Max-Cut and Goemans-Williamson*

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Abstract

This is a short summary of approximation algorithms for the Max-Cut problem of finding a maximum cut of a graph. After introducing the problem and trivial $\frac{1}{2}$-approximation, we summarize the famous semidefinite programming relaxation and hyperplane rounding technique from [Goemans and Williamson(1995)] that gives the best known approximation ratio for Max-Cut. We then take a look at the dual problem and some previous results. Taking the dual approach further, using so-called Sum-of-Squares hierarchy framework with Gaussian rounding technique, we arrive at the same approximation ratio for Max-Cut. Finally, we discuss some possible generalizations.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph with $V := \{1, \dots, n\}$ and $|E| = m$. For a subset $S \subset V$, we define $cut(S)$ as a subset of edges E having one vertex in S and the other one in $V \setminus S$. We denote the cut size by $f(S) = |cut(S)|$. The *Max-Cut problem* is to find a subset $S \subseteq V$ that maximizes the cut size $f(S)$. Max-Cut problem is NP-complete [Karp(1972)].

Given a graph $G = (V, E)$, denote the maximum cut size by $mc(G)$. Given some algorithm that returns the subset $S \subseteq V$, we say that S is an *α-approximation* of Max-Cut if

$$f(S) \geq \alpha \cdot mc(G) \tag{1.1}$$

for all graphs $G = (V, E)$ and some approximation ratio $\alpha \in [0, 1]$. If algorithm employed is randomized, meaning S that it returns is a random variable, then we say the same, if 1.1 holds with an expectation taken on the left-hand side. For example, for an algorithm, that assigns each vertex of V to S and $V \setminus S$ independently uniformly at random, we get

$$\mathbb{E}[f(S)] = \sum_{i,j} \mathbb{P}[(i, j) \in \mathrm{cut}(S)] = \frac{1}{2} \cdot m \geq \frac{1}{2} \cdot mc(G)$$

This implies that S is a $\frac{1}{2}-$approximation for Max-Cut. This approximation was first proposed and analyzed by [Erdős(1967)]. In [Goemans and Williamson(1995)] they improved this by proposing α_{GW}-approximation algorithm with $\alpha_{GW} \geq 0.878$ using semidefinite programming (SDP) and hyperplane rounding technique. The proposed algorithm is also simple to implement [1] using off-the-shelf SDP solver.

2 SDP Relaxation

Cuts in the complete graph K_n can be represented by a set of 2^{n-1} matrices xx^T of rank one with $x \in \{-1, 1\}^n$. The *elliptope* $\mathcal{E}_n := \{X \in S_n : X_{ii} = 1, \ \forall i\}$ is a set of all $n \times n$ correlation matrices, giving the formulation

$$mc(G) = \max_{X \in \mathcal{E}_n, rk(X)=1} \sum_{i,j} \frac{1 - X_{ij}}{2}$$

Letting go of rank one constraint, we get an SDP

$$sdp(G) := \max_{X \in \mathcal{E}_n} \sum_{i,j} \frac{1 - X_{ij}}{2}$$

Since $\mathcal{E}_n$ includes all the matrices of rank one, we have $mc(G) \leq sdp(G)$, implying that SDP is a *relaxation.*

3 Hyperplane Rounding

Solution X that realizes the maximum $sdp(G)$ is a positive semidefinite matrix, that can be decomposed as $X = \sum_{i=1} v_i v_i^T$ for some unit vectors v_i. To construct a subset $S \subset V$, select a random unit vector $r \in \mathbb{R}^n$ and let $S = \{i \in V : v_i^T r \geq 0\}$. Denote the angle between vectors v_i, v_j by $\theta_{i,j}$.

Since v_i are unit vectors $v_i^T v_j = \cos \theta_{i,j}$, we can bound the contribution of an edge $(i, j) \in E$ by

$$\mathbb{P}[(i, j) \in cut(S)] = \frac{\theta_{i,j}}{\pi} \geq \alpha_{GW} \cdot \frac{1 - v_i^T v_j}{2}$$

where $\alpha_{GW} = \min_{\theta_{i,j} \in [0,\pi]} \{ \frac{2}{\pi} \frac{\theta_{i,j}}{1 - \cos(\theta_{i,j})} \} \geq 0.878$.

Summing up the contributions of all edges, we arrive at the famous result

$$\mathbb{E}[f(S)] = \sum_{i,j} \mathbb{P}[(i, j) \in cut(S)] \geq \alpha_{GW} \cdot sdp(G)$$

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[1] Source code: https://github.com/markolalovic/max-cut-sdp

4 Dual Problem

Let L be the Laplacian matrix of a graph $G = (V, E)$. The dual problem of SDP relaxation is equivalent to finding maximum eigenvalue of L with smallest added correction $u \in \mathbb{R}^n$, under constraint that $\sum_i u_i = 0$

$$\frac{n}{4} \min_{u:\mathbf{1}^T u=0} \lambda_{\max}(L + diag(u))$$

Setting $u = \mathbf{0}$ and by weak duality we get an upper bound

$$mc(G) \leq sdp(G) \leq \frac{n}{4}\lambda_{\max}(L)$$

given earlier in [Mohar and Poljak(1990)]. For 5-cycle C_5, $\lambda_{\max} = \frac{1}{5}(5 + \sqrt{5})$, giving an upper bound

$$\frac{1}{2}(5 + \sqrt{5})/4 \geq 0.9 \cdot mc(C_5)$$

This bound was studied in [Delorme and Poljak(1993)].

5 SoS Relaxation

Now let $x \in \{0, 1\}^n$ and let $\mathcal{P}_n$ be a set of real valued polynomials $p(x)$ of degree at most $d/2$. Let $\tilde{\mathbb{E}} : \mathcal{P}_n \to \mathbb{R}$ be some operator over the probability distribution on x. We formulate the following optimization problem

$$sos(G) := \max_{\tilde{\mathbb{E}}} \ \tilde{\mathbb{E}}[\sum_{i,j}(x_i - x_j)^2]$$

subject to:

$$\begin{array}{ll} (1)\ \tilde{\mathbb{E}} \text{ is linear} & (2)\ \tilde{\mathbb{E}}[1] = 1 \\ (3)\ \tilde{\mathbb{E}}[p^2] \geq 0 & (4)\ \tilde{\mathbb{E}}[x_i^2 p] = \tilde{\mathbb{E}}[x_i p] \\ \forall p \in \mathcal{P}_n & \end{array}$$

An operator $\tilde{\mathbb{E}}$ that realizes the maximum $sos(G)$ is called *pseudo-expectation*, because it only meets a subset of constraints (1)-(3) required to be an actual expectation. The fourth constraint requires x to be a vector $x \in \{0, 1\}^n$. Given a subset S with maximum cut size $mc(G)$, let a be the indicator vector of S and set $\tilde{\mathbb{E}} = a$. Then $\tilde{\mathbb{E}}$ satisfies the constraints (1)-(4) and achieves objective value $mc(G)$. In other words, given an expectation a, we can build a pseudo-expectation $\tilde{\mathbb{E}}$ just by setting $\tilde{\mathbb{E}} = a$. Therefore $mc(G) \leq sos(G)$.

This framework is called *Sum-of-Squares hierarchy* introduced by [Parrilo(2000)] and [Lasserre(2001)]. Increasing the degree d, increases the size of the corresponding SDP problem. For $d = n$, the relaxation is exact. We can choose $d = 2$ and show that we can get the same α_{GW}-approximation as before by rounding the solution of SoS relaxation.

6 Gaussian Rounding

Let $\tilde{\mathbb{E}}$ be solution that realizes the maximum $sos(G)$. Select y to be a Gaussian vector with the mean $\mu = \tilde{\mathbb{E}}[x] = \frac{1}{2}\mathbf{1}$ and covariance matrix $\Sigma = \tilde{\mathbb{E}}[(x-\mu)(x-\mu)^T]$ and construct a subset $S = \{i \in V : y_i \leq \frac{1}{2}\}$. For each edge $(i, j) \in E(G)$, define $\rho_{i,j} = 4\tilde{\mathbb{E}}[x_i x_j] - 1$. From $(s, t) \overset{\text{i.i.d.}}{\sim} N(0, I)$, setting $u = \rho_{i,j}s + \sqrt{1 - \rho_{i,j}^2}t$, gives $\rho_{i,j} = \mathbb{E}[us] = \cos(\theta_{i,j})$. Summing up the contributions, we can show that

$$\mathbb{E}[f(S)] = \sum_{i,j} \mathbb{P}[(i, j) \in cut(S)] \geq \alpha_{GW} \cdot sos(G)$$

7 Generalizations

Given a non-negative weight function $w \in \mathbb{R}_+^E$ on the edges, $f(S) = \sum_{e\in \text{cut}(S)} w_e$, with no effect on analysis of approximation algorithms presented here. Given arbitrary weight function $w \in \mathbb{R}^E$, the analysis has to be adjusted but it is possible to derive a generalization of guarantees presented here.

Whether increasing degree from $d = 2$ to $d = 4$ of SoS Relaxation, also improves the approximation, still remains unresolved question.

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