An Invitation to Knot Theory

Problem Solving Maths Group

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Knots and Celtic Knots

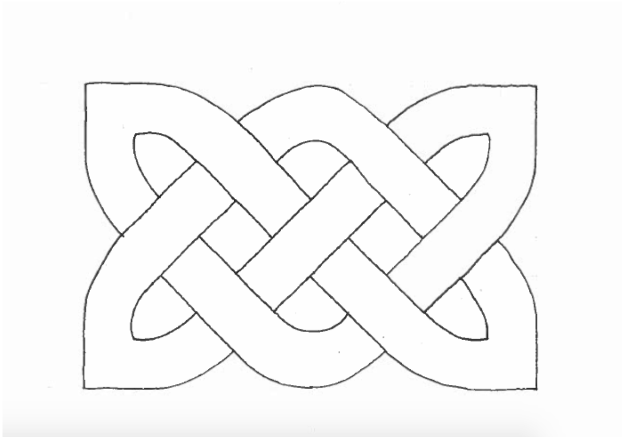


Figure 1: A Celtic knot diagram

Definition (Knot)**.** A mathematical knot is a closed loop in 3-dimensional space, which may be tangled.

Definition (Trivial knot)**.** The trival knot (or unknot) is the untangled circle.

Definition (Link)**.** A link is a collection of knots, which may or may not be intertwined. We will call some links 'knots', even though this is incorrect, because it is not always easy to tell the difference.

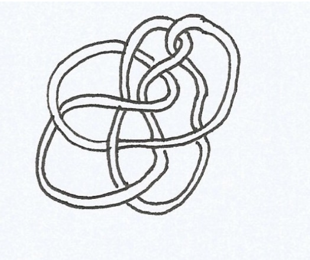
Definition (Knot diagram)**.** A knot diagram is an embedding of a knot into the plane. This may have self-intersections, and where is does, it will be indicated which strand passes above, and which strand passes below.

Definition (Link diagram)**.** A link diagram is an embedding of a link into the plane.

Definition (Alternating)**.** If a strand of an alternating knot or link is chosen, and followed, consecutive crossings along this strand will alternate between 'over' and 'under'. Alternating knots are also sometimes called **Celtic knots**.

Exercise 1. Spot the links! (See handouts).

Exercise 2. Prove the knot depicted by the following diagram is alternating.



The Reidemeister Moves

Reidemeister moves are manipulations of the knot diagram which do not change the knot it represents. They correspond to the ways we can twist and untwist our knot without cutting or breaking it.

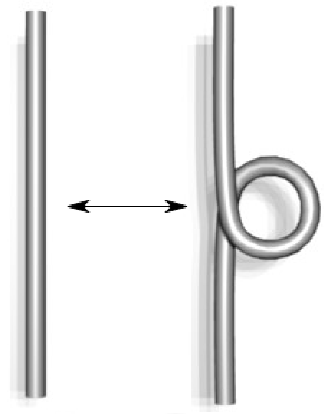


Figure 2: First Reidemeister Move [1]

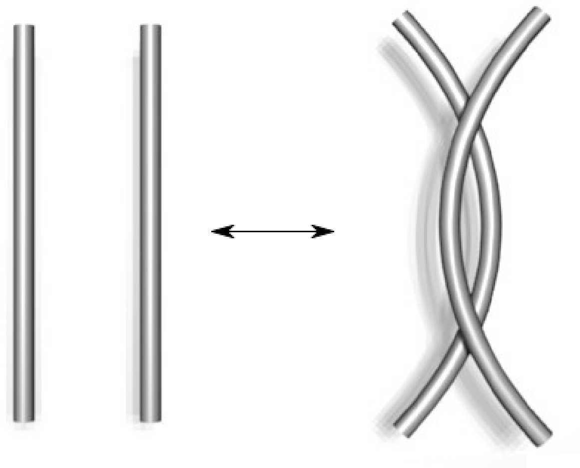


Figure 3: Second Reidemeister Move [1]

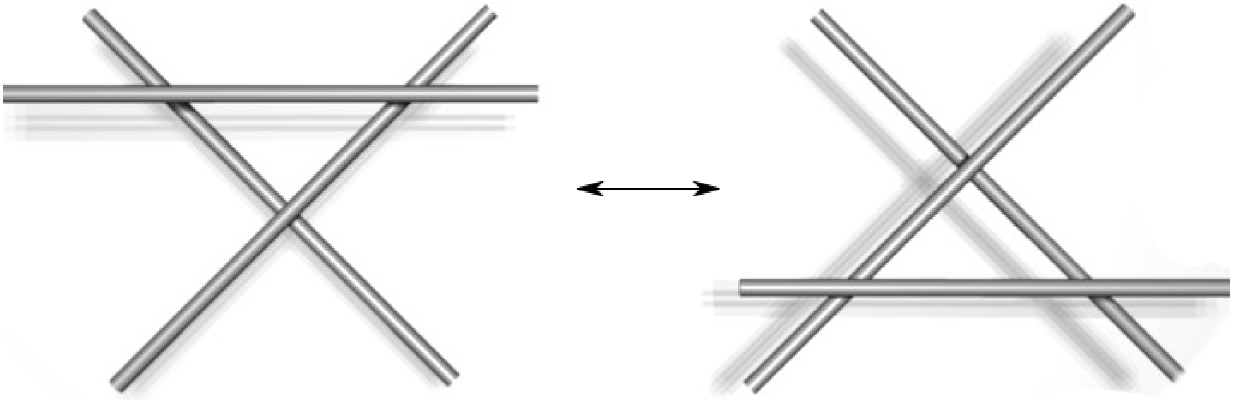


Figure 4: Third Reidemeister Move [1]

Exercise 3. Show that the following pairs of knot diagrams depict the same knots by using Reidemeister moves. You may find it helpful to use a piece of string.

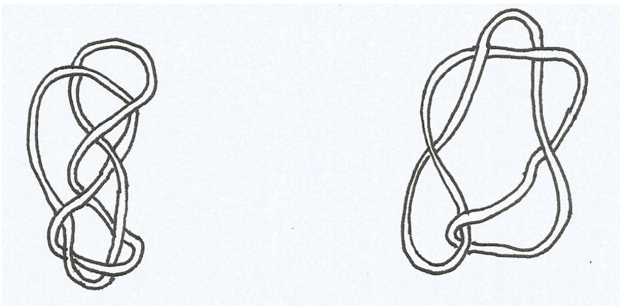


Figure 5:

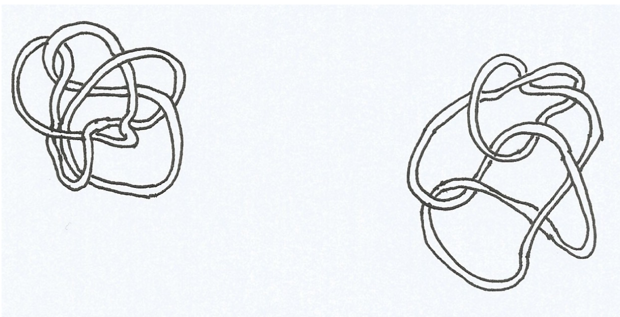


Figure 6:

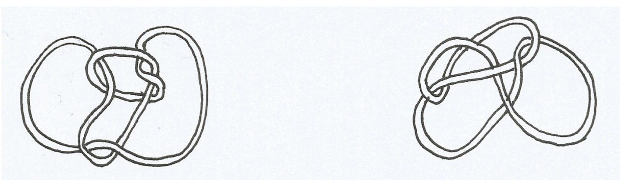


Figure 7:

Figure 8:

Knot Invariants

Knot invariants are properties of a knot that do not depend on the particular diagram or representation. We know that every knot diagram of a knot can be reached from any other by a sequence of Reidemeister moves.This motivates the following definition:

Definition (Knot invariant)**.** Knot invariants are properties of a knot that are preserved by Reidemeister moves. We can think of them as a function from knots to values.

A word of warning: if two knots share some of the same knot invariant properties, that does not necessarily mean they are the same knot. If you can think of a computable knot invariant that has a distinct value for each distinct knot, you will have solved the central problem of knot theory!

Crossing Number

The crossing number is one of the simplest, and least powerful of all knot invariants. However, it is useful for classifying knots.

Definition (Crossing number)**.** The crossing number of a knot is the minimum number of self-intersections in any of the knot's diagrams.

Exercise 4. What values can the crossing number take?

Exercise 5. What is the crossing number of the knot in Figure 9?

Exercise 6. Justify why crossing number is preserved by Reidemeister moves.

Tricolourability

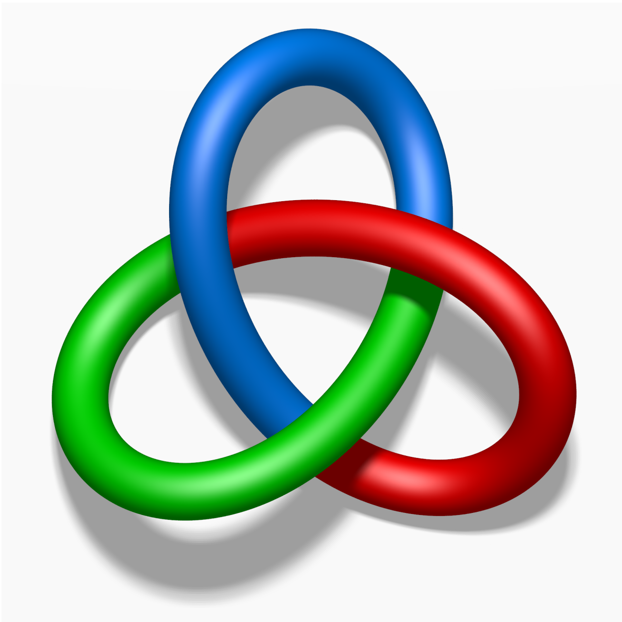
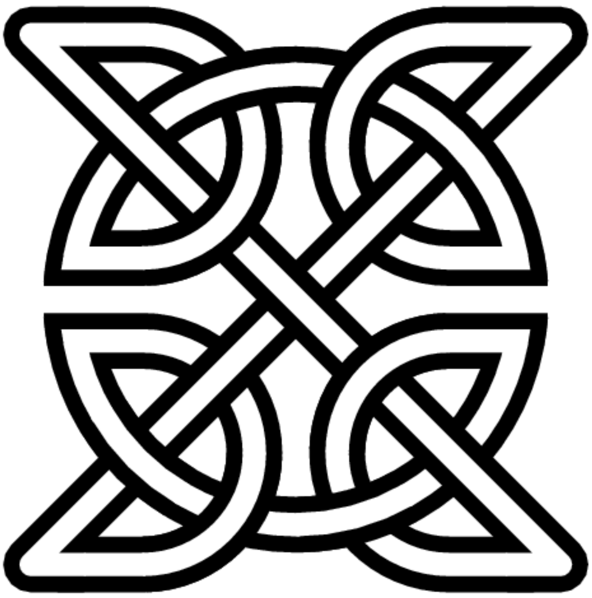


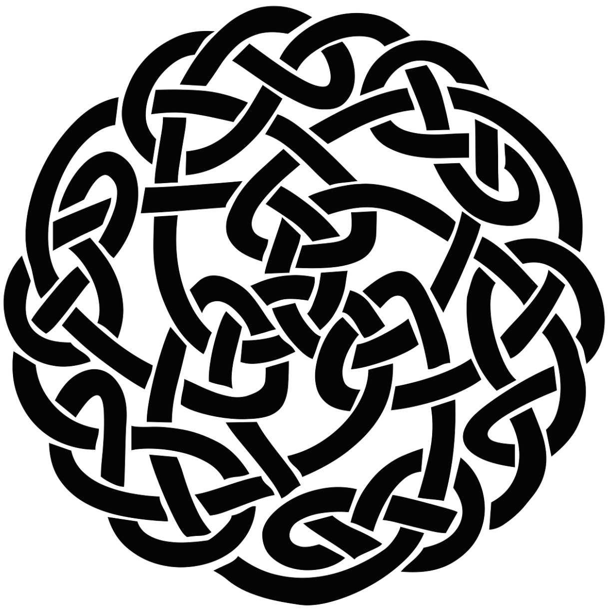
Figure 9: The trefoil knot is tricolourable.

Definition (Tricolourability)**.** A knot is tricolourable if it can be coloured with three colours such that at each crossing, either one or three colours meet.

Exercise 7. What values can tricolourability take?

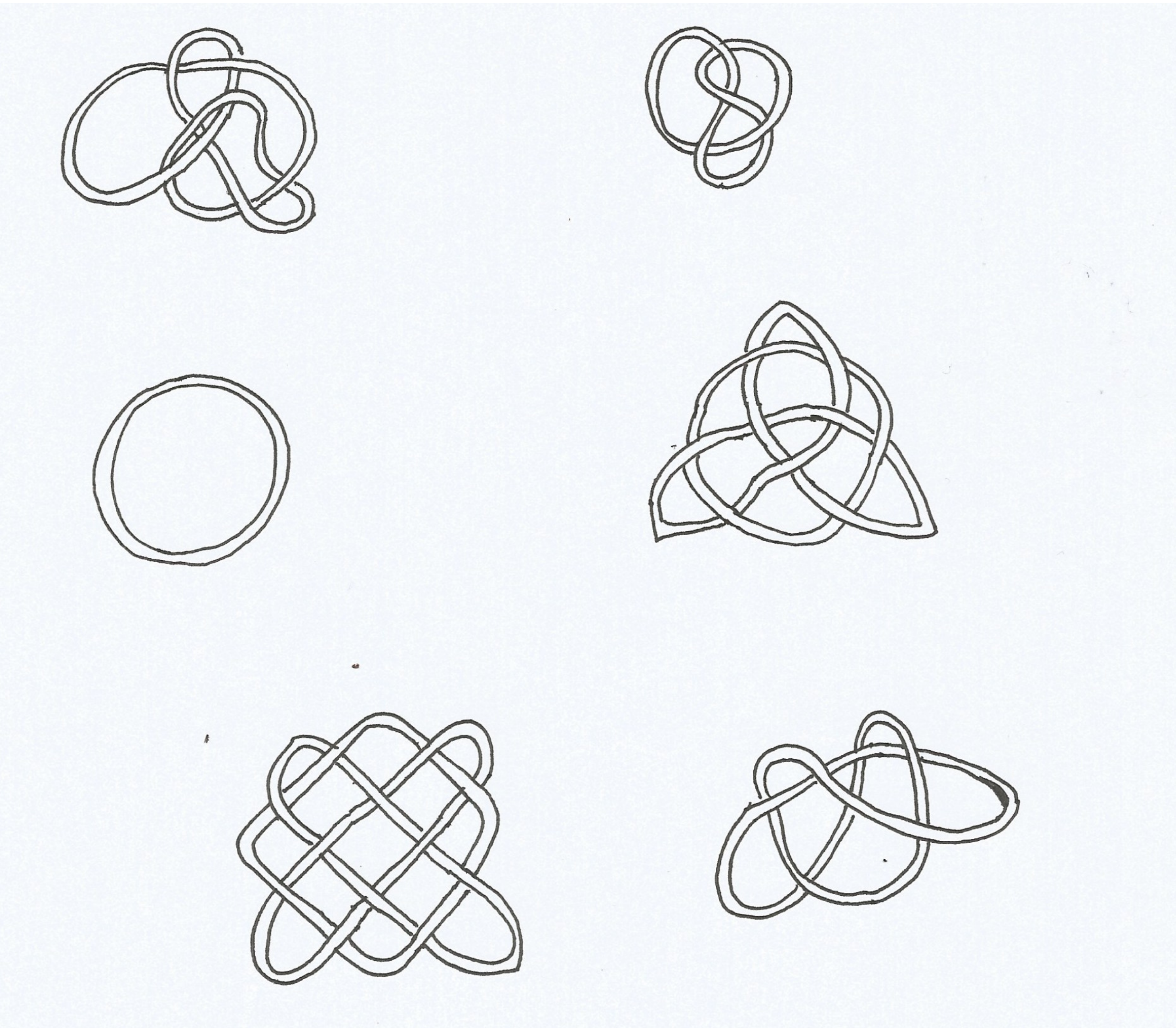
Exercise 8. Are these knots tricolourable?





Exercise 9. Show that tricolourability is preserved by Reidemeister moves.

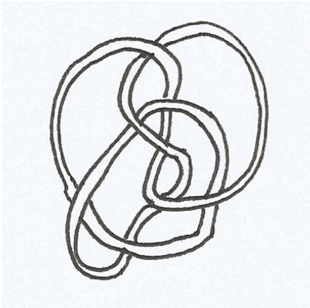
Exercise 10. Prove that none of these knot diagrams depict the same knot.



Further Exercises

Exercise 11. Can a knot diagram ever be perfectly symmetric? Which knots/links can have a symmetric knot diagram? If you considered reflectional symmetry, now consider rotational symmetry.

Exercise 12. Prove this knot is nonalternating.



Exercise 13. Invent your own knot invariant and try it out.

Unsolved Problem. What knots can be made by knitting an object from one strand and joining the ends? Can the same knots be made with crochet?

Further Reading

An Overview of Knot Invariants by Will Adkisson

`http://math.uchicago.edu/~may/REU2015/REUPapers/Adkisson`.

Knot Atlas

A database of knots and links.

`http://katlas.org/wiki/Main_Page`

The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots by Colin C. Adams

References

[1] Reidemeister diagrams are by Makoto Yamashita (Own work) CC-BY-SA-3.0 `http://creativecommons.org/licenses/by-sa/3.0/`, via Wikimedia Commons