Computing broadband magnitudes of a redshifted source with a spectrum

Problem: consider a source at a distance d, which corresponds to a redshift z. Given the spectral energy distribution (SED) at the source in form of f_λ and a bandpass filter $A(\lambda)$ for a band B, what is the observed magnitude of the source in AB system?
Solution. The simple answer is

$$m_{\mathrm{B}} = -2.5 \log_{10} \langle \hat{f}_\nu \rangle_{\mathrm{B}} - 48.6, \tag{1}$$

where $\langle \hat{f}_\nu \rangle_{\mathrm{B}}$ is the averaged observed flux through the filter. Here and below, the hat labels observed quantities, and no hat means quantities in the source frame of reference. The angle brackets $\langle \dots \rangle_{\mathrm{B}}$ mean weighted average over the band. Clearly, the averaging should be performed over the photons, because light is quantized. In fact, a standard bandpass filter $A(\nu)$ encodes fraction of photons that goes through, rather than fraction of energy or any other physical quantity. For an SED $\hat{f}_\nu$, the number of photons in an infinitesimal interval of frequencies $d\nu$ is

$$d\hat{N}_\nu = \hat{f}_\nu \frac{h d\nu}{h\nu} = \hat{f}_\nu \frac{d\nu}{\nu}. \tag{2}$$

The formula for statistical averaging over the band is, then,

$$\langle \dots \rangle_{\mathrm{B}} = \frac{\int_{\nu_i}^{\nu_f} (\dots) A(\nu) \frac{d\nu}{\nu}}{\int_{\nu_i}^{\nu_f} A(\nu) \frac{d\nu}{\nu}}, \tag{3}$$

where integration is performed over the interval of frequencies (ν_i, ν_f) where the bandpass is nonzero.
This gives the following expression for the observed averaged broadband flux:

$$\langle \hat{f}_\nu \rangle_{\mathrm{B}} = \frac{\int_{\nu_i}^{\nu_f} \hat{f}_\nu A(\nu) \frac{d\nu}{\nu}}{\int_{\nu_i}^{\nu_f} A(\nu) \frac{d\nu}{\nu}}. \tag{4}$$

We can also rewrite it as a function of wavelength, because since $\nu = c/\lambda$,

$$\frac{d\nu}{\nu} = -\frac{d\lambda}{\lambda}, \tag{5}$$

and therefore

$$\langle \hat{f}_\nu \rangle_{\mathrm{B}} = \frac{\int_{\lambda_i}^{\lambda_f} \hat{f}_\nu(\lambda) A(\lambda) \frac{d\lambda}{\lambda}}{\int_{\lambda_i}^{\lambda_f} A(\lambda) \frac{d\lambda}{\lambda}}, \tag{6}$$

where now $\lambda_i = c/\nu_f$ and $\lambda_f = c/\nu_i$ are inverted boundaries of the band in wavelength space. Note also that, in a somewhat confusing notation, $\hat{f}_\nu(\lambda)$ is the spectral energy distribution in frequency space, written as a function of wavelength. Similarly, $A(\lambda) = A(\nu(\lambda))$ can be a written as a function of wavelength. This does not change the meaning of it as a filter function: it denotes a relative fraction of photons with either wavelength λ or $\nu = c/\lambda$ that pass though.
For consistency, we can also express SED $\hat{f}_\nu$ in frequency space through an SED $\hat{f}_\lambda$ in wavelength space:

$$\hat{f}_\nu = \hat{f}_\lambda \frac{\lambda^2}{c}, \tag{7}$$

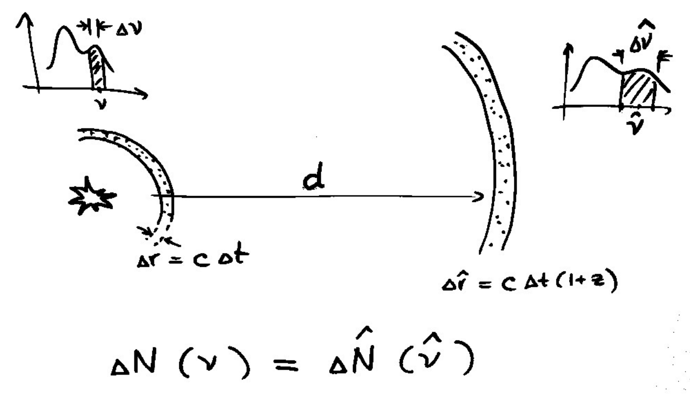

Figure 1: Invariant of the problem: the number of photons departed from the source equals the number of photons which reached the observer distance.

which is easy to show by equating the number of photons per corresponding frequency and wavelength bins. Note that $\hat{f}_\nu$ and $\hat{f}_\lambda$ are different distribution functions, and either of them can be expressed as functions of ν or λ.
Consider how the observed flux $\hat{f}_\nu$ relates to the given source flux f_ν. We can equate the number of photons in frequency bin $\Delta\nu$, crossing the sphere of radius r near the source over the time period Δt, with the number of photons in frequency bin $\Delta\hat{\nu}$, crossing the sphere of radius d at the observer location over the time period $\Delta\hat{t}$ (see Fig. 1):

$$\Delta N(\nu) = \Delta \hat{N}(\hat{\nu}) \tag{8}$$

We can multiply left and right hand sides by $\frac{h\nu}{4\pi r^2 \Delta t \Delta\nu}$ to relate this to the SED in frequency space:

$$\frac{h\nu \Delta N(\nu)}{4\pi r^2 \Delta t \Delta\nu} = \frac{h\nu \Delta\hat{N}(\hat{\nu})}{4\pi r^2 \Delta t \Delta\nu} = \frac{h\hat{\nu} \Delta\hat{N}(\hat{\nu})}{4\pi d^2 \Delta\hat{t} \Delta\hat{\nu}} \cdot \frac{d^2}{r^2} \cdot \frac{\Delta\hat{t}}{\Delta t}, \tag{9}$$

$$f_\nu(\nu) = \hat{f}_\nu(\hat{\nu}) \cdot \frac{d^2}{r^2} \frac{\Delta\hat{t}}{\Delta t}. \tag{10}$$

We can now use the redshift and substitute $\Delta\hat{t}/\Delta t \to (1+z)$ and $\hat{\nu} \to \nu/(1+z)$ to obtain

$$r^2 f_\nu(\nu) = d^2 (1+z) \hat{f}_\nu \left(\frac{\nu}{1+z} \right) \tag{11}$$

Introducing *luminosity distance* $d_L := d(1+z)$:

$$r^2 f_\nu(\nu) = \frac{d_L^2}{1+z} \hat{f}_\nu \left(\frac{\nu}{1+z} \right), \tag{12}$$

we can finally express the SED in observer frame via the SED at the source:

$$\hat{f}_\nu[\hat{\nu}] = \frac{r^2}{d_L^2} (1+z) f_\nu \left[\hat{\nu}(1+z) \right]. \tag{13}$$

This can be converted to the SED in wavelengths,

$$\hat{f}_\lambda[\hat{\lambda}] \hat{\lambda}^2 = \frac{r^2}{d_L^2} (1+z) f_\lambda \left[\frac{\hat{\lambda}}{1+z} \right] \left(\frac{\hat{\lambda}}{1+z} \right)^2 \tag{14}$$

such that

$$\hat{f}_\lambda(\hat{\lambda}) = \frac{r^2}{d_L^2}\frac{1}{1+z} f_\lambda\left(\frac{\hat{\lambda}}{1+z}\right) \tag{15}$$

The averaged broadband flux is:

$$\langle \hat{f}_\nu \rangle_{\mathrm{B}} = \frac{r^2}{d_L^2}(1+z)\frac{\int_{\nu_i}^{\nu_f} f_\nu(\nu(1+z))A(\nu)\frac{d\nu}{\nu}}{\int_{\nu_i}^{\nu_f} A(\nu)\frac{d\nu}{\nu}} \tag{16}$$

$$= \frac{r^2}{d_L^2}(1+z)\frac{\int_{\lambda_i}^{\lambda_f} f_\lambda\left(\frac{\lambda}{1+z}\right)\left(\frac{\lambda}{1+z}\right)^2 A(\lambda)\frac{d\lambda}{\lambda}}{c\int_{\lambda_i}^{\lambda_f} A(\lambda)\frac{d\lambda}{\lambda}} \tag{17}$$

$$= \frac{r^2}{d_L^2}\frac{\int_{\lambda_i}^{\lambda_f} f_\lambda\left(\frac{\lambda}{1+z}\right)\frac{\lambda}{1+z}A(\lambda)d\lambda}{c\int_{\lambda_i}^{\lambda_f} A(\lambda)\frac{d\lambda}{\lambda}}, \tag{18}$$

which is *exactly* the expression Eve uses.
We could instead make a substitution $\lambda/(1+z) \to \lambda'$, which would be equivalent to applying a blueshifted filter to the spectrum at the source and multiplying the result with the $(1+z)$ factor:

$$\langle \hat{f}_\nu \rangle_{\mathrm{B}} = \frac{r^2}{d_L^2}(1+z)\frac{\int_{\lambda'_i}^{\lambda'_f} f_\lambda(\lambda)A[\lambda(1+z)]\lambda d\lambda}{c\int_{\lambda'_i}^{\lambda'_f} A[\lambda(1+z)]\frac{d\lambda}{\lambda}}, \tag{19}$$

where $\lambda'_{i/f} = \lambda_{i/f}/(1+z)$. This is what I've been using, but *without* the factor of $(1+z)$. Oopsies..

O.K. - 2020.11.20