^2+2^2} dt = \int_0^1 \sqrt{5} dt = \sqrt{5}. \tag{1.229}$$

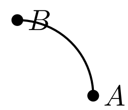
Silu možemo zapisati preko parametrizacije kao:

$$\mathbf{F} = 3x\hat{x} + \hat{y} = \begin{pmatrix} 3x \\ 1 \end{pmatrix} = \begin{pmatrix} 3(1+2t) \\ 1 \end{pmatrix}. \tag{1.230}$$

Rad je onda dan po definiciji kao:

$$W = \int \mathbf{F} d\mathbf{s} = \int_0^1 (3(1+2t) \quad 1) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} dt = \int_0^1 (12t+7)dt = 13 \tag{1.231}$$

Primjer 1.4.2. Neka su zadane točke $A = (1, 0)$ i $B = (0, 1)$ kroz koje prolazi kružnica. Naći duljinu luka krivulje.



Rješenje. Možemo se poslužiti polarnim sustavom za opis krivulje:

$$\mathbf{s} = \begin{pmatrix} x = \rho \cos \phi \\ y = \rho \sin \phi \end{pmatrix} \tag{1.232}$$

relacije $\rho = \sqrt{x^2 + y^2}$ vidimo da je $\rho = 1$. Za granice integrala koristimo izraz:

$$\phi\Big|_A = \arctan\left(\frac{y}{x}\right)\Big|_A = \arctan\left(\frac{0}{1}\right) = 0, \tag{1.233}$$

$$\phi\Big|_B = \arctan\left(\frac{y}{x}\right)\Big|_B = \arctan\frac{1}{0} = \frac{\pi}{2}. \tag{1.234}$$

Diferencijal linije (s obzirom da ρ držimo konstantnim) glasi:

$$d\mathbf{s} = \begin{pmatrix} -\sin \phi \\ \cos \phi \end{pmatrix} d\phi. \tag{1.235}$$

Rezultat je:

$$l = \int \sqrt{d\mathbf{s}^2} = \int_0^{\frac{\pi}{2}} dt = \frac{\pi}{2}. \tag{1.236}$$

Primjer 1.4.3. Pronaći duljinu luka parabole $y = x^2$ za $x \in [-1, 1]$.

Rješenje. Krivulju parametriziramo na sljedeći način $\mathbf{s}(t) = t\hat{x} + t^2\hat{y}$. Pripadni tangentni vektor je

$$\frac{d}{dt}\mathbf{s} = \hat{x} + 2t\hat{y}, \tag{1.237}$$

te je stoga diferencijal dužine luka

$$ds = \sqrt{1^2 + (2t)^2}dt. \tag{1.238}$$

Duljina luka krivulje je

$$
\begin{aligned}
L &= \int ds &(1.239)\\
&= \int_{-1}^{1} \sqrt{1+4t^2}dt &(1.240)\\
&= 2\int_{-1}^{1} \sqrt{\frac{1}{4}+t^2}dt &(1.241)\\
&= 2\frac{1}{2}\left(t\sqrt{\frac{1}{4}+t^2}+\frac{1}{4}\ln\left(t+\sqrt{\frac{1}{4}+t^2}\right)\right)\Big|_{-1}^{1} &(1.242)\\
&= \left(\sqrt{\frac{1}{4}+1^2}+\frac{1}{4}\ln\left(1+\sqrt{\frac{1}{4}+1^2}\right)\right) &(1.243)\\
&- \left(-\sqrt{\frac{1}{4}+1^2}+\frac{1}{4}\ln\left(-1+\sqrt{\frac{1}{4}+1^2}\right)\right) &(1.244)\\
&= 2\sqrt{\frac{1}{4}+1^2}+\frac{1}{4}\ln\frac{1+\sqrt{\frac{1}{4}+1^2}}{-1+\sqrt{\frac{1}{4}+1^2}} &(1.245)\\
&= 2\sqrt{5/4}+\frac{1}{4}\ln\frac{1+\sqrt{5}/2}{-1+\sqrt{5}/2} &(1.246)\\
&= \sqrt{5}+\frac{1}{4}\ln\frac{\sqrt{5}+2}{\sqrt{5}-2} &(1.247)
\end{aligned}
$$

Primjer 1.4.4. Nađite duljinu luka heliksa za $r=1$ te $t\in[0,2n\pi]$. Također nađite rad ukoliko je dana sila $\mathbf{F}=-y\hat{x}$. Koordinate su dane kao:

$$
\begin{aligned}
x &= r\cos t\\
y &= r\sin t\\
z &= ct.
\end{aligned} \tag{1.248}
$$

Rješenje. Zapišemo $\mathbf{s}$ i uvrstimo da $r=1$:

$$
\mathbf{s}=\begin{pmatrix}\cos t\\ \sin t\\ ct\end{pmatrix}. \tag{1.249}
$$

Diferencijal linije glasi:

$$
d\mathbf{s}=\begin{pmatrix}-\sin t\\ \cos t\\ c\end{pmatrix}dt. \tag{1.250}
$$

Duljina luka je:

$$
l=\int\sqrt{d\mathbf{s}^2}=\int_0^{2n\pi}\sqrt{\sin^2 t+\cos^2 t+c^2}dt=\int_0^{2n\pi}\sqrt{1+c^2}dt=2n\pi\sqrt{1+c^2}. \tag{1.251}
$$

Zapišimo **F** u cilindričnoj reprezentaciji:

$$\mathbf{F} = -\sin t\hat{x}. \tag{1.252}$$

Rad je dan kao:

$$W = \int_0^{2n\pi} \begin{pmatrix} -\sin\phi & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ c \end{pmatrix} dt = \int_0^{2n\pi} \sin^2 t dt = \frac{1}{2}\int_0^{2n\pi} (1-\cos 2t) dt = n\pi \tag{1.253}$$

1.4.2 Parametrizacije površine

Krivulju u prostoru parametriziramo s dva parametra, u oznaci u i v kao $\mathbf{r}(u, v)$. Diferencijal duž krivulje je tada

$$d\mathbf{r} = \partial_u \mathbf{r} du + \partial_v \mathbf{r} dv. \tag{1.254}$$

Želimo li se držati ideje diferencijala (što je korisno za generalizacije), kao pomoć možemo uvesti

$$[\partial_u \mathbf{r}] = \partial_u \mathbf{r} du \tag{1.255}$$
$$[\partial_v \mathbf{r}] = \partial_v \mathbf{r} dv. \tag{1.256}$$

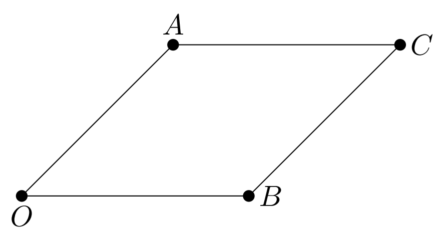
Diferencijalni element površine dobit ćemo iz vektorskog produkta

$$d^2\mathbf{r} = [\partial_u \mathbf{r}] \times [\partial_v \mathbf{r}] = \partial_u \mathbf{r} \times \partial_v \mathbf{r} du dv, \tag{1.257}$$

a skalarnu površinu dobivamo računajući normu

$$d^2 r = |\partial_u \mathbf{r} \times \partial_v \mathbf{r}| du dv. \tag{1.258}$$

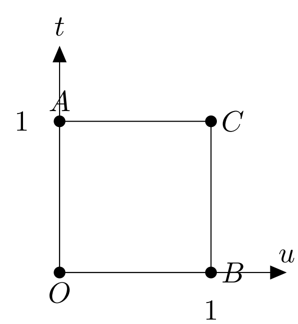
Primjer 1.4.5. Zadan je paralelogram s koordinatama $O = (0, 0)$, $A = (2, 2)$, $B = (3, 0)$ i $C = (5, 2)$. Nađite površinu.



Rješenje. Zapišimo radijvektor preko parametara t i u koji su takvi da $t, u \in [0, 1]$:

$$\mathbf{r} = (\overrightarrow{A} - \overrightarrow{O})t + (\overrightarrow{B} - \overrightarrow{O})u + \overrightarrow{O} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} t + \begin{pmatrix} 3 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.259}$$

.



Konstruiramo element površine:

$$\partial_t \mathbf{r} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} dt, \tag{1.260}$$

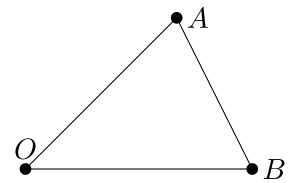
$$\partial_u \mathbf{r} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} du, \tag{1.261}$$

$$d\mathbf{A} = \partial_t \mathbf{r} \times \partial_u \mathbf{r} dt du = -6\hat{z} dt du. \tag{1.262}$$

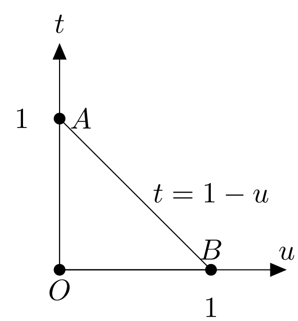
Površina je onda dana:

$$P = \int \int \sqrt{d\mathbf{A}^2} = 6 \int_0^1 \int_0^1 dt du = 6. \tag{1.263}$$

Primjer 1.4.6. Iz prošlog primjera nađite površinu trokuta zadanu sa A, B i O.



Rješenje. Pogledajmo kako ovise u $t-u$ prostoru parametri. Vidimo da t za gornju granicu ovisi o u.



Stoga imamo:

$$P = \int \sqrt{d\mathbf{A}^2} = \int_0^1 \int_0^{1-u} 6dudt = 6 \int_0^1 (1-u)du = 3 \tag{1.264}$$

Primjer 1.4.7. Izračunati tok koji prolazi kroz kvadrat stranice a na $y = 1$ za vektorsko polje $\mathbf{F} = 2xy\hat{x} + yx\hat{y} + xz^2\hat{z}$.

Rješenje. Zapišimo prvo radijvektor za lokaciju kvadrata:

$$\mathbf{r} = x\hat{x} + \hat{y} + z\hat{z}. \tag{1.265}$$

Diferencijal površine glasi:

$$\partial_x \mathbf{r} = \hat{x}dx, \tag{1.266}$$
$$\partial_z \mathbf{r} = \hat{z}dz, \tag{1.267}$$
$$d\mathbf{A} = \partial_z \mathbf{r} \times \partial_x \mathbf{r} dzdx = \hat{y}dzdx. \tag{1.268}$$

Tok je onda jednak:

$$\Phi\Big|_{y=1} = \int_0^a \int_0^a (2xy\hat{x}+yx\hat{y}+xz^2\hat{z})\cdot\hat{y}dzdx\Big|_{y=1} = \int_0^a \int_0^a (2x\hat{x}+x\hat{y}+xz^2\hat{z})\cdot\hat{y}dzdx = \int\limits_0^a \int\limits_0^a xdxdz = \frac{a^3}{2}. \tag{1.269}$$

Primjer 1.4.8. Izračunajte površinu sferne kapice koja se nalazi na radijusu R od ishodišta i u intervalu $\theta \in [0, \theta_0]$.

Rješenje. Zapišimo prvo radijvektor za lokaciju sferne kapice:

$$\mathbf{r} = R(\cos\phi\sin\theta\hat{x} + \sin\phi\sin\theta\hat{y} + \cos\theta\hat{z}). \tag{1.270}$$

Potom moramo naći diferencijalni element površine:

$$\partial_\theta \mathbf{r} = R(\cos\phi\cos\theta\hat{x} + \sin\phi\cos\theta\hat{y} - \sin\theta\hat{z})d\theta \tag{1.271}$$

$$\partial_\phi \mathbf{r} = R(-\sin\phi\sin\theta\hat{x} + \cos\phi\sin\theta\hat{y})d\phi, \tag{1.272}$$

$$d\mathbf{A} = \partial_\theta \mathbf{r} \times \partial_\phi \mathbf{r} d\theta d\phi = R^2 \sin\theta\hat{r} d\theta d\phi. \tag{1.273}$$

Konačno, površina je:

$$P = \int\int \sqrt{d\mathbf{A}^2} = R^2 \int_0^{\theta_0} \int_0^{2\pi} \sin\theta d\theta d\phi = 2\pi R^2(1-\cos\theta_0) = 4\pi R^2 \sin^2\frac{\theta_0}{2}. \tag{1.274}$$

Primjer 1.4.9. Izračunajte površinu stošca definiranog kutom θ_0 za $r \in [0, R]$.

Rješenje. Zapišimo prvo radijvektor za lokaciju stošca:

$$\mathbf{r} = r(\cos\phi\sin\theta_0\hat{x} + \sin\phi\sin\theta_0\hat{y} + \cos\theta_0\hat{z}). \tag{1.275}$$

Potom moramo naći diferencijalni element površine:

$$\partial_\phi \mathbf{r} = r(-\sin\phi\sin\theta_0\hat{x} + \cos\phi\sin\theta_0\hat{y})d\phi, \tag{1.276}$$

$$\partial_r \mathbf{r} = (\cos\phi\sin\theta_0\hat{x} + \sin\phi\sin\theta_0\hat{y} + \cos\theta_0\hat{z})dr, \tag{1.277}$$

$$d\mathbf{A} = \partial_\phi \mathbf{r} \times \partial_r \mathbf{r} d\phi dr = r\sin\theta_0\hat{\theta} dr d\phi. \tag{1.278}$$

Površina je onda:

$$P = \int\int \sqrt{d\mathbf{A}^2} = \int_0^{2\pi} \int_0^R r\sin\theta_0 dr d\phi = \pi\sin\theta_0 R^2 \tag{1.279}$$

Primjer 1.4.10. Naći tok koji prolazi kroz plašt cilindra radijusa R i visine h ako je vektorsko polje $\mathbf{F} = x\hat{x}$ te je $z \in [0, h]$.

Rješenje. Zapišimo prvo radijvektor za lokaciju plašta cilindra:

$$\mathbf{r} = R(\cos\phi\hat{x} + \sin\phi\hat{y}) + z\hat{z}. \tag{1.280}$$

Izračunamo diferencijal površine:

$$\partial_\phi \mathbf{r} = R(-\sin\phi\hat{x} + \cos\phi\hat{y})d\phi, \tag{1.281}$$

$$\partial_z \mathbf{r} = \hat{z}dz, \tag{1.282}$$

$$d\mathbf{A} = \partial_\phi \mathbf{r} \times \partial_z \mathbf{r} d\phi dz = R\hat{\rho} dz d\phi. \tag{1.283}$$

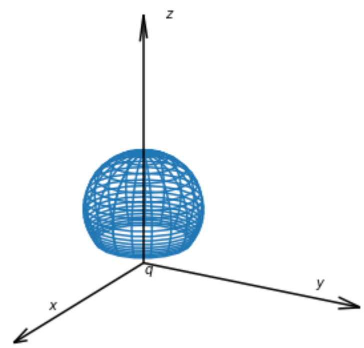
Zapišimo $\mathbf{F}$ u cilindričnoj reprezentaciji:

$$\mathbf{F} = \rho\cos\phi\hat{x}. \tag{1.284}$$

Tok je onda:

$$\Phi\Big|_R = \int_0^{2\pi}\int_0^h R^2\cos\phi\hat{x}\cdot\hat{\rho} dz d\phi = R^2 h\int_0^{2\pi}\cos^2\phi = R^2 h\pi. \tag{1.285}$$

Primjer 1.4.11. Izračunati površinu opisanu koordinatama u tangentnom sfernom sustavu $x = \frac{\mu\cos\psi}{\mu^2+\nu^2}$, $y = \frac{\mu\sin\psi}{\mu^2+\nu^2}$, $z = \frac{\nu}{\mu^2+\nu^2}$ za $\mu \in [0, \mu_0]$, $\nu = \nu_0$, $\psi \in [0, 2\pi]$.



Rješenje. Tangentni vektori su u smjerovima ∂_μ i ∂_ψ te imamo

$$\mathbf{r} = \frac{\mu\cos\psi}{\mu^2+\nu^2}\hat{x} + \frac{\mu\sin\psi}{\mu^2+\nu^2}\hat{y} + \frac{\nu}{\mu^2+\nu^2}\hat{z} \tag{1.286}$$

$$\partial_\mu\mathbf{r} = \frac{\nu^2-\mu^2}{(\mu^2+\nu^2)^2}(\cos\psi\hat{x} + \sin\psi\hat{y}) - \frac{2\mu\nu}{(\mu^2+\nu^2)^2}\hat{z} \tag{1.287}$$

$$\partial_\psi\mathbf{r} = \frac{\mu}{\mu^2+\nu^2}(-\sin\psi\hat{x} + \cos\psi\hat{y}) \tag{1.288}$$

$$\partial_\mu\mathbf{r} \times \partial_\psi\mathbf{r} = \frac{\mu}{(\mu^2+\nu^2)^3}(2\mu\nu\cos\psi\hat{x} + 2\mu\nu\sin\psi\hat{y} + (\nu^2-\mu^2)\hat{z}) \tag{1.289}$$

$$|d\mathbf{A}|\Big|_{\nu_0} = \frac{\mu d\mu d\psi}{(\mu^2+\nu_0^2)^3}\sqrt{4\mu^2\nu_0^2 + (\nu_0^2-\mu^2)^2} = \frac{\mu d\mu d\psi}{(\mu^2+\nu_0^2)^3}\sqrt{2\mu^2\nu_0^2 + \nu_0^4 + \mu^4} = \frac{\mu d\mu d\psi}{(\mu^2+\nu_0^2)^2} \tag{1.290}$$

Površina je tada:

$$P = \int_0^{\mu_0}\int_0^{2\pi} \frac{\mu d\mu d\psi}{(\mu^2+\nu_0^2)^2} = \left| \begin{matrix} x = \mu^2+\nu_0^2 \\ dx = 2\mu d\mu \\ \nu_0^2 \leq x \leq \nu_0^2+\mu_0^2 \end{matrix} \right| = \frac{1}{2}\int_{\nu_0^2}^{\mu_0^2+\nu_0^2} \frac{dx}{x^2} 2\pi = \pi\left(\frac{1}{\nu_0} - \frac{1}{\mu_0^2+\nu_0^2}\right) = \pi\frac{\mu_0^2}{\nu_0^2}\frac{1}{\mu_0^2+\nu_0^2} \tag{1.291}$$

1.4.3 Parametrizacije volumena

Primjer 1.4.12. Izračunati volumen paralelopipeda koji prolazi točkama $A = (0, 0, 0)$, $B = (1, 1, 0)$, $C = (0, 1, 1)$, $D = (1, 1, 1)$.

Rješenje. Parametriziramo pomoću tri parametra u, v i w

$$\begin{aligned}\mathbf{r} &= (\overrightarrow{B} - \overrightarrow{A})u + (\overrightarrow{C} - \overrightarrow{A})v + (\overrightarrow{D} - \overrightarrow{A})w + \overrightarrow{A} &\quad (1.292)\\ &= u \begin{pmatrix}1\\1\\0\end{pmatrix} + v \begin{pmatrix}0\\1\\1\end{pmatrix} + w \begin{pmatrix}1\\1\\1\end{pmatrix} + w \begin{pmatrix}0\\0\\0\end{pmatrix}. &\quad (1.293)\end{aligned}$$

Parcijalne derivacije su tada

$$\partial_u \mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} \quad (1.294)$$

$$\partial_v \mathbf{r} = \begin{pmatrix}0\\1\\1\end{pmatrix} \quad (1.295)$$

$$\partial_w \mathbf{r} = \begin{pmatrix}1\\1\\1\end{pmatrix}. \quad (1.296)$$

Diferencijalni element volumena dobivamo kao Jacobijan

$$\begin{aligned}dV &= |J| du dv dw &\quad (1.297)\\ &= \left|\begin{vmatrix}1 & 1 & 1\\1 & 1 & 0\\0 & 1 & 1\end{vmatrix}\right| du dv dw &\quad (1.298)\\ &= \left|\begin{vmatrix}1 & 0\\1 & 1\end{vmatrix} - \begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} + \begin{vmatrix}1 & 1\\0 & 1\end{vmatrix}\right| du dv dw &\quad (1.299)\\ &= |1 - 0 - (1 - 0) + 1 - 0| du dv dw &\quad (1.300)\\ &= du dv dw. &\quad (1.301)\end{aligned}$$

Volumen je

$$\begin{aligned}V &= \int dV &\quad (1.302)\\ &= \int_0^1 \int_0^1 \int_0^1 du dv dw &\quad (1.303)\\ &= 1. &\quad (1.304)\end{aligned}$$

Primjer 1.4.13. Izračunati volumen piramide koji prolazi točkama $A = (0,0,0)$, $B = (1,1,0)$, $C = (0,1,1)$, $D = (1,1,1)$.

Rješenje. Parametriziramo pomoću tri parametra u, v i w

$$\begin{aligned}
\mathbf{r} &= (\overrightarrow{B} - \overrightarrow{A})u + (\overrightarrow{C} - \overrightarrow{A})v + (\overrightarrow{D} - \overrightarrow{A})w + \overrightarrow{A} && (1.305)\\
&= u\begin{pmatrix}1\\1\\0\end{pmatrix} + v\begin{pmatrix}0\\1\\1\end{pmatrix} + w\begin{pmatrix}1\\1\\1\end{pmatrix} + w\begin{pmatrix}0\\0\\0\end{pmatrix}. && (1.306)
\end{aligned}$$

Parcijalne derivacije su tada

$$\partial_u \mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} \tag{1.307}$$

$$\partial_v \mathbf{r} = \begin{pmatrix}0\\1\\1\end{pmatrix} \tag{1.308}$$

$$\partial_w \mathbf{r} = \begin{pmatrix}1\\1\\1\end{pmatrix}. \tag{1.309}$$

Diferencijalni element volumena dobivamo kao Jacobijan

$$\begin{aligned}
dV &= |J| dudvdw && (1.310)\\
&= \left|\begin{vmatrix}1 & 1 & 1\\1 & 1 & 0\\0 & 1 & 1\end{vmatrix}\right| dudvdw && (1.311)\\
&= \left|\begin{vmatrix}1 & 0\\1 & 1\end{vmatrix} - \begin{vmatrix}1 & 0\\0 & 1\end{vmatrix} + \begin{vmatrix}1 & 1\\0 & 1\end{vmatrix}\right| dudvdw && (1.312)\\
&= |1 - 0 - (1-0) + 1 - 0| dudvdw && (1.313)\\
&= dudvdw. && (1.314)
\end{aligned}$$

Za volumen moramo biti pažljivi, $u \in [0,1]$, $v \in [0, 1-u]$, $w \in [0, 1-u-v]$

$$\begin{aligned}
V &= \int dV && (1.315)\\
&= \int_0^1 \int_0^{1-u} \int_0^{1-u-v} dudvdw && (1.316)\\
&= \int_0^1 \int_0^{1-u} (1-u-v) dudv && (1.317)\\
& && (1.318)
\end{aligned}$$

$$= \int_0^1 \left((1-u)v - \frac{v^2}{2} \right) \Big|_0^{1-u} du \tag{1.319}$$

$$= \int_0^1 \left((1-u)(1-u) - \frac{(1-u)^2}{2} \right) du \tag{1.320}$$

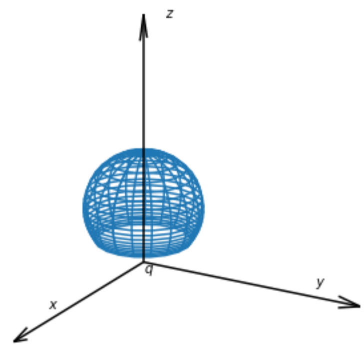
$$= \int_0^1 \frac{(1-u)^2}{2} du \tag{1.321}$$

$$\{t = 1-u,\ dt = -du\} \tag{1.322}$$

$$= \int_0^1 \frac{t^2}{2} dt = \frac{t^3}{6} \Big|_0^1 \tag{1.323}$$

$$= \frac{1}{6} \tag{1.324}$$

Primjer 1.4.14. Izračunati volumen opisan koordinatama u tangentnom sfernom sustavu $x = \frac{\mu\cos\psi}{\mu^2+\nu^2}$, $y = \frac{\mu\sin\psi}{\mu^2+\nu^2}$, $z = \frac{\nu}{\mu^2+\nu^2}$ za $\mu \in [0, \mu_0]$, $\nu \in [\mu_0, \nu_0]$, $\psi \in [0, 2\pi]$.



Rješenje. Tangentni vektori su u smjerovima ∂_μ, ∂_ν i ∂_ψ te imamo

$$\mathbf{r} = \frac{\mu\cos\psi}{\mu^2+\nu^2}\hat{x} + \frac{\mu\sin\psi}{\mu^2+\nu^2}\hat{y} + \frac{\nu}{\mu^2+\nu^2}\hat{z} \tag{1.325}$$

$$\partial_\mu\mathbf{r} = \frac{\nu^2-\mu^2}{(\mu^2+\nu^2)^2}(\cos\psi\hat{x} + \sin\psi\hat{y}) - \frac{2\mu\nu}{(\mu^2+\nu^2)^2}\hat{z} \tag{1.326}$$

$$\partial_\nu\mathbf{r} = \frac{-2\nu\mu}{(\mu^2+\nu^2)^2}(\cos\psi\hat{x} + \sin\psi\hat{y}) + \frac{\mu^2-\nu^2}{(\mu^2+\nu^2)^2}\hat{z} \tag{1.327}$$

$$\partial_\psi\mathbf{r} = \frac{\mu}{\mu^2+\nu^2}(-\sin\psi\hat{x} + \cos\psi\hat{y}) \tag{1.328}$$

$$\partial_\mu\mathbf{r} \times \partial_\psi\mathbf{r} = \frac{\mu}{(\mu^2+\nu^2)^3}(2\mu\nu\cos\psi\hat{x} + 2\mu\nu\sin\psi\hat{y} + (\nu^2-\mu^2)\hat{z}) \tag{1.329}$$

$$|\partial_\nu\mathbf{r}\cdot(\partial_\mu\mathbf{r} \times \partial_\psi\mathbf{r})| = \left|\frac{\mu}{(\mu^2+\nu^2)^5}\left(\frac{-4\mu^2\nu^2-(\mu^2-\nu^2)^2}{\mu^2+\nu^2}\right)\right| \tag{1.330}$$

$$= \frac{\mu}{(\mu^2+\nu^2)^5}\left(4\mu\nu^2+(\mu^2-\nu^2)^2\right) \tag{1.331}$$

$$= \frac{\mu}{(\mu^2+\nu^2)^5}\left((\mu^2+\nu^2)^2\right) \tag{1.332}$$

$$= \frac{\mu}{(\mu^2+\nu^2)^3}. \tag{1.333}$$

Volumen je tada

$$V = \int_0^{\mu_0}\int_{\mu_0}^{\nu_0}\int_0^{2\pi} \frac{\mu}{(\mu^2+\nu^2)^3} d\mu d\nu d\psi \tag{1.334}$$

$$= 2\pi \int_{\mu_0}^{\nu_0} d\nu \int_0^{\mu_0} \frac{\mu d\mu}{(\mu^2+\nu^2)^3} \tag{1.335}$$

$$= \{t = \mu^2 + \nu^2\} \tag{1.336}$$

$$= \pi \int_{\mu_0}^{\nu_0} d\nu \int_{\nu^2}^{\nu^2+\mu_0^2} \frac{dt}{t^3} \tag{1.337}$$

$$= \frac{\pi}{2} \int_{\mu_0}^{\nu_0} d\nu \left(\frac{1}{\mu_0^4} - \frac{1}{(\nu^2+\mu_0^2)^2}\right) \tag{1.338}$$

$$= \frac{\pi}{2}\left(\frac{\nu_0-\mu_0}{\mu_0^4} - \frac{1}{2\mu_0^3}\left(\frac{\mu_0\nu}{\mu_0^2+\nu^2} + \arctan\frac{\nu}{\mu_0}\right)\Big|_{\mu_0}^{\nu_0}\right) \tag{1.339}$$

$$= \frac{\pi}{2}\left(\frac{\nu_0-\mu_0}{\mu_0^4} - \frac{1}{2\mu_0^3}\left(\frac{\mu_0\nu_0}{\mu_0^2+\nu_0^2} + \arctan\frac{\nu_0}{\mu_0} - \frac{\mu_0^2}{2\mu_0^2} - \arctan 1\right)\right) \tag{1.340}$$

$$= \frac{\pi}{2}\left(\frac{\nu_0-\mu_0}{\mu_0^4} - \frac{1}{2\mu_0^3}\left(\frac{\mu_0\nu_0}{\mu_0^2+\nu_0^2} + \arctan\frac{\nu_0}{\mu_0} - \frac{1}{2} - \frac{\pi}{4}\right)\right) \tag{1.341}$$

Primjer 1.4.15. Izračunati volumen omeđen krivuljama $x^2+y^2+z^2 = 3a^2$ i $x^2+y^2 = 2az$ za $z > 0$.

Rješenje. Najlakše je koristiti cilindrični sustav.

$$\rho^2 + z^2 = 3a^2 \tag{1.342}$$

$$\rho^2 = 2az. \tag{1.343}$$

Ove krivulje se spajaju u točki

$$2az + z^2 - 3a^2 = 0 \tag{1.344}$$

$$z_{1,2} = -a \pm \sqrt{a^2+3a^2} \tag{1.345}$$

$$= a(-1 \pm 2). \tag{1.346}$$

Za $z > 0$ dobivamo $z_1 = a$, $\rho_1 = \sqrt{2}a$ te možemo jednostavno pronaći volumen koristeći jakobijan

$$\mathbf{r} = \rho(\cos\phi\hat{x} + \sin\phi\hat{y}) + z\hat{z} \tag{1.347}$$
$$\partial_\rho\mathbf{r} = \cos\phi\hat{x} + \sin\phi\hat{y} \tag{1.348}$$
$$\partial_\phi\mathbf{r} = \rho(-\sin\phi\hat{x} + \cos\phi\hat{y}) \tag{1.349}$$
$$\partial_z\mathbf{r} = \hat{z} \tag{1.350}$$
$$J = \begin{vmatrix} 0 & 0 & 1 \\ \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \end{vmatrix} \tag{1.351}$$
$$= \rho. \tag{1.352}$$

Tada je volumen

$$V = \int\int\int |J| d\rho d\phi dz \tag{1.353}$$
$$= \int_0^{2\pi}\int_0^{\sqrt{2}a}\int_{\frac{\rho^2}{2a}}^{\sqrt{3a^2-\rho^2}} \rho d\rho d\phi dz \tag{1.354}$$
$$= 2\pi \int_0^{\sqrt{2}a} \rho d\rho \left(\sqrt{3a^2-\rho^2} - \frac{\rho^2}{2a}\right) \tag{1.355}$$
$$= 2\pi \left(\int_0^{\sqrt{2}a} \rho\sqrt{3a^2-\rho^2} - \frac{\rho^4}{8a}\Big|_0^{\sqrt{2}a}\right) \tag{1.356}$$
$$= \{t = 3a^2 - \rho^2\} \tag{1.357}$$
$$= 2\pi \left(\frac{1}{2}\int_{a^2}^{3a^2} \sqrt{t} - \frac{16a^4}{8a}\Big|_0^{\sqrt{2}a}\right) \tag{1.358}$$
$$= 2\pi \left(\frac{2}{6}t^{3/2}\Big|_{a^2}^{3a^2} - \frac{16a^4}{8a}\Big|_0^{\sqrt{2}a}\right) \tag{1.359}$$
$$= 2\pi a^3 \left(\frac{3\sqrt{3}-1}{3} - \frac{1}{2}\right) \tag{1.360}$$
$$= 2\pi a^3 \left(\frac{3\sqrt{3}}{3} - \frac{5}{6}\right) \tag{1.361}$$
$$= \frac{\pi a^3}{3}(6\sqrt{3} - 5). \tag{1.362}$$

1.5 Stokesov i Gaussov teorem

Primjer 1.5.1. Primjeniti Stokesov teorem za vektorsko polje $\mathbf{F} = -y\hat{x} + x\hat{y} + c\hat{z}$ na jediničnu kružnicu smještenu u $z = 0$.

Rješenje. Primijetimo da zbog date krivulje je pogodnije koristiti cilindrični sustav. U njemu dano vektorsko polje ima oblik:

$$\mathbf{F} = -\rho\sin\phi\hat{x} + \rho\cos\phi + c\hat{z} = \rho\hat{\phi} + c\hat{z} \tag{1.363}$$

Kružnicu parametriziramo na sljedeći način:

$$\mathbf{r}\Big|_{\rho=1} = \cos\phi\hat{x} + \sin\phi\hat{y} + 0\hat{z} = \begin{pmatrix}\cos\phi\\ \sin\phi\\ 0\end{pmatrix}. \tag{1.364}$$

Diferencijalni element jest:

$$\mathbf{dr} = \begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix} d\phi = \hat{\phi}d\phi. \tag{1.365}$$

Konačno, integral je:

$$\int_C \mathbf{F}d\mathbf{r} = \int_0^{2\pi} (\hat{\phi} + c\hat{z})\hat{\phi}d\phi = \int_0^{2\pi} d\phi = 2\pi. \tag{1.366}$$

Za drugi način moramo izračunati rotaciju polja $\mathbf{F}$:

$$\nabla\times\mathbf{F} = \frac{1}{\rho}\begin{vmatrix}\hat{\rho} & \rho\hat{\phi} & \hat{z}\\ \partial_\rho & \partial_\phi & \partial_z\\ F_\rho & \rho F_\phi & F_z\end{vmatrix} = \frac{1}{\rho}\begin{vmatrix}\hat{\rho} & \rho\hat{\phi} & \hat{z}\\ \partial_\rho & \partial_\phi & \partial_z\\ 0 & \rho^2 & c\end{vmatrix} = 2\hat{z}. \tag{1.367}$$

Napravimo diferencijalni element površine:

$$\mathbf{r} = \rho\hat{\rho} + 0\hat{z} \tag{1.368}$$

$$\partial_\rho\mathbf{r} = \hat{\rho}, \tag{1.369}$$

$$\partial_\phi\mathbf{r} = \rho\hat{\phi}, \tag{1.370}$$

$$d\mathbf{A} = \partial_\rho\mathbf{r}\times\partial_\phi\mathbf{r} = \rho\hat{z}. \tag{1.371}$$

Integral sada glasi:

$$\int\int_S \mathbf{F}\cdot d\mathbf{A} = \int_0^{2\pi}\int_0^1 2\rho d\rho d\phi = 2\pi. \tag{1.372}$$

Primjer 1.5.2. Primjeniti Stokesov teorem za vektorsko polje $\mathbf{F} = z\hat{x} + x\hat{y}$ na trokut definiran vrhovima $A = (4, -2, 0), B = (3, 4, 0), C = (1, 2, 0)$ u $z = 0$.

Rješenje. Integral po rubu ćemo rješavati dio po dio. Za L1 radijvektor je:

$$\mathbf{r_1} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} (1-t) + \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} t = \begin{pmatrix} 4-t \\ 6t-2 \\ 0 \end{pmatrix}, \quad t \in [0,1]. \tag{1.373}$$

Diferencijal glasi:

$$d\mathbf{r_1} = \begin{pmatrix} -1 \\ 6 \\ 0 \end{pmatrix} dt. \tag{1.374}$$

Prvi integral je:

$$I_1 = \int \mathbf{F} \cdot d\mathbf{r_1} = \int_0^1 (0, \quad 4-t, \quad 0) \cdot \begin{pmatrix} -1 \\ 6 \\ 0 \end{pmatrix} dt = 21. \tag{1.375}$$

Za L2:

$$\mathbf{r_2} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} (1-t) + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} t = \begin{pmatrix} 3-2t \\ 4-2t \\ 0 \end{pmatrix}. \tag{1.376}$$

Diferencijal glasi:

$$d\mathbf{r_2} = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} dt. \tag{1.377}$$

Drugi integral je:

$$I_2 = \int \mathbf{F} \cdot d\mathbf{r_2} = \int_0^1 (0, \quad 3-2t, \quad 0) \cdot \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} dt = -4. \tag{1.378}$$

Za treći integral imamo:

$$\mathbf{r_3} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} (1-t) + \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} t = \begin{pmatrix} 1+3t \\ 2-4t \\ 0 \end{pmatrix}, \quad t \in [0,1]. \tag{1.379}$$

Diferencijal glasi:

$$d\mathbf{r_3} = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} dt. \tag{1.380}$$

$$I_3 = \int \mathbf{F} \cdot d\mathbf{r_3} = \int_0^1 (0, \quad 1+3t, \quad 0) \cdot \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} dt = -10. \tag{1.381}$$

Ukupni rezultat je dan sumom tih dijelova:

$$I = \sum_i I_i = 7. \tag{1.382}$$

Provjerimo drugim načinom. Parametrizirajmo površinu:

$$\mathbf{r} = (\overrightarrow{A} - \overrightarrow{C})t + (\overrightarrow{B} - \overrightarrow{C})u + \overrightarrow{C} = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} t + \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} u + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}. \tag{1.383}$$

Diferencijal površine je:

$$\partial_t \mathbf{r} = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} dt, \quad \partial_u \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} du \tag{1.384}$$

$$d\mathbf{A} = \begin{vmatrix} 3 & -4 \\ 2 & 2 \end{vmatrix} \hat{z} dt du = 14 dt du \hat{z}. \tag{1.385}$$

Rotacija $\mathbf{F}$ je:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ z & x & 0 \end{vmatrix} = \hat{y} + \hat{z}. \tag{1.386}$$

Kako rješavamo integral u $t - u$ prostoru, ne smijemo imati išta što ovisi o x i/ili y. Pronađimo tangentne vektore

$$\hat{t} = \frac{\partial_t \mathbf{r}}{|\partial_t \mathbf{r}|} \tag{1.387}$$

$$= \frac{3\hat{x} - 4\hat{y}}{\sqrt{3^2 + 4^2}} \tag{1.388}$$

$$= \frac{3\hat{x} - 4\hat{y}}{\sqrt{25}} \tag{1.389}$$

$$= \frac{3\hat{x} - 4\hat{y}}{5} \tag{1.390}$$

$$\hat{u} = \frac{\partial_u \mathbf{r}}{|\partial_u \mathbf{r}|} \tag{1.391}$$

$$= \frac{2\hat{x} + 2\hat{y}}{\sqrt{2^2 + 2^2}} \tag{1.392}$$

$$= \frac{2\hat{x} + 2\hat{y}}{2\sqrt{2}} \tag{1.393}$$

$$= \frac{\hat{x} + \hat{y}}{\sqrt{2}}. \tag{1.394}$$

Primijetimo da vektori $\hat{u}$ i $\hat{t}$ čine bazu, iako nisu ortogonalni

$$\hat{t} \cdot \hat{u} = \left(\frac{3\hat{x} - 4\hat{y}}{5} \right) \cdot \left(\frac{\hat{x} + \hat{y}}{\sqrt{2}} \right) \tag{1.395}$$

$$= \frac{3 - 4}{5\sqrt{2}} \tag{1.396}$$

$$= \frac{-1}{5\sqrt{2}}. \tag{1.397}$$

Prema tome, ako bismo željeli dobiti $\hat{y}$ preko $\hat{u}$ i $\hat{t}$, moramo invertirati transformaciju

$$\begin{pmatrix}\hat{t}\\ \hat{u}\end{pmatrix} = \begin{pmatrix}3/5 & -4/5\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} \tag{1.398}$$

$$\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = \begin{pmatrix}3/5 & -4/5\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}^{-1}\begin{pmatrix}\hat{t}\\ \hat{u}\end{pmatrix} \tag{1.399}$$

$$= \frac{1}{\begin{vmatrix}3/5 & -4/5\\ 1/\sqrt{2} & 1/\sqrt{2}\end{vmatrix}}\begin{pmatrix}1/\sqrt{2} & 4/5\\ -1/\sqrt{2} & 3/5\end{pmatrix}\begin{pmatrix}\hat{t}\\ \hat{u}\end{pmatrix} \tag{1.400}$$

$$= \frac{1}{\frac{3}{5\sqrt{2}} - \frac{-4}{5\sqrt{2}}}\begin{pmatrix}1/\sqrt{2} & 4/5\\ -1/\sqrt{2} & 3/5\end{pmatrix}\begin{pmatrix}\hat{t}\\ \hat{u}\end{pmatrix} \tag{1.401}$$

$$= \frac{5\sqrt{2}}{7}\begin{pmatrix}1/\sqrt{2} & 4/5\\ -1/\sqrt{2} & 3/5\end{pmatrix}\begin{pmatrix}\hat{t}\\ \hat{u}\end{pmatrix} \tag{1.402}$$

$$= \frac{1}{7}\begin{pmatrix}5 & 4\sqrt{2}\\ -5 & 3\sqrt{2}\end{pmatrix}\begin{pmatrix}\hat{t}\\ \hat{u}\end{pmatrix}. \tag{1.403}$$

Sada lako možemo provjeriti

$$\hat{y}\cdot\hat{y} = \frac{1}{49}\left(25\hat{t}\cdot\hat{t} + 18\hat{u}\cdot\hat{u} - 2\hat{t}\cdot\hat{u}15\sqrt{2}\right) \tag{1.404}$$

$$= \frac{1}{49}\left(25 + 18 - 2\frac{-1}{5\sqrt{2}}15\sqrt{2}\right) \tag{1.405}$$

$$= \frac{25+6+18}{49} \tag{1.406}$$

$$= 1. \tag{1.407}$$

Integral glasi:

$$\int\int_S \nabla\times\mathbf{F}\cdot d\mathbf{A} = 14\int_0^1\int_0^{1-u}\left(-\frac{5}{7}\hat{t} + \frac{3\sqrt{2}}{7}\hat{u} + \hat{z}\right)\cdot\hat{z}dudt \tag{1.408}$$

$$= 14\int_0^1\int_0^{1-u} dudt \tag{1.409}$$

$$= 14\int_0^1 (1-u)du \tag{1.410}$$

$$= 7. \tag{1.411}$$

Primjer 1.5.3. Primjeniti Gaussov teorem za vektorsko polje $\mathbf{F} = xz, yz, x^2 + y^2 + z^2$ za područje omeđeno polusferom radijusa a i $z = 0$ ravninom.

Rješenje. Prvi način je da nađemo tok polja kroz polusferu i disk. Za polusferu moramo naći diferencijalni element površine:

$$\mathbf{r} = a(\cos\phi\sin\theta\hat{x} + \sin\phi\sin\theta\hat{y} + \cos\theta\hat{z}, \tag{1.412}$$

$$\partial_\theta\mathbf{r} = a(\cos\phi\cos\theta\hat{x} + \sin\phi\cos\theta\hat{y} - \sin\theta\hat{z})d\theta, \tag{1.413}$$

$$\partial_\phi\mathbf{r} = a(-\sin\phi\sin\theta\hat{x} + \cos\phi\sin\theta\hat{y})d\phi, \tag{1.414}$$

$$d\mathbf{A} = \partial_\theta\mathbf{r} \times \partial_\phi\mathbf{r} = a^2\sin\theta\hat{r}d\theta d\phi. \tag{1.415}$$

Isto tako moramo prebaciti $\mathbf{F}$ u sferne koordinate:

$$\mathbf{F} = \begin{pmatrix} r^2\cos\phi\sin\theta\cos\theta \\ r^2\sin\phi\sin\theta\cos\theta \\ r^2 \end{pmatrix}. \tag{1.416}$$

Prvi integral glasi:

$$I_1\Big|_{r=a} = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} a^2(\cos\phi\sin\theta\cos\theta \quad \sin\phi\sin\theta\cos\theta \quad 1)\cdot\begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix} a^2\sin\theta d\theta d\phi \tag{1.417}$$

$$= a^4\int_0^{\frac{\pi}{2}}\int_0^{2\pi}(\sin^2\theta + 1)\sin\theta\cos\theta d\theta d\phi = \begin{vmatrix} \sin\theta = x \\ \cos\theta d\theta = dx \\ 0 \le x \le 1 \end{vmatrix} \tag{1.418}$$

$$= 2\pi a^4\int_0^1(x^2+1)xdx = \frac{3}{2}\pi a^4. \tag{1.419}$$

Drugi integral se radi po disku u $z = 0$:

$$\mathbf{r} = \rho\hat{\rho} + 0\hat{z} \tag{1.420}$$

$$\partial_\rho\mathbf{r} = \hat{\rho}d\rho \tag{1.421}$$

$$\partial_\phi\mathbf{r} = \rho\hat{\phi}d\phi \tag{1.422}$$

$$d\mathbf{A} = \partial_\phi\mathbf{r} \times \partial_\rho\mathbf{r}d\phi d\rho = -\hat{z}\rho d\rho d\phi. \tag{1.423}$$

Također, moramo prebaciti $\mathbf{F}$ u cilindrične koordinate:

$$\mathbf{F} = \begin{pmatrix} \rho\cos\phi z \\ \rho\sin\phi z \\ \rho^2 + z^2 \end{pmatrix}. \tag{1.424}$$

Integral glasi:

$$I_2\Big|_{z=0} = \int_0^a\int_0^{2\pi}(0 \quad 0 \quad \rho^2)\cdot\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\rho d\rho d\phi = -\frac{\pi}{2}a^4. \tag{1.425}$$

Znači da je tok:

$$I = \sum_i I_i = \pi a^2. \tag{1.426}$$

Drugi način je da izračunamo divergenciju vektorskog polja $\mathbf{F}$:

$$\nabla \cdot \mathbf{F} = 4z = 4r\cos\theta. \tag{1.427}$$

Integral je po formuli:

$$\int_V \nabla \cdot \mathbf{F} dV = \int_0^a \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 4r \cdot r^2 \sin\theta dr d\theta d\phi = \pi a^4 \tag{1.428}$$

čime je jednakost dokazana.

Primjer 1.5.4. Primjeniti Gaussov teorem za vektorsko polje $\mathbf{F} = xz\hat{x}$ za područje omeđeno kvadrom stranica $x \in [-a, a], y \in [0, a], z \in [0, a]$.

Rješenje. Definirajmo radijvektore stranica:

$$\mathbf{r_1}\Big|_{x=a} = a\hat{x} + y\hat{y} + z\hat{z}, \tag{1.429}$$

$$\mathbf{r_2}\Big|_{x=-a} = -a\hat{x} + y\hat{y} + z\hat{z}, \tag{1.430}$$

$$\mathbf{r_3}\Big|_{y=a} = x\hat{x} + a\hat{y} + z\hat{z}, \tag{1.431}$$

$$\mathbf{r_4}\Big|_{y=0} = x\hat{x} + z\hat{z}, \tag{1.432}$$

$$\mathbf{r_5}\Big|_{z=a} = x\hat{x} + y\hat{y} + a\hat{z}, \tag{1.433}$$

$$\mathbf{r_6}\Big|_{z=0} = x\hat{x} + y\hat{y}. \tag{1.434}$$

Diferencijalni elementi površine:

$$d\mathbf{A_1} = \partial_y \mathbf{r_1} \times \partial_z \mathbf{r_1} dydz = \hat{x} dydz, \tag{1.435}$$

$$d\mathbf{A_2} = \partial_z \mathbf{r_2} \times \partial_y \mathbf{r}_2 dydz = -\hat{x} dydz. \tag{1.436}$$

Primijetimo da nam ostale stranice ne trebaju jer nam vektorsko polje ima samo x smjer pa pri skalarnom množenju ostali dijelovi nestaju. Ono što sada treba integrirati jest:

$$I_1|_{x=a} = \int_0^a \int_0^a az\hat{x} \cdot \hat{x} dydz = \frac{a^4}{2}, \tag{1.437}$$

$$I_2|_{x=-a} = \int_0^a \int_0^a (-a)z\hat{x} \cdot (-\hat{x}) dydz = \frac{a^4}{2}, \tag{1.438}$$

$$I = a^4. \tag{1.439}$$

Drugi način je izračunati volumni integral po divergenciji od $\mathbf{F}$:

$$\nabla \cdot \mathbf{F} = z. \tag{1.440}$$

Integral je onda jednak:

$$\int_{-a}^{a} \int_{0}^{a} \int_{0}^{a} z dx dy dz = a^4. \tag{1.441}$$

Primjer 1.5.5. Izračunati tok polja $\mathbf{F} = 2\sqrt{x^2 + y^2}\hat{x}$ kroz plohu omeđenu krivuljama $x^2 + y^2 + z^2 = 3a^2$ i $x^2 + y^2 = 2az$ za $x, y, z > 0$.

Rješenje.

2. Elektrostatika

2.1 Koncept električnog naboja

Električni naboj Q je skalarna varijabla koju pridružujemo nabijenom objektu. Kao i masa, električni naboj je intrizično svojstvo materije. Zasad nije otkrivena niti jedna slobodna čestica takva da nije njezin naboj neki cijelobrojni iznos naboja elektrona: $e = 1.60217733(49) \times 10^{-19}$ C. Unatoč dikretiziranoj vrijednosti naboja, često se u elektrodinamici koristi volumna gustoća naboja $\rho(r)$ čija veza sa ukupnim nabojem je dana preko relacije:

$$Q = \int_V \rho(\mathbf{r}) dV. \tag{2.1}$$

Također, koristi se još površinska gustoća naboja:

$$Q = \int_S \sigma(\mathbf{r_S}) dS, \tag{2.2}$$

kao i linijska:

$$Q = \int \lambda(\mathbf{l}) dl. \tag{2.3}$$

U situacijama kada je sistem sastavljen od točkastih naboja, gustoću naboja prikazujemo pomoću delta distribucije:

$$Q = \sum_i \delta(\mathbf{r} - \mathbf{r_i}) q_i. \tag{2.4}$$

2.2 Coulombov zakon

Eksperimentalni radovi Priestleyja, Cavendisha i Coulomba krajem 18. stoljeća pokazali su da kada se radi o slučaju N stacionarnih točkastih naboja, sila na naboj q u točki $\mathbf{r}$ zbog N točkastih naboja q_k u točkama $\mathbf{r_k}$ dana je relacijom:

$$\mathbf{F} = \frac{q}{4\pi\epsilon_0}\sum_{k=1}^{N} q_k \frac{\mathbf{r}-\mathbf{r}_k}{|\mathbf{r}-\mathbf{r_k}|^3}, \tag{2.5}$$

poznatom još kao i Coulombov zakon.

2.3 Električno polje

Umjesto određenog naboja q, možemo zamisliti da računamo silu na neki probni naboj te sve što ga množi definirati kao električno polje. Eksperimentalni radovi Priestleyja, Cavendisha i Coulomba krajem 18. stoljeća pokazali su da kada se radi o slučaju N stacionarnih točkastih naboja, sila na naboj q u točki $\mathbf{r}$ zbog N točkastih naboja q_k u točkama $\mathbf{r_k}$ dana je relacijom:

$$\mathbf{F} = \frac{q}{4\pi\epsilon_0}\sum_{k=1}^{N} q_k \frac{\mathbf{r}-\mathbf{r_k}}{|\mathbf{r}-\mathbf{r_k}|^3} = q\mathbf{E}, \tag{2.6}$$

odnosno

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{N} q_k \frac{\mathbf{r}-\mathbf{r_k}}{|\mathbf{r}-\mathbf{r_k}|^3}. \tag{2.7}$$

U slučaju linijske gustoće λ, površinske gustoće σ i volumne gustoće ρ, ovaj izraz postaje

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{r}-\mathbf{r'}}{|\mathbf{r}-\mathbf{r'}|^{\mathbf{3}}}\lambda dr' \tag{2.8}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{r}-\mathbf{r'}}{|\mathbf{r}-\mathbf{r'}|^{\mathbf{3}}}\sigma d^2r' \tag{2.9}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{r}-\mathbf{r'}}{|\mathbf{r}-\mathbf{r'}|^{\mathbf{3}}}\rho d^3r'. \tag{2.10}$$

Za električno polje vrijede Maxwellove jednadžbe

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2.11}$$

$$\nabla\times\mathbf{E} = 0. \tag{2.12}$$

Integrirajući prvu jednadžbu u dijelu prostora Ω, dobivamo Gaussov zakon

$$\int_\Omega \nabla\cdot\mathbf{E}dV = \oint_{\partial\Omega}\mathbf{E}\cdot d\mathbf{A} = \frac{q}{\epsilon_0}. \tag{2.13}$$

U izrazu se pojavljuje fluks električnog polja, kojeg možemo izračunati za proizvoljnu plohu formulom

$$\Phi_E = \int \mathbf{E}\cdot d\mathbf{A}. \tag{2.14}$$

Primjer 2.3.1. Pronaći električno polje uniformno nabijene kugle.

Rješenje. Budući da je problem ima sfernu simetriju, električno polje je isto za sve radijalne smjerove. Prema tome, možemo iskoristiti Gaussov zakon koristeći kao plohu sferu konstantnog radijusa. Izvan sfere dobivamo

$$E4\pi r^2 = \frac{q}{\epsilon_0} \to \mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}, \tag{2.15}$$

dok unutar sfere vrijedi:

$$q = \int_V \rho dV \to \rho = \frac{3q}{4\pi R^3}, \tag{2.16}$$

$$E4\pi r^2 = \frac{4\pi\rho r^3}{3\epsilon_0} \to \mathbf{E} = \frac{qr}{4\pi\epsilon_0 R^3}\hat{r} = \frac{q}{4\pi\epsilon_0 R^3}\mathbf{r}. \tag{2.17}$$

Primjer 2.3.2. Kugla ukupnog naboja q ima gustoću koja linearno raste s radijusom, $\rho(r) = kr$. Pronaći električno polje u prostoru. Što se događa za $\rho(r) = k/r^2$ za $R_b > r > R_a$?

Rješenje. Ukupni naboj možemo jednostavno povezati s konstanom k

$$q = \int_0^R krr^2 dr4\pi = kR^4\pi \to k = \frac{q}{\pi R^4}. \tag{2.18}$$

Uvrstimo k u izraz za gustoću:

$$\rho(r) = \frac{qr}{\pi R^4}. \tag{2.19}$$

Izvan sfere, električno polje je jednako električnom polju točkastog naboja q, dok je unutar sfere

$$4\pi r^2 E = \frac{1}{\epsilon_0}\int_0^r r^2 dr4\pi\frac{qr}{\pi R^4} \tag{2.20}$$

$$\mathbf{E} = \frac{qr^2\hat{r}}{4\pi\epsilon_0 R^4}. \tag{2.21}$$

U slučaju $\rho = k/r^2$ ukupni naboj je dan relacijom:

$$q = \int_{R_a}^{R_b} k\frac{r^2}{r^2} dr4\pi = 4\pi k(R_b - R_a) \to k = \frac{q}{4\pi(R_b - R_a)}, \tag{2.22}$$

pa je gustoća dana:

$$\rho(r) = \frac{q}{4\pi(R_b - R_a)r^2}. \tag{2.23}$$

Električno polje unutar kugle je tada

$$4\pi r^2 E = \frac{1}{\epsilon_0}\int_{R_a}^r r^2 dr4\pi\frac{q}{4\pi(R_b - R_a)r^2} \tag{2.24}$$

$$\mathbf{E} = \frac{q\hat{r}(r - R_a)}{4\pi\epsilon_0(R_b - R_a)r^2}. \tag{2.25}$$

Primjer 2.3.3. Pronaći električno polje beskonačne žice nabijene linijskom gustoćom naboja λ.

Rješenje. Koristeći Gaussov zakon cilindrom duljine L i radijusa ρ dobivamo

$$E\rho 2\pi L = \frac{\lambda L}{\epsilon_0} \tag{2.26}$$

$$\mathbf{E} = \frac{\lambda\hat{\rho}}{2\pi\rho\epsilon_0} \tag{2.27}$$

Primjer 2.3.4. Točkasti naboj q nalazi se u ishodištu, dok je u $z = a$ ravnini kvadrat s koordinatama $b > x > 0$, $b > y > 0$. Izračunati tok električnog polja kroz kvadrat.

Rješenje. Lokacija kvadrata je:

$$\mathbf{r} = x\hat{x} + y\hat{y} + a\hat{z}. \tag{2.28}$$

Električno polje točkastog naboja na lokaciji kvadrata (z=a) je:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{x\hat{x} + y\hat{y} + a\hat{z}}{(x^2 + y^2 + a^2)^{3/2}}. \tag{2.29}$$

Naboj nalazi na koordinati

$$\mathbf{r}' = 0. \tag{2.30}$$

Da bi izračunali diferencijal površine trebamo derivirati po dvije nefiksirane koordinate radij-vektor kvadrata:

$$\partial_x \mathbf{r} = \hat{x} \tag{2.31}$$

$$\partial_y \mathbf{r} = \hat{y} \tag{2.32}$$

Diferencijal površine je onda:

$$d\mathbf{A} = \partial_x \mathbf{r} \times \partial_y \mathbf{r} dxdy = \hat{z} dxdy \tag{2.33}$$

te je tok jednak

$$\Phi_E = \int_0^b \int_0^b dxdy \frac{q}{4\pi\epsilon_0}\frac{a}{(x^2 + y^2 + a^2)^{3/2}}. \tag{2.34}$$

Koristimo integral

$$\int \frac{dy}{(y^2 + A^2)^{3/2}} = \frac{y}{A^2(y^2 + A^2)^{1/2}}, \tag{2.35}$$

uz $A = x^2 + a^2$ te dobivamo

$$\Phi_E = \frac{qa}{4\pi\epsilon_0}\int_0^b \frac{ydx}{(x^2 + a^2)\sqrt{(x^2 + y^2 + a^2)}}\Big|_0^b \tag{2.36}$$

$$= \int_0^b dx \frac{qab}{4\pi\epsilon_0}\frac{1}{(x^2 + a^2)\sqrt{x^2 + a^2 + b^2}}. \tag{2.37}$$

Sada koristimo integral

$$\int dx \frac{1}{(x^2+A^2)\sqrt{x^2+B^2}} = \begin{cases} \frac{1}{A\sqrt{A^2-B^2}} \operatorname{arctanh} \frac{\sqrt{A^2-B^2}x}{A\sqrt{x^2+B^2}} & A > B \\ \frac{1}{A\sqrt{B^2-A^2}} \arctan \frac{\sqrt{B^2-A^2}x}{A\sqrt{x^2+B^2}} & A < B, \end{cases} \tag{2.38}$$

uz $B = \sqrt{a^2+b^2} > a = A$ te dobivamo

$$\Phi_E = \frac{qab}{4\pi\epsilon_0}\frac{1}{ab} \arctan \frac{bx}{a\sqrt{x^2+a^2+b^2}}\Big|_0^b \tag{2.39}$$

$$= \frac{q}{4\pi\epsilon_0} \arctan\left(\frac{b^2}{a\sqrt{a^2+2b^2}}\right). \tag{2.40}$$

Primjer 2.3.5. Točkasti naboj q nalazi se u ishodištu. Pronaći tok električnog polja kroz plašt cilindra radijusa R koji se proteže od $z = -h$ do $z = h$.

Rješenje. Naboj nalazi na koordinati

$$\mathbf{r}' = 0. \tag{2.41}$$

Lokacija plašta cilindra je:

$$\mathbf{r} = R\hat{\rho} + z\hat{z}. \tag{2.42}$$

Diferencijal površine cilindra je:

$$\partial_z \mathbf{r} = \hat{z} \tag{2.43}$$

$$\partial_\phi \mathbf{r} = R\partial_\phi(\cos(\phi)\hat{x} + \sin(\phi)\hat{y}) = R\hat{\phi} \tag{2.44}$$

$$d\mathbf{A} = \partial_z \mathbf{r} \times \partial_\phi \mathbf{r} dz d\phi = R d\phi dz \hat{\rho} \tag{2.45}$$

Tok je jednak

$$\Phi_E = \frac{q}{4\pi\epsilon_0} \int_{-h}^{h} \int_0^{2\pi} \frac{R dz d\phi (R\hat{\rho} + z\hat{z}) \cdot \hat{\rho}}{(R^2+z^2)^{3/2}} \tag{2.46}$$

$$= \frac{qR^2}{2\epsilon_0} \int_{-h}^{h} \frac{dz}{(R^2+z^2)^{3/2}} \tag{2.47}$$

$$\tag{2.48}$$

Koristimo integral

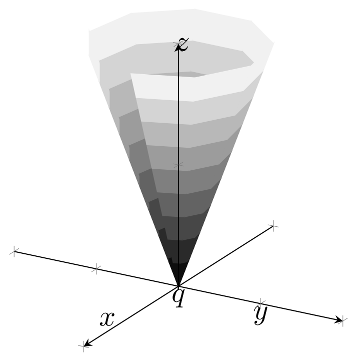
$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}}, \tag{2.49}$$

te je tok električnog polja

$$\Phi_E = \frac{qR^2}{2\epsilon_0} \frac{z}{R^2\sqrt{R^2+z^2}}\Big|_{-h}^{h} \tag{2.50}$$

$$= \frac{qh}{\epsilon_0\sqrt{R^2+h^2}}. \tag{2.51}$$

Primjer 2.3.6. Naboj je postavljen u ishodište koordinatnog sustava. Pronaći tok električnog polja kroz plašt stošca duljine brida a i otvorenog pod kutem θ_0.



Rješenje. Povoljno je koristiti sferni koordinatni sustav. Vektor na stošcu je opisan s radijusom r i kutem ϕ, dok je $\theta = \theta_0$ fiksan

$$\mathbf{r} = r(\sin\theta_0 \cos\phi\hat{x} + \sin\theta_0 \sin\phi\hat{y} + \cos\theta_0\hat{z}). \tag{2.52}$$

Parcijalne derivacije ovog vektora su

$$\partial_r \mathbf{r} = (\sin\theta_0 \cos\phi\hat{x} + \sin\theta_0 \sin\phi\hat{y} + \cos\theta_0\hat{z}) = \hat{r} \tag{2.53}$$

$$\partial_\phi \mathbf{r} = r(-\sin\theta_0 \sin\phi\hat{x} + \sin\theta_0 \cos\phi\hat{y}) = r\sin\theta_0\hat{\phi}. \tag{2.54}$$

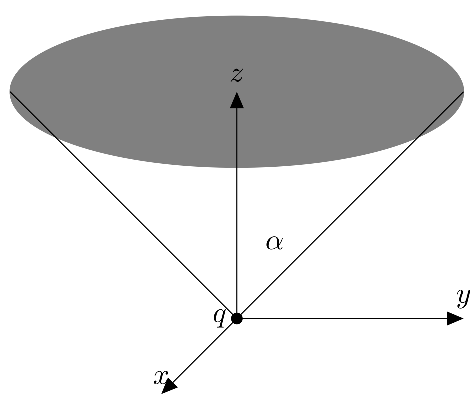
Normala je tada

$$\partial_\phi \mathbf{r} \times \partial_r \mathbf{r} = r\sin\theta_0\hat{\phi} \times \hat{r} = r\sin\theta_0\hat{\theta}. \tag{2.55}$$

Tok električnog polja $\mathbf{E} = E(r)\hat{r}$ je

$$\Phi_E = \int \mathbf{E} \cdot (\partial_\phi \mathbf{r} \times \partial_r \mathbf{r}) dr d\phi = \int E(r)\hat{r}(r\sin\theta_0) \cdot \hat{\theta} dr d\phi = 0 \tag{2.56}$$

Primjer 2.3.7. Naboj je postavljen u ishodište koordinatnog sustava. Pronaći tok električnog polja kroz bazu stošca visine h i otvorenog pod kutem α.



Rješenje. Za ovaj primjer, korisno je koristiti cilindrični sustav. Baza stošca nalazi se na koordinatama

$$\mathbf{r} = \rho\hat{\rho} + h\hat{z} = \rho(\cos\phi\hat{x} + \sin\phi\hat{y}) + h\hat{z}, \tag{2.57}$$

dok se naboj nalazi na koordinatama

$$\mathbf{r}' = 0. \tag{2.58}$$

Električno polje naboja na koordinatama baze je

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{q}{4\pi\epsilon_0}\frac{\rho\hat{\rho}+h\hat{z}}{(\rho^2+h^2)^{3/2}}. \tag{2.59}$$

Normalu dobivamo vektorskim produktom tangentnih vektora

$$\partial_\rho\mathbf{r} = (\cos\phi\hat{x} + \sin\phi\hat{y}) \tag{2.60}$$

$$\partial_\phi\mathbf{r} = \rho(-\sin\phi\hat{x} + \cos\phi\hat{y}) \tag{2.61}$$

$$\partial_\rho\mathbf{r}\times\partial_\phi\mathbf{r} = \rho(\cos^2\phi + \sin^2\phi)\hat{z} = \rho\hat{z} \tag{2.62}$$

Prema tome, tok je

$$\Phi_E = \int_0^{2\pi}\int_0^{h\tan\alpha} \frac{q}{4\pi\epsilon_0}\frac{(\rho\hat{\rho}+h\hat{z})\cdot\rho\hat{\rho}d\rho d\phi}{(\rho^2+h^2)^{3/2}} \tag{2.63}$$

$$= \frac{qh}{2\epsilon_0}\int_0^{h\tan\alpha}\frac{\rho d\rho}{(\rho^2+h^2)^{3/2}} \tag{2.64}$$

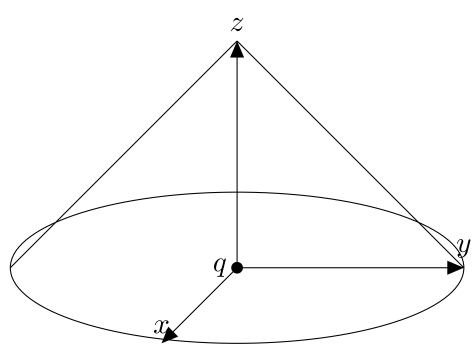
$$= \left|\begin{matrix}\rho^2+h^2 = x\\ 2\rho d\rho = dx\\ h^2\leq x\leq h^2(1+\tan^2\alpha)\end{matrix}\right| \tag{2.65}$$

$$= \frac{qh}{2\epsilon_0}\int_{h^2}^{h^2(1+\tan^2\alpha)}\frac{dx}{2x^{3/2}} \tag{2.66}$$

$$= \frac{qh}{2\epsilon_0}\left(\frac{1}{h} - \frac{1}{h\sqrt{1+\tan^2\alpha}}\right) \tag{2.67}$$

$$= \frac{q}{2\epsilon_0}(1-\cos\alpha) = \frac{q}{\epsilon_0}\sin^2\frac{\alpha}{2} \tag{2.68}$$

Primjer 2.3.8. Naboj je postavljen u ishodište koordinatnog sustava. Pronaći tok električnog polja kroz plašt stošca visine h i radijusa R kao na slici



Rješenje. Neka je kut stošca jednak $\theta_0 = \arctan(R/h)$. Koordinatu položaja točke na plaštu stošca, tangentne i normalne vektore, možemo jednostavno izračunati

$$\mathbf{r} = z\tan\theta_0\hat{\rho} + (h-z)\hat{z} \tag{2.69}$$
$$\partial_\phi \mathbf{r} = z\tan\theta_0\hat{\phi} \tag{2.70}$$
$$\partial_z \mathbf{r} = \tan\theta_0\hat{\rho} - \hat{z} \tag{2.71}$$
$$\partial_z \mathbf{r} \times \partial_\phi \mathbf{r} = z\tan\theta_0(\hat{\rho} + \tan\theta_0\hat{z}). \tag{2.72}$$

Električno polje točkastog naboja je

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{z\tan\theta_0\hat{\rho} + (h-z)\hat{z}}{(\tan^2\theta_0 z^2 + (h-z)^2)^{3/2}} \tag{2.73}$$

te je tok električnog polja

$$\Phi_E = \frac{q}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^h \frac{z\tan\theta_0\hat{\rho} + (h-z)\hat{z}}{(\tan^2\theta_0 z^2 + (h-z)^2)^{3/2}} \cdot z\tan\theta_0(\hat{\rho} + \tan\theta_0\hat{z})dzd\phi \tag{2.74}$$
$$= \frac{q2\pi}{4\pi\epsilon_0}\int_0^h \frac{z\tan\theta_0(z\tan\theta_0 + (h-z)\tan\theta_0)}{(z^2\tan^2\theta_0 + (h-z)^2)^{3/2}}dz \tag{2.75}$$

Koristimo tablični integral

$$\int \frac{xdx}{(ax^2+bx+c)^{3/2}} = -\frac{2bx+4c}{(4ac-b^2)(ax^2+bx+c)^{1/2}}, \tag{2.76}$$

uz

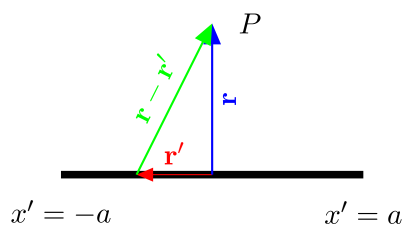
$$a = 1 + \tan^2\theta_0 \tag{2.77}$$
$$b = -2h \tag{2.78}$$
$$c = h^2 \tag{2.79}$$

$$= \frac{qh\tan^2\theta_0}{2\epsilon_0}\int_0^h \frac{zdz}{(z^2(1+\tan^2\theta_0) - 2zh + h^2)^{3/2}} \tag{2.80}$$
$$= \frac{qh\tan^2\theta_0}{2\epsilon_0}\left(-\frac{4h^2-4hz}{4(1+\tan^2\theta_0)h^2 - 4h^2}\frac{1}{\sqrt{z^2(1+\tan^2\theta_0) - 2zh + h^2}}\right)\Bigg|_{z=0}^{z=h} \tag{2.81}$$
$$= \frac{qh\tan^2\theta_0}{2\epsilon_0}\left(-\frac{4h(h-z)}{4h^2\tan^2\theta_0\sqrt{z^2(1+\tan^2\theta_0) - 2zh + h^2}}\right)\Bigg|_{z=0}^{z=h} \tag{2.82}$$
$$= \frac{qh\tan^2\theta_0}{2\epsilon_0}\frac{h}{h\tan^2\theta_0\sqrt{h^2}} \tag{2.83}$$
$$= \frac{q}{2\epsilon_0} \tag{2.84}$$

Primjer 2.3.9. Nađi električno polje na visini d iznad središta uniformno nabijenog segmenta linije dužine 2a.

Rješenje. Vektor točke koju opisujemo je $\mathbf{r} = d\hat{z}$, a vektor bilo koje točke na linijskom segmentu je $\mathbf{r}' = (-a, 0, 0)(1-t) + (a, 0, 0)t = (a(2t-1), 0, 0) = a(2t-1)\hat{x}$, pri čemu je t∈[0,1]. Udaljenost između ovih točaka je:

$$|\mathbf{r} - \mathbf{r}'| = |d\hat{z} - a(2t-1)\hat{x}| = \sqrt{a^2(2t-1)^2 + d^2}. \tag{2.85}$$



Diferencijalni element dužine je

$$dr' = |d\mathbf{r}| = \left| \begin{pmatrix} 2a \\ 0 \\ 0 \end{pmatrix} \right| dt = 2adt. \tag{2.86}$$

Ukupni naboj određuje gustoću naboja λ

$$q = \int \lambda dr' = \lambda \int_0^1 2adt\lambda = 2a\lambda. \tag{2.87}$$

Električno polje je prema formuli:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{2a} \int_0^1 \frac{(d\hat{z} - a(2t-1)\hat{x})2adt}{(a^2(2t-1)^2 + d^2)^{3/2}} = \left| \begin{matrix} u = 2t-1 \\ du = 2dt \\ -1 \leq u \leq 1 \end{matrix} \right|. \tag{2.88}$$

Koristimo integral

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}}, \tag{2.89}$$

dok je drugi integral neparna funkcija na parnom intervalu te stoga jednak 0. Električno polje je

$$\mathbf{E} = \frac{q}{8\pi\epsilon_0 a^2} \int_{-1}^1 \frac{\frac{d}{a}\hat{z} - u\hat{x}}{\left(u^2 + \left(\frac{d}{a}\right)^2\right)^{3/2}} du \tag{2.90}$$

$$= \frac{q\hat{z}}{8\pi\epsilon_0 a^2} \frac{\frac{2d}{a}}{\left(\frac{d}{a}\right)^2 \sqrt{\left(\frac{d}{a}\right)^2 + 1}} \tag{2.91}$$

$$= \frac{q\hat{z}}{4\pi\epsilon_0 d\sqrt{d^2+a^2}}. \tag{2.92}$$

Primjer 2.3.10. Nađi električno polje u točki P=(d,0,0) koja leži na osi uniformno nabijenog segmenta linije dužine a.

Rješenje. Vektor točke koju opisujemo je $\mathbf{r} = d\hat{x}$, a vektor bilo koje točke na linijskom segmentu je $\mathbf{r}' = (0,0,0)(1-t)+(a,0,0)t = (at,0,0) = at\hat{x}$, pri čemu je $t \in [0,1]$. Udaljenost između ovih točaka je:

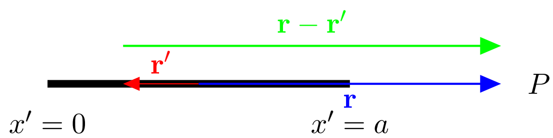
$$|\mathbf{r}-\mathbf{r}'| = |(d-at)\hat{x}| = d - at. \tag{2.93}$$

Diferencijalni element je

$$dr' = |d\mathbf{r}| = adt, \tag{2.94}$$

te je gustoća naboja određena ukupnim nabojem

$$q = \int \lambda dr' = \int_0^1 \lambda adt = a\lambda \tag{2.95}$$



Električno polje je prema formuli:

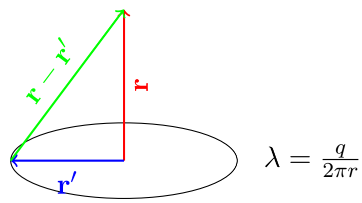
$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{q}{a}\int_0^1 \frac{(d-at)\hat{x}adt}{(d-at)^3} && (2.96)\\
&= \frac{q}{4\pi\epsilon_0}\hat{x}\int_0^1 \frac{dt}{(d-at)^2} && (2.97)\\
&= \left|\begin{matrix} u = d-at \\ du = -adt \\ d-a \le u \le d \end{matrix}\right| && (2.98)\\
&= \frac{q}{4\pi\epsilon_0 a}\hat{x}\int_{d-a}^{d}\frac{du}{u^2} && (2.99)\\
&= \frac{q\hat{x}}{4\pi\epsilon_0 a}\left(\frac{1}{d-a}-\frac{1}{d}\right) && (2.100)\\
&= \frac{q\hat{x}}{4\pi\epsilon_0 a}\frac{a}{d(d-a)} && (2.101)\\
&= \frac{q\hat{x}}{4\pi\epsilon_0}\frac{1}{d(d-a)}. && (2.102)
\end{aligned}$$

Primjer 2.3.11. Nađi električno polje na osi uniformno nabijene kružne petlje radijusa r na visini d od njegovog središta.

Rješenje. Vektor točke koju opisujemo je $\mathbf{r} = d\hat{z}$, a vektor bilo koje točke na prstenu je $\mathbf{r}' = r\cos(\phi)\hat{x} + r\sin(\phi)\hat{y}$. Udaljenost između ovih točaka je

$$|\mathbf{r}-\mathbf{r}'| = |d\hat{z} - r\cos(\phi)\hat{x} - r\sin(\phi)\hat{y}| = \sqrt{r^2\cos^2\phi + r^2\sin^2\phi + d^2} = \sqrt{r^2+d^2}, \tag{2.103}$$

kao što je prikazano na sljedećoj slici.



Električno polje je dano izrazom

$$\mathbf{E} = \int_0^{2\pi} \frac{\lambda(\phi)}{4\pi\epsilon_0} \frac{\mathbf{r}-\mathbf{r'}}{|\mathbf{r}-\mathbf{r'}|^3} r d\phi \tag{2.104}$$

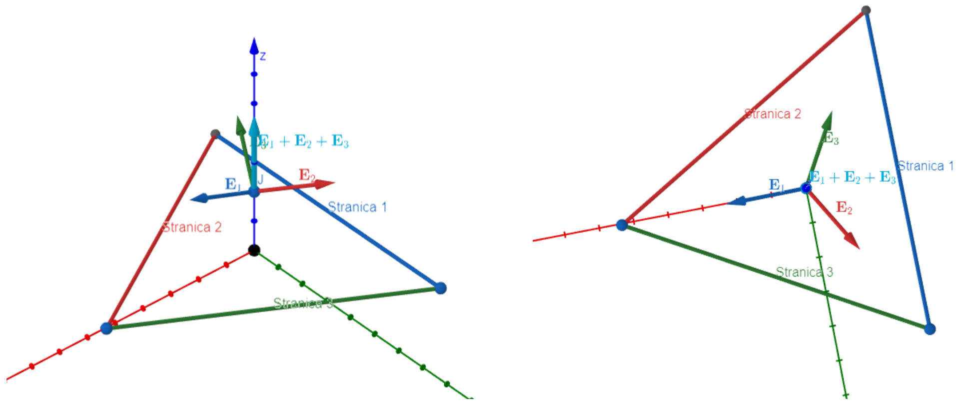
$$= \frac{q}{2\pi r} \frac{r}{4\pi\epsilon_0 (r^2+d^2)^{3/2}} \int_0^{2\pi} (d\hat{z} - r\cos(\phi)\hat{x} - r\sin(\phi)\hat{y}) d\phi \tag{2.105}$$

$$\tag{2.106}$$

Integrali kosinusa i sinusa na punom periodu su 0 i preostaje

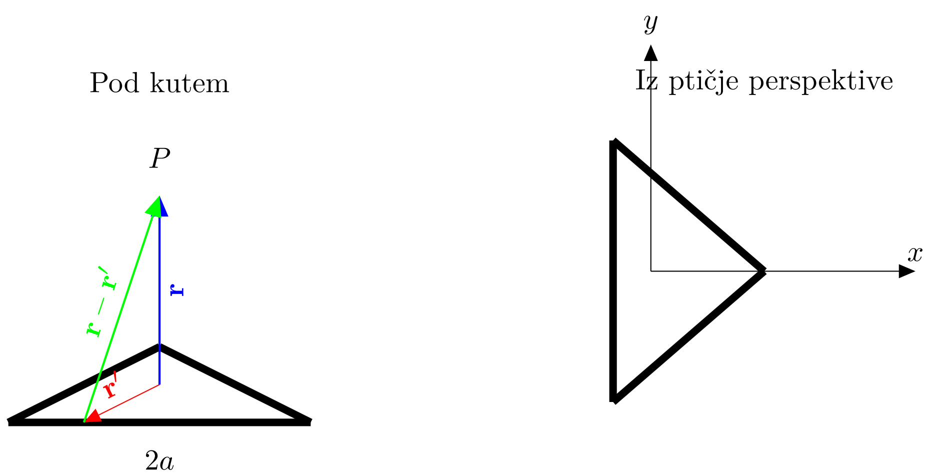
$$= \frac{q}{2\pi} \frac{2\pi d}{4\pi\epsilon_0 (r^2+d^2)^{3/2}} \hat{z} \tag{2.107}$$

$$= \frac{qd}{4\pi\epsilon_0 (r^2+d^2)^{3/2}} \hat{z}. \tag{2.108}$$



Slika 2.1: Vizualizacija električnih polja trokuta. Puni model je moguće pogledati na `https://www.geogebra.org/3d/cvwph6s3`.

Primjer 2.3.12. Nađi električno polje na visini d iznad središta uniformno linijski nabijenog jednakostraničnog trokuta stranice dužine 2a.



Rješenje. Rješenje dobivamo pomoću zadatka 2.3.9. Zapišimo električno polje iz zadatka 2.3.9 koristeći oznake x_0, y_0, z_0

$$\mathbf{E}_0 = \frac{q\hat{z}_0}{4\pi\epsilon_0 z_0\sqrt{z_0^2 + a^2}}. \tag{2.109}$$

Budući da je polje simetrično oko $\hat{x}$ osi, možemo zamijeniti $z_0 = \cos\phi\hat{y} + \sin\phi\hat{z}$. Na z osi udaljeni mo za visinu d, a na y osi na visini $h = a/\sqrt{3}$.

$$\hat{z}_0 = \frac{d\hat{z} + h\hat{y}}{\sqrt{d^2 + h^2}}, \tag{2.110}$$

dok je udaljenost $z_0 = \sqrt{d^2 + h^2}$. Trokut i električna polja prikazani su na slici 2.1. Također, ukupan naboj q raspodijeljen je na 3 štapa duljine $2a$, tako da naboj jednog štapa mora biti trećina ukupnog. Polje štapa na x osi je tada

$$\mathbf{E}_0 = \frac{q}{12\pi\epsilon_0(d^2+h^2)\sqrt{d^2+h^2+a^2}}\begin{pmatrix}0\\h\\d\end{pmatrix}. \tag{2.111}$$

Polja štapova 1, 2 i 3 dobivamo rotacijom za $-90°$, $150°$ i $30°$ oko z osi. Matrice rotacije su

$$R_1 = \begin{pmatrix}\cos(-90) & -\sin(-90) & 0\\ \sin(-90) & \cos(-90) & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{2.112}$$

$$R_2 = \begin{pmatrix}\cos(150) & -\sin(150) & 0\\ \sin(150) & \cos(150) & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}-\sqrt{3}/2 & -1/2 & 0\\ 1/2 & -\sqrt{3}/2 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{2.113}$$

$$R_3 = \begin{pmatrix}\cos(30) & -\sin(30) & 0\\ \sin(30) & \cos(30) & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 & -1/2 & 0\\ 1/2 & \sqrt{3}/2 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{2.114}$$

Ukupno električno polje je

$$\begin{aligned}
\mathbf{E} &= (R_1+R_2+R_3)\mathbf{E}_0 && (2.115)\\
&= \begin{pmatrix}0+\sqrt{3}/2-\sqrt{3}/2 & 1-1/2-1/2 & 0\\ 1-1/2-1/2 & \sqrt{3}/2-\sqrt{3}/2 & 0\\ 0 & 0 & 3\end{pmatrix}\mathbf{E}_0 && (2.116)\\
&= \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 3\end{pmatrix}\frac{q}{4\pi\epsilon_0(d^2+h^2)\sqrt{d^2+h^2+a^2}}\begin{pmatrix}0\\h\\d\end{pmatrix} && (2.117)\\
&= \frac{3qd\hat{z}}{12\pi\epsilon_0(d^2+h^2)\sqrt{d^2+h^2+a^2}} && (2.118)\\
&= \frac{3qd\hat{z}}{4\pi\epsilon_0(3d^2+a^2)\sqrt{d^2+4a^2/3}}. && (2.119)
\end{aligned}$$

Primjer 2.3.13. Nađi električno polje na visini d iznad središta uniformno linijski nabijenog kvadrata stranice dužine 2a.

Rješenje. Rješenje dobivamo pomoću zadatka 2.3.9. Zapišimo električno polje iz zadatka 2.3.9 koristeći oznake x_0, y_0, z_0

$$\mathbf{E}_0 = \frac{q\hat{z}_0}{4\pi\epsilon_0 z_0\sqrt{z_0^2+a^2}}. \tag{2.120}$$

Budući da je polje simetrično oko $\hat{x}$ osi, možemo zamijeniti $z_0 = \cos\phi\hat{y} + \sin\phi\hat{z}$. Na z osi udaljeni mo za visinu d, a na y osi za visinu koja odgovara polovici stranice kvadrata, $h = a$.

$$\hat{z}_0 = \frac{d\hat{z}+h\hat{y}}{\sqrt{d^2+h^2}}, \tag{2.121}$$

dok je udaljenost $z_0 = \sqrt{d^2 + h^2}$. Također, ukupan naboj q raspodijeljen je na 4 štapa duljine $2a$, tako da naboj jednog štapa mora biti četvrtina ukupnog. Polje štapa na x osi je

$$\mathbf{E}_0 = \frac{q}{16\pi\epsilon_0(d^2 + h^2)\sqrt{d^2 + h^2 + a^2}} \begin{pmatrix} 0 \\ h \\ d \end{pmatrix}. \tag{2.122}$$

Polja štapova dobivamo rotacijom za 90°, ±180°, i 270° oko z osi. Matrice rotacije su

$$R_1 = \begin{pmatrix} \cos(90) & -\sin(90) & 0 \\ \sin(90) & \cos(90) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.123}$$

$$R_2 = \begin{pmatrix} \cos(180) & -\sin(180) & 0 \\ \sin(180) & \cos(180) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.124}$$

$$R_3 = \begin{pmatrix} \cos(270) & -\sin(270) & 0 \\ \sin(270) & \cos(270) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.125}$$

$$\tag{2.126}$$

Ukupno električno polje je

$$\mathbf{E} = (1 + R_1 + R_2 + R_3)\mathbf{E}_0 \tag{2.127}$$

$$= \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \mathbf{E}_0 \tag{2.128}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} \frac{q}{16\pi\epsilon_0(d^2 + h^2)\sqrt{d^2 + h^2 + a^2}} \begin{pmatrix} 0 \\ h \\ d \end{pmatrix} \tag{2.129}$$

$$= \frac{q}{4\pi\epsilon_0(d^2 + h^2)\sqrt{d^2 + h^2 + a^2}} \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix} \tag{2.130}$$

$$= \frac{q}{4\pi\epsilon_0(d^2 + a^2)\sqrt{d^2 + 2a^2}} \hat{z} \tag{2.131}$$

$$\tag{2.132}$$

2.4 Elektrostatički potencijal

Budući da su integrali za nalaženje električnog polja ponekad teški, uvodimo elektrostatski potencijal, V, kojeg je puno lakše pronaći. Njega definiramo izrazom

$$\mathbf{E} = -\nabla V. \tag{2.133}$$

Izraz za elektrostatski potencijal je

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{N}\frac{q_k}{|\mathbf{r}-\mathbf{r}_k|} \tag{2.134}$$
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda dr'}{|\mathbf{r}-\mathbf{r}'|} \tag{2.135}$$
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma d^2r'}{|\mathbf{r}-\mathbf{r}'|} \tag{2.136}$$
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho d^3r'}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.137}$$

Jednostavnim uvrštavanjem možemo se uvjeriti da ovakav potencijal zadovoljava jednadžbu 2.133.

$$-\nabla V = -\nabla_{\mathbf{r}}\frac{1}{4\pi\epsilon_0}\int\frac{\rho d^3r'}{|\mathbf{r}-\mathbf{r}'|} \tag{2.138}$$
$$= -\frac{1}{4\pi\epsilon_0}\int\rho d^3r'\nabla_{\mathbf{r}-\mathbf{r}'}\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{2.139}$$
$$= -\frac{1}{4\pi\epsilon_0}\int\rho d^3r'\left(-\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right) \tag{2.140}$$
$$= \mathbf{E}. \tag{2.141}$$

Primjer 2.4.1. Električno polje je oblika $\mathbf{E} = (y+1)\hat{x} + (x-1)\hat{y} + 2\hat{z}$. Pronaći razliku potencijala između točaka $A = (2,-2,-1)$ i $B = (0,0,0)$. Ponoviti izračun za $A = (3,2,-1)$ i $B = (-2,-3,4)$.

Rješenje. Provjerimo uvjet elektrostatike

$$\nabla\times\mathbf{E} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial_x & \partial_y & \partial_z\\ y+1 & x-1 & 2\end{vmatrix} \tag{2.142}$$
$$= 0\hat{x} - 0\hat{y} + \begin{vmatrix}\partial_x & \partial_y\\ y+1 & x-1\end{vmatrix}\hat{z} \tag{2.143}$$
$$= \hat{z}(1-1) \tag{2.144}$$
$$= \vec{0}. \tag{2.145}$$

Budući da razlika potencijala ne ovisi o putu, možemo izabrati najjednostavniju spojnicu: ravnu liniju:

$$\mathbf{r} = \mathbf{A}(1-t) + \mathbf{B}t = (2-2t)\hat{x} + (2t-2)\hat{y} + (t-1)\hat{z}. \tag{2.146}$$

Diferencijal je tada

$$d\mathbf{r} = -2dt\hat{x} + 2dt\hat{y} + dt\hat{z}, \tag{2.147}$$

a električno polje duž pravca je

$$\mathbf{E}(t) = (2t-1)\hat{x} + (1-2t)\hat{y} + 2\hat{z}. \tag{2.148}$$

Razlika potencijala je

$$\begin{aligned}
\Delta V &= -\int \mathbf{E} \cdot d\mathbf{r} && (2.149)\\
&= -\int_0^1 ((2t-1)\hat{x} + (1-2t)\hat{y} + 2\hat{z}) \cdot (-2dt\hat{x} + 2dt\hat{y} + dt\hat{z}) && (2.150)\\
&= -\int_0^1 (6-8t)dt && (2.151)\\
&= -2. && (2.152)
\end{aligned}$$

U drugom slučaju vidimo da vrijedi

$$\begin{aligned}
\mathbf{r} &= \mathbf{A}(1-t) + \mathbf{B}t = (3-5t)\hat{x} + (2-5t)\hat{y} + (-1+5t)\hat{z} && (2.153)\\
d\mathbf{r} &= (-5\hat{x} - 5\hat{y} + 5\hat{z})dt && (2.154)\\
\mathbf{E}(t) &= (3-5t)\hat{x} + (2-5t)\hat{y} + 2\hat{z}. && (2.155)
\end{aligned}$$

Za razliku potencijala dobivamo

$$\begin{aligned}
\Delta V &= -\int_0^1 ((3-5t)\hat{x} + (2-5t)\hat{y} + 2\hat{z}) \cdot (-5\hat{x} - 5\hat{y} + 5\hat{z})dt && (2.156)\\
&= -\int_0^1 (50t-15)dt = -(25-15) = -10. && (2.157)
\end{aligned}$$

Primjer 2.4.2. Zadano je električno polje $\mathbf{E} = \frac{\sin^2\theta}{r^2}\hat{r} - \frac{2\sin\theta\cos\theta}{r^2}\hat{\theta}$. Provjeriti zadovoljava li ovo polje uvjet elektrostatike i izračunati razliku potencijala između točaka $A = (r, \theta, \phi) = (2, \pi/4, 0)$ i $B = (r, \theta, \phi) = (1, \pi/4, \pi/2)$.

Rješenje. Uvjet elektrostatike je da je rotacija vektorskog polja jednaka 0 te računamo

$$\nabla \times \mathbf{E} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ E_r & rE_\theta & r\sin\theta E_\phi \end{vmatrix} \tag{2.158}$$

$$= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ \frac{\sin^2\theta}{r^2} & -\frac{2\sin\theta\cos\theta}{r} & 0 \end{vmatrix} \tag{2.159}$$

$$= \frac{\hat{r}\begin{vmatrix} \partial_\theta & \partial_\phi \\ -\frac{2\sin\theta\cos\theta}{r} & 0 \end{vmatrix} - r\hat{\theta}\begin{vmatrix} \partial_r & \partial_\phi \\ \frac{\sin^2\theta}{r^2} & 0 \end{vmatrix} + r\sin\theta\hat{\phi}\begin{vmatrix} \partial_r & \partial_\theta \\ \frac{\sin^2\theta}{r^2} & -\frac{2\sin\theta\cos\theta}{r} \end{vmatrix}}{r^2\sin\theta} \tag{2.160}$$

$$= \frac{0\hat{r} - 0r\hat{\theta} + r\sin\theta\hat{\phi}\left(\frac{2\sin\theta\cos\theta}{r^2} - \frac{2\sin\theta\cos\theta}{r^2}\right)}{r^2\sin\theta} \tag{2.161}$$

$$= \vec{0}, \tag{2.162}$$

odnosno vidimo da su ispunjeni uvjeti elektrostatike. Parametrizirajmo krivulju i električno polje

$$\begin{pmatrix} r(t) \\ \theta(t) \\ \phi(t) \end{pmatrix} = t\begin{pmatrix} 1 \\ \pi/4 \\ \pi/2 \end{pmatrix} + (1-t)\begin{pmatrix} 2 \\ \pi/4 \\ 0 \end{pmatrix} \tag{2.163}$$

$$= \begin{pmatrix} 2-t \\ \pi/4 \\ t\pi/2 \end{pmatrix} \tag{2.164}$$

$$\mathbf{r}(t) = r(t)\left(\sin\theta(t)\cos\phi(t)\hat{x} + \sin\theta(t)\sin\phi(t)\hat{y} + \cos\theta(t)\hat{z}\right) \tag{2.165}$$

$$= (2-t)\left(\sin\frac{\pi}{4}\cos\frac{\pi t}{2}\hat{x} + \sin\frac{\pi}{4}\sin\frac{\pi t}{2}\hat{y} + \cos\frac{\pi}{4}\hat{z}\right) \tag{2.166}$$

$$= \frac{2-t}{\sqrt{2}}\left(\cos\frac{\pi t}{2}\hat{x} + \sin\frac{\pi t}{2}\hat{y} + \hat{z}\right) \tag{2.167}$$

$$d\mathbf{r} = \frac{-dt}{\sqrt{2}}\left(\cos\frac{\pi t}{2}\hat{x} + \sin\frac{\pi t}{2}\hat{y} + \hat{z}\right) + \frac{2-t}{\sqrt{2}}\frac{\pi}{2}dt\left(-\sin\frac{\pi t}{2}\hat{x} + \cos\frac{\pi t}{2}\hat{y}\right) \tag{2.168}$$

$$= dt\frac{-\hat{x}\left(\cos\frac{\pi t}{2} + (2-t)\frac{\pi}{2}\sin\frac{\pi t}{2}\right) + \hat{y}\left(-\sin\frac{\pi t}{2} + (2-t)\frac{\pi}{2}\cos\frac{\pi t}{2}\right) - \hat{z}}{\sqrt{2}} \tag{2.169}$$

$$\mathbf{E}(t) = \frac{\sin^2\theta(t)}{r^2(t)}\hat{r}(t) - \frac{2\sin\theta(t)\cos\theta(t)}{r^2(t)}\hat{\theta}(t) \tag{2.170}$$

$$= \frac{\sin\theta(t)(\cos\phi(t)\hat{x} + \sin\phi(t)\hat{y}) + \cos\theta(t)\hat{z} - 2(\cos\theta(t)(\cos\phi(t)\hat{x} + \sin\phi(t)\hat{y}) - \sin\theta(t)\hat{z})}{2(2-t)^2} \tag{2.171}$$

$$= \frac{-\cos\frac{\pi t}{2}\hat{x} - \sin\frac{\pi t}{2}\hat{y} + 3\hat{z}}{2\sqrt{2}(2-t)^2}. \tag{2.172}$$

Računamo razliku potencijala

$$V_B - V_A = -\int\limits_0^1 \mathbf{E}(t) \cdot d\mathbf{r}(t) \tag{2.173}$$

$$= \frac{-1}{4} \int\limits_0^1 \frac{-\cos\frac{\pi t}{2}\hat{x} - \sin\frac{\pi t}{2}\hat{y} + 3\hat{z}}{(2-t)^2} \tag{2.174}$$

$$\cdot \frac{-\hat{x}\left(\cos\frac{\pi t}{2} + (2-t)\frac{\pi}{2}\sin\frac{\pi t}{2}\right) + \hat{y}\left(-\sin\frac{\pi t}{2} + (2-t)\frac{\pi}{2}\cos\frac{\pi t}{2}\right) - \hat{z}}{\sqrt{2}} dt \tag{2.175}$$

$$= \frac{-1}{4} \int\limits_0^1 \frac{\cos^2\frac{\pi t}{2} + (2-t)\frac{\pi}{2}\cos\frac{\pi t}{2}\sin\frac{\pi t}{2} + \sin^2\frac{\pi t}{2} - (2-t)\frac{\pi}{2}\cos\frac{\pi t}{2}\sin\frac{\pi t}{2} - 3}{(2-t)^2} dt \tag{2.176}$$

$$= \frac{-1}{4} \int\limits_0^1 \frac{1-3}{(2-t)^2} dt \tag{2.177}$$

$$= \begin{vmatrix} u = 2-t \\ du = -dt \\ u \in [1,2] \end{vmatrix} \tag{2.178}$$

$$= \frac{2}{4} \int\limits_1^2 \frac{du}{u^2} \tag{2.179}$$

$$= \frac{1}{2}\left(1 - \frac{1}{2}\right) \tag{2.180}$$

$$= \frac{1}{4}. \tag{2.181}$$

Primjer 2.4.3. Naboj je jednoliko raspodijeljen po krivulji opisanoj koordinatama $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$. Pronaći potencijal na visini $z = d$.

Rješenje. Krivulju opisujemo vektorom

$$\mathbf{r}' = a(\cos t + t\sin t)\hat{x} + a(\sin t - t\cos t)\hat{y}, \tag{2.182}$$

a diferencijal duž krivulje računamo iz norme

$$dr' = \sqrt{d\mathbf{r}'^2} dt = atdt \tag{2.183}$$

te potencijal računamo u točki $\mathbf{r} = d\hat{z}$. Udaljenost između $\mathbf{r}$ i $\mathbf{r}'$ je

$$|\mathbf{r} - \mathbf{r}'| = a\sqrt{\frac{d^2}{a^2} + t^2 + 1}, \tag{2.184}$$

te je potencijal dan sa

$$V = \frac{\lambda}{4\pi\epsilon_0}\int_0^{2\pi}\frac{dr'}{|\mathbf{r}-\mathbf{r}'|} \tag{2.185}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\int_0^{2\pi}\frac{atdt}{a\sqrt{\left(\frac{d}{a}\right)^2+t^2+1}}. \tag{2.186}$$

Koristimo integral

$$\int\frac{udu}{(u^2+c^2)^{1/2}} = (u^2+c^2)^{1/2}, \tag{2.187}$$

uz $c^2 = 1+\left(\frac{d}{a}\right)^2$ te dobivamo

$$= \frac{\lambda}{4\pi\epsilon_0}\left(\sqrt{\left(\frac{d}{a}\right)^2+4\pi^2+1}-\sqrt{\left(\frac{d}{a}\right)^2+1}\right). \tag{2.188}$$

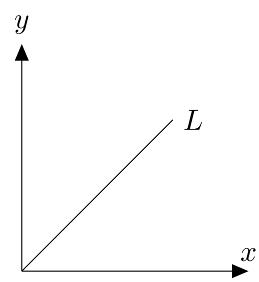
Ukupni naboj računamo po formuli

$$q = \int\lambda dr' = \lambda\int_0^{2\pi}atdt = 2\pi^2 a\lambda. \tag{2.189}$$

U konačnici, dobivamo potencijal

$$V = \frac{q}{8\pi^3\epsilon_0 a}\left(\sqrt{\left(\frac{d}{a}\right)^2+4\pi^2+1}-\sqrt{\left(\frac{d}{a}\right)^2+1}\right) \tag{2.190}$$

Primjer 2.4.4. Pronaći potencijal uniformno nabijenog štapa duljine L koji se nalazi pod kutem od 45° u x-y ravnini. Koliki je potencijal za $x = y = 0$, $|z| >> L$?



Rješenje. Točka u kojoj računamo potencijal je

$$\mathbf{r} = x\hat{x}+y\hat{y}+z\hat{z}. \tag{2.191}$$

Početna točka štapa je $(0,0,0)$ i završava u $L/\sqrt{2}(1,1,0)$ te su stoga naboji na koordinatama

$$\mathbf{r}' = \frac{L}{\sqrt{2}}t\begin{pmatrix}1\\1\\0\end{pmatrix}. \tag{2.192}$$

Diferencijalni element i duljina luka krivulje

$$dr' = \sqrt{1+1}\frac{L}{\sqrt{2}}dt = Ldt, \tag{2.193}$$

gdje ntegriramo od $t = 0$ do $t = L/\sqrt{2}$. Gustoća naboja je dana preko ukupnog naboja

$$q = \lambda \int dr' = \lambda \int_0^1 Ldt = \lambda L. \tag{2.194}$$

Potencijal je tada

$$V = \frac{q}{4\pi\epsilon_0 L} \int_0^1 \frac{Ldt}{\sqrt{\left(x - t\frac{L}{\sqrt{2}}\right)^2 + \left(y - t\frac{L}{\sqrt{2}}\right)^2 + z^2}} \tag{2.195}$$

$$= \frac{q}{4\pi\epsilon_0} \int_0^1 \frac{dt}{\sqrt{L^2t^2 - \sqrt{2}Lt(x+y) + x^2 + y^2 + z^2}} \tag{2.196}$$

Koristimo tablični integral

$$\int \frac{dt}{(at^2+bt+c)^{1/2}} = \frac{1}{\sqrt{a}} \ln \left| 2\sqrt{a}\sqrt{at^2+bt+c} + 2at + b \right|, \tag{2.197}$$

uz $a = L^2$, $b = -\sqrt{2}L(x+y)$ i $c = x^2+y^2+z^2$ te dobivamo

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{L} \ln |2\sqrt{x^2+y^2+z^2 - \sqrt{2}L(x+y) + L^2t^2} + 2Lt - 2(x+y)| \Big|_0^1 \tag{2.198}$$

$$= \frac{q}{4\pi\epsilon_0 L} \ln \left| \frac{\sqrt{x^2+y^2+z^2+L^2-\sqrt{2}L(x+y)} + L - (x+y)}{\sqrt{x^2+y^2+z^2} - (x+y)} \right|. \tag{2.199}$$

U posebnom slučaju kada je $x = 0$ i $y = 0$ potencijal poprima oblik

$$V = \frac{q}{4\pi\epsilon_0 L} \ln \left| \frac{\sqrt{z^2+L^2} + L}{|z|} \right| \tag{2.200}$$

$$= \frac{q}{4\pi\epsilon_0 L} \ln \left| \sqrt{1 + \frac{L^2}{z^2}} + \frac{L}{|z|} \right|, \tag{2.201}$$

budući da je $|z| >> L$, član $(L/|z|)^2$ možemo zanemariti i dobivamo

$$V \approx \frac{q}{4\pi\epsilon_0 L} \ln \left(1 + \frac{L}{|z|}\right) \tag{2.202}$$

Koristeći razvoj logaritma $\ln(1+t) \approx t$ izraz se pojednostavljuje

$$V \approx \frac{q}{4\pi\epsilon_0 L}\frac{L}{|z|} \tag{2.203}$$

$$= \frac{q}{4\pi\epsilon_0 |z|} \tag{2.204}$$

Primjer 2.4.5. Naboj je uniformno distribuiran duž prvog luka spirale opisanog koordinatama $x = R\cos t$, $y = R\sin t$, $z = ct$. Pronaći potencijal na z osi i z komponentu električnog polja duž z osi.

Rješenje. Osnovne izračune dobijemo jednostavnim uvrštavanjem

$$\mathbf{r} = z\hat{z} \tag{2.205}$$

$$\mathbf{r}' = R\cos t\hat{x} + R\sin t\hat{y} + ct\hat{z} \tag{2.206}$$

$$d\mathbf{r}' = -R\sin t\hat{x} + R\cos t\hat{y} + c\hat{z} \tag{2.207}$$

$$dr' = \sqrt{d\mathbf{r}'^2} = \sqrt{R^2 + c^2}dt \tag{2.208}$$

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{R^2\cos^2\phi + R^2\sin^2\phi + (z - ct)^2} \tag{2.209}$$

$$= \sqrt{R^2 + (z - ct)^2}. \tag{2.210}$$

Ukupni naboj je

$$q = \int_0^{2\pi} \lambda dr' = \int_0^{2\pi} \lambda\sqrt{R^2 + c^2}dt = 2\pi\lambda\sqrt{R^2 + c^2}, \tag{2.211}$$

a potencijal je dan izrazom

$$V = \frac{\lambda}{4\pi\epsilon_0}\int_0^{2\pi} \frac{\lambda dr'}{|\mathbf{r} - \mathbf{r}'|} \tag{2.212}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\int_0^{2\pi} \frac{\sqrt{R^2 + c^2}dt}{\sqrt{R^2 + (z - ct)^2}} \tag{2.213}$$

$$= \frac{q}{8\pi^2\epsilon_0}\int_0^{2\pi} \frac{dt}{\sqrt{R^2 + (z - ct)^2}} \tag{2.214}$$

$$= \left| \begin{matrix} u = z - ct \\ du = -cdt \\ u \in [z - 2\pi c, z] \end{matrix} \right| \tag{2.215}$$

$$= \frac{q}{8\pi^2\epsilon_0 c}\int_{z-2\pi c}^{z} \frac{du}{\sqrt{u^2 + R^2}}. \tag{2.216}$$

Koristimo tablični integral

$$\int \frac{du}{\sqrt{u^2 + R^2}} = \ln\frac{u + \sqrt{u^2 + R^2}}{R}, \tag{2.217}$$

te dobivamo

$$V = \frac{q}{8\pi^2\epsilon_0 c}\ln\frac{u + \sqrt{u^2 + R^2}}{R}\Big|_{z-2\pi c}^{z} \tag{2.218}$$

$$= \frac{q}{8\pi^2\epsilon_0 c}\ln\frac{z + \sqrt{z^2 + R^2}}{z - 2\pi c + \sqrt{(z - 2\pi c)^2 + R^2}}. \tag{2.219}$$

Komponentu električnog polja u Z smjeru dobivamo jednostavnim deriviranjem

$$
\begin{aligned}
E^z &= -\partial_z V && (2.220)\\
&= -\partial_z \frac{q}{8\pi^2\epsilon_0 c} \ln \frac{z+\sqrt{z^2+R^2}}{z-2\pi c+\sqrt{(z-2\pi c)^2+R^2}} && (2.221)\\
&= \frac{q}{8\pi^2\epsilon_0 c}\partial_z \left(\ln\left(z-2\pi c+\sqrt{(z-2\pi c)^2+R^2}\right)-\ln\left(z+\sqrt{z^2+R^2}\right)\right) && (2.222)\\
&= \frac{q}{8\pi^2\epsilon_0 c}\left(\frac{1+\frac{1}{2}\frac{2(z-2\pi c)}{\sqrt{(z-2\pi c)^2+R^2}}}{z-2\pi c+\sqrt{(z-2\pi c)^2+R^2}}-\frac{1+\frac{1}{2}\frac{2z}{\sqrt{z^2+R^2}}}{z+\sqrt{z^2+R^2}}\right) && (2.223)\\
&= \frac{q}{8\pi^2\epsilon_0 c}\left(\frac{\frac{\sqrt{(z-2\pi c)^2+R^2}+(z-2\pi c)}{\sqrt{(z-2\pi c)^2+R^2}}}{z-2\pi c+\sqrt{(z-2\pi c)^2+R^2}}-\frac{\frac{\sqrt{z^2+R^2}+z}{\sqrt{z^2+R^2}}}{z+\sqrt{z^2+R^2}}\right) && (2.224)\\
&= \frac{q}{8\pi^2\epsilon_0 c}\left(\frac{1}{\sqrt{(z-2\pi c)^2+R^2}}-\frac{1}{\sqrt{z^2+R^2}}\right). && (2.225)
\end{aligned}
$$

2.5 Multipolni razvoj elektrostatičkog potencijala

Potencijal je ponekad lakše naći rastavljajući potencijal u red. U ovom poglavlju promotrit ćemo razvoj u sfernom sustavu.

Ovaj razvoj se temelji na razvoju korijena

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\alpha}} = \sum_l P_l(\cos\alpha)\frac{r_<^l}{r_>^{l+1}}, \tag{2.226}$$

gdje se u brojniku nalazi manji od r i r', a u nazivniku veći od r i r'. Pogledajmo preciznije kut između vektora $\mathbf{r}$ i $\mathbf{r}'$, α

$$rr'\cos\alpha = \mathbf{r}\cdot\mathbf{r}' = rr'(\hat{x}\sin\theta\cos\phi+\hat{y}\sin\theta\sin\phi+\hat{z}\cos\theta)\cdot(\hat{x}\sin\theta'\cos\phi'+\hat{y}\sin\theta'\sin\phi'+\hat{z}\cos\theta') \tag{2.227}$$

$$= rr'(\sin\theta\sin\theta'\cos(\phi-\phi')+\cos\theta\cos\theta') \tag{2.228}$$

$$\cos\alpha = \sin\theta\sin\theta'\cos(\phi-\phi')+\cos\theta\cos\theta'. \tag{2.229}$$

Po adicijskom teoremu, ovaj bi nas izraz vodio na

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi')\frac{r_<^l}{r_>^{l+1}}, \tag{2.230}$$

Budući da ovaj izraz sadrži kugline funkcije, često ga nećemo koristiti u punome obliku jer su nam dovoljni problemi s aksijalnom simetrijom. U tom slučaju, $\cos\alpha$ postaje puno jednostavniji $\cos\alpha = \cos(\theta-\theta')$ te možemo koristiti samo Legendreove polinome

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_l P_l(\cos\theta)P_l(\cos\theta')\frac{r_<^l}{r_>^{l+1}}. \tag{2.231}$$

Pogledajmo što bismo dobili ako bismo ovaj izraz uvrstili u izraz za elektrostatski potencijal

$$V = \frac{1}{4\pi\epsilon_0}\sum_l P_l(\cos\theta)\frac{1}{r^{l+1}}\int P_l(\cos\theta')r'^{l+2}\rho(r')dr'\sin\theta' d\theta' d\phi'. \tag{2.232}$$

Pogledajmo prvih nekoliko članova ovog razvoja

$$V = \frac{1}{4\pi\epsilon_0}\frac{1}{r^1}\int\rho(r')d^3r' \tag{2.233}$$

$$+\frac{1}{4\pi\epsilon_0}\cos\theta\frac{1}{r^2}\int r'\cos\theta'\rho(r')d^3r' \tag{2.234}$$

$$+\frac{1}{4\pi\epsilon_0}P_2(\cos\theta)\frac{1}{r^3}\int r'^2\frac{3\cos^2\theta'-1}{2}\rho(r')d^3r' \tag{2.235}$$

$$+\frac{1}{4\pi\epsilon_0}\sum_l P_l(\cos\theta)\frac{1}{r^{l+1}}\int P_l(\cos\theta')r'^l\rho(r')d^3r' \tag{2.236}$$

$$= \frac{q}{4\pi\epsilon_0 r}+\frac{\mathbf{p}\cdot\widehat{\mathbf{r}}}{4\pi\epsilon_0 r^2}+\frac{1}{4\pi\epsilon_0}\frac{Q_{ij}\hat{r}^i\hat{r}^j}{r^3}+\cdots, \tag{2.237}$$

gdje smo iskoristiti izraze za prvih nekoliko Legendreovih polinoma i uveli monopolni, dipolni i kvadrupolni moment

$$q = \int \rho(r')d^3r' \tag{2.238}$$

$$\mathbf{p} = \int \rho(r')\mathbf{r}'d^3r' \tag{2.239}$$

$$Q_{ij} = \int (3r'_i r'_j - r'^2\delta_{ij})\rho(r')d^3r'. \tag{2.240}$$

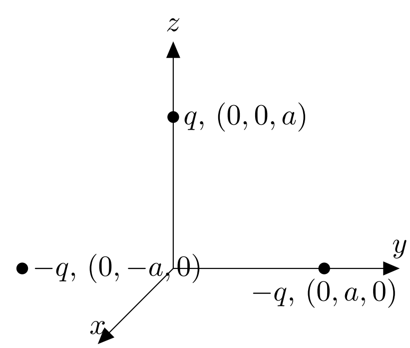
Električno polje dipola je

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r} - \mathbf{p}\mathbf{r}^2}{r^5} = \frac{1}{4\pi\epsilon_0}\frac{3(\mathbf{p}\cdot\hat{r})\hat{r} - \mathbf{p}}{r^3}. \tag{2.241}$$

Silu na dipol računamo po formuli

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}. \tag{2.242}$$

Primjer 2.5.1. Naboji se nalaze na pozicijama kao na slici. Pronaći monopolni i dipolni moment potencijala.



Rješenje. Monopolni moment je samo suma naboja $Q = q - 2q = -q$, dok je dipolni moment dan izrazom

$$\mathbf{p} = q(a\hat{z}) - q(-a\hat{y}) - q(a\hat{y}) = qa\hat{z}. \tag{2.243}$$

Potencijal dipola je približno

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{-q}{r} + \frac{qa\cos\theta}{r^2}\right). \tag{2.244}$$

Električno polje dobivamo također direktnim uvrštavanjem

$$\mathbf{E} = \frac{Q\hat{r}}{4\pi\epsilon_0 r^2} + \frac{1}{4\pi\epsilon_0}\frac{3(\mathbf{p}\cdot\hat{r})\hat{r} - \mathbf{p}}{r^3} \tag{2.245}$$

$$= \frac{-q\hat{r}}{4\pi\epsilon_0 r^2} + \frac{1}{4\pi\epsilon_0 r^3}(3qa\cos\theta\hat{r} - qa\hat{z}) \tag{2.246}$$

$$= \frac{-q\hat{r}}{4\pi\epsilon_0 r^2} + \frac{qa}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}). \tag{2.247}$$

Primjer 2.5.2. U ishodištu koordinatnog sustava nalazi se naboj q, a na z osi na udaljenosti d nalazi se dipol $\mathbf{p}$. Pronaći silu na dipol ako je on usmjeren u $\hat{z}$ smjeru te ako je usmjeren u $\hat{x}$ smjeru. Ponoviti ovaj izračun u sfernom sustavu.

Rješenje. Ako je dipol u $\hat{z}$ smjeru, sila je dana izrazom

$$\mathbf{F} = \frac{q}{4\pi\epsilon_0} p\partial_z \frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2+y^2+z^2)^{3/2}}\Big|_{x=0,y=0,z=d} \tag{2.248}$$

$$= \frac{q}{4\pi\epsilon_0} p\partial_z \frac{\hat{z}}{z^2}\Big|_{z=d} \tag{2.249}$$

$$= \frac{qp}{4\pi\epsilon_0} \frac{-2\hat{z}}{d^3}. \tag{2.250}$$

U sfernom sustavu dipol pokazuje u radijalnom smjeru ako je na z osi i usmjeren je u z smjeru. Koristeći $\hat{z}\cdot\hat{r} = \cos\theta$, $\hat{z}\cdot\hat{\theta} = -\sin\theta$ i $\hat{z}\cdot\hat{\phi} = 0$ dobivamo

$$\mathbf{F} = \frac{q}{4\pi\epsilon_0} p\hat{z}\cdot\left(\hat{r}\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \frac{\hat{\phi}}{r\sin\theta}\partial_\phi\right)\frac{\hat{r}}{r^2}\Big|_{\theta=0,\,r=d} \tag{2.251}$$

$$= \frac{qp}{4\pi\epsilon_0}\left(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\right)\Big|_{\theta=0,\,r=d} \tag{2.252}$$

$$= \frac{qp}{4\pi\epsilon_0}\left(\cos\theta\partial_r\frac{\hat{r}}{r^2} - \frac{1}{r^2}\frac{\sin\theta}{r}\partial_\theta\hat{r}\right)\frac{\hat{r}}{r^2}\Big|_{\theta=0,\,r=d} \tag{2.253}$$

$$= \frac{qp}{4\pi\epsilon_0}\partial_r\frac{\hat{r}}{r^2}\Big|_{\theta=0,\,r=d} \tag{2.254}$$

$$= \frac{qp}{4\pi\epsilon_0}\partial_r\frac{\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}}{r^2}\Big|_{\theta=0,\,r=d} \tag{2.255}$$

$$= \frac{qp}{4\pi\epsilon_0}\partial_r\frac{\hat{z}}{r^2}\Big|_{r=d} \tag{2.256}$$

$$= \frac{qp}{4\pi\epsilon_0}\frac{-2\hat{z}}{r^3}\Big|_{r=d} \tag{2.257}$$

$$= \frac{qp}{4\pi\epsilon_0}\frac{-2\hat{z}}{d^3}. \tag{2.258}$$

Ako je dipol usmjeren u $\hat{x}$ smjeru, dobivamo

$$\begin{aligned}
\mathbf{F} &= \frac{q}{4\pi\epsilon_0} p\partial_x \frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2+y^2+z^2)^{3/2}}\Big|_{x=0,y=0,z=d} && (2.259)\\
&= \frac{q}{4\pi\epsilon_0} p\partial_x \frac{x\hat{x} + d\hat{z}}{(x^2+d^2)^{3/2}}\Big|_{x=0} && (2.260)\\
&= \frac{qp}{4\pi\epsilon_0}\left(\frac{x\hat{x}+z\hat{z}}{(x^2+d^2)^{5/2}}(-3/2)2x + \frac{\hat{x}}{(x^2+d^2)^{3/2}}\right)\Big|_{x=0} && (2.261)\\
&= \frac{qp}{4\pi\epsilon_0(x^2+d^2)^{5/2}}\left(-3x(x\hat{x}+d\hat{z}) + (x^2+d^2)\hat{x}\right)\Big|_{x=0} && (2.262)\\
&= \frac{qp}{4\pi\epsilon_0(x^2+d^2)^{5/2}}\left(-3xd\hat{z} + (d^2-2x^2)\hat{x}\right)\Big|_{x=0} && (2.263)\\
&= \frac{qp}{4\pi d^5} d^2\hat{x} && (2.264)\\
&= \frac{qp}{4\pi d^3}\hat{x}. && (2.265)
\end{aligned}$$

Koristeći $\hat{x}\cdot\hat{r} = \sin\theta\cos\phi$, $\hat{x}\cdot\hat{\theta} = \cos\theta\cos\phi$ i $\hat{x}\cdot\hat{\phi} = -\sin\phi$ rastav u sfernom sustavu daje

$$\begin{aligned}
\mathbf{F} &= \frac{q}{4\pi\epsilon_0} p\hat{x}\cdot\left(\hat{r}\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \frac{\hat{\phi}}{r\sin\theta}\partial_\phi\right)\frac{\hat{r}}{r^2}\Big|_{\theta=0,\,r=d} && (2.266)\\
&= \frac{qp}{4\pi\epsilon_0}\left(\sin\theta\cos\phi\partial_r + \frac{\cos\theta\cos\phi}{r}\partial_\theta - \frac{\sin\phi}{r\sin\theta}\partial_\phi\right)\frac{\hat{r}}{r^2}\Big|_{\theta=0,\,r=d} && (2.267)\\
&= \frac{qp}{4\pi\epsilon_0}\left(-2\frac{\sin\theta\cos\phi\hat{r}}{r^3} + \frac{\cos\theta\cos\phi}{r^3}\partial_\theta\hat{r} - \frac{\sin\phi}{r^3\sin\theta}\partial_\phi\hat{r}\right)\Big|_{\theta=0,\,r=d} && (2.268)
\end{aligned}$$

Koristimo

$$\begin{aligned}
\hat{r} &= (\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}) && (2.269)\\
&= \hat{r} && (2.270)\\
\partial_\theta\hat{r} &= (\cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}) && (2.271)\\
&= \hat{\theta} && (2.272)\\
\partial_\phi\hat{r} &= (-\sin\theta\sin\phi\hat{x} + \sin\theta\cos\phi\hat{y}) && (2.273)\\
&= \sin\theta\left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right) && (2.274)\\
&= \sin\theta\hat{\phi}, && (2.275)
\end{aligned}$$

Izraz za silu se pojednostavljuje u

$$\begin{aligned}
\mathbf{F} &= \frac{qp}{4\pi\epsilon_0}\left(-2\frac{\sin\theta\cos\phi\hat{r}}{r^3} + \frac{\cos\theta\cos\phi}{r^3}\hat{\theta} - \frac{\sin\phi}{r^3\sin\theta}\sin\theta\hat{\phi}\right)\Big|_{\theta=0,\,r=d} && (2.276)\\
&= \frac{qp}{4\pi\epsilon_0 r^3}\left(-2\sin\theta\hat{z} + \cos\theta\cos\phi(\cos\phi\hat{x} + \sin\phi\hat{y}) - \sin\phi\left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right)\right)\Big|_{\theta=0,\,r=d} && (2.277)
\end{aligned}$$

Sada možemo uvrstiti $r = d$ i $\theta = 0$ u jedinične vektore

$$\hat{r}\Big|_{\theta=0} = (\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z})\Big|_{\theta=0} \tag{2.278}$$
$$= \hat{z} \tag{2.279}$$
$$\hat{\theta}\Big|_{\theta=0} = (\cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z})\Big|_{\theta=0} \tag{2.280}$$
$$= \cos\phi\hat{x} + \sin\phi\hat{y} \tag{2.281}$$
$$\tag{2.282}$$

pa uvrštavanjem dobivamo

$$\mathbf{F} = \frac{qp}{4\pi\epsilon_0 d^3}\left(0\hat{z} + \cos\phi(\cos\phi\hat{x} + \sin\phi\hat{y}) - \sin\phi\left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right)\right) \tag{2.283}$$
$$= \frac{qp}{4\pi\epsilon_0 d^3}\left(\cos^2\phi\hat{x} + \sin\phi\cos\phi\hat{y} + \sin^2\phi\hat{x} - \sin\phi\cos\phi\hat{y}\right) \tag{2.284}$$
$$= \frac{qp}{4\pi\epsilon_0 d^3}\hat{x}. \tag{2.285}$$

Primjer 2.5.3. Izračunati približni potencijal uniformno nabijenog jednakostraničnog trokuta ukupnog naboja q i duljine stranice a s jednim vrhom u ishodištu. Također izračunati pripadno električno polje. U točki $(b, 0, 0)$ nalazi se naboj Q. Izračunati silu trokuta na naboj Q i silu naboja Q na trokut.

Rješenje. Trokut ima vrhove na koordinatama $\mathbf{A} = (0, 0)$, $\mathbf{B} = (a, 0)$ i $\mathbf{C} = (a/2, a\sqrt{3}/2)$. Točku $\mathbf{r}$ unutar trokuta parametriziramo parametrima $u \in [0, 1]$ i $v \in [0, 1-u]$

$$\mathbf{r}(u, v) = \mathbf{A} + (\mathbf{B} - \mathbf{A})u + (\mathbf{C} - \mathbf{A})v \tag{2.286}$$
$$= au\begin{pmatrix}1\\0\end{pmatrix} + \frac{a}{2}v\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}. \tag{2.287}$$

Pronađimo diferencijalni element površine

$$\mathbf{r} = a\hat{x}u + \frac{a}{2}(\hat{x} + \sqrt{3}\hat{y})v \tag{2.288}$$
$$\partial_u\mathbf{r} = a\hat{x}du \tag{2.289}$$
$$\partial_v\mathbf{r} = \frac{a}{2}(\hat{x} + \sqrt{3}\hat{y})dv \tag{2.290}$$
$$\partial_u\mathbf{r} \times \partial_v\mathbf{r} = \frac{a^2\sqrt{3}}{2}(\hat{z})dudv \tag{2.291}$$
$$|\partial_u\mathbf{r} \times \partial_v\mathbf{r}| = \frac{a^2\sqrt{3}}{2}dudv. \tag{2.292}$$

Računamo monopolni moment

$$q = \int_0^1 \int_0^{1-u} \frac{a^2\sqrt{3}}{2} \sigma du dv \tag{2.293}$$
$$= \sigma \frac{a^2\sqrt{3}}{2} \int_0^1 (1-u) du \tag{2.294}$$
$$= \sigma \frac{a^2\sqrt{3}}{2} \left(u - \frac{u^2}{2}\right)\Big|_0^1 \tag{2.295}$$
$$= \sigma \frac{a^2\sqrt{3}}{2} \left(1 - \frac{1}{2}\right) \tag{2.296}$$
$$= \sigma \frac{a^2\sqrt{3}}{4} \tag{2.297}$$

Stoga vidimo da je površina trokuta $a^2\sqrt{3}/4$ te je površinska gustoća $\sigma = \frac{4q}{a^2\sqrt{3}}$. Monopolni moment je zadan ukupnim nabojem q, dok dipolni računamo

$$\mathbf{p} = \sigma \int_0^1 \int_0^{1-u} \frac{a^2\sqrt{3}}{2} (x\hat{x} + y\hat{y}) du dv \tag{2.298}$$
$$= \sigma \frac{a^3\sqrt{3}}{2} \int_0^1 \int_0^{1-u} \left(\left(u + \frac{v}{2}\right)\hat{x} + \frac{\sqrt{3}v}{2}\hat{y} \right) du dv \tag{2.299}$$
$$= \sigma \frac{a^3\sqrt{3}}{2} \int_0^1 \left(\left(u(1-u) + \frac{(1-u)^2}{4}\right)\hat{x} + \frac{\sqrt{3}(1-u)^2}{4}\hat{y} \right) du \tag{2.300}$$
$$= \sigma \frac{a^3\sqrt{3}}{8} \int_0^1 \left(\left(4u - 4u^2 + 1 - 2u + u^2\right)\hat{x} + \sqrt{3}(1 - 2u + u^2)\hat{y} \right) du \tag{2.301}$$
$$= \sigma \frac{a^3\sqrt{3}}{8} \int_0^1 \left(\left(2u - 3u^2 + 1\right)\hat{x} + \sqrt{3}(1 - 2u + u^2)\hat{y} \right) du \tag{2.302}$$
$$= \frac{4q}{a^2\sqrt{3}} \frac{a^3\sqrt{3}}{8} \left((1 - \frac{3}{3} + 1)\hat{x} + \frac{\sqrt{3}}{3}\left(1 - \frac{2}{2} + \frac{1}{3}\right)\hat{y} \right) \tag{2.303}$$
$$= \frac{qa}{2}(\hat{x} + \frac{\sqrt{3}\hat{y}}{3}) \tag{2.304}$$
$$= \frac{qa}{2}(\hat{x} + \frac{\hat{y}}{\sqrt{3}}). \tag{2.305}$$

Približni potencijal je tada

$$V = \frac{q}{4\pi\epsilon_0 r} + \frac{qa(x + y/\sqrt{3})}{8\pi\epsilon_0 r^3}. \tag{2.306}$$

Pripadno električno polje dobivamo deriviranjem

$$\mathbf{E} = -\nabla V \tag{2.307}$$

$$= -\frac{q}{4\pi\epsilon_0}\left(\frac{-1}{2}\frac{2x\hat{x}+2y\hat{y}+2z\hat{z}}{(x^2+y^2+z^2)^{3/2}}\right) \tag{2.308}$$

$$-\frac{q}{4\pi\epsilon_0}\left(\frac{a}{2}\left(\frac{\hat{x}+\hat{y}/\sqrt{3}}{(x^2+y^2+z^2)^{3/2}}+\frac{x+y/\sqrt{3}}{(x^2+y^2+z^2)^{3/2}}\frac{-3}{2}(2x\hat{x}+2y\hat{y}+2z\hat{z})\right)\right) \tag{2.309}$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}}+\frac{a}{2}\left(3\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{5/2}}(x+y/\sqrt{3})-\frac{\hat{x}+\hat{y}/\sqrt{3}}{(x^2+y^2+z^2)^{3/2}}\right)\right). \tag{2.310}$$

Na poziciji točkastog naboja, $b\hat{x}$, silu dobivamo uvrštavanjem $x = b$, $y = z = 0$ i množenjem s Q

$$\mathbf{F} = Q\mathbf{E}(b,0,0) \tag{2.311}$$

$$= \frac{qQ}{4\pi\epsilon_0}\left(\frac{b\hat{x}}{b^3}+\frac{a}{2}\left(3\frac{b\hat{x}}{b^5}(b+0)-\frac{\hat{x}+\hat{y}/\sqrt{3}}{b^3}\right)\right) \tag{2.312}$$

$$= \frac{qQ}{4\pi\epsilon_0}\left(\frac{\hat{x}}{b^2}+\frac{a}{2b^3}\left(2\hat{x}-\frac{\hat{y}}{\sqrt{3}}\right)\right). \tag{2.313}$$

Silu na trokut dobivamo kombinirajući silu na dipolni i monopolni član električnog polja točkastog naboja

$$\mathbf{E}_Q = \frac{Q}{4\pi\epsilon_0}\frac{(x-b)\hat{x}+y\hat{y}+z\hat{z}}{((x-b)^2+y^2+z^2)^{3/2}}. \tag{2.314}$$

Monopolni i dipolni članovi su samo

$$\mathbf{F}_m = q\mathbf{E}_Q\Big|_{0,0,0} = \frac{qQ}{4\pi\epsilon_0}\frac{-b\hat{x}}{b^3} \tag{2.315}$$

$$= -\frac{qQ}{4\pi\epsilon_0}\frac{\hat{x}}{b^2} \tag{2.316}$$

$$\mathbf{F}_D = (\mathbf{p}\cdot\nabla)\mathbf{E}_Q\Big|_{0,0,0} \tag{2.317}$$

$$= \frac{qa}{2}\left(\partial_x+\frac{\partial_y}{\sqrt{3}}\right)\frac{Q}{4\pi\epsilon_0}\frac{(x-b)\hat{x}+y\hat{y}+z\hat{z}}{((x-b)^2+y^2+z^2)^{3/2}} \tag{2.318}$$

$$= \frac{Qqa}{8\pi\epsilon_0}\left(\frac{\hat{x}+\hat{y}/\sqrt{3}}{((x-b)^2+y^2+z^2)^{3/2}}-\frac{3}{2}\frac{(x-b)\hat{x}+y\hat{y}+z\hat{z}}{((x-b)^2+y^2+z^2)^{5/2}}\left(2(x-b)+2\frac{y}{\sqrt{3}}\right)\right)\Bigg|_{0,0,0} \tag{2.319}$$

$$= \frac{Qqa}{8\pi\epsilon_0}\left(\frac{\hat{x}+\hat{y}/\sqrt{3}}{b^3}-3\frac{-b\hat{x}}{b^5}(-b)\right) \tag{2.320}$$

$$= \frac{Qqa}{8\pi\epsilon_0}\left(\frac{\hat{x}+\hat{y}/\sqrt{3}}{b^3}-3\frac{\hat{x}}{b^3}\right) \tag{2.321}$$

$$= \frac{Qqa}{8\pi\epsilon_0 b^3}\left(\hat{y}/\sqrt{3}-2\hat{x}\right). \tag{2.322}$$

Ukupna sila na trokut je tada

$$\mathbf{F} = \mathbf{F}_m + \mathbf{F}_D = -\frac{qQ}{4\pi\epsilon_0}\left(\frac{\hat{x}}{b^2} + \frac{a}{2b^3}\left(2\hat{x} - \frac{\hat{y}}{\sqrt{3}}\right)\right). \tag{2.323}$$

Primjer 2.5.4. Izračunati približni potencijal uniformno ispunjenog paralelepipeda naboja q koji prolazi kroz točke $A = (-a, 0, 0)$, $B = (a, 0, a)$, $C = (0, a, 0)$, $D = (a, a, 0)$. Također izračunati pripadno električno polje. U točki $(b, 0, 0)$ nalazi se naboj Q. Izračunati silu paralelepipeda na naboj Q i silu naboja Q na paralelepiped.

Rješenje. Položaj bilo koje točke unutar paralelepipeda parametriziramo vektorom

$$\mathbf{r} = A + (B - A)u + (C - A)v + (D - A)w \tag{2.324}$$

$$= a\left[\begin{pmatrix}-1\\0\\0\end{pmatrix} + \begin{pmatrix}2\\0\\1\end{pmatrix}u + \begin{pmatrix}1\\1\\0\end{pmatrix}v + \begin{pmatrix}2\\1\\0\end{pmatrix}w\right] \tag{2.325}$$

Parcijalne derivacije i jakobijan su

$$\partial_u\mathbf{r} = a\begin{pmatrix}2\\0\\1\end{pmatrix} \tag{2.326}$$

$$\partial_v\mathbf{r} = a\begin{pmatrix}1\\1\\0\end{pmatrix} \tag{2.327}$$

$$\partial_w\mathbf{r} = a\begin{pmatrix}2\\1\\0\end{pmatrix} \tag{2.328}$$

$$J = \begin{vmatrix}2a & a & 2a\\0 & a & a\\a & 0 & 0\end{vmatrix} \tag{2.329}$$

$$= 2a\begin{vmatrix}a & a\\0 & 0\end{vmatrix} - a\begin{vmatrix}0 & a\\a & 0\end{vmatrix} + 2a\begin{vmatrix}0 & a\\a & 0\end{vmatrix} \tag{2.330}$$

$$= -a^3 + 2a^3 \tag{2.331}$$

$$= a^3 \tag{2.332}$$

Ukupni naboj nam omogućava izračun gustoće

$$q = \rho\int_0^1\int_0^1\int_0^1 |J| du dv dw \tag{2.333}$$

$$= \rho a^3\int_0^1\int_0^1\int_0^1 du dv dw = a^3\rho \tag{2.334}$$

$$\rho = \frac{q}{a^3}. \tag{2.335}$$

Dipolni moment je

$$\mathbf{p} = \frac{q}{a^3}a^3 \int_0^1 \int_0^1 \int_0^1 a \left[\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} v + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} w \right] dudvdw \tag{2.336}$$

$$= qa \left[\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} uvw + \frac{u^2vw}{2} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \frac{uv^2w}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{uvw^2}{2} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right] \Bigg|_{0,0,0}^{1,1,1} \tag{2.337}$$

$$= \frac{qa}{2} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}. \tag{2.338}$$

Potencijal je tada približno

$$V = \frac{q}{4\pi\epsilon_0} + \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3} = \frac{q}{4\pi\epsilon_0 r} + \frac{qa}{8\pi\epsilon_0}\frac{3x+2y+z}{r^3}. \tag{2.339}$$

Pripadno električno polje dobivamo deriviranjem

$$\mathbf{E} = -\nabla V = -\frac{q}{4\pi\epsilon_0}\left(\frac{-1}{2}\frac{2x\hat{x}+2y\hat{y}+2z\hat{z}}{(x^2+y^2+z^2)^{3/2}} \right) \tag{2.340}$$

$$-\frac{q}{4\pi\epsilon_0}\left(\frac{a}{2}\left(\frac{3\hat{x}+2\hat{y}+\hat{z}}{(x^2+y^2+z^2)^{3/2}} + \frac{3x+2y+z}{(x^2+y^2+z^2)^{3/2}}\frac{-3}{2}(2x\hat{x}+2y\hat{y}+2z\hat{z}) \right) \right) \tag{2.341}$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}} + \frac{a}{2}\left(3\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{5/2}}(3x+2y+z) - \frac{3\hat{x}+2\hat{y}+\hat{z}}{(x^2+y^2+z^2)^{3/2}} \right) \right). \tag{2.342}$$

Na poziciji točkastog naboja, $b\hat{x}$, silu dobivamo uvrštavanjem $x = b$, $y = z = 0$ i množenjem s Q

$$\mathbf{F} = Q\mathbf{E}(b,0,0) \tag{2.343}$$

$$= \frac{qQ}{4\pi\epsilon_0}\left(\frac{b\hat{x}}{b^3} + \frac{qa}{2}\left(3\frac{b\hat{x}}{b^5}(3b+0) - \frac{3\hat{x}+2\hat{y}+\hat{z}}{b^3} \right) \right) \tag{2.344}$$

$$= \frac{qQ}{4\pi\epsilon_0}\left(\frac{\hat{x}}{b^2} + \frac{a}{2b^3}(6\hat{x}-2\hat{y}-\hat{z}) \right). \tag{2.345}$$

Silu na paralelepiped dobivamo kombinirajući silu na dipolni i monopolni član električnog polja točkastog naboja

$$\mathbf{E}_Q = \frac{Q}{4\pi\epsilon_0}\frac{(x-b)\hat{x}+y\hat{y}+z\hat{z}}{((x-b)^2+y^2+z^2)^{3/2}}. \tag{2.346}$$

Monopolni i dipolni članovi su samo

$$\mathbf{F}_m = q\mathbf{E}_Q\Big|_{0,0,0} = \frac{qQ}{4\pi\epsilon_0}\frac{-b\hat{x}}{b^3} \tag{2.347}$$

$$= -\frac{qQ}{4\pi\epsilon_0}\frac{\hat{x}}{b^2} \tag{2.348}$$

$$\mathbf{F}_D = (\mathbf{p}\cdot\nabla)\mathbf{E}_Q\Big|_{0,0,0} \tag{2.349}$$

$$= \frac{qa}{2}\left(3\partial_x + 2\partial_y + \partial_z\right)\frac{Q}{4\pi\epsilon_0}\frac{(x-b)\hat{x} + y\hat{y} + z\hat{z}}{((x-b)^2 + y^2 + z^2)^{3/2}} \tag{2.350}$$

$$= \frac{Qqa}{8\pi\epsilon_0}\left(\frac{3\hat{x} + 2\hat{y} + \hat{z}}{((x-b)^2 + y^2 + z^2)^{3/2}} - \frac{3}{2}\frac{(x-b)\hat{x} + y\hat{y} + z\hat{z}}{((x-b)^2 + y^2 + z^2)^{5/2}}(6(x-b) + 4y + 2z)\right)\Big|_{0,0,0} \tag{2.351}$$

$$= \frac{Qqa}{8\pi\epsilon_0}\left(\frac{3\hat{x} + 2\hat{y} + \hat{z}}{b^3} - 3\frac{-b\hat{x}}{b^5}(-3b)\right) \tag{2.352}$$

$$= \frac{Qqa}{8\pi\epsilon_0}\left(\frac{3\hat{x} + 2\hat{y} + \hat{z}}{b^3} - 9\frac{\hat{x}}{b^3}\right) \tag{2.353}$$

$$= \frac{Qqa}{8\pi\epsilon_0 b^3}\left(2\hat{y} + \hat{z} - 6\hat{x}\right). \tag{2.354}$$

Ukupna sila na paralelepiped je tada

$$\mathbf{F} = \mathbf{F}_m + \mathbf{F}_D = -\frac{qQ}{4\pi\epsilon_0}\left(\frac{\hat{x}}{b^2} + \frac{a}{2b^3}\left(6\hat{x} - 2\hat{y} - \hat{z}\right)\right). \tag{2.355}$$

2.6 Laplaceova i Poissonova jednadžba. Rubni uvjeti.

Umjesto integriranja, možemo rješavati Maxwellovu jednadžu za elektrostatski potencijal

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}. \tag{2.356}$$

Ako možemo izolirati područje gdje nema naboja, potencijal u njemu dobivamo Laplaceovom jednadžbom

$$\nabla^2 V = 0. \tag{2.357}$$

Na rubovima tog područja, Ω, možemo imati dva tipa rubnih uvjeta. U slučaju kada je potencijal na rubu područja zadan imamo Dirichletov rubni uvjet

$$V|_{\partial\Omega} = V_0(\partial\Omega). \tag{2.358}$$

U slučaju da nam je zadan naboj na rubu područja, s normalom $\hat{n}$, možemo izračunati razliku električnog polja s obje strane ruba, što nas dovodi do izraza

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{n} = \frac{\sigma}{\epsilon_0}. \tag{2.359}$$

Potencijal u tom slučaju zadovoljava Neumannov rubni uvjet

$$\partial_n V_1 - \partial_n V_2 = \frac{\sigma}{\epsilon_0}. \tag{2.360}$$

Primjer 2.6.1. Pronaći potencijal u prostoru između dvaju beskonačnih ploča. Ploča na $x = 0$ je na potencijalu V_1, a ploča na $x = L$ je na potencijalu V_2. Kako izgleda električno polje? Uz pretpostavku da električno polje za $x < 0$ i $x > L$ iščezava, koliki je naboj na svakoj ploči?

Rješenje. Radi se o jednodimenzionalnom problemu, te Laplaceova jednadžba, zajedno s rubnim uvjetima, glasi

$$\partial_x^2 V = 0 \tag{2.361}$$
$$V(0) = V_1 \tag{2.362}$$
$$V(L) = V_2. \tag{2.363}$$

Običnim integriranjem dolazimo do oblika

$$V(x) = Ax + B. \tag{2.364}$$

Iz $V(0) = V_1$ dobivamo $B = V_1$, a iz drugog uvjeta dobivamo $A = (V_2 - V_1)/L$ te je potencijal i pripadno električno polje

$$V(x) = \frac{V_2 - V_1}{L} x + V_1 \tag{2.365}$$
$$\mathbf{E}(x) = -\frac{V_2 - V_1}{L} \hat{x}. \tag{2.366}$$

Normala na $x = 0$ je $\hat{n} = -\hat{x}$, a u $x = L$ normala je $\hat{n} = \hat{x}$ Prema tome, naboji na plohama su

$$\sigma_1 = \epsilon_0(\vec{0} - \mathbf{E}(0)) \cdot (-\hat{x}) = -\epsilon_0 \frac{V_2 - V_1}{L} \tag{2.367}$$
$$\sigma_2 = \epsilon_0(\vec{0} - \mathbf{E}(L)) \cdot (\hat{x}) = \epsilon_0 \frac{V_2 - V_1}{L}. \tag{2.368}$$

Primjer 2.6.2. Prostor je ispunjen gustoćom naboja $\rho = \rho_0 x$ između ploha $x = 0$ i $x = a$. Na rubovima su zadani potencijali $V(x = 0) = 0$ i $V(x = a) = V_0$. Kako izgleda potencijal $V(x)$?

Rješenje. Ovaj zadatak svodi se na rješavanje Poissonove jednadžbe

$$\nabla^2 V = \partial_x^2 V = -\frac{\rho_0}{\epsilon_0} x. \tag{2.369}$$

Rješenje dobivamo dvostrukom integracijom

$$\frac{dV}{dx} = -\frac{\rho_0 x^2}{2\epsilon_0} + A \tag{2.370}$$

$$V = -\frac{\rho_0 x^3}{6\epsilon_0} + Ax + B. \tag{2.371}$$

Rubni uvjeti daju

$$B = 0 \tag{2.372}$$

$$V_0 = -\frac{\rho_0 a^3}{6\epsilon_0} + Aa. \tag{2.373}$$

Rješenje je

$$A = \frac{V_0 + \frac{\rho_0 a^3}{6\epsilon_0}}{a} = \frac{V_0}{a} + \frac{\rho_0 a^2}{6\epsilon_0}. \tag{2.374}$$

Uvrštavanjem dobivamo

$$V = -\frac{\rho_0 x^3}{6\epsilon_0} + \left(\frac{V_0}{a} + \frac{\rho_0 a^2}{6\epsilon_0}\right) x \tag{2.375}$$

$$V = \frac{\rho_0}{6\epsilon_0}(a^2 x - x^3) + V_0 \frac{x}{a}. \tag{2.376}$$

Primjer 2.6.3. Između ploča kondenzatora površine S međusobne udaljenosti $d = a+b$ na udaljenosti a od jedne od njih nalazi se treća ploča ukupnog naboja Q. Koliko je električno polje između ploča kondenzatora ako je jedna od ploča kondenzatora na potencijalu V_0? Kolika je sila na unutarnju ploču?

Rješenje. Prostor možemo podijeliti na dva dijela, $0 < x < a$ i $a + b > x > a$. U oba područja potencijal linearno ovisi o x, uz rubne uvjete $V(0) = 0$, $V(a + b) = V_0$

$$V_1(x) = Ax + B \tag{2.377}$$

$$V_2(x) = Cx + D \tag{2.378}$$

$$V_1(0) = 0 \tag{2.379}$$

$$V_2(a + b) = V_0 \tag{2.380}$$

$$V_2(a) = V_1(a) \tag{2.381}$$

$$\partial_x V_1(a) - \partial_x V_2(a) = \frac{\sigma}{\epsilon_0} \tag{2.382}$$

Uvrštavanjem prva dva rubna uvjeta dobivamo

$$V_1(x) = Ax \tag{2.383}$$
$$V_2(x) = C(x - (a + b)) + V_0. \tag{2.384}$$

Druga dva rubna uvjeta daju jednadžbe

$$Aa + Cb = V_0 \tag{2.385}$$
$$A - C = \frac{\sigma}{\epsilon_0} \tag{2.386}$$

Te vrijedi $A = \frac{1}{a+b}(\frac{\sigma b}{\epsilon_0} + V_0)$ i $C = \frac{1}{a+b}(V_0 - \frac{\sigma a}{\epsilon_0})$, odnosno električna polja su

$$\mathbf{E}_1 = \left(-\frac{V_0}{d} - \frac{b}{d}\frac{Q}{\epsilon_0 S}\right)\hat{x} \tag{2.387}$$
$$\mathbf{E}_2 = \left(-\frac{V_0}{d} + \frac{a}{d}\frac{Q}{\epsilon_0 S}\right)\hat{x} \tag{2.388}$$

Ukupna sila je umnožak naboja na ploči i srednje vrijednosti električnih polja s obje strane ploče

$$\mathbf{F} = \frac{Q}{2}\left(-\frac{2V_0}{d} + \frac{a-b}{d}\frac{Q}{\epsilon_0 S}\right)\hat{x}. \tag{2.389}$$

2.7 Elektrostatika u sredstvu. Polarizacija

U sredstvu uz slobodne naboje ρ_f i σ_f postoje i vezani naboji, ρ_b i σ_b. Vezani naboji određeni su ako poznajemo polarizaciju, $\mathbf{P}$ - dipolni moment po jedinici volumena - na analogni način s električnim poljem

$$\rho_b = -\nabla \cdot \mathbf{P} \tag{2.390}$$

$$\sigma_b = \mathbf{P} \cdot \hat{n}. \tag{2.391}$$

Ako rastavimo gustoću naboja na slobodnu i vezanu, uviđamo da su električno polje i polarizacija povezani s trećim vektorom, $\mathbf{D}$ - dielektričnim pomakom, koji opisuje slobodni naboj

$$\rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P} = \epsilon_0 \nabla \cdot \mathbf{E} \rightarrow \rho_f = \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \nabla \cdot \mathbf{D}. \tag{2.392}$$

Generalno, skup jednadžbi koji opisuju elektrostatiku u sredstvu je

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{2.393}$$

$$\nabla \cdot \mathbf{D} = \rho_f \tag{2.394}$$

$$\nabla \cdot \mathbf{P} = -\rho_b \tag{2.395}$$

$$\nabla \times \mathbf{D} = \nabla \times \mathbf{P}, \tag{2.396}$$

dok su rubni uvjeti na granici sredstava 1 i 2

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{n} = \sigma_f \tag{2.397}$$

$$(\mathbf{D}_2 - \mathbf{D}_1) \times \hat{n} = (\mathbf{P}_2 - \mathbf{P}_1) \times \hat{n} \tag{2.398}$$

U slučaju linearnih dielektrika, polarizacija je povezana s električnim poljem električnom susceptibilnošću χ_e

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}. \tag{2.399}$$

Uz ovaj uvjet, možemo definirati relativnu permitivnost uvrštavajući izraz za linearne dielektrike u definiciju dielektričnog pomaka

$$\mathbf{D} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E}. \tag{2.400}$$

U ovom slučaju vrijedi $\nabla \times \mathbf{P} = \epsilon \nabla \times \mathbf{E} = 0$ te imamo pojednostavljeni skup jednadžbi i rubnih uvjeta

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \tag{2.401}$$

$$\mathbf{D} = \epsilon \mathbf{E} \tag{2.402}$$

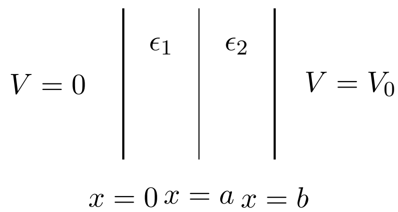
$$\nabla \cdot \mathbf{D} = \rho_f \tag{2.403}$$

$$\nabla \times \mathbf{D} = 0 \tag{2.404}$$

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \hat{n} = (\epsilon_2 \mathbf{E}_2 - \epsilon_1 \mathbf{E}_1) \cdot \hat{n} = \sigma_f \tag{2.405}$$

$$(\mathbf{D}_2 - \mathbf{D}_1) \times \hat{n} = 0. \tag{2.406}$$

Primjer 2.7.1. Prostor između dviju ploča površina S, na udaljenosti b, ravninskog kondenzatora ispunjen je polovično dielektrikom ϵ_1 do udaljenosti a te dielektrikom ϵ_2 do udaljenosti b paralelno s pločama. Pronaći potencijal, električno polje, dielektrični pomak, polarizaciju, slobodne i vezane naboje te kapacitet ovog sustava.



Rješenje. Potencijal za $x < a$ prozovimo V_1, a potencijal za $x > a$ prozovimo V_2. Budući da je Laplacevoa jednadžba u jednoj dimenziji

$$\nabla^2 V = \partial_x^2 V = 0, \tag{2.407}$$

rješenja su

$$V_1(x) = Ax + B \tag{2.408}$$
$$V_2(x) = Cx + D, \tag{2.409}$$

uz rubne uvjete

$$V_1(0) = 0 \tag{2.410}$$
$$V_2(b) = V_0 \tag{2.411}$$
$$V_1(a) = V_2(a) \tag{2.412}$$
$$\epsilon_1 \partial_x V_1(a) = \epsilon_2 \partial_x V_2(a). \tag{2.413}$$

Prvi uvjet daje $B = 0$, dok nakon primjene drugog uvjeta dobivamo

$$V_1(x) = Ax \tag{2.414}$$
$$V_2(x) = C(x - b) + V_0. \tag{2.415}$$

Posljednja dva uvjeta daju

$$Aa = C(a - b) + V_0 \tag{2.416}$$
$$\epsilon_1 A = \epsilon_2 C, \tag{2.417}$$

odnosno

$$V_1(x) = \frac{\epsilon_2 V_0}{\epsilon_2 a + \epsilon_1 (b - a)} x \tag{2.418}$$
$$V_2(x) = V_0 + \frac{\epsilon_1 V_0}{\epsilon_2 a + \epsilon_1 (b - a)} (x - b). \tag{2.419}$$

Električno polje dobivamo deriviranjem

$$\mathbf{E}_1(x) = -\nabla V_1 = -\frac{\epsilon_2 V_0}{\epsilon_2 a + \epsilon_1 (b - a)} \hat{x} \tag{2.420}$$
$$\mathbf{E}_2(x) = -\nabla V_2 = -\frac{\epsilon_1 V_0}{\epsilon_2 a + \epsilon_1 (b - a)} \hat{x}. \tag{2.421}$$

Kao što je očekivano, dielektrični pomaci su isti s u oba područja. Dielektrični pomaci i polarizacije su

$$\mathbf{D}_1 = \epsilon_1 \mathbf{E}_1 = -\frac{\epsilon_1 \epsilon_2 V_0}{\epsilon_2 a + \epsilon_1 (b-a)} \hat{x} \tag{2.422}$$

$$\mathbf{D}_2 = \epsilon_2 \mathbf{E}_2 = -\frac{\epsilon_1 \epsilon_2 V_0}{\epsilon_2 a + \epsilon_1 (b-a)} \hat{x} \tag{2.423}$$

$$\mathbf{P}_1 = \mathbf{D}_1 - \epsilon_0 \mathbf{E}_1 = -\frac{(\epsilon_1 - \epsilon_0) \epsilon_2 V_0}{\epsilon_2 a + \epsilon_1 (b-a)} \hat{x} \tag{2.424}$$

$$\mathbf{P}_2 = \mathbf{D}_2 - \epsilon_0 \mathbf{E}_2 = -\frac{(\epsilon_2 - \epsilon_0) \epsilon_1 V_0}{\epsilon_2 a + \epsilon_1 (b-a)} \hat{x}. \tag{2.425}$$

Slobodni površinski naboji na pozicijama $x = 0$, $x = a$ i $x = b$ te volumne gustoće slobodnih naboja su

$$\rho_{f1} = \nabla \cdot \mathbf{D}_1 = 0 \tag{2.426}$$

$$\rho_{f2} = \nabla \cdot \mathbf{D}_2 = 0 \tag{2.427}$$

$$\sigma_{f,x=0} = \hat{x} \cdot (\mathbf{D}_1 - \vec{0}) = -\frac{\epsilon_1 \epsilon_2 V_0}{\epsilon_2 a + \epsilon_1 (b-a)} \tag{2.428}$$

$$\sigma_{f,x=a} = \hat{x} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = 0 \tag{2.429}$$

$$\sigma_{f,x=b} = \hat{x} \cdot (\vec{0} - \mathbf{D}_2) = \frac{\epsilon_1 \epsilon_2 V_0}{\epsilon_2 a + \epsilon_1 (b-a)}. \tag{2.430}$$

Vezani naboji su

$$\rho_{b1} = -\nabla \cdot \mathbf{P}_1 = 0 \tag{2.431}$$

$$\rho_{b2} = -\nabla \cdot \mathbf{P}_2 = 0 \tag{2.432}$$

$$\sigma_{b,x=0} = (-\hat{x}) \cdot \mathbf{P}_1 = \frac{(\epsilon_1 - \epsilon_0) \epsilon_2 V_0}{\epsilon_2 a + \epsilon_1 (b-a)} \tag{2.433}$$

$$\sigma_{b1,x=a} = \hat{x} \cdot \mathbf{P}_1 = -\frac{(\epsilon_1 - \epsilon_0) \epsilon_2 V_0}{\epsilon_2 a + \epsilon_1 (b-a)} \tag{2.434}$$

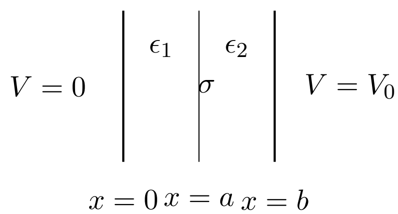
$$\sigma_{b2,x=a} = -\hat{x} \cdot \mathbf{P}_2 = \frac{(\epsilon_2 - \epsilon_0) \epsilon_1 V_0}{\epsilon_2 a + \epsilon_1 (b-a)} \tag{2.435}$$

$$\sigma_{b,x=b} = \hat{x} \cdot \mathbf{P}_2 = -\frac{(\epsilon_2 - \epsilon_0) \epsilon_1 V_0}{\epsilon_2 a + \epsilon_1 (b-a)}. \tag{2.436}$$

Kapacitet dobivamo uvrštavanjem na ploči $x = b$

$$C = \frac{\sigma_{f,x=b} S}{V(x=b) - V(x=0)} = S \frac{\epsilon_1 \epsilon_2}{\epsilon_2 a + \epsilon_1 (b-a)} \tag{2.437}$$

Primjer 2.7.2. Prostor između dviju ploča površina S, na udaljenosti b, ravninskog kondenzatora ispunjen je polovično dielektrikom ϵ_1 do udaljenosti a te dielektrikom ϵ_2 do udaljenosti b paralelno s pločama. Na $x = a$ postavljen je površinski naboj σ. Pronaći potencijal i električno polje ovog sustava, dielektrični pomak, polarizaciju, slobodne i vezane naboje.



Rješenje. Potencijal za $x < a$ prozovimo V_1, a potencijal za $x > a$ prozovimo V_2. Budući da je Laplacevoa jednadžba u jednoj dimenziji

$$\nabla^2 V = \partial_x^2 V = 0, \tag{2.438}$$

rješenja su

$$V_1(x) = Ax + B \tag{2.439}$$
$$V_2(x) = Cx + D, \tag{2.440}$$

uz rubne uvjete

$$V_1(0) = 0 \tag{2.441}$$
$$V_2(b) = V_0 \tag{2.442}$$
$$V_1(a) = V_2(a) \tag{2.443}$$
$$\epsilon_1 \partial_x V_1(a) - \epsilon_2 \partial_x V_2(a) = \sigma \tag{2.444}$$

Prvi uvjet daje $B = 0$, dok nakon primjene drugog uvjeta dobivamo

$$V_1(x) = Ax \tag{2.445}$$
$$V_2(x) = C(x - b) + V_0. \tag{2.446}$$

Posljednja dva uvjeta daju

$$Aa = C(a - b) + V_0 \tag{2.447}$$
$$\epsilon_1 A = \epsilon_2 C + \sigma, \tag{2.448}$$

odnosno

$$V_1(x) = \frac{\epsilon_2 V_0 + \sigma(b - a)}{\epsilon_2 a + \epsilon_1 (b - a)} x \tag{2.449}$$
$$V_2(x) = V_0 + \frac{\epsilon_1 V_0 - \sigma a}{\epsilon_2 a + \epsilon_1 (b - a)} (x - b). \tag{2.450}$$

Električno polje dobivamo deriviranjem

$$\mathbf{E}_1(x) = -\nabla V_1 = -\frac{\epsilon_2 V_0 + \sigma(b-a)}{\epsilon_2 a + \epsilon_1 (b-a)}\hat{x} \tag{2.451}$$

$$\mathbf{E}_2(x) = -\nabla V_2 = -\frac{\epsilon_1 V_0 - \sigma a}{\epsilon_2 a + \epsilon_1 (b-a)}\hat{x}. \tag{2.452}$$

Dielektrični pomaci se razlikuju. Dielektrični pomaci i polarizacije su

$$\mathbf{D}_1 = \epsilon_1 \mathbf{E}_1 = -\frac{\epsilon_1\epsilon_2 V_0 + \epsilon_1\sigma(b-a)}{\epsilon_2 a + \epsilon_1 (b-a)}\hat{x} \tag{2.453}$$

$$\mathbf{D}_2 = \epsilon_2 \mathbf{E}_2 = -\frac{\epsilon_1\epsilon_2 V_0 - \epsilon_2\sigma a}{\epsilon_2 a + \epsilon_1 (b-a)}\hat{x} \tag{2.454}$$

$$\mathbf{P}_1 = \mathbf{D}_1 - \epsilon_0 \mathbf{E}_1 = -\frac{(\epsilon_1 - \epsilon_0)(\epsilon_2 V_0 + \sigma(b-a))}{\epsilon_2 a + \epsilon_1 (b-a)}\hat{x} \tag{2.455}$$

$$\mathbf{P}_2 = \mathbf{D}_2 - \epsilon_0 \mathbf{E}_2 = -\frac{(\epsilon_2 - \epsilon_0)(\epsilon_1 V_0 - \sigma a)}{\epsilon_2 a + \epsilon_1 (b-a)}\hat{x}. \tag{2.456}$$

Slobodni površinski naboji na pozicijama $x = 0$, $x = a$ i $x = b$ te volumne gustoće slobodnih naboja su

$$\rho_{f1} = \nabla \cdot \mathbf{D}_1 = 0 \tag{2.457}$$

$$\rho_{f2} = \nabla \cdot \mathbf{D}_2 = 0 \tag{2.458}$$

$$\sigma_{f,x=0} = \hat{x} \cdot (\mathbf{D}_1 - \vec{0}) = -\frac{\epsilon_1\epsilon_2 V_0 + \epsilon_1\sigma(b-a)}{\epsilon_2 a + \epsilon_1 (b-a)} \tag{2.459}$$

$$\sigma_{f,x=a} = \hat{x} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma \tag{2.460}$$

$$\sigma_{f,x=b} = \hat{x} \cdot (\vec{0} - \mathbf{D}_2) = \epsilon_2 \mathbf{E}_2 = -\frac{\epsilon_1\epsilon_2 V_0 - \epsilon_2\sigma a}{\epsilon_2 a + \epsilon_1 (b-a)}. \tag{2.461}$$

Vezani naboji su

$$\rho_{b1} = -\nabla \cdot \mathbf{P}_1 = 0 \tag{2.462}$$

$$\rho_{b2} = -\nabla \cdot \mathbf{P}_2 = 0 \tag{2.463}$$

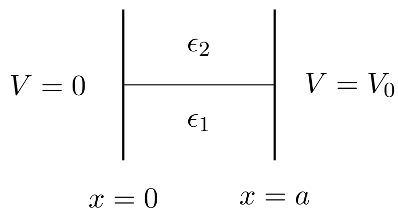
$$\sigma_{b,x=0} = (-\hat{x}) \cdot \mathbf{P}_1 = \frac{(\epsilon_1 - \epsilon_0)(\epsilon_2 V_0 + \sigma(b-a))}{\epsilon_2 a + \epsilon_1 (b-a)} \tag{2.464}$$

$$\sigma_{b1,x=a} = \hat{x} \cdot \mathbf{P}_1 = -\frac{(\epsilon_1 - \epsilon_0)(\epsilon_2 V_0 + \sigma(b-a))}{\epsilon_2 a + \epsilon_1 (b-a)} \tag{2.465}$$

$$\sigma_{b2,x=a} = -\hat{x} \cdot \mathbf{P}_2 = \frac{(\epsilon_1 - \epsilon_0)(\epsilon_2 V_0 - \sigma a)}{\epsilon_2 a + \epsilon_1 (b-a)} \tag{2.466}$$

$$\sigma_{b,x=b} = \hat{x} \cdot \mathbf{P}_2 = -\frac{(\epsilon_2 - \epsilon_0)(\epsilon_1 V_0 - \sigma a)}{\epsilon_2 a + \epsilon_1 (b-a)}. \tag{2.467}$$

Primjer 2.7.3. Prostor između dvije ploče površina S, na udaljenosti a, ravninskog kondenzatora ispunjen je dielektrikom ϵ_1 do polovice visine, dok je ostatak ispunjen dielektrikom ϵ_2. Pronaći potencijal, električno polje, dielektrični pomak, polarizaciju, slobodne i vezane naboje te kapacitet ovog sustava.



Rješenje. Potencijal u donjem prozovimo V_1, a potencijal u gornjem prozovimo V_2. Budući da je Laplacevoa jednadžba u jednoj dimenziji

$$\nabla^2 V = \partial_x^2 V = 0, \tag{2.468}$$

rješenja su

$$V_1(x) = Ax + B \tag{2.469}$$
$$V_2(x) = Cx + D, \tag{2.470}$$

uz rubne uvjete

$$V_1(0) = 0 \tag{2.471}$$
$$V_2(0) = 0 \tag{2.472}$$
$$V_1(a) = V_0 \tag{2.473}$$
$$V_2(a) = V_0 \tag{2.474}$$
$$\mathbf{E}_1 \cdot \hat{x} = \mathbf{E}_2 \cdot \hat{x} \tag{2.475}$$

Prvi i drugi uvjet daju $B = D = 0$, dok nakon primjene posljednjeg uvjeta dobivamo $A = C$. Primjenom uvjeta u $x = a$ dobivamo

$$V_1(x) = V_2(x) = V_0 \frac{x}{a} \tag{2.476}$$
$$\tag{2.477}$$

Električno polje dobivamo deriviranjem

$$\mathbf{E}_1(x) = -\nabla V_1 = -\frac{V_0}{a}\hat{x} \tag{2.478}$$
$$\mathbf{E}_2(x) = -\nabla V_2 = -\frac{V_0}{a}\hat{x}. \tag{2.479}$$

Dielektrični pomaci i polarizacije su

$$\mathbf{D}_1 = \epsilon_1 \mathbf{E}_1 = -\epsilon_1 \frac{V_0}{a}\hat{x} \tag{2.480}$$
$$\mathbf{D}_2 = \epsilon_2 \mathbf{E}_2 = -\epsilon_2 \frac{V_0}{a}\hat{x} \tag{2.481}$$
$$\mathbf{P}_1 = \mathbf{D}_1 - \epsilon_0 \mathbf{E}_1 = -(\epsilon_1 - \epsilon_0)\frac{V_0}{a}\hat{x} \tag{2.482}$$
$$\mathbf{P}_2 = \mathbf{D}_2 - \epsilon_0 \mathbf{E}_2 = -(\epsilon_2 - \epsilon_0)\frac{V_0}{a}\hat{x}. \tag{2.483}$$

Slobodni površinski naboji na pozicijama $x = 0$ i $x = a$ te volumne gustoće slobodnih naboja su

$$\rho_{f1} = \nabla \cdot \mathbf{D}_1 = 0 \tag{2.484}$$

$$\rho_{f2} = \nabla \cdot \mathbf{D}_2 = 0 \tag{2.485}$$

$$\sigma_{f1,x=0} = \hat{x} \cdot (\mathbf{D}_1 - \vec{0}) = -\epsilon_1 \frac{V_0}{a} \hat{x} \tag{2.486}$$

$$\sigma_{f1,x=a} = \hat{x} \cdot (\vec{0} - \mathbf{D}_1) = \epsilon_1 \frac{V_0}{a} \hat{x} \tag{2.487}$$

$$\sigma_{f2,x=0} = \hat{x} \cdot (\mathbf{D}_2 - \vec{0}) = -\epsilon_2 \frac{V_0}{a} \hat{x} \tag{2.488}$$

$$\sigma_{f2,x=a} = \hat{x} \cdot (\vec{0} - \mathbf{D}_2) = \epsilon_2 \frac{V_0}{a} \hat{x} \tag{2.489}$$

$$\tag{2.490}$$

Vezani naboji su

$$\rho_{b1} = -\nabla \cdot \mathbf{P}_1 = 0 \tag{2.491}$$

$$\rho_{b2} = -\nabla \cdot \mathbf{P}_2 = 0 \tag{2.492}$$

$$\sigma_{b1,x=0} = (-\hat{x}) \cdot \mathbf{P}_1 = (\epsilon_1 - \epsilon_0) \frac{V_0}{a} \tag{2.493}$$

$$\sigma_{b1,x=a} = \hat{x} \cdot \mathbf{P}_1 = -(\epsilon_1 - \epsilon_0) \frac{V_0}{a} \tag{2.494}$$

$$\sigma_{b2,x=0} = (-\hat{x}) \cdot \mathbf{P}_2 = (\epsilon_2 - \epsilon_0) \frac{V_0}{a} \tag{2.495}$$

$$\sigma_{b2,x=a} = \hat{x} \cdot \mathbf{P}_2 = -(\epsilon_2 - \epsilon_0) \frac{V_0}{a} \tag{2.496}$$

Kapacitet dobivamo uvrštavanjem

$$C = \frac{\sigma_{f1,x=a} \frac{S}{2} + \sigma_{f2,x=a} \frac{S}{2}}{V(x=a) - V(x=0)} = \frac{S}{a} \frac{\epsilon_1 + \epsilon_2}{2}. \tag{2.497}$$

3. Magnetostatika

3.1 Stacionarne struje

Struje su naboj u gibanju. Analogno gustoćama naboja, možemo definirati struju I i gustoće struja $\mathbf{K}$ i $\mathbf{J}$

$$\mathbf{I} = \lambda \mathbf{v} \tag{3.1}$$

$$\mathbf{K} = \sigma \mathbf{v} \tag{3.2}$$

$$\mathbf{J} = \rho \mathbf{v}, \tag{3.3}$$

dimenzije ovih veličina su $\mathrm{A} = \mathrm{C/s}$, $\mathrm{A}/m = \mathrm{Cs}^{-1}\mathrm{m}^{-1}$ i $\mathrm{A}/m^2 = \mathrm{Cs}^{-1}\mathrm{m}^{-2}$, respektivno. Primijetimo da smo struju prikazali kao vektor. Ovaj smjer odgovara smjeru brzine te ako integriramo duž linijski raspodijeljene struje, dobivamo naboj koji je protekao u jedinici vremena kroz neki dio prostora

$$\frac{dq}{dt} = \mathbf{I} \cdot \hat{v} = I \tag{3.4}$$

$$= \int \mathbf{K} \cdot d\mathbf{r} \tag{3.5}$$

$$= \int \mathbf{J} \cdot d^2\mathbf{r}. \tag{3.6}$$

Budući da je naboj sačuvam vrijedi

$$\frac{d\rho}{dt} = 0. \tag{3.7}$$

Budući da je ρ općenito funkcija položaja i vremena, ovaj diferencijal možemo raspisati koristeći parcijalne derivacije

$$\frac{d\rho}{dt} = \partial_t \rho + \nabla \rho \cdot \frac{d\mathbf{r}}{dt} \tag{3.8}$$

$$= \partial_t \rho + \nabla \cdot \mathbf{J} = 0. \tag{3.9}$$

Ograničimo se na slučaj stacionarnih struja - kada vrijedi da se naboj ne gomila u bilo kojoj točki - $\partial_t \rho = 0$. Ovaj se uvjet svodi na

$$\nabla \cdot \mathbf{J} = 0. \tag{3.10}$$

U integralnom obliku, uvjet stacionarnih struja nam govori da struja niti u jednoj točci prostora ne izvire niti ponire

$$\oint \mathbf{J} \cdot d^2\mathbf{r} = 0. \tag{3.11}$$

U slučaju nekoliko žica $I_1, \cdots, I_n$ koje se spajaju u nekoj točki, uvjet stacionarnosti je zapravo Kirchoffov zakon

$$\sum_i I_i = 0. \tag{3.12}$$

3.2 Magnetsko polje

Ako je poznata distribucija struja, magnetsko polje računamo po Biot-Savartovom zakonu

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{I d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{3.13}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^2 r' \tag{3.14}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r' \tag{3.15}$$

U magnetostatici, divergencija i rotacija magnetskog polja su

$$\nabla \cdot \mathbf{B} = 0 \tag{3.16}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{3.17}$$

U integralnom obliku, dolazimo do Ampereovog zakona

$$\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 I. \tag{3.18}$$

Možemo također definirati i tok magnetskog polja kroz neku površinu, S

$$\Phi_M = \int_S \mathbf{B} \cdot d\mathbf{A}. \tag{3.19}$$

Primjer 3.2.1. Štap se proteže duž z osi od točke z_1 do z_2 i njime teče struja I. Pronaći magnetsko polje u prostoru. Što se događa u slučaju $z_1 \to -\infty$ i $z_2 \to \infty$?

Rješenje. Koordinata točke u kojoj računamo polje je $\mathbf{r} = \rho\hat{\rho} + z\hat{z}$, a koordinata točke na štapu je $\mathbf{r}' = z'\hat{z}$. Primijetimo da smo prekrili cijeli prostor jer štap počinje od proizvoljnog z_1. Računamo

$$d\mathbf{r}' = dz'\hat{z} \tag{3.20}$$

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(\rho\hat{\rho} + (z - z')\hat{z}) \cdot (\rho\hat{\rho} + (z - z')\hat{z})} \tag{3.21}$$

$$= \sqrt{\rho^2 + (z - z')^2} \tag{3.22}$$

Magnetsko polje je

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int\limits_{z_1}^{z_2} I\frac{dz'\hat{z}\times(\rho\hat{\rho}+(z-z')\hat{z})}{(\rho^2+(z-z')^2)^{3/2}} \tag{3.23}$$

$$= \frac{\mu_0 I\rho}{4\pi}\hat{\phi}\int\limits_{z_1}^{z_2}\frac{dz'}{(\rho^2+(z-z')^2)^{3/2}} \tag{3.24}$$

$$= \left|\begin{matrix} u = z'-z \\ du = dz' \\ u\in[z_1-z, z_2-z]\end{matrix}\right| \tag{3.25}$$

$$= \frac{\mu_0 I\rho}{4\pi}\hat{\phi}\int\limits_{z_1-z}^{z_2-z}\frac{du}{(\rho^2+u^2)^{3/2}} \tag{3.26}$$

Koristimo integral

$$\int\frac{du}{(u^2+\rho^2)^{3/2}} = \frac{u}{\rho^2(u^2+\rho^2)^{1/2}}, \tag{3.27}$$

te dobivamo

$$\mathbf{B} = \frac{\mu_0 I\rho}{4\pi}\hat{\phi}\frac{u}{\rho^2\sqrt{\rho^2+u^2}}\Bigg|_{z_1-z}^{z_2-z} \tag{3.28}$$

$$= \frac{\mu_0 I\hat{\phi}}{4\pi\rho}\left(\frac{z_2-z}{\sqrt{(z_2-z)^2+\rho^2}} - \frac{z_1-z}{\sqrt{(z_1-z)^2+\rho^2}}\right). \tag{3.29}$$

U slučaju $z_1\to-\infty$ i $z_2\to\infty$ za $|z|<\infty$ vrijedi $z_1-z\to-\infty$ i $z_2-z\to\infty$ te dobivamo polje beskonačne žice, što možemo vidjeti ako riješimo limese

$$\lim_{z\to\infty}\frac{z}{\sqrt{\rho^2+z^2}} = \lim_{z\to\infty}\frac{z}{|z|\sqrt{1+\frac{\rho^2}{z^2}}} \tag{3.30}$$

$$= \lim_{z\to\infty}\frac{1}{\sqrt{1+\frac{\rho^2}{z^2}}} \tag{3.31}$$

$$= 1 \tag{3.32}$$

$$\lim_{z\to-\infty}\frac{z}{\sqrt{\rho^2+z^2}} = \lim_{z\to-\infty}\frac{z}{|z|\sqrt{1+\frac{\rho^2}{z^2}}} \tag{3.33}$$

$$= \lim_{z\to-\infty}\frac{-1}{\sqrt{1+\frac{\rho^2}{z^2}}} \tag{3.34}$$

$$= -1. \tag{3.35}$$

Uvrštavanjem dobivamo

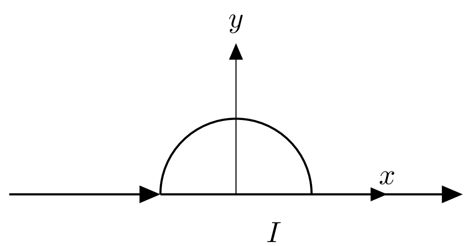
$$\mathbf{B} = \frac{\mu_0 I\hat{\phi}}{4\pi\rho}\left(\lim_{z_2 - z\to\infty}\frac{z_2 - z}{\sqrt{(z_2-z)^2+\rho^2}} - \lim_{(z_1 - z)\to-\infty}\frac{z_1 - z}{\sqrt{(z_1-z)^2+\rho^2}}\right). \tag{3.36}$$

$$= \frac{\mu_0 I\hat{\phi}}{4\pi\rho}(1-(-1)) \tag{3.37}$$

$$= \frac{\mu_0 I\hat{\phi}}{4\pi\rho}2 \tag{3.38}$$

$$= \frac{\mu_0 I}{2\pi\rho}\hat{\phi}. \tag{3.39}$$

Primjer 3.2.2. Struja I teče žicom postavljenom duž x osi. Žica je savijena tako da čini polukrug radijusa a. Izračunati magnetsko polje na z osi. Izračunati posebno za $z = 0$.



Rješenje. Žicu podijelimo na tri dijela. Točka u kojoj računamo polje je $\mathbf{r} = z\hat{z}$

$$
\begin{aligned}
\mathbf{r}_1 &= u\hat{x} && (3.40)\\
d\mathbf{r}_1 &= du\hat{x} && (3.41)\\
\mathbf{r} - \mathbf{r}_1 &= z\hat{z} - u\hat{x} && (3.42)\\
|\mathbf{r} - \mathbf{r}_1| &= \sqrt{(z\hat{z} - u\hat{x}) \cdot (z\hat{z} - u\hat{x})} && (3.43)\\
&= \sqrt{u^2 + z^2} && (3.44)\\
d\mathbf{r}_1 \times (\mathbf{r} - \mathbf{r}_1) &= -zdu\hat{y} && (3.45)\\
\mathbf{r}_2 &= a\cos\phi\hat{x} + a\sin\phi\hat{y} && (3.46)\\
d\mathbf{r}_2 &= (-\sin\phi\hat{x} + \cos\phi\hat{y})ad\phi && (3.47)\\
\mathbf{r} - \mathbf{r}_2 &= z\hat{z} - a\cos\phi\hat{x} - a\sin\phi\hat{y} && (3.48)\\
&= z\hat{z} - a\hat{\rho} && (3.49)\\
|\mathbf{r} - \mathbf{r}_2| &= \sqrt{(z\hat{z} - a\hat{\rho}) \cdot (z\hat{z} - a\hat{\rho})} && (3.50)\\
&= \sqrt{z^2 + a^2} && (3.51)\\
d\mathbf{r}_2 \times (\mathbf{r} - \mathbf{r}_2) &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -a\sin\phi & a\cos\phi & 0 \\ -a\cos\phi & -a\sin\phi & z \end{vmatrix} d\phi && (3.52)\\
&= (z\cos\phi\hat{x} + z\sin\phi\hat{y} + a\hat{z})ad\phi && (3.53)\\
\mathbf{r}_3 &= u\hat{x} && (3.54)\\
d\mathbf{r}_3 &= du\hat{x} && (3.55)\\
\mathbf{r} - \mathbf{r}_3 &= z\hat{z} - u\hat{x} && (3.56)\\
|\mathbf{r} - \mathbf{r}_3| &= \sqrt{(z\hat{z} - u\hat{x}) \cdot (z\hat{z} - u\hat{x})} && (3.57)\\
&= \sqrt{u^2 + z^2} && (3.58)\\
d\mathbf{r}_3 \times (\mathbf{r} - \mathbf{r}_3) &= -zdu\hat{y}. && (3.59)
\end{aligned}
$$

Koristeći integral

$$
\int \frac{du}{(z^2 + u^2)^{3/2}} = \frac{u}{z^2\sqrt{z^2 + u^2}}, \tag{3.60}
$$

magnetsko polje je

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\left(\int\limits_{-\infty}^{-a}\frac{-z\hat{y}du}{(z^2+u^2)^{3/2}} + a\int\limits_{\pi}^{0}\frac{(z\cos\phi\hat{x}+z\sin\phi\hat{y}+a\hat{z})d\phi}{(z^2+a^2)^{3/2}} + \int\limits_{a}^{\infty}\frac{-z\hat{y}du}{(z^2+u^2)^{3/2}}\right) \tag{3.61}$$

$$= \frac{\mu_0 I}{4\pi}\left(2\int\limits_{a}^{\infty}\frac{-z\hat{y}du}{(z^2+u^2)^{3/2}} + \frac{a}{(z^2+a^2)^{3/2}}\left(z\hat{x}\int\limits_{\pi}^{0}\cos\phi d\phi + z\hat{y}\int\limits_{\pi}^{0}\cos\phi d\phi + a\hat{z}\int\limits_{\pi}^{0}d\phi\right)\right) \tag{3.62}$$

$$= \frac{\mu_0 I}{4\pi}\left(-2z\hat{y}\int\limits_{a}^{\infty}\frac{du}{(z^2+u^2)^{3/2}} + \frac{a}{(z^2+a^2)^{3/2}}(0\hat{x}-2z\hat{y}-\pi a\hat{z})\right) \tag{3.63}$$

$$= \frac{\mu_0 I}{4\pi}\left(-2z\hat{y}\frac{u}{z^2\sqrt{z^2+u^2}}\Big|_a^\infty - \frac{\pi a^2\hat{z}+2za\hat{y}}{(z^2+a^2)^{3/2}}\right) \tag{3.64}$$

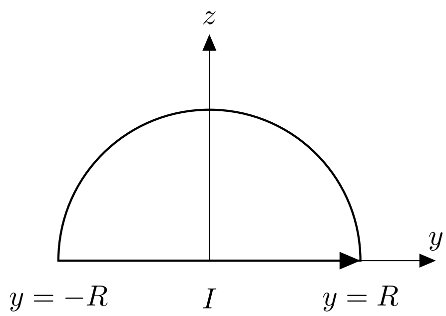
$$= \frac{\mu_0 I}{4\pi}\left(\frac{-2\hat{y}}{z}\left(\lim_{u\to\infty}\frac{1}{\sqrt{1+\frac{x^2}{u^2}}} - \frac{a}{\sqrt{z^2+a^2}}\right) - \frac{\pi a^2\hat{z}+2za\hat{y}}{(z^2+a^2)^{3/2}}\right) \tag{3.65}$$

$$= \frac{\mu_0 I}{4\pi}\left(\frac{-2\hat{y}}{z}\left(1-\frac{a}{\sqrt{z^2+a^2}}\right) - \frac{\pi a^2\hat{z}+2za\hat{y}}{(z^2+a^2)^{3/2}}\right). \tag{3.66}$$

Za $z=0$ komponenta u $\hat{y}$ jednaka je 0 te preostaje

$$\mathbf{B} = -\frac{\mu_0 I\hat{z}}{4a}. \tag{3.67}$$

Primjer 3.2.3. Struja I prolazi petljom prikazanom na slici. Izračunati magnetsko polje na x osi.



Rješenje. Kružni dio, označen s $\mathbf{r}_1$, parametriziramo kutem ϕ, dok ravni dio, označen s $\mathbf{r}_2$, parametriziramo parametrom t. Točka u kojoj računamo polje je $\mathbf{r} = x\hat{x}$

$$\begin{aligned}
\mathbf{r}_1 &= R(\cos\phi\hat{y} + \sin\phi\hat{z}) && (3.68)\\
d\mathbf{r}_1 &= R(-\sin\phi\hat{y} + \cos\phi\hat{z})d\phi && (3.69)\\
\mathbf{r} - \mathbf{r}_1 &= x\hat{x} - R(\cos\phi\hat{y} + \sin\phi\hat{z}) && (3.70)\\
|\mathbf{r} - \mathbf{r}_1| &= \sqrt{(x\hat{x} - R(\cos\phi\hat{y} + \sin\phi\hat{z}))\cdot(x\hat{x} - R(\cos\phi\hat{y} + \sin\phi\hat{z}))} && (3.71)\\
&= \sqrt{x^2 + R^2(\cos^2\phi + \sin^2\phi)} && (3.72)\\
&= \sqrt{x^2 + R^2} && (3.73)\\
\mathbf{r}_2 &= -R\hat{y}(1-t) + tR\hat{y} = R\hat{y}(2t-1) && (3.74)\\
d\mathbf{r}_2 &= 2Rdt\hat{y} && (3.75)\\
\mathbf{r} - \mathbf{r}_2 &= x\hat{x} - R\hat{y}(2t-1) && (3.76)\\
|\mathbf{r} - \mathbf{r}_2| &= \sqrt{(x\hat{x} - R\hat{y}(2t-1))\cdot(x\hat{x} - R\hat{y}(2t-1))} && (3.77)\\
&= \sqrt{x^2 + R^2(2t-1)^2}. && (3.78)
\end{aligned}$$

Vektorski produkti su

$$\begin{aligned}
d\mathbf{r}_1 \times (\mathbf{r} - \mathbf{r}_1) &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & -Rd\phi\sin\phi & Rd\phi\cos\phi \\ x & -R\cos\phi & -R\sin\phi \end{vmatrix} && (3.79)\\
&= (R\hat{x} + x\cos\phi\hat{y} + x\sin\phi\hat{z})Rd\phi && (3.80)\\
d\mathbf{r}_2 \times (\mathbf{r} - \mathbf{r}_2) &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 2Rdt & 0 \\ x & -R(2t-1) & 0 \end{vmatrix} = -2xRdt\hat{z}. && (3.81)
\end{aligned}$$

Koristeći integral

$$\int \frac{du}{R^3\left(\frac{x^2}{R^2} + u^2\right)^{3/2}} = \frac{u}{\frac{x^2}{R^2}\sqrt{\frac{x^2}{R^2} + u^2}}, \tag{3.82}$$

magnetsko polje je tada

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \left(\frac{d\mathbf{r}_1 \times (\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3} + \frac{d\mathbf{r}_2 \times (\mathbf{r} - \mathbf{r}_2)}{|\mathbf{r} - \mathbf{r}_2|^3} \right) \tag{3.83}$$

$$= \frac{\mu_0 I}{4\pi} \left(\int_0^\pi \frac{Rd\phi(R\hat{x} + x\cos\phi\hat{y} + x\sin\phi\hat{z})}{(x^2+R^2)^{3/2}} + \int_0^1 \frac{-2xR\hat{z}dt}{(x^2+(2t-1)^2R^2)^{3/2}} \right) \tag{3.84}$$

$$= \begin{vmatrix} u = 2t-1 \\ du = 2dt \\ -1 \leq u \leq 1 \end{vmatrix} \tag{3.85}$$

$$= \frac{\mu_0 I}{4\pi} \left(\frac{R^2\pi\hat{x} + Rx(\sin\pi - \sin 0)\hat{y} + Rx(\cos 0 - \cos\pi)\hat{z}}{(x^2+R^2)^{3/2}} - xR\hat{z} \int_{-1}^{1} \frac{du}{R^3\left(\frac{x^2}{R^2} + u^2\right)^{3/2}} \right) \tag{3.86}$$

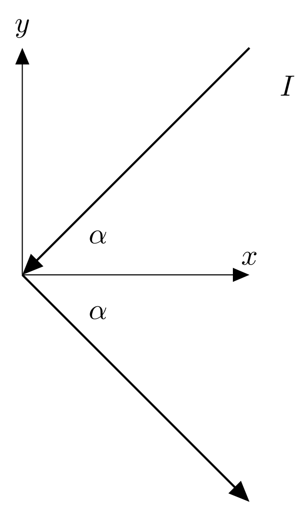
$$= \frac{\mu_0 I}{4\pi} \left(\frac{R^2\pi\hat{x} + 2Rx\hat{z}}{(x^2+R^2)^{3/2}} - \frac{2x\hat{z}}{R^2} \frac{u}{\frac{x^2}{R^2}\sqrt{\frac{x^2}{R^2}+u^2}} \Bigg|_{-1}^{1} \right) \tag{3.87}$$

$$= \frac{\mu_0 I}{4\pi} \left(\frac{R^2\pi\hat{x} + 2Rx\hat{z}}{(x^2+R^2)^{3/2}} - \frac{2R\hat{z}}{x\sqrt{x^2+R^2}} \right) \tag{3.88}$$

$$= \frac{\mu_0 I}{4\pi(x^2+R^2)^{3/2}} \left(R^2\pi\hat{x} + \frac{2R\hat{z}}{x}\left(x^2 - x^2 - R^2\right) \right) \tag{3.89}$$

$$= \frac{\mu_0 I}{4} \frac{R^2}{(x^2+R^2)^{3/2}} \left(\hat{x} - \frac{2R}{\pi x}\hat{z} \right) \tag{3.90}$$

Primjer 3.2.4. Žica kroz koju prolazi struja I zakrivljena je kao na slici. Izračunati magnetsko polje duž x osi.



Rješenje. Gornji dio žice označimo s $\mathbf{r}_1$, a donji s $\mathbf{r}_2$. Točka u kojoj računamo polje je

$\mathbf{r} = x\hat{x}$.

$$\mathbf{r}_1 = R\cos\alpha\hat{x} + R\sin\alpha\hat{y} \tag{3.91}$$
$$d\mathbf{r}_1 = (\cos\alpha\hat{x} + \sin\alpha\hat{y})dR \tag{3.92}$$
$$\mathbf{r} - \mathbf{r}_1 = (x - R\cos\alpha)\hat{x} - R\sin\alpha\hat{y} \tag{3.93}$$
$$|\mathbf{r} - \mathbf{r}_1| = \sqrt{((x - R\cos\alpha)\hat{x} - R\sin\alpha\hat{y}) \cdot ((x - R\cos\alpha)\hat{x} - R\sin\alpha\hat{y})} \tag{3.94}$$
$$= \sqrt{(x - R\cos\alpha)^2 + R^2\sin^2\alpha} \tag{3.95}$$
$$= \sqrt{x^2 - 2xR\cos\alpha + R^2} \tag{3.96}$$
$$d\mathbf{r}_1 \times (\mathbf{r} - \mathbf{r}_1) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos\alpha & \sin\alpha & 0 \\ x - R\cos\alpha & -R\sin\alpha & 0 \end{vmatrix} dR \tag{3.97}$$
$$= -\hat{z}x\sin\alpha dR \tag{3.98}$$
$$\mathbf{r}_2 = R\cos\alpha\hat{x} - R\sin\alpha\hat{y} \tag{3.99}$$
$$d\mathbf{r}_2 = (\cos\alpha\hat{x} - \sin\alpha\hat{y})dR \tag{3.100}$$
$$\mathbf{r} - \mathbf{r}_2 = (x - R\cos\alpha)\hat{x} + R\sin\alpha\hat{y} \tag{3.101}$$
$$|\mathbf{r} - \mathbf{r}_2| = \sqrt{((x - R\cos\alpha)\hat{x} + R\sin\alpha\hat{y}) \cdot ((x - R\cos\alpha)\hat{x} + R\sin\alpha\hat{y})} \tag{3.102}$$
$$= \sqrt{(x - R\cos\alpha)^2 + R^2\sin^2\alpha} \tag{3.103}$$
$$= \sqrt{x^2 - 2Rx\cos\alpha + R^2\cos^2\alpha + R^2\sin^2\alpha} \tag{3.104}$$
$$= \sqrt{x^2 - 2xR\cos\alpha + R^2} \tag{3.105}$$
$$d\mathbf{r}_2 \times (\mathbf{r} - \mathbf{r}_2) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos\alpha & -\sin\alpha & 0 \\ x - R\cos\alpha & R\sin\alpha & 0 \end{vmatrix} dR \tag{3.106}$$
$$= \hat{z}x\sin\alpha dR. \tag{3.107}$$

U gornjem dijelu integriramo od $R = \infty$ do $R = 0$, a u donjem od $R = 0$ do $R = \infty$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\left(-\int_\infty^0 \frac{\hat{z}x\sin\alpha dR}{(x^2 - 2xR\cos\alpha + R^2)^{3/2}} + \int_0^\infty \frac{\hat{z}x\sin\alpha dR}{(x^2 - 2xR\cos\alpha + R^2)^{3/2}}\right) \tag{3.108}$$
$$= \frac{\mu_0 I}{2\pi}\hat{z}x\sin\alpha\int_0^\infty \frac{dR}{(x^2 - 2xR\cos\alpha + R^2)^{3/2}}. \tag{3.109}$$

Koristimo integral

$$\int \frac{dR}{(aR^2 + bR + c)^{3/2}} = \frac{4aR + 2b}{(4ac - b^2)(aR^2 + bR + c)^{1/2}}, \tag{3.110}$$

uz pokrate

$$a = 1 \tag{3.111}$$
$$b = -2x\cos\alpha \tag{3.112}$$
$$c = x^2 \tag{3.113}$$
$$4ac - b^2 = 4x^2 - 4x^2\cos^2\alpha \tag{3.114}$$
$$4aR + 2b = 4R - 4x\cos\alpha, \tag{3.115}$$

dobivamo

$$= \frac{\mu_0 I \sin\alpha x\hat{z}}{2\pi}\frac{4R - 4x\cos\alpha}{4x^2 - 4x^2\cos^2\alpha}\frac{1}{\sqrt{x^2 - 2xR\cos\alpha + R^2}}\Big|_0^\infty \tag{3.116}$$
$$= \frac{\mu_0 I \sin\alpha\hat{z}}{2\pi x(1-\cos^2\alpha)}\frac{R - x\cos\alpha}{\sqrt{x^2 - 2xR\cos\alpha + R^2}}\Big|_0^\infty \tag{3.117}$$
$$= \frac{\mu_0 I \sin\alpha\hat{z}}{2\pi x(1-\cos^2\alpha)}\left(\lim_{R\to\infty}\frac{1 - \frac{x}{R}\cos\alpha}{\sqrt{\frac{x^2}{R^2} + 1 - 2\frac{x}{R}\cos\alpha}} + \frac{x\cos\alpha}{|x|}\right) \tag{3.118}$$
$$= \frac{\mu_0 I \sin\alpha\hat{z}}{2\pi x(1-\cos^2\alpha)}\begin{cases}1+\cos\alpha, & x>0\\ 1-\cos\alpha, & x<0\end{cases} \tag{3.119}$$
$$= \frac{\mu_0 I\hat{z}}{2\pi x}\begin{cases}\frac{\sin\alpha}{1-\cos\alpha}, & x>0\\ \frac{\sin\alpha}{1+\cos\alpha}, & x<0\end{cases} \tag{3.120}$$
$$= \frac{\mu_0 I\hat{z}}{2\pi}\begin{cases}\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1-1+2\sin^2\frac{\alpha}{2}}\frac{1}{x}, & x>0\\ \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1+2\cos^2\frac{\alpha}{2}-1}\frac{1}{x}, & x<0\end{cases} \tag{3.121}$$
$$= \frac{\mu_0 I\hat{z}}{2\pi}\begin{cases}\cot\frac{\alpha}{2}\frac{1}{|x|}, & x>0\\ \tan\frac{\alpha}{2}\frac{-1}{|x|}, & x<0\end{cases} \tag{3.122}$$
$$= \frac{\mu_0 I\hat{z}}{2\pi|x|}\begin{cases}\cot\frac{\alpha}{2}, & x>0\\ -\tan\frac{\alpha}{2}, & x<0\end{cases}. \tag{3.123}$$

3.3 Vektorski potencijal magnetskog polja

Za magnetsko polje također možemo uvesti potencijal, $\mathbf{A}$ definiran izrazom

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{3.124}$$

Uvrštavanjem u Maxwellove jednadžbe, dobivamo sljedeći izraz za vektorski potencijal raspodjele struje

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{I d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \tag{3.125}$$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{|\mathbf{r} - \mathbf{r}'|} d^2 r' \tag{3.126}$$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{|\mathbf{r} - \mathbf{r}'|} d^3 r' \tag{3.127}$$

Približni vektorski potencijal i magnetsko polje možemo izračunati koristeći magnetski dipolni moment. On je jednak

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r} \times I d\mathbf{r} \tag{3.128}$$

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r} \times \mathbf{K} d^2 r \tag{3.129}$$

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r} \times \mathbf{J} d^3 r. \tag{3.130}$$

Ovi izrazi nam omogućavaju jednostavno računanje približnog vektorskog potencijala i magnetskog polja

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3} \tag{3.131}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{3(\mathbf{m} \cdot \mathbf{r})\mathbf{r} - \mathbf{m} r^2}{r^5} \tag{3.132}$$

Primjer 3.3.1. Žicom koja se proteže od $z = -L/2$ do $z = L/2$ teče struja I. Pronaći vektorski potencijal i magnetsko polje u prostoru za $z = 0$. Pokazati da se za veliki L magnetsko polje svodi na magnetsko polje beskonačne žice.

Rješenje. Izrazi korisni u proračunu su

$$\mathbf{r} = z\hat{z} + \rho\hat{\rho} \tag{3.133}$$

$$\mathbf{r}' = z'\hat{z}' \tag{3.134}$$

$$\mathbf{r} - \mathbf{r}' = \rho\hat{\rho} + (z - z')\hat{z} \tag{3.135}$$

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{\rho^2 + (z - z')^2} \tag{3.136}$$

$$d\mathbf{r}' = z'\hat{z} \tag{3.137}$$

Vektorski potencijal dan je izrazom

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int\limits_{-L/2}^{L/2} \frac{I dz' \hat{z}}{\sqrt{\rho^2 + (z - z')^2}} \tag{3.138}$$

$$= \frac{\mu_0 I \hat{z}}{4\pi} \int\limits_{z-L/2}^{z+L/2} \frac{d\zeta}{\sqrt{\zeta^2 + \rho^2}} \tag{3.139}$$

$$= \frac{\mu_0 I \hat{z}}{4\pi} \ln \frac{\sqrt{\left(z + \frac{L}{2}\right)^2 + \rho^2} + z + \frac{L}{2}}{\sqrt{\left(z - \frac{L}{2}\right)^2 + \rho^2} + z - \frac{L}{2}}. \tag{3.140}$$

Magnetsko polje dobivamo rotacijom

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3.141}$$

$$= \hat{\rho}\left(\frac{1}{\rho}\partial_\phi A^z - \partial_z A^\phi\right) + \hat{\phi}\left(\partial_z A^\rho - \partial_\rho A^z\right) + \frac{\hat{z}}{\rho}\left(\partial_\rho(\rho A^\phi) - \partial_\phi A^\rho\right) \tag{3.142}$$

$$= -\partial_\rho A^z \hat{\phi} \tag{3.143}$$

$$= \frac{-\mu_0 I \hat{\phi}}{4\pi}\left(\frac{1}{\sqrt{\left(z + \frac{L}{2}\right)^2 + \rho^2} + z + \frac{L}{2}} \frac{\rho}{\sqrt{\left(z + \frac{L}{2}\right)^2 + \rho^2}} - \frac{1}{\sqrt{\left(z - \frac{L}{2}\right)^2 + \rho^2} + z - \frac{L}{2}} \frac{\rho}{\sqrt{\left(z - \frac{L}{2}\right)^2 + \rho^2}}\right). \tag{3.144}$$

Na $z = 0$ imamo

$$\mathbf{B}(z=0) = \frac{-\mu_0 I \hat{\phi}}{4\pi}\left(\frac{1}{\sqrt{\left(\frac{L}{2}\right)^2 + \rho^2} + \frac{L}{2}} \frac{\rho}{\sqrt{\left(\frac{L}{2}\right)^2 + \rho^2}} - \frac{1}{\sqrt{\left(\frac{L}{2}\right)^2 + \rho^2} - \frac{L}{2}} \frac{\rho}{\sqrt{\left(\frac{L}{2}\right)^2 + \rho^2}}\right) \tag{3.145}$$

$$= \frac{-\mu_0 I \hat{\phi}}{4\pi} \frac{\rho}{\sqrt{\left(\frac{L}{2}\right)^2 + \rho^2}} \left(\frac{\sqrt{\left(\frac{L}{2}\right)^2 + \rho^2} - \frac{L}{2} - \sqrt{\left(\frac{L}{2}\right)^2 + \rho^2} - \frac{L}{2}}{\rho^2 + \frac{L^2}{4} - \frac{L^2}{4}}\right) \tag{3.146}$$

$$= \frac{\mu_0 I \hat{\phi}}{2\pi\rho} \frac{1}{\sqrt{1 + \left(\frac{2\rho}{L}\right)^2}}. \tag{3.147}$$

U slučaju jako duge žice ($\rho << L/2$) dobivamo magnetsko polje beskonačno duge žice

$$\mathbf{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}. \tag{3.148}$$

Primjer 3.3.2. Izračunati magnetske dipolne momente kružne petlje radijusa R i eliptične petlje poluosi a i b. Izračunati polja $\mathbf{A}$ i $\mathbf{B}$.

Rješenje. U prvom slučaju relevantne veličine su

$$\mathbf{r} = R(\cos\phi\hat{x} + \sin\phi\hat{y}) = R\hat{\rho} \tag{3.149}$$

$$d\mathbf{r} = R(-\sin\phi\hat{x} + \cos\phi\hat{y}) = R\hat{\phi} \tag{3.150}$$

te je magnetski moment dan izrazom

$$\mathbf{m} = \frac{1}{2}\int_0^{2\pi} IR\hat{\rho} \times Rd\phi\hat{\phi} \tag{3.151}$$

$$= \frac{R^2 I}{2} 2\pi\hat{z} \tag{3.152}$$

$$= R^2\pi I\hat{z}. \tag{3.153}$$

Vektorski potencijal i magnetsko polje računamo u cilindričnom sustavu

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3} \tag{3.154}$$

$$= \frac{\mu_0 m}{4\pi}\frac{\hat{z}\times(\rho\hat{\rho} + z\hat{z})}{(\rho^2+z^2)^{3/2}} \tag{3.155}$$

$$= \frac{\mu_0 R^2 I}{4}\frac{\rho\hat{\phi}}{(\rho^2+z^2)^{3/2}} \tag{3.156}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r} - \mathbf{m}r^2}{r^5} \tag{3.157}$$

$$= \frac{\mu_0 m}{4\pi}\frac{3z(\rho\hat{\rho}+z\hat{z}) - \hat{z}(z^2+\rho^2)}{(\rho^2+z^2)^{5/2}} \tag{3.158}$$

$$= \frac{\mu_0 R^2 I}{4}\frac{3z\rho\hat{\rho} + (2z^2-\rho^2)\hat{z}}{(\rho^2+z^2)^{5/2}}. \tag{3.159}$$

U drugom slučaju imamo

$$\mathbf{r} = (a\cos\phi\hat{x} + b\sin\phi\hat{y}) \tag{3.160}$$

$$d\mathbf{r} = (-a\sin\phi\hat{x} + b\cos\phi\hat{y})d\phi, \tag{3.161}$$

iz čega zaključujemo

$$\mathbf{m} = \frac{1}{2}\int_0^{2\pi}(a\cos\phi\hat{x} + b\sin\phi\hat{y}) \times I(-a\sin\phi\hat{x} + b\cos\phi\hat{y})d\phi \tag{3.162}$$

$$= \frac{1}{2}\int_0^{2\pi}\left(ab\cos^2\phi\hat{z} + ab\sin^2\phi\hat{z}\right)d\phi \tag{3.163}$$

$$= ab\pi I\hat{z}. \tag{3.164}$$

Vektorski potencijal i magnetsko polje računamo u cilindričnom sustavu

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3} \tag{3.165}$$

$$= \frac{\mu_0 m}{4\pi}\frac{\hat{z}\times(\rho\hat{\rho}+z\hat{z})}{(\rho^2+z^2)^{3/2}} \tag{3.166}$$

$$= \frac{\mu_0 abI}{4}\frac{\rho\hat{\phi}}{(\rho^2+z^2)^{3/2}} \tag{3.167}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}-\mathbf{m}r^2}{r^5} \tag{3.168}$$

$$= \frac{\mu_0 m}{4\pi}\frac{3z(\rho\hat{\rho}+z\hat{z})-\hat{z}(z^2+\rho^2)}{(\rho^2+z^2)^{5/2}} \tag{3.169}$$

$$= \frac{\mu_0 abI}{4}\frac{3z\rho\hat{\rho}+(2z^2-\rho^2)\hat{z}}{(\rho^2+z^2)^{5/2}}. \tag{3.170}$$

4. Elektrodinamika

4.1 Ohmov zakon

Ako na naboj djeluju električna i magnetska polja, dolazi do pojave struje. Proporcionalnost između gustoće struje i sile po jediničnom naboju $\mathbf{f}$ je vodljivost σ

$$\mathbf{J} = \sigma \mathbf{f} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{4.1}$$

Ovaj izraz zove se Ohmov zakon. Vodljivost ima jedinicu S/m = A/V/m. Ekvivalentno, definiramo otpornost

$$\rho = \frac{1}{\sigma}. \tag{4.2}$$

Otpornost mjerimo u mjernim jedinicama Ωm. Promotrimo žicu presjeka A i duljine L. U ovom jednostavnom slučaju kada je električno polje razlika potencijala kroz duljina žice $E = \Delta V/L$. Gustoća struje je $J = I/A$ te dobivamo

$$\frac{I}{A} = \sigma \frac{\Delta V}{L} \rightarrow V = \frac{L}{\sigma A} I = RI, \tag{4.3}$$

gdje smo uveli otpor

$$R = \rho \frac{L}{A}, \tag{4.4}$$

koji se mjeri u jedinicama Ω.

Budući da smo povezali električno polje i gustoću struje, možemo rješavati Laplaceovu jednadžbu za potencijal, pronaći električno polje te množenjem s vodljivošću pronaći struju. Na granici dvaju vodljivosti, rubni uvjet je da nema nakupljanja naboja, što možemo ekvivalentno izraziti

$$\hat{n} \cdot \mathbf{J}_1 = \hat{n} \cdot \mathbf{J}_2 \tag{4.5}$$

$$\sigma_1 \hat{n} \cdot \mathbf{E}_1 = \sigma_2 \hat{n} \cdot \mathbf{E}_2 \tag{4.6}$$

$$\sigma_1 \partial_n \phi_1 = \sigma_2 \partial_n \phi_2. \tag{4.7}$$

U slučaju da struja nalijeće na nevodljivi dio prostora, rubni uvjet je $\partial_n \phi_1 = 0$.

Promotrimo najjednostavniji slučaj, točkasti izvor struje I, primjerice žicu spojenu na savršeno vodljivu kuglu na poziciji $\mathbf{r}'$, koja se tada širi cijelim prostorom vodljivosti σ. Gustoća struje će biti jednaka u svim smjerovima te vrijedi

$$\oint \mathbf{J} \cdot d^2\mathbf{r} = 4\pi J r^2 = I, \tag{4.8}$$

odnosno

$$\mathbf{J} = \frac{I}{4\pi} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \tag{4.9}$$

$$\mathbf{E} = \frac{I}{4\pi\sigma} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \tag{4.10}$$

$$\phi = \frac{I}{4\pi\sigma|\mathbf{r} - \mathbf{r}'|}. \tag{4.11}$$

4.2 Strujni krugovi

U strujnim krugovima možemo imati otpornike, kondenzatore i zavojnice. Promjena napona na zavojnici jednaka je $-LdI/dt$, dok je promjena napona na kondenzatoru Q/C. Sveukupno za RLC strujni krug s izvorom $V_0 = const$ vrijedi

$$V_0 - L\frac{dI}{dt} - \frac{Q}{C} - IR = 0. \tag{4.12}$$

Uvrštavanjem $I = dQ/dt$ dobivamo jednadžbu za naboj

$$V_0 - \frac{d^2Q}{dt^2} - \frac{R}{L}\frac{dQ}{dt} - \frac{1}{LC}Q = 0. \tag{4.13}$$

Ako pak izvor ima vremensku ovisnost, $V_0 = V_0 e^{i\omega t}$, možemo pretpostaviti da struja i napon ovise u vremenu kao $e^{i\omega t}$. Izražavajući sve preko struja, dobivamo

$$\frac{d(Ie^{i\omega t})}{dt} = i\omega I(t) \tag{4.14}$$

$$Q = \int I(t)dt = \frac{1}{i\omega}I(t). \tag{4.15}$$

Odnosno, jednadžbu za napon možemo zapisati

$$V_0 - (i\omega L + \frac{1}{i\omega C} + R)I = V_0 - IZ, \tag{4.16}$$

gdje smo uveli impedanciju zavojnice i kondenzatora

$$Z_R = R \tag{4.17}$$

$$Z_L = i\omega L \tag{4.18}$$

$$Z_C = \frac{1}{i\omega C}. \tag{4.19}$$

Primjer 4.2.1. U strujnom krugu nalaze se zavonjica induktiviteta L, kondenzator kapaciteta C i naboja q_0. Kako izgleda vremenska ovisnost naboja i struje u ovom strujnom krugu nakon zatvaranja strujnog kruga?

Rješenje. Jednadžba strujnog kruga je

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0. \tag{4.20}$$

Radi se o jednadžbi harmoničkog oscilatora

$$Q'' = -\omega_0^2 Q, \tag{4.21}$$

sa

$$\omega_0 = \frac{1}{\sqrt{LC}}. \tag{4.22}$$

Općenito rješenje je

$$Q(t) = A\cos\omega_0 t + B\sin\omega_0 t \tag{4.23}$$

$$I(t) = -\omega_0 A\sin\omega_0 t + B\omega_0\cos\omega_0 t \tag{4.24}$$

U $t = 0$ vrijedi $Q = q_0$, $I = 0$ te je

$$q_0 = A \tag{4.25}$$

$$0 = B\omega_0, \tag{4.26}$$

odnosno

$$Q(t) = q_0\cos\omega_0 t \tag{4.27}$$

$$I(t) = -\omega_0 q_0\sin\omega_0 t. \tag{4.28}$$

Primjer 4.2.2. U strujnom krugu nalaze se otpornik otpora R, kondenzator kapaciteta C i naboja q_0. Kako izgleda vremenska ovisnost naboja i struje u ovom strujnom krugu nakon zatvaranja strujnog kruga?

Rješenje. Jednadžba strujnog kruga je

$$R\frac{dQ}{dt} + \frac{Q}{C} = 0, \tag{4.29}$$

koju možemo jednostavno riješiti

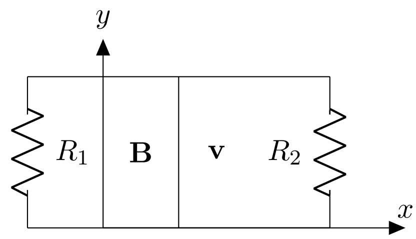
$$\frac{dQ}{Q} = -\frac{dt}{RC} \tag{4.30}$$

$$\ln\frac{Q(t)}{q_0} = \frac{-t}{RC} \tag{4.31}$$

$$Q(t) = q_0 e^{-\frac{t}{RC}} \tag{4.32}$$

$$I(t) = \frac{-q_0}{RC}e^{-\frac{t}{RC}}. \tag{4.33}$$

Primjer 4.2.3. Štap duljine l klizi po strujnom krugu na slici, $\mathbf{B} = -B\hat{z}$. Izračunati struju koja prolazi kroz štap koji klizi brzinom $\mathbf{v}$ prema otporniku 2.



Rješenje. Štap u trenutku t opisuje površinu

$$\mathbf{r} = wvt\hat{x} + lu\hat{y} \tag{4.34}$$

Tok magnetskog polja i elektromotorna sila su

$$\begin{aligned}
\Phi &= \int \mathbf{B} \cdot d^2\mathbf{r} &(4.35)\\
&= \int_0^1 \int_0^1 (-B\hat{z}) \cdot (\partial_w \mathbf{r} \times \partial_u \mathbf{r}) dwdu &(4.36)\\
&= B(-\hat{z}) \cdot (vt\hat{x} \times l\hat{y}) \int_0^1 \int_0^1 dwdu &(4.37)\\
&= -vtlB &(4.38)\\
\mathcal{E} &= -\frac{d\Phi}{dt} = vlB. &(4.39)
\end{aligned}$$

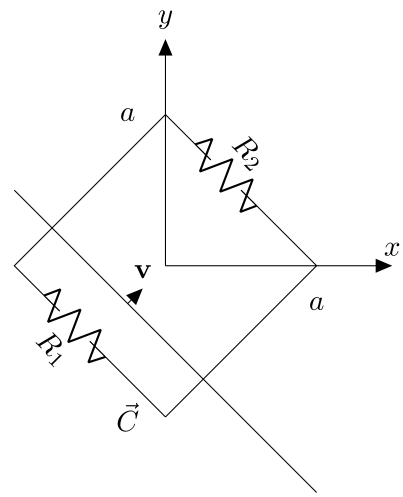
Otpornici su zapravo spojeni paralelno s elektromotornom silom kao izvorom. Tada vrijedi

$$\begin{aligned}
\mathcal{E} - I_i R_i &= 0 &(4.40)\\
I_i &= \frac{\mathcal{E}}{R_i}. &(4.41)
\end{aligned}$$

Struja kroz štap je po Kirchoffovom zakonu zbroj struja

$$I = I_1 + I_2 = Blv \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \tag{4.42}$$

Primjer 4.2.4. Štap klizi po strujnom krugu na slici te se nalazi u magnetskom polju $\mathbf{B} = -B\frac{x+y}{a}\hat{z}$. Napisati parametrizaciju sustava, tok magnetskog polja elektromotornu silu te struju koja prolazi kroz štap koji klizi brzinom $\mathbf{v} = v(\hat{x} + \hat{y})$ prema otporniku 2, a u $t = 0$ prolazi točkom $\vec{C}$. U kojim je trenucima struja maksimalna (u apsolutnom iznosu), a u kojima je jednaka nuli?



Rješenje. Kao pomoć, parametrizirajmo prvo kvadrat počevši od donjeg vrha

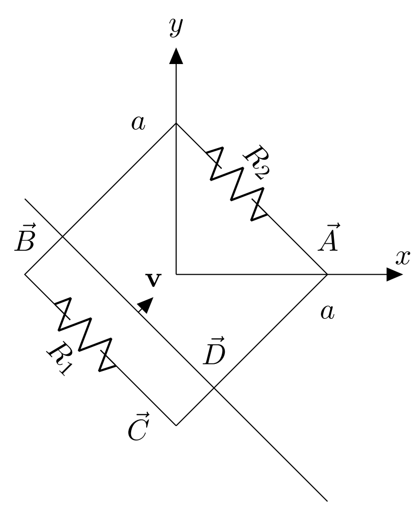
$$\vec{A} = a\hat{x} \tag{4.43}$$

$$\vec{B} = -a\hat{x} \tag{4.44}$$

$$\vec{C} = -a\hat{y} \tag{4.45}$$

$$\mathbf{r}_K = (\vec{A} - \vec{C})w + (\vec{B} - \vec{C})u + \vec{C} \tag{4.46}$$

$$= a(\hat{x} + \hat{y})w + a(-\hat{x} + \hat{y})u - a\hat{y}. \tag{4.47}$$



Kako bismo opisali štap, promotrimo točku $\vec{D}$ koja se u $t = 0$ poklapa s točkom $\vec{C}$, a općenito je možemo zapisati kao $\vec{D}(t) = \mathbf{v}t + \vec{C}$. Stoga vrijedi

$$\begin{aligned} \mathbf{r} &= (\vec{D}(t) - \vec{C})w + (\vec{B} - \vec{C})u + \vec{C} && (4.48) \\ &= vt(\hat{x} + \hat{y})w + a(-\hat{x} + \hat{y})u - a\hat{y}. && (4.49) \end{aligned}$$

Tok magnetskog polja i elektromotorna sila su

$$\mathbf{r} = vt(\hat{x} + \hat{y})w + a(-\hat{x} + \hat{y})u - a\hat{y} = \begin{pmatrix} x = vtw - au \\ y = vtw + au - a \\ z = 0 \end{pmatrix} \tag{4.50}$$

$$\partial_w \mathbf{r} = vt(\hat{x} + \hat{y}) \tag{4.51}$$

$$\partial_u \mathbf{r} = a(-\hat{x} + \hat{y}) \tag{4.52}$$

$$d^2\mathbf{r} = atv(\hat{x} + \hat{y}) \times (-\hat{x} + \hat{y})dudw = 2atv\hat{z}dudw \tag{4.53}$$

$$\mathbf{B} = -B\frac{vtw - au + vtw + au - a}{a}\hat{z} = -B\left(\frac{2vt}{a}w - 1\right)\hat{z} \tag{4.54}$$

$$\begin{aligned} \Phi &= \int_0^1 \int_0^1 -B\left(\frac{2vt}{a}w - 1\right)\hat{z} \cdot 2atv\hat{z}dudw && (4.55) \\ &= -2atvB \int_0^1 \left(\frac{2vt}{a}w - 1\right)dw \int_0^1 du && (4.56) \\ &= -2atvB\left(\frac{vt}{a} - 1\right) && (4.57) \\ &= -2avB\left(\frac{v}{a}t^2 - t\right) && (4.58) \end{aligned}$$

$$\mathcal{E} = 2avB\left(\frac{2v}{a}t - 1\right) \tag{4.59}$$

Otpornici su zapravo spojeni paralelno sa štapom kao izvorom. Tada vrijedi

$$\mathcal{E} - I_i R_i = 0 \tag{4.60}$$

$$I_i = \frac{\mathcal{E}}{R_i}. \tag{4.61}$$

Struja kroz štap (ukupna struja) je po Kirchoffovom zakonu zbroj struja kroz otpornike

$$I = I_1 + I_2 = 2avB\left(\frac{2v}{a}t - 1\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{4.62}$$

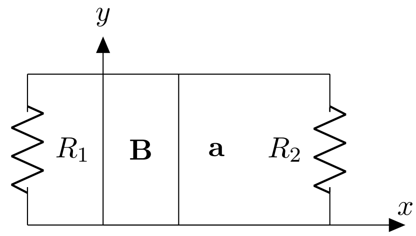
Štap se giba dok ne prođe pravokutnik, odnosno dok točka $\vec{D}$ ne dođe u $a\hat{x}$: $t = a/v$. Struja je jednaka nuli kada je $t = \frac{a}{2v}$ te dobivamo

$$I(t = 0) = -2avB\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{4.63}$$

$$I\left(t = \frac{a}{2v}\right) = 0 \tag{4.64}$$

$$I\left(t = \frac{a}{v}\right) = 2avB\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{4.65}$$

Primjer 4.2.5. Štap duljine l klizi po strujnom krugu na slici, $\mathbf{B} = -B \cdot (y/l) \sin \omega t \hat{z}$. Napisati parametrizaciju sustava, tok magnetskog polja, elektromotornu silu te struju koja prolazi kroz štap koji klizi akceleracijom **a** prema otporniku 2.



Rješenje. Štap u trenutku t opisuje površinu

$$\mathbf{r} = w\frac{at^2}{2}\hat{x} + lu\hat{y} \tag{4.66}$$

Tok magnetskog polja i elektromotorna sila su

$$\mathbf{r} = \frac{at^2}{2}w\hat{x} + l\hat{y}u \tag{4.67}$$

$$\partial_w \mathbf{r} = \frac{at^2}{2}\hat{x} \tag{4.68}$$

$$\partial_u \mathbf{r} = l\hat{y} \tag{4.69}$$

$$d^2\mathbf{r} = \frac{at^2}{2}l\hat{z}dudw \tag{4.70}$$

$$\Phi = \int_0^1 \int_0^1 -B \sin \omega t \frac{lu}{l} \frac{at^2}{2} ldudw \tag{4.71}$$

$$= -Bl \sin \omega t \frac{at^2}{2}\frac{1}{2} \tag{4.72}$$

$$\mathcal{E} = \frac{Bla}{4}(2t \sin \omega t + \omega t^2 \cos \omega t) \tag{4.73}$$

Otpornici su zapravo spojeni paralelno sa štapom kao izvorom. Tada vrijedi

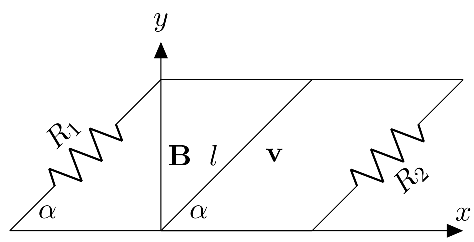
$$\mathcal{E} - I_i R_i = 0 \tag{4.74}$$

$$I_i = \frac{\mathcal{E}}{R_i}. \tag{4.75}$$

Struja kroz štap (ukupna struja) je po Kirchoffovom zakonu zbroj struja kroz otpornike

$$I = I_1 + I_2 = \frac{Bla}{4}(2t \sin \omega t + \omega t^2 \cos \omega t)\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{4.76}$$

Primjer 4.2.6. Štap ukupne duljine l klizi po strujnom krugu na slici, $\mathbf{B} = -B \cdot (y/l)\hat{z}$. Napisati parametrizaciju sustava, tok magnetskog polja, elektromotornu silu te struju koja prolazi kroz štap koji klizi brzinom $\mathbf{v}$ prema otporniku 2.



Rješenje. Štap u trenutku t opisuje površinu koju opisuje vektor

$$\mathbf{r} = vtw\hat{x} + l(\cos\alpha\hat{x} + \sin\alpha\hat{y})u \tag{4.77}$$

Tok magnetskog polja i elektromotorna sila su

$$\mathbf{r} = vtw\hat{x} + l(\cos\alpha\hat{x} + \sin\alpha\hat{y})u \tag{4.78}$$
$$\partial_w \mathbf{r} = vt\hat{x} \tag{4.79}$$
$$\partial_u \mathbf{r} = l(\cos\alpha\hat{x} + \sin\alpha\hat{y}) \tag{4.80}$$
$$d^2\mathbf{r} = vtl\sin\alpha\hat{z}dudw \tag{4.81}$$

$$\Phi = \int_0^1 \int_0^1 -B\frac{l\sin\alpha u}{l} vtl\sin\alpha dudw \tag{4.82}$$
$$= -Bl\sin^2\alpha vt\frac{1}{2} \tag{4.83}$$

$$\mathcal{E} = \frac{Bl\sin^2\alpha v}{2} \tag{4.84}$$

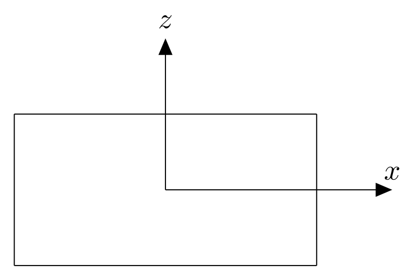
Otpornici su zapravo spojeni paralelno sa štapom kao izvorom. Tada vrijedi

$$\mathcal{E} - I_i R_i = 0 \tag{4.85}$$
$$I_i = \frac{\mathcal{E}}{R_i}. \tag{4.86}$$

Struja kroz štap (ukupna struja) je po Kirchoffovom zakonu zbroj struja kroz otpornike

$$I = I_1 + I_2 = \frac{Bl\sin^2\alpha v}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{4.87}$$

Primjer 4.2.7. Pravokutnik stranica a i b rotira kutnom brzinom $\omega\hat{z}$ kao na slici. Pronaći tok magnetskog polja usmjerenog u $\mathbf{B} = B\hat{y}$ smjeru.



Rješenje. Točke unutar petlje opisujemo koordinatama u i v. Pravokutnik se proteže od $-a/2$ do $a/2$ duž x osi i od $-b/2$ do $b/2$ duž z osi. U trenutku t točka pravokutnika nalazi se u ravnini definiranoj vektorima $\hat{n}(t)$ i $\hat{z}$. Uvedemo li bezdimenzionalne varijable $\xi \in [-1, 1]$ i $\eta \in [-1, 1]$ dobivamo

$$\mathbf{r} = \frac{a}{2}\xi\hat{n}(t) + \frac{b}{2}\eta\hat{z}. \tag{4.88}$$

Budući da je u $t = 0$ pravokutnik u $x - z$ ravnini, vrijedi

$$\hat{n}(t) = \cos\omega t\hat{x} + \sin\omega t\hat{y}. \tag{4.89}$$

Radi jednostavnosti uvedimo, $u \in [0, 1]$ i $v \in [0, 1]$

$$\xi = 2u - 1 \tag{4.90}$$
$$\eta = 2v - 1. \tag{4.91}$$

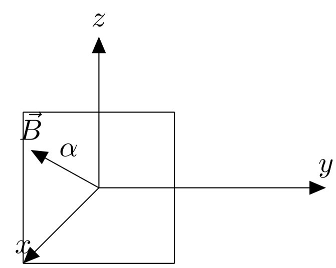
Ovim postupkom dobivamo

$$\mathbf{r} = a(u - 1/2)(\cos\omega t\hat{x} + \sin\omega t\hat{y}) + b(v - 1/2)\hat{z} \tag{4.92}$$
$$\partial_u\mathbf{r} = a(\cos\omega t\hat{x} + \sin\omega t\hat{y}) \tag{4.93}$$
$$\partial_v\mathbf{r} = b\hat{z} \tag{4.94}$$
$$d^2\mathbf{r} = (a(\cos\omega t\hat{x} + \sin\omega t\hat{y}) \times b\hat{z})dudv = ab(-\cos\omega t\hat{y} + \sin\omega t\hat{x})dudv. \tag{4.95}$$

Tok i elektromotorna sila su tada

$$\Phi = \int_0^1 \int_0^1 (B\hat{y}) \cdot d^2\mathbf{r} \tag{4.96}$$
$$= -\cos\omega t abB \int_0^1 \int_0^1 dudv \tag{4.97}$$
$$= -\cos\omega t abB \tag{4.98}$$
$$\mathcal{E} = -\omega ab \sin\omega t B. \tag{4.99}$$

Primjer 4.2.8. Kvadrat stranice a rotira kutnom brzinom $\omega\hat{z}$ oko središta kvadrata, dok se magnetsko polje nalazi pod kutem α u odnosu na z os u $x-z$ ravnini. Izračunati elektromotornu silu. Kolika je srednja snaga ako kvadrat ima otpor R?



Rješenje. Točke unutar petlje opisujemo koordinatama u i v. Kvadrat se proteže od $-a/2$ do $a/2$ duž x osi i od $-a/2$ do $a/2$ duž z osi. U trenutku t točka pravokutnika nalazi se u ravnini definiranoj vektorima $\hat{n}(t)$ i $\hat{z}$. Uvedemo li bezdimenzionalne varijable $\xi \in [-1,1]$ i $\eta \in [-1,1]$ dobivamo

$$\mathbf{r} = \frac{a}{2}\xi\hat{n}(t) + \frac{a}{2}\eta\hat{z}. \tag{4.100}$$

Budući da je u $t=0$ kvadrat u $y-z$ ravnini, vrijedi

$$\hat{n}(t) = -\sin\omega t\hat{x} + \cos\omega t\hat{y}. \tag{4.101}$$

Radi jednostavnosti uvedimo, $u \in [0,1]$ i $v \in [0,1]$

$$\xi = 2u - 1 \tag{4.102}$$
$$\eta = 2v - 1. \tag{4.103}$$

Pomoću u i v dobivamo

$$\mathbf{r} = a(u-1/2)(\cos\omega t\hat{y} - \sin\omega t\hat{x}) + a(v-1/2)\hat{z} \tag{4.104}$$
$$\partial_u\mathbf{r} = a(\cos\omega t\hat{y} - \sin\omega t\hat{x}) \tag{4.105}$$
$$\partial_v\mathbf{r} = a\hat{z} \tag{4.106}$$
$$d^2\mathbf{r} = (a(\cos\omega t\hat{y} - \sin\omega t\hat{x}) \times a\hat{z})dudv = a^2(\cos\omega t\hat{x} + \sin\omega t\hat{y})dudv. \tag{4.107}$$

Magnetsko polje je $\mathbf{B} = B(\cos\alpha\hat{z} + \sin\alpha\hat{x})$. Tok i elektromotorna sila su

$$\Phi = \int_0^1\int_0^1 B(\cos\alpha\hat{z} + \sin\alpha\hat{x}) \cdot a^2(\cos\omega t\hat{x} + \sin\omega t\hat{y})dudv \tag{4.108}$$
$$= a^2 B\sin\alpha\cos\omega t \tag{4.109}$$
$$\mathcal{E} = \omega a^2 B\sin\alpha\sin\omega t \tag{4.110}$$

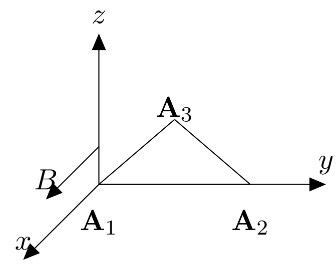
Srednju snagu računamo za jedan period $T = 2\pi/\omega$

$$\langle P \rangle = \frac{\langle \mathcal{E}^2 \rangle}{R} \tag{4.111}$$
$$= \frac{1}{TR} \int_0^T \mathcal{E}^2 dt \tag{4.112}$$
$$= \frac{1}{TR} \int_0^{2\pi} \frac{dx}{\omega} (\omega a^2 B \sin\alpha)^2 \sin^2 x \tag{4.113}$$
$$= \frac{\omega}{2\pi R} \int_0^{2\pi} \frac{dx}{\omega} (\omega^2 a^4 \sin^2 \alpha B^2) \frac{1 - \cos 2x}{2} \tag{4.114}$$
$$= \frac{a^4 \sin^2 \alpha B^2 \omega^2}{2\pi R} \frac{2\pi}{2} \tag{4.115}$$
$$= \frac{\omega^2 B^2 a^4 \sin^2 \alpha}{2R}. \tag{4.116}$$

Primijetimo da vrijedi

$$\langle \cos^2 \omega t \rangle = \frac{1}{2} \tag{4.117}$$
$$\langle \sin^2 \omega t \rangle = \frac{1}{2} \tag{4.118}$$

Primjer 4.2.9. Jednakostranični trokut stranice a rotira kutnom brzinom $\omega\hat{z}$ dok se magnetsko polje nalazi u $\hat{x}$ smjeru. Izračunati elektromotornu silu. Kolika je srednja snaga ako trokut ima otpor R?



Rješenje. U trenutku t točke trokuta nalaze se u ravnini definiranoj vektorima $\hat{n}(t)$ i $\hat{z}$. Koordinate točaka trokuta su

$$\mathbf{A}_1 = \vec{0} \tag{4.119}$$

$$\mathbf{A}_2 = a\hat{n}(t) \tag{4.120}$$

$$\mathbf{A}_3 = \frac{a}{2}\hat{n}(t) + \frac{a\sqrt{3}}{2}\hat{z} \tag{4.121}$$

$$\tag{4.122}$$

Točke unutar petlje opisujemo koordinatama u i v, a trokut je područje opisano sa $u \in [0, 1]$ i $v \in [0, 1 - u]$

$$\mathbf{r} = (\mathbf{A}_2 - \mathbf{A}_1)u + (\mathbf{A}_3 - \mathbf{A}_1)v + \mathbf{A}_1 \tag{4.123}$$

$$= au\hat{n}(t) + \frac{av}{2}\hat{n}(t) + \frac{a\sqrt{3}v}{2}\hat{z} \tag{4.124}$$

$$= a\hat{n}(t)\left(u + \frac{v}{2}\right) + \frac{a\sqrt{3}v}{2}\hat{z} \tag{4.125}$$

Budući da je u $t = 0$ trokut u $y - z$ ravnini, vrijedi

$$\hat{n}(t) = -\sin\omega t\hat{x} + \cos\omega t\hat{y}. \tag{4.126}$$

Uvrštavanjem dobivamo diferencijal površine

$$\partial_u\mathbf{r} = a\hat{n} \tag{4.127}$$

$$\partial_v\mathbf{r} = \frac{\hat{n}a}{2} + \frac{a\sqrt{3}}{2}\hat{z} \tag{4.128}$$

$$d^2\mathbf{r} = a\hat{n} \times \left(\frac{\hat{n}a}{2} + \frac{a\sqrt{3}}{2}\hat{z}\right)dudv \tag{4.129}$$

$$= \frac{a^2\sqrt{3}}{2}\hat{n} \times \hat{z}dudv \tag{4.130}$$

$$= \frac{a^2\sqrt{3}}{2}(\cos\omega t\hat{x} + \sin\omega t\hat{y})dudv. \tag{4.131}$$

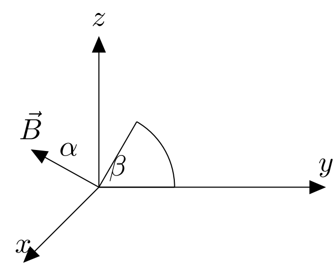
Magnetsko polje je $\mathbf{B} = B\hat{x}$. Tok i elektromotorna sila su

$$\begin{aligned}
\Phi &= \int_0^1 \int_0^{1-u} B\hat{x} \cdot \left(\frac{a^2\sqrt{3}}{2} (\cos\omega t\hat{x} + \sin\omega t\hat{y}) \right) dudv && (4.132)\\
&= \frac{a^2\sqrt{3}}{2} B\cos\omega t \int_0^1 \int_0^{1-u} dudv && (4.133)\\
&= \frac{a^2\sqrt{3}}{2} B\cos\omega t \int_0^1 (1-u)du && (4.134)\\
&= \frac{a^2\sqrt{3}}{2} B\cos\omega t(1-1/2) && (4.135)\\
&= \frac{a^2\sqrt{3}}{4} B\cos\omega t && (4.136)\\
\mathcal{E} &= \omega \frac{a^2\sqrt{3}}{4} B\sin\omega t && (4.137)
\end{aligned}$$

Srednja snaga je:

$$\begin{aligned}
\langle P \rangle &= \frac{\langle \mathcal{E}^2 \rangle}{R} && (4.138)\\
&= \frac{\left(\omega \frac{a^2\sqrt{3}}{4} B\right)^2}{R} \langle \cos^2 \omega t \rangle && (4.139)\\
&= \frac{\left(\omega \frac{a^2\sqrt{3}}{4} B\right)^2}{R} \frac{1}{2} && (4.140)\\
&= \frac{3a^4\omega^2 B^2}{32R}. && (4.141)
\end{aligned}$$

Primjer 4.2.10. Kružni isječak radijusa a definiran kutem β rotira kutnom brzinom $\omega\hat{z}$ oko središta, dok se magnetsko polje nalazi pod kutem α u odnosu na z os u $x-z$ ravnini. Izračunati elektromotornu silu. Kolika je srednja snaga ako kružni isječak ima otpor R?



Rješenje. Točku unutar kružnog isječka opisujemo koordinatama ρ i ϕ, $\rho \in [0, a]$ i $\phi \in [0, \beta]$ U trenutku t točka isječka nalazi se u ravnini definiranoj vektorima $\hat{n}(t)$ i $\hat{z}$.

$$\mathbf{r} = \rho\cos\phi\hat{n}(t) + \rho\sin\phi\hat{z}. \tag{4.142}$$

Budući da je u $t = 0$ kružni isječak u $y - z$ ravnini, vrijedi

$$\hat{n}(t) = -\sin\omega t\hat{x} + \cos\omega t\hat{y}. \tag{4.143}$$

$$\begin{aligned}
\mathbf{r} &= \rho\cos\phi(\cos\omega t\hat{y} - \sin\omega t\hat{x}) + \rho\sin\phi\hat{z} && (4.144)\\
\partial_\rho\mathbf{r} &= \cos\phi(\cos\omega t\hat{y} - \sin\omega t\hat{x}) + \sin\phi\hat{z} && (4.145)\\
\partial_\phi\mathbf{r} &= -\rho\sin\phi(\cos\omega t\hat{y} - \sin\omega t\hat{x}) + \rho\cos\phi\hat{z} && (4.146)\\
d^2\mathbf{r} &= \partial_\rho\mathbf{r} \times \partial_\phi\mathbf{r} d\rho d\phi && (4.147)\\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\cos\phi\sin\omega t & \cos\phi\cos\omega t & \sin\phi \\ \rho\sin\phi\sin\omega t & -\rho\sin\phi\cos\omega t & \rho\cos\phi \end{vmatrix} d\rho d\phi && (4.148)\\
&= \left(\hat{x}\begin{vmatrix} \cos\phi\cos\omega t & \sin\phi \\ -\rho\sin\phi\cos\omega t & \rho\cos\phi \end{vmatrix} - \hat{y}\begin{vmatrix} -\cos\phi\sin\omega t & \sin\phi \\ \rho\sin\phi\sin\omega t & \rho\cos\phi \end{vmatrix} + \hat{z}\begin{vmatrix} -\cos\phi\sin\omega t & \cos\phi\cos\omega t \\ \rho\sin\phi\sin\omega t & -\rho\sin\phi\cos\omega t \end{vmatrix}\right) d\rho d\phi && (4.149)\\
&= \rho\left(\hat{x}(\cos\phi^2)\cos\omega t - \hat{y}(-\cos^2\phi - \sin^2\phi)\sin\omega t + \hat{z}(\sin\phi\sin\omega t\cos\phi\cos\omega t - \sin\phi\sin\omega t\cos\phi\cos\omega t\right) d\rho d\phi && (4.150)\\
&= (\hat{x}\cos\omega t + \hat{y}\sin\omega t)\rho d\rho d\phi. && (4.151)
\end{aligned}$$

Magnetsko polje je $\mathbf{B} = B(\cos\alpha\hat{z} + \sin\alpha\hat{x})$. Tok i elektromotorna sila su

$$\Phi = \int_0^a \int_0^\beta B(\cos\alpha\hat{z} + \sin\alpha\hat{x}) \cdot (\hat{x}\cos\omega t + \hat{y}\sin\omega t)\rho d\rho d\phi \tag{4.152}$$

$$= B\sin\alpha\cos\omega t \int_0^a \rho d\rho \int_0^\beta d\phi \tag{4.153}$$

$$= B\sin\alpha\cos\omega t \frac{\beta a^2}{2} \tag{4.154}$$

$$\mathcal{E} = B\sin\alpha\sin\omega t \frac{\omega\beta a^2}{2} \tag{4.155}$$

$$\tag{4.156}$$

Srednja snaga je:

$$\langle P\rangle = \frac{\langle\mathcal{E}^2\rangle}{R} \tag{4.157}$$

$$= \left(B\sin\alpha\frac{\omega\beta a^2}{2}\right)^2 \frac{1}{R}\langle\sin^2\omega t\rangle \tag{4.158}$$

$$= \left(B\sin\alpha\frac{\omega\beta a^2}{2}\right)^2 \frac{1}{2R} \tag{4.159}$$

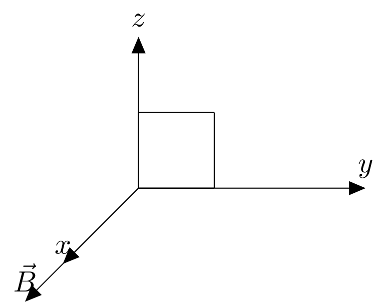
$$= \frac{\omega^2\beta^2a^4\sin^2\alpha B^2}{8R}, \tag{4.160}$$

gdje smo iskoristili

$$\langle\cos^2\omega t\rangle = \frac{1}{2} \tag{4.161}$$

$$\langle\sin^2\omega t\rangle = \frac{1}{2}. \tag{4.162}$$

Primjer 4.2.11. Kvadrat stranice a rotira kutnom brzinom $\omega\hat{z}$ oko jedne od stranica, dok je magnetsko polje dano izrazom $\mathbf{B} = B(y/a)\hat{x}$. Izračunati elektromotornu silu. Kolika je srednja snaga ako kvadrat ima otpor R?



Rješenje. Točke unutar petlje opisujemo koordinatama u i v. Kvadrat se proteže od 0 do a duž x osi i od 0 do a duž z osi. U trenutku t točka pravokutnika nalazi se u ravnini definiranoj vektorima $\hat{n}(t)$ i $\hat{z}$. Uvedemo li bezdimenzionalne varijable $u \in [0,1]$ i $v \in [0,1]$ dobivamo

$$\mathbf{r} = au\hat{n}(t) + av\hat{z}. \tag{4.163}$$

Budući da je u $t = 0$ kvadrat u $y - z$ ravnini, vrijedi

$$\hat{n}(t) = -\sin\omega t\hat{x} + \cos\omega t\hat{y}. \tag{4.164}$$

Dobivamo

$$\begin{aligned}
\mathbf{r} &= au(\cos\omega t\hat{y} - \sin\omega t\hat{x}) + av\hat{z} && (4.165)\\
\partial_u\mathbf{r} &= a(\cos\omega t\hat{y} - \sin\omega t\hat{x}) && (4.166)\\
\partial_v\mathbf{r} &= a\hat{z} && (4.167)\\
d^2\mathbf{r} &= a(\cos\omega t\hat{y} - \sin\omega t\hat{x}) \times a\hat{z}dudv = a^2(\cos\omega t\hat{x} + \sin\omega t\hat{y})dudv. && (4.168)
\end{aligned}$$

Magnetsko polje ovisi o koordinati y koju pročitamo iz vektora $\mathbf{r}$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -au\sin\omega t \\ au\cos\omega t \\ av \end{pmatrix} \tag{4.169}$$

pa slijedi $\mathbf{B} = B(y/a)\hat{x} = Bu\cos(\omega t)\hat{x}$. Tok i elektromotorna sila su

$$\begin{aligned}
\Phi &= \int_0^1\int_0^1 Bu\cos(\omega t)\hat{x}\cdot a^2(\cos\omega t\hat{x} + \sin\omega t\hat{y})dudv && (4.170)\\
&= B\cos^2(\omega t)a^2\frac{u^2}{2}\Big|_0^1 v\Big|_0^1 && (4.171)\\
&= a^2B\cos^2\omega t\frac{1}{2} && (4.172)\\
\mathcal{E} &= \omega a^2B\sin\omega t\cos\omega t && (4.173)\\
&= \frac{\omega a^2 B}{2}\sin 2\omega t && (4.174)
\end{aligned}$$

Srednju snagu računamo za jedan period $T = 2\pi/\omega$

$$\langle P \rangle = \frac{\langle \mathcal{E}^2 \rangle}{R} \tag{4.175}$$

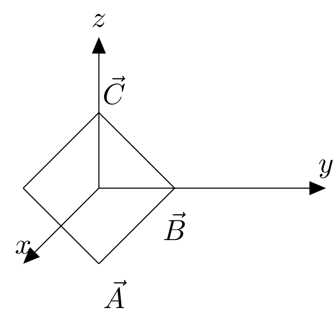
$$= \frac{1}{TR} \int_0^T \left(\frac{\omega a^2 B}{2} \right)^2 \sin^2 2\omega t dt \tag{4.176}$$

$$= \left(\frac{\omega a^2 B}{2} \right)^2 \frac{1}{RT} \int_0^T \left(\frac{1 - \cos 4\omega t}{2} \right) dt \tag{4.177}$$

$$= \left(\frac{\omega a^2 B}{2} \right)^2 \frac{1}{RT} (T/2 - 0) \tag{4.178}$$

$$= \frac{\omega^2 a^4 B^2}{8R}. \tag{4.179}$$

Primjer 4.2.12. Kvadrat stranice a rotira, po dijagonali kutnom brzinom $\omega\hat{z}$ u smjeru suprotnom od kazaljke na satu dok se magnetsko polje B_0 nalazi u $\hat{x}$ smjeru. Napisati parametrizaciju sustava ako se u $t = 0$ kvadrat nalazi u yz ravnini. Izračunati elektromotornu silu i srednju snagu.



Rješenje. U trenutku t točka kvadrata nalazi se u ravnini definiranoj vektorima $\hat{n}(t)$ i $\hat{z}$. Budući da je u $t = 0$ pravokutnik u $y - z$ ravnini, točka $\mathbf{B}$ je u smjeru

$$\hat{n}(t) = -\sin\omega t\hat{x} + \cos\omega t\hat{y}. \tag{4.180}$$

Tok možemo pronaći koristeći nekoliko parametrizacija. Jedna mogućnost je da parametriziramo 2 pravokutna trokuta. Polovica dijagonale kvadrata je $\sqrt{2}a/2 = a/\sqrt{2}$ te koristeći točke $\vec{B} = \frac{a}{\sqrt{2}}\hat{n}$ i $\vec{C} = \frac{a}{\sqrt{2}}\hat{z}$ te ishodište možemo pisati

$$\mathbf{r} = (\vec{B} - \vec{0})u + (\vec{C} - \vec{0})v + \vec{0} \tag{4.181}$$

$$= \frac{a}{\sqrt{2}}\left(u\hat{n}(t) + v\hat{z}\right). \tag{4.182}$$

Parametri u i v moraju opisivati čitavi kvadrat, te stoga znamo da vrijedi, $|u| + |v| \leq 1$, odnosno:

$$u \in [0, 1] \to v \in [u - 1, 1 - u] \tag{4.183}$$

$$u \in [-1, 0] \to v \in [-1 - u, u + 1]. \tag{4.184}$$

Zapišimo diferencijalni element površine u ovoj parametrizaciji

$$\mathbf{r} = \frac{a}{\sqrt{2}}\left(u\hat{n}(t) + v\hat{z}\right) \tag{4.185}$$

$$\partial_u\mathbf{r} = \frac{a}{\sqrt{2}}\hat{n}(t) \tag{4.186}$$

$$\partial_v\mathbf{r} = \frac{a}{\sqrt{2}}\hat{z} \tag{4.187}$$

$$d^2\mathbf{r} = \partial_u\mathbf{r} \times \partial_v\mathbf{r}dudv \tag{4.188}$$

$$= \frac{a^2}{2}\left(-\sin\omega t\hat{x} + \cos\omega t\hat{y}\right) \times \hat{z}dudv \tag{4.189}$$

$$= \frac{a^2}{2}\left(\cos\omega t\hat{x} + \sin\omega t\hat{y}\right)dudv \tag{4.190}$$

Magnetsko polje je $\mathbf{B} = B_0\hat{x}$. Tok i elektromotorna sila su

$$\Phi = \iint\limits_{|u|+|v|\leq 1} B_0\hat{x}\cdot(\hat{x}\cos\omega t + \hat{y}\sin\omega t)\frac{a^2}{2}dudv \tag{4.191}$$

$$= \frac{B_0 a^2}{2}\cos\omega t\left(\int\limits_{-1}^{0}\int\limits_{-u-1}^{1+u} dudv + \int\limits_{0}^{1}\int\limits_{u-1}^{1-u} dudv\right) \tag{4.192}$$

$$= \frac{B_0 a^2}{2}\cos\omega t\left(\int\limits_{-1}^{0} 2(1+u)du + \int\limits_{0}^{1} 2(1-u)du\right) \tag{4.193}$$

$$= \frac{B_0 a^2}{2}\cos\omega t\left(2(u+u^2/2)|_{-1}^{0} + 2(u-u^2/2)|_{0}^{1}\right) \tag{4.194}$$

$$= \frac{B_0 a^2}{2}\cos\omega t\,(2(0+0-(-1+1/2)) + 2(1-1/2-0-0)) \tag{4.195}$$

$$= \frac{B_0 a^2}{2}\cos\omega t\,(1+1) \tag{4.196}$$

$$= B_0 a^2\cos\omega t \tag{4.197}$$

$$\mathcal{E} = B_0 a^2\omega\sin\omega t \tag{4.198}$$

$$\tag{4.199}$$

Srednja snaga je:

$$\langle P\rangle = \frac{\langle\mathcal{E}^2\rangle}{R} \tag{4.200}$$

$$= \left(B_0 a^2\omega\right)^2\frac{1}{R}\langle\sin^2\omega t\rangle \tag{4.201}$$

$$= \left(B_0 a^2\omega\right)^2\frac{1}{2R} \tag{4.202}$$

$$= \frac{\omega^2 a^4 B_0^2}{2R}. \tag{4.203}$$

Alternativno, mogli smo odabrati točku $\vec{B}$ kao ishodište, te točke $\vec{A}$ i $\vec{C}$

$$\mathbf{r} = (\vec{C} - \vec{B})u + (\vec{A} - \vec{B})v + \vec{B} \tag{4.204}$$

$$= \frac{a}{\sqrt{2}}\left((\hat{z} - \hat{n})u + (-\hat{z} - \hat{n})v + \hat{n}\right) \tag{4.205}$$

$$= \frac{a}{\sqrt{2}}\left((\hat{z} - \hat{n})u - (\hat{z} + \hat{n})v + \hat{n}\right) \tag{4.206}$$

$$\partial_u \mathbf{r} = \frac{a}{\sqrt{2}}(\hat{z} - \hat{n}) \tag{4.207}$$

$$\partial_v \mathbf{r} = \frac{-a}{\sqrt{2}}(\hat{z} + \hat{n}) \tag{4.208}$$

$$\partial_u \mathbf{r} \times \partial_v \mathbf{r} = \frac{-a^2}{2}(\hat{z} + \sin\omega t\hat{x} - \cos\omega t\hat{y}) \times (\hat{z} - \sin\omega t\hat{x} + \cos\omega t\hat{y}) \tag{4.209}$$

$$= \frac{-a^2}{2}(-\sin\omega t\hat{y} - \cos\omega t\hat{x} - \sin\omega t\hat{y} \tag{4.210}$$

$$+ \sin\omega t\cos\omega t\hat{z} - \cos\omega t\hat{x} - \sin\omega t\cos\omega t\hat{z}) \tag{4.211}$$

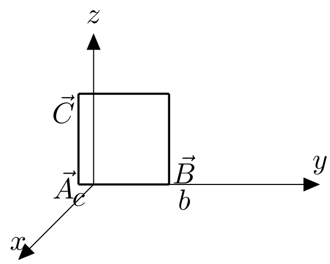
$$= a^2(\cos\omega t\hat{x} + \sin\omega t\hat{y}) \tag{4.212}$$

$$\Phi = \int_0^1 \int_0^1 B_0\hat{x} \cdot (\hat{x}\cos\omega t + \hat{y}\sin\omega t)a^2 dudv \tag{4.213}$$

$$= B_0 \cos\omega t a^2 \int_0^1 \int_0^1 dudv \tag{4.214}$$

$$= B_0 a^2 \cos\omega t. \tag{4.215}$$

Primjer 4.2.13. Kvadrat stranice a rotira oko z osi kutnom brzinom $\omega\hat{z}$ u smjeru suprotnom od kazaljke na satu dok se magnetsko polje B_0 nalazi u $\hat{x}$ smjeru. Os rotacije dijeli kvadrat tako da se proteže od $y = -c$ do $y = b$. Napisati parametrizaciju sustava ako se u $t = 0$ kvadrat nalazi u yz ravnini. Izračunati elektromotornu silu i srednju snagu.



Rješenje. U trenutku t točka kvadrata nalazi se u ravnini definiranoj vektorima $\hat{n}(t)$ i $\hat{z}$. Budući da je u $t = 0$ pravokutnik u $y - z$ ravnini, točka $\vec{B}$ je u smjeru

$$\hat{n}(t) = -\sin\omega t\hat{x} + \cos\omega t\hat{y}. \tag{4.216}$$

Pri parametrizaciji koristimo točke $\vec{A} = -c\hat{n}$, $\vec{B} = b\hat{n}$ i $\vec{C} = a\hat{z} - c\hat{n}$ te pišemo

$$\mathbf{r} = (\vec{B} - \vec{A})u + (\vec{C} - \vec{A})v + \vec{A} \tag{4.217}$$
$$= (b - (-c))u\hat{n} + (a\hat{z} - c\hat{n} - (-c)\hat{n})v - c\hat{n} \tag{4.218}$$
$$= au\hat{n} + av\hat{z} - c\hat{n} \tag{4.219}$$

Parametri u i v moraju opisivati čitavi kvadrat, te stoga znamo da vrijedi $u \in [0, 1]$ i $v \in [0, 1]$. Zapišimo diferencijalni element površine u ovoj parametrizaciji

$$\partial_u \mathbf{r} = a\hat{n}(t) \tag{4.220}$$
$$\partial_v \mathbf{r} = a\hat{z} \tag{4.221}$$
$$d^2\mathbf{r} = \partial_u \mathbf{r} \times \partial_v \mathbf{r} dudv \tag{4.222}$$
$$= a^2 \left(-\sin\omega t\hat{x} + \cos\omega t\hat{y}\right) \times \hat{z} dudv \tag{4.223}$$
$$= a^2 \left(\cos\omega t\hat{x} + \sin\omega t\hat{y}\right) dudv \tag{4.224}$$

Magnetsko polje je $\mathbf{B} = B_0\hat{x}$. Tok i elektromotorna sila su

$$\Phi = \int_0^1 \int_0^1 B_0\hat{x} \cdot (\hat{x}\cos\omega t + \hat{y}\sin\omega t)a^2 dudv \tag{4.225}$$
$$= B_0 a^2 \cos\omega t \int_0^1 \int_0^1 dudv \tag{4.226}$$
$$= B_0 a^2 \cos\omega t \tag{4.227}$$
$$\mathcal{E} = B_0 a^2 \omega \sin\omega t \tag{4.228}$$
$$\tag{4.229}$$

Srednja snaga je:

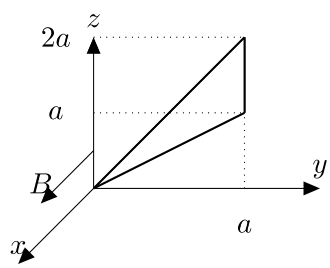
$$\langle P\rangle = \frac{\langle \mathcal{E}^2\rangle}{R} \tag{4.230}$$

$$= \left(B_0 a^2 \omega\right)^2 \frac{1}{R}\langle \sin^2 \omega t\rangle \tag{4.231}$$

$$= \left(B_0 a^2 \omega\right)^2 \frac{1}{2R} \tag{4.232}$$

$$= \frac{\omega^2 a^4 B_0^2}{2R}. \tag{4.233}$$

Primjer 4.2.14. Trokut prikazan na slici, rotira kutnom brzinom $\omega\hat{z}$ dok se magnetsko polje oblika $\mathbf{B} = B\frac{z}{a}\hat{x}$ nalazi u $\hat{x}$ smjeru. Izračunati tok magnetskog polja u trenutku t koristeći parametrizaciju, elektromotornu silu. Kolika je srednja snaga ako trokut ima otpor R?



Rješenje. U trenutku t točke trokuta nalaze se u ravnini definiranoj vektorima $\hat{n}(t)$ i $\hat{z}$. Koordinate točaka trokuta su

$$\mathbf{A}_1 = \vec{0} \tag{4.234}$$
$$\mathbf{A}_2 = a\hat{n}(t) + a\hat{z} \tag{4.235}$$
$$\mathbf{A}_3 = a\hat{n}(t) + 2a\hat{z} \tag{4.236}$$
$$\tag{4.237}$$

Točke unutar petlje opisujemo koordinatama u i v, a trokut je područje opisano sa $u \in [0, 1]$ i $v \in [0, 1 - u]$

$$\mathbf{r} = (\mathbf{A}_2 - \mathbf{A}_1)u + (\mathbf{A}_3 - \mathbf{A}_1)v + \mathbf{A}_1 \tag{4.238}$$
$$= au(\hat{n}(t) + \hat{z}) + av(\hat{n} + 2\hat{z}) \tag{4.239}$$
$$= a\,(u + v)\,\hat{n}(t) + a(u + 2v)\hat{z} \tag{4.240}$$

Budući da je u $t = 0$ trokut u $y - z$ ravnini, vrijedi

$$\hat{n}(t) = -\sin\omega t\hat{x} + \cos\omega t\hat{y}. \tag{4.241}$$

Možemo očitati

$$\mathbf{r} = \begin{pmatrix} x = -a(u + v)\sin\omega t \\ y = a(u + v)\cos\omega t \\ z = a(u + 2v) \end{pmatrix} \tag{4.242}$$

Uvrštavanjem dobivamo diferencijal površine

$$\partial_u\mathbf{r} = a(\hat{n} + \hat{z}) \tag{4.243}$$
$$\partial_v\mathbf{r} = a(\hat{n} + 2\hat{z}) \tag{4.244}$$
$$d^2\mathbf{r} = a^2(\hat{n} + \hat{z}) \times (\hat{n} + 2\hat{z})dudv \tag{4.245}$$
$$= a^2(2\hat{n} \times \hat{z} - \hat{n} \times \hat{z})dudv \tag{4.246}$$
$$= a^2(\cos\omega t\hat{x} + \sin\omega t\hat{y})dudv. \tag{4.247}$$

Magnetsko polje je $\mathbf{B} = B\frac{z}{a}\hat{x}$. Tok i elektromotorna sila su

$$\Phi = \int_0^1 \int_0^{1-u} B\frac{a(u+2v)}{a}\hat{x}\cdot\left(a^2(\cos\omega t\hat{x} + \sin\omega t\hat{y})\right) dudv \tag{4.248}$$

$$= a^2 B\cos\omega t \int_0^1 \int_0^{1-u} (u+2v)dudv \tag{4.249}$$

$$= a^2 B\cos\omega t \int_0^1 (u(1-u) + (1-u)^2)du \tag{4.250}$$

$$= a^2 B\cos\omega t \int_0^1 (1-u)(u+1-u)du \tag{4.251}$$

$$= a^2 B\cos\omega t \int_0^1 (1-u)du \tag{4.252}$$

$$= a^2 B\cos\omega t(1-1/2) \tag{4.253}$$

$$= \frac{a^2 B}{2}\cos\omega t \tag{4.254}$$

$$\mathcal{E} = \omega\frac{a^2}{2}B\sin\omega t \tag{4.255}$$

Srednja snaga je:

$$\langle P\rangle = \frac{\langle\mathcal{E}^2\rangle}{R} \tag{4.256}$$

$$= \frac{\left(\omega\frac{a^2}{2}B\right)^2}{R}\langle\sin^2\omega t\rangle \tag{4.257}$$

$$= \frac{\left(\omega\frac{a^2}{2}B\right)^2}{R}\frac{1}{2} \tag{4.258}$$

$$= \frac{a^4\omega^2 B^2}{8R}. \tag{4.259}$$

5. Elektromagnetski valovi

5.1 Elektromagnetski val u vakuumu

Promotrimo Maxwellove jednadžbe za rotacije u vakuumu i primijenimo rotacije.

$$
\begin{aligned}
\nabla \times \mathbf{E} &= -\partial_t \mathbf{B} && (5.1)\\
\nabla \times \nabla \times \mathbf{E} &= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} && (5.2)\\
&= -\nabla^2 \mathbf{E} && (5.3)\\
&= -\partial_t \nabla \times \mathbf{B} && (5.4)\\
&= -\mu_0 \epsilon_0 \partial_t^2 \mathbf{E} && (5.5)\\
\nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \partial_t \mathbf{E} && (5.6)\\
\nabla \times \nabla \times \mathbf{B} &= \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} && (5.7)\\
&= -\nabla^2 \mathbf{B} && (5.8)\\
&= \mu_0 \epsilon_0 \partial_t \nabla \times \mathbf{E} && (5.9)\\
&= -\mu_0 \epsilon_0 \partial_t^2 \mathbf{B}, && (5.10)
\end{aligned}
$$

odnosno, dobili smo dvije valne jednadžbe

$$
\begin{aligned}
\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right) \mathbf{E} &= 0 && (5.11)\\
\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right) \mathbf{B} &= 0 && (5.12)\\
c &= \frac{1}{\sqrt{\mu_0 \epsilon_0}}. && (5.13)
\end{aligned}
$$

Rješenja tražimo u obliku ravnih valova $e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$.

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right) e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} = \left((i\mathbf{k})^2 - \frac{(-i\omega)^2}{c^2}\right) e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \tag{5.14}$$

$$= -e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\left(\mathbf{k}^2 - \frac{\omega^2}{c^2}\right) \tag{5.15}$$

$$\mathbf{k} = \pm\hat{k}\frac{\omega}{c}. \tag{5.16}$$

Općenito rješenje je tada

$$\mathbf{E} = \mathbf{E}_0^+ e^{i\omega\left(\frac{\hat{k}\cdot\mathbf{r}}{c}-t\right)} + \mathbf{E}_0^- e^{i\omega\left(-\frac{\hat{k}\cdot\mathbf{r}}{c}-t\right)} \tag{5.17}$$

$$\mathbf{B} = \mathbf{B}_0^+ e^{i\omega\left(\frac{\hat{k}\cdot\mathbf{r}}{c}-t\right)} + \mathbf{B}_0^- e^{i\omega\left(-\frac{\hat{k}\cdot\mathbf{r}}{c}-t\right)}, \tag{5.18}$$

gdje općenito možemo imati val koji se giba na pozitivnu i na negativnu stranu.

Više informacija o amplitudama dobivamo uvrštavanjem oblika ravnog vala u Maxwellove jednažbe

$$\mathbf{k}\cdot\mathbf{E}_0 = 0 \tag{5.19}$$

$$\mathbf{k}\cdot\mathbf{B}_0 = 0 \tag{5.20}$$

$$\mathbf{k}\times\mathbf{E}_0 = \omega\mathbf{B}_0 \tag{5.21}$$

$$\mathbf{k}\times\mathbf{B}_0 = \frac{-\omega}{c^2}\mathbf{E}_0. \tag{5.22}$$

Prve dvije jednadžbe nam govore da električno i magnetsko polje mora biti okomito na vektor $\mathbf{k}$, odnosno moraju ležati u ravnini u 3-D prostoru. Budući da je ravnina 2-D potprostor, postoje dva linearno nezavisna bazna vektora polarizacije koja ju opisuju, $\mathbf{e}_\lambda$, $\lambda = 1, 2$. S druge strane, druge dvije jednadžbe nam govore da ako znamo električno polje, znamo smjer i iznos magnetskog polja

$$\mathbf{E}_\lambda = E_\lambda \mathbf{e}_\lambda e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{5.23}$$

$$\mathbf{B}_\lambda = E_\lambda \frac{\mathbf{k}\times\mathbf{e}_\lambda}{\omega} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = \frac{\hat{k}}{c}\times\mathbf{E}_\lambda. \tag{5.24}$$

Posebno koristan oblik dobivamo ako umjesto magnetskog polja koristimo polje $\mathbf{H}$

$$\mathbf{H} = \frac{\hat{k}}{\mu_0 c}\times\mathbf{E} = \sqrt{\frac{\epsilon_0}{\mu_0}}\hat{k}\times\mathbf{E} = \frac{1}{Z}\hat{k}\times\mathbf{E}, \tag{5.25}$$

gdje smo uveli impedanciju

$$Z = \sqrt{\frac{\mu_0}{\epsilon}} \tag{5.26}$$

5.2 Elektromagnetski val u nevodljivom sredstvu

Ponovimo isti izvod za materijal opisan permitivnošću i permeabilnošću. Maxwellove jednadžbe izrazimo preko $\mathbf{E}$ i $\mathbf{H}$

$$\nabla \cdot \mathbf{E} = 0 \tag{5.27}$$
$$\nabla \cdot \mathbf{H} = 0 \tag{5.28}$$
$$\nabla \times \mathbf{E} = -\mu \partial_t \mathbf{H} \tag{5.29}$$
$$\nabla \times \mathbf{H} = \epsilon \partial_t \mathbf{E}. \tag{5.30}$$

Koristimo pretpostavku ravnih valova i dobivamo sustav

$$\mathbf{k} \cdot \mathbf{E} = 0 \tag{5.31}$$
$$\mathbf{k} \cdot \mathbf{H} = 0 \tag{5.32}$$
$$\mathbf{k} \times \mathbf{E} = \mu\omega \mathbf{H} \tag{5.33}$$
$$\mathbf{k} \times \mathbf{H} = -\epsilon\omega \mathbf{E}. \tag{5.34}$$

Množenjem druge dvije jednadžbe s $\mathbf{k}\times$, dobivamo

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = \mathbf{k}(\mathbf{k} \cdot \mathbf{E}) - \mathbf{k}^2 \mathbf{E} \tag{5.35}$$
$$= -\mu\omega^2 \epsilon \mathbf{E} \tag{5.36}$$
$$\mathbf{k} \times (\mathbf{k} \times \mathbf{H}) = \mathbf{k}(\mathbf{k} \cdot \mathbf{H}) - \mathbf{k}^2 \mathbf{H} \tag{5.37}$$
$$= -\mu\omega^2 \epsilon \mathbf{H}, \tag{5.38}$$

odnosno

$$\left(\nabla^2 - \frac{1}{v^2}\partial_t^2\right)\mathbf{E} = 0 \tag{5.39}$$
$$\left(\nabla^2 - \frac{1}{v^2}\partial_t^2\right)\mathbf{H} = 0 \tag{5.40}$$
$$v = \frac{1}{\sqrt{\mu\epsilon}} \tag{5.41}$$
$$k = \frac{\omega}{v}. \tag{5.42}$$

Možemo također uvesti indeks loma

$$n = \frac{c}{v} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \sqrt{\mu_r\epsilon_r}, \tag{5.43}$$

te tada valne jednadžbe postaju

$$\left(\nabla^2 - \frac{n^2}{c^2}\partial_t^2\right)\mathbf{E} = 0 \tag{5.44}$$
$$\left(\nabla^2 - \frac{n^2}{c^2}\partial_t^2\right)\mathbf{H} = 0. \tag{5.45}$$

Ako znamo električno polje, možemo izračunati magnetsko - i obratno koristeći jednadžbe 5.33 i 5.34

$$\mathbf{H} = \frac{1}{Z}\hat{k} \times \mathbf{E} \tag{5.46}$$

$$\mathbf{E} = -Z\hat{k} \times \mathbf{H}, \tag{5.47}$$

gdje je Z impedancija

$$Z = \sqrt{\frac{\mu}{\epsilon}}. \tag{5.48}$$

Pri pronalaženju rješenja sustava između dva sredstva, moramo koristiti rubne uvjete

$$\hat{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0 \tag{5.49}$$

$$\hat{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0 \tag{5.50}$$

$$\hat{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{0} \tag{5.51}$$

$$\hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{0}, \tag{5.52}$$

uz $\mathbf{D} = \epsilon\mathbf{E}$ i $\mathbf{B} = \mu\mathbf{H}$. Ako na granici sredstava postoji slobodan naboj, σ_f i struja, $\mathbf{K}_f$, tada rubni uvjeti glase

$$\hat{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \sigma_f \tag{5.53}$$

$$\hat{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0 \tag{5.54}$$

$$\hat{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{0} \tag{5.55}$$

$$\hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{K}_f. \tag{5.56}$$

Možemo izračunati i srednju vrijednost Poyntingovog vektora

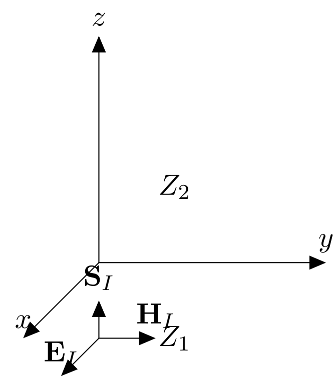
$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}(\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2Z}\mathrm{Re}(\mathbf{E} \times (\hat{k} \times \mathbf{E}^*)) = \frac{1}{2Z}\mathrm{Re}(|\mathbf{E}|^2\hat{k} - (\hat{k} \cdot \mathbf{E})\mathbf{E}^*) = \frac{|\mathbf{E}|^2}{2Z}\hat{k}. \tag{5.57}$$

U slučaju granice sredstava, uz ulazni val, opisan usrednjenim Poyntingovim vektorom $\langle \mathbf{S}_I \rangle$, imamo i reflektirani val, opisan Poyntingovim vektorom $\langle \mathbf{S}_R \rangle$ te transmitirani val, opisan s $\langle \mathbf{S}_T \rangle$. Koeficijenti refleksije i transmisije na plohi normale $\hat{n}$ su

$$R = \left| \frac{\langle \mathbf{S}_R \rangle \cdot \hat{n}}{\langle \mathbf{S}_I \rangle \cdot \hat{n}} \right| = \frac{|\mathbf{E}_R|^2}{|\mathbf{E}_I|^2} \tag{5.58}$$

$$T = \left| \frac{\langle \mathbf{S}_T \rangle \cdot \hat{n}}{\langle \mathbf{S}_I \rangle \cdot \hat{n}} \right| = \frac{Z_I}{Z_T} \left| \frac{\hat{k}_T \cdot \hat{n}}{\hat{k}_I \cdot \hat{n}} \right| \frac{|\mathbf{E}_T|^2}{|\mathbf{E}_I|^2}. \tag{5.59}$$

Primjer 5.2.1. Elektromagnetski val upada okomito iz materijala opisanog sa ϵ_1 i μ_1 na materijal opisan sa ϵ_2 i μ_2. Pronaći koeficijente refleksije i transmisije. Provjeriti relaciju $R + T = 1$.



Rješenje. Rješenje u područjima 1 i 2 opisujemo s

$$\mathbf{E}_I = \hat{x} E_I e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} \tag{5.60}$$
$$\mathbf{E}_R = \hat{x} E_R e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} \tag{5.61}$$
$$\mathbf{E}_T = \hat{x} E_T e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}. \tag{5.62}$$

Valni vektor je u $\hat{z}$ smjeru te ima sljedeći oblik

$$\mathbf{k}_I = \frac{n_1 \omega}{c} \hat{z} \tag{5.63}$$
$$\mathbf{k}_R = -\frac{n_1 \omega}{c} \hat{z} \tag{5.64}$$
$$\mathbf{k}_T = \frac{n_2 \omega}{c} \hat{z} \tag{5.65}$$

U $z = 0$ preostaje $\mathbf{k} \cdot \mathbf{r} = 0$ i nemamo dodatnih uvjeta na $\mathbf{k}$. Pripadna magnetska polja su

$$\mathbf{H}_I = \frac{\hat{z}}{Z_1} \times \mathbf{E}_I = \frac{\hat{y}}{Z_1} E_I e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} \tag{5.66}$$
$$\mathbf{H}_R = \frac{-\hat{z}}{Z_1} \times \mathbf{E}_R = \frac{-\hat{y}}{Z_1} E_I e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} \tag{5.67}$$
$$\mathbf{H}_T = \frac{\hat{z}}{Z_2} \times \mathbf{E}_T = \frac{\hat{y}}{Z_2} E_T e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}. \tag{5.68}$$

Po jednadžbama rubnih uvjeta, imamo samo komponente okomite na vektor normale te u $z = 0$ vrijedi

$$E_I + E_R = E_T \tag{5.69}$$
$$\frac{E_I - E_R}{Z_1} = \frac{E_T}{Z_2}. \tag{5.70}$$

Sustav možemo jednostavno riješiti

$$E_I + E_R = E_T \tag{5.71}$$
$$E_I - E_R = \frac{Z_1}{Z_2} E_T \tag{5.72}$$
$$E_I = \frac{E_T}{2}\left(1 + \frac{Z_1}{Z_2}\right) \tag{5.73}$$
$$E_R = \frac{E_T}{2}\left(1 - \frac{Z_1}{Z_2}\right) \tag{5.74}$$
$$= E_I \frac{1 - \frac{Z_1}{Z_2}}{1 + \frac{Z_1}{Z_2}} \tag{5.75}$$
$$= \frac{Z_2 - Z_1}{Z_2 + Z_1} \tag{5.76}$$
$$E_T = E_I \frac{2Z_2}{Z_2 + Z_1}. \tag{5.77}$$

Koeficijenti refleksije i transmisije su

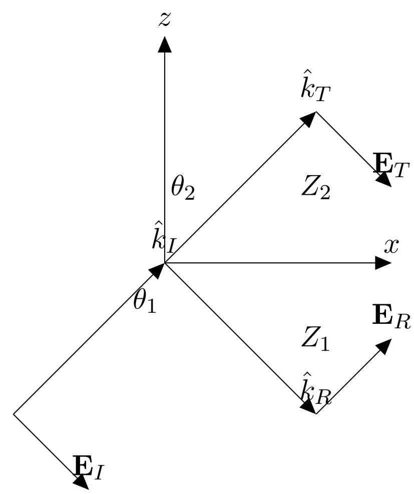
$$R = \frac{|\mathbf{E_R}|^2}{|\mathbf{E_I}|^2} = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2 \tag{5.78}$$
$$T = \frac{Z_1}{Z_2}\frac{|\mathbf{E_T}|^2}{|\mathbf{E_I}|^2} = \frac{4Z_1Z_2}{(Z_1 + Z_2)^2}. \tag{5.79}$$

Zbroj koeficijenata refleksije i transmisije je

$$R + T = \frac{(Z_2 - Z_1)^2 + 4Z_2Z_1}{(Z_1 + Z_2)^2} = \frac{(Z_1 + Z_2)^2}{(Z_1 + Z_2)^2} = 1. \tag{5.80}$$

Primjer 5.2.2. Elektromagnetski val nalijeće iz sredstva opisanog s ϵ_1 i μ_1 na sredstvo opisano sa ϵ_2 i μ_2 pod kutem θ_1. Pronaći kut pod kojim izlazi transmitirani val (Snellov zakon) te izračunati koeficijente refleksije i transmisije u TM i TE modu. Provjeriti $R+T = 1$. Izraziti koeficijente refleksije i transmisije pomoću permeabilnosti i indeksa loma.

Rješenje. U TM modu, električno polje nailazi pod kutem u odnosu na granicu između sredstava, dok je magnetsko polje paralelno s njom. Postavimo koordinatni sustav tako da se električno polje nalazi u $x - z$ ravnini.



Raspišimo ulazni, reflektirani i transmitirani valni vektor

$$\hat{k}_I = \sin\theta_1\hat{x} + \cos\theta_1\hat{z} \tag{5.81}$$
$$\hat{k}_R = \sin\theta_1\hat{x} - \cos\theta_1\hat{z} \tag{5.82}$$
$$\hat{k}_T = \sin\theta_2\hat{x} + \cos\theta_2\hat{z} \tag{5.83}$$

Budući da u $z = 0$ paralelne komponente valnog vektora naspram plohe moraju biti iste, mora vrijediti

$$k_1 \sin\theta_1 = k_2 \sin\theta_2 \to n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{5.84}$$

Odnosno, kut transmitiranog vala je

$$\theta_2 = \arcsin\left(\frac{n_1}{n_2}\sin\theta_1\right). \tag{5.85}$$

Vrijedi

$$\mathbf{E}_I = (\cos\theta_1\hat{x} - \sin\theta_1\hat{z})E_I \tag{5.86}$$
$$\mathbf{E}_R = (\cos\theta_1\hat{x} + \sin\theta_1\hat{z})E_R \tag{5.87}$$
$$\mathbf{E}_T = (\cos\theta_2\hat{x} - \sin\theta_2\hat{z})E_T \tag{5.88}$$
$$\mathbf{H}_I = \frac{E_I}{Z_1}(\sin\theta_1\hat{x} + \cos\theta_1\hat{z}) \times (\cos\theta_1\hat{x} - \sin\theta_1\hat{z}) = \frac{E_I}{Z_1}\hat{y} \tag{5.89}$$
$$\mathbf{H}_R = \frac{E_R}{Z_1}(\sin\theta_1\hat{x} - \cos\theta_1\hat{z}) \times (\cos\theta_1\hat{x} + \sin\theta_1\hat{z}) = -\frac{E_R}{Z_1}\hat{y} \tag{5.90}$$
$$\mathbf{H}_T = \frac{E_T}{Z_2}(\sin\theta_2\hat{x} + \cos\theta_2\hat{z}) \times (\cos\theta_2\hat{x} - \sin\theta_2\hat{z}) = \frac{E_T}{Z_2}\hat{y}. \tag{5.91}$$

Jednadžbe su tada

$$(\mathbf{E}_I + \mathbf{E}_R)\cdot\hat{x} = \mathbf{E}_T\cdot\hat{x} \tag{5.92}$$
$$(\mathbf{H}_I + \mathbf{H}_R)\cdot\hat{y} = \mathbf{H}_T\cdot\hat{y} \tag{5.93}$$
$$E_I + E_R = \frac{\cos\theta_2}{\cos\theta_1}E_T \tag{5.94}$$
$$E_I - E_R = \frac{Z_1}{Z_2}E_T \tag{5.95}$$
$$E_I = \frac{E_T}{2}\left(\frac{\cos\theta_2}{\cos\theta_1} + \frac{Z_1}{Z_2}\right) \tag{5.96}$$
$$E_R = \frac{E_T}{2}\left(\frac{\cos\theta_2}{\cos\theta_1} - \frac{Z_1}{Z_2}\right) \tag{5.97}$$
$$E_R = \frac{\cos\theta_2 Z_2 - \cos\theta_1 Z_1}{\cos\theta_2 Z_2 + \cos\theta_1 Z_1}E_I \tag{5.98}$$
$$E_T = \frac{2\cos\theta_1 Z_2}{\cos\theta_2 Z_2 + \cos\theta_1 Z_1}E_I \tag{5.99}$$
$$R = \frac{|\mathbf{E_R}|^2}{|\mathbf{E_I}|^2} = \left(\frac{\cos\theta_2 Z_2 - \cos\theta_1 Z_1}{\cos\theta_2 Z_2 + \cos\theta_1 Z_1}\right)^2 \tag{5.100}$$
$$= \left(\frac{\cos\theta_2 Z_2 - \cos\theta_1 Z_1}{\cos\theta_2 Z_2 + \cos\theta_1 Z_1}\right)^2 \tag{5.101}$$
$$T = \frac{Z_1|\mathbf{E}_T|^2}{Z_2|\mathbf{E}_I|^2} = \left|\frac{Z_1}{Z_2}\frac{\hat{k}_T\cdot\hat{z}}{\hat{k}_I\cdot\hat{z}}\right| \frac{4\cos\theta_1^2 Z_2^2}{(\cos\theta_2 Z_2 + \cos\theta_1 Z_1)^2} \tag{5.102}$$
$$= \frac{4\cos\theta_2\cos\theta_1 Z_2 Z_1}{(\cos\theta_2 Z_2 + \cos\theta_1 Z_1)^2} \tag{5.103}$$
$$R + T = \frac{(\cos\theta_2 Z_2 - \cos\theta_1 Z_1)^2 + 4Z_1 Z_2\cos\theta_1\cos\theta_2}{(\cos\theta_2 Z_2 + \cos\theta_1 Z_1)^2} = 1. \tag{5.104}$$

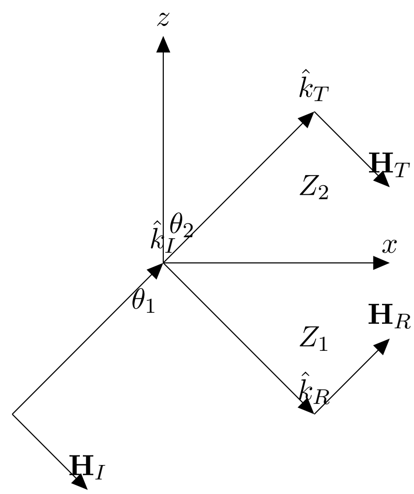
Primijetimo da vrijedi

$$Z_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \sqrt{\frac{\mu_0}{\epsilon_0}\frac{\mu_{r1}^2}{n_1^2}} = Z_0\frac{\mu_{r1}}{n_1} \tag{5.105}$$
$$Z_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \sqrt{\frac{\mu_0}{\epsilon_0}\frac{\mu_{r2}^2}{n_2^2}} = Z_0\frac{\mu_{r2}}{n_2}, \tag{5.106}$$

pri čemu je Z_0 impedancija vakuuma, $Z_0 = \sqrt{\mu_0/\epsilon_0}$. Koeficijenti refleksije i transmisije su tada

$$R = \left(\frac{\cos\theta_2 n_1\mu_{r2} - \cos\theta_1 n_2\mu_{r1}}{\cos\theta_2 n_1\mu_{r2} - \cos\theta_1 n_2\mu_{r1}}\right)^2 \tag{5.107}$$
$$T = \frac{4\cos\theta_2\cos\theta_1\mu_{r1}\mu_{r2}n_1 n_2}{(\cos\theta_2\mu_{r2}n_1 + \cos\theta_1\mu_{r1}n_2)^2}. \tag{5.108}$$

U TE modu, magnetsko polje nailazi pod kutem u odnosu na granicu između sredstava, dok je električno polje paralelno s njom. Postavimo koordinatni sustav tako da se magnetsko polje nalazi u $x - z$ ravnini.



Vrijedi

$$\mathbf{H}_I = (\cos\theta_1\hat{x} - \sin\theta_1\hat{z})H_I \tag{5.109}$$

$$\mathbf{H}_R = (\cos\theta_1\hat{x} + \sin\theta_1\hat{z})H_R \tag{5.110}$$

$$\mathbf{H}_T = (\cos\theta_2\hat{x} - \sin\theta_2\hat{z})H_T \tag{5.111}$$

$$\mathbf{E}_I = -Z_1H_I(\sin\theta_1\hat{x} + \cos\theta_1\hat{z}) \times (\cos\theta_1\hat{x} - \sin\theta_1\hat{z}) = -H_IZ_1\hat{y} \tag{5.112}$$

$$\mathbf{E}_R = -Z_1H_R(\sin\theta_1\hat{x} - \cos\theta_1\hat{z}) \times (\cos\theta_1\hat{x} + \sin\theta_1\hat{z}) = Z_1H_R\hat{y} \tag{5.113}$$

$$\mathbf{E}_T = -Z_2E_T(\sin\theta_2\hat{x} + \cos\theta_2\hat{z}) \times (\cos\theta_2\hat{x} - \sin\theta_2\hat{z}) = -Z_2H_T\hat{y}. \tag{5.114}$$

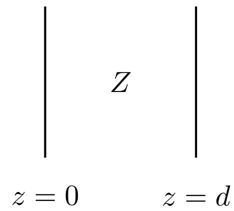
Jednadžbe su tada

$$
\begin{aligned}
(\mathbf{H}_I + \mathbf{H}_R) \cdot \hat{x} &= \mathbf{H}_T \cdot \hat{x} && (5.115)\\
(\mathbf{E}_I + \mathbf{E}_R) \cdot \hat{y} &= \mathbf{E}_T \cdot \hat{y} && (5.116)\\
H_I + H_R &= \frac{\cos\theta_2}{\cos\theta_1} H_T && (5.117)\\
H_I - H_R &= \frac{Z_2}{Z_1} H_T && (5.118)\\
H_I &= \frac{H_T}{2}\left(\frac{\cos\theta_2}{\cos\theta_1} + \frac{Z_2}{Z_1}\right) && (5.119)\\
H_R &= \frac{H_T}{2}\left(\frac{\cos\theta_2}{\cos\theta_1} - \frac{Z_2}{Z_1}\right) && (5.120)\\
&= \frac{\cos\theta_2 Z_1 - \cos\theta_1 Z_2}{\cos\theta_2 Z_1 + \cos\theta_1 Z_2} H_I && (5.121)\\
H_T &= \frac{2\cos\theta_1 Z_1}{\cos\theta_2 Z_1 + \cos\theta_1 Z_2} H_I && (5.122)\\
E_R = Z_1 H_R &= -\frac{\cos\theta_2 Z_1 - \cos\theta_1 Z_2}{\cos\theta_2 Z_1 + \cos\theta_1 Z_2} E_I && (5.123)\\
&= \frac{\cos\theta_1 Z_2 - \cos\theta_2 Z_1}{\cos\theta_2 Z_1 + \cos\theta_1 Z_2} E_I && (5.124)\\
E_T = -Z_2 H_T &= \frac{2\cos\theta_1 Z_2}{\cos\theta_2 Z_1 + \cos\theta_1 Z_2} E_I && (5.125)\\
R = \frac{|\mathbf{E}_R|^2}{|\mathbf{E}_I|^2} &= \left(\frac{\cos\theta_1 Z_2 - \cos\theta_2 Z_1}{\cos\theta_1 Z_2 + \cos\theta_2 Z_1}\right)^2 && (5.126)\\
&= \left(\frac{\cos\theta_1 Z_2 - \cos\theta_2 Z_1}{\cos\theta_1 Z_2 + \cos\theta_2 Z_1}\right)^2 && (5.127)\\
&= \left(\frac{\cos\theta_1 \mu_{r2} n_1 - \cos\theta_2 \mu_{r1} n_2}{\cos\theta_1 \mu_{r2} n_1 + \cos\theta_2 \mu_{r1} n_2}\right)^2 && (5.128)\\
T &= \left|\frac{Z_1}{Z_2}\frac{\hat{k}_T \cdot \hat{z}}{\hat{k}_I \cdot \hat{z}}\right| \frac{4\cos\theta_1^2 Z_2^2}{(\cos\theta_2 Z_2 + \cos\theta_1 Z_1)^2} && (5.129)\\
&= \frac{4\cos\theta_2\cos\theta_1 Z_2 Z_1}{(\cos\theta_2 Z_2 + \cos\theta_1 Z_1)^2} && (5.130)\\
&= \frac{4\cos\theta_2\cos\theta_1 \mu_{r1}\mu_{r2} n_1 n_2}{(\cos\theta_2 \mu_{r2} n_1 + \cos\theta_1 \mu_{r1} n_2)^2} && (5.131)\\
R + T &= \frac{(\cos\theta_1 Z_2 - \cos\theta_2 Z_1)^2 + 4 Z_1 Z_2 \cos\theta_1\cos\theta_2}{(\cos\theta_1 Z_2 + \cos\theta_2 Z_1)^2} = 1. && (5.132)
\end{aligned}
$$

Primjer 5.2.3. Materijal debljine d proteže se duž z osi. Povezati iznose električnog i magnetskog polja na $z = d$ s njihovim vrijednostima u $z = 0$ (matrica transfera).

Rješenje. Kao što vidimo na slici materijal je beskonačan u x i y smjeru, samo u z smjeru ima konačnu dimenziju, a to je da je debljena materijala jednaka d.

Za početak promotrimo elektromagnetski val koji upada okomito na granicu sredstava u ravnini $z = 0$ i neka smjer širenja neka bude $\hat{z}$.



Problem ne promatramo u početnom trenutku kada svjetlost tek počinje interagirati sa atomima materijala jer taj formalizam nadilazi gradivo ovog kolegija već opisujemo nastalo ravnotežno stanje kada se materijal polarizirao od utjecaja vanjskog elektromagnetskog vala. Unutar materijala će tada postojati reflektirani i transmitirani val kao odgovor na vanjsko zračenje. Možemo si orijentirati električno polje u $\hat{x}$, a polje H u $\hat{y}$ smjeru. Onda imamo za ukupno el. polje u materijalu:

$$E(z) = E_I e^{ikz} + E_R e^{-ikz}, \tag{5.133}$$

pri čemu E_I je ovdje oznaka za transmitirani val koji se nalazi u materijalu. Generalno, unutar materijala postoje reflektirani i transmitirani val. U $z = 0$ vrijedi

$$E(z = 0) = E_I + E_R \tag{5.134}$$

$$H(z = 0) = \frac{E_I - E_R}{Z} \tag{5.135}$$

dok u $z = d$ vrijedi

$$E(z = d) = E_I e^{ikd} + E_R e^{-ikd} \tag{5.136}$$

$$H(z = d) = \frac{e^{ikd} E_I - E_R e^{-ikd}}{Z}. \tag{5.137}$$

Matrično, ovi izrazi poprimaju oblik

$$\begin{pmatrix} E \\ H \end{pmatrix} (z = 0) = \begin{pmatrix} 1 & 1 \\ \frac{1}{Z} & \frac{-1}{Z} \end{pmatrix} \begin{pmatrix} E_I \\ E_R \end{pmatrix} \tag{5.138}$$

$$\begin{pmatrix} E \\ H \end{pmatrix} (z = d) = \begin{pmatrix} e^{ikd} & e^{-ikd} \\ \frac{e^{ikd}}{Z} & \frac{-e^{-ikd}}{Z} \end{pmatrix} \begin{pmatrix} E_I \\ E_R \end{pmatrix}. \tag{5.139}$$

Uvrštavanjem dobivamo:

$$\begin{pmatrix} E \\ H \end{pmatrix} (z = d) = \begin{pmatrix} e^{ikd} & e^{-ikd} \\ \frac{e^{ikd}}{Z} & \frac{-e^{-ikd}}{Z} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{1}{Z} & \frac{-1}{Z} \end{pmatrix}^{-1} \begin{pmatrix} E \\ H \end{pmatrix} (z = 0) \tag{5.140}$$

$$= \begin{pmatrix} e^{ikd} & e^{-ikd} \\ \frac{e^{ikd}}{Z} & \frac{-e^{-ikd}}{Z} \end{pmatrix} \frac{1}{-\frac{1}{Z} - \frac{1}{Z}} \begin{pmatrix} -\frac{1}{Z} & -1 \\ \frac{-1}{Z} & 1 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} (z = 0) \tag{5.141}$$

$$= \begin{pmatrix} e^{ikd} & e^{-ikd} \\ \frac{e^{ikd}}{Z} & \frac{-e^{-ikd}}{Z} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & Z \\ 1 & -Z \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} (z = 0) \tag{5.142}$$

$$= \frac{1}{2} \begin{pmatrix} 2\cos kd & 2iZ \sin kd \\ \frac{2i \sin kd}{Z} & 2\cos kd \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} (z = 0) \tag{5.143}$$

$$= \begin{pmatrix} \cos kd & iZ \sin kd \\ \frac{i \sin kd}{Z} & \cos kd \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} (z = 0), \tag{5.144}$$

gdje smo iskoristili formule:

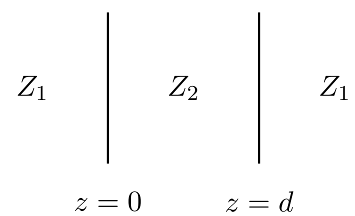
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{5.145}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}. \tag{5.146}$$

Matrica transfera je tada

$$M = \begin{pmatrix} \cos kd & iZ\sin kd \\ \frac{i\sin kd}{Z} & \cos kd \end{pmatrix}. \tag{5.147}$$

Primjer 5.2.4. U područjima $z < 0$ i $z > d$ nalazi se materijal opisan s impedancijom Z_1, a u području $z \in [0, d]$ nalazi se materijal opisan impedancijom Z_2. Pronaći koeficijente refleksije i transmisije za elektromagnetski val koji upada okomito na granice sredstava. Izvrijedniti R, T i $R + T$ za $k_2 d = n\pi$. Također, izračunati R i T ako je $n_1 = 1.5$, $n_2 = 1$, $\lambda = 600\,\mathrm{nm}$ te $d = 30.1\,\mu\mathrm{m}$.



Rješenje. Najteži dio zadatka smo već riješili u prethodnom zadatku uvodeći matricu transfera kojom povezujemo amplitude ukupnih polja E i H izvrijednjena unutar materijala 2 u točki $z = d$ s poljima u $z = 0$ koristeći matricu transfera

$$\begin{pmatrix} E \\ H \end{pmatrix}_2 (z = d) = M \begin{pmatrix} E \\ H \end{pmatrix}_2 (z = 0), \tag{5.148}$$

jer su i električno polje i magnetsko polje paralelni s granicama sredstava te stoga vrijede jednostavni rubni uvjeti (kao što su opisani u izvodu matrice transfera) samo sada u točkama $z = 0$ i $z = d$

$$\begin{pmatrix} E \\ H \end{pmatrix}_1 (z = 0) = \begin{pmatrix} E \\ H \end{pmatrix}_2 (z = 0) \tag{5.149}$$

$$\begin{pmatrix} E \\ H \end{pmatrix}_2 (z = d) = \begin{pmatrix} E \\ H \end{pmatrix}_3 (z = d). \tag{5.150}$$

Koristeći ove rubne uvjete možemo jednostavno povezati ukupna polja E_1 i H_1 u $z = 0$ s E_3 i H_3 u $z = d$

$$\begin{pmatrix} E \\ H \end{pmatrix}_3 (z = d) = \begin{pmatrix} E \\ H \end{pmatrix}_2 (z = d) = M \begin{pmatrix} E \\ H \end{pmatrix}_2 (z = 0) \tag{5.151}$$

$$= M \begin{pmatrix} E \\ H \end{pmatrix}_1 (z = 0). \tag{5.152}$$

U području 1 imamo upadni I i reflektirani R val, dok elektromagnetski val izlazi iz materijala 2 u područje 3 te postoji samo transmitirani val. Područje 2 također ima upadni i reflektirani val, no njega smo već razriješili matricom transfera izvedenom u prethodnom zadatku[1]. Poslužimo se već izvedenim matričnim zapisom za prebacivanje iz reprezentacije

[1] U drugom području transmitirani val smatramo upadnim.

upadnih i reflektiranih valova u reprezentaciju ukupnih polja:

$$\begin{pmatrix} E \\ H \end{pmatrix}_1 (z=0) = \begin{pmatrix} E_I^1 + E_R^1 \\ \frac{E_I^1 - E_R^1}{Z_1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{Z_1} & -\frac{1}{Z_1} \end{pmatrix} \begin{pmatrix} E_I^1 \\ E_R^1 \end{pmatrix} \tag{5.153}$$

$$\begin{pmatrix} E \\ H \end{pmatrix}_3 (z=d) = \begin{pmatrix} E_T^3 \\ \frac{E_T^3}{Z_1} \end{pmatrix} = E_T^3 \begin{pmatrix} 1 \\ \frac{1}{Z_1} \end{pmatrix}. \tag{5.154}$$

I sada iz ove jednadžbe:

$$\begin{pmatrix} E \\ H \end{pmatrix}_3 (z=d) = M \begin{pmatrix} E \\ H \end{pmatrix}_1 (z=0), \tag{5.155}$$

lijevu stranu zapišemo preko 5.238, a desnu preko dobivamo:

$$E_T^3 \begin{pmatrix} 1 \\ \frac{1}{Z_1} \end{pmatrix} = M \begin{pmatrix} E_1 \\ H_1 \end{pmatrix} (0) \tag{5.156}$$

$$= M \begin{pmatrix} 1 & 1 \\ \frac{1}{Z_1} & \frac{-1}{Z_1} \end{pmatrix} \begin{pmatrix} E_I^1 \\ E_R^1 \end{pmatrix}. \tag{5.157}$$

Primijetimo da je nepoznanica E_T^3 samo multiplikativni faktor te si možemo pomoći invertiranjem kako bi izrazili E_I^1 i E_R^1 preko E_T^3 :

$$\begin{pmatrix} E_I^1 \\ E_R^1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{Z_1} & \frac{-1}{Z_1} \end{pmatrix}^{-1} M^{-1} E_T^3 \begin{pmatrix} 1 \\ \frac{1}{Z_1} \end{pmatrix} \tag{5.158}$$

$$= E_T^3 \frac{1}{\frac{-2}{Z_1}} \begin{pmatrix} \frac{-1}{Z_1} & -1 \\ \frac{-1}{Z_1} & 1 \end{pmatrix} \frac{1}{\det M} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{Z_1} \end{pmatrix} \tag{5.159}$$

$$= \frac{E_T^3}{2 \det M} \begin{pmatrix} 1 & Z_1 \\ 1 & -Z_1 \end{pmatrix} \begin{pmatrix} M_{22} - \frac{M_{12}}{Z_1} \\ -M_{21} + \frac{M_{11}}{Z_1} \end{pmatrix} \tag{5.160}$$

$$= \frac{E_T^3}{2 \det M} \begin{pmatrix} M_{22} - \frac{M_{12}}{Z_1} - Z_1 M_{21} + M_{11} \\ M_{22} - \frac{M_{12}}{Z_1} + Z_1 M_{21} - M_{11} \end{pmatrix}. \tag{5.161}$$

Vidimo iz prvog retka vektora stupca da je E_I^1 jednak:

$$E_I^1 = \frac{E_T^3}{2 \det M} \left(M_{22} + M_{11} - \frac{M_{12}}{Z_1} - Z_1 M_{21} \right). \tag{5.162}$$

Drugi redak vektora stupca će dati da je E_R jednak:

$$E_R^1 = \frac{E_T^3}{2 \det M} \left(M_{22} - \frac{M_{12}}{Z_1} + Z_1 M_{21} - M_{11} \right). \tag{5.163}$$

I sada trebamo izraziti E_T^3 i E_R^1 preko E_I^1 da bi mogli izračunati omjere koji su nam potrebni za koeficijente refleksije i transmisije.:

$$t = \frac{E_T^3}{E_I^1} = \frac{2\det M}{M_{22} + M_{11} - \frac{M_{12}}{Z_1} - Z_1 M_{21}} \tag{5.164}$$

$$r = \frac{E_R^1}{E_I^1} = \frac{E_T^3}{E_I^1}\frac{M_{22} - \frac{M_{12}}{Z_1} + Z_1 M_{21} - M_{11}}{2\det M} \tag{5.165}$$

$$= \frac{M_{22} - \frac{M_{12}}{Z_1} + Z_1 M_{21} - M_{11}}{M_{22} + M_{11} - \frac{M_{12}}{Z_1} - Z_1 M_{21}}. \tag{5.166}$$

Matrica transfera je

$$M = \begin{pmatrix} \cos k_2 d & iZ_2 \sin k_2 d \\ \frac{i \sin k_2 d}{Z_2} & \cos k_2 d \end{pmatrix}, \tag{5.167}$$

te je njezina determinanta

$$\det M = \cos^2 k_2 d - i^2 \sin^2 k_2 d = 1. \tag{5.168}$$

Izračunamo brojčane iznose matričnih elemenata i potom ih uvrstimo u formule za r i t. Kako su u pitanju kompleksni brojevi, a trebaju nam kvadrati modula, poslužit ćemo se formulom:

$$\left|\frac{A + iB}{C + iD}\right|^2 = \frac{A^2 + B^2}{C^2 + D^2}. \tag{5.169}$$

Koeficijente refleksije i transmisije možemo lako izračunati koristeći r i t

$$R = \left|\frac{\mathbf{E}_R}{\mathbf{E}_I}\right|^2 \tag{5.170}$$

$$= |r|^2 \tag{5.171}$$

$$T = \left|\frac{Z_1}{Z_1}\frac{\hat{k}_T \cdot \hat{z}}{\hat{k}_I \cdot \hat{z}}\right| |t|^2 \tag{5.172}$$

$$= |t|^2, \tag{5.173}$$

gdje smo uvrstili da je impedancija u području 3 ista kao u području 1.

Za $k_2 d = n\pi$ vrijedi:

$$M = \begin{pmatrix} (-1)^n & 0 \\ 0 & (-1)^n \end{pmatrix} \tag{5.174}$$

$$t = \frac{2 \det M}{M_{22} + M_{11} - \frac{M_{12}}{Z_1} - Z_1 M_{21}} \tag{5.175}$$

$$= \frac{2}{2(-1)^n - 0 - 0} \tag{5.176}$$

$$= (-1)^n \tag{5.177}$$

$$r = \frac{M_{22} - \frac{M_{12}}{Z_1} + Z_1 M_{21} - M_{11}}{M_{22} + M_{11} - \frac{M_{12}}{Z_1} - Z_1 M_{21}} \tag{5.178}$$

$$= \frac{(-1)^n - (-1)^n}{2(-1)^n - 0 - 0} \tag{5.179}$$

$$R = |r|^2 = 0 \tag{5.180}$$

$$T = |t|^2 = 1 \tag{5.181}$$

$$R + T = 1. \tag{5.182}$$

Ukoliko bi u pitanju bila svjetlost koja je vidljiva za ljudske oči intrepretacija preko fizikalne optike je da se u ovom slučaju ispunio uvjet destruktivne interferencije jer se ništa ne bi reflektiralo natrag prema promatraču od granice koja se nalazi na $z = 0$. Za $k_2 d = (2n-1)\frac{\pi}{2}$ bi u slučaju vidljive svjetlosti imali konstruktivnu interferenciju jer se svjetlo najjače reflektiralo od materijala prema promatraču. Maksimumi refleksije na slici 5.1 nam ukazuju da nije riječ o potpunoj konstruktivnoj interferenciji. Za posebni uvjet, koji odgovara staklu, dobivamo

$$Z_1 = 251.153\,\Omega \tag{5.183}$$

$$Z_2 = 376.730\,\Omega \tag{5.184}$$

$$k_2 d = 2\pi \frac{30100}{600} \tag{5.185}$$

$$= \frac{301\pi}{3} = 100\pi + \frac{\pi}{3} \tag{5.186}$$

$$M = \begin{pmatrix} \cos\frac{\pi}{3} & iZ_2 \sin\frac{\pi}{3} \\ \frac{i \sin\frac{\pi}{3}}{Z_2} & \cos\frac{\pi}{3} \end{pmatrix} \tag{5.187}$$

$$= \begin{pmatrix} 0.5 & 326.258i \\ 0.002i & 0.5 \end{pmatrix} \tag{5.188}$$

$$r = 0.3 - 0.16i \tag{5.189}$$

$$R = 0.115 \tag{5.190}$$

$$t = 0.442 + 0.83i \tag{5.191}$$

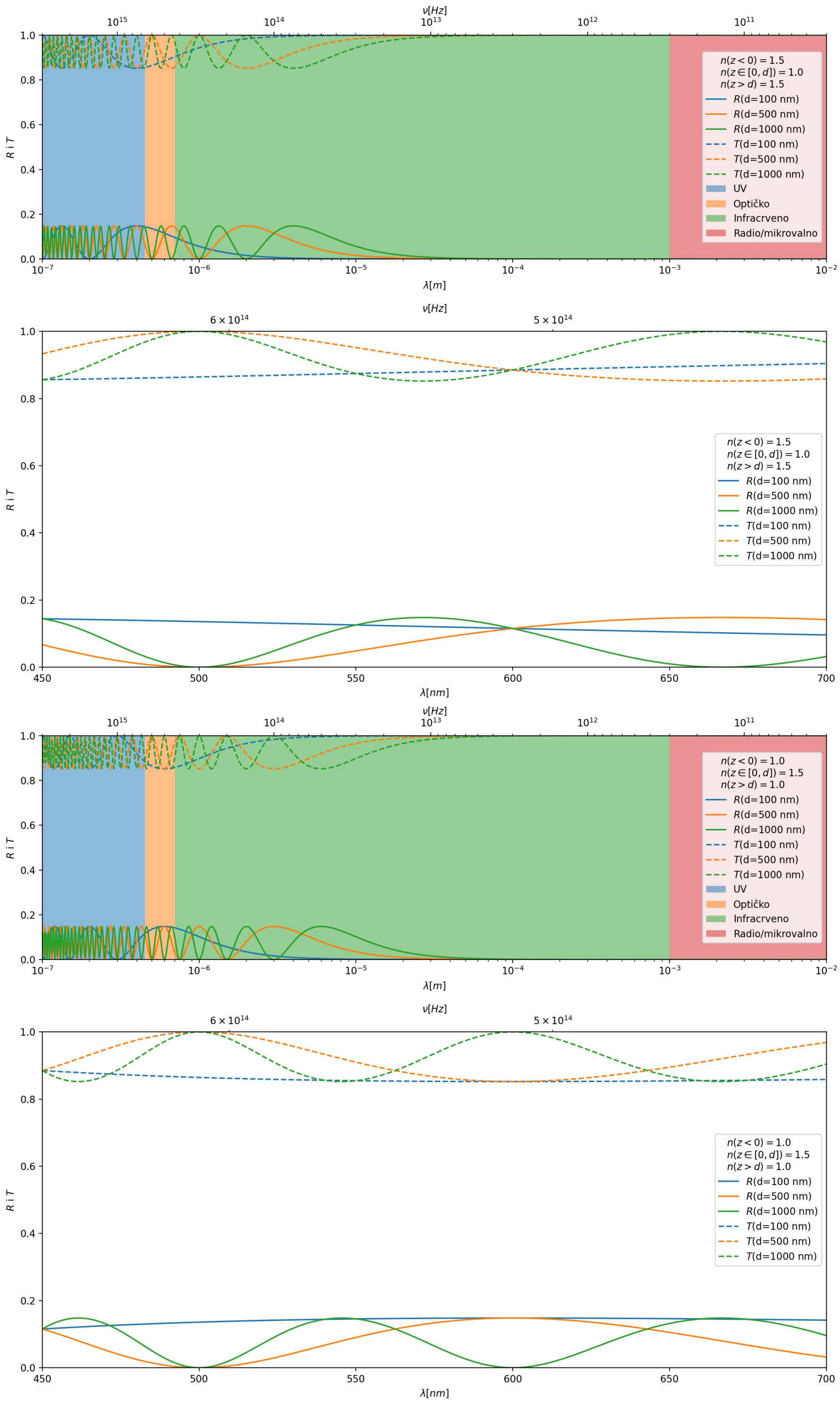
$$T = 0.885 \tag{5.192}$$

$$R + T = 1. \tag{5.193}$$

Rezultat je taj da zbog visokog koeficijenta transmisije praktički se ništa ne reflektira s najbliže granične plohe prema promatraču, odnosno staklo jest za spektar vidljivog elektromagnetskog zračenja prozirno. Na slici 5.1 je prikazano poopćenje ovog zadatka za različite valne duljine, za slučaj staklo-zrak-staklo i zrak-staklo zrak, dok su u tablici 5.1 dani izračuni za različite materijale i debljinu od 30.1 μm i $\lambda = 600$ nm.

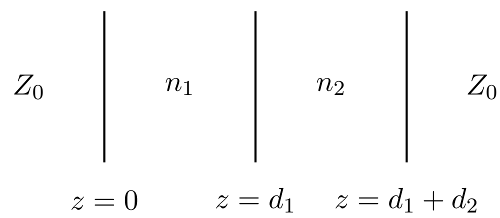
Tablica 5.1: Izračuni za različite materijale, debljinu od 30.1 μm i $\lambda = 600$ nm.

Sredstvo 1	Sredstvo 2	n_1	n_2	r	t	R	T	$R+T$
Staklo	Zrak	1.5	1.0	$0.300 - 0.160i$	$0.442 + 0.830i$	0.115	0.885	1.000
Voda	Zrak	1.33	1.0	$0.212 - 0.118i$	$0.471 + 0.848i$	0.059	0.941	1.000
Dijamant	Zrak	2.42	1.0	$0.607 - 0.248i$	$0.285 + 0.699i$	0.430	0.570	1.000
Zrak	Staklo	1.0	1.5	$-0.385 - 0.000i$	$-0.000 + 0.923i$	0.148	0.852	1.000
Voda	Staklo	1.33	1.5	$-0.120 - 0.000i$	$-0.000 + 0.993i$	0.014	0.986	1.000
Dijamant	Staklo	2.42	1.5	$0.445 + 0.000i$	$-0.000 + 0.896i$	0.198	0.802	1.000
Zrak	Voda	1.0	1.33	$-0.270 + 0.047i$	$-0.164 - 0.948i$	0.075	0.925	1.000
Staklo	Voda	1.5	1.33	$0.116 - 0.021i$	$-0.175 - 0.978i$	0.014	0.986	1.000
Dijamant	Voda	2.42	1.33	$0.524 - 0.080i$	$-0.127 - 0.838i$	0.281	0.719	1.000
Zrak	Dijamant	1.0	2.42	$-0.349 - 0.354i$	$-0.618 + 0.609i$	0.247	0.753	1.000
Staklo	Dijamant	1.5	2.42	$-0.167 - 0.215i$	$-0.760 + 0.590i$	0.074	0.926	1.000
Voda	Dijamant	1.33	2.42	$-0.217 - 0.263i$	$-0.726 + 0.598i$	0.116	0.884	1.000



Slika 5.1: Koeficijenti transmisije i refleksije za materijal indeksa loma n_2 različitih debljina.

Primjer 5.2.5. Materijali debljina $d_1 = 1\,\mu$m i $d_2 = 1\,\mu$m i indeksa loma $n_1 = 1.5$ i $n_2 = 2$ i s $\mu_{1r} = \mu_{2r} = 1$ postavljeni su duž z osi te na njih upada elektromagnetski val valne duljine 550 nm, usmjeren duž z osi. Izračunati matricu transfera i koeficijente refleksije i transmisije ako se materijali nalaze u zraku.



Rješenje. Prijelaz kroz materijal 1 dobivamo matricom transfera M_1, a kroz materijal 2 matricom transfera M_2. Budući da val upada okomito na sredstva, ne trebamo se brinuti o matricama kuteva za TE i TM modove te samo množimo

$$M_1 = \begin{pmatrix} \cos k_1 d_1 & iZ_1 \sin k_1 d_1 \\ \frac{i \sin k_1 d_1}{Z_1} & \cos k_1 d_1 \end{pmatrix} \tag{5.194}$$

$$M_2 = \begin{pmatrix} \cos k_2 d_2 & iZ_2 \sin k_2 d_2 \\ \frac{i \sin k_2 d_2}{Z_2} & \cos k_2 d_2 \end{pmatrix} \tag{5.195}$$

$$\begin{aligned}
\begin{pmatrix} E \\ H \end{pmatrix}_3 (z = d_1 + d_2) &= \begin{pmatrix} E \\ H \end{pmatrix}_2 (z = d_1 + d_2) && (5.196)\\
&= M_2 \begin{pmatrix} E \\ H \end{pmatrix}_2 (z = d_1) && (5.197)\\
&= M_2 \begin{pmatrix} E \\ H \end{pmatrix}_1 (z = d_1) && (5.198)\\
&= M_2 M_1 \begin{pmatrix} E \\ H \end{pmatrix}_1 (z = 0) && (5.199)\\
&= M_2 M_1 \begin{pmatrix} E \\ H \end{pmatrix}_0 (z = 0) && (5.200)\\
&= M \begin{pmatrix} E \\ H \end{pmatrix}_0 (z = 0). && (5.201)
\end{aligned}$$

Kao i u prethodnim primjerima prvo zapišemo ukupna polja E i H u sredstvu 0 na $z < 0$ na upadni i reflektirani val, a za $z \geq d_1 + d_2$ (sredstvo 3) imamo samo transmitirani val:

$$\begin{pmatrix} E \\ H \end{pmatrix}_0 (z = 0) = \begin{pmatrix} E_I + E_R \\ \frac{E_I - E_R}{Z_0} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1/Z_0 & -1/Z_0 \end{pmatrix} \begin{pmatrix} E_I \\ E_R \end{pmatrix} \tag{5.202}$$

$$\begin{pmatrix} E \\ H \end{pmatrix}_3 (z = d_1 + d_2) = \begin{pmatrix} E_T \\ E_T/Z_0 \end{pmatrix}. \tag{5.203}$$

Preko matrice transfera povežemo ukupna polja E i H u sredstvu 3 na granici $z = d_1 + d_2$ sa ukupnim poljima E i H u u sredstvu 3 za $z = 0$

$$\begin{pmatrix} E \\ H \end{pmatrix}_3 (z = d_1 + d_2) = M \begin{pmatrix} E \\ H \end{pmatrix}_0 (z = 0), \tag{5.204}$$

te lijevu stranu zapišemo preko 5.203, a desnu preko 5.202:

$$\begin{pmatrix} E_T \\ E_T/Z_0 \end{pmatrix} = M \begin{pmatrix} 1 & 1 \\ 1/Z_0 & -1/Z_0 \end{pmatrix} \begin{pmatrix} E_I \\ E_R \end{pmatrix}. \tag{5.205}$$

Invertirajući dobivamo upadni i reflektirani preko transmitiranog:

$$\begin{aligned}
\begin{pmatrix} E_I \\ E_R \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ \frac{1}{Z_0} & \frac{-1}{Z_0} \end{pmatrix}^{-1} M^{-1} E_T \begin{pmatrix} 1 \\ \frac{1}{Z_0} \end{pmatrix} && (5.206)\\
&= E_T \frac{1}{\frac{-2}{Z_0}} \begin{pmatrix} \frac{-1}{Z_0} & -1 \\ \frac{-1}{Z_0} & 1 \end{pmatrix} \frac{1}{\det M} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{Z_0} \end{pmatrix} && (5.207)\\
&= \frac{E_T}{2\det M} \begin{pmatrix} 1 & Z_0 \\ 1 & -Z_0 \end{pmatrix} \begin{pmatrix} M_{22} - \frac{M_{12}}{Z_0} \\ -M_{21} + \frac{M_{11}}{Z_0} \end{pmatrix} && (5.208)\\
&= \frac{E_T}{2\det M} \begin{pmatrix} M_{22} - \frac{M_{12}}{Z_0} - Z_0 M_{21} + M_{11} \\ M_{22} - \frac{M_{12}}{Z_0} + Z_0 M_{21} - M_{11} \end{pmatrix}. && (5.209)\\
& && (5.210)
\end{aligned}$$

Vidimo iz prvog retka vektora stupca da je E_I jednak:

$$E_I = \frac{E_T}{2\det M} \left(M_{22} + M_{11} - \frac{M_{12}}{Z_0} - Z_0 M_{21} \right). \tag{5.211}$$

Iz toga dobivamo da je t:

$$t = \frac{E_T}{E_I} = \frac{2\det M}{M_{22} + M_{11} - \frac{M_{12}}{Z_0} - Z_0 M_{21}} \tag{5.212}$$

$$\tag{5.213}$$

Drugi redak vektora stupca će dati da je E_R jednak:

$$E_R = \frac{E_T}{2\det M} \left(M_{22} - \frac{M_{12}}{Z_0} + Z_0 M_{21} - M_{11} \right), \tag{5.214}$$

pa za r imamo:

$$\begin{aligned}
r &= \frac{E_R}{E_I} && (5.215)\\
&= \frac{E_T}{E_I} \frac{M_{22} - \frac{M_{12}}{Z_0} + Z_0 M_{21} - M_{11}}{2\det M} && (5.216)\\
&= \frac{M_{22} - \frac{M_{12}}{Z_0} + Z_0 M_{21} - M_{11}}{M_{22} + M_{11} - \frac{M_{12}}{Z_0} - Z_0 M_{21}}. && (5.217)
\end{aligned}$$

Direktnim računom dobivamo

$$k_1 = \frac{2\pi n_1}{\lambda} = 1.71 \times 10^7\,\mathrm{m}^{-1} \tag{5.218}$$

$$k_2 = \frac{2\pi n_2}{\lambda} = 2.28 \times 10^7\,\mathrm{m}^{-1} \tag{5.219}$$

$$k_1 d_1 = 17.136 \tag{5.220}$$

$$k_2 d_2 = 22.848 \tag{5.221}$$

$$Z_1 = Z_0 \frac{\mu_{1r}}{n_1} = 251.153\,\Omega \tag{5.222}$$

$$Z_2 = Z_0 \frac{\mu_{2r}}{n_2} = 188.365\,\Omega \tag{5.223}$$

$$M_1 = \begin{pmatrix} -0.142 & -248.597i \\ -0.004i & -0.142 \end{pmatrix} \tag{5.224}$$

$$M_2 = \begin{pmatrix} -0.655 & -142.357i \\ -0.004i & -0.655 \end{pmatrix} \tag{5.225}$$

$$M = M_2 M_1 = \begin{pmatrix} -0.468 & 183.056i \\ 0.003i & -0.904 \end{pmatrix} \tag{5.226}$$

$$\det M = 1 \tag{5.227}$$

$$r = -0.123 - 0.361i \tag{5.228}$$

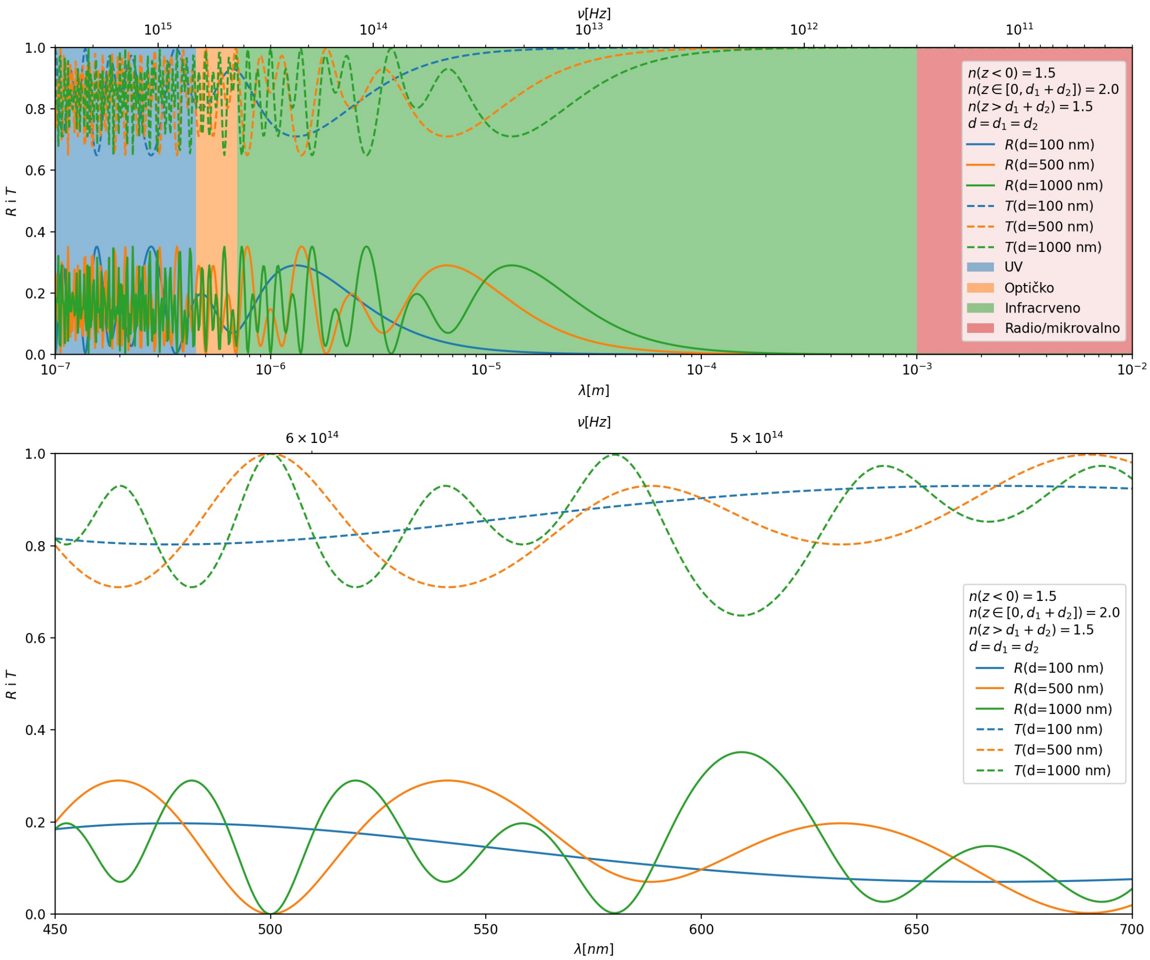
$$t = -0.586 + 0.715i \tag{5.229}$$

$$R = |r|^2 = 0.146 \tag{5.230}$$

$$T = |t|^2 = 0.854 \tag{5.231}$$

$$R + T = 1.000 \tag{5.232}$$

Na slici 5.2 je prikazan izračun za različite frekvencije i debljine materijala.



Slika 5.2: Koeficijenti refleksije i transmisije za različite valne duljine i dva sredstva.

Primjer 5.2.6. Svjetlost valne duljine 300 nm upada okomito na granicu sredstva $x=0$.

- Koristeći matricu tranfera izračunajte koeficijente refleksije i transmisije na granici sredstva u $x=15.05\,\mu$m te na granici u $x=30.1\,\mu$m.
- Sredstva 1 i 2 zamijene mjesta. Koristeći matricu tranfera izračunajte koeficijente refleksije i transmisije za granicu sredstva $x=15.05\,\mu$m te za granicu $x=30.1\,\mu$m.

Relativna permeabilnost je svagdje jednaka jedinici.

$$n_0=1 \quad\Big|\quad n_1=1.5 \quad\Big|\quad n_2=2 \quad\Big|\quad n_3=1$$

$$x=0 \quad x=15.05\mu\text{m} \quad x=30.1\mu\text{m}$$

Rješenje. Povezujemo ukupna polja u $x=a_1+a_2$ i $x=0$ u području koje se nalazi nakon sredstva,

$$\begin{pmatrix} E \\ H \end{pmatrix}_3 (x = a_1 + a_2) = M_2 \begin{pmatrix} E \\ H \end{pmatrix}_2 (x = a_1) = M_2 M_1 \begin{pmatrix} E \\ H \end{pmatrix}_1 (x = 0) = M^{0\to3} \begin{pmatrix} E \\ H \end{pmatrix}_0 (x = 0) \tag{5.233}$$

jer su i električno polje i magnetsko polje paralelni s granicama sredstava te stoga vrijede jednostavni rubni uvjeti. Pritom smo iskoristili rubne uvjete

$$\begin{pmatrix} E \\ H \end{pmatrix}_0 (x = 0) = \begin{pmatrix} E \\ H \end{pmatrix}_1 (x = 0) \tag{5.234}$$

$$\begin{pmatrix} E \\ H \end{pmatrix}_2 (x = a_1) = \begin{pmatrix} E \\ H \end{pmatrix}_1 (x = a_1) \tag{5.235}$$

$$\begin{pmatrix} E \\ H \end{pmatrix}_3 (x = a_1 + a_2) = \begin{pmatrix} E \\ H \end{pmatrix}_2 (x = a_1 + a_2) \tag{5.236}$$

U području 1 imamo upadni I i reflektirani R val, dok elektromagnetski val izlazi iz materijala 2 u područje 3 te postoji samo transmitirani val. Poslužimo se već izvedenim matričnim zapisom za prebacivanje iz reprezentacije upadnih i reflektiranih valova u reprezentaciju ukupnih polja:

$$\begin{pmatrix} E \\ H \end{pmatrix}_0 (x = 0) = \begin{pmatrix} E_I^0 + E_R^0 \\ \frac{E_I^0 - E_R^0}{Z_0} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{Z_0} & -\frac{1}{Z_0} \end{pmatrix} \begin{pmatrix} E_I^0 \\ E_R^0 \end{pmatrix} \tag{5.237}$$

$$\begin{pmatrix} E \\ H \end{pmatrix}_3 (x = a_1 + a_2) = \begin{pmatrix} E_T^3 \\ \frac{E_T^3}{Z_3} \end{pmatrix} = E_T^3 \begin{pmatrix} 1 \\ \frac{1}{Z_3} \end{pmatrix}. \tag{5.238}$$

Dobivamo

$$E_T^3 \begin{pmatrix} 1 \\ \frac{1}{Z_3} \end{pmatrix} = M^{0\to3} \begin{pmatrix} E \\ H \end{pmatrix}_0 (x = 0) \tag{5.239}$$

$$= M^{0\to3} \begin{pmatrix} 1 & 1 \\ \frac{1}{Z_0} & \frac{-1}{Z_0} \end{pmatrix} \begin{pmatrix} E_I^0 \\ E_R^1 \end{pmatrix}. \tag{5.240}$$

Primijetimo da je nepoznanica E_T^3 samo multiplikativni faktor te si možemo pomoći invertiranjem kako bi izrazili E_I^0 i E_R^0 preko E_T^3 :

$$\begin{pmatrix} E_I^0 \\ E_R^0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{Z_0} & \frac{-1}{Z_0} \end{pmatrix}^{-1} (M^{0\to3})^{-1} E_T^3 \begin{pmatrix} 1 \\ \frac{1}{Z_3} \end{pmatrix} \tag{5.241}$$

$$= E_T^3 \frac{1}{\frac{-2}{Z_0}} \begin{pmatrix} \frac{-1}{Z_0} & -1 \\ \frac{-1}{Z_0} & 1 \end{pmatrix} \frac{1}{\det M} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{Z_3} \end{pmatrix} \tag{5.242}$$

$$= \frac{E_T^3}{2 \det M} \begin{pmatrix} 1 & Z_0 \\ 1 & -Z_0 \end{pmatrix} \begin{pmatrix} M_{22} - \frac{M_{12}}{Z_3} \\ -M_{21} + \frac{M_{11}}{Z_3} \end{pmatrix} \tag{5.243}$$

$$= \frac{E_T^3}{2 \det M} \begin{pmatrix} M_{22} - \frac{M_{12}}{Z_3} - Z_0 M_{21} + M_{11} \frac{Z_0}{Z_3} \\ M_{22} - \frac{M_{12}}{Z_3} + Z_0 M_{21} - M_{11} \frac{Z_0}{Z_3} \end{pmatrix}. \tag{5.244}$$

Vidimo iz prvog retka vektora stupca da je E_I^0 jednak:

$$E_I^0 = \frac{E_T^3}{2\det M}\left(M_{22} + M_{11}\frac{Z_0}{Z_3} - \frac{M_{12}}{Z_3} - Z_0 M_{21}\right). \tag{5.245}$$

Drugi redak vektora stupca će dati da je E_R jednak:

$$E_R^0 = \frac{E_T^3}{2\det M}\left(M_{22} - \frac{M_{12}}{Z_3} + Z_0 M_{21} - M_{11}\frac{Z_0}{Z_3}\right). \tag{5.246}$$

I sada trebamo izraziti E_T^3 i E_R^1 preko E_I^1 da bi mogli izračunati omjere koji su nam potrebni za koeficijente refleksije i transmisije.:

$$t = \frac{E_T^3}{E_I^0} = \frac{2\det M}{M_{22} + M_{11}\frac{Z_0}{Z_3} - \frac{M_{12}}{Z_3} - Z_0 M_{21}} \tag{5.247}$$

$$r = \frac{E_R^1}{E_I^1} = \frac{E_T^3}{E_I^1}\frac{M_{22} - \frac{M_{12}}{Z_3} + Z_0 M_{21} - M_{11}\frac{Z_0}{Z_3}}{2\det M} \tag{5.248}$$

$$= \frac{M_{22} - \frac{M_{12}}{Z_3} + Z_0 M_{21} - M_{11}\frac{Z_0}{Z_3}}{M_{22} + M_{11}\frac{Z_0}{Z_3} - \frac{M_{12}}{Z_3} - Z_0 M_{21}}. \tag{5.249}$$

Izračunamo brojčane iznose matričnih elemenata i potom ih uvrstimo u formule za r i t. Kako su u pitanju kompleksni brojevi, a trebaju nam kvadrati modula, poslužit ćemo se formulom:

$$\left|\frac{A + iB}{C + iD}\right|^2 = \frac{A^2 + B^2}{C^2 + D^2}. \tag{5.250}$$

Koeficijente refleksije i transmisije možemo lako izračunati koristeći r i t

$$R = \left|\frac{\mathbf{E}_R}{\mathbf{E}_I}\right|^2 \tag{5.251}$$

$$= |r|^2 \tag{5.252}$$

$$T = \left|\frac{\hat{k}_T \cdot (\mathbf{E}_T \times \mathbf{H}_T)}{\hat{k}_I \cdot (\mathbf{E}_I \times \mathbf{H}_I)}\right| |t|^2 \tag{5.253}$$

$$= \left|\frac{\frac{\hat{k}_T \cdot \hat{x}}{Z_3}}{\frac{\hat{k}_I \cdot \hat{x}}{Z_0}}\right| |t|^2 \tag{5.254}$$

$$= \frac{Z_0}{Z_3}|t|^2, \tag{5.255}$$

Ove izraze možemo iskoristiti za račun na svakoj granici sredstava, uz odgovarajuće zamjene indeksa. Uvrstimo brojke

$$
\begin{aligned}
n_1 &= 1.5 && (5.256)\\
n_2 &= 2 && (5.257)\\
n_3 &= 1 && (5.258)\\
Z_0 &= 376.73\,\Omega && (5.259)\\
Z_1 &= 251.15\,\Omega && (5.260)\\
Z_2 &= 188.37\,\Omega && (5.261)\\
Z_3 &= 376.73\,\Omega && (5.262)\\
k_1 a &= n_1 2\pi \frac{15050}{300} && (5.263)\\
&= \frac{3}{2}\frac{301\pi}{3} = 150\pi + \frac{\pi}{2} && (5.264)\\
k_2 a &= n_2 2\pi \frac{15050}{300} && (5.265)\\
&= 2\frac{301\pi}{3} = 200\pi + \frac{2\pi}{3} && (5.266)
\end{aligned}
$$

Izračunamo matrice transfera za sve kombinacije

$$
\begin{aligned}
M_1 &= \begin{pmatrix} 0 & 251.153i \\ i0.004 & 0 \end{pmatrix} && (5.267)\\
M_2 &= \begin{pmatrix} -0.5 & 163.129i \\ 0.005i & -0.5 \end{pmatrix} && (5.268)\\
M^{0\to 3} &= M_2 M_1 = \begin{pmatrix} -0.649 & -125.577i \\ -0.002i & -1.155 \end{pmatrix} && (5.269)\\
M^{3\to 0} &= M_1 M_2 = \begin{pmatrix} -1.155 & -125.577i \\ -0.002i & -0.649 \end{pmatrix} && (5.270)
\end{aligned}
$$

Kroz sredstvo 1

$$
\begin{aligned}
r &= \frac{0.000\frac{376.73}{188.37} - 0.000 + \frac{-251.153i}{188.37}0.004i \cdot 376.73}{-0.000\frac{376.73}{188.37} - 0.000 + \frac{-251.153i}{188.37} - 0.004i \cdot 376.73} = \frac{0.166i}{-2.833i} && (5.271)\\
R &= \frac{(0.0)^2 + (0.1661)^2}{(0.0)^2 + (-2.8327)^2} = \frac{0.0276^2}{8.0242^2} = 0.003 && (5.272)\\
t &= \frac{2}{-0.000\frac{376.73}{188.37} - 0.000 + \frac{-251.153i}{188.365} - 0.004i \cdot 376.730} = \frac{2}{-2.833i} && (5.273)\\
T &= \frac{376.730}{188.365}\frac{(2)^2 + (0)^2}{(0.0)^2 + (-2.8327)^2} = \frac{4^2}{8.0242^2} = 0.997 && (5.274)
\end{aligned}
$$

Kroz oba sredstva

$$r = \frac{0.649\frac{376.73}{376.73} - 1.155 + \frac{125.577i}{376.73} - 0.002i \cdot 376.73}{-0.649\frac{376.73}{376.73} - 1.155 + \frac{125.577i}{376.73} + 0.002i \cdot 376.73} = \frac{-0.506 - 0.416i}{-1.805 + 1.083i} \tag{5.275}$$

$$R = \frac{(-0.5061)^2 + (-0.4164)^2}{(-1.8046)^2 + (1.083)^2} = \frac{0.4295^2}{4.4295^2} = 0.097 \tag{5.276}$$

$$t = \frac{2}{-0.649\frac{376.73}{376.73} - 1.155 + \frac{125.577i}{376.730} + 0.002i \cdot 376.730} = \frac{2}{-1.805 + 1.083i} \tag{5.277}$$

$$T = \frac{376.730}{376.730}\frac{(2)^2 + (0)^2}{(-1.8046)^2 + (1.083)^2} = \frac{4^2}{4.4295^2} = 0.903 \tag{5.278}$$

Obratnjem dobivamo

$$r = \frac{0.500\frac{376.73}{251.15} - 0.500 + \frac{-163.129i}{251.15} + 0.005i \cdot 376.73}{-0.500\frac{376.73}{251.15} - 0.500 + \frac{-163.129i}{251.15} - 0.005i \cdot 376.73} = \frac{0.250 + 1.083i}{-1.250 - 2.382i} \tag{5.279}$$

$$R = \frac{(0.25)^2 + (1.0834)^2}{(-1.25)^2 + (-2.3825)^2} = \frac{1.2363^2}{7.2388^2} = 0.171 \tag{5.280}$$

$$t = \frac{2}{-0.500\frac{376.73}{251.15} - 0.500 + \frac{-163.129i}{251.153} - 0.005i \cdot 376.730} = \frac{2}{-1.250 + -2.382i} \tag{5.281}$$

$$T = \frac{376.730}{251.153}\frac{(2)^2 + (0)^2}{(-1.25)^2 + (-2.3825)^2} = \frac{4^2}{7.2388^2} = 0.829 \tag{5.282}$$

Za oba sredstva

$$r = \frac{1.155\frac{376.73}{376.73} - 0.649 + \frac{125.577i}{376.73} - 0.002i \cdot 376.73}{-1.155\frac{376.73}{376.73} - 0.649 + \frac{125.577i}{376.73} + 0.002i \cdot 376.73} = \frac{0.506 - 0.416i}{-1.805 + 1.083i} \tag{5.283}$$

$$R = \frac{(0.5061)^2 + (-0.4164)^2}{(-1.8046)^2 + (1.083)^2} = \frac{0.4295^2}{4.4295^2} = 0.097 \tag{5.284}$$

$$t = \frac{2}{-1.155\frac{376.73}{376.73} - 0.649 + \frac{125.577i}{376.730} + 0.002i \cdot 376.730} = \frac{2}{-1.805 + 1.083i} \tag{5.285}$$

$$T = \frac{376.730}{376.730}\frac{(2)^2 + (0)^2}{(-1.8046)^2 + (1.083)^2} = \frac{4^2}{4.4295^2} = 0.903 \tag{5.286}$$

6. Specijalna teorija relativnosti

6.1 Lorentzove transformacije

Ako se prebacujemo između koordinatnog sustava, S, opisanog koordinatama t, x, y, z i sustava S. U sustavu S se sustav S' giba brzinom v u $\hat{x}$ smjeru. Vremena i položaje objekata opisujemo koordinatama

$$t' = \gamma\left(t - \frac{v}{c}\frac{x}{c}\right) \tag{6.1}$$

$$x' = \gamma\left(x - vt\right) \tag{6.2}$$

$$y' = y \tag{6.3}$$

$$z' = z \tag{6.4}$$

Ovaj izraz možemo zapisati matrično, ako uvedemo veličinu ct, koja se mjeri u metrima te $\beta = v/c$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}. \tag{6.5}$$

Analogno, ako želimo saznati obrnutu transformaciju, vrijedi izraz

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}. \tag{6.6}$$

Možemo uvesti matricu $\Lambda(\beta)$ kojome opisujemo transformacije

$$\Lambda(\beta) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.7}$$

$$\Lambda(-\beta) = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.8}$$

Prema tome, transformacije vektora s 4 komponente (četverovektora), možemo kompaktnije matrično zapisati[1]

$$r' = \Lambda(\beta) r \tag{6.9}$$

$$r = \Lambda(-\beta) r' \tag{6.10}$$

Primijetimo da se, gledano iz sustava S', sustav S giba brzinom $-v\hat{x}$ pa možemo skraćeno zapisati tranformacije u sva tri smjera

$$\Lambda(\beta\hat{x}) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.11}$$

$$\Lambda(\beta\hat{y}) = \begin{pmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.12}$$

$$\Lambda(\beta\hat{z}) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}. \tag{6.13}$$

Napomena: općenita Lorentzova transformacija nije samo umnožak ovih matrica zato što ne komutiraju! Promotrimo dvije posebne situacije. U sustavu S' možemo mjeriti period

[1] Na četverovektore se po konvenciji ne pišu strijelice.

između 2 događaja, 1 i 2, koji se zbivaju na istim prostornim koordinatama, $\mathbf{r}'_1 = \mathbf{r}'_2$:

$$\begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} ct_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} - \begin{pmatrix} ct_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} = \Lambda(-\beta) \begin{pmatrix} ct'_2 \\ x'_2 \\ y'_2 \\ z'_2 \end{pmatrix} - \Lambda(-\beta) \begin{pmatrix} ct'_1 \\ x'_1 \\ y'_1 \\ z'_1 \end{pmatrix} \tag{6.14}$$

$$= \Lambda(-\beta) \begin{pmatrix} c(t'_2 - t'_1) \\ x'_2 - x'_1 \\ y'_2 - y'_1 \\ z'_2 - z'_1 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t' \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{6.15}$$

$$= \begin{pmatrix} \gamma(c\Delta t' + 0\beta) \\ \beta\gamma c\Delta t' + \gamma 0 \\ 0 \\ 0 \end{pmatrix} \tag{6.16}$$

$$= \begin{pmatrix} \gamma c\Delta t' \\ \beta\gamma c\Delta t' \\ 0 \\ 0 \end{pmatrix} \tag{6.17}$$

Tada vrijedi

$$c\Delta t = c(t_2 - t_1) = \gamma(c\Delta t' - 0) \rightarrow \Delta t = \gamma\Delta t', \tag{6.18}$$

odnosno vremenski interval mjeren iz sustava S je zapravo puno duži. Primijetimo iz druge komponente vektora da u sustavu $\mathcal{S}$ događaji koji su se dogodili na istom mjestu u $\mathcal{S}'$ se događaju na poziciji $\Delta x = \beta c\gamma\Delta t' = \beta c\Delta t$.

Isti problem možemo promatrati i obrnutu transformaciju koordinata:

$$\begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} c\Delta t' \\ 0 \\ 0 \\ 0 \end{pmatrix} = \Lambda(\beta) \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \tag{6.19}$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \tag{6.20}$$

$$= \begin{pmatrix} \gamma(c\Delta t - \beta\Delta x) \\ \gamma(\Delta x - \beta c\Delta t) \\ \Delta y \\ \Delta z \end{pmatrix}, \tag{6.21}$$

iz čega možemo eliminirati nepoznati Δx koristeći $0 = \gamma(\Delta x - \beta c\Delta t)$

$$\begin{aligned} c\Delta t' &= \gamma(c\Delta t - \beta\Delta x) &(6.22)\\ &= \gamma(c\Delta t - c\Delta t\beta^2) &(6.23)\\ &= c\Delta t\gamma(1-\beta^2) &(6.24)\\ &= c\Delta t\frac{1-\beta^2}{\sqrt{1-\beta^2}} &(6.25)\\ &= c\Delta t\sqrt{1-\beta^2} &(6.26)\\ &= c\Delta t/\gamma &(6.27)\end{aligned}$$

vidimo da smo dobili isti izraz.

S druge strane, ako u sustavima S i S' mjerimo prostornu udaljenost između dva događaja, dobit ćemo također dugačiji rezultat. Naime, ako mjerimo u sustavu S' i neka je $\Delta t' = 0$ (uvjet istovremenosti), vrijedi:

$$\begin{aligned} \begin{pmatrix} c\Delta t\\ \Delta x\\ \Delta y\\ \Delta z\end{pmatrix} &= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0\\ \beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 0\\ \Delta x'\\ \Delta y'\\ \Delta z'\end{pmatrix} &(6.28)\\ &= \begin{pmatrix} \gamma(0+\beta\Delta x')\\ \gamma(0+\Delta x')\\ \Delta y'\\ \Delta z'\end{pmatrix} &(6.29)\\ &= \begin{pmatrix} \gamma\beta\Delta x'\\ \gamma\Delta x'\\ \Delta y'\\ \Delta z'\end{pmatrix}. &(6.30)\end{aligned}$$

Primjer za štap koji se proteže u x smjeru duljine $L = \Delta x$ (znači $\Delta y = 0$, $\Delta z = 0$) u sustavu $\mathcal{S}$, u sustavu $\mathcal{S}'$ dobivamo:

$$\begin{aligned} \begin{pmatrix} 0\\ \Delta x'\\ \Delta y'\\ \Delta z'\end{pmatrix} &= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} c\Delta t\\ \Delta x\\ 0\\ 0\end{pmatrix} &(6.31)\\ &= \begin{pmatrix} \gamma(c\Delta t - \beta\Delta x)\\ \gamma(\Delta x - \beta c\Delta t)\\ 0\\ 0\end{pmatrix}, &(6.32)\end{aligned}$$

pri čemu iz prvog reda vektora možemo eliminirati nepoznati Δt

$$0 = \gamma(c\Delta t - \beta\Delta x) \to c\Delta t = \beta\Delta x. \tag{6.33}$$

Drugi redak vektora daje jednadžbu:

$$L' = \Delta x' = \gamma(L - \beta c\Delta t) = \gamma(L - \beta^2 L) = L\frac{1-\beta^2}{\sqrt{1-\beta^2}} = \frac{L}{\gamma}. \tag{6.34}$$

Znači da duljinu L' koju ćemo dobiti u sustavu $\mathcal{S}'$ je dana izrazom $L' = L/\gamma$. Također, odmah iz prvog reda matrice vidimo da je trenutačno mjerenje[2] u sustavu $\mathcal{S}'$ gledano iz sustava $\mathcal{S}$ trajalo konačno vrijeme

$$\Delta t = \frac{\gamma\beta L'}{c} = \frac{L}{c}\beta. \tag{6.35}$$

Ukratko, vremenski interval u sustavu koji se giba, $\mathcal{S}'$ opisan koordinatama $(c\Delta t', 0)$ u mirujućem sustavu $\mathcal{S}$ je opisan s $(c\Delta t, \Delta x) = (\gamma c\Delta t', \gamma\beta c\Delta t')$ i u mirujućem sustavu vremenski interval traje duže. Druga situacija je ako štapu izmjerimo duljinu L (i to mjerenje traje Δt) u mirujućem sustavu $\mathcal{S}$, mjerenje njegove duljine u sustavu $\mathcal{S}'$ će biti opisano koordinatama $(0, L')$. Uvjet da je $\Delta t'$ jednak 0 nam je bitan fizikalno jer fiksiramo vrijeme mjerenja da bismo mogli očitati duljinu L'. Analogija bi bila da smo snimili štap i sa fotografije očitali njegovu duljinu. Prema tome, štap ćemo mjereći iz S' izmjeriti kraćim, $L' = L/\gamma$.

Također možemo izračunati izraz za zbrajanje brzina

$$\Lambda(\beta_{12}) = \Lambda(\beta_1)\Lambda(\beta_2) \tag{6.36}$$

$$\begin{pmatrix} \gamma_{12} & -\beta_{12}\gamma_{12} & 0 & 0 \\ -\beta_{12}\gamma_{12} & \gamma_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 & 0 & 0 \\ -\beta_2\gamma_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.37}$$

$$= \begin{pmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & -\gamma_1\gamma_2(\beta_1+\beta_2) & 0 & 0 \\ -\gamma_1\gamma_2(\beta_1+\beta_2) & \gamma_1\gamma_2(1+\beta_1\beta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.38}$$

Iz prva dva retka matrice očitamo jednadžbe:

$$\gamma_{12} = \gamma_1\gamma_2(1+\beta_1\beta_2) \tag{6.39}$$

$$\gamma_{12}\beta_{12} = \gamma_1\gamma_2(\beta_1+\beta_2), \tag{6.40}$$

ako ih podijelimo dobivamo

$$\beta_{12} = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}. \tag{6.41}$$

Primjer 6.1.1. Mion se giba kroz Zemljinu atmosferu brzinom od $0.999c$ te ima poluvrijeme života od $1.4\,\mu$s. Koliko je poluvrijeme života miona gledano sa Zemlje?

Rješenje.

$$\begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t' \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(c\Delta t' + 0) \\ \gamma(\beta c\Delta t' + 0) \\ 0 \\ 0 \end{pmatrix} \tag{6.42}$$

$$= \begin{pmatrix} \gamma c\Delta t' \\ \gamma\beta c\Delta t' \\ 0 \\ 0 \end{pmatrix}. \tag{6.43}$$

[2] uvjet istovremenosti $\Delta t' = 0$

Vidimo da vrijedi

$$c\Delta t = c\gamma\Delta t'. \tag{6.44}$$

Uvrštavanjem vremena poluživota koji se označava sa τ dobivamo:

$$\tau = \frac{\tau'}{\sqrt{1-\beta^2}} = \frac{1.5\,\mu\text{s}}{\sqrt{1-0.999^2}} = 34\,\mu\text{s}. \tag{6.45}$$

Primjer 6.1.2. Štap duljine L' nalazi se pod kutem θ' u odnosu na x os u xy ravnini. Pod kolikim se kutem nalazi štap u sustavu u kojemu se štap giba brzinom $v\hat{x}$? Kolika je duljina štapa u ovom sustavu?

Rješenje. U sustavu u kojemu se štap giba, x komponenta se mijenja, dok y ostaje ista. Mjerenje je izvršeno u sustavu bez crtice

$$\begin{pmatrix} 0 \\ L\cos\theta \\ L\sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t' \\ L'\cos\theta' \\ L'\sin\theta' \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(c\Delta t' + \beta L'\cos\theta') \\ \gamma(L'\cos\theta' + \beta c\Delta t') \\ L'\sin\theta' \\ 0 \end{pmatrix} \tag{6.46}$$

Primijetimo da vrijedi

$$\begin{aligned} 0 &= c\Delta t' + \beta L'\cos\theta' && (6.47) \\ c\Delta t' &= -\beta L'\cos\theta' && (6.48) \\ L\cos\theta &= \gamma(L'\cos\theta' + \beta c\Delta t') && (6.49) \\ &= L'\cos\theta'\gamma(1-\beta^2) && (6.50) \\ &= \frac{L'\cos\theta'}{\gamma}, && (6.51) \end{aligned}$$

odnosno možemo zapisati

$$\begin{pmatrix} L\cos\theta \\ L\sin\theta \end{pmatrix} = \begin{pmatrix} L'\cos\theta'/\gamma \\ L'\sin\theta' \end{pmatrix}. \tag{6.52}$$

Dijeljenjem dobivamo

$$\tan\theta = \gamma\tan\theta'. \tag{6.53}$$

Duljina štapa je

$$\begin{aligned} L &= L'\sqrt{\gamma^{-2}\cos^2\theta' + \sin^2\theta'} && (6.54) \\ &= L'\sqrt{\cos^2\theta'(\gamma^{-2}-1)+1} && (6.55) \\ &= L'\sqrt{1+\cos^2\theta'(1-\beta^2-1)} && (6.56) \\ &= L'\sqrt{1-\beta^2\cos^2\theta'}. && (6.57) \end{aligned}$$

Primjer 6.1.3. U sustavu $\mathcal{S}'$, čestica se giba brzinom $\mathbf{u} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z}$. U sustavu $\mathcal{S}$, sustav $\mathcal{S}'$ giba se brzinom $v\hat{x}$. Koja je brzina čestice mjerena iz sustava S?

Rješenje. U sustavu $\mathcal{S}'$ čestica u vremenu $\Delta t'$ prijeđe put $\Delta\mathbf{r}' = (u_x\hat{x} + u_y\hat{y} + u_z\hat{z})\Delta t'$. Primijenimo Lorentzovu transformaciju

$$\begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t' \\ u_x\Delta t' \\ u_y\Delta t' \\ u_z\Delta t' \end{pmatrix} \tag{6.58}$$

$$= \Delta t' \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix} \tag{6.59}$$

$$= \Delta t' \begin{pmatrix} \gamma(c + \beta u_x) \\ \gamma(c\beta + u_x) \\ u_y \\ u_z \end{pmatrix}. \tag{6.60}$$

Uspoređujući početni i krajnji vektor vidimo

$$V_x = \frac{\Delta x}{\Delta t} = c\frac{\gamma(c\beta + u_x)}{\gamma(c + \beta u_x)} = \frac{\beta c + u_x}{1 + \beta\frac{u_x}{c}} \tag{6.61}$$

$$V_y = \frac{\Delta y}{\Delta t} = \frac{u_y}{\gamma(1 + \beta\frac{u_x}{c})} \tag{6.62}$$

$$V_z = \frac{\Delta z}{\Delta t} = \frac{u_z}{\gamma(1 + \beta\frac{u_x}{c})} \tag{6.63}$$

Primjer 6.1.4. U inercijalnom sustavu S jedan događaj odvija se na x-osi u točki A, a T sekundi kasnije odvije se događaj B, također na x osi. U sustavu S događaji A i B udaljeni su L metara. Postoji li drugi inercijalni sustav $\mathcal{S}$ u kojemu će događaji A iB biti istovremeni?

Rješenje. Opišimo događaje u sustavu $\mathcal{S}$

$$r_A = \begin{pmatrix} 0 \\ x_A \\ 0 \\ 0 \end{pmatrix} \tag{6.64}$$

$$r_B = \begin{pmatrix} T \\ x_B \\ 0 \\ 0 \end{pmatrix}. \tag{6.65}$$

Izračunajmo četverovektor razlike točaka

$$\Delta r = \begin{pmatrix} T \\ L \\ 0 \\ 0 \end{pmatrix}. \tag{6.66}$$

Transformirajmo se sada u neki sustav S' koji se giba duž x osi

$$\Delta r' = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cT \\ L \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(cT - \beta L) \\ \gamma(L - \beta cT) \\ 0 \\ 0 \end{pmatrix}. \tag{6.67}$$

Želimo li da događaji budu istovremeni u $\mathcal{S}'$, moramo staviti $\Delta t' = 0$.

$$\Delta t' = 0 = \gamma(cT - \beta L) \tag{6.68}$$

$$\beta = \frac{cT}{L}, \tag{6.69}$$

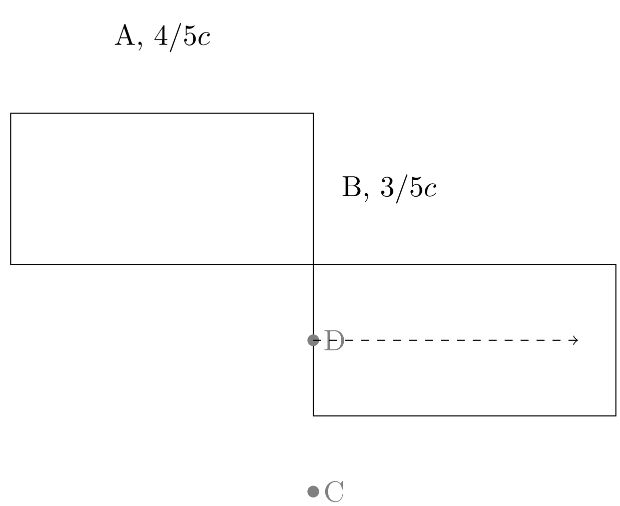
odnosno postoji takav sustav. Točke A i B su u njemu udaljene

$$x'_B - x'_A = \gamma(L - \beta cT) \tag{6.70}$$

$$= \gamma\left(L - \frac{c^2T^2}{L}\right) \tag{6.71}$$

$$= \gamma L\left(1 - \frac{c^2T^2}{L^2}\right) \tag{6.72}$$

Primjer 6.1.5. Vlakovi A i B, duljine L (u njihovim sustavima) gibaju se u istom smjeru. Vlak A giba se brzinom $4/5c$, dok se vlak B giba brzinom od $3/5c$. Opažač C promatra vlakove sa strane. U trenutku $t = 0$ vlak A je neposredno iza vlaka B. Koliko dugo će trebati vlaku A da prestigne (i prođe) vlak B? Koliko dugo mu treba, gledano iz vlakova A i B? Osoba D nalazi se na zadnjem sjedalu vlaka B. U trenutku kada vidi da je vlak A došao do prozora, pojuri prema vratima koja se nalaze na početku vlaka B kako bi uskočila na vrata koja se nalaze na kraju vlaka A. Koliko vremena osoba D ima za pretrčavanje?



Rješenje. Rješenje iz sustava C. Izračunajmo Lorentzove matrice

$$\gamma_A = \frac{1}{\sqrt{1-(4/5)^2}} = \frac{5}{3} \tag{6.73}$$

$$\gamma_A\beta_A = \frac{5}{3}\frac{4}{5} = \frac{4}{3} \tag{6.74}$$

$$\Lambda_A^{(C)} = \begin{pmatrix} \gamma_A & \gamma_A\beta_A & 0 & 0 \\ \gamma_A\beta_A & \gamma_A & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5/3 & 4/3 & 0 & 0 \\ 4/3 & 5/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.75}$$

$$\gamma_B = \frac{1}{\sqrt{1-(3/5)^2}} = \frac{5}{4} \tag{6.76}$$

$$\gamma_B\beta_B = \frac{5}{4}\frac{3}{5} = \frac{3}{4} \tag{6.77}$$

$$\Lambda_B^{(C)} = \begin{pmatrix} \gamma_B & \gamma_B\beta_B & 0 & 0 \\ \gamma_B\beta_B & \gamma_B & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5/4 & 3/4 & 0 & 0 \\ 3/4 & 5/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.78}$$

Mjerenje duljine vlakova radimo u sustavu C $(\Delta t|_C = 0)$

$$\begin{pmatrix} 0 \\ L_A^{(C)} \\ 0 \\ 0 \end{pmatrix} = \Lambda_A^{(C)} \begin{pmatrix} c\Delta t \\ L \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5c\Delta t - 4L}{3} \\ \frac{5L - 4c\Delta t}{3} \\ 0 \\ 0 \end{pmatrix} \tag{6.79}$$

$$c\Delta t = \frac{4L}{5} \tag{6.80}$$

$$L_A^{(C)} = \frac{5L - 4c\Delta t}{3} = L\frac{5-16/5}{3} = L\frac{9}{15} = L\frac{3}{5} \tag{6.81}$$

$$\begin{pmatrix} 0 \\ L_B^{(C)} \\ 0 \\ 0 \end{pmatrix} = \Lambda_B^{(C)} \begin{pmatrix} c\Delta t \\ L \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5c\Delta t + 3L}{4} \\ \frac{5L + 3c\Delta t}{4} \\ 0 \\ 0 \end{pmatrix} \tag{6.82}$$

$$c\Delta t = -\frac{3L}{5} \tag{6.83}$$

$$L_B^{(C)} = \frac{5L + 3c\Delta t}{3} = L\frac{5-9/5}{4} = L\frac{16}{20} = \frac{4}{5}L. \tag{6.84}$$

Postavimo koordinatni sustav tako da je u $t = 0$ početak vlaka B u točki $x = L_B$, a početak vlaka A u $x = 0$. Početna točka vlaka A nalazi se na koordinati $X_{B,p} = L_B^{(C)} + v_B t$, dok kraj vlaka A ima položaj $x_{A,k}(t) = -L_A^{(C)} + v_A t$. Tražimo vremenski trenutak t kada se ove krivulje poklope

$$x_{A,k}(t) = x_{B,p}(t) - L_A + v_A t = L_B^{(C)} + v_B t \tag{6.85}$$

$$t(v_A - v_B) = L_A^{(C)} + L_B \tag{6.86}$$

$$t = \frac{L_A^{(C)} + L_B^{(C)}}{v_A - v_B} \tag{6.87}$$

$$= \frac{\frac{3}{5} + \frac{4}{5}}{\frac{4}{5} - \frac{3}{5}} \frac{L}{c} = \frac{7L}{c}. \tag{6.88}$$

Osoba D mora prijeći put $L_B^{(C)}$ prije nego što vlak A pretekne vlak B te stoga ima na raspolaganju upravo vrijeme $t = \frac{7L}{c}$.

Rješenje iz sustava A. Pronađimo Lorentzovu transformaciju vlaka B

$$\Lambda_B^{(A)} = (\Lambda_A^{(C)})^{-1} \Lambda_B^{(C)} = \begin{pmatrix} 5/3 & -4/3 & 0 & 0 \\ -4/3 & 5/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5/4 & 3/4 & 0 & 0 \\ 3/4 & 5/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.89}$$

$$= \begin{pmatrix} \frac{5}{3}\frac{5}{4} - \frac{4}{3}\frac{3}{4} & \frac{5}{3}\frac{3}{4} - \frac{4}{3}\frac{5}{4} & 0 & 0 \\ -\frac{4}{3}\frac{5}{4} + \frac{5}{3}\frac{3}{4} & -\frac{4}{3}\frac{3}{4} + \frac{5}{3}\frac{5}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.90}$$

$$= \begin{pmatrix} \frac{13}{12} & -\frac{5}{12} & 0 & 0 \\ -\frac{5}{12} & \frac{13}{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.91}$$

Duljina vlaka B iz sustava A je

$$\begin{pmatrix} 0 \\ L_B^{(A)} \\ 0 \\ 0 \end{pmatrix} = \Lambda_B^{(A)} \begin{pmatrix} c\Delta t \\ L \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{13c\Delta t - 5L}{12} \\ \frac{13L - 5c\Delta t}{12} \\ 0 \\ 0 \end{pmatrix} \tag{6.92}$$

$$c\Delta t = \frac{5L}{13} \tag{6.93}$$

$$L_B^{(A)} = \frac{13L - 5c\Delta t}{12} = L\frac{13 - 25/13}{12} = L\frac{144}{12 \cdot 13} = \frac{12}{13}L. \tag{6.94}$$

Vlak B ima brzinu $v_B^{(A)} = c\beta\gamma/\gamma = (-5/12)/(13/12)c = -5/13c$ i kreće iz točke $x(0) = L_B^A$ tako da se u trenutku t nalazi u $x_B^{(A)}(t) = L_B^{(A)} + v_B^{(A)} t$, dok kraj vlaka A miruje u $x = -L$.

Vrijeme t dobivamo rješavajući $x_B^{(A)}(t) = -L$

$$x_B^{(A)}(t) = -L \tag{6.95}$$
$$\frac{12L}{13} - \frac{5c}{13}t = -L \tag{6.96}$$
$$t = L\frac{\frac{12}{13}+1}{\frac{5c}{13}} \tag{6.97}$$
$$= \frac{L}{c}\frac{\frac{25}{13}}{\frac{5}{13}} \tag{6.98}$$
$$= \frac{5L}{c}, \tag{6.99}$$

a to je ujedno i vrijeme koje putnik D ima na raspolaganju.

Rješenje iz sustava B. Pronađimo matricu transformacije za vlak A

$$\Lambda_A^{(B)} = (\Lambda_B^{(C)})^{-1}\Lambda_A^{(C)} = \begin{pmatrix} 5/4 & -3/4 & 0 & 0 \\ -3/4 & 5/4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 5/3 & 4/3 & 0 & 0 \\ 4/3 & 5/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.100}$$
$$= \begin{pmatrix} \frac{5}{4}\frac{5}{3} - \frac{3}{4}\frac{4}{3} & \frac{5}{4}\frac{4}{3} - \frac{3}{4}\frac{5}{3} & 0 & 0 \\ -\frac{3}{4}\frac{5}{3} + \frac{5}{4}\frac{4}{3} & -\frac{3}{4}\frac{4}{3} + \frac{5}{4}\frac{5}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{25-12}{12} & \frac{20-15}{12} & 0 & 0 \\ \frac{20-15}{12} & \frac{25-12}{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.101}$$
$$= \begin{pmatrix} \frac{13}{12} & \frac{5}{12} & 0 & 0 \\ \frac{5}{12} & \frac{13}{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.102}$$
$$v_A^{(B)} = c\frac{5/12}{13/12} = \frac{5}{13}c. \tag{6.103}$$

Duljina vlaka A iz sustava B je

$$\begin{pmatrix} 0 \\ L_A^{(B)} \\ 0 \\ 0 \end{pmatrix} = \Lambda_A^{(B)}\begin{pmatrix} c\Delta t \\ L \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{13c\Delta t+5L}{12} \\ \frac{13L+5c\Delta t}{12} \\ 0 \\ 0 \end{pmatrix} \tag{6.104}$$
$$c\Delta t = -\frac{5L}{13} \tag{6.105}$$
$$L_A^{(B)} = \frac{13L + 5c\Delta t}{12} = L\frac{13 - 25/13}{12} = L\frac{144}{12 \cdot 13} = \frac{12}{13}L. \tag{6.106}$$

U ovom sustavu, kraj vlaka A nalazi se na $x_A^{(B)}(t) = -L_A^{(B)} + v_A^{(B)}t$ dok je početak vlaka B na $x = L$. Rješavamo

$$x_A^{(B)}(t) = L \tag{6.107}$$

$$t = \frac{L + L_A^{(B)}}{v_A^{(B)}} \tag{6.108}$$

$$= \frac{\frac{12+13}{12}L}{\frac{5}{13}} = \frac{\frac{25}{13}}{\frac{5}{13}}\frac{L}{c} = \frac{5L}{c}, \tag{6.109}$$

a to je vrijeme dostupno putniku D. Putnik D u sustavu B mora imati brzinu

$$v_D^{(B)} = \frac{L}{t} = \frac{L}{5\frac{L}{c}} = \frac{1}{5}c. \tag{6.110}$$

Njegov Lorentzov faktor i Lorentzova matrica su

$$\gamma_D = 1/\sqrt{1 - \frac{1}{25}} = \frac{5}{\sqrt{24}} = \frac{5}{2\sqrt{6}} \tag{6.111}$$

$$\gamma_D \beta_D = \frac{5}{2\sqrt{6}}\frac{1}{5} = \frac{1}{2\sqrt{6}} \tag{6.112}$$

$$\Lambda_D^{(B)} = \begin{pmatrix} \frac{5}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & 0 & 0 \\ \frac{1}{2\sqrt{6}} & \frac{5}{2\sqrt{6}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.113}$$

Rješenje iz sustava D. Izračunajmo sada brzine vlakova u sustavu D

$$\Lambda_A^{(D)} = (\Lambda_D^{(B)})^{-1}\Lambda_A^{(B)} \tag{6.114}$$

$$= \begin{pmatrix} \frac{5}{2\sqrt{6}} & -\frac{1}{2\sqrt{6}} & 0 & 0 \\ -\frac{1}{2\sqrt{6}} & \frac{5}{2\sqrt{6}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{13}{12} & \frac{5}{12} & 0 & 0 \\ \frac{5}{12} & \frac{13}{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.115}$$

$$= \begin{pmatrix} \frac{13\cdot5-5}{24\sqrt{6}} & \frac{5\cdot5-13}{24\sqrt{6}} & 0 & 0 \\ \frac{5\cdot5-13}{24\sqrt{6}} & \frac{13\cdot5-5}{24\sqrt{6}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{12\cdot5}{24\sqrt{6}} & \frac{12}{24\sqrt{6}} & 0 & 0 \\ \frac{12}{24\sqrt{6}} & \frac{12\cdot5}{24\sqrt{6}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.116}$$

$$= \begin{pmatrix} \frac{5}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & 0 & 0 \\ \frac{1}{2\sqrt{6}} & \frac{5}{2\sqrt{6}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.117}$$

$$v_A^{(D)} = c\frac{\frac{1}{2\sqrt{6}}}{\frac{5}{2\sqrt{6}}} = \frac{c}{5} \tag{6.118}$$

$$\Lambda_B^{(D)} = (\Lambda_D^{(B)})^{-1} = \begin{pmatrix} \frac{5}{2\sqrt{6}} & -\frac{1}{2\sqrt{6}} & 0 & 0 \\ -\frac{1}{2\sqrt{6}} & \frac{5}{2\sqrt{6}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.119}$$

$$v_B^{(D)} = c\frac{-\frac{1}{2\sqrt{6}}}{\frac{5}{2\sqrt{6}}} = \frac{-c}{5}. \tag{6.120}$$

Sada možemo pronaći duljine vlakova u D

$$\begin{pmatrix} 0 \\ L_A^{(D)} \\ 0 \\ 0 \end{pmatrix} = \Lambda_D^{(A)} \begin{pmatrix} c\Delta t \\ L \\ 0 \\ 0 \end{pmatrix} \tag{6.121}$$

$$= \begin{pmatrix} \frac{5}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & 0 & 0 \\ \frac{1}{2\sqrt{6}} & \frac{5}{2\sqrt{6}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ L \\ 0 \\ 0 \end{pmatrix} \tag{6.122}$$

$$= \begin{pmatrix} \frac{5c\Delta t+L}{2\sqrt{6}} \\ \frac{5L+c\Delta t}{2\sqrt{6}} \\ 0 \\ 0 \end{pmatrix} \tag{6.123}$$

$$c\Delta t = -\frac{L}{5} \tag{6.124}$$

$$L_A^{(D)} = \frac{5L + c\Delta t}{2\sqrt{6}} = L\frac{5-1/5}{2\sqrt{6}} = L\frac{24}{\sqrt{24}\cdot 5} = \frac{2\sqrt{6}}{5}L \tag{6.125}$$

$$\begin{pmatrix} 0 \\ L_B^{(D)} \\ 0 \\ 0 \end{pmatrix} = (\Lambda_D^{(B)})^{-1} \begin{pmatrix} c\Delta t \\ L \\ 0 \\ 0 \end{pmatrix} \tag{6.126}$$

$$= \begin{pmatrix} \frac{5}{2\sqrt{6}} & -\frac{1}{2\sqrt{6}} & 0 & 0 \\ -\frac{1}{2\sqrt{6}} & \frac{5}{2\sqrt{6}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ L \\ 0 \\ 0 \end{pmatrix} \tag{6.127}$$

$$= \begin{pmatrix} \frac{5c\Delta t-L}{2\sqrt{6}} \\ \frac{5L-c\Delta t}{2\sqrt{6}} \\ 0 \\ 0 \end{pmatrix} \tag{6.128}$$

$$c\Delta t = \frac{L}{5} \tag{6.129}$$

$$L_B^{(D)} = \frac{5L - c\Delta t}{2\sqrt{6}} = L\frac{5-1/5}{2\sqrt{6}} = L\frac{24}{\sqrt{24}\cdot 5} = \frac{\sqrt{24}}{5}L = \frac{2\sqrt{6}}{5}L. \tag{6.130}$$

U sustavu D, nalazimo se na $x^D = 0$. Koordinate odgovarajućih točaka vlakova su

$$x_A^{(D)} = -L_A^{(D)} + v_A^{(D)}t \tag{6.131}$$

$$x_B^{(D)} = L_B^{(D)} + v_B^{(D)}t. \tag{6.132}$$

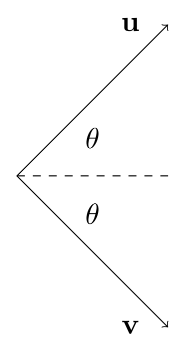
Zanima nas kada kraj vlaka A dođe u ishodište,

$$x_A^{(D)} = 0 \tag{6.133}$$

$$t = \frac{L_A^{(D)}}{v_A^{(D)}} = \frac{\frac{2\sqrt{6}}{5}L}{\frac{c}{5}} = \frac{2\sqrt{6}}{c}. \tag{6.134}$$

U tom trenutku, početak vlaka B se nalazi u istoj točki (jer je D došao do početka vlaka B) $x_B^{(D)}(t) = \frac{2\sqrt{6}}{5}L - \frac{c}{5}\frac{2\sqrt{6}}{c} = 0$.

Primjer 6.1.6. Čestice se gibaju brzinama u i v kao na slici u sustavu opažača. Kolika je brzina jedne čestice, iz sustava druge?

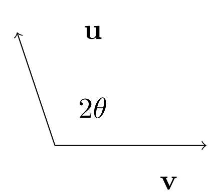


Rješenje. U sustavu prikazanom na slici, čestice imaju koordinate

$$r_u = \begin{pmatrix} c\Delta t \\ |\mathbf{u}|\cos\theta\Delta t \\ |\mathbf{u}|\sin\theta\Delta t \\ 0 \end{pmatrix} \tag{6.135}$$

$$r_v = \begin{pmatrix} c\Delta t \\ |\mathbf{v}|\cos\theta\Delta t \\ -|\mathbf{v}|\sin\theta\Delta t \\ 0 \end{pmatrix}. \tag{6.136}$$

Nije jednostavno provesti Lorentzove transformacije pri gibanju pod kutem, stoga ćemo zarotirati sustav opažača gdje se čestica $\mathbf{v}$ giba u $\hat{x}$ smjeru, kao na slici



U ovom sustavu, koordinate su

$$r'_u = \begin{pmatrix} c\Delta t \\ |\mathbf{u}|\cos 2\theta\Delta t \\ |\mathbf{u}|\sin 2\theta\Delta t \\ 0 \end{pmatrix} \tag{6.137}$$

$$r'_v = \begin{pmatrix} c\Delta t \\ |\mathbf{v}| \\ 0 \\ 0 \end{pmatrix}. \tag{6.138}$$

Prebacimo četverovektor r'_u u sustav od čestice koja se opažaču giba sa brzinom $\mathbf{v}$:

$$r''_v = \begin{pmatrix} \gamma & -\gamma\frac{|\mathbf{v}|}{c} & 0 & 0 \\ -\gamma\frac{|\mathbf{v}|}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ |\mathbf{u}|\cos 2\theta\Delta t \\ |\mathbf{u}|\sin 2\theta\Delta t \\ 0 \end{pmatrix} \tag{6.139}$$

$$= \begin{pmatrix} \gamma(c\Delta t - \frac{|\mathbf{v}|}{c}|\mathbf{u}|\cos 2\theta\Delta t) \\ \gamma(-|\mathbf{v}|\Delta t + |\mathbf{u}|\cos 2\theta\Delta t) \\ |\mathbf{u}|\sin 2\theta\Delta t \\ 0 \end{pmatrix}. \tag{6.140}$$

Očitamo komponente brzine od u

$$u''_x = \frac{\Delta x''}{\Delta t''} = c\frac{-|\mathbf{v}|\Delta t + |\mathbf{u}|\cos 2\theta\Delta t}{c\Delta t - \frac{|\mathbf{v}||\mathbf{u}|}{c}\cos 2\theta\Delta t} \tag{6.141}$$

$$= \frac{-|\mathbf{v}| + |\mathbf{u}|\cos 2\theta}{1 - \frac{|\mathbf{v}||\mathbf{u}|\cos 2\theta}{c^2}} \tag{6.142}$$

$$u''_y = \frac{\Delta y''}{\Delta t''} = c\frac{|\mathbf{u}|\sin 2\theta\Delta t}{c\Delta t - \frac{|\mathbf{v}|}{c}|\mathbf{u}|\cos 2\theta\Delta t} \tag{6.143}$$

$$= \frac{1}{\gamma}\frac{|\mathbf{u}|\sin 2\theta}{1 - \frac{|\mathbf{v}||\mathbf{u}|\cos 2\theta}{c^2}} \tag{6.144}$$

$$|\mathbf{u}''|^2 = \frac{(-|\mathbf{v}| + |\mathbf{u}|\cos 2\theta)^2 + \frac{|\mathbf{u}|^2\sin^2 2\theta}{\gamma^2}}{(1 - \frac{|\mathbf{v}||\mathbf{u}|\cos 2\theta}{c^2})^2} \tag{6.145}$$

$$= \frac{(-|\mathbf{v}| + |\mathbf{u}|\cos 2\theta)^2 + (1 - \frac{|\mathbf{v}|^2}{c^2})|\mathbf{u}|^2\sin^2 2\theta}{(1 - \frac{|\mathbf{v}||\mathbf{u}|\cos 2\theta}{c^2})^2} \tag{6.146}$$

$$= \frac{|\mathbf{v}|^2 - |\mathbf{u}|^2\cos^2 2\theta - 2|\mathbf{u}||\mathbf{v}|\cos 2\theta + \mathbf{u}|^2\sin^2 2\theta - \left(\frac{|\mathbf{u}||\mathbf{v}|\sin 2\theta}{c}\right)^2}{(1 - \frac{|\mathbf{v}||\mathbf{u}|\cos 2\theta}{c^2})^2} \tag{6.147}$$

$$= \frac{|\mathbf{v}|^2 + |\mathbf{u}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos 2\theta - \left(\frac{|\mathbf{u}||\mathbf{v}|\sin 2\theta}{c}\right)^2}{(1 - \frac{|\mathbf{v}||\mathbf{u}|\cos 2\theta}{c^2})^2}. \tag{6.148}$$

6.2 Relativistička energija i impuls

Drugi četverovektor je vektor energije i impulsa

$$p = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} \tag{6.149}$$

Za čestice mase m generalno vrijedi

$$p^2 = \frac{E^2}{c^2} - \mathbf{p}^2 = m^2c^2. \tag{6.150}$$

U slučaju bezmasenih čestica tada možemo jednostavno izraziti

$$E = |\mathbf{p}|c. \tag{6.151}$$

Primjer 6.2.1. Svjetlost frekvencije ν_0 emitirana je pod kutem θ u odnosu na x os. Opažač se giba brzinom $v\hat{x}$. Koliku frekvenciju fotona će ovaj opažač izmjeriti (Dopplerov efekt)? Izvrijediti posebno u slučajevima $\theta = 0$, $\theta = \pi/2$ i $\theta = \pi$.

Rješenje. Koristimo Lorentzove transformacije. U sustavu gdje je emitiran, energija fotona je E_0 te on ima impuls $\mathbf{p_0}$. U sustavu opažača vrijedi

$$\begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_0/c \\ p_{x0} \\ p_{y0} \\ p_{z0} \end{pmatrix} = \begin{pmatrix} \gamma(E_0/c - \beta p_{x0}) \\ \gamma(p_{x0} - \beta E_0/c) \\ p_{y0} \\ p_{z0} \end{pmatrix}, \tag{6.152}$$

Iskoristimo izraze za energiju i impuls fotona $E_0 = h\nu_0$ i $\mathbf{p}_0 = h\nu_0/c(\cos\theta\hat{x} + \sin\theta\hat{y})$.

$$\begin{pmatrix} \frac{h\nu}{c} \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \gamma(\frac{h\nu_0}{c} - \beta\frac{h\nu_0}{c}\cos\theta) \\ \gamma(\frac{h\nu_0}{c}\cos\theta - \beta\frac{h\nu_0}{c}) \\ \frac{h\nu_0}{c}\sin\theta \\ 0 \end{pmatrix} = \frac{h\nu_0}{c} \begin{pmatrix} \gamma(1 - \beta\cos\theta) \\ \gamma(\cos\theta - \beta) \\ \frac{h\nu_0}{c}\sin\theta \\ 0 \end{pmatrix}. \tag{6.153}$$

odnosno izmjerena frekvencija fotona se može iščitati iz prvog retka vektor stupca:

$$\nu = \nu_0\gamma(1 - \beta\cos\theta). \tag{6.154}$$

Za $\theta = 0$ (udaljavanje) vrijedi

$$\nu = \nu_0\frac{1-\beta}{\sqrt{1-\beta^2}} = \nu_0\sqrt{\frac{1-\beta}{1+\beta}}. \tag{6.155}$$

Za $\theta = \pi/2$ vrijedi

$$\nu = \frac{\nu_0}{\sqrt{1-\beta^2}}. \tag{6.156}$$

Za $\theta = \pi$ (približavanje) vrijedi

$$\nu = \nu_0\frac{1+\beta}{\sqrt{1-\beta^2}} = \nu_0\sqrt{\frac{1+\beta}{1-\beta}}. \tag{6.157}$$

Primjer 6.2.2. Zvijezda emitira foton pod kutem θ' u odnosu na spojnicu zvijezda-Zemlja, a detektor na Zemlji ga detektira pod kutem θ. Ako se zvijezda udaljava brzinom v, koliki je kut θ?

Rješenje. Postavimo Lorentzove transformacije

$$\begin{pmatrix} \frac{E}{c} \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{E'}{c} \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma\left(\frac{E'}{c} - \beta p'_x\right) \\ \gamma\left(p'_x - \beta\frac{E'}{c}\right) \\ p'_y \\ p'_z \end{pmatrix}. \tag{6.158}$$

Budući da se radi o fotonu, vrijedi $|\mathbf{p}'| = E'/c$ i $|\mathbf{p}| = E/c$ te možemo pisati

$$\begin{pmatrix} \frac{E}{c} \\ \frac{E}{c}\cos\theta \\ \frac{E}{c}\sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma\left(\frac{E'}{c} - \beta\frac{E'}{c}\cos\theta'\right) \\ \gamma\left(\frac{E'}{c}\cos\theta' - \beta\frac{E'}{c}\right) \\ \frac{E'}{c}\sin\theta' \\ 0 \end{pmatrix}, \tag{6.159}$$

te stoga možemo pisati

$$\frac{E}{c} = \gamma\left(\frac{E'}{c} - \frac{E'}{c}\beta\cos\theta'\right) \tag{6.160}$$

$$= \gamma\frac{E'}{c}\left(1 - \beta\cos\theta'\right) \tag{6.161}$$

$$\frac{E}{c}\cos\theta = \gamma\frac{E'}{c}\left(\cos\theta' - \beta\right). \tag{6.162}$$

Dijeljenjem dobivamo

$$\cos\theta = \frac{\cos\theta' - \beta}{1 - \beta\cos\theta'}. \tag{6.163}$$

Primjer 6.2.3. Čestica se giba u smjeru $\hat{x}$. Pokazati da je komponenta akceleracije u $\hat{x}$ smjeru

$$F = \gamma^3 m\frac{dv}{dt}. \tag{6.164}$$

Rješenje. Rješenje dobivamo deriviranjem

$$F = \frac{dp}{dt} = \frac{d}{dt}\frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{6.165}$$

$$= m(\gamma\dot{v} - \frac{1}{2c^2}\gamma^3(-2v\dot{v})) \tag{6.166}$$

$$= m(\gamma\dot{v} + \frac{v^2}{c^2}\gamma^3\dot{v}) \tag{6.167}$$

$$= \dot{v}\gamma^3(\gamma^{-2} + \frac{v^2}{c^2}) \tag{6.168}$$

$$= m\dot{v}\gamma^3. \tag{6.169}$$

Primjer 6.2.4. Odrediti generalni izraz za silu u relativističkom slučaju.

Rješenje. Deriviranjem dobivamo

$$\frac{\mathbf{F}}{m} = \dot{\mathbf{v}}\gamma + \mathbf{v}\dot{\gamma} \tag{6.170}$$

$$= \gamma\dot{\mathbf{v}} + \mathbf{v}\frac{-\gamma^3}{2c^2}(-2)(\mathbf{v}\cdot\dot{\mathbf{v}}) \tag{6.171}$$

$$= \gamma\dot{\mathbf{v}} + \gamma^3\frac{\mathbf{v}\cdot\dot{\mathbf{v}}}{c^2}\mathbf{v}. \tag{6.172}$$

Rastavimo li silu na komponentu paralelnu s brzinom, smjera $\mathbf{e}_\parallel$ i na komponentu okomitu na brzinu, $\mathbf{e}_\perp$ dobivamo

$$\mathbf{e}_\parallel\cdot\frac{\mathbf{F}}{m} = \gamma\mathbf{e}_\parallel\cdot\dot{\mathbf{v}} + \gamma^3\frac{v^2}{c^2}\mathbf{e}_\parallel\cdot\dot{\mathbf{v}} \tag{6.173}$$

$$= \gamma^3\mathbf{e}_\parallel\cdot\dot{\mathbf{v}}\left(\gamma^{-2}+\frac{v^2}{c^2}\right) \tag{6.174}$$

$$= \gamma^3\mathbf{e}_\parallel\cdot\dot{\mathbf{v}} \tag{6.175}$$

$$\mathbf{e}_\perp\cdot\frac{\mathbf{F}}{m} = \gamma\mathbf{e}_\perp\cdot\dot{\mathbf{v}}. \tag{6.176}$$

Prema tome, silu možemo zapisati kao

$$\mathbf{F} = m\gamma^3(\mathbf{e}_\parallel\cdot\dot{\mathbf{v}})\mathbf{e}_\parallel + m\gamma(\mathbf{e}_\perp\cdot\dot{\mathbf{v}})\mathbf{e}_\perp. \tag{6.177}$$

Primjer 6.2.5. Na česticu naboja q i brzine v_0 djeluje konstantno električno polje u smjeru njezine brzine. Izračunati brzinu i položaj u ovisnosti o vremenu.

Rješenje. Rješavamo jednadžbu

$$m\gamma^3\frac{dv}{dt} = qE, \tag{6.178}$$

odnosno

$$\frac{dv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} = \frac{qEdt}{m} \tag{6.179}$$

$$\int_{v_0}^{v}\frac{dv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} = \int_0^t\frac{qE}{m}dt \tag{6.180}$$

$$\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{v_0}{\sqrt{1-\frac{v_0^2}{c^2}}} = \frac{qEt}{m}. \tag{6.181}$$

Uvedemo li izraz

$$f(t) = \frac{v_0}{\sqrt{1-\frac{v_0^2}{c^2}}} + \frac{qEt}{m} = \gamma_0 v_0 + \frac{qEt}{m} \tag{6.182}$$

Dobivamo

$$\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} = f(t) \tag{6.183}$$

$$v^2 + f^2(t)\frac{v^2}{c^2} = f^2(t) \tag{6.184}$$

$$v^2(1+\frac{f^2(t)}{c^2}) = f^2(t) \tag{6.185}$$

$$v = \frac{f(t)}{\sqrt{1+\frac{f^2(t)}{c^2}}} \tag{6.186}$$

$$= \frac{\gamma_0 v_0 + \frac{qEt}{m}}{\sqrt{1+\left(\gamma_0\frac{v_0}{c}+\frac{qEt}{mc}\right)^2}}. \tag{6.187}$$

Prijeđeni put je

$$x(t) = x_0 + \int_0^t \frac{\gamma_0 v_0 + \frac{qEt}{m}}{\sqrt{1+\left(\gamma_0\frac{v_0}{c}+\frac{qEt}{mc}\right)^2}}dt \tag{6.188}$$

$$\begin{pmatrix} z = \gamma_0\frac{v_0}{c}+\frac{qEt}{mc} \\ dz = \frac{qE}{mc}dt \end{pmatrix} \tag{6.189}$$

$$= x_0 + \frac{mc^2}{qE}\int_{z_0}^{z(t)} \frac{zdz}{\sqrt{1+z^2}} \tag{6.190}$$

$$= x_0 + \frac{mc^2}{qE}\sqrt{1+z^2}\Big|_{z_0}^{z} \tag{6.191}$$

$$= x_0 + \frac{mc^2}{qE}\left(\sqrt{1+\left(\gamma_0\frac{v_0}{c}+\frac{qEt}{mc}\right)^2} - \sqrt{1+\left(\gamma_0\frac{v_0}{c}\right)^2}\right) \tag{6.192}$$

$$= x_0 + \frac{mc^2}{qE}\left(\sqrt{\gamma_0^2\left(\gamma_0^{-2}+\frac{v_0^2}{c^2}\right)+2\gamma_0\frac{v_0tqE}{mc^2}+\left(\frac{qEt}{mc}\right)^2} - \sqrt{\gamma_0^2\left(\gamma_0^{-2}+\frac{v_0^2}{c^2}\right)}\right) \tag{6.193}$$

$$= x_0 + \frac{\gamma_0 mc^2}{qE}\left(\sqrt{1+\frac{2qE}{\gamma_0 mc^2}v_0t+\left(\frac{qEt}{\gamma_0 mc}\right)^2}-1\right). \tag{6.194}$$

Primjer 6.2.6. Nabijena čestica kruži konstantnom brzinom v na radijusu R u magnetskom polju. Kolika je frekvencija kruženja?

Rješenje. Čestica ima konstantnu brzinu, te je jedina akceleracija u smjeru okomitom na brzinu. Tada vrijedi

$$m\gamma a = m\gamma\frac{v^2}{R} = qvB, \tag{6.195}$$

Odnosno radijus kruženja je

$$2\pi fR = v = \frac{qBR}{\gamma m}, \tag{6.196}$$

tj.

$$f = \frac{qB}{2\pi m}\sqrt{1 - \frac{v^2}{c^2}}. \tag{6.197}$$

Primijetimo da impuls čestice ima isti izraz kao u nerelativističkom slučaju, $p = qBR/m$.

7. Dodatak

7.1 Korisni integrali

Zbog jednostavnosti, izostavljene su konstante integracije

$$\int \frac{dx}{(ax^2+bx+c)^{1/2}} = \frac{1}{\sqrt{a}} \ln \left|2\sqrt{a}\sqrt{ax^2+bx+c}+2ax+b\right| \tag{7.1}$$

$$\int \frac{xdx}{(ax^2+bx+c)^{1/2}} = \frac{(ax^2+bx+c)^{1/2}}{a} - \frac{b}{2a^{3/2}} \operatorname{arctanh} \frac{2ax+b}{2\sqrt{a}(ax^2+bx+c)^{1/2}} \tag{7.2}$$

$$\int \frac{dx}{(ax^2+bx+c)^{3/2}} = \frac{4ax+2b}{(4ac-b^2)(ax^2+bx+c)^{1/2}} \tag{7.3}$$

$$\int \frac{xdx}{(ax^2+bx+c)^{3/2}} = -\frac{2bx+4c}{(4ac-b^2)(ax^2+bx+c)^{1/2}} \tag{7.4}$$

$$\int \frac{dx}{(x^2+a^2)^{1/2}} = \operatorname{arctanh} \frac{x}{(x^2+a^2)^{1/2}} = \ln \frac{x+\sqrt{x^2+a^2}}{a} \tag{7.5}$$

$$\int \frac{xdx}{(x^2+a^2)^{1/2}} = (x^2+a^2)^{1/2} \tag{7.6}$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}} \tag{7.7}$$

$$\int \frac{xdx}{(x^2+a^2)^{3/2}} = -\frac{1}{(x^2+a^2)^{1/2}} \tag{7.8}$$

$$\int dx \frac{1}{(x^2+a^2)\sqrt{x^2+b^2}} = \begin{cases} \frac{1}{a\sqrt{a^2-b^2}} \operatorname{arctanh} \frac{\sqrt{a^2-b^2}x}{a\sqrt{x^2+b^2}} & a > b \\ \frac{1}{a\sqrt{b^2-a^2}} \arctan \frac{\sqrt{b^2-a^2}x}{a\sqrt{x^2+b^2}} & a < b \end{cases} \tag{7.9}$$