Geometry II.

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Week 9

1 Theory

After having looked into angle-chasing problem solving approaches in geometry last week, the following pages contain simple tools that will enable us to augment angle-chasing with algebraic tools.
More precisely, we will try to reason about angles using lengths and vice versa using Power of a point and Radical lines.

Theorem (Power of a point). *Let s denote distance of point P from center O of a circle k with radius r. The* Power of a point *with respect to k is a real number h defined as:*

$$h = s^2 - r^2$$

This number reflects the relative distance of P from k.
Furthermore, let an arbitrary line through P intersect k in (not necessarily distinct) points X, Y. Then

$$|PX| \cdot |PY| = h$$

Proof. We will prove only the case where the point lies outside of a circle - other cases can be proved analogically using similar tirangles. Consider line through P that intersects circle k in points X, Y and point T also on k such that line PT is tangent to circle k. We know that $\angle PAT = \angle PTB$ (circumscribed angles) and thus by uu triangles $\triangle PAT, \triangle PTB$ are similar. This implies that $\frac{|PA|}{|PT|} = \frac{|PT|}{|PB|}$ or equivalently $|PA| \cdot |PB| = |PT|^2$. However, by the Pythagoras theorem also $|PT|^2 = s^2 - r^2$. $\square$

Exercise 1. *Prove that Power of a point theorem holds for any n-dimensional sphere, $n \geq 2$.*

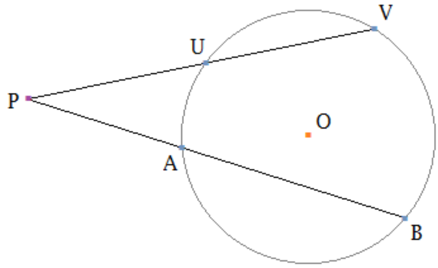


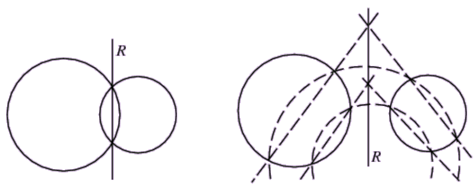
Figure 1: Here, $|PA| \cdot |PB| = |PU| \cdot |PV| = h$

Theorem. *Two circles are orthogonal if they intersect in right angles. Equivalently:*

$$r_1^2 + r_2^2 = d^2$$

where r_1, r_2 are radii of the circles and d distance of their centers.

Theorem (Chordal theorem)**.** *Given two circles k, l with distinct centers, the set of points such that their power is same with respect to k and with respect to l is a line $R_{k,l}$ perpendicular to the line joining centers of k, l. It is known as* radical axis *of the two circles.*
Furthermore, the radical axis is set of centers of all circles that are orthogonal to both k, l.

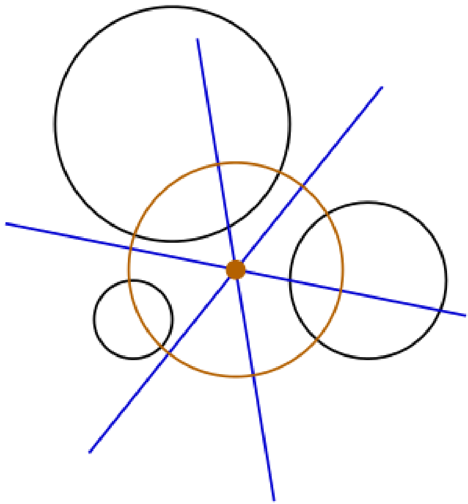


Exercise 2. *Considering the definition of orthogonal circles above, prove that a radical axis is indeed a set of centers of all circles that are orthogonal to k, l.*

Theorem (Radical axis theorem)**.** *Let there three circles a, b, c given such that no two are concentric. The three radical axes (for each pair of circles) intersect in a single point called the* radical center, *or are parallel.*

This also means that there is a unique circle with its center at the radical center that is orthogonal to all three circles.

Proof. Consider radical axes $R_{a,b}, R_{b,c}$ intersecting in point C. By definition of a radical axis the tangents to a and b through C are equal in length, as well as tangents to b and c through C are. By the transitivity of equality, also tangents to a and c through C are equal in length, which means that C also lies on $R_{a,c}$. □



Exercise 3. *Prove by similar argument that the unique circle orthogonal to all three circles is indeed centered in radical center.*

Let's finish the theory section by mentioning that radical lines play important role in solution of a Problem of Apollonius [1].

2 Problems

Easy

1. Prove that quadrilateral $ABCD$ with $M = AC \cap BD$ is cyclic if and only if $|MA| \cdot |MC| = |MB| \cdot |MD|$.

[1]Given three objects, each of which may be a point, line, or circle, draw a circle that is tangent to each

2. Given line segments of lenghts x and y, construct a line segment of length $\sqrt{xy}$.

3. We are given line p and points A and B on the same side of p but not on p. How can we construct circle tangent to p through points A and B?

4. We're given two circles with a common tangent line. Let the tangent points be A and B. Show that chordal of the two circles divides AB in half.

5. Prove that all three altitudes of a triangle intersect in a single point.

Medium

6. We are given line p and points A and B on the opposite sides of p. Find circle through A and B, such that the length of p inside the circle is minimal.

7. Let $ABCD$ be a quadrilateral with $AB \parallel CD$, $AB > CD$, $AC \perp BD$. Let O be circumcenter of $\triangle ABC$ and $E = OB \cap CD$. Show that $BC^2 = CD \cdot CE$.

8. On lines CA and CB of acute triangle ABC there are points X and Y. Let P be intersection of AX and BY and Q be intersection of circumcircles of triangles AXC and BYC (other than C). Prove that C, P, and Q are on a line if and only if $ABXY$ is cyclic.

Difficult

9. ABC is an acute triangle and H its orthocenter. Circle with diameter AH intersects circumcircle in points A and K. Line KV intersects BC in M. Prove that M is midpoint of BC.

10. Let circles k and l be externaly tangent at point T. Let P be any point on l. Tangent lines to k through P intersect k at A and B. Let $AT \cap l = C$ and $BT \cap l = D$. Let t be tangent to l at P and $M = t \cap CD$. Find set of all possible positions of M as we move P along l.

11. On lines AB and AC of acute triangle ABC there are points M and N. Circles with diameters BN, CM intersect at P and Q. Prove that P, Q, and orthocenter H are colinear.

References

[1] Ondrej Budáč, Tomáš Jurík, and Ján Mazák. *Zbierka úloh KMS*. Trojsten, Bratislava, 2010.

[2] Matematický korespondenční seminář. Knihovna. [ONLINE] Available at: `https://mks.mff.cuni.cz/library/library.php`. [Accessed November 10].