Partial Drawings of Complete Graphs

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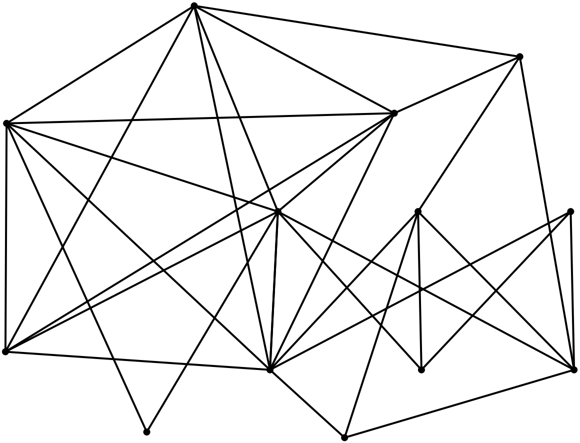
Outline

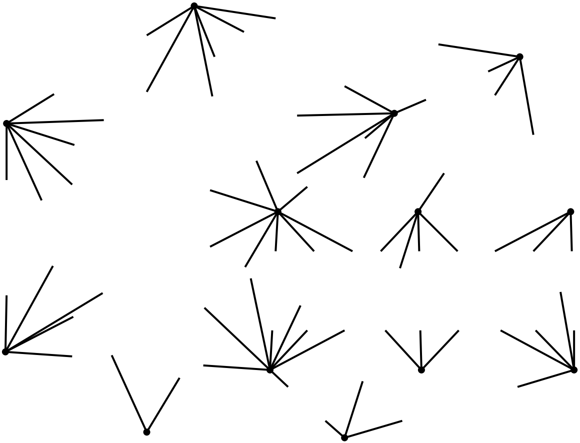
Only two obstacles to planarity



Drawing of a graph that contains subgraphs K_5 and $K_{3,3}$.

Kuratowski's theorem.

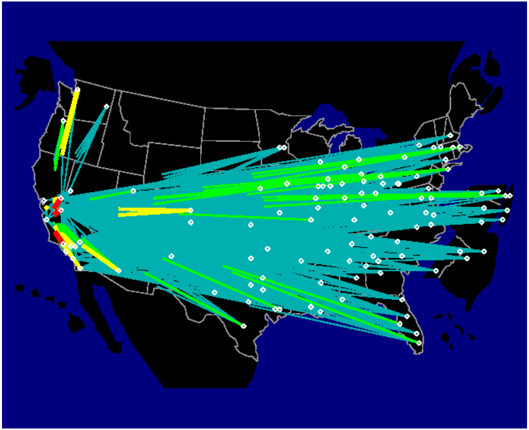
Idea of Partial Drawings



Partial drawing of a graph that contains subgraphs K_5 and $K_{3,3}$.

User study [M. Burch et. al., 2012].

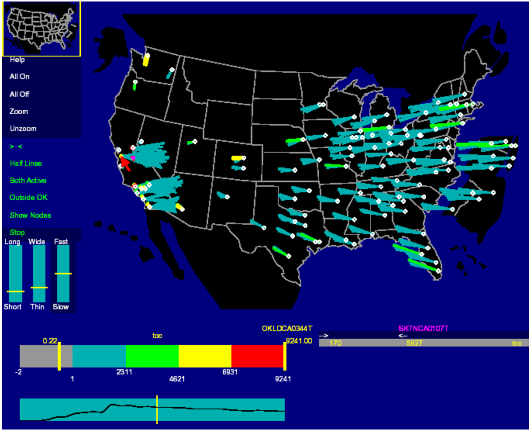
Idea of Partial Drawings



Calls between locations after the earthquake on 17. October 1989.

[R. A. Becker et. al., 1995].

Idea of Partial Drawings



Calls between locations presented with partial edges.

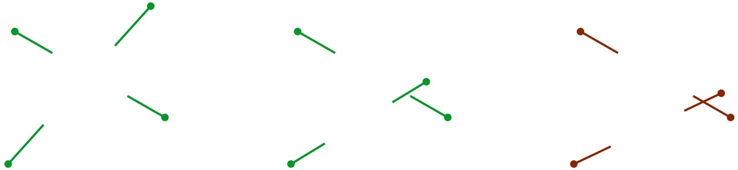
[R. A. Becker et. al., 1995].

What is a Partial Drawing

Partial edge is a pair of quarter-lines called stubs and we treat them as closed sets.

Stubs.

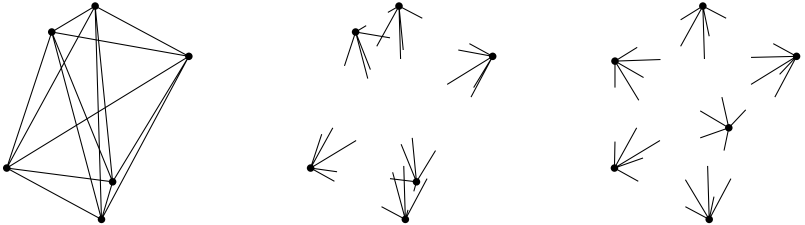
In addition, we require that the drawing is without crossings of partial edges or stubs.



(a) ✓ (b) ✓ (c) ×

What is a Partial Drawing

Partial drawing depends only on the relative positions of points.



(a) Complete edges. (b) Partial edges. (c) Partial drawing.

Various drawings of K_6.

What is the Problem We Were Trying to Solve

Problem

For how big complete graph the partial drawing exists?

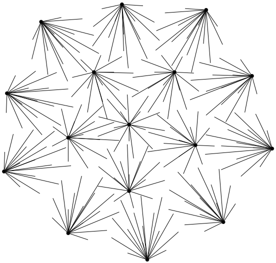
In other words

Let M denote the maximum number of points in a complete graph that we can draw as partial drawing. We want to estimate the upper bound of M:

$$M < \ ?$$

Known Work

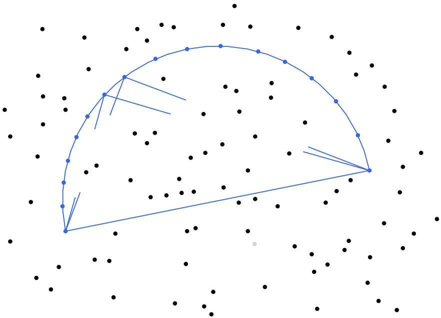
The formalization of the problem and estimate of the "lower bound" $M \geq 16$ [T. Bruckdorfer, M. Kaufmann, 2012].



Partial drawing of K_{16}

Known Work

It is not possible to draw a partial drawing of the complete graph on seventeen points which lie in one-sided convex position [T. Bruckdorfer et. al., 2013]. According to the result of Erdős and Szekeres we obtain $M < \binom{30}{15} = 155\,117\,520$.



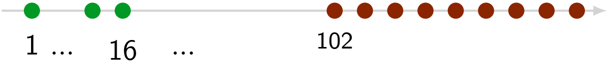
Known Work

[Bruckdorfer et. al., 2013]: $M < 241$.

Our Contribution

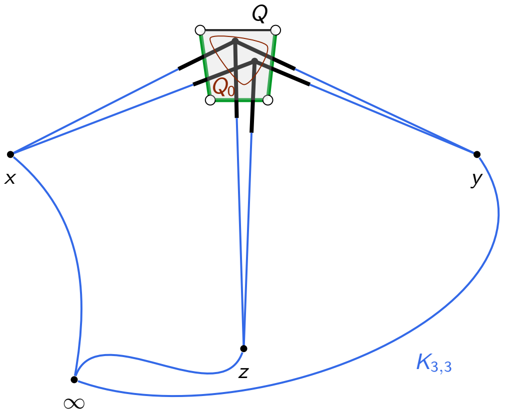
[Bruckdorfer et. al., 2013]: $M < 241$.

Our result: $M < 102$.



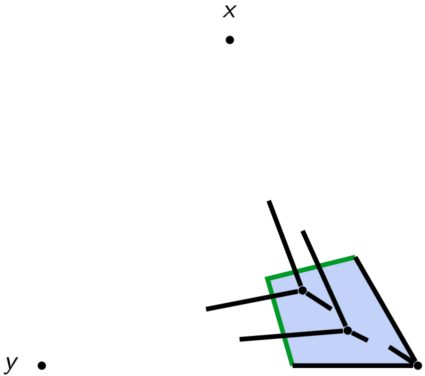
First Tool

If the region is small enough, it doesn't contain two points of the partial edge drawing.

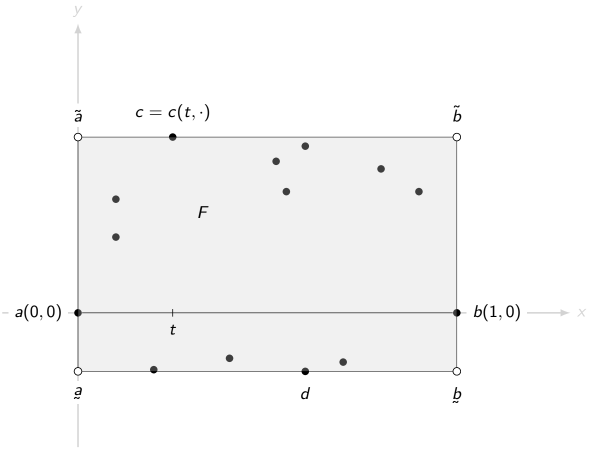


Second Tool

If a corner region is small enough, it doesn't contain many points of the partial drawing.



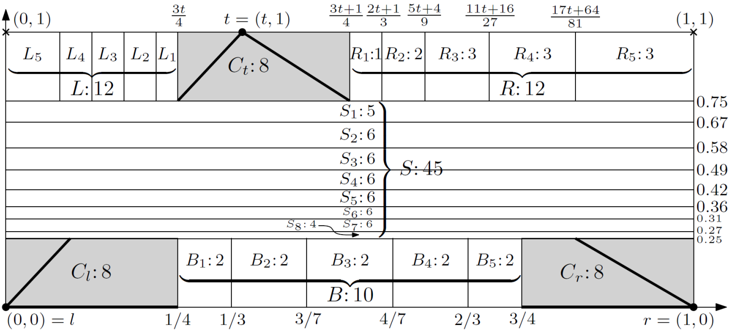
Frame of the Drawing



Frame F

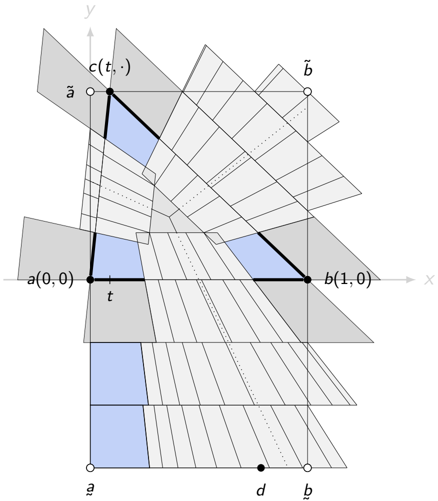
Tessellation of the Frame

[Bruckdorfer et. al., 2013]:

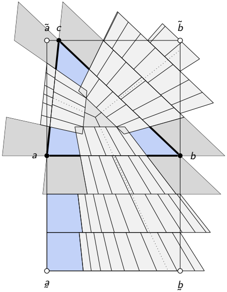


Tessellation of the Frame

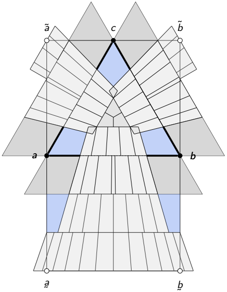
Our tessellation:



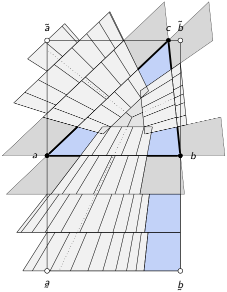
The Tessellation Dependents on the Positions of Points



(a) $t = \frac{1}{11}, M < 102.$

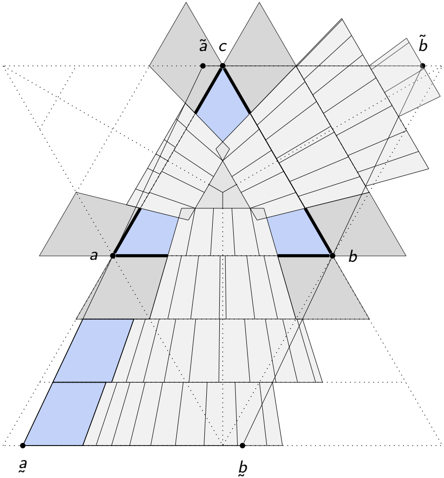


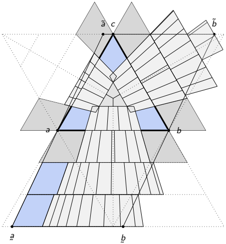
(b) $t = \frac{1}{2}, M < 99.$



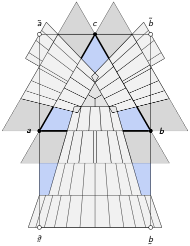
(c) $t = \frac{10}{11}, M < 102.$

Transformation of the Drawing

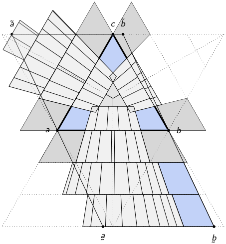




(a) $t = \frac{1}{11}, M < 102.$

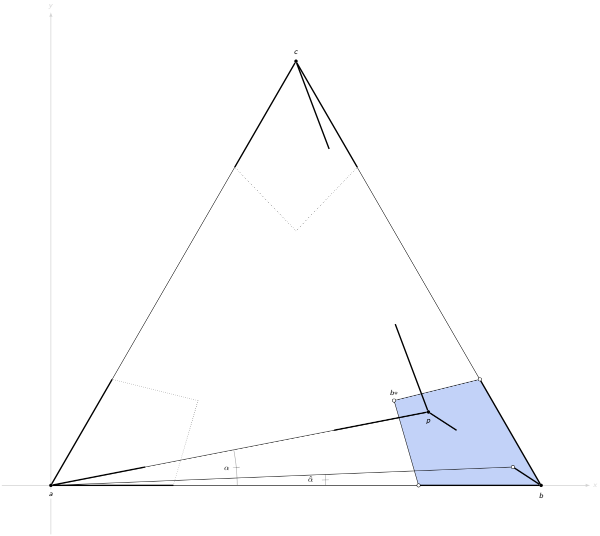


(b) $t = \frac{1}{2}, M < 99.$



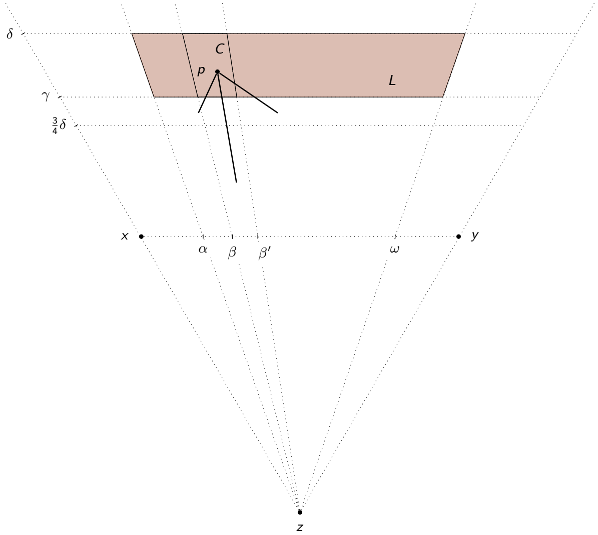
(c) $t = \frac{10}{11}, M < 102.$

Treatment of Regions

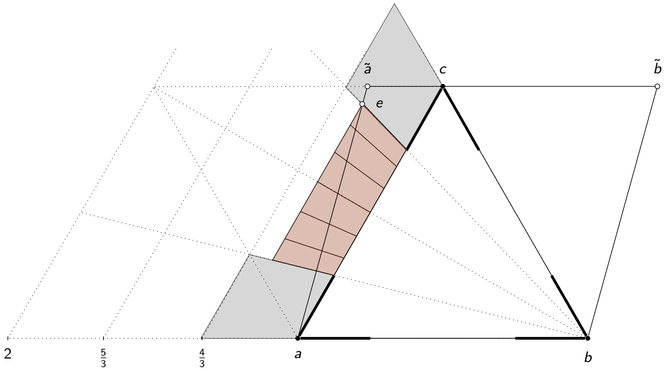


The corner regions.

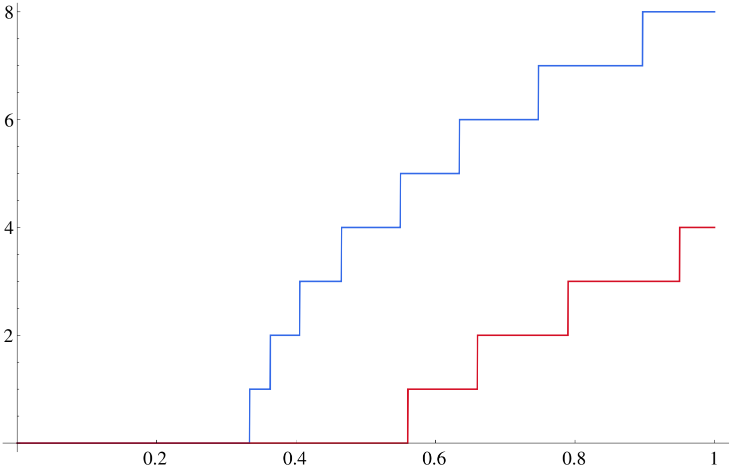
Treatment of Regions



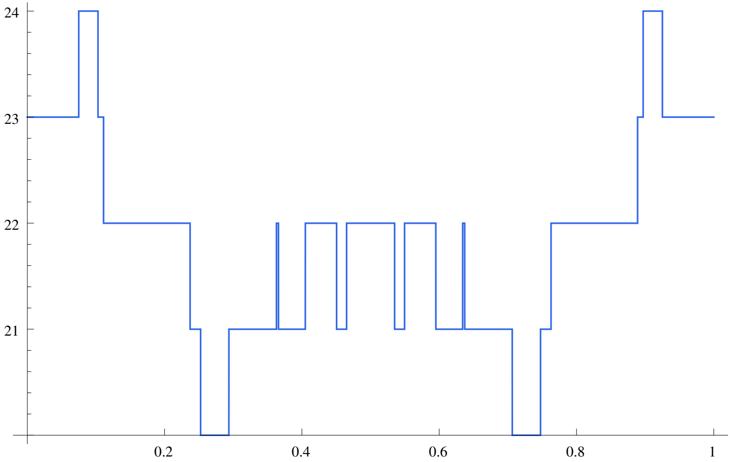
Maximal cell C in layer L.



First layer for $t = 0.24$.



Plot of the function $k_{L_2}(t)$ in blue and plot of the function $j_{L_2}(t)$ in red as functions of t.



The estimate for the number of points in the left and right triangle as a function of t.

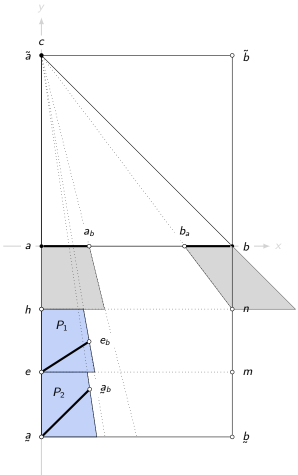


Illustration for very small values of the parameter t.

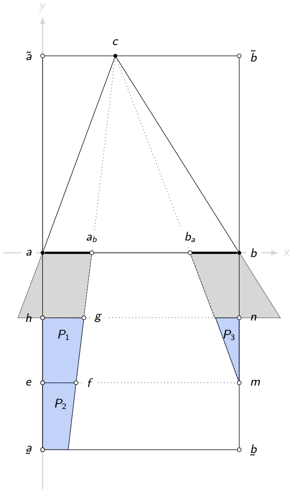
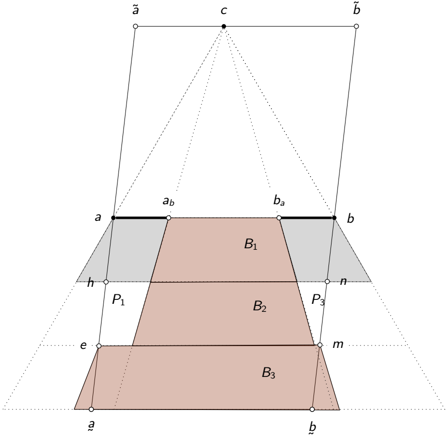


Illustration for $t = \frac{3}{8}$.

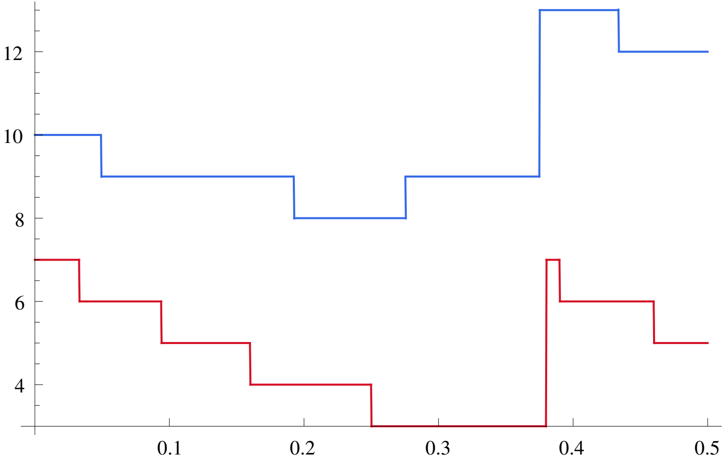
Example

$$\frac{|pc|}{\left|\mathrm{proj}_s\left(\overline{p'c}\right)\right|} \le \frac{\max\left\{|ec|, |fc|\right\}}{\min\left\{\left|\mathrm{proj}_{\overline{gc}}\left(\overline{hc}\right)\right|, \left|\mathrm{proj}_{\overline{hc}}\left(\overline{gc}\right)\right|\right\}} = q_c(t) =$$

$$= \begin{cases} \dfrac{\max\left\{\sqrt{\frac{25}{9}+\frac{49}{576}(1-4t)^2}, \sqrt{\frac{25}{9}+t^2}\right\}}{\min\left\{\frac{128+15t(4t-1)}{24\sqrt{16+9t^2}}, \frac{128+15t(4t-1)}{3\sqrt{1049+200t(2t-1)}}\right\}}, & \text{if } t \le \frac{1}{4}, \\ \dfrac{\max\left\{\frac{5}{3}\sqrt{1+\left(t-\frac{1}{4}\right)^2}, \sqrt{\frac{25}{9}+t^2}\right\}}{\min\left\{\frac{16+3t(4t-1)}{3\sqrt{16+9t^2}}, \frac{16+3t(4t-1)}{3\sqrt{17+8t(2t-1)}}\right\}}, & \text{if } t \ge \frac{1}{4}. \end{cases}$$



The bottom layers for $t = 0.4$.



Plot of the function $k_{B_3}(t)$ in blue and plot of $j_{B_3}(t)$ in red as functions of t.

How to find critical values of t

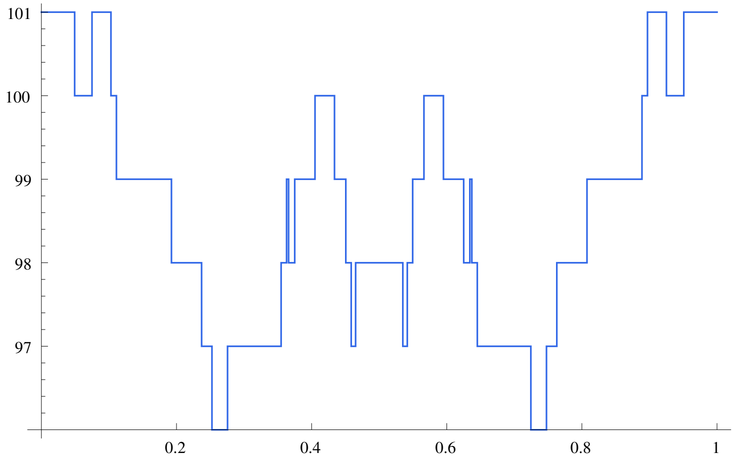
$$\beta_i = \frac{3\delta}{1+6\delta},$$

$$\left(\frac{1+3\delta}{3\delta}\right)^i \beta_0 = \frac{3\delta}{1+6\delta},$$

$$\alpha(t) = \frac{3\delta}{1+6\delta}\left(\frac{3\delta}{1+3\delta}\right)^i,$$

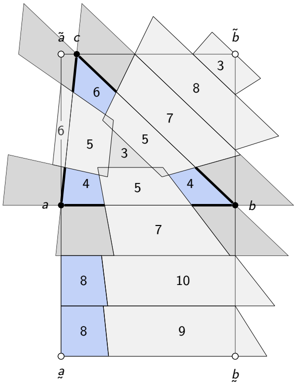
$$t = \overset{-1}{\alpha}\left(\frac{(3\delta)^{i+1}}{(1+6\delta)(1+3\delta)^i}\right).$$

Evaluation of the Result



Plot of the estimate M.

Evaluation of the Result

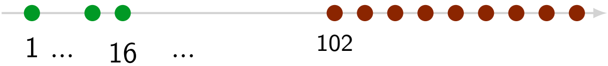


Estimate $M < 102$ at $t = \frac{1}{11}$.

The Result

We improved the upper bound by more than twice. We have shown that it is not possible to draw a partial drawing of the complete graph on 102 or more points:

$$M < 102.$$



We believe that the right estimate is much closer to 16 than to 102.