TSP: Text Searching and Processing

Important Notes About These Notes

These notes were written by me, Mark Ormesher, during my revision for Computer Science exams at King's College London. I am publishing them here to help my classmates and other students in the years below me. To the best of my knowledge they are correct.

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These notes were relevant for my year (2016/17) but **the content for your course may be different**. Check your lecture slides, syllabi, tutorial guides, etc.

These notes were produced for my own personal use (it's part of how I study and revise). That means that some annotations may by irrelevant to you, and **some topics might be skipped** entirely.

Feel free to **share** these notes, however please only share a link to the repo (see the link below), not individual files.

Notes are originally from **https://github.com/markormesher/CS-Notes**.

Contents

Terms and Definitions

Alphabets

An alphabet Σ is a finite, **non-empty** set of elements called **letters** or **characters**.

String Basics

A string on an alphabet Σ is a finite, **possible empty** sequence of letters from Σ.

The **empty string** is denoted by ϵ.

The set of **all possibly strings** on an alphabet Σ is denoted by Σ^*.

The **length** of a string x is the length of its sequence of letters, denoted by $|x|$.

The letter at the i**-th position** in the string x is denoted by $x[i]$, for all $0 \leq i < |x|$.

Identity Between Strings

Two strings x and y are **identical** if and only if $|x| = |y|$ and $x[i] = y[i]$ for all $0 \leq i < |x|$.

Concatenation/Product of Strings

The concatenation or product of two strings x and y is the string formed from all letters of x followed by all letters of y. It is denoted by xy.

Factor/Substring of a String

A string x is a factor or substring of a string y if two strings u and v exist such that $y = uxv$. Note that u and/or v may be the empty string ϵ.

A factor or substring x of a string y is **proper** if $x \neq y$.

Superstring of a String

A string x is a superstring of a string y if two strings u and v exist such that $x = uyv$. Note that u and/or v may be the empty string ϵ.

Suffix and Prefix of a String

Let the strings x, y, u and v exist such that $y = uxv$...

- If $u = \epsilon$ then x is a prefix of y.
- If $v = \epsilon$ then x is a suffix of y.

Occurrence of a String

If x and y are two strings and $x \neq \epsilon$, we can say that x **occurs in** y if x is a factor of y.

Every occurrence of x can be identified by a position within y. We say that x occurs in y starting at position i if $y[i...i + |x| - 1] = x$.

It is sometimes more useful to refer to the end position of the occurrence, $i + |x| - 1$.

Example, in the string `abracadabra`, the string `abr` occurs starting from positions 0 and 7.

Power of a String

For a string x and a natural number n, the n-th power of x, denoted by x^n, is defined as follows:

- $x^0 = \epsilon$
- $x^n = x^{n-1}x$

Example: if $a =$ `ha` then $a^4 =$ `hahahaha`.

If two strings x and y and two natural numbers n and m exist such that $x^m = y^n$ then x and y are powers of some string z.

Primitivity of Strings

A string is **primitive** if it is not the power of any other string.

A string x is **primitive** if and only if it exists as a factor of x^2 as a prefix and a suffix.

Example: given $x =$ `abaab` and $x^2 =$ `abaababaab`, the string x is primitive because x only appears at the very start and very end of x^2.

String Roots and Exponents

If $x \neq \epsilon$, a **primitive** string z and natural number n exist such that $x = z^n$. z is the **root** of x and n is the **exponent** of x. If x is primitive then $z = x$ and $n = 1$.

Example: $x =$ `abab`, $z =$ `ab`, $n = 2$.

Example: $x =$ `abcd`, $z =$ `abcd`, $n = 1$.

Conjugate Strings

Two non-empty strings x and y are conjugate if two strings u and v exist such that $x = uv$ and $y = vu$.

Example: `goldfish` and `fishgold` are conjugate.

Two non-empty strings are conjugate if and only if their **roots are also conjugate**.

Two non-empty strings are conjugate if and only if a string z exists such that $xz = zy$ (for the above example, $z =$ `gold`).

Periods of a String

For a non-empty string x, an integer p such that $0 < p \leq |x|$ is a period of x if $x[i] = x[i + p]$ for all $0 \leq i < |x| - p$.

A string may have multiple periods, and every string has at least one period (the length of the string itself). We define *the* period of a string as its **smallest** period, denoted by $per(x)$.

Example: the string $x =$ `aabaabaa` has periods of 3, 6, 7 and 8, and $per(x) = 3$.

Borders of a String

For a non-empty string x, a border is simultaneously a **proper factor**, **prefix** and **suffix** of x.

We define *the* border of a string as its **longest** border. $border(x)$ denotes the **length** of the longest border of x.

Example: the string $x =$ `aabaabaa` has borders of ϵ, `a`, `aa` and `aabaa`, so $border(x) = 5$.

Relationship Between Periodicity and Borders

For any non-empty string x it holds that $per(x) + border(x) = |x|$.

String Matching

Objective: given two strings, $pattern$ and $text$, find the **starting positions of all occurrences** of $pattern$ within $text$.

Example: within the text `abracadabra`, the pattern `abr` starts at positions 0 and 7.

Notation

- The **pattern** is denoted as x with length m.
- The **text** is denoted as y with length n.

Online and Offline Algorithms

- In an online algorithm, only the **pattern** is known and can be pre-processed.
- In an offline algorithm, only the **text** is known and can be pre-processed.

Näive Matching

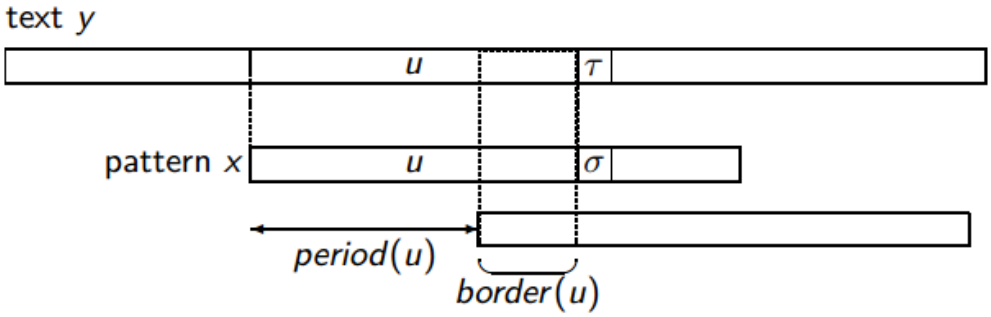
Matches are found by comparing the pattern to the text string repeatedly, incrementing the starting character each time. In the example below, the text is `abracadabra`, the pattern is `abr`, and matching portions are shown in red.

Index i:	0	1	2	3	4	5	6	7	8	9	10
Text y:	a	b	r	a	c	a	d	a	b	r	a
Successful try from 0:	a	b	r								
Failed try from 1:		a	b	r							
Failed try from 2:			a	b	r						
Failed try from 3:				a	b	r					
Failed try from 4:					a	b	r				
Failed try from 5:						a	b	r			
Failed try from 6:							a	b	r		
Successful try from 7:								a	b	r	
Failed try from 8:									a	b	r

The runtime of this algorithm is $O(nm)$ in the worst case; this is **too slow**.

Morris-Pratt (MP) Algorithm

This algorithm improves the runtime by 'shifting' the pattern along the text by **more than one position** when a mismatch occurs.



In the diagram above, u shows the **prefix** of x that has been successfully matched within the text y; the mismatch is caused by the non-equal characters τ and σ.

When the mismatch occurs, it is safe to move x along so that it lines up with the **end border** of u and resume searching from the position of τ (because we already know that the portion within the border already matches). The size of this movement is $period(u)$, because $|u| - border(u) = period(u)$.

The general structure of the algorithm is as follows:

- Memorise the borders of all prefixes of the pattern.
- Perform straightforward online search.
- When a mismatch occurs:
  - Let u be the string matched so far.
  - Shift by $period(u)$.
  - Resume scanning from $border(u)$.

```
1  fun MP_MATCHING(string y, string x, int n, int m):
2
3      mpNext = COMPUTE_MP_NEXT(x, m)
4
5      i = 0 // index of current comparison, relative to pattern x
6      j = 0 // start of window in text y
7
8      while (j < n) do:
9          while ((i == m) || (i >= 0 and x[i] != y[j])) do:
10             i = mpNext[i]
11         i++
12         j++
13         if (i == m):
14             output(j − i)
```

In the above code, $mpNext[i]$ is the length of the longest border of the first i characters of the pattern x.

MP Pre-processing

It is possible to efficiently compute the border size of pattern prefixes using one key property: **a border of a border of u is also a border of u.** Using this, the algorithm will compute the array $border$ (a.k.a $mpNext$) such that $border[i]$ is the length of the longest border of the first i characters of the pattern x.

After defining $border[0] = -1$ for the empty string, for each x-prefix of length $1 \leq i \leq m$ the algorithm starts by assuming the length of the previous border. If the i^{th} character matches then the while loop is skipped and the border length is increased by 1. When a mismatch occurs, the algorithm tries successively shorter 'borders of borders' until a match is found.

```
1  fun COMPUTE_MP_NEXT(string x, int m):
2      border[0] = -1
3
4      for (i from 1 to m) do:
5          j = border[i - 1]
6          while (j >= 0 and x[i - 1] != x[j]) do:
7              j = border[j]
8          border[i] = j + 1
9
10     return border
```

Note: `border = mpNext`.

In the outer loop j is only incremented exactly m times. In the inner loop j can only decrease but $j \geq 0$ at all times. This means that the maximum amount of 'travel' for the value of j is $2m$ and therefore the algorithm's runtime is $O(m)$.

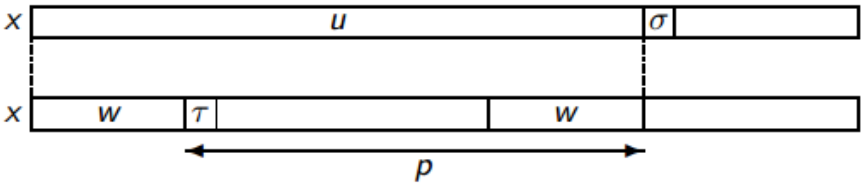
Knuth-Morris-Pratt (KMP) Algorithm

This algorithm improves slightly on the runtime of the MP matching algorithm (*see more: Morris-Pratt (MP) Algorithm, page 8*) by changing only the pre-processing stage to give slightly longer shifts.

The improvement relies on **strict borders** and their corresponding **interrupted periods**.

w is a strict border of the prefix u if:

- w is a border of u, and
- $w\tau$ is a prefix of x but $u\tau$ is not.



In the diagram above, w is a strict border of the prefix u because $\tau \neq \sigma$. Therefore p is the interrupted period of u.

KMP Pre-processing

This algorithm assumes that $mpNext$ is already available. Assuming $k = mpNext[i]$, it is defined as follows:

$$kmpNext[i] = \begin{cases} k & x[i] \neq x[k] \text{ or } i = m \\ kmpNext[k] & x[i] = x[k] \end{cases}$$

This states that $kmpNext[i]$ is equal to $mpNext[i]$ when there is a mismatch between the next character after the prefix and the next character after the border (i.e. a strict border). If the letters do match then the strict border of the border is taken.

```
fun COMPUTE_KMP_NEXT(string x, int m):

    kmpNext[0] = −1
    k = 0 // k will be mpNext[i]

    for (i from 1 to m − 1) do:

        if (x[i] == x[k]):
            kmpNext[i] = kmpNext[k]
        else:
            kmpNext[i] = k
            do:
                k = kmpNext[k]
            while (k >= 0 and x[i] != x[k])

        k++

    kmpNext[m] = k
    return kmpNext
```

As before, in the outer loop k is only incremented exactly m times. In the inner loop k can only decrease but $k \geq 0$ at all times. This means that the maximum amount of 'travel' for the value of k is $2m$ and therefore the algorithm's runtime is $O(m)$.

Runtime of MP and KMP Matching

On a text of length n, the MP and KMP algorithms have runtimes of $O(n)$. In particular, they make no more than $2n$ character comparisons. This is because positive comparisons will increase the value of j (which ranges from 0 to n and does not decrease), and negative comparisons will increase the value of the starting position $j - i$ (which also ranges from 0 to n and does not decrease).

Including the pre-processing, the runtime of both algorithms is $O(n + m)$. In most situations, KMP matching is faster than MP matching.

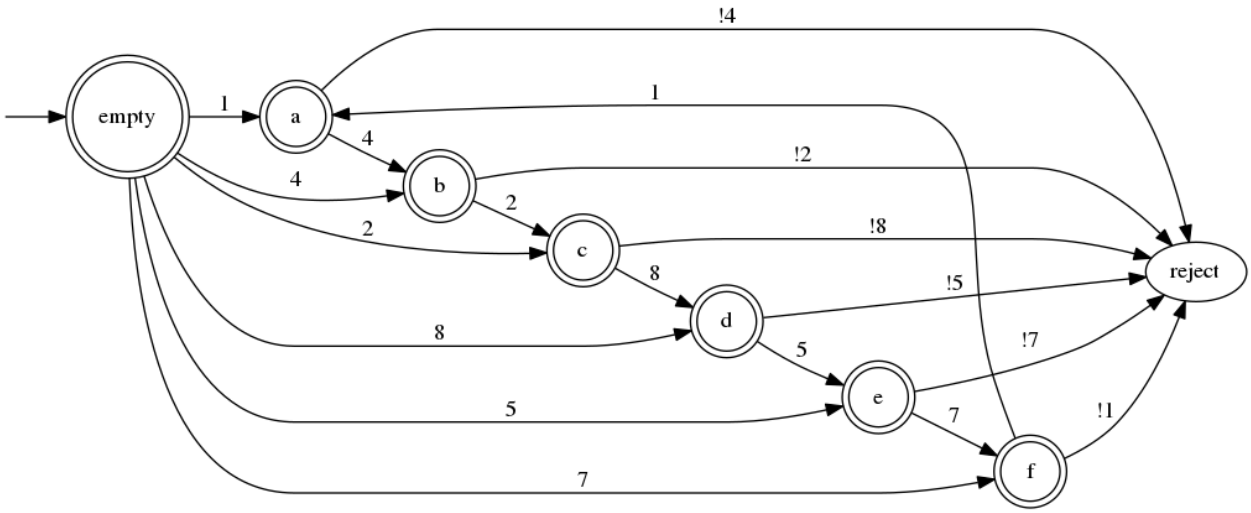
Matching with Automata

Designing Automata to Match Periodic Strings

A **non-empty string** is periodic if it has the form $uuu...uu'$, such that u' is a prefix of u. For example, the string `abbabbab` is periodic, where $u =$ `abb` and $u' =$ `ab` (*see more: Periods of a String, page 7*).

Automata can be easily designed to **accept any substring of a repeating string**, including the empty string ϵ.

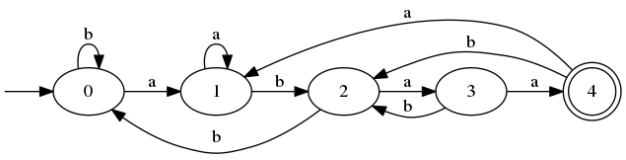
For example, the following automaton operates over the alphabet $\Sigma = \{1, 2, 4, 5, 7, 8\}$ and accepts any substring of the decimal expansion of $\frac{1}{7} = 0.142857142857...$ (ignoring the $0.$ prefix).



Designing Substring-Matching Automata (SMAs)

String-matching automata (SMAs) should reach an accepting state every time a given pattern is encountered as a substring of the input. For example, with the pattern `abb` and the input `cabbabb`, the automaton should reach an accepting state after processing characters 3 and 6 (note: zero-indexed).

For example, the automaton below accepts $u =$ `abaa` in the alphabet $\Sigma = \{a, b\}$.



Text y		b	a	b	b	a	a	b	a	a	b	a	a	b	b	a
State	0	0	1	2	0	1	1	2	3	**4**	2	3	**4**	2	0	1

Matching Using SMAs

A simple online parsing of the text y with $SMA(u)$ is sufficient:

```
fun SMA_MATCHING(string y, sma):
    q = sma.initial_state

    if  (q is terminal):
        report occurrence of pattern

    while (y has remaining characters) do:
        a = next character of y
        q = q.successors[a]
        if (q is terminal):
            report occurrence of pattern
```

Systematic Construction

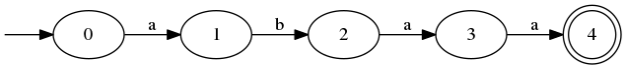
An SMA that matches u (where $|u| = m$) will contain $m + 1$ states numbered $0..m$. In general, if the automaton is in state k then the k most recently read characters match the first k characters of the pattern. The final state (numbered m) is the only accepting state.

The SMA can be constructed with the following process:

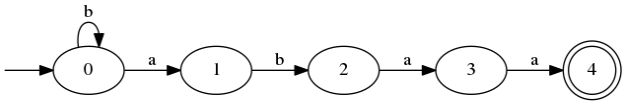
1. Create the initial $m + 1$ states, numbered $0..m$.
    - Declare state 0 as the initial state.
    - Declare state m as the accepting state.
2. For each $0 \leq k < m$, create a forwards arc from state k to state $k + 1$ labelled by $u[k]$.
    - This is the successful path that simply spells u.
3. At each state, create a backwards arc for each character in Σ that breaks the pattern u.
    - The backwards arc should lead to state k, where the k most recently read characters match the first k characters of the pattern.
    - Another way of stating this is to find the longest suffix of processed input that matches a prefix of u. The length of this suffix/prefix is k.
    - k can be found by shifting the pattern along the read characters to the right, until a match occurs. An example follows.

Systematic Construction Example

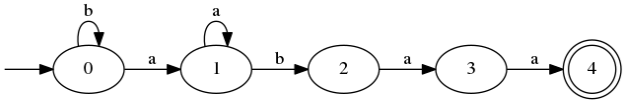
Consider the alphabet $\Sigma = \{a, b\}$ and the pattern $u =$ `abaa`. After steps 1 and 2, the following SMA will be created:



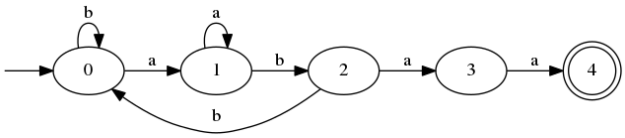
Applying step 3 to state 0, a break in the pattern occurs after reading `b`. This shares no characters with the start of u, so the arc should lead to state 0:



Applying step 3 to state 1, a break in the pattern occurs after reading `aa`. The longest suffix of `aa` that matches a prefix of `abaa` is `a`, so the backwards arc leads to state 1:



Applying step 3 to state 2, a break in the pattern occurs after reading `abb`. No suffix of `abb` matches a prefix of `abaa`, so the backwards arc leads to state 0:

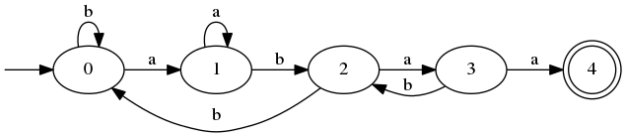


Applying step 3 to state 3, a break in the pattern occurs after reading `abab`. The longest suffix of `abab` that matches a prefix of `abaa` is `ab`, so the backwards arc leads to state 2. This can be seen by shifting the pattern to the right along the input:

```
Input:   abab
Pattern: abaa
```

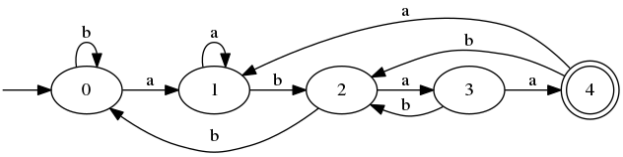
```
Input:   abab
Pattern:  abaa
```

```
Input:   abab
Pattern:   abaa
             ^^ Match!
```



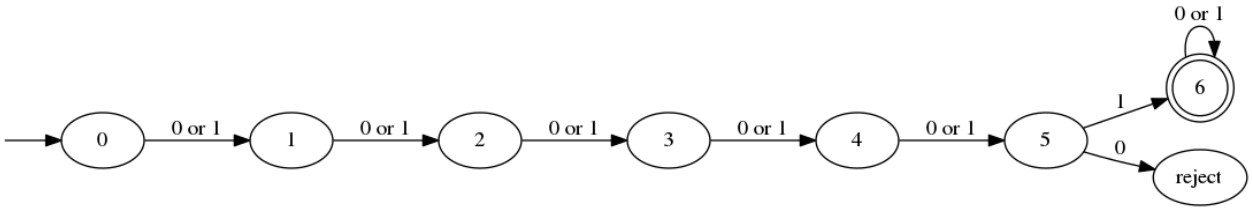
Finally, at state 4, backwards arcs for `a` and `b` need to be created.

The longest suffix of `abaaa` matching a prefix of `abaa` is `a`, so a backwards arc for `a` leads to state 1. The longest suffix of `abaab` matching a prefix of `abaa` is `ab`, so a backwards arc for `b` leads to state 2:

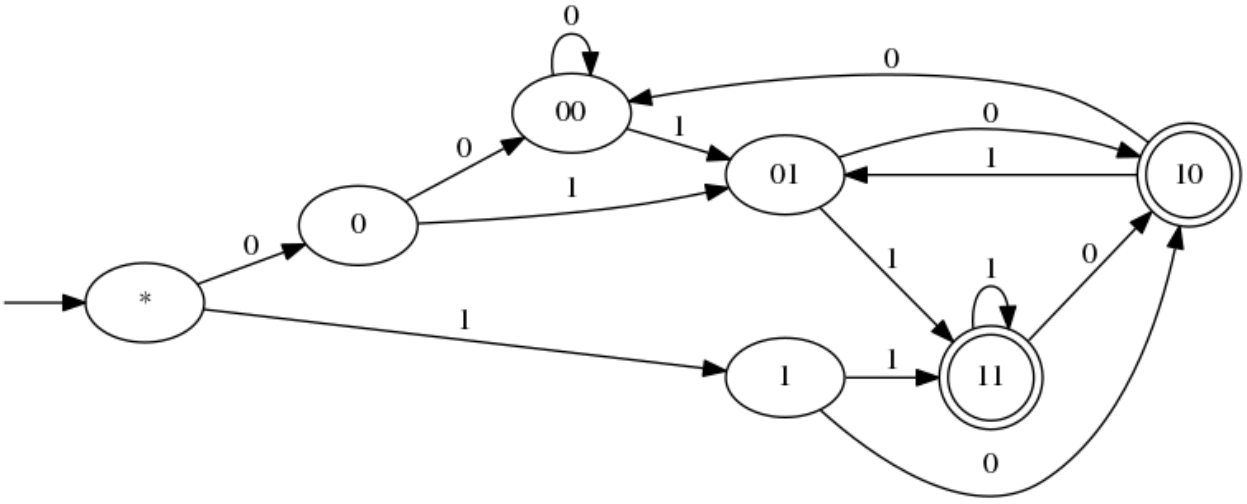


Matching n^{th} Character from the Left and Right

Matching **from the left** is trivial: the first $n-1$ character are skipped, and the n^{th} character must match. For example, the following automaton accepts any string from $\Sigma = \{0, 1\}$ where the 6th character from the left is 1:



Matching **from the right** is more complex: the automaton must be 'tricked' into keeping a memory. This can be achieved by making a state for every combination of the n right-most characters and accepting states that begin with the required character. For example, the following automaton accepts any string from $\Sigma = \{0, 1\}$ where the 2nd character from the right is 1:



Dictionary Matching

Formally, a **dictionary** is a set of strings $X = \{x_0, x_1, ...\}$ such that $\epsilon \notin X$.

Objective: given a **dictionary** of strings X and a text y, find all occurrences of strings from the dictionary within y. The output should be the **list of positions** in y that are **end positions** of some string in X.

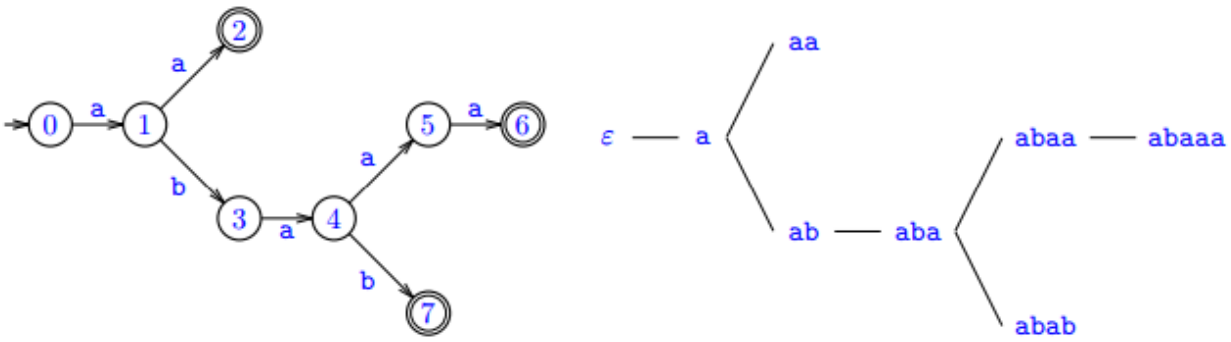
Example: if $X = \{\texttt{abc}, \texttt{cba}\}$ and $y = \texttt{aabcbabc}$, then the output should be $[3, 5, 7]$.

The standard approach for this is to use a **dictionary matching automaton (DMA)**.

Tries of Dictionaries

A trie $T(X)$ is a tree whose leaves are labelled by strings of X. Internal nodes are prefixes of strings in X. As an automaton, the trie $T(X)$ accepts all strings in X.

For example, $T(\{\texttt{aa}, \texttt{abaaa}, \texttt{abab}\})$:



Dictionary Matching Automata (DMAs)

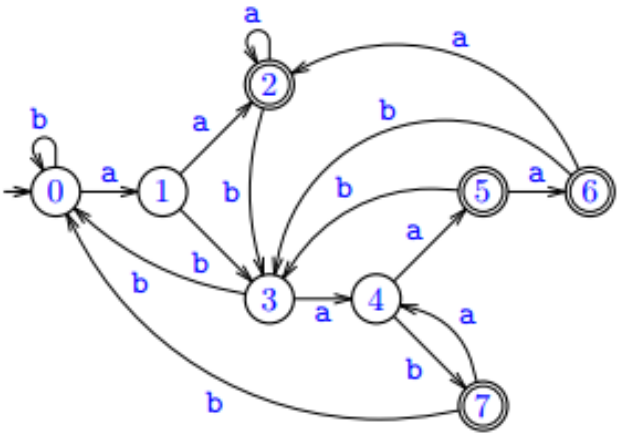
DMAs reach an accepting state every time a word from the dictionary is read in full (very similar to the SMAs from earlier). Formally, a DMA for the dictionary X (denoted $D(X)$) operates over the alphabet A and accepts any word in the form A^*X (i.e. any string from the language, suffixed with a string from X).

A dictionary trie forms the basis of a DMA. DMAs are constructed as follows:

- The set of states is $pref(X)$.
  - i.e. all prefixes of strings in the dictionary X.
- The initial state is the empty string ϵ.
- The set of terminal/accepting states is $pref(X) \cap A^*X$.
  - i.e. all prefixes that are suffixed with strings from X.

- Edges are in the form $\langle u, a, h(ua)\rangle$, where u is the text read so far, a is the character labelling the edge, and $h(ua)$ is the longest suffix of ua that belongs to $pref(X)$.

For example, $D(\{\texttt{aa}, \texttt{abaaa}, \texttt{abab}\})$:



$0 = \epsilon$, $1 = \texttt{a}$, $2 = \texttt{aa}$, $3 = \texttt{ab}$ etc.

Note: state $5 = \texttt{abaa}$ is an accepting state because `abaa` is suffixed by `aa`, which is in the dictionary.

The size of DMAs is $O(|A| \cdot \Sigma(|x| : x \in X))$ (i.e. one arc for each letter in the alphabet, for each of the possible prefix states).

Matching Using DMAs

Almost identical to the SMA searching algorithm (*see more: Matching Using SMAs, page 14*).

```
fun DMA_MATCHING(string y, dma):
    q = dma.initial_state

    for each character a in y, do:
        q = q.successors[a]
        if (q is terminal):
            report occurrence of dictionary string
```

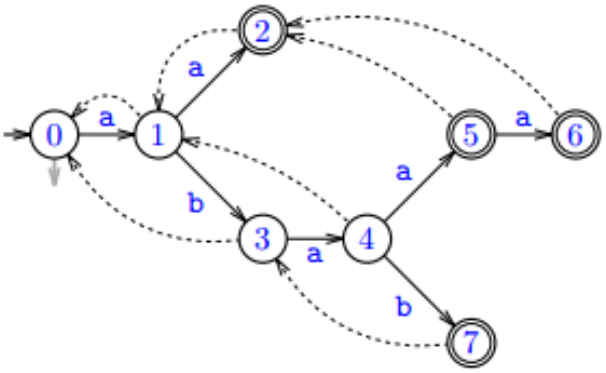
The running time of this matching process on a text y is $\Theta(|y|)$.

DMAs With Failure Links

This adaptation of DMAs reduces the size to $O(\Sigma(|x| : x \in X))$, independent of the alphabet size. It works by keeping the 'forward' arcs between prefixes and replacing all 'backward' arcs with one **failure link** per prefix node.

Failure links are determined by the **failure function** f, such that $f(u) =$ the **longest proper suffix** of u that is in $pref(X)$. The failure link for a node labelled by u leads to $f(u)$.

For example, $D(\{\texttt{aa}, \texttt{abaaa}, \texttt{abab}\})$:



This can be represented with a compact successor table, as below. Note that the 'label' column is not needed and is only for clarity.

State	Label	Successors	Failure
0	ϵ	$(a, 1)$	nil
1	a	$(a, 2)$, $(b, 3)$	0
2	aa		1
3	ab	$(a, 4)$	0
4	aba	$(a, 5)$, $(b, 7)$	1
5	abaa	$(a, 6)$	2
6	abaaa		2
7	abab		3

Matching Using DMAs with Failure Links

This matching process works as before when successors (forward arcs) exist, but when no successors exist the algorithm will **follow failure links** until it reaches a state containing an appropriate successor (or the initial state).

```
fun DMA2_MATCHING(string y, dma):
    q = dma.initial_state

    for each character a in y, do:
        q = TARGET_BY_FAILURE(q, a, dma)
        if (q is terminal):
            report occurrence of dictionary string
```

```
fun TARGET_BY_FAILURE(p, a, dma):
    while (p != nil and p.successors[a] = nil) do:
        p = p.failure

    if (p == nil):
        return dma.initial_state
    else:
        return p.successors[a]
```

For example, using the DMA with failure links above and the text `abbabaababababb`:

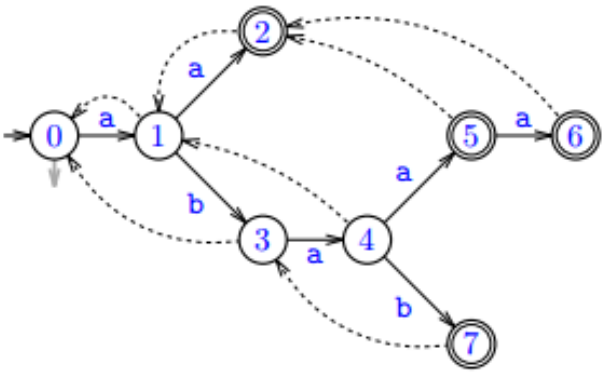
Text y		a	b	b		a	b	a	a	b		a	b	a		b	b
State	0	1	3	0, 0	1	3	4	**5**	2, 1, 3	4	**7**	3, 4	**7**	3, 0, 0

Computing Failure Links

Failure links can be efficiently computed via a **breadth-first traversal** of the trie, as described below. Breadth-first traversal is efficient because failure links for a given state can be computed using the failure links for 'earlier' states, which will have already been computed.

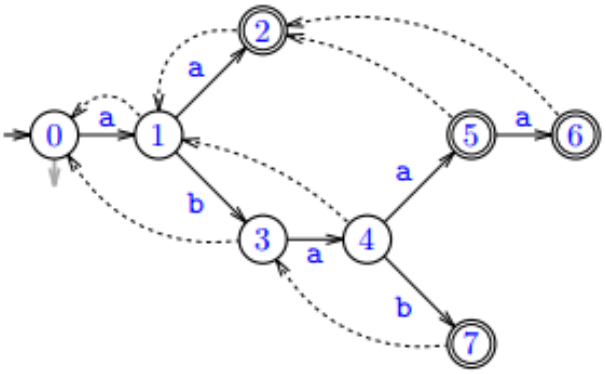
```
fun BUILD_DMA_WITH_FAILURE_LINKS(X):
    M = BUILD_TRIE(X)
    fail[M.initial_node] = nil

    Q = empty queue
    Q.enqueue(M.initial_state)
    while (Q is not empty) do:
        t = Q.dequeue()

        for each pair (a, p) in t.successors, do:

            // successor failure is built on the parent's failure
            r = TARGET_BY_FAILURE(fail[t], a)
            p.failure = r

            // if the failure goes to a terminal state,
            // the success becomes terminal
            if (r.terminal):
                p.terminal = true

            Q.enqueue(p)

return M
```

For example, using $D(\{\texttt{aa}, \texttt{abaaa}, \texttt{abab}\})$:



- State 5 is a child of state 4 along an arc labelled `a`.
  - $fail[4] = 1$.
  - `TARGET_BY_FAILURE(1, a)` = 2, so $fail[5] = 2$.
  - State 2 is terminal, so state 5 is marked terminal.
- State 6 is a child of state 5 along an arc labelled `a`.
  - $fail[5] = 2$.
  - `TARGET_BY_FAILURE(2, a)` = 2, so $fail[6] = 2$.
  - This is because 2 'fails to' 1, which then has an edge back to 2 via `a`.

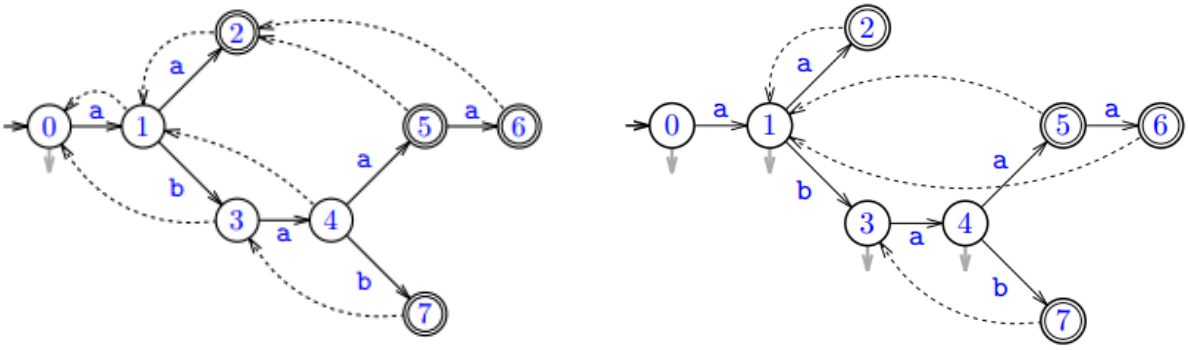
Optimising Failure Links



In the example above, $fail[4] = 1$. However, the outgoing edges from state 1 are a subset of the outgoing edges from state 4, so any input that failed on state 4 is guaranteed to also fail on state 1.

Therefore, the initial set of failure links can be optimised: at a state k, follow failure links until a state is reached with outgoing edges that are not a subset of k's outgoing edges (i.e. a state with 'new' outgoing options).

This is shown in the following example, where the optimised version is on the right:



Delay

The delay is the **maximum time spent on any letter** of the input y. Even with optimised failure links, it is $max\{|x| : x \in X\}$ for worst-case strings.

Searching in a List of Strings

Objective: find the position of a string within a sorted list, or an indication of where it should be added if it is not present.

- Input:
  - A list L from n strings of Σ^*, sorted in lexicographic order.
  - A string $x \in \Sigma^*$ of length m.
- Simple searching output:
  - $x \in L$: a position i such that $0 \leq i < n$, where $L_i = x$.
  - $x \notin L$: positions d, f such that $d+1 = f$ and $-1 \leq d < f \leq n$, where $L_d < x < L_f$.
  - i.e. i is the position of x, or d and f are indexes of the strings either side of where x *would* be if it was in the list.
- Interval output:
  - Positions d, f such that $-1 \leq d < f \leq n$, where x is a prefix of L_i for all $d < i < f$.

For example:

$$L = \begin{array}{c|l} L_0 & \texttt{aaabaa} \\ L_1 & \texttt{aaabb} \\ L_2 & \texttt{aabbbb} \\ L_3 & \texttt{ab} \\ L_4 & \texttt{baaa} \\ L_5 & \texttt{bb} \end{array}$$

Simple Search:

$x = \texttt{aaabb} \longrightarrow 1$

$x = \texttt{aaba} \longrightarrow (1, 2)$

Interval:

$x = \texttt{aa} \longrightarrow (-1, 3)$

Simple Searching Algorithm

The simple search algorithm works in a similar fashion to **binary search** over a sorted array, using a 'mid-point' i and choosing to search in the first or second half of the remaining list. At each i the length of the **longest common prefix** (LCP) of x and L_i is recorded as l, then one of a few things can happen:

- If $|lcp(x, L_i)| = |L_i| = |x|$, the search term has been found so i is returned.
- If $|lcp(x, L_i)| = |L_i| \neq |x|$, the search term must exist in the second half of the list.
- If $|lcp(x, L_i)| \neq |x|$ **and** the next character after the LCP is lower in L_i than it is in x, the search term must exist in the second half of the list.
- Otherwise, the search term must exist in the first half of the list.

```
fun SIMPLE_LIST_SEARCH(L, n, x, m):
    d = −1
    f = n

    while (d + 1 < f) do:
        i = floor((d + f) / 2)
        lcp = LONGEST_COMMON_PREFIX(x, L_i).length

        if (lcp == L_i.length and lcp == m):
            return i

        else if (lcp == L_i.length):
            d = i

        else if (lcp != m and L_i[lcp] < x[lcp]):
            d = i

        else:
            f = i

    return (d, f)
```

The binary search takes $O(log_2(n))$ and computing the LCP at each step takes $O(m)$, so this has a total running time of $O(m \cdot log_2(n))$.

The worst case running time can be created with $L = \{\mathtt{a}^{m-1}\mathtt{b}, \mathtt{a}^{m-1}\mathtt{c}, \mathtt{a}^{m-1}\mathtt{d}, ...\}$ and $x = \mathtt{a}^m$.

Improving Search with LCPs

Objective: reduce running time to $O(m + log_2(n))$.

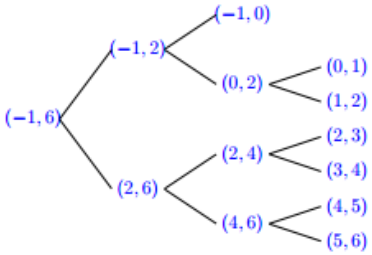
This improvement relies on $lcp(L_d, L_f)$ being known for any for (d, f) considered in the binary search. This requires $O(n)$ extra space to store the $2n + 1$ LCPs associated with the nodes in the binary search tree (see below).

The algorithm is based on properties that arise in three cases (and their symmetric cases).

Side Note: Binary Search Tree

- Nodes:
  - $n + 1$ external nodes, $(-1, 0), (0, 1), (1, 2), ..., (n - 1, n)$.
  - n internal nodes in the form (d, f) with children $(d, \lfloor (d+f)/2 \rfloor)$ and $(\lfloor (d+f)/2 \rfloor, f)$.
  - Root of $(-1, n)$.
- Size: $2n + 1$ for a list of n strings.

Example for $n = 6$:



Notation

Throughout the algorithm...

- $ld = |lcp(x, L_d)|$
- $lf = |lcp(x, L_f)|$
- $i = \lfloor (d + f)/2 \rfloor$

Case 1

Hypothesis:

- $L_d < x < L_f$
- $ld \leq |lcp(L_i, L_f)| < lf$.

Example:

L_d	aaaca	$x =$ aabbbaa
	aaacba	
L_i	aabbaba	
	aabbabb	
L_f	aabbbab	$x =$ aabbbaa

Conclusion:

- $L_i < x < L_f$
  - x is in the **second half** of the list.
- $|lcp(x, L_i)| = |lcp(L_i, L_f)|$
  - See underlined portions above.

Case 2

Hypothesis:

- $L_d < x < L_f$
- $ld \leq lf < |lcp(L_i, L_f)|$.

Example:

L_d	aaaca	$x =$ aabacb
	aaacba	
L_i	aabbaba	
	aabbabb	
L_f	aabbbab	$x =$ aabacb

Conclusion:

- $L_d < x < L_i$
  - x is in the **first half** of the list.
- $|lcp(x, L_i)| = |lcp(x, L_f)|$
  - See underlined portions above.

Case 3

Hypothesis:

- $L_d < x < L_f$
- $ld \leq lf = |lcp(L_i, L_f)|$.

Example:

L_d	aaaca	$x =$ aabbab
	aaacba	
L_i	aabbaba	
	aabbabb	
L_f	aabbbab	$x =$ aabbab

Conclusion:

- Compare x and L_i from position lf.

Improved Search Algorithm

```
fun SEARCH(L, n, x, m, lcp):
    d = −1, ld = 0
    f = n, lf = 0

    while (d + 1 < f) do:
        i = floor((d + f) / 2)

        if (ld <= lcp(i, f) < lf):        // case 1
            d = i
            ld = lcp(i, f)

        else if (ld <= lf < lcp(i, f)):   // case 2
            f = i

        else if (lf <= lcp(d, i) < ld):   // case 1 (symmetry)
            f = i
            lf = lcp(d, i)

        else if (lf < ld < lcp(d, i)):    // case 2 (symmetry)
            d = i

        else:                             // case 3
            li = max(ld, lf)
            li = li + LCP(x[li...], L_i[li...]).length

            if (li == L_i.length and li == m):
                return i

            else if (li == L_i.length):
                d = i
                ld = li

            else if (li != m and L_i[li] < x[li]):
                d = i
                ld = li

            else:
                f = i
                lf = li

    return (d, f)
```

The complexity of this algorithm is $O(m + log_2(n))$, as explained:

- Each positive comparison increases l, which can happen no more than m times.

- Each negative comparison halves the value of $f - d$, giving no more than $\lceil log_2(n+2) \rceil$ comparisons.
- After preprocessing, `LCP` can run in constant time.

Improved Interval Algorithm

TODO: Improved Interval Algorithm

Preprocessing the List

Notation: $||L|| = \sum_{i=0}^{n-1} |L_i|$ (i.e. the total length of all strings in the list).

- The list can be sorted in time $O(||L||)$ using repetitive bucket sorting.
- Computing LCPs for L_i and L_{i-1} for $0 \leq i \leq n$ can be done in time $O(||L||)$.
- Computing LCPs for other nodes inside the tree can be done in constant time:
  - Let $L_0 \leq L_1 \leq L_2 \leq ... \leq L_n$.
  - Let $-1 < d < i < f < n$.
  - $|lcp(L_d, L_f)| = min\{|lcp(L_d, L_i)|, |lcp(L_i, L_d)|\}$.

Therefore, the entire preprocessing stage can be done in time $O(||L||)$.

Regular Expressions

Regular expressions are used for **matching strings** or **substrings**. A regular expression is simply a string that is augmented with special characters that have specific meanings:

- (and) - used for grouping and/or nesting expressions.
- $*$ (Kleene star) - indicates zero or more repetitions of the preceding expression.
- $\cup$ (union) - indicates a choice.
- Two expressions side-by-side implies the concatenation of those expressions.

For example, the expression $1*011*(0 \cup 1)*111$ will match any string with the pattern 'any number of 1s, then 01, then any number of 1s, then any number of 1s and 0, then 111'.

Regular Expressions and Automata

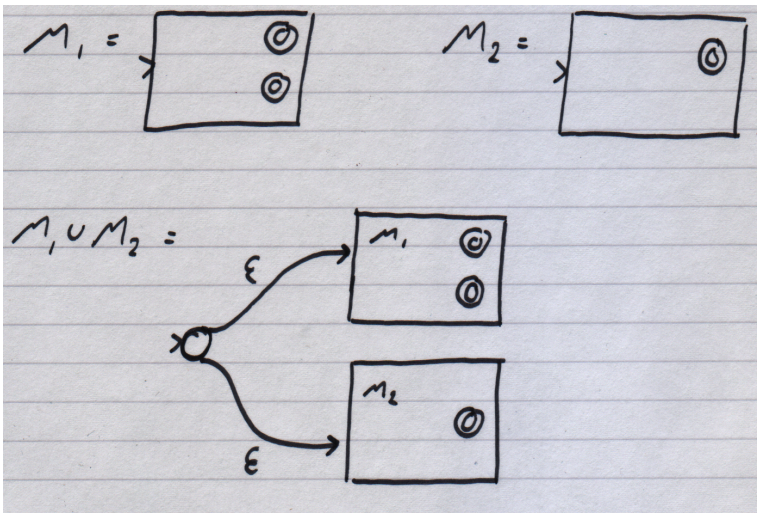
Regular expressions and automata are **equally powerful** and any one can be converted into the other (although not always easily). This is proved in the following sections, which show that every function of regular expressions can be implemented with automata.

Union of Two Automata

If M is the union of two automata, then M should accept any string that would be accepted by either of the two original automata.

Example: an automaton that accepts x when x has a substring `ab` or has a substring `bbb`.

- Create a new initial state.
- Create arcs from the new initial state to the previous initial states, labelled with ϵ.

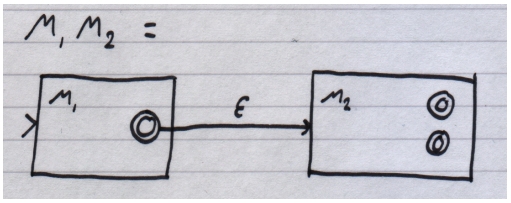


Concatenation of Two Automata

If M is the concatenation of two automata, then M should accept any string that can be broken into two strings, each of which satisfies the first and second original automata respectively.

Example: an automaton that accepts x when x is made up of a string that has `ab` as a substring, followed by another string that has `bbb` as a substring.

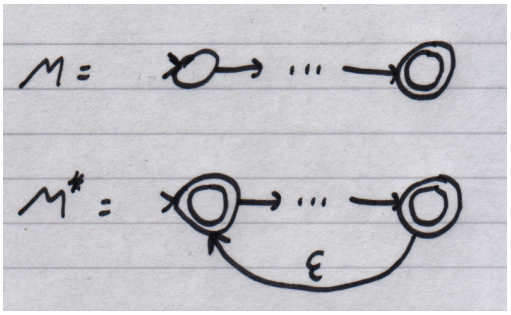
- Join the final state(s) of the first automaton to the initial state of the second automaton with an arc labelled by ϵ.



Kleene Star of an Automaton

If M is the Kleene star of an automaton, then M should accept any number of repetitions of a string that would be accepted by the original automaton (including zero repetitions).

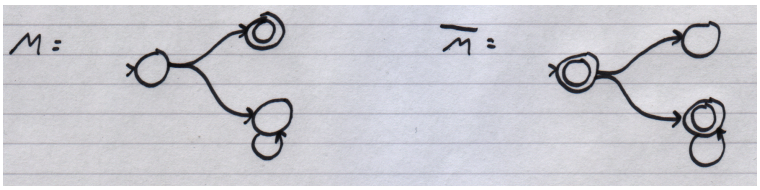
- Join the final state(s) of the automaton back to the initial state with arcs labelled by ϵ.
- Convert the initial state into a final/accepting state (to allow for zero repetitions).



Compliment of an Automata

If M is the compliment an automaton, then M should accept anything that would be rejected by the original automaton.

- Swap the type of all states, so that final states become non-final and vice versa.



Intersection of Two Automata

If M is the intersection an automaton, then M should accept anything that would be accepted by both of the original automata.

This can be derived from previous cases:

- Let M_1 and M_2 be the two original automata.
- Let $L(n)$ be the language accepted by some automaton, n.
- $L(M_1) \cap L(M_2) = \overline{\overline{L(M_1)} \cup \overline{L(M_2)}}$

Example Automata Questions

Given a finite automaton M and a string w, check $w \in L(M)$.

Solution: trace execution of the automaton, letter by letter.

Given a finite automaton M, check $L(M) = \varnothing$.

Solution: check for a path from the initial state to any final state. If such a path exists, a string exists in the language that matches it; if there is no path, $L(M)$ is empty.

Given a finite automaton M, check $L(M) = \sigma^*$.

i.e. does it accept every possible string in the language?

Solution: yes, if $\overline{L(M)} = \varnothing$.

Given two finite automata M_1 and M_2, check $L(M_1) \subseteq L(M_2)$.

Solution: yes, if $L(M_1) \cap \overline{L(M_2)} = \varnothing$.

Given two finite automata M_1 and M_2, check $L(M_1) = L(M_2)$.

Solution: yes, if $L(M_1) \subseteq L(M_2)$ and $L(M_2) \subseteq L(M_2)$.

Checking if a Language is Regular

A language regular if there exists an **automaton/regular expression that accepts it**.

The Pumping Lemma

If L is an **infinite**, regular language, then there exists strings x, y and z such that $y \neq \epsilon$ and $xy^nz \in L$ for any n.

If this lemma is violated then L cannot be an infinite regular language. Note that this only works one way: satisfying this lemma does not guarantee that L is infinite and regular.

Example 1: Proof by Contradiction

$L = \{0^i1^i \mid i > 0\}$

We assume towards contradiction that the language L is an infinite regular language, and therefore strings x, y and z should exist such that $y \neq \epsilon$ and $xy^nz \in L$.

- y could be a number of 0s.
  - This won't work, because xyz and xy^2z will have different numbers of 0s.
- y could be a number of 1s.
  - This won't work, because xyz and xy^2z will have different numbers of 1s.
- y could be a mixture of 0s and 1s.
  - This won't work, because any $n > 1$ is guaranteed to break the pattern of the language.

The pumping lemma cannot be satisfied, so therefore the assumption cannot hold.

Example 2: Proof by Contradiction

$L = \{a^iaba^i \mid i > 0\}$

We assume towards contradiction that the language L is an infinite regular language, and therefore strings x, y and z should exist such that $y \neq \epsilon$ and $xy^nz \in L$.

- y could be a^n.
  - This won't work, because different values of n will create different, non-equal quantities of a on either side.
- y could be b.
  - This won't work, because there must be only one b.
- y could be a mixture of as and bs.
  - This won't work, because there must be only one b.

The pumping lemma cannot be satisfied, so therefore the assumption cannot hold.

Example 3: Proof by Restriction and Contradiction

$L = \{ww \mid w \in \{a, b\}^*\}$

The pumping lemma could be satisfied now: $x = a$, $y = aa$, $z = a$.

We can still prove that L is not an infinite regular language by using a known regular language to **create a restriction**, then using proof by contradiction on that. This works because **the intersection of two languages will only be regular if both of the original languages were**.

$P = L \cap \{a^i b a^k b \mid i > 0\} = \{a^m b a^m b \mid m > 0\}$

We assume towards contradiction that the language P is an infinite regular language, and therefore strings x, y and z should exist such that $y \neq \epsilon$ and $xy^n z \in P$.

- y could be a number of as.
  - This won't work, because varying values of n will change the number of as on one side.
- y could be a number of bs.
  - This won't work, because there must be only one b on each side.
- y could be a mixture of as and bs.
  - This won't work, because there must be only one b on each side.

The pumping lemma cannot be satisfied for P, so therefore the assumption cannot hold (and therefore both P and L are not infinite regular languages).

Example 4: Proof by Restriction and Contradiction

$L = \{w\overline{w} \mid w \in \{a, b\}^*\}$

$P = L \cap \{a^k b b^l a \mid i > 0\} = \{a^m b b^m a \mid m > 0\}$

We assume towards contradiction that the language P is an infinite regular language, and therefore strings x, y and z should exist such that $y \neq \epsilon$ and $xy^n z \in P$.

- y could be a number of as.
  - This won't work, because varying values of n will change the number of as on one side.
- y could be a number of bs.
  - This won't work, because varying values of n will change the number of bs on one side.
- y could be a mixture of as and bs.
  - This won't work, because only a few carefully chosen values of n will preserve the correct format.

The pumping lemma cannot be satisfied for P, so therefore the assumption cannot hold (and therefore both P and L are not infinite regular languages).

Example 5: Proof by Restriction and Contradiction

$L = \{ww^R \mid w \in \{a, b\}^*\}$

Note: w^R is the reverse of w.

$P = L \cap \{a^k bba^l \mid i > 0\} = \{a^m bba^m \mid m > 0\}$

We assume towards contradiction that the language P is an infinite regular language, and therefore strings x, y and z should exist such that $y \neq \epsilon$ and $xy^nz \in P$.

- y could be a number of as.
  - This won't work, because varying values of n will change the number of as on one side.
- y could be b.
  - This won't work, because there must be only one b in each w.
- y could be a mixture of as and bs.
  - This won't work, because there must be only one b in each w.

The pumping lemma cannot be satisfied for P, so therefore the assumption cannot hold (and therefore both P and L are not infinite regular languages).

Example 6: Proof by Restriction and Contradiction

$L = \{w\overline{w}w \mid w \in \{a, b\}^*\}$

$P = L \cap \{a^j bba^k a^l b \mid i > 0\} = \{a^m bba^m a^m b \mid m > 0\}$

We assume towards contradiction that the language P is an infinite regular language, and therefore strings x, y and z should exist such that $y \neq \epsilon$ and $xy^nz \in P$.

- y could be a number of as.
  - This won't work, because varying values of n will change the number of as in one section.
- y could be b.
  - This won't work, because there must be only one b in each w.
- y could be a mixture of as and bs.
  - This won't work, because there must be only one b in each w.

The pumping lemma cannot be satisfied for P, so therefore the assumption cannot hold (and therefore both P and L are not infinite regular languages).

Example 7: Proof by Restriction and Contradiction

$L = \{x \in \{0,1\}^* \mid \text{equal number of 1s and 0s}\}$

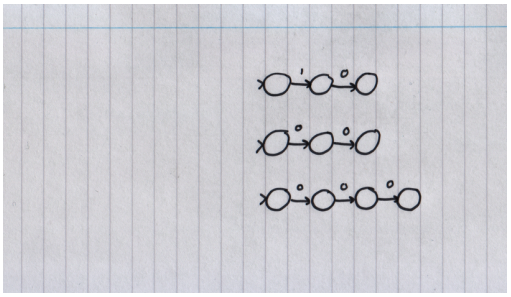
$P = L \cap \{0^j 1^k\} = \{0^m 1^m \mid m > 0\}$ which was already proved to be non-regular.

Converting Regular Expressions to Automata

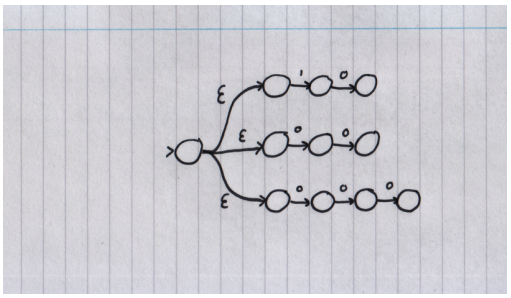
Regular expressions can be converted to automata by drawing the trivial automata for the most deeply-nested concatenated strings, then working outwards to combine automata as described earlier (*see more: Regular Expressions and Automata, page 29*).

For example, consider the construction of the automata for $101(10 \cup 00 \cup 000) * 111$.

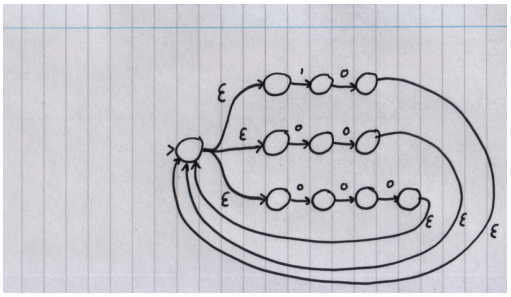
The three primitive automata for 10, 00 and 000 are drawn:



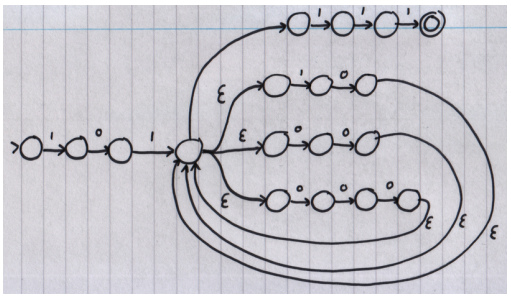
They are combined to create $10 \cup 00 \cup 000$:



They are wrapped to create $(10 \cup 00 \cup 000)*$:



101 and 111 are added as a prefix and suffix respectively:



During this process, it is valid to **compress** multiple empty ϵ transitions.

Automata to Regular Expressions

To convert automata to regular expressions, we must find **all strings that drive the automaton from the initial state to accepting state(s)**.

We use $R(i, j, k)$ to denote the set of all strings that can drive the automaton in question from the state q_i to the state q_j without **passing through** any state q_x where $x \geq k$.

For example, $R(2, 4, 2)$ is the set of all strings that can drive the automaton from q_2 to q_4, only passing through q_1.

The **base step** of computation must be done manually, to determine $R(i, j, 1)$ for all pairs of states, q_i and q_j. The equation for $k = 1$ is as follows:

$$R(i,j,1) = \begin{cases} \{\sigma \in \Sigma : \delta(q_i, \sigma) = q_j\} & i \neq j \\ \{\epsilon\} \cup \{\sigma \in \Sigma : \delta(q_i, \sigma) = q_j\} & i = j \end{cases}$$

The first term states that $R(i, j, 1)$ is the set of characters that will drive the automaton from q_i to q_j **directly** when $i \neq j$. The second term is similar but adds the empty string when $i = j$ (i.e. you can apply no characters and stay in the state).

The **inductive step** then allows the final output for $R(q_{initial}, q_{final}, n + 1)$ (where n is the number of states) to be built with the following rule:

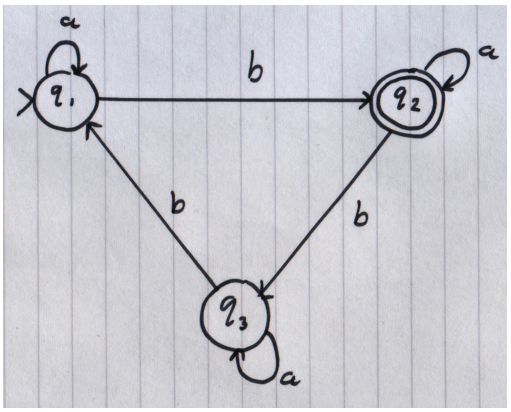
$$R(i,j,k+1) = \quad R(i,j,k) \quad \cup \quad R(i,k,k)R(k,k,k)^*R(k,j,k)$$

This states that to drive the automaton from q_i to q_j without passing through a state q_x where $x \geq k + 1$, there are two options:

- Go from q_i to q_j without passing through any state q_x where $x \geq k$.

- Go...
  - ...from q_i to q_k,
  - then from q_k to q_k zero or more times,
  - then from q_k to q_j,
  - all without passing through any state q_x where $x \geq k$.

For example:



$R(1,1,1) = a \cup \epsilon$ $R(1,2,1) = b$ $R(1,3,1) = \varnothing$

$R(2,1,1) = \varnothing$ $R(2,2,1) = a \cup \epsilon$ $R(2,3,1) = b$

$R(3,1,1) = b$ $R(3,2,1) = \varnothing$ $R(3,3,1) = a \cup \epsilon$

We want to get from q_1 to q_2, so...

$$\begin{aligned} R(1,2,2) &= R(1,2,1) \cup R(1,1,1)R(1,1,1)^*R(1,2,1) \\ &= b \cup (a \cup \epsilon)(a \cup \epsilon)^*b \end{aligned}$$

The final result, $R(1,2,4)$, would take several more rounds of computation.

Tries

Text Indexing Problem

What if the text doesn't change, but we want to find all occurrences of the pattern? This problem is called the **text indexing problem**, and can be solved efficiently with tries, suffix trees, suffix automata, and suffix arrays. All of these are **structures built around the text** with **online processing** techniques.

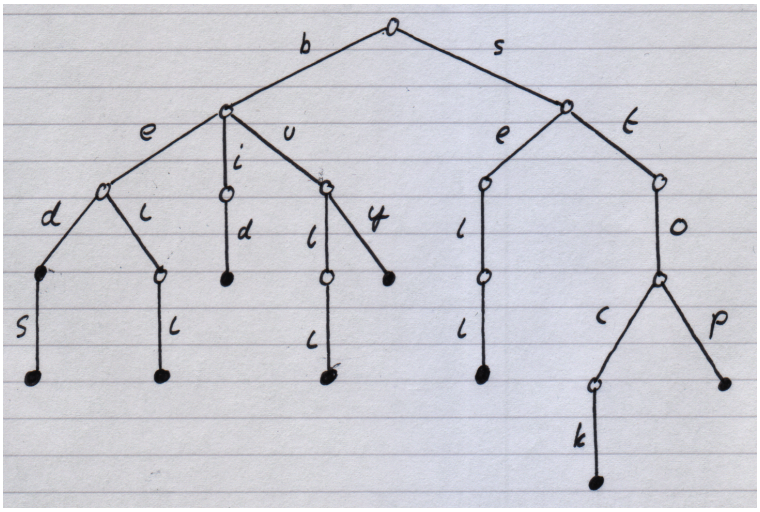
Trie Definition

A trie (pronounced 'try') is a simple structure for storing a **compact representation** of a **set of strings**. It is a rooted tree with the following properties:

- Edges are labelled with letters from the alphabet, Σ.
- At each node, all outgoing edges are uniquely labelled.
- **Terminating nodes** are marked, and represent the end of a string formed by concatenating the letters on the path from the root to that node.

The trie that represents all of the strings in the set R is denoted $trie(R)$. For example:

$$R = \{\texttt{bed}, \texttt{beds}, \texttt{bell}, \texttt{bid}, \texttt{buy}, \texttt{bull}, \texttt{sell}, \texttt{stock}, \texttt{stop}\}$$



Using a trie to check whether a string exists in the set of strings is easy: start at the root, and follow the edges letter by letter.

- If you finish on a terminating node, the string is in the set.
- If you finish on a non-terminating node, the string is a prefix of at least one word in the set.

- If you ‘fall out of the trie’, the string is not in the set and is not a prefix.

Suffix Trees

Suffix trees provide fast exact pattern matching on a text T.

Notation

- $T = T[0..n-1]$ is the text (a string with zero-indexed letters).
- For any single integer $i \in [0..n]$, we define T_i as the suffix $T[i..n-1]$.
  - T_i is the string with the first i letters removed.
  - T_0 is the entire string.
  - T_{n-1} is the last letter.
  - T_n is the empty string ϵ.
- For any set of integers $c \subseteq [0..n]$, we define $T_C = \{T_i | i \in c\}$.
  - $T_{\{0\}}$ is the set containing the entire string.
  - $T_{\{0,1,2\}}$ is the set of the entire string, plus the suffix with the first letter missing, and the suffix with the first two letters missing.
  - $T_{[0..n]}$ is the **set of all suffixes**, including the empty string ϵ.

Properties

A suffix trie is a **compact trie**[1] for the set $T_{[0..n]}$ (i.e. **the set of all suffixes**).

We assume that an extra letter is added to the end of the text. This letter **cannot appear in the alphabet**. `$` is typically used, such that $T[n] =$ `$` and the empty string no longer appears in $T_{[0..n]}$ (it is replaced by the string `$`). This has two consequences:

- No suffix is the prefix of another suffix.
- All nodes in the trie representing suffixes are leaves.

For example:

$T =$ `banana`
$T_{[0..n]} = \{$`banana$`, `anana$`, `nana$`, `ana$`, `na$`, `a$`, `$`$\}$

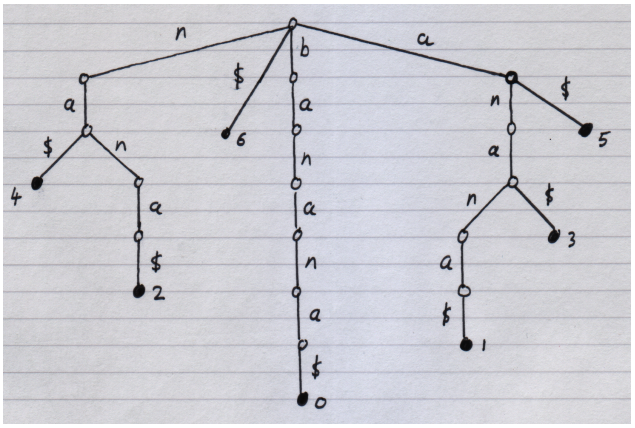
The set $T_{[0..n]}$ contains $|T_{[0..n]}| = n + 1$ strings with a total length of $||T_{[0..n]}|| = \Theta(n^2)$.

[1] Path compression converts non-branching path segments into single edges. In the example on page 38, the path for `sell` could be compressed into the edges {`s`, `ell`}.

Suffix Tries

A basic suffix trie uses $O(n^2)$ nodes to store the suffixes for most texts and the brute force approach for its construction takes $O(n^2)$ time.

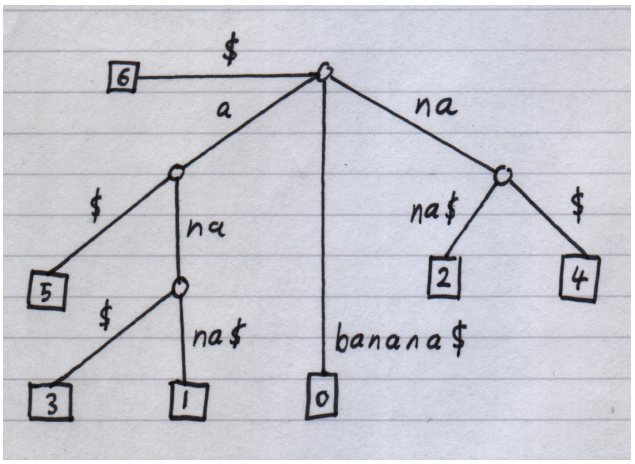
The suffix trie is constructed by adding suffixes one at a time to a standard trie, starting with the largest.



Suffix Trees

Compressing the **non-branching edges** of a suffix trie produces a suffix tree, which will use only $O(n)$ space:

- We assume that the size of the alphabet is a constant.
- There will be exactly $n + 1$ leaves and at most n internal nodes.
- There will be at most $2n$ edges (basic tree properties).
- Edge labels are substrings of the text and can be represented by two integers (starting position and end position or substring length).

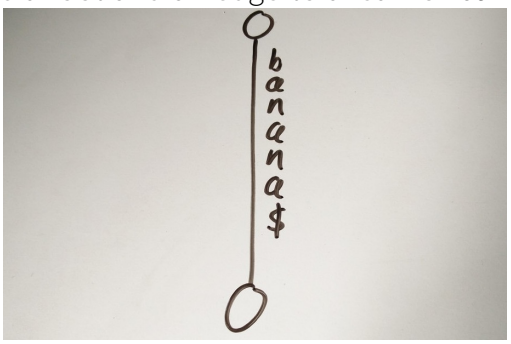


This type of suffix tree can be constructed systematically, without creating the $O(n^2)$ version first:

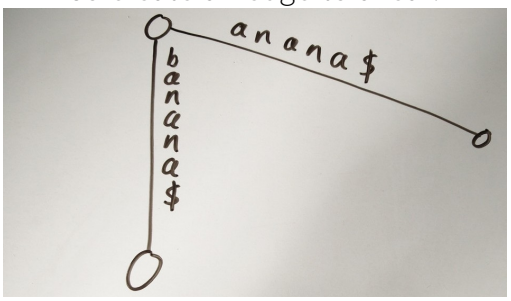
- Create a root node.
- Start with the longest suffix (the entire word) and create one edge from the root to represent it.
- For each remaining suffix, longest first:
  - Use the existing edges to start spelling the suffix.
  - **Create a new edge to a leaf** if no edge exists to keep spelling the suffix. Use the rest of the suffix to label this edge.
  - **Split an existing edge** if the suffix only matches part of it, then create a new edge to a leaf as before.

Construction for `banana$` is shown below (note: the edges are labelled with substrings for clarity, instead of start/end indexes).

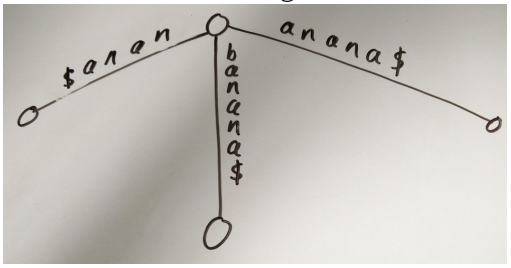
Create a root and an edge to a leaf for `banana$`:



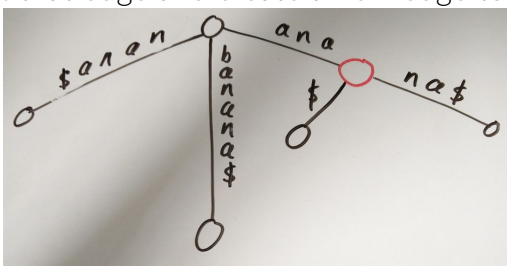
There's no path from the root for `anana$`, so create an edge to a leaf:



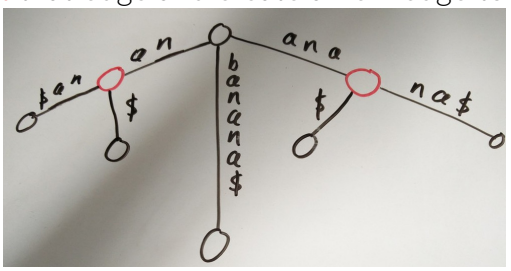
There's no path from the root for `nana$`, so create an edge to a leaf:



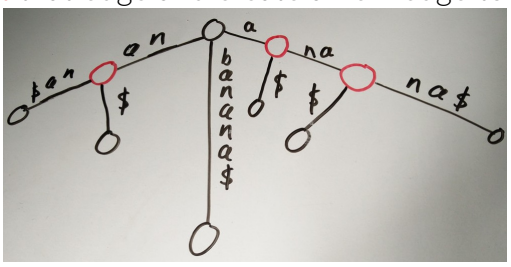
The path for `ana$` matches the start of `anana$`,
so split that edge and create a new edge to a leaf:



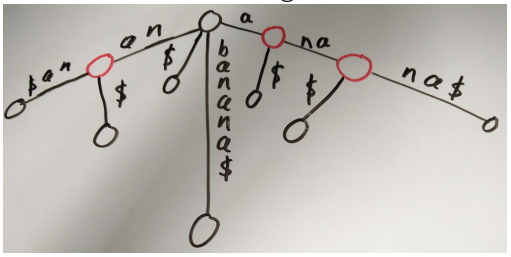
The path for `na$` matches the start of `nana$`,
so split that edge and create a new edge to a leaf:



The path for `a$` matches the start of `ana`,
so split that edge and create a new edge to a leaf:



There's no path from the root for `$`,
so create an edge to a leaf:



String Matching

Given a suffix trie/tree for T, all occurrences of a pattern P can be found in $O(|P| + occ)$ where occ is the number of occurrences. Searching is based on the following observation:

A pattern P has an occurrence in T at position i if and only if P is a prefix of T_i. Example:

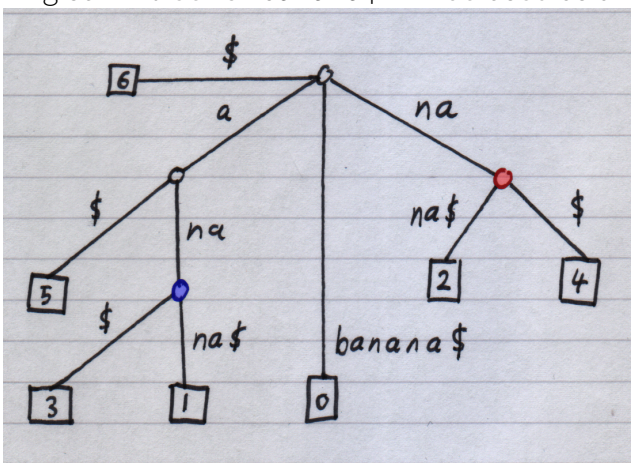
$T =$ `horizon`
$P =$ `riz`
$T_2 =$ `rizon`
P is a prefix of T_2, therefore P occurs in T starting at $i = 2$.

Therefore, we can find all occurrences of P in T by a **prefix search** in $T_{[0..n]}$:

- From the root, follow edges that spell out P.
- If you 'fall out' of the suffix tree, P does not occur in T.
- If you finish inside the tree...
  - The number of leaves in the sub-tree below the finishing point gives the number of occurrences.
  - The suffix numbers of all leaves below the finishing point give the starting positions of all occurrences of P in T.
  - If there is an outgoing edge with the label `$`, P is a suffix of T.

Suffix Links

The following suffix tree for `banana$` will be used as an example:



There are three types of node in a suffix tree: the **root** node, **internal** nodes, and **leaf** nodes. There are two important properties of these node types:

- Internal nodes always have more than one outgoing edge, which means they mark the points where **branching** occurs.
- Branching occurs whenever a **repeated string** is involved, and only then. For any internal node u, the string leading from the root to u must appear in the text at least as many times as u's branching factor (its number of outgoing edges).
  - For example, the blue node in the example spells `ana` from the root. It has a branching factor of 2, so therefore `ana` must appear in the text `banana$` at least twice.

These properties lead to the definition of suffix links:

- If the string S_u leading from the root to u is longer than 1 character, the same string minus it's first character (call it S_v) must be in the tree too.
  - $S_u =$ `ana`, so $S_v =$ `na`. We can see that `na` appears in the tree.
- If some string S_u leads to an internal node u, it's shorter version S_v must also lead to an internal node. Why? S_u must appear at least twice in the text (because u is a branching node), so S_v must also appear at least twice (because it's a part of S_u; wherever S_u appears, S_v also appears). Because S_v appears more than once, it must have an internal node.
  - We can see that an internal node (in red) for $S_v =$ `na` does appear in the tree.
- In this situation, a suffix link connects u to v.

Suffix Tree Construction with Suffix Links

Suffix links are the key to **efficient suffix tree construction**. They can be used as 'shortcuts' during construction so that each iteration does not necessarily have to start again from the root.

Manual, informal construction of a suffix tree using suffix links can be done as follows:

- Create a root.
- Insert each suffix, longest first, with the following logic:
  - Determine the starting position.
    * If a suffix link was set for the previous suffix, start there.
    * Otherwise, start from the root.
  - Let d be the depth of your starting position (i.e. the length of the string from root to your start).
  - Skip d letters from the start of the suffix you are inserting.
  - Insert the rest of your suffix as before, working from your starting position.
    * This may involve creating a new branching point.
  - After the suffix is inserted, go back to the branching point.
  - Determine the suffix link that would be created from the branching point.
    * Use the S_u and S_v definition from earlier.
  - If the suffix link goes to somewhere other than the root, add it to the suffix tree.
  - Rinse and repeat for the next suffix.

Algorithmic Approach

Algorithms exist that take advantage of suffix links for efficient suffix tree construction, including **McCreight's** algorithm and **Ukkonen's** algorithm, both of which run in $O(n)$.

Suffix Arrays

Suffix arrays provide **exact pattern matching over a given text**, as do suffix trees. Suffix arrays are more **space-efficient**: although both structures require linear space, a suffix array usually requires $\approx O(4n)$, whereas a suffix tree usually requires at least $\approx O(20n)$.

Definition: Lexicographic Order

$s \leq t$ if s is a prefix of t, or $s[0] = t[0], s[1] = t[1], ..., s[i] = t[i], s[i+1] < t[i+1]$.

`gold` is lexicographically smaller than `goldfish`, which is lexicographically smaller than `zebra`.

Note: throughout this section, terms like 'ordered' and 'sorted' mean 'lexicographically ordered', unless explicitly stated otherwise.

Suffix Array Structure

With a properly-constructed suffix tree, the leaves from left to right represent the **suffixes of the text in lexicographic order** (assuming edges are drawn in the same order). The aim of a suffix array is to construct this ordering of suffixes **without constructing the tree**.

A suffix array is simply **an ordered array of all suffixes**:

$W =$	`CATTATTAGGA`
$SA[0] = 10$	$=$ `A`
$SA[1] = 7$	$=$ `AGGA`
$SA[2] = 4$	$=$ `ATTAGGA`
$SA[3] = 1$	$=$ `ATTAATTAGGA`
$SA[4] = 0$	$=$ `CATTAATTAGGA`
$SA[5] = 9$	$=$ `GA`
$SA[6] = 8$	$=$ `GGA`
$SA[7] = 6$	$=$ `TAGGA`
$SA[8] = 3$	$=$ `TATTAGGA`
$SA[9] = 5$	$=$ `TTAGGA`
$SA[10] = 2$	$=$ `TTATTAGGA`

Note that only the number would be stored, the suffixes are shown above just for clarity. Assuming 32-bit integers, storing the suffix array only requires $4n$ bytes in memory.

Finding Pattern Occurrences (Naive)

All occurrences of x as a prefix (i.e. an occurrence in the text) must appear in a **contiguous fragment**, because the suffix array is ordered. For example, if $x =$ `AT`:

$$
\begin{aligned}
W &= \texttt{CATTATTAGGA} \\
SA[0] = 10 &= \texttt{A} \\
SA[1] = 7 &= \texttt{AGGA} \\
SA[2] = 4 &= \texttt{ATTAGGA} \\
SA[3] = 1 &= \texttt{ATTAATTAGGA} \\
SA[4] = 0 &= \texttt{CATTAATTAGGA} \\
SA[5] = 9 &= \texttt{GA} \\
SA[6] = 8 &= \texttt{GGA} \\
SA[7] = 6 &= \texttt{TAGGA} \\
SA[8] = 3 &= \texttt{TATTAGGA} \\
SA[9] = 5 &= \texttt{TTAGGA} \\
SA[10] = 2 &= \texttt{TTATTAGGA}
\end{aligned}
$$

With this knowledge, it's clear that occurrences could be found with a simple **binary search** over the suffix array. This would take $O(log_2(n))$ iterations, each with an $O(m)$ comparison, so $O(m \cdot log_2(n))$ in total.

Suffix Array Construction in $O(n^2 \cdot log_2(n))$

A brute-force approach of creating all substrings and sorting them would take $O(n^2 \cdot log_2(n))$ using an $O(n \cdot log_2(n))$ sorting algorithm (merge sort, quick sort, etc.) on substrings that take $O(n)$ time to compare.

Improving Construction Time

We will show, step by step, how this run time can be reduced to $O(n^2)$, then $O(n \cdot log_2(n))$, then finally $O(n)$. Two key building blocks are required first:

Building Block: Radix Sort

With radix sort, which is based on **bucket sort**, a set of n numbers in the range $[1..k]$ can be sorted in $O(n + k)$ time.

A sequence of n **pairs** from $[1..k] \times [1..k]$ (i.e. the Cartesian product of the ranges) can also be sorted in $O(n + k)$ time by sorting for the first number in the pair, then sorting again by the second number (within each block where the first number is the same).

Building Block: Merging

Two sorted sequences of sizes n and m can be merged into a single sorted sequence in time $O(n + m)$.

Alphabet Size

Previously, we have considered a constant-sized alphabet. We now consider an alphabet where each letter in W is an integer in the range $[1...|W|]$, so the alphabet size is now $O(|W|)$, instead of a constant. The objective is to find a construction algorithm that runs in $O(n)$, where $n = |W| = |\Sigma|$.

Plain English: the algorithm run-time should not assume a fixed-sized alphabet, so **every letter in the text may be unique**, regardless of the text's length.

Suffix Array Construction in $O(n^2)$

For each $i = n-1, n-2, ..., 0$, construct a sorted list L_i containing all $W[i..n-1], W[i+1..n-1], ..., W[n-1, n-1]$. In English: start with a list containing the smallest suffix, the add the next-longest suffix until you have all of them.

From any L_{i+1} we can construct L_i (i.e. the list with the next-longest suffix) in linear time:

For each $j = i, i+1, ..., n-1$, we construct a pair $(W[j], nr_{i+1}(j+1))$, where $nr_i(j)$ is the position of $W[j..n-1]$ on the list L_i. These pairs are then sorted with radix sort, giving the order for L_i.

For example, assume $W = \texttt{CATTATTAGGA}$ and we already have $L_{n-3} = \langle \texttt{A}, \texttt{GA}, \texttt{GGA} \rangle$. We need to construct L_{n-4}.

For each $j = n-4, n-3, n-2, n-1$, we construct the pairs:

$$
\begin{array}{lllllll}
j = n-4: & (W[n-4], nr_{n-3}(n-3)) & \mapsto & (\texttt{A}, nr_{i+1}(\texttt{GGA})) & \mapsto & (\texttt{A}, 3) \\
j = n-3: & (W[n-3], nr_{n-3}(n-2)) & \mapsto & (\texttt{G}, nr_{i+1}(\texttt{GA})) & \mapsto & (\texttt{G}, 2) \\
j = n-2: & (W[n-2], nr_{n-3}(n-1)) & \mapsto & (\texttt{G}, nr_{i+1}(\texttt{A})) & \mapsto & (\texttt{G}, 1) \\
j = n-1: & (W[n-1], nr_{n-3}(n)) & \mapsto & (\texttt{A}, nr_{i+1}(\epsilon)) & \mapsto & (\texttt{A}, 0)
\end{array}
$$

The resulting pairs are then sorted with radix sort, which gives the following result:

$$
\begin{array}{lll}
(\texttt{A}, 0) & \mapsto & \texttt{A} \\
(\texttt{A}, 3) & \mapsto & \texttt{AGGA} \\
(\texttt{G}, 1) & \mapsto & \texttt{GA} \\
(\texttt{G}, 2) & \mapsto & \texttt{GGA}
\end{array}
$$

This process is repeated until L_0, which will contain all suffixes in sorted order.

Each list construction (creating pairs and sorting them with radix sort) takes $O(n)$ and $\Theta(n)$ lists must be constructed, giving an overall run-time of $O(n^2)$.

Suffix Array Construction in $O(n \cdot log_2(n))$

The $O(n^2)$ algorithm meant that a lot of chunks were **sorted repeatedly**, which wastes time. This algorithm avoids that by sorting all prefixes at once, with respect to their first letter, then

first 2 letters, then first 4 letters, etc.

Note: we need to conceptually **append n null characters** to the end of the text, making W a string of length $2n$. This avoids 'reading past the end' of the string in the algorithm, but note also that null characters will not actually be shown in examples. Null characters sort before all other characters.

For each $i = 0, 1, ..., log_2(n)$, construct a sorted list L_i containing all $W[0..0+2^i-1], W[1..1+2^i-1], ..., W[n-1..n-1+2^i-1]$. In English: for each $i = 0, 1, ..., log_2(n)$, the list L_i will contain the first 2^i characters of each suffix (i.e. 1 character, then 2, then 4, etc.), in sorted order.

As before, from any L_i we can construct L_{i+1} (i.e. the list with suffix prefixes that are twice as long) in linear time: for each $j = 0, 1, ..., n-1$, we construct a pair $(nr_i(j), nr_i(j+2^i))$, where $nr_i(j)$ is the position of $W[j..j+2^i-1]$ on the list L_i.

For example, to construct the list containing suffixes of length 8 (i.e. to build $L_{i+1} = L_3$ from L_2), we make a pair for each position j consisting of the ranks of the 4-character prefixes starting at j and $j+4$. These two chunks will make the 8-character prefix starting at j, and will have already been sorted in the previous step.

This approach works, because in each step the **pairs are made up of two previously-sorted components**, and so the result will also be sorted.

This construction has $O(log_2(n))$ stages, because the i_{th} stage **sorts all suffixes with respect to their first 2^i letters**. Each stage takes $O(n)$ for the pair construction and radix sort, so the total run-time is $O(n \cdot log_2(n))$.

Suffix Array Construction in $O(n)$

This is a recursive algorithm. The aim is to design an approach with $T(n) = T(\alpha n) + O(n)$, where $\alpha < 1$, so that the recursion resolves to $T(n) = O(n)$.

The algorithm depends on a partitioning of suffixes and **three key steps**, each of which must be understood separately.

Suffix Partitioning

We partition all suffixes into three groups:

- $S_0 = \{W[0..n-1], W[3..n-1], W[6..n-1], ...\}$
- $S_1 = \{W[1..n-1], W[4..n-1], W[7..n-1], ...\}$
- $S_2 = \{W[2..n-1], W[5..n-1], W[8..n-1], ...\}$

The suffixes in the group S_r start at positions in the form $3k + r$.

Key Step 1

How could we sort just one group?

We can **split W into blocks of length** 3, treat each block as a single letter, then recursively solve for a smaller problem of size $n/3$.

We need to be careful though: the word W contains only letters from $[1..|W|]$, and we are creating triples of letters. We can apply a **renaming trick**: create a sorted list of the triples, then rename them with their rank in that list.

This can be done in $O(n)$ using radix sort.

Key Step 2

Assuming we have sorted $S_1 \cup S_2$, how can we sort $S_0 \cup S_1 \cup S_2$ in $O(n)$?

The trick is to represent each suffix as a pair:

- $W[3k+0...n-1]$ becomes $(W[3k+0], W[3k+1..n-1])$
- $W[3k+1...n-1]$ becomes $(W[3k+1], W[3k+2..n-1])$
- $W[3k+2...n-1]$ becomes $(W[3k+2], W[3k+3..n-1])$
- And so on...

The order on pairs is the same as the order on suffixes.

We can quickly radix sort S_0 by replacing each suffix with the corresponding pair, the second item of which can be replaced with it's rank in the already sorted $S_1 \cup S_2$ (the substring will be in there). This means we're just sorting pairs of integers.

We then just merge the two sorted sequences, S_0 and $S_1 \cup S_2$, which have lengths of $n/3$ and $2n/3$. This can be done in $O(n)$, assuming we can make comparisons of any two elements in constant time.

How do we compare any $W[3i..n-1] \in S_0$ with any $W[j..n-1] \in S_1 \cup S_2$ in constant time?

- If $j = 3k+1$
  - Represent $W[3i..n-1]$ as the pair $(W[3i], W[3i+1..n-1])$
  - Represent $W[3k+1..n-1]$ as the pair $(W[3k+1], W[3k+2..n-1])$
  - The first elements in both pairs are single characters.
  - The second elements in both pairs appear in $S_1 \cup S_2$, so we can just compare the ranks.
- If $j = 3k+2$
  - Represent $W[3i..n-1]$ as the triple $(W[3i], W[3i+1], W[3i+2..n-1])$

- Represent $W[3k+2..n-1]$ as the triple $(W[3k+2], W[3k+3], W[3k+4..n-1])$
- The first and second elements in both triples are single characters.
- The third elements in both triples appear in $S_1 \cup S_2$, so we can just compare the ranks.

Therefore, we can compare any $W[3i..n-1] \in S_0$ with any $W[j..n-1] \in S_1 \cup S_2$ in constant time.

Key Step 3

How do we sort $S_1 \cup S_2$ so that it can be used for key step 2?

The order on all suffixes of $S_1 \cup S_2$ can be computed by sorting all suffixes of:

$$W' = nr(1)nr(4)nr(7)...nr(2)nr(5)nr(8)...$$

This is because every suffix of W' corresponds to some suffix of W (maybe with some extra letters at the end).

Combining Steps

Let $W =$ `MISSISSIPPI$$`.

- From key point 3, we construct $W' =$ `ISS|ISS|IPP|I$$|SSI|SSI|PPI`.
- We sort $S_1 \cup S_2$ by sorting W' using key point 1.
- We sort $S_0 \cup S_1 \cup S_2$ using key point 2.

Therefore, we get an algorithm with running time in the form $T(n) = T(2n/3) + O(n)$ which resolves to $O(n)$.

- $T(2n/3)$ for the recursive calls to sort $S_1 \cup S_2$.
- $O(n)$ to sort S_0.
- $O(n)$ to merge S_0 with $S_1 \cup S_2$.

Augmenting the Suffix Array

Two additional arrays are also used to execute efficient algorithms over the suffix array, and both can be computed in linear time.

Inverse Suffix Array

- The suffix array answers 'where does the i^{th} ordered suffix start in the text?'
- The inverse suffix array answers 'what is the ordered rank of the suffix starting at i?'

The inverse suffix array is denoted as SA^{-1}, and $SA^{-1}[i]$ is the rank of $W[i..n-1]$ in SA. Notice that $SA[SA^{-1}[i]] = i$.

The inverse suffix array can be computed in a trivial linear scan.

Longest Common Prefix (LCP) Array

The $lcp(i, j)$ function is defined as the length of the longest prefix common to $W[i..n-1]$ and $W[j..n-1]$. It is also the length of the labels on the shared branches in the suffix tree leading to the two suffixes at i and j.

The LCP array is defined as follows:

- $LCP[0] = 0$
- $LCP[i] = lcp(SA[i-1], SA[i])$
  - i.e. the length of the common prefix for item i and the one before it.

For example: $W =$ `CATTATTAGGA`

i	$SA[i]$	Suffix	$LCP[i]$	Common Prefix
0	10	`A`	0	ϵ
1	7	`AGGA`	1	`A`
2	4	`ATTAGGA`	1	`A`
3	1	`ATTATTAGGA`	4	`ATTA`
4	0	`CATTATTAGGA`	0	ϵ
5	9	`GA`	0	ϵ
6	8	`GGA`	1	`G`
		...		

The LCP array only gives the LCP **between adjacent suffixes**, so how do we get the LCP **between any two suffixes**?

Observe that because the suffix array is sorted, **the LCP of any two suffixes must be shared by all suffixes between them**.

This means that $LCP[i, j] = \min \{LCP[k] \mid k = SA^{-1}[i] + 1, SA^{-1}[i] + 2, ..., SA^{-1}[j]\}$ (assuming i is before j in the suffix array).

The minimum LCP in a given interval **can be found in constant time**, but the algorithm to do so is outside the scope of this course.

Constructing the LCP Array

The LCP array can be constructed trivially in $O(n^2)$ by doing an $O(n)$ comparison for each of the $O(n)$ adjacent pairs, but this can be improved to $O(n)$:

If $i < n - 1$ and neither $SA^{-1}[i]$ nor $SA^{-1}[i + 1]$ are the first element of the suffix array, it holds that:

$$LCP[SA^{-1}[i]] - 1 \leq LCP[SA^{-1}[i + 1]] \text{ for any suffix } W[i..n - 1]$$

In English: after computing the LCP for a given suffix starting at i ($LCP[SA^{-1}[i]]$), the LCP for the next-shorter suffix starting at $i + 1$ ($LCP[SA^{-1}[i + 1]]$) is at least $LCP[SA^{-1}[i]] - 1$.

This means that we can speed up computation working from the longest ($i = 0$) suffix to the shortest ($i = n - 1$), skipping the first $LCP[SA^{-1}[i]]$ letters of the comparison at each step. This gives the following algorithm:

```
fun MAKE_LCP(text y, integer n, array SA, array SA_INV):

    // L will be the running total of the LCP for the current suffix
    // being worked on, and will also tell us how many characters
    // can be skipped when comparing in the next loop
    L = 0

    // for each suffix, starting from the longest...
    for (i from 0 to n — 1) do:

        // SA rank of the suffix starting at i
        r = SA_INV[i]

        // start pos. of the suffix before the one starting at i
        j = SA[r — 1]

        // i is the start pos. of the suffix at i
        // j is the start pos. of the suffix before the suffix at i

        // count matching letters in the suffixes starting at i and j,
        // note: we can skip the first L letters
        while (y[i + L] = y[j + L]) do:
            L = L + 1

        // store the LCP counter
        LCP[i] = L

        // decrement L
        if (L > 0):
            L = L — 1

    return LCP
```

This algorithm is **linear**, because L is bounded by $0 \leq L < n$ and L is only decreased by at most 1 in each loop, so L can make $O(2n)$ steps.

Putting it All Together

We let SA be the suffix array of the text y. We want to find occurrences of the pattern x in y.

Using binary search without LCP information we can search within SA using $O(log_2(n))$ steps, each with $O(m)$ comparison, giving a total run-time of $O(m \cdot log_2(n))$. However, LCP information can make this faster.

In the binary search, we consider the interval $[L..R]$ and the mid-point M, and decide whether to move to $[L..M]$ or $[M..R]$. To decide, we compare x to the suffix at M. We then consider

either $[L..M]$ or $[M..R]$ and a new mid-point M', but we already know $LCP[M, M']$, so there's no need to compare x and M' from scratch! We can skip the first $LCP[M, M']$ letters, because they were already compared.

The effect of this is that no character of x is compared to another character more than once. This reduces $O(m \cdot log_2(n))$ to $O(m + log_2(n))$, which is substantial.

In summary, given SA, SA^{-1} and LCP (pre-processed for range-minimum queries), we can:

- Find **some** occurrence in $O(m + log_2(n))$.
- Find **all** occurrences in $O(m + log_2(n) + occ)$.

Practically Efficient String Matching

The algorithms discussed so far are optimal in the worst case, but better algorithms work for average-case (i.e. practical) string matching, including some that are **sub-linear**. These algorithms are often used in the real world, even though their worst-case performance is not optimal.

Boyer-Moore Algorithm

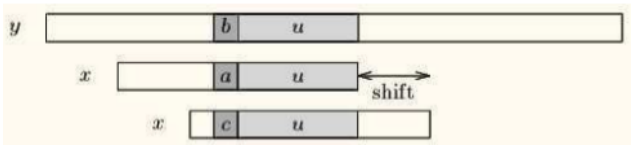
The Boyer-Moore algorithm uses a left-to-right scanning window to move the pattern along the text, but **compares letters from right to left**. In the case of a mismatch, two pre-computed functions are used to move the window to the right: the **good-suffix shift** and the **bad-character shift**. Both of these can be computed in linear time, $O(m)$.

Intuition

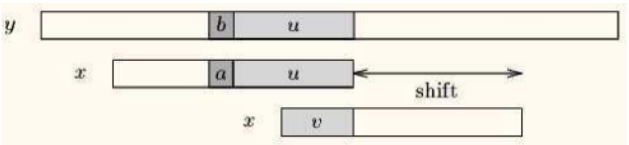
Good-suffix shift (1): when a mismatch occurs after matching u, move the window to the right so that the right-most recurrence of u in the pattern that is preceded by a different letter lines up with u in the text. For example:

```
Text:      abccaababacd
Pattern:   abcababa
Shift:       abcababa
```

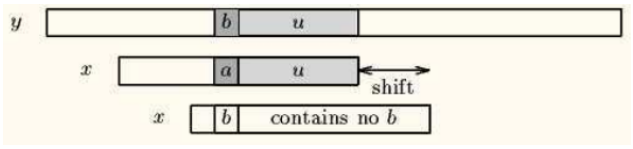
`aba` matches in the text, but a mismatch occurs between `a` and `b`. The right-most non-suffix recurrence of `aba` is preceded by `c`, so the pattern is shifted right to line it up with the already-matched `aba` in the text.



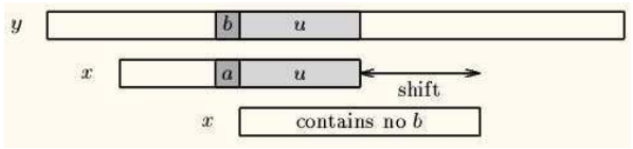
Good-suffix shift (2): if no such recurrence of u exists, shift the pattern to the right to line up the longest prefix of x that is a suffix of u.



Bad-character shift (1): if the mismatch occurs between $x[i] = \texttt{a}$ of the pattern and $y[j+i] = \texttt{b}$ of the text, move the pattern so that $y[j+i] = \texttt{b}$ lines up with the right-most occurrence of `b` in $x[0..m-2]$.



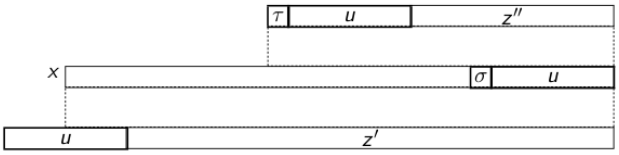
Bad-character shift (2): if no such occurrence of the mismatching letter exists, shift the pattern to immediately after the mismatched position.



Good-Suffix Shift

We define a function $d(u) = min\{|z'|, |z''|\} > 0$ where either of the following are true:

- x is a suffix of uz'
  - i.e. a prefix of x lines up with a suffix of u.
- $\tau uz''$ is a suffix of x and τu is not a suffix of x
  - i.e. $|z''|$ is the distance x should be moved to line up with the right-most recurrence of u that has a different preceding letter.



The good-suffix shift function $d(u)$ is stored in the **displacement table** D of size m, defined as follows:

$$D[i] = d(x[i+1..m-1]) \text{ for } i = 0, 1, ...m-1$$

This means that $D[i]$ is the $d(u)$ value for the suffix starting at position $i+1$. D **indicates how far the window can be shifted when a mismatch occurs.**

The computation of D is built from a table $suff$, which records for each position how far to the left a factor ending at that position can be extended and still match a suffix of x. That is:

$$suff[i] = max\{k : x[i-k+1..i] = x[m-k..m-1]\}$$

The computation of $suff$ is covered later (*see more: Table of Prefixes, page 65*).

For example:

i	0	1	2	3	4	5	6	7	8	9	10
$x[i]$	a	b	a	a	a	b	a	b	a	b	a
$suff[i]$	1	0	3	1	1	0	3	0	5	0	11

$suff[8] = 5$ shows that the factor in $x[4..8]$ matches the 5-letter suffix of x.

D is computed using two phases:

- Initialise with the periods of x.
  - If $suff[i] = i + 1$ then we have a border that corresponds to a period.
- Then the right-most recurrences of x's suffixes are accounted for by working left-to-right, overwriting non-right-most recurrences.

```
1  fun COMPUTE_D(m, suff):
2      j = 0
3      for (i from m − 2 to −1), do:
4          if (i == −1 or suff[i] = i + 1):
5              while (j < m − i − 1), do:
6                  D[j] = m − i − 1
7                  j = j + 1
8      for (i from 0 to m − 2), do:
9          D[m − suff[i] − 1] = m − i − 1
10
11     return D
```

This construction can be done in $O(m)$.

Bad-Character Shift

The bad-character shift function is stored in a table DA of size $|\Sigma|$. For all $c \in \Sigma$, $DA[c]$ stores the distance from the end of the right-most occurrence of the character c if c occurs in x, otherwise it stores m.

For example: $x =$ GCAGAGAG.

c	A	C	G	T
$DA[c]$	1	6	2	8

Computation of DA is simple:

```
fun COMPUTE_DA(x):
    for each a in alphabet, do:
        DA[a] = m

    for (k from 0 to m − 2), do:
        DA[x[k]] = m − 1 − k

    return DA
```

Computation on paper is even easier: for each letter, check how many positions you have to jump backwards from the last character of the pattern to reach the letter in question. If it's not there, use m as the value.

Complete Algorithm

```
fun BOYER_MOORE(pattern x, text y):
    // start of the matching window
    pos = 0

    // while the window is still within the text
    while (pos <= n − m), do:
        // start matching from the right of the window
        i = m − 1

        // move i to the left for each matching letter
        while (i >= 0 and x[i] = y[pos + i]), do:
            i = i − 1

        if (i == −1):
            // every letter matched
            output("x occurs in y at position " + pos)
            pos = pos + period(x)
        else:
            // mismatch at letter pos + i
            pos = pos + max( D[i], DA[y[pos + i]] − m + i + 1 )
```

The algorithm uses the maximum safe shift to gain the best performance.

Efficiency

The preprocessing stage runs in $O(m + |\Sigma|)$ time and requires $O(m + |\Sigma|)$ space.

In the worst case, the searching phase can take $O(nm)$ time.

$3n$ letter comparisons will be made in the worst case when searching for a non-period pattern.

$O(n/m)$ is the best performance offered, for instance when searching for $\texttt{a}^{m-1}\texttt{b}$ in $\texttt{b}^{n}$.

Reverse Factor Algorithm

The reverse factor algorithm uses the suffix tree of x^R (i.e. the reverse of x). It is very fast, especially for long patterns, and is optimal in the average case. On average it performs $O(\frac{n \cdot log_\sigma(m)}{m})$ inspections of text letters - this is sub-linear.

Similar to the Boyer-Moore algorithm, it uses a left-to-right window and makes comparisons from right to left, but uses the suffix tree of x^R to maximise the length of shifts.

Preprocessing

In the preprocessing phase, the algorithm constructs the suffix free for x^R as previously described, which takes $O(m)$ time and space.

Searching

When scanning the comparison window from right to left, the suffix that matches is 'spelled out' in the suffix tree. This continues until there is no edge defined for the current letter of the text for the current node in the tree.

At this point it is easy to know the length of the longest factor of the pattern that has been matched: it corresponds to the length of the path taken in the suffix tree from the root to the final visited node (explicit or implicit). Using the length of this longest factor, it is easy to compute a safe shift.

Parameter: q

The efficiency of the algorithm is contingent on the choice of value for q, which is the number of letters that will be read backwards in any given comparison window. In other words, we will aim to find a mismatch in the first q positions that gives a significant shift. From a starting position i, we will read to $i + m - q + 1$.

Why does this matter? Because if we run out of edges to follow within the q characters, then the q-length factor cannot appear anywhere in the pattern (because the suffix tree stores *all* suffixes).

So how do we estimate a good value for q, given a pattern of length m and an alphabet of size σ?

Observe the following:

- Verifying all starting positions of a window of length m using the suffix tree takes time $O(m^2)$.

- The probability that a factor of length q in the text matches a factor of length q in the pattern is no more than m/σ^q.
- The cost of spelling q letters backwards in the suffix tree is $\Theta(q)$.

We want the expected cost of verifying a window to be not more than the cost of spelling q letters. This means that we want $O(m^3/\sigma^q)$ to be bounded by $O(q)$.

This works out to $q \geq 3 \cdot log_\sigma(m)$.

Efficiency

We take a value of $q = 3 \cdot log_\sigma(m)$. There are $O(n/m)$ non-overlapping windows of length m, and the expected cost per window is $O(q) = O(log_\sigma(m))$.

On average, the reverse factor algorithm therefore takes time proportional to:

$$O(\frac{n \cdot log_\sigma(m)}{m}) = O(\frac{nq}{m})$$

Karp-Rabin Algorithm

The Karp-Rabin algorithm uses a **rolling hash** to check whether the text in the window is *likely* to be a match for the pattern, followed by an every-letter check on windows where the hash indicates a possible match.

To be effective, this algorithm requires a hash function that is:

- Highly discriminating for strings.
- Efficiently computable.
- Designed so that $h(y[j+1..j+m])$ can be easily computed from $h(y[j..j+m-1])$ and $y[j+m]$.
  - For example, a 3-character rolling hash for the string `abcd` must be able to easily compute $h(\texttt{bcd})$ from $h(\texttt{abc})$ and `d`.

The Algorithm (Summary)

```
fun RK_SEARCH(x, y):
    hx = HASH(x)
    for (pos from 0 to n − m), do:
        h = HASH(y[pos...pos + m − 1])
        if (hy == hx):
            i = 0
            while (i < m and x[i] = y[pos + i]), do:
                i = i + 1
            if (i == m):
                output("x occurs in y at position " + pos)
```

The Hash Function

The hash function $h : \Sigma^m \mapsto \mathbb{N}$ represents all letters as integers and uses two parameters: the base d and the modulo q. It is defined as follows:

$$\begin{aligned} h_i = h(y[i..i+m-1]) &= (y[i]d^{m-1} + y[i+1]d^{m-2} + ... + y[i+m-1]d^0) \bmod q \\ &= ((...((((y[i]d) + y[i+1])d + y[i+2])d... + y[i+m-1]) \bmod q \end{aligned}$$

The modulus operator can also be used at intermediate stages to avoid dealing with huge numbers, such as:

$$((...((((y[i]d \bmod q) + y[i+1])d \bmod q + y[i+2])d \bmod q... + y[i+m-1]) \bmod q$$

There is a catch to using modulus though: different input may map to the same hash value, creating **false positives**.

h_i can also be converted into h_{i+1} with a **constant** number of operations:

$$h_{i+1} = ((h_i - y[i]d^{m-1})d + y[i+m]) \bmod q$$

The Algorithm (Full)

```
fun RK_SEARCH(x, y):
    hx = 0
    hy = 0
    D = d^(m—1) mod q

    // build the hash of the initial m letters
    for (i from 0 to m — 1), do:
        hx = ((hx * d) + x[i]) mod q
        hy = ((hy * d) + y[i]) mod q

    // scan from left to right, updating the hash as we go
    for (pos from 0 to n — m), do:

        // do a full check for a possible match
        if (hx == hy):
            i = 0
            while (i < m and x[i] = y[pos + i]), do:
                i = i + 1
            if (i == m):
                output("x occurs in y at position " + pos)

        // update the hash
        if (pos < n — m):
            hy = ((hy — (y[pos] * D)) + y[pos + m]) mod q
```

Efficiency

The pre-processing phase takes $O(m)$ time and **constant** space.

The searching phase has a worst case of $O(nm)$, when all hashes match (either because of a string like `aaa...`, a poorly chosen hash function, or unbelievable bad luck).

With a sufficiently large $q \geq m$ and the assumption that $h(...)$ mod q distributes evenly in $[0, q)$, the average number of false positives is $O(n/q)$.

The average runtime is therefore $O(n + nm/q) = O(n)$ (because $q \geq m$, so $m/q < 1$).

Table of Prefixes

Similar to the $suff$ table from the previous topic, the $pref$ table is defined as the longest distance to the right a window from a given position can be extended whilst still being a prefix of the string.

For example, in the string `abcacabcaca`, position 5 can be extended 6 characters to the right to give `abcaca`, which is a prefix of the whole string. Therefore, $pref[5] = 6$.

The $pref$ table is useful, because it can be used to compute the $suff$ table needed for the Boyer-Moore algorithm by simply reversing the string.

Brute-Force Solution

The brute-force solution is trivial: for each position, keep checking characters to the right until a mismatch (or the end of the string) is reached. This approach would take time proportional to $O(m^2)$, which is far too slow.

Linear Solution

The algorithm can be made to run in time proportional to $O(m)$ by taking advantage of previously-computed values.

To compute the $pref$ array, we maintain two values as we go along - f and g - defined as follows:

If $i > 1$,

$$g = max\{j + pref[j] \quad : \quad 0 < j < i\}$$

$$f \in \{j \quad : \quad 0 < j < i \text{ and } j + pref[j] = g\}$$

In other words, f is the starting position of the match that pushes furthest to the right in the string (u in the diagram below), and g is the position immediately after the end of this match.

The algorithm is efficient because f and g can be easily after the computation each position, and they can be used to calculate the $pref$ value of a new position with minimal effort.

If $i < g$, then following holds:

$$pref[i] = \begin{cases} pref[i-f] & \text{if } pref[i-f] < g-i \\ g-i & \text{if } pref[i-f] > g-i \\ g-i+l & \text{otherwise} \end{cases}$$

where $l = |lcp(x[g-i..m-1], x[g..m-1])|$ (that is, the the manually-compared prefix match, skipping the first $g-i$ comparisons).

Case 1 The start of the factor starting at $i-f$ is the same as the start of the factor starting at i, because they are both at the same position within u. If $pref[i-f]$ is less than $g-i$, the factor starting at $i-f$ must suffer a mismatch before the end of u. Because u is repeated, the factor starting at i must suffer the same mismatch, and therefore matches the same length of prefix, such that $pref[i] = pref[i-f]$.

Case 2 The factors starting at $i-f$ matches past the end of u, because $pref[i-f] > g-i$. The factor stating at i has the same first characters as the factor starting at $i-f$, but it cannot match the same prefix because it encounters the `a` vs. `b` mismatch, and therefore $pref[i] = g-1$ (i.e. the amount between i and `b`).

Case 3 If the factor starting at $i-f$ matches directly to the end of u, we cannot be sure what caused the mismatch, so we must compared characters maually.

For brevity (and to make memorising easier) can also be stated as follows:

$$pref[i] = \begin{cases} g-i+l & \text{if } pref[i-f] = g-i \\ min\{pref[i-f], g-i\} & \text{otherwise} \end{cases}$$

Algorithm

```
1  fun MAKE_PREF(string x, length m):
2
3      // the first position trivially matches the entire string
4      pref[0] = m
5
6      g = 0
7
8      for i in 1 to m − 1, do:
9          if (i < g and pref[i − f] != g − i):
10             // no manual comparison needed
11             pref[i] = min( pref[i − f], g − i )
12
13         else:
14             // manual matching
15             g = max(g, i)
16             f = i
17             while (g < m and x[g] = x[g − f]), do:
18                 g = g + 1
19             pref[i] = g − f
20
21     return pref
```

The loop looks at each character in turn, and either performs an $\Theta(1)$ assignment (line 11) or does manual comparison (lines 17 and 18). Note however that g is initialised to 0, is bounded by m, and never decreases. This is why the algorithm is guaranteed to run in $O(m)$ time.