EXPLORING FRACTIONS
Farey sequence

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Convention. In the whole text, by a *simple fraction* I will mean a fraction in reduced terms (that is, a fraction that has nominator and denominator coprime).

Farey sequence

Definition 1. The n-th Farey sequence F_n is a sequence of simple fractions in $[0, 1]$ with denominator less than or equal to n, ordered by size.

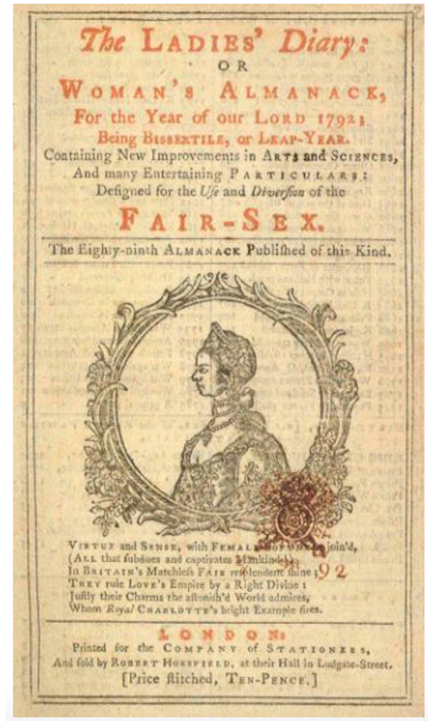
The first 8 Farey sequences are:

$F_1 = \{0/1,\ 1/1\}$
$F_2 = \{0/1,\ 1/2,\ 1/1\}$
$F_3 = \{0/1,\ 1/3,\ 1/2,\ 2/3,\ 1/1\}$
$F_4 = \{0/1,\ 1/4,\ 1/3,\ 1/2,\ 2/3,\ 3/4,\ 1/1\}$
$F_5 = \{0/1,\ 1/5,\ 1/4,\ 1/3,\ 2/5,\ 1/2,\ 3/5,\ 2/3,\ 3/4,\ 4/5,\ 1/1\}$
$F_6 = \{0/1,\ 1/6,\ 1/5,\ 1/4,\ 1/3,\ 2/5,\ 1/2,\ 3/5,\ 2/3,\ 3/4,\ 4/5,\ 5/6,\ 1/1\}$
$F_7 = \{0/1,\ 1/7,\ 1/6,\ 1/5,\ 1/4,\ 2/7,\ 1/3,\ 2/5,\ 3/7,\ 1/2,\ 4/7,\ 3/5,\ 2/3,\ 5/7,\ 3/4,\ 4/5,\ 5/6,\ 6/7,\ 1/1\}$
$F_8 = \{0/1,\ 1/8,\ 1/7,\ 1/6,\ 1/5,\ 1/4,\ 2/7,\ 1/3,\ 3/8,\ 2/5,\ 3/7,\ 1/2,\ 4/7,\ 3/5,\ 5/8,\ 2/3,\ 5/7,\ 3/4,\ 4/5,\ 5/6,\ 6/7,\ 7/8,\ 1/1\}$

Brief history

Before we dive into exploring these simply looking sequences, let me first include a few paragraphs about their background. As often happens in Mathematics, the history behind the Farey sequences turns out to be really curious.
We will begin the story with an 18-th century British magazine called The Ladies' Diary, also known as the Woman's Almanack. As opposed to what you may consider to be the Woman's almanacks nowadays, the Ladies's Diary contents could be described (as the cover says):
"Containing New Improvements in ARTS and SCIENCES, and many entertaining PARTICULARS: Designed for the USE AND DIVERSION OF THE FAIR SEX.".

The magazine was published yearly in between 1704 and 1841, its cover always featured a current famous or successful British woman and in general was meant to intellectually stimulate the members of the fair sex. Among its contents, you could find charades, puzzles, scientific queries and various mathematical questions.



In 1747, Mr. J. May of Amsterdam posed the following mathematical query in the magazine:

Query. It is required to find (by a general theorem) the number of fractions of different values, each less than unity, so that the greatest denominator be less than 100?

Problem 1. Solve the query.

Spending a few minutes thinking about it, you might have solved the problem by now, or at least you might have thought about a way of how you would find the required number. In any case, you should know that it took people 4 years to find the answer. Moreover, out of the 3 proposed solutions, only one turned out to be correct.

The correct solution made use of the Euler's totient function $\phi(n) = |\{1 \leq k \leq n \ : \ \gcd(k, n) = 1\}|$, which somehow seems trivial today. In any case, it was not trivial back in the day, as the Euler's totient function was published only a few years ago. The query was also not solved by any general method, but rather by constructing an extensive mathematical table. Part of it looked like this:

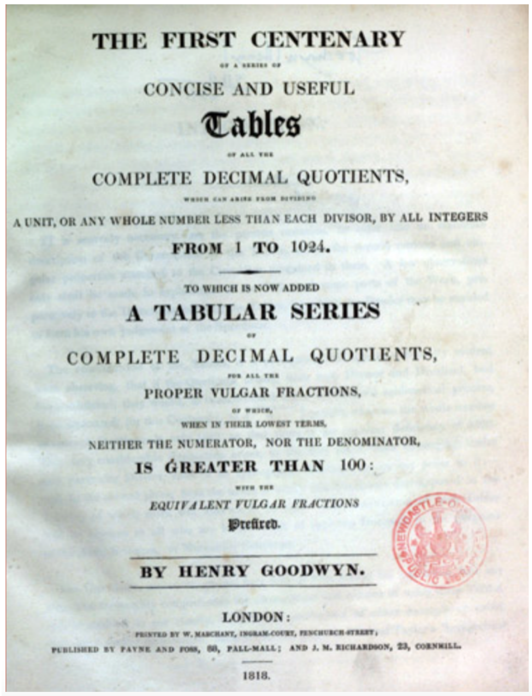
II.

Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions	Denominator	Component primes	Diff. fractions
2	P	1	16	—	8	30	2·35	8	44	—	20	58	2·23	28	72	—	24	86	2·43	42
3	P	2	17	P	16	31	P	30	45	—	24	59	P	58	73	P	72	87	3·29	56
4	—	2	18	—	6	32	—	16	46	2·23	22	60	—	16	74	2 37	36	88	—	40
5	P	4	19	P	18	33	5·11	20	47	P	46	61	P	60	75	—	40	89	P	88
6	2·3	2	20	—	8	34	2·17	16	48	—	16	62	2 31	30	76	—	36	90	—	24
7	P	6	21	3·7	12	35	5·7	24	49	—	42	63	—	36	77	7·11	60	91	7·13	72
8	—	4	22	2 11	10	36	—	12	50	—	20	64	—	32	78	23·13	24	92	—	44
9	—	6	23	P	22	37	P	36	51	3·17	32	65	5·13	48	79	P	78	93	3·31	60
10	2·5	4	24	—	8	38	2 19	18	52	—	24	66	2·3·11	20	80	—	32	94	2·47	46
11	P	10	25	—	20	39	3·13	24	53	P	52	67	P	66	81	—	54	95	5·19	72
12	—	4	26	2·13	12	40	—	16	54	—	18	68	—	32	82	2·41	40	96	—	32
13	P	12	27	—	18	41	P	40	55	5·11	40	69	2 23	44	83	P	82	97	P	9
14	2·7	6	28	—	12	42	2·37	12	56	—	24	70	2·5·7	24	84	—	24	98	—	2
15	3·5	8	29	P	28	43	P	42	57	3·19	56	71	P	70	85	5·17	64	99	—	6
		71			198			314			416			564			666			774

Third columns collected.	The sum of all the different fractions.
71, 198, 314, 416, 564, 666, 774	3003

Exploring fractions was in general a very popular during those times and new findings on "vulgar"[1] fractions were published regularly. One of the publications that played a role in the discovery of the Farey sequence was (brace yourselves):

"The first centenary of a series of concise and useful tables of all the complete decimal quotients which can arise from dividing a unit, or any whole number less than each divisor, by all integers from 1 to 1024, to which is added a tabular series of complete decimal quotients, for all the proper vulgar fractions of which, when in their lowest terms, neither the numerator, neither the denominator, is greater than 100: with the equivalent vulgar fractions prefired"



THE FIRST CENTENARY

OF A SERIES OF

CONCISE AND USEFUL

Tables

OF ALL THE

COMPLETE DECIMAL QUOTIENTS,

WHICH CAN ARISE FROM DIVIDING

A UNIT, OR ANY WHOLE NUMBER LESS THAN EACH DIVISOR, BY ALL INTEGERS

FROM 1 TO 1024.

TO WHICH IS NOW ADDED

A TABULAR SERIES

OF

COMPLETE DECIMAL QUOTIENTS,

FOR ALL THE

PROPER VULGAR FRACTIONS,

OF WHICH,

WHEN IN THEIR LOWEST TERMS,

NEITHER THE NUMERATOR, NOR THE DENOMINATOR,

IS GREATER THAN 100:

WITH THE

EQUIVALENT VULGAR FRACTIONS

Prefixed.

BY HENRY GOODWYN.

LONDON:

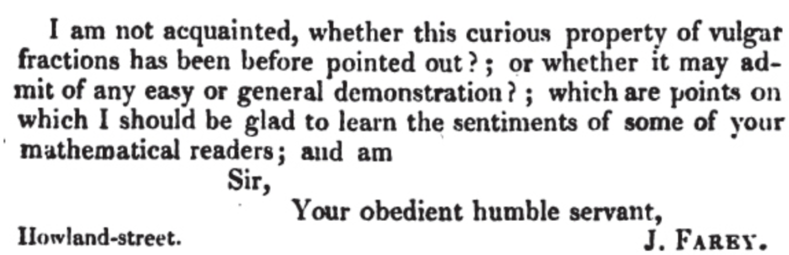
PRINTED BY W. MARCHANT, INGRAM-COURT, FENCHURCH-STREET;

PUBLISHED BY PAYNE AND FOSS, 86, PALL-MALL; AND J. M. RICHARDSON, 23, CORNHILL.

1818.

Yes, all of this is a title and it was written by a curious character of Henry Goodwyn, who owned a brewery and only made mathematical tables in his spare time[2].

A month later, after reading the above publication, and English geologist John Farey wrote a short note into The Philosophical Magazine and Journal. His not was simplistically titled "On a curious Property of the vulgar Fractions" and it only contained 4 paragraphs, describing that he had noted a curious property of the vulgar fractions (the mediant property, which we will mention soon). His note was concluded by the following paragraph:



I am not acquainted, whether this curious property of vulgar fractions has been before pointed out?; or whether it may admit of any easy or general demonstration?; which are points on which I should be glad to learn the sentiments of some of your mathematical readers; and am

Sir,

Your obedient humble servant,

Howland-street. J. Farey.

Now all that was needed was a proof of the curious property that Farey noted. Luckily, not long after, Cauchy read the Farey's note in the French magazine "Bulletin de la Société Philomatique" and, promptly, in August 1816 saved the day by providing a proof. The sequence connected to this curious property was named after Farey afterwards.

[1] a fraction that has both integral nominator and denominator

[2] this is, by all means, a dream life

Curious properties of the Farey sequence

Definition 2 (mediant)**.** The mediant of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a}{b} \bigoplus \frac{c}{d} = \frac{a+c}{b+d}$.

The curious property that Farey noted was the following.

Problem 2. Show that $\frac{a}{b} < \frac{a}{b} \bigoplus \frac{c}{d} < \frac{c}{d}$.

Problem 3 (mediant construction)**.** Starting with the initial sequence of fractions $\frac{0}{1}$ and $\frac{1}{1}$, successively include the mediant of two neighbouring fractions into the sequence. What is the connection to the Farey sequences?

Problem 4. Given F_n, find a way to construct F_{n+1} using only the mediant construction (you don't need to prove that the construction always works).

Problem 5. Notice that the mediant construction only generates simple fractions. Argue that this is always the case.

Problem 6. Show that the middle in any 3 consecutive terms in F_n is a mediant of the other two.

Problem 7. Show that two fractions $\frac{a}{b}$, $\frac{c}{d}$ are neighbours in some F_n if and only if $|bc - ad| = 1$

Problem 8. Show that two neighbouring fractions in a Farey sequence will remain neighbours (with increasing n in F_n) until their mediant separates them in a later Farey sequence.

Problem 9. Show that when making a transition from F_n to F_{n+1} by the mediant construction, all of the simple fractions with $n+1$ as a denominator will be covered.

Problem 10. Given two consecutive terms $\frac{a}{b}$, $\frac{c}{d}$ in F_n, find a formula for the next term. Deduce a simple algorithm to generate F_n.

Problem 11. Show that $|F_n| = |F_{n-1}| + \phi(n)$ and $|F_n| = 1 + \sum_{m=1}^{n} \phi(m)$.

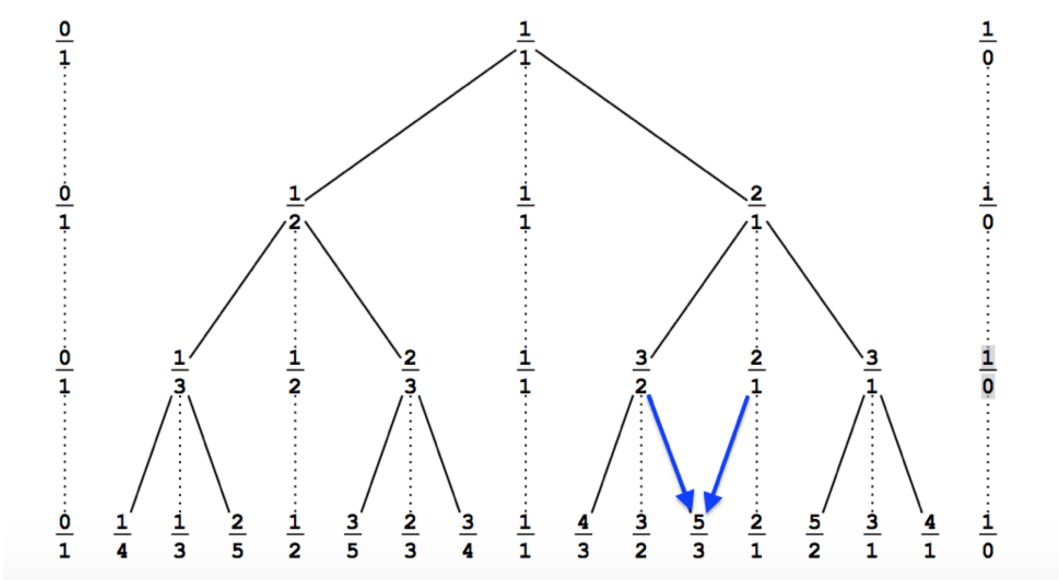
To say something more about the lengths of the Farey sequences, the Möbius inversion may be used to derive:

$$|F_n| = \frac{1}{2}(n+3)n - \sum_{d=2}^{n} |F_{\lfloor n/d \rfloor}|$$

which gives some insight about how the lenght of the Farey sequence behaves assymptotically: $|F_n| \sim \frac{3n^2}{\pi^2}$

Stern-Brocot tree

Stern-Brocot tree is a structured closely related to the Farey sequences. It uses the mediant construction, and as oposed to the Farey sequence that only lists fractions in $[0, 1]$, it contains all of the positive rationals.
It constructions is also fairly similar to that of the Farey sequence. Rather than $\frac{0}{1}$ and $\frac{1}{1}$, we will start the mediant construction with $\frac{0}{1}$ and $\frac{1}{0}$ (note that this represents infinity). Also, rather than constructing a sequence, we will use the mediants to construct a binary tree. This surely deserves a picture:



Problem 12. Try to go through the construction of the depicted first levels of the Stern-Brocot tree by hand.

Problem 13. Argue that the Stern-Brocot tree is a *binary search tree*. That is, argue that the left child of any node is always less than itself, whereas the right child is always greater.

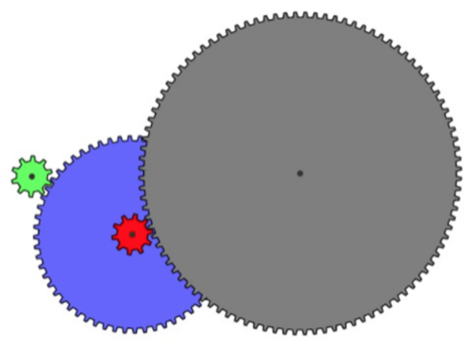
Problem 14 (hard)**.** Show that every rational appears exactly once in the tree.

Problem 15. Deduce that any *real* number can be represented as a (possibly infinite) sequence of L's and R's, by following the path in the Stern-Brocot tree.

Problem 16. Given any number, how can we approximate it to arbitrary precision using the Stern-Brocot tree?

Clockmaking

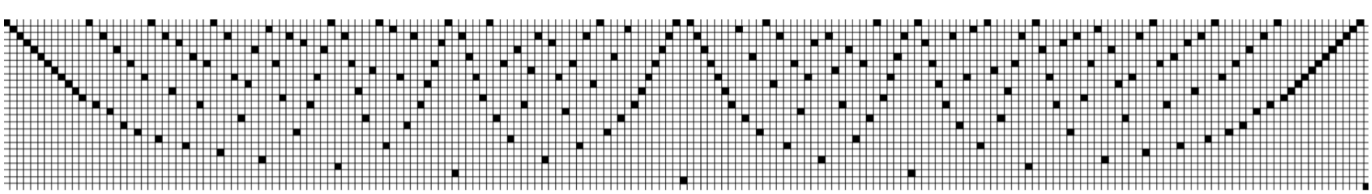
Both the Farey sequence and the Stern-Brocot tree can be used to approximate a number with a rational that has denominator bounded by a given bound. Let me now describe one interesting application of such approximations[3] - clockmaking. This will also partially explain the name of the Stern-Brocot tree - Brocot was a clockmaker.
While it is not a surprise that we can measure time quite precisely nowadays, it involves a big amount of ingenuity to create mechanical clocks that are precise. To see why this is the case, suppose that you want your clock to take count of the years - although it sounds simple, it is not. A mean length of an astronomical year is roughly 365 days, 5 hours and 39 minutes and if you were to design the clock gears to account for this, it would result in a need of having a system of gears that can somehow work with the ratio $\frac{720}{525949}$.



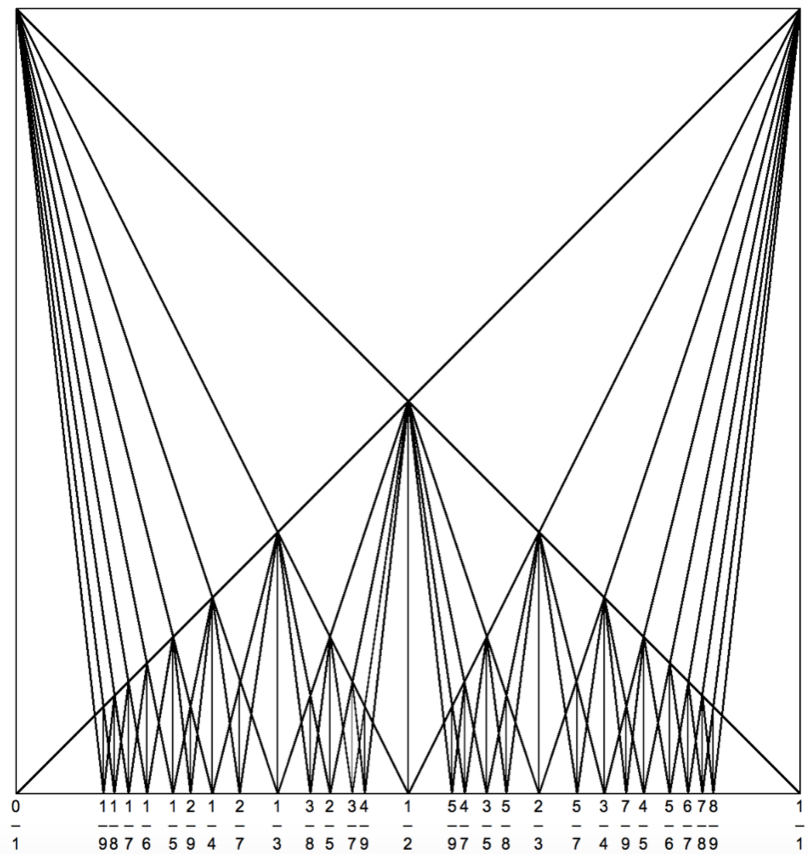
This is (most probably) not in human abilities, as the complexity of the system and the number of teeth needed would have to be insane. How to overcome this problem? Simply approximate $\frac{720}{525949}$ by a fraction with smaller denominator and design a system of gears for the approximation (making some small error)! And the Stern-Brocot tree (provably) yields very good approximations.
And as it turns out, using the Stern-Brocot tree, it is possible to design a system of gears that will only have 83 teeth in the largest gear and makes only a 1 second error per year.

Some visualizations

Let me now conclude the exploration of the Farey sequences by some visual exercises. For each of the diagrams/pictures below, try to unravel it's connection to the Farey sequence and try to find out how it is constructed.



[3]there are, of course, others - for example such approximations can be found in the standard library of Python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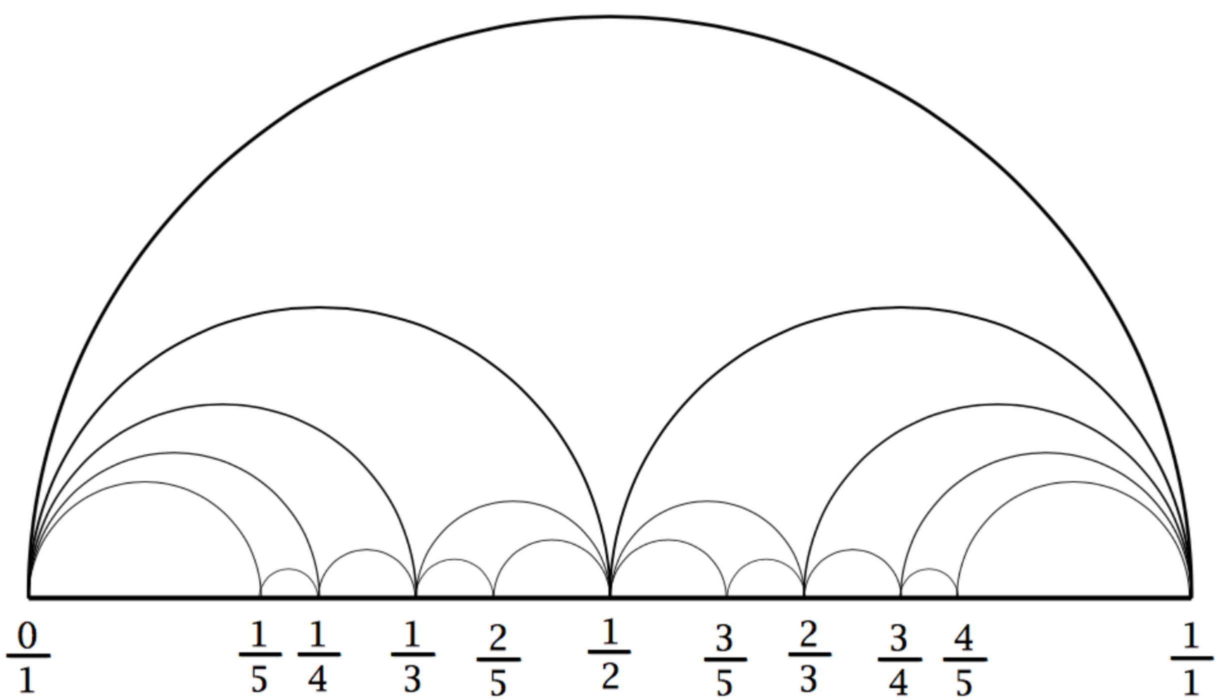
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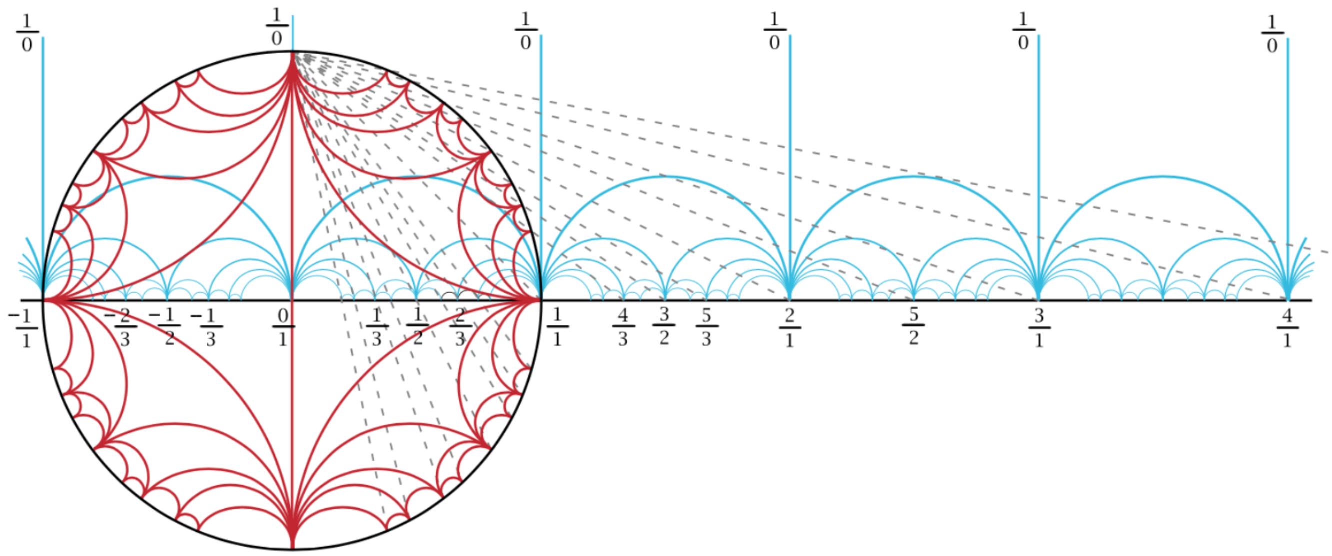
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