Everything

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Week 10

The following pages contain problems that are a mixture of everything[1] we discussed so far on Problem Solving Maths Group, focused on the most important part of each topic.
This whole sheet can be regarded as a sun beam aiming to ripen the fruit of all gained concepts and connections.

1 Invariance Principle

Property which does not change is called *invariant*, property that only changes in one direction, e.g. always increases, is called *monovariant*. These properties are often used in problems involving different states and transitions between them.

1. *Show that if every room in a house has an even number of doors, then the number of outside entrance doors must be even as well.*

2. *Let $d(n)$ denote the digit sum of number n. Find all solutions of the equation:*

$$n + d(n) + d(d(n)) = 2015$$

3. *In an $n \times n$ board the squares are painted black or white. Three of the squares in the corners are white and one is black. Show that there is a 2×2 square with an odd number of white unit squares.*

[1]42

2 Extreme Principle

One of basic problem solving strategies is *Extreme principle.* It uses some extreme property of the problem (e.g. largest element, longest path, or smallest sum) and uses this to construct proof or show contradiction.

Good indicator that Extreme Principle might be useful is when problem has bounded set of possible values, or there is invariant or monovariant.

4. *Prove that for every pair of positive integers* x, y

$$x^2 \neq 35y^2$$

5. *All plane sections of a solid are circles. Prove that the solid is a ball.*

6. *Consider string created by concatinating numbers* $1, 2, \ldots, n$ *for* $n > 1$ *in order. Can such string be a palindrome*[2]*?*

3 Pigeonhole Principle

Also known as *Dirichlet's principle*, it comes from the fact, that if you have more than mn pigeons and n holes, there must be a hole with at least $m + 1$ pigeons. It is often used to prove existance of some object (with given property).

7. *Prove that given a* 10×10 *cm square with* 101 *points marked inside, there exists a triangle of area* 1 *cm*^{2} *containing at least two of these points.*

8. *Show, that in a party there are always two persons who have shaken hands with the same number of people.*

9. *We are given* 1985 *positive integers such that none has a prime divisor greater than* 23*. Show that there are* 4 *of them whose product is the fourth power of an integer.*

[2] *palindrome* is string that is read backwards and forwards in the same way, e.g. civic, 12321 or aibohphobia.

4 AG-Inequality

AG states, that for nonegative real numbers $x_1, x_2, \dots x_n$:

$$\sqrt[n]{x_1 x_2 \dots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

It can be used to solve many inequality problems, often summing multiple AGs together.

10. *Prove that* $\forall x \geq 0$*:* $1 + x \geq 2\sqrt{x}$.

11. *Prove that* $\forall x \geq 0$*:* $2 + 3x^5 \geq 5x^3$.

12. *Prove that for possitive* a, b, c *with* $abc = 1$ *and* $a+b+c \geq a^{-1}+b^{-1}+c^{-1}$ *that also* $a^n + b^n + c^n \geq a^{-n} + b^{-n} + c^{-n}$ *for every possitive integer* n.

5 Geometry

One of most useful properties in geometry, is that given points A, B, C on a circle, angle $\angle ABC$ does not change as B moves along the arc. Consequence of this quality are properties of cyclic quadrilaterals:
- Any side is viewed with same angle from any of the remaining vertices.
- The opposite angles add up to $180°$.

One of the methods how to find and make use of cyclic quadraliterals is so-called Power of a point. it states, that for any circle k (centred at S with radius r) and point P: If line through P intersects k in A and B, then $PA \cdot PB$ is always same and equal to $PS^2 - r^2$.

13. *Let the altitudes of an acute triangle* $\triangle ABC$ *intersect the opposite sides at points* D, E, F*. Show that the altitudes of* $\triangle ABC$ *are angle bisectors in* $\triangle DEF$*.*

14. *Let line* p *intersect circle* k *centered at point* O *at two distinct points* K, L*. Let* X *be the point on* p *outside of* k*. The tangent lines to* k *through* X *are touching the circle at points* A, B*. Prove, that if* M *is the middle of segment* AB *then quadrilateral* $KLMO$ *is*

15. *Look at parabolas given by* $x^2+px+q=0$ *which intersect* x*- and* y*-axis in three different points - A, B, nad C. Prove that circumcircles of all possible triangles* ABC *have a common point.*

16. *Let incircle of triangle* ABC *touch sides* AC, BC *in* E, D. *Let* AD *and* BE *intersect at* G, *and let's have* U *and* V *such that* $ABUE$ *and* $ABDV$ *are parallelograms. Prove that* $GU = GV$.