TSP: Text Searching and Processing

Important Notes About These Notes

These notes were written by me, Mark Ormesher, during my revision for Computer Science exams at King's College London. I am publishing them here to help my classmates and other students in the years below me. To the best of my knowledge they are correct.

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These notes were relevant for my year (2016/17) but **the content for your course may be different**. Check your lecture slides, syllabi, tutorial guides, etc.

These notes were produced for my own personal use (it's part of how I study and revise). That means that some annotations may by irrelevant to you, and **some topics might be skipped** entirely.

Feel free to **share** these notes, however please only share a link to the repo (see the link below), not individual files.

Notes are originally from **https://github.com/markormesher/CS-Notes**.

Contents

Terms and Definitions

Alphabets

An alphabet Σ is a finite, **non-empty** set of elements called **letters**.

String Basics

A string on an alphabet Σ is a finite, **possible empty** sequence of letters from Σ.

The **empty string** is denoted by ϵ.

The set of **all possible strings** on an alphabet Σ is denoted by Σ^*.

The **length** of a string x is the length of its sequence of letters, denoted by $|x|$.

The letter at the i**-th position** in the string x is denoted by $x[i]$, for all $0 \leq i < |x| - 1$.

Identity Between Strings

Two strings x and y are **identical** if and only if $|x| = |y|$ and $x[i] = y[i]$ for all $0 \leq i < |x| - 1$.

Concatenation/Product of Strings

The concatenation or product of two strings x and y is the string formed from all letters of x followed by all letters of y. It is denoted by xy.

Factor/Substring of a String

A string x is a factor or substring of a string y if two strings u and v exist such that $y = uxv$. Note that u and/or v may be the empty string ϵ.

A factor or substring x of a string y is **proper** if $x \neq y$.

Superstring of a String

A string x is a superstring of a string y if two strings u and v exist such that $x = uyv$. Note that u and/or v may be the empty string ϵ.

Suffix and Prefix of a String

Let the strings x, y, u and v exist such that $y = uxv$...

- If $u = \epsilon$ then x is a prefix of y.
- If $v = \epsilon$ then x is a suffix of y.

Occurrence of a String

If x and y are two strings and $x \neq \epsilon$, we can say that x **occurs in** y if x is a factor of y.

Every occurrence of x can be identified by a position within y. We say that x occurs in y starting at position i if $y[i...i + |x| - 1] = x$.

It is sometimes more useful to refer to the end position of the occurrence, $i + |x| - 1$.

Example, in the string `abracadabra`, the string `abr` occurs starting from positions 0 and 7.

Power of a String

For a string x and a natural number n, the n-th power of x, denoted by x^n, is defined as follows:

- $x^0 = \epsilon$
- $x^n = x^{n-1}x$

Example: if $a =$ `ha` then $a^4 =$ `hahahaha`.

If two strings x and y and two natural numbers n and m exist such that $x^m = y^n$ then x and y are powers of some string z.

Primitivity of Strings

A string is **primitive** if it is not the power of any other string.

A string x is **primitive** if and only if it exists as a factor of x^2 as a prefix and a suffix.

Example: given $x =$ `abaab` and $x^2 =$ `abaababaab`, the string x is primitive because x only appears at the very start and very end of x^2.

String Roots and Exponents

If $x \neq \epsilon$, a **primitive** string z and natural number n exist such that $x = z^n$. z is the **root** of x and n is the **exponent** of x. If x is primitive then $z = x$ and $n = 1$.

Example: $x =$ `abab`, $z =$ `ab`, $n = 2$.

Example: $x =$ `abcd`, $z =$ `abcd`, $n = 1$.

Conjugate Strings

Two non-empty strings x and y are conjugate if two string u and v exist such that $x = uv$ and $y = vu$.

Example: `goldfish` and `fishgold` are conjugate.

Two non-empty strings are conjugate if and only if their **roots are also conjugate**.

Two non-empty strings are conjugate if and only if a string z exists such that $xz = zy$ (for the above example, $z =$ `gold`).

Periods of a String

For a non-empty string x, an integer p such that $0 < p \leq |x|$ is a period of x if $x[i] = x[i + p]$ for all $0 \leq i < |x| - p$.

A string may have multiple periods, and every string has at least one period (the length of the string itself). We define *the* period of a string as its **smallest** period, denoted by $per(x)$.

Example: the string $x =$ `aabaabaa` has periods of 3, 6, 7 and 8, and $per(x) = 3$.

Borders of a String

For a non-empty string x, a border is simultaneously a **proper factor**, **prefix** and **suffix** of x.

We define *the* border of a string as its **longest** border. $border(x)$ denotes the **length** of the longest border of x.

Example: the string $x =$ `aabaabaa` has borders of ϵ, `a`, `aa` and `aabaa`, so $border(x) = 5$.

Relationship Between Periodicity and Borders

For any non-empty string x it holds that $per(x) + border(x) = |x|$.

String Matching

Objective: given two strings, $pattern$ and $text$, find the **starting positions of all occurrences** of $pattern$ within $text$.

Example: within the text `abracadabra`, the pattern `abr` starts at positions 0 and 7.

Notation

- The **pattern** is denoted as x with length m.
- The **text** is denoted as y with length n.

Online and Offline Algorithms

- In an online algorithm, only the **pattern** is known and can be pre-processed.
- In an offline algorithm, only the **text** is known and can be pre-processed.

Näive Matching

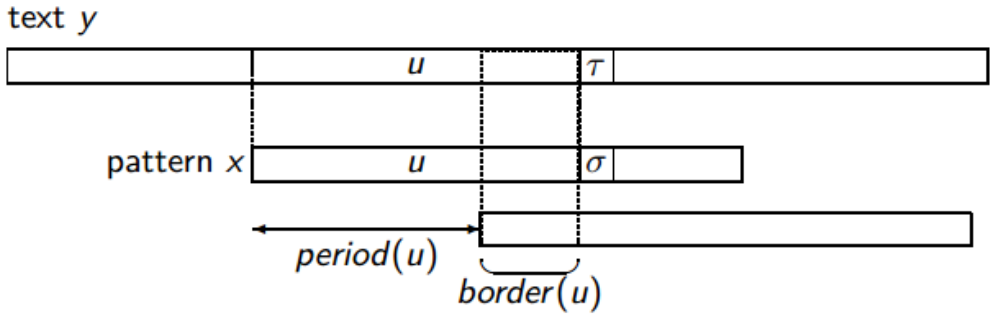
Matches are found by comparing the pattern to the text string repeatedly, incrementing the starting character each time. In the example below, the text is `abracadabra`, the pattern is `abr`, and matching portions are shown in red.

Index i:	0	1	2	3	4	5	6	7	8	9	10
Text y:	a	b	r	a	c	a	d	a	b	r	a
Successful try from 0:	a	b	r								
Failed try from 1:		a	b	r							
Failed try from 2:			a	b	r						
Failed try from 3:				a	b	r					
Failed try from 4:					a	b	r				
Failed try from 5:						a	b	r			
Failed try from 6:							a	b	r		
Successful try from 7:								a	b	r	
Failed try from 8:									a	b	r

The runtime of this algorithm is $O(nm)$ in the worst case; this is **too slow**.

Morris-Pratt (MP) Matching

This algorithm improves the runtime by 'shifting' the pattern along the text by **more than one position** when a mismatch occurs.



In the diagram above, u shows the **prefix** of x that has been successfully matched within the text y; the mismatch is caused by the non-equal characters τ and σ.

When the mismatch occurs, it is safe to move x along so that it lines up with the **end border** of u and resume searching from the position of τ (because we already know that the portion within the border already matches). The size of this movement is $period(u)$, because $|u| - border(u) = period(u)$.

The general structure of the algorithm is as follows:

- Memorise the borders of all prefixes of the pattern.
- Perform straightforward online search.
- Shift by $period(u)$ when a mismatch occurs.
- Resume scanning from $border(u)$.

```
1  fun MP_MATCHING(string y, string x, int n, int m):
2
3      mpNext = COMPUTE_MP_NEXT(x, m)
4
5      i = 0 // index of current comparison, relative to pattern x
6      j = 0 // start of window in text y
7
8      while (j < n) do:
9          while (i >= 0 and x[i] != y[i]) do:
10             i = mpNext[i]
11         i++
12         j++
13         if (i >= m):
14             output(j − i)
15             i = mpNext[i]
16
17     return matches
```

In the above code, $mpNext[i]$ is the length of the longest border of the first i characters of the pattern x.

MP Pre-processing

It is possible to efficiently compute the border size of pattern prefixes using one key property: **a border of a border of u is also a border of u.** Using this, the algorithm will compute the array $border$ (a.k.a $mpNext$) such that $border[i]$ is the length of the longest border of the first i characters of the pattern x.

After defining $border[0] = -1$ for the empty string, for each x-prefix of length $1 \leq i \leq m$ the algorithm starts by assuming the length of the previous border. If the i^{th} character matches then the while loop is skipped and the border length is increased by 1. When a mismatch occurs, the algorithm tries successively shorter 'borders of borders' until a match is found.

```
1  fun COMPUTE_MP_NEXT(string x, int m):
2      border[0] = -1
3
4      for (i from 1 to m) do:
5          j = border[i - 1]
6          while (j > 0 and x[i - 1] != x[j]) do:
7              j = border[j]
8          border[i] = j + 1
9
10     return border
```

Note: `border = mpNext`.

In the outer loop j is only incremented exactly m times. In the inner loop j can only decrease but $j \geq 0$ at all times. This means that the maximum amount of 'travel' for the value of j is $2m$ and therefore the algorithm's runtime is $O(m)$.

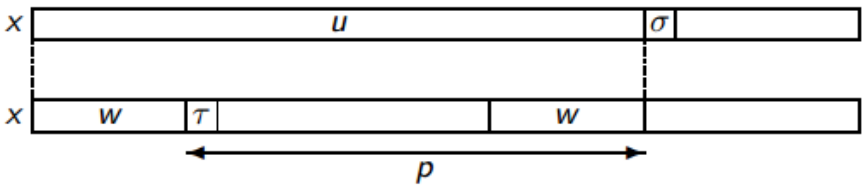
Knuth-Morris-Pratt (KMP) Matching

This algorithm improves slightly on the runtime of the MP matching algorithm (*see more: Morris-Pratt (MP) Matching, page 7*) by changing only the pre-processing stage to give slightly longer shifts.

The improvement relies on **strict borders** and their corresponding **interrupted periods**.

w is a strict border of the prefix u if:

- w is a border of u, and
- $w\tau$ is a prefix of x but $u\tau$ is not.



In the diagram above, w is a strict border of the prefix u because $\tau \neq \sigma$. Therefore p is the interrupted period of u.

KMP Pre-processing

This algorithm assumes that $mpNext$ is already available. Assuming $k = mpNext[i]$, it is defined as follows:

$$kmpNext[i] = \begin{cases} k & x[i] \neq x[k] \text{ or } i = m \\ kmpNext[k] & x[i] = x[k] \end{cases}$$

This states that $kmpNext[i]$ is equal to $mpNext[i]$ when there is a mismatch between the next character after the prefix and the next character after the border (i.e. a strict border). If the letters do match then the strict border of the border is taken.

```
1  fun COMPUTE_KMP_NEXT(string x, int m):
2
3      kmpNext[0] = −1
4      k = 0 // k will be mpNext
5
6      for (i from 1 to m − 1) do:
7
8          if (x[i] == x[k]):
9              kmpNext[i] = kmpNext[k]
10         else:
11             kmpNext[i] = k
12             do:
13                 k = kmpNext[k]
14             while (k >= 0 and x[i] != x[k])
15
16         k++
17
18     kmpNext[m] = k
```

As before, in the outer loop k is only incremented exactly m times. In the inner loop k can only decrease but $k \geq 0$ at all times. This means that the maximum amount of 'travel' for the value of k is $2m$ and therefore the algorithm's runtime is $O(m)$.

Runtime of MP and KMP Matching

On a text of length n, the MP and KMP algorithms have runtimes of $O(n)$. In particular, they make no more than $2n$ character comparisons. This is because positive comparisons will increase the value of j (which ranges from 0 to n and does not decrease), and negative comparisons will increase the value of the starting position $j - i$ (which also ranges from 0 to n and does not decrease).

Including the pre-processing, the runtime of both algorithms is $O(n + m)$.

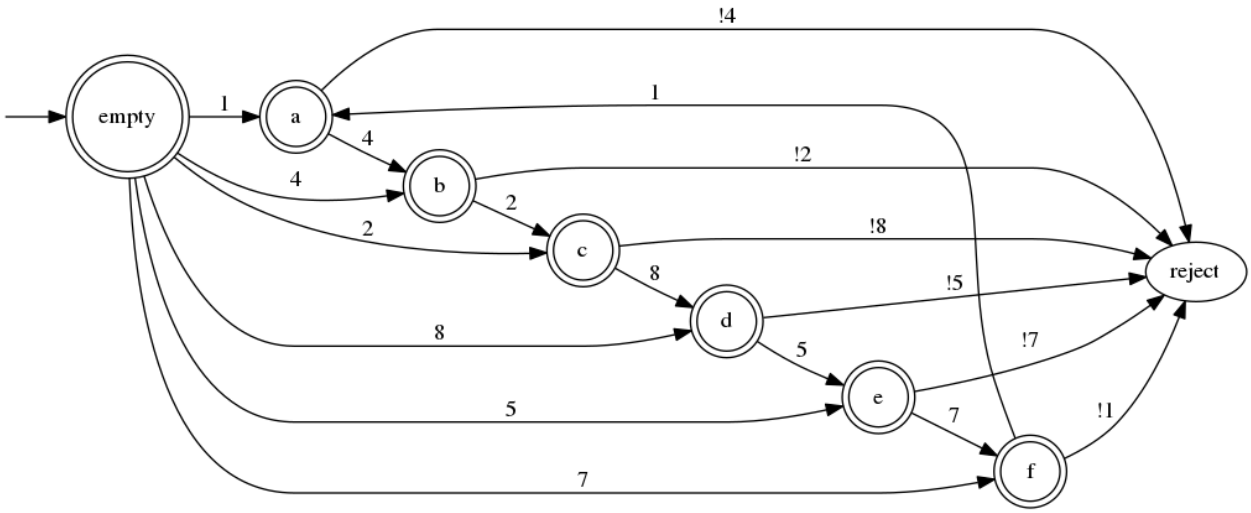
Matching with Automata

Designing Automata to Match Periodic Strings

A **non-empty string** is periodic if it has the form $uuu...uu'$, such that u' is a prefix of u. For example, the string `abbabbab` is periodic, where $u = \texttt{abb}$ and $u' = \texttt{ab}$ (*see more: Periods of a String, page 6*).

Automata can be easily designed to **accept any substring of a repeating string**, including the empty string ϵ.

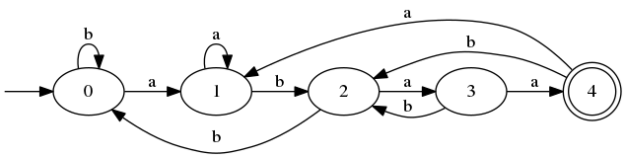
For example, the following automaton operates over the alphabet $\Sigma = \{1, 2, 4, 5, 7, 8\}$ and accepts any substring of the decimal expansion of $\frac{1}{7} = 0.142857142857...$ (ignoring the $0.$ prefix).



Designing Substring-Matching Automata (SMAs)

String-matching automata (SMAs) should reach an accepting state every time a given pattern is encountered as a sub-string of the input. For example, with the pattern `abb` and the input `cabbabb`, the automaton should reach an accepting state after processing characters 3 and 6 (note: zero-indexed).

For example, the automaton below accepts $u = \texttt{abaa}$ in the alphabet $\Sigma = \{a, b\}$.



Text y		b	a	b	b	a	a	b	a	a	b	a	a	b	b	a
State	0	0	1	2	0	1	1	2	3	**4**	2	3	**4**	2	0	1

Matching Using SMAs

A simple online parsing of the text y with $SMA(u)$ is sufficient:

```
fun SMA_MATCHING(string y, sma):
    q = sma.initial_state

    if  (q is terminal):
        report occurrence of pattern

    while (y has remaining characters) do:
        a = next character of y
        q = q.successors[a]
        if (q is terminal):
            report occurrence of pattern
```

Systematic Construction

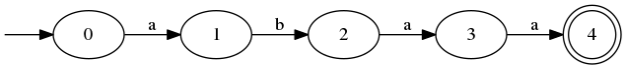
An SMA that matches u (where $|u| = m$) will contain $m + 1$ states numbered $0..m$. In general, if the automaton is in state k then the k most recently read characters match the first k characters of the pattern. The final state (numbered m) is the only accepting state.

The SMA can be constructed with the following process:

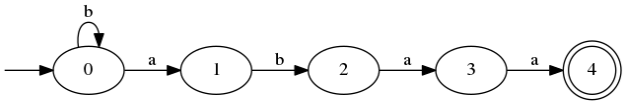
1. Create the initial $m + 1$ states, numbered $0..m$.
    - Declare state 0 as the initial state.
    - Declare state m as the accepting state.
2. For each $0 \leq k < m$, create a forwards arc from state k to state $k + 1$ labelled by $u[k]$.
    - This is the successful path that simply spells u.
3. At each state, create a backwards arc for each character in Σ that breaks the pattern u.
    - The backwards arc should lead to state k, where the k most recently read characters match the first k characters of the pattern.
    - Another way of stating this is to find the longest suffix of processed input that matches a prefix of u. The length of this suffix/prefix is k.
    - k can be found by shifting the pattern along the read characters to the right, until a match occurs. An example follows.

Systematic Construction Example

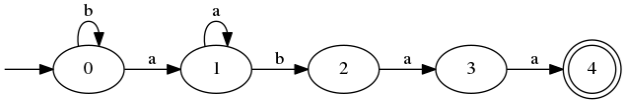
Consider the alphabet $\Sigma = \{a, b\}$ and the pattern $u =$ `abaa`. After steps 1 and 2, the following SMA will be created:



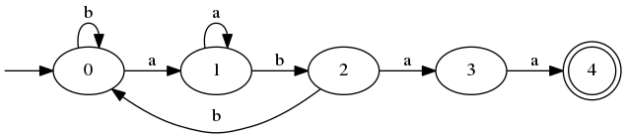
Applying step 3 to state 0, a break in the pattern occurs after reading `b`. This shares no characters with the start of u, so the arc should lead to state 0:



Applying step 3 to state 1, a break in the pattern occurs after reading `aa`. The longest suffix of `aa` that matches a prefix of `abaa` is `a`, so the backwards arc leads to state 1:



Applying step 3 to state 2, a break in the pattern occurs after reading `abb`. No suffix of `abb` matches a prefix of `abaa`, so the backwards arc leads to state 0:

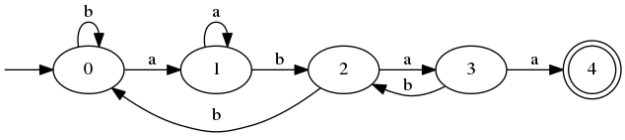


Applying step 3 to state 3, a break in the pattern occurs after reading `abab`. The longest suffix of `abab` that matches a prefix of `abaa` is `ab`, so the backwards arc leads to state 2. This can be seen by shifting the pattern to the right along the input:

```
Input:   abab
Pattern: abaa
```

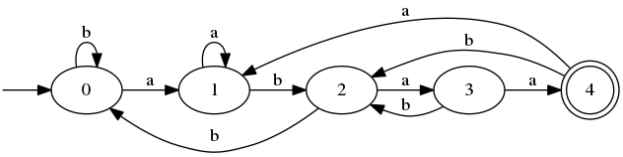
```
Input:   abab
Pattern:  abaa
```

```
Input:   abab
Pattern:   abaa
             ^^ Match!
```



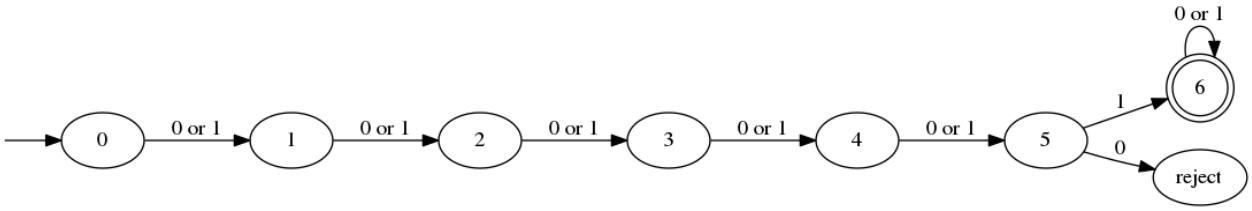
Finally, at state 4, backwards arcs for `a` and `b` need to be created.

The longest suffix of `abaaa` matching a prefix of `abaa` is `a`, so a backwards arc for `a` leads to state 1. The longest suffix of `abaab` matching a prefix of `abaa` is `ab`, so a backwards arc for `b` leads to state 2:

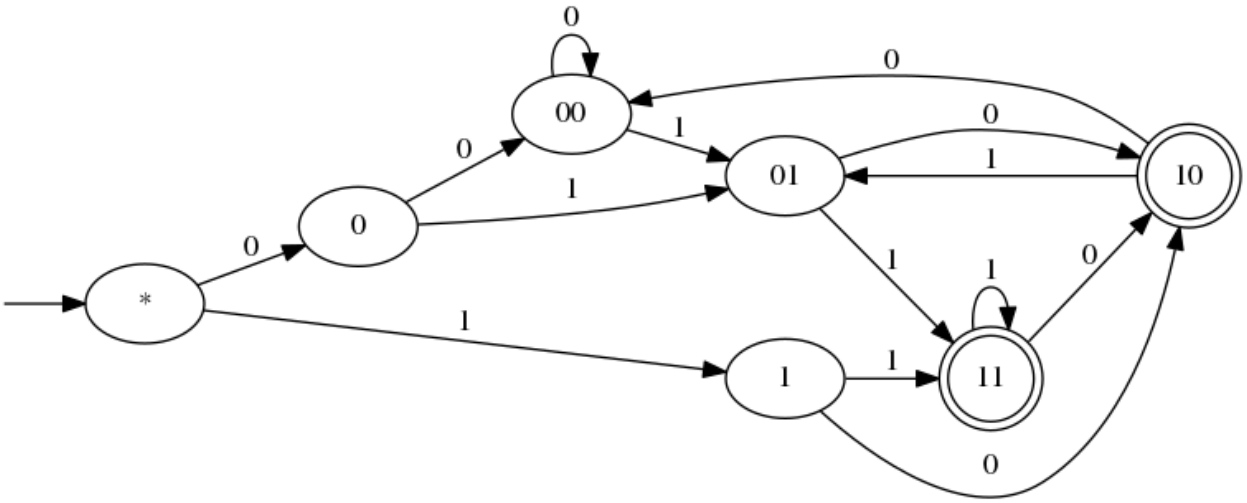


Matching n^{th} Character from the Left and Right

Matching **from the left** is trivial: the first $n-1$ character are skipped, and the n^{th} character must match. For example, the following automaton accepts any string from $\Sigma = \{0, 1\}$ where the 6th character from the left is 1:



Matching **from the right** is more complex: the automaton must be 'tricked' into keeping a memory. This can be achieved by making a state for every combination of the n right-most characters and accepting states that begin with the required character. For example, the following automaton accepts any string from $\Sigma = \{0, 1\}$ where the 2nd character from the left is 1:



Dictionary Matching

Formally, a **dictionary** is a set of strings $X = \{x_0, x_1, ...\}$ such that $\epsilon \notin X$.

Objective: given a **dictionary** of strings X and a text y, find all occurrences of strings from the dictionary within y. The output should be the **list of positions** in y that are **end positions** of some string in X.

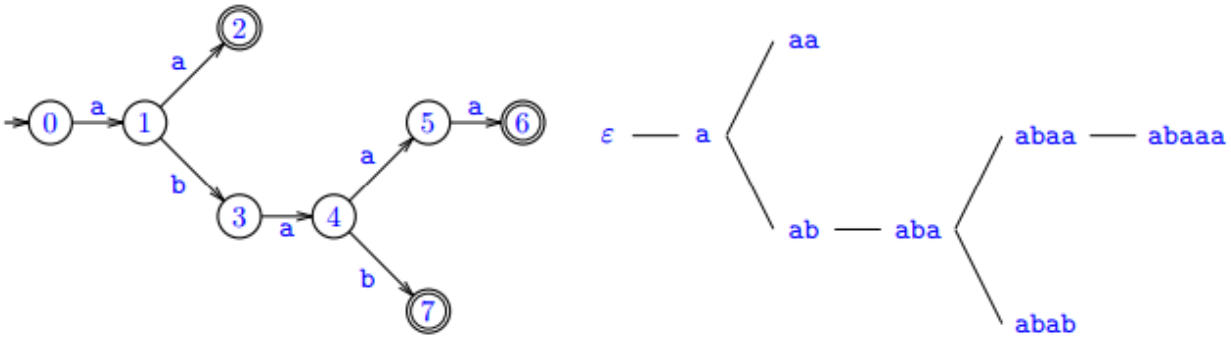
Example: if $X = \{\texttt{abc}, \texttt{cba}\}$ and $y = \texttt{aabcbabc}$, then the output should be $[3, 5, 7]$.

The standard approach for this is to use a **dictionary matching automaton (DMA)**.

Tries of Dictionaries

A trie $T(X)$ is a tree whose leaves are labelled by strings of X. Internal nodes are prefixes of strings in X. As an automaton, the trie $T(X)$ accepts all strings in X.

For example, $T(\{\texttt{aa}, \texttt{abaaa}, \texttt{abab}\})$:



Dictionary Matching Automata (DMAs)

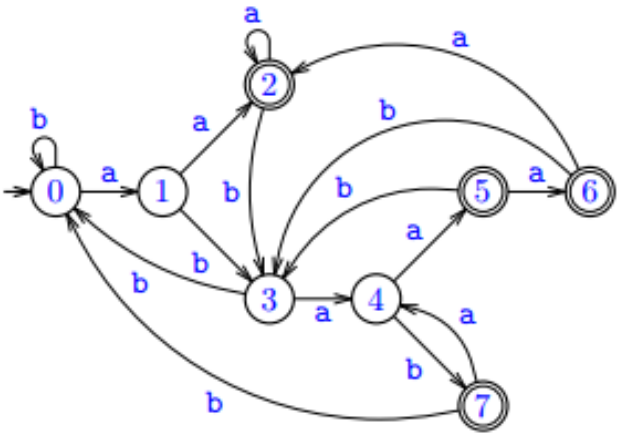
DMAs reach an accepting state every time a word from the dictionary is read in full (very similar to the SMAs from earlier). Formally, a DMA for the dictionary X (denoted $D(X)$) operates over the alphabet A and accepts any word in the form A^*X (i.e. any string from the language, suffixed with a string from X).

A dictionary trie forms the basis of a DMA. DMAs are constructed as follows:

- The set of states is $pref(X)$.
  - i.e. all prefixes of strings in the dictionary X.
- The initial state is the empty string ϵ.
- The set of terminal/accepting states is $pref(X) \cap A^*X$.
  - i.e. all prefixes that are suffixed with strings from X.

- Edges are in the form $\langle u, a, h(ua)\rangle$, where u is the text read so far, a is the character labelling the edge, and $h(ua)$ is the longest suffix of ua that belongs to $pref(X)$.

For example, $D(\{\texttt{aa}, \texttt{abaaa}, \texttt{abab}\})$:



$0 = \epsilon$, $1 = \texttt{a}$, $2 = \texttt{aa}$, $3 = \texttt{ab}$ etc.

Note: state $5 = \texttt{abaa}$ is an accepting state because `abaa` is suffixed by `aa`, which is in the dictionary.

The size of DMAs is $O(|A| \cdot \Sigma(|x| : x \in X))$ (i.e. one arc for each letter in the alphabet, for each of the possible prefix states).

Matching Using DMAs

Almost identical to the SMA searching algorithm (*see more: Matching Using SMAs, page 13*).

```
fun DMA_MATCHING(string y, dma):
    q = dma.initial_state

    for each character a in y, do:
        q = q.successors[a]
        if (q is terminal):
            report occurrence of dictionary string
```

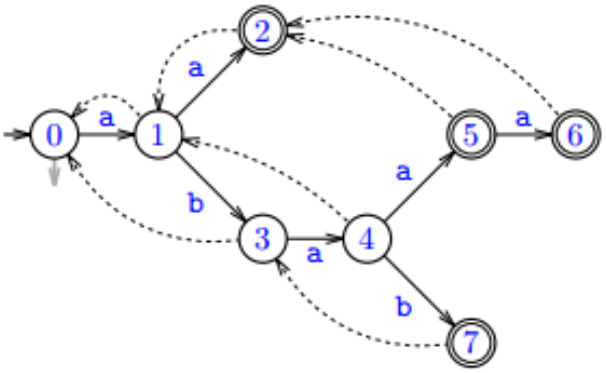
The running time of this matching process on a text y is $\Theta(|y|)$.

DMAs With Failure Links

This adaptation of DMAs reduces the size to $O(\Sigma(|x| : x \in X))$, independent of the alphabet size. It works by keeping the 'forward' arcs between prefixes and replacing all 'backward' arcs with one **failure link** per prefix node.

Failure links are determined by the **failure function** f, such that $f(u) =$ the **longest proper suffix** of u that is in $pref(X)$. The failure link for a node labelled by u leads to $f(u)$.

For example, $D(\{\texttt{aa}, \texttt{abaaa}, \texttt{abab}\})$:



This can be represented with a compact successor table, as below. Note that the 'label' column is not needed and is only for clarity.

State	Label	Successors	Failure
0	ϵ	$(a, 1)$	nil
1	a	$(a, 2)$, $(b, 3)$	0
2	aa		1
3	ab	$(a, 4)$	0
4	aba	$(a, 5)$, $(b, 7)$	1
5	abaa	$(a, 6)$	2
6	abaaa		2
7	abab		3

Matching Using DMAs with Failure Links

This matching process works as before when successors (forward arcs) exist, but when no successors exist the algorithm will **follow failure links** until it reaches a state containing and appropriate successor (or the initial state).

```
fun DMA2_MATCHING(string y, dma):
    q = dma.initial_state

    for each character a in y, do:
        q = TARGET_BY_FAILURE(q, a, dma)
        if (q is terminal):
            report occurrence of dictionary string
```

```
fun TARGET_BY_FAILURE(p, a, dma):
    while (p != nil and p.successors[a] = nil) do:
        p = p.failure

    if (p == nil):
        return dma.initial_state
    else:
        return p.successors[a]
```

For example, using the DMA with failure links above and the text `abbabaababababb`:

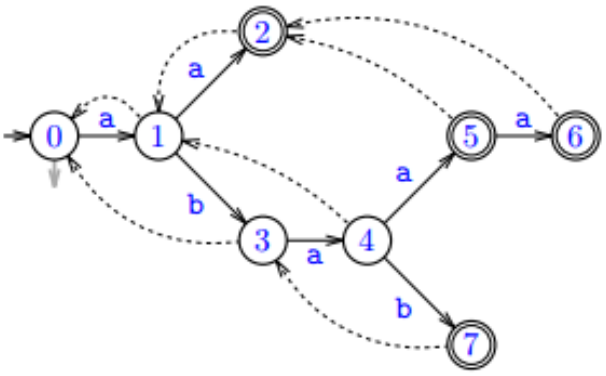
Text y		a	b	b		a	b	a	a	b		a	b	a		b	b
State	0	1	3	0, 0	1	3	4	**5**	2, 1, 3	4	**7**	3, 4	**7**	3, 0, 0

Computing Failure Links

Failure links can be efficiently computed via a **breadth-first traversal** of the trie, as described below. Breadth-first traversal is efficient because failure links for a given state can be computed using the failure links for 'earlier' states, which will have already been computed.

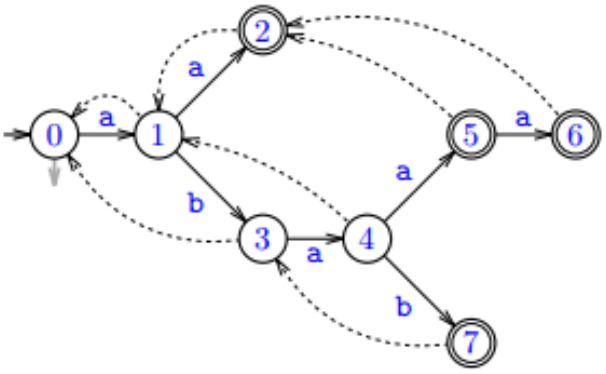
```
fun BUILD_DMA_WITH_FAILURE_LINKS(X):
    M = BUILD_TRIE(X)
    fail[M.initial_node] = nil

    Q = empty queue
    Q.enqueue(M.initial_state)
    while (Q is not empty) do:
        t = Q.dequeue()

        for each pair (a, p) in t.successors, do:

            // successor failure is built on the parent's failure
            r = TARGET_BY_FAILURE(fail[t], a)
            p.failure = r

            // if the failure goes to a terminal state,
            // the success becomes terminal
            if (r.terminal):
                p.terminal = true

            Q.enqueue(p)

    return M
```

For example, using $D(\{\texttt{aa}, \texttt{abaaa}, \texttt{abab}\})$:



- State 5 is a child of state 4 along an arc labelled `a`.
  - $fail[4] = 1$.
  - `TARGET_BY_FAILURE(1, a)` = 2, so $fail[5] = 2$.
  - State 2 is terminal, so state 5 is marked terminal.
- State 6 is a child of state 5 along an arc labelled `a`.
  - $fail[5] = 2$.
  - `TARGET_BY_FAILURE(2, a` = 2, so $fail[6] = 2$.

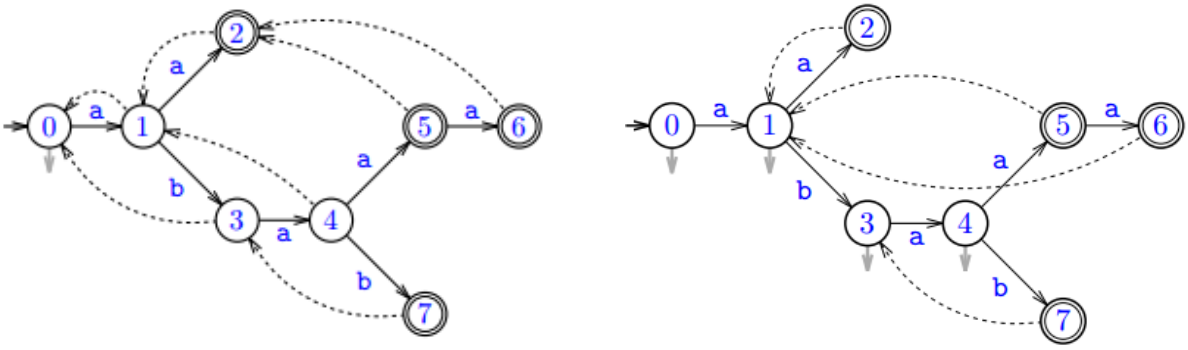
Optimising Failure Links



In the example above, $fail[4] = 1$. However, the outgoing edges from state 1 are a subset of the outgoing edges from state 4, so any input that failed on state 4 is guaranteed to also fail on state 1.

Therefore, the initial set of failure links can be optimised: at a state k, follow failure links until a state is reached with outgoing edges that are not a subset of k's outgoing edges (i.e. a state with 'new' outgoing options).

This is shown in the following example, where the optimised version is on the right:



Delay

The delay is the **maximum time spent on any letter** of the input y. Even with optimised failure links, it is $max\{|x| : x \in X\}$.

Searching in a List of Strings

- Input:
  - A list L of n strings of Σ^*, sorted in lexicographic order.
  - A string $x \in \Sigma^*$ of length m.
- Simple searching output:
  - $x \in L$: a position i such that $0 \leq i < n$, where $L_i = x$.
  - $x \notin L$: positions d, f such that $d+1 = f$ and $-1 \leq d < f \leq n$, where $L_d < x < L_f$.
  - i.e. i is the position of x, or d and f are indexes of the strings either side of where x *would* be if it was in the list.
- Interval output:
  - Positions d, f such that $-1 \leq d < f \leq n$, where x is a prefix of L_i for all $d < i < f$.

For example:

$$L = \begin{array}{c|l} L_0 & \texttt{aaabaa} \\ L_1 & \texttt{aaabb} \\ L_2 & \texttt{aabbbb} \\ L_3 & \texttt{ab} \\ L_4 & \texttt{baaa} \\ L_5 & \texttt{bb} \end{array}$$

Simple Search:

$x = \texttt{aaabb} \longrightarrow 1$

$x = \texttt{aaba} \longrightarrow (1, 2)$

Interval:

$x = \texttt{aa} \longrightarrow (-1, 3)$

Simple Searching Algorithm

The simple search algorithm works in a similar fashion to **binary search** over a sorted array, using a 'mid-point' i and choosing to search in the first or second half of the remaining list. At each i the length of the **longest common prefix** (LCP) of x and L_i is recorded as l, then one of a few things can happen:

- If $|lcp(x, L_i)| = |L_i| = |x|$, the search term has been found so i is returned.
- If $|lcp(x, L_i)| = |L_i| \neq |x|$, the search term must exist in the second half of the list.
- If $|lcp(x, L_i)| \neq |x|$ **and** the next character after the LCP is lower in L_i than it is in x, the search term must exist in the second half of the list.
- Otherwise, the search term must exist in the first half of the list.

```
fun SIMPLE_LIST_SEARCH(L, n, x, m):
    d = −1
    f = n

    while (d + 1 < f) do:
        i = floor((d + f) / 2)
        lcp = LONGEST_COMMON_PREFIX(x, L_i).length

        if (lcp == L_i.length and lcp == m):
            return i

        else if (lcp == L_i.length):
            d = i

        else if (lcp != m and L_i[lcp] < x[lcp]):
            d = i

        else:
            f = i

    return (d, f)
```

The binary search takes $O(log_2(n))$ and computing the LCP at each step takes $O(m)$, so this has a total running time of $O(m \cdot log_2(n))$.

The worst case running time can be created with $L = \{\mathtt{a}^{m-1}\mathtt{b}, \mathtt{a}^{m-1}\mathtt{c}, \mathtt{a}^{m-1}\mathtt{d}, ...\}$ and $x = \mathtt{a}^m$.

Improving Search with LCPs

Objective: reduce running time to $O(m + log_2(n))$.

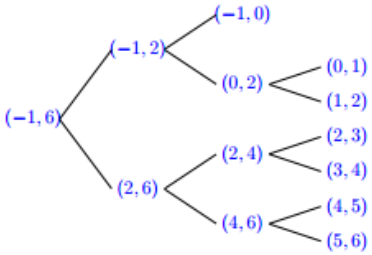
This improvement relies on $lcp(L_d, L_f)$ being known for any for (d, f) considered in the binary search. This requires $O(n)$ extra space to store the $2n + 1$ LCPs associated with the nodes in the binary search tree (see below).

The algorithm is based on properties that arise in three cases (and their symmetric cases).

Side Note: Binary Search Tree

- Nodes:
  - $n + 1$ external nodes, $(-1, 0), (0, 1), (1, 2), ..., (n - 1, n)$.
  - n internal nodes in the form (d, f) with children $(d, \lfloor (d+f)/2 \rfloor)$ and $(\lfloor (d+f)/2 \rfloor, f)$.
  - Root of $(-1, n)$.
- Size: $2n + 1$ for a list of n strings.

Example for $n = 6$:



Notation

Throughout the algorithm...

- $ld = |lcp(x, L_d)|$
- $lf = |lcp(x, L_f)|$
- $i = \lfloor (d + f)/2 \rfloor$

Case 1

Hypothesis:

- $L_d < x < L_f$
- $ld \leq |lcp(L_i, L_f)| < lf$.

Example:

L_d	aaaca	$x =$ aabbbaa
	aaacba	
L_i	aabbaba	
	aabbabb	
L_f	aabbbab	$x =$ aabbbaa

Conclusion:

- $L_i < x < L_f$
  - x is in the **second half** of the list.
- $|lcp(x, L_i)| = |lcp(L_i, L_f)|$
  - See underlined portions above.

Case 2

Hypothesis:

- $L_d < x < L_f$
- $ld \leq lf < |lcp(L_i, L_f)|$.

Example:

L_d	aaaca	$x =$ aabacb
	aaacba	
L_i	aabbaba	
	aabbabb	
L_f	aabbbab	$x =$ aabacb

Conclusion:

- $L_d < x < L_i$
  - x is in the **first half** of the list.
- $|lcp(x, L_i)| = |lcp(x, L_f)|$
  - See underlined portions above.

Case 3

Hypothesis:

- $L_d < x < L_f$
- $ld \leq lf = |lcp(L_i, L_f)|$.

Example:

L_d	aaaca	$x =$ aabbab
	aaacba	
L_i	aabbaba	
	aabbabb	
L_f	aabbbab	$x =$ aabbab

Conclusion:

- Compare x and L_i from position lf.

Improved Search Algorithm

```
fun SEARCH(L, n, x, m, lcp):
    d = −1, ld = 0
    f = n, lf = 0

    while (d + 1 < f) do:
        i = floor((d + f) / 2)

        if (ld <= lcp(i, f) < lf):        // case 1
            d = i
            ld = lcp(i, d)

        else if (ld <= lf < lcp(i, f)):   // case 2
            f = i

        else if (lf <= lcp(d, i) < ld):   // case 1 (symmetry)
            f = i
            lc = lcp(d, i)

        else if (lf < ld < lcp(d, i)):    // case 2 (symmetry)
            d = i

        else:                             // case 3
            li = max(ld, lf)
            li = li + LCP(x[li...], L_i[li...]).length

            if (li == L_i.length and li == m):
                return i

            else if (li == L_i.length):
                d = i
                ld = li

            else if (li != m and L_i[li] < x[li]):
                d = i
                ld = li

            else:
                f = i
                lf = li

    return (d, f)
```

The complexity of this algorithm is $O(m + log_2(n))$, as explained:

- Each positive comparison increases l, which can happen no more than m times.

- Each negative comparison halves the value of $f - d$, giving no more than $\lceil log_2(n+2) \rceil$ comparisons.
- After preprocessing, `LCP` can run in constant time.

Improved Interval Algorithm

TODO: Improved Interval Algorithm

Preprocessing the List

Notation: $||L|| = \sum_{i=0}^{n-1} |L_i|$ (i.e. the total length of all strings in the list).

- The list can be sorted in time $O(||L||)$ using repetitive bucket sorting.
- Computing LCPs for L_i and L_{i-1} for $0 \leq i \leq n$ can be done in time $O(||L||)$.
- Computing LCPs for other nodes inside the tree can be done in constant time:
  - Let $L_0 \leq L_1 \leq L_2 \leq ... \leq L_n$.
  - Let $-1 < d < i < f < n$.
  - $|lcp(L_d, L_f)| = min\{|lcp(L_d, L_i)|, |lcp(L_i, L_d)|\}$.

Therefore, the entire preprocessing stage can be done in time $O(||L||)$.

Regular Expressions

Regular expressions are used for **matching strings** or **substrings**. A regular expression is simply a string that is augmented with special characters that have specific meanings:

- (and) - used for grouping and/or nesting expressions.
- $*$ (Kleene star) - indicates zero or more repetitions of the preceding expression.
- $\cup$ (union) - indicates a choice.
- Two expressions side-by-side implies the concatenation of those expressions.

For example, the expression $1*011*(0 \cup 1)*111$ will match and string with the pattern 'any number of 1s, then 01, then any number of 1s, then any number of 1s and 0, then 111'.

Regular Expressions and Automata

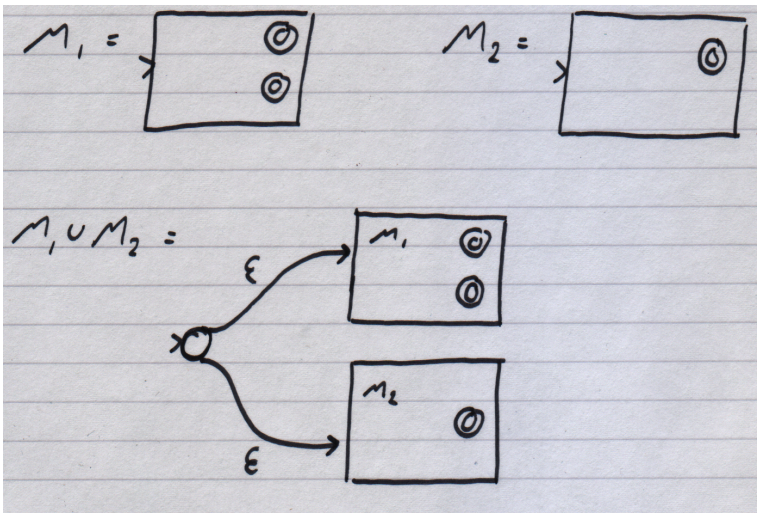
Regular expressions and automata are **equally powerful** and any one can be converted into the other (although not always easily). This is proved in the following sections, that show that every function of regular expressions can be implemented with automata.

Union of Two Automata

If M is the union of two automata, then M should accept any string that would be accepted by either of the two original automata.

Example: an automaton that accepts x when x has a substring `ab` or has a substring `bbb`.

- Create a new initial state.
- Create arcs from the new initial state to the previous initial states, labelled with ϵ.

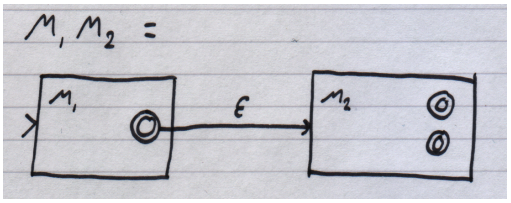


Concatenation of Two Automata

If M is the concatenation of two automata, then M should accept any string that can be broken into two strings, each of which satisfies the first and second original automata respectively.

Example: an automaton that accepts x when x is made up of a string that has `ab` as a substring, followed by another string that has `bbb` as a substring.

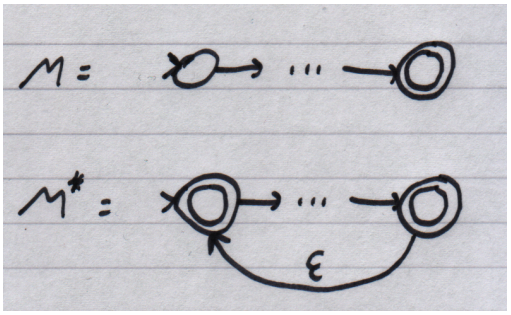
- Join the final state(s) of the first automaton to the initial state of the second automaton with an arc labelled by ϵ.



Kleene Star of an Automaton

If M is the Kleene star of an automaton, then M should accept any number of repetitions of a string that would be accepted by the original automaton.

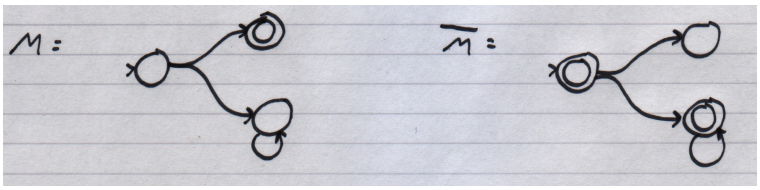
- Join the final state(s) of the automaton back to the initial state with arcs labelled by ϵ.
- Convert the initial state into a final/accepting state (to allow for zero repetitions).



Compliment of an Automata

If M is the compliment an automaton, then M should accept anything that would be rejected by the original automaton.

- Swap the type of all states, so that final states become non-final and vice versa.



Intersection of Two Automata

If M is the intersection an automaton, then M should accept anything that would be accepted by both of the original automata.

This can be derived from previous cases:

- Let M_1 and M_2 be the two original automata.
- Let $L(n)$ be the language accepted by some automaton, n.
- $L(M_1) \cap L(M_2) = \overline{\overline{L(M_1)} \cup \overline{L(M_2)}}$

Example Automata Questions

Given a finite automaton M and a string w, check $w \in L(M)$.

Solution: trace execution of the automaton, letter by letter.

Given a finite automaton M, check $L(M) = \varnothing$.

Solution: check for a path from the initial state to any final state. If such a path exists, a string exists in the language that matches it; if there is no path, $L(M)$ is empty.

Given a finite automaton M, check $L(M) = \sigma^*$.

i.e. does it accept every possible string in the language?

Solution: yes, if $\overline{L(M)} = \varnothing$.

Given two finite automata M_1 and M_2, check $L(M_1) \subseteq L(M_2)$.

Solution: yes, if $L(M_1) \cap \overline{L(M_2)} = \varnothing$.

Given two finite automata M_1 and M_2, check $L(M_1) = L(M_2)$.

Solution: yes, if $L(M_1) \subseteq L(M_2)$ and $L(M_2) \subseteq L(M_2)$.

Checking if a Language is Regular

A language regular if there exists an **automaton/regular expression that accepts it**.

The Pumping Lemma

If L is an **infinite**, regular language, then there exists strings x, y and z such that $y \neq \epsilon$ and $xy^nz \in L$ for any n.

If this lemma is violated then L cannot be an infinite regular language. Note that this only works one way: satisfying this lemma does not guarantee that L is infinite and regular.

Example 1: Proof by Contradiction

$L = \{0^i1^i \mid i > 0\}$

We assume towards contradiction that the language L is an infinite regular language, and therefore strings x, y and z should exist such that $y \neq \epsilon$ and $xy^nz \in L$.

- y could be a number of 0s.
  - This won't work, because xyz and xy^2z will have different numbers of 0s.
- y could be a number of 1s.
  - This won't work, because xyz and xy^2z will have different numbers of 1s.
- y could be a mixture of 0s and 1s.
  - This won't work, because any $n > 1$ is guaranteed to break the pattern of the language.

The pumping lemma cannot be satisfied, so therefore the assumption cannot hold.

Example 2: Proof by Contradiction

$L = \{a^iaba^i \mid i > 0\}$

We assume towards contradiction that the language L is an infinite regular language, and therefore strings x, y and z should exist such that $y \neq \epsilon$ and $xy^nz \in L$.

- y could be a^n.
  - This won't work, because different values of n will create different, non-equal quantities of a on either side.
- y could be b.
  - This won't work, because there must be only one b.
- y could be a mixture of as and bs.
  - This won't work, because there must be only one b.

The pumping lemma cannot be satisfied, so therefore the assumption cannot hold.

Example 3: Proof by Restriction and Contradiction

$L = \{ww \mid w \in \{a, b\}^*\}$

The pumping lemma could be satisfied now: $x = a$, $y = aa$, $z = a$.

We can still prove that L is not an infinite regular language by using a known regular language to **create a restriction**, then using proof by contradiction on that. This works because the intersection of two languages will only be regular if both of the original languages were.

$P = L \cap \{a^i b a^k b \mid i > 0\} = \{a^m b a^m b \mid m > 0\}$

We assume towards contradiction that the language P is an infinite regular language, and therefore strings x, y and z should exist such that $y \neq \epsilon$ and $xy^n z \in P$.

- y could be a number of as.
  - This won't work, because varying values of n will change the number of as on one side.
- y could be a number of bs.
  - This won't work, because there must be only one b on each side.
- y could be a mixture of as and bs.
  - This won't work, because there must be only one b on each side.

The pumping lemma cannot be satisfied for P, so therefore the assumption cannot hold (and therefore both P and L are not infinite regular languages).

Example 4: Proof by Restriction and Contradiction

$L = \{w\overline{w} \mid w \in \{a, b\}^*\}$

$P = L \cap \{a^k b b^l a \mid i > 0\} = \{a^m b b^m a \mid m > 0\}$

We assume towards contradiction that the language P is an infinite regular language, and therefore strings x, y and z should exist such that $y \neq \epsilon$ and $xy^n z \in P$.

- y could be a number of as.
  - This won't work, because varying values of n will change the number of as on one side.
- y could be a number of bs.
  - This won't work, because varying values of n will change the number of bs on one side.
- y could be a mixture of as and bs.
  - This won't work, because only a few carefully chosen values of n will preserve the correct format.

The pumping lemma cannot be satisfied for P, so therefore the assumption cannot hold (and therefore both P and L are not infinite regular languages).

Example 5: Proof by Restriction and Contradiction

$L = \{ww^R \mid w \in \{a, b\}^*\}$

Note: w^R is the reverse of w.

$P = L \cap \{a^k bba^l \mid i > 0\} = \{a^m bba^m \mid m > 0\}$

We assume towards contradiction that the language P is an infinite regular language, and therefore strings x, y and z should exist such that $y \neq \epsilon$ and $xy^nz \in P$.

- y could be a number of as.
  - This won't work, because varying values of n will change the number of as on one side.
- y could be b.
  - This won't work, because there must be only one b in each w.
- y could be a mixture of as and bs.
  - This won't work, because there must be only one b in each w.

The pumping lemma cannot be satisfied for P, so therefore the assumption cannot hold (and therefore both P and L are not infinite regular languages).

Example 6: Proof by Restriction and Contradiction

$L = \{w\overline{w}w \mid w \in \{a, b\}^*\}$

$P = L \cap \{a^j bba^k a^l b \mid i > 0\} = \{a^m bba^m a^m b \mid m > 0\}$

We assume towards contradiction that the language P is an infinite regular language, and therefore strings x, y and z should exist such that $y \neq \epsilon$ and $xy^nz \in P$.

- y could be a number of as.
  - This won't work, because varying values of n will change the number of as in one section.
- y could be b.
  - This won't work, because there must be only one b in each w.
- y could be a mixture of as and bs.
  - This won't work, because there must be only one b in each w.

The pumping lemma cannot be satisfied for P, so therefore the assumption cannot hold (and therefore both P and L are not infinite regular languages).

Example 7: Proof by Restriction and Contradiction

$L = \{x \in \{0,1\}^* \mid \text{equal number of 1s and 0s}\}$

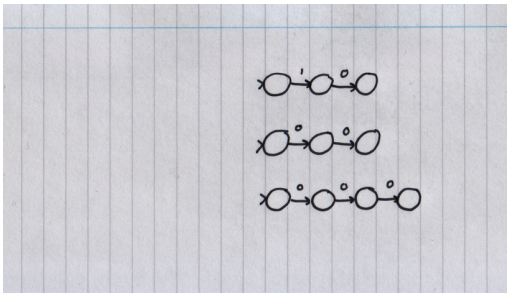
$P = L \cap \{0^j 1^k\} = \{0^m 1^m \mid m > 0\}$ which was already proved to be non-regular.

Converting Regular Expressions to Automata

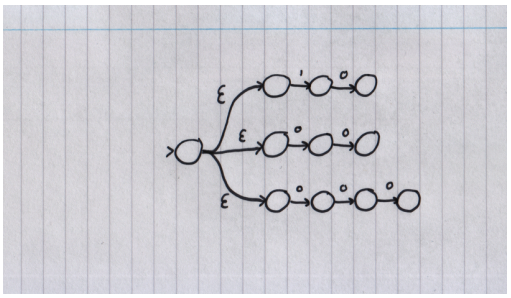
Regular expressions can be converted to automata by drawing the trivial automata for the most deeply-nested concatenated strings, then working outwards to combine automata as described earlier (*see more: Regular Expressions and Automata, page 28*).

For example, consider the construction of the automata for $101(10 \cup 00 \cup 000) * 111$.

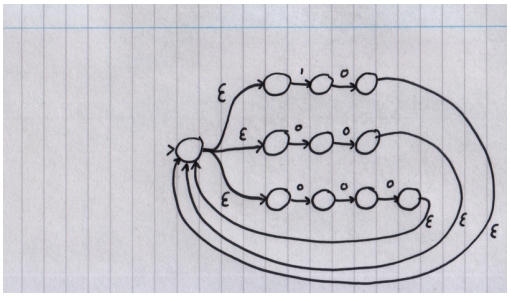
The three primitive automata for 10, 00 and 000 are drawn:



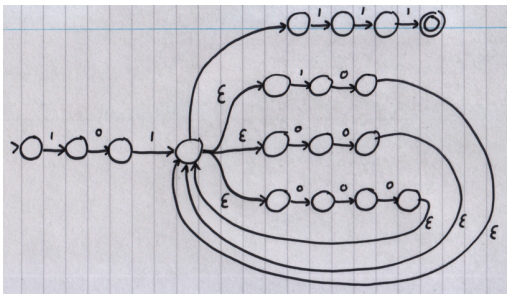
They are combined to create $10 \cup 00 \cup 000$:



They are wrapped to create $(10 \cup 00 \cup 000)*$:



101 and 111 are added as a prefix and suffix respectively:



During this process, it is valid to **compress** multiple empty ϵ transitions.

Automata to Regular Expressions

To convert automata to regular expressions, we must find **all strings that drive the automaton from the initial state to accepting state(s)**.

We use $R(i, j, k)$ to denote the set of all strings that can drive the automaton in question from the state q_i to the state q_j without **passing through** any state q_x where $x \geq k$.

For example, $R(2, 4, 2)$ is the set of all strings that can drive the automaton from q_2 to q_4, only passing through q_1.

The **base step** of computation must be done manually, to determine $R(i, j, 1)$ for all pairs of states, q_i and q_j. The equation for $k = 1$ is as follows:

$$R(i,j,1) = \begin{cases} \{\sigma \in \Sigma : \delta(q_i, \sigma) = q_j\} & i \neq j \\ \{\epsilon\} \cup \{\sigma \in \Sigma : \delta(q_i, \sigma) = q_j\} & i = j \end{cases}$$

The first term states that $R(i, j, 1)$ is the set of characters that will drive the automaton from q_i to q_j **directly** when $i \neq j$. The second term is similar but adds the empty string when $i = j$ (i.e. you can apply no characters and stay in the state).

The **inductive step** then allows the final output for $R(q_{initial}, q_{final}, n + 1)$ (where n is the number of states) to be built with the following rule:

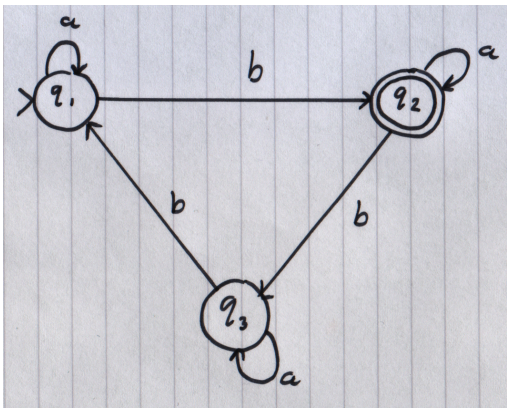
$$R(i,j,k+1) = \quad R(i,j,k) \quad \cup \quad R(i,k,k)R(k,k,k)^*R(k,j,k)$$

This states that to drive the automaton from q_i to q_j without passing through a state q_x where $x \geq k + 1$, there are two options:

- Go from q_i to q_j without passing through any state q_x where $x \geq k$.

- Go...
  - ...from q_i to q_k,
  - then from q_k to q_k zero or more times,
  - then from q_k to q_j,
  - all without passing through any state any state q_x where $x \geq k$.

For example:



$R(1,1,1) = a \cup \epsilon$ $R(1,2,1) = b$ $R(1,3,1) = \varnothing$

$R(2,1,1) = \varnothing$ $R(2,2,1) = a \cup \epsilon$ $R(2,3,1) = b$

$R(3,1,1) = b$ $R(3,2,1) = \varnothing$ $R(3,3,1) = a \cup \epsilon$

We want to get from q_1 to q_2, so...

$$\begin{aligned} R(1,2,2) &= R(1,2,1) \cup R(1,1,1)R(1,1,1)^*R(1,2,1) \\ &= b \cup (a \cup \epsilon)(a \cup \epsilon)^*b \end{aligned}$$

The final result, $R(1,2,4)$, would take several more rounds of computation.

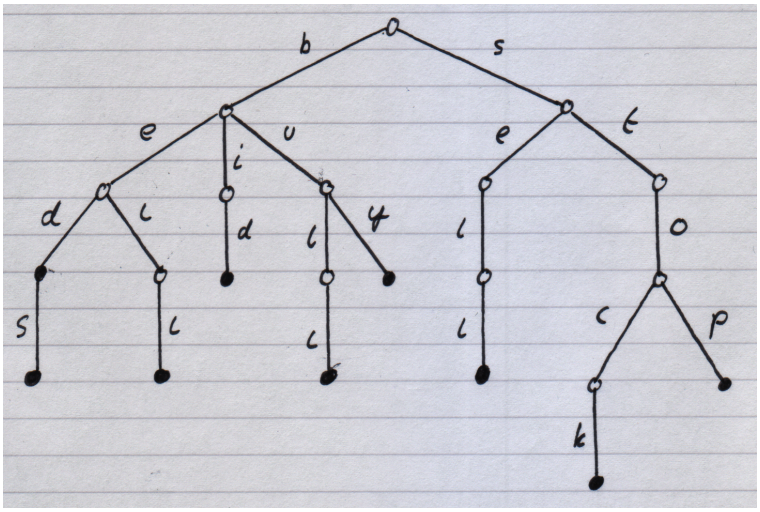
Tries

A trie (pronounced 'try') is a simple structure for storing a **compact representation** of a **set of strings**. It is a rooted tree with the following properties:

- Edges are labelled with letters from the alphabet, Σ.
- At each node, all outgoing edges are uniquely labelled.
- **Terminating nodes** are marked, and represent the end of a string formed by concatenating the letters on the path from the root to that node.

The trie that represents all of the strings in the set R is denoted $trie(R)$. For example:

$$R = \{\texttt{bed}, \texttt{beds}, \texttt{bell}, \texttt{bid}, \texttt{buy}, \texttt{bull}, \texttt{sell}, \texttt{stock}, \texttt{stop}\}$$



Using a trie to check whether a string exists in the set of strings is easy: start at the root, and follow the edges letter by letter.

- If you finish on a terminating node, the string is in the set.
- If you finish on a non-terminating node, the string is a prefix of at least one word in the set.
- If you 'fall out of the trie', the string is not in the set and is not a prefix.

Suffix Trees

Suffix trees provide fast exact pattern matching on a text T.

Notation

- $T = T[0..n-1]$ is the text (a string with zero-indexed letters).
- For any single integer $i \in [0..n]$, we define T_i as the suffix $T[i..n-1]$.
  - T_i is the string with the first i letters removed.
  - T_0 is the entire string.
  - T_{n-1} is the last letter.
  - T_n is the empty string ϵ.
- For any set of integers $c \subseteq [0..n]$, we define $T_C = \{T_i | i \in c\}$.
  - $T_{\{0\}}$ is the set containing the entire string.
  - $T_{\{0,1,2\}}$ is the set of the entire string, plus the suffix with the first letter missing, and the suffix with the first two letters missing.
  - $T_{[0..n]}$ is the **set of all suffixes**, including the empty string ϵ.

Properties

A suffix trie is a **compact trie**[1] for the set $T_{[0..n]}$ (i.e. **the set of all suffixes**).

We assume that an extra letter is added to the end of the text. This letter **cannot appear in the alphabet**. `$` is typically used, such that $T[n] =$ `$` and the empty string no longer appears in $T_{[0..n]}$ (it is replaced by the string `$`). This has two consequences:

- No suffix is the prefix of another suffix.
- All nodes in the trie representing suffixes are leaves.

For example:

$T =$ `banana`
$T_{[0..n]} = \{$`banana$`, `anana$`, `nana$`, `ana$`, `na$`, `a$`, `$`$\}$

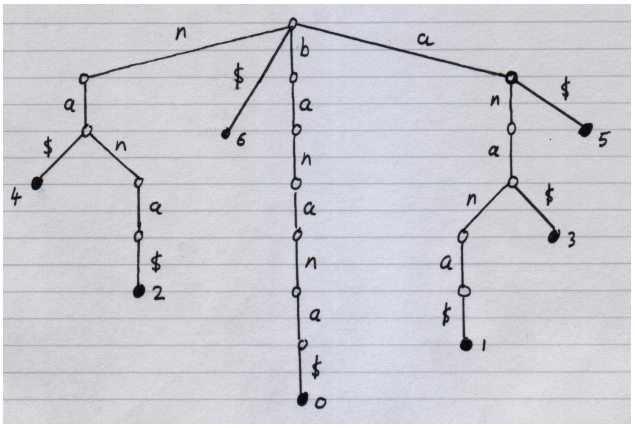
The set $T_{[0..n]}$ contains $|T_{[0..n]}| = n + 1$ strings with a total length of $||T_{[0..n]}|| = \Theta(n^2)$.

[1] Path compression converts non-branching path segments into single edges. In the example on page 37, the path for `sell` could be compressed into the edges {`s`, `ell`}.

Suffix Tries

A basic suffix trie uses $O(n^2)$ nodes to store the suffixes for most texts and the brute force approach for its construction takes $O(n^2)$ time.

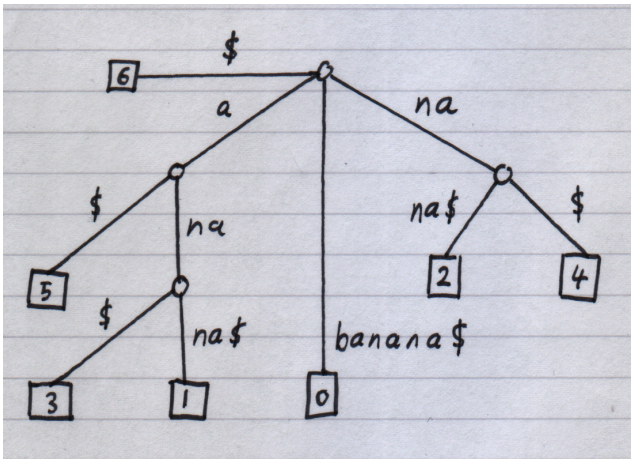
The suffix trie is constructed by adding suffixes one at a time to a standard trie, starting with the largest.



Suffix Trees

Compressing the **non-branching edges** of a suffix trie produces a suffix tree, which will use only $O(n)$ space:

- We assume that the size of the alphabet is a constant.
- There will be exactly $n + 1$ leaves and at most n internal nodes.
- There will be at most $2n$ edges (basic tree properties).
- Edge labels are substrings of the text and can be represented by two integers (starting position and end position or substring length).

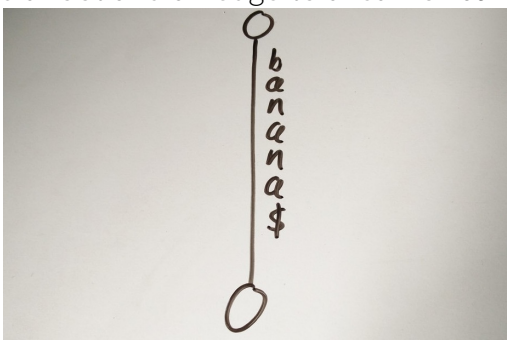


This type of suffix tree can be constructed systematically, without creating the $O(n^2)$ version first:

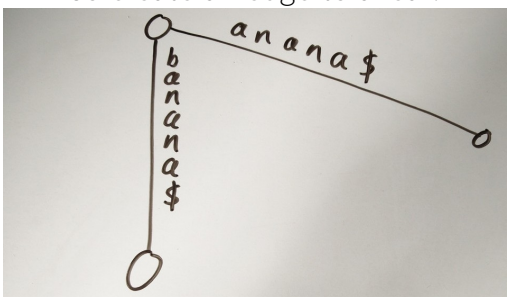
- Create a root node.
- Start with the longest suffix (the entire word) and create one edge from the root to represent it.
- For each remaining suffix, longest first:
  - Use the existing edges to start spelling the suffix.
  - **Create a new edge to a leaf** if no edge exists to keep spelling the suffix. Use the rest of the suffix to label this edge.
  - **Split an existing edge** if the suffix only matches part of it, then create a new edge to a leaf as before.

Construction for `banana$` is shown below (note: the edges are labelled with substrings for clarity, instead of start/end indexes).

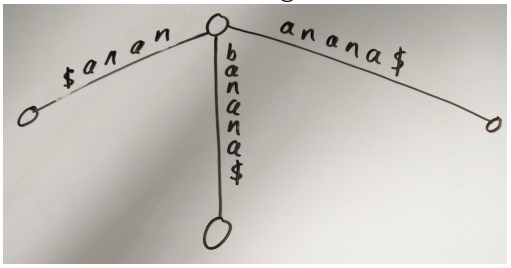
Create a root and an edge to a leaf for `banana$`:



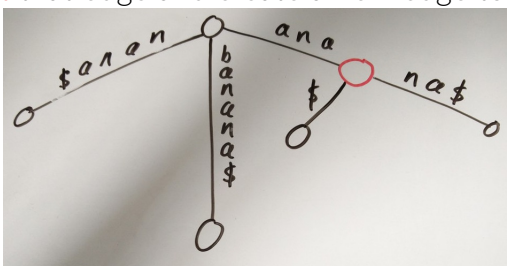
There's no path from the root for `anana$`, so create an edge to a leaf:



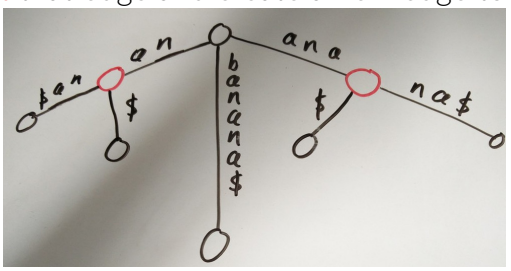
There's no path from the root for `nana$`, so create an edge to a leaf:



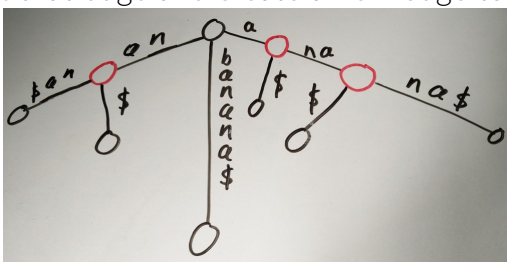
The path for `ana$` matches the start of `anana$`,
so split that edge and create a new edge to a leaf:



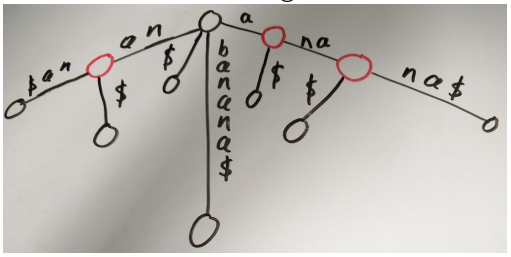
The path for `na$` matches the start of `nana$`,
so split that edge and create a new edge to a leaf:



The path for `a$` matches the start of `ana`,
so split that edge and create a new edge to a leaf:



There's no path from the root for `$`,
so create an edge to a leaf:



String Matching

Given a suffix trie/tree for T, all occurrences of a pattern P can be found in $O(|P| + occ)$ where occ is the number of occurrences. Searching is based on the following observation:

A pattern P has an occurrence in T at position i if and only if P is a prefix of T_i. Example:

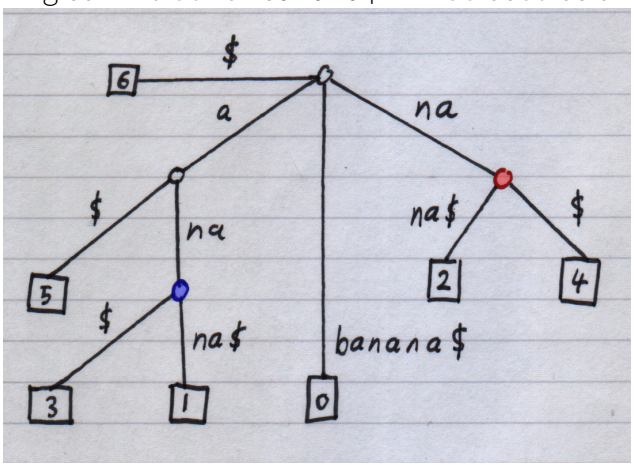
$T =$ `horizon`
$P =$ `riz`
$T_2 =$ `rizon`
P is a prefix of T_2, therefore P occurs in T starting at $i = 2$.

Therefore, we can find all occurrences of P in T by a **prefix search** in $T_{[0..n]}$:

- From the root, follow edges that spell out P.
- If you 'fall out' of the suffix tree, P does not occur in T.
- If you finish inside the tree...
  - The number of leaves in the sub-tree below the finishing point gives the number of occurrences.
  - The suffix numbers of all leaves below the finishing point give the starting positions of all occurences of P in T.
  - If there is an outgoing edge with the label `$`, P is a suffix of T.

Suffix Links

The following suffix tree for `banana$` will be used as an example:



There are three types of node in a suffix tree: the **root** node, **internal** nodes, and **leaf** nodes. There are two important properties of these node types:

- Internal nodes always have more than one outgoing edge, which means they mark the points where **branching** occurs.
- Branching occurs whenever a **repeated string** is involved, and only then. For any internal node u, the string leading from the root to u must appear in the text at least as many times as u's branching factor (its number of outgoing edges).
  - For example, the blue node in the example spells `ana` from the root. It has a branching factor of 2, so therefore `ana` must appear in the text `banana$` at least twice.

These properties lead to the definition of suffix links:

- If the string S_u leading from the root to u is longer than 1 character, the same string minus it's first character (call it S_v) must be in the tree too.
  - $S_u =$ `ana`, so $S_v =$ `na`. We can see that `na` appears in the tree.
- If some string S_u leads to an internal node u, it's shorter version S_v must also lead to an internal node. Why? S_u must appear at least twice in the text (because u is a branching node), so S_v must also appear at least twice (because it's a part of S_u; wherever S_u appears, S_v also appears). Because S_v appears more than once, it must have an internal node.
  - We can see that an internal node (in red) for $S_v =$ `na` does appear in the tree.
- In this situation, a suffix link connects u to v.

Suffix Tree Construction with Suffix Links

Suffix links are the key to **efficient suffix tree construction**. They can be used as 'shortcuts' during construction so that each iteration does not necessarily have to start again from the root.

Manual, informal construction of a suffix tree using suffix links can be done as follows:

- Create a root.
- Insert each suffix, longest first, with the following logic:
  - Determine the starting position.
    - If $slink$ was set for the previous suffix, start there.
    - Otherwise, start from the root.
  - Let d be the depth of your starting position (i.e. the length of the string from root to your start).
  - Skip d letters from the start of the suffix you are inserting.
  - Insert the rest of your suffix as before, working from your starting position.
    - This may involve creating a new branching point.
  - After the suffix is inserted, go back to the branching point.
  - Determine the suffix link that would be created from the branching point.
    - Use the S_u and S_v definition from earlier.
  - If the suffix link goes to somewhere other than the root, add it to the suffix tree.
  - Rinse and repeat for the next suffix.

Algoritmic Approach

Algorithms exist that take advantage of suffix links for efficient suffix tree construction, including **McCreight's** algorithm and **Ukkonen's** algorithm, both of which run in $O(n)$.