ENPM667 - Project 1: Unscented Kalman Filter

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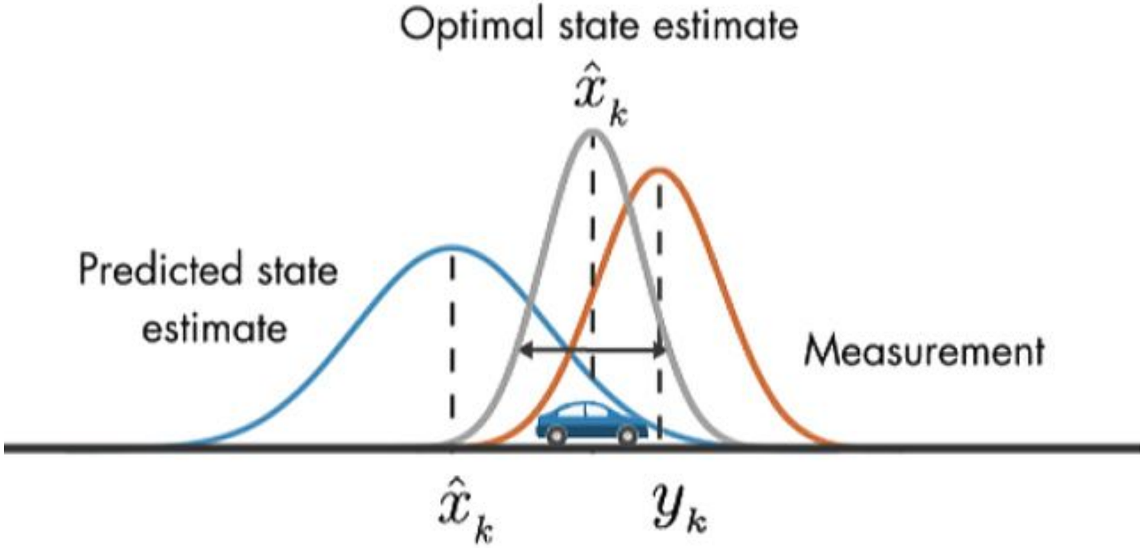
Introduction

- A Kalman filter is a method used to estimate information using noisy input data.
- Useful for applications (such as robotics) involving environments with disturbances, or sensors with error.
- All consist of a state variable x and a covariance matrix P.
- All consist of prediction and measurement steps, in which x and P are updated based on current state vector and uncertainty.
- The state vector x contains information that we wish to track, such as location, angle, mass, fuel, etc.

Introduction (continued)

- The covariance matrix P indicates how sure we are of the true value of the state vector. The larger the uncertainty, the more the filter relies on new measurements.
- When the state vector is updated, uncertainty always increases. When new measurements are retrieved, uncertainty always decreases.



Organization

- Linear Kalman filter
- Extended Kalman filter
  - Complex Step Derivative
- Unscented Kalman filter
- Results from paper
  - Figure 1a and 1b
  - Figure 9
- References

Linear Kalman Filter

- State transition function A and measurement function H
- Covariance matrices Q for process noise (e.g. environmental disturbances) and R for measurement noise (e.g. sensor error)
- Linear relationship between state variables

Prediction

$$\boldsymbol{x}_n^- = \boldsymbol{A}x_{n-1}$$
$$\boldsymbol{P}_n^- = \boldsymbol{A}\boldsymbol{P}_{n-1}\boldsymbol{A}^T + \boldsymbol{Q}$$

Measurement

$$\boldsymbol{K}_n = \boldsymbol{P}_n^- \boldsymbol{H}^T (\boldsymbol{H}\boldsymbol{P}_n^- \boldsymbol{H}^T + \boldsymbol{R})^{-1}$$
$$\boldsymbol{x}_n = \boldsymbol{x}_n^- + \boldsymbol{K}_n (z_n - H\boldsymbol{x}_n^-)$$
$$\boldsymbol{P}_n = (\boldsymbol{I} - \boldsymbol{K}_n \boldsymbol{H})\boldsymbol{P}_n^-$$

Extended Kalman Filter

- Uses function vectors in place of state transition matrix and measurement matrix. Works well for nonlinear systems with Gaussian noise.
- Matrices are defined as follows:

$$\boldsymbol{A}_{ij} = \frac{\partial f_i}{\partial x_j}(\boldsymbol{x}_{n-1}) \qquad \boldsymbol{W}_{ij} = \frac{\partial f_i}{\partial w_j}(\boldsymbol{x}_{n-1}) \qquad \boldsymbol{H}_{ij} = \frac{\partial h_i}{\partial x_j}(\boldsymbol{x}_n) \qquad \boldsymbol{V}_{ij} = \frac{\partial h_i}{\partial v_j}(\boldsymbol{x}_n)$$

Prediction

$$\boldsymbol{x}_n^- = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

$$\boldsymbol{P}_n^- = \boldsymbol{A}_n \boldsymbol{P}_{n-1} \boldsymbol{A}_n^T + \boldsymbol{W}_n \boldsymbol{Q}_{n-1} \boldsymbol{W}_n^T$$

Measurement

$$\boldsymbol{K}_n = \boldsymbol{P}_n^- \boldsymbol{H}_n^T (\boldsymbol{H}_n \boldsymbol{P}_n \boldsymbol{H}_n^T + \boldsymbol{V}_n \boldsymbol{R}_n \boldsymbol{V}_n^T)^{-1}$$

$$\boldsymbol{x}_n = \boldsymbol{x}_n^- + \boldsymbol{K}_n [z_n - h(\boldsymbol{x}_n^-)]$$

$$\boldsymbol{P}_n = (\boldsymbol{I} - \boldsymbol{K}_n \boldsymbol{H}_n) \boldsymbol{P}_n^-$$

Unscented Kalman Filter

- The unscented Kalman filter computes sigma points from the mean and standard deviation of the data.
- The unscented transformation is the actual conversion used to change from one coordinate system to another.
- The variables are assigned weights for transforming the mean and covariance.

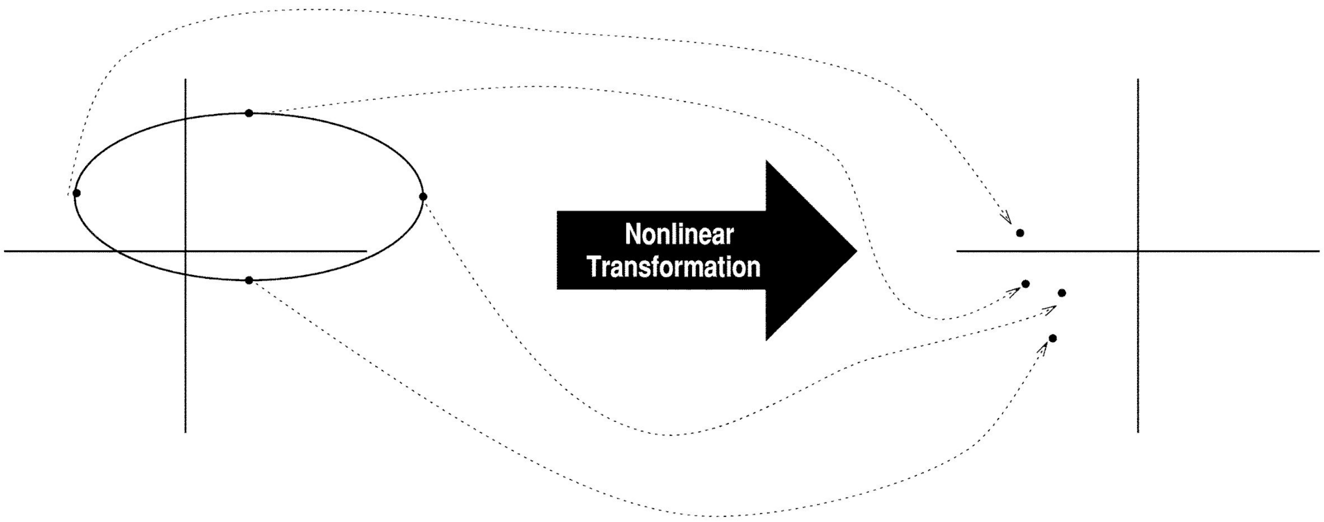
Unscented Kalman Filter

Calculation steps:

1) Creating the sigma points through a sigma point selection algorithm.
2) Calculating the transformed set:
   $\hat{x}_{a,n} = f[x_{a,n}]$
3) Predicting the mean:
   $\hat{\mu}_{a,n} = \sum_{i=0}^{p} W^{(i)} \hat{x}_{a,n}^{(i)}$
4) Predicting the covariance:
   $\hat{K}_{a,n} = \sum_{i=0}^{p} W^{(i)} (\hat{x}_{a,n}^{(i)} - \hat{\mu}_{a,n})(\hat{x}_{a,n}^{(i)} - \hat{\mu}_{a,n})^T$
5) Instantiating prediction points:
   $y_n^{(i)} = g[x_{a,n}^{(i)}]$
6) Predicting the observation:
   $y_n = g[y_n^{(i)}]$
7) Calculating the innovation covariance:
   $\hat{S}_n = \sum_{i=0}^{p} W^{(i)} (\hat{y}_n^{(i)} - \hat{y}_n)(\hat{y}_n^{(i)} - \hat{y}_n)^T$
8) Calculating the cross covariance matrix:
   $\hat{K}_n^{xy} = \sum_{i=0}^{p} W^{(i)} (\hat{x}_n^{(i)} - \hat{\mu}_n)(\hat{x}_n^{(i)} - \hat{\mu}_n)^T$
9) The update is performed as per the typical Kalman filter operation:
   $\mu_n = \hat{\mu}_n + W_n \nu_n$
   $K_n = \hat{K}_n - W_n \hat{S}_n W_n^T$
   $\nu_n = y_n - \hat{y}_n$
   $W_n = \hat{K}_n^{xy} \hat{S}_n^{-1}$

Unscented Kalman filter

Sigma points are transformed by the unscented transformation to the new coordinate system, along with the data.



Complex Step Derivative

- We used the complex step method to calculate the Jacobian matrices used in the extended Kalman filter.
- This method was used to avoid "catastrophic cancellation" resulting from subtracting extremely small values (≅1e-7).
- Derivative is normally approximated by:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Complex Step Derivative (continued)

We used the following calculations:

1. $$\frac{\partial f}{\partial x} = \frac{\Im[f(x+ih)]}{h}$$

2. $$f(x+ih) = f(x) + ihf'(x) - \frac{h^2 f''(x)}{2!} - \frac{ih^3 f'''(x)}{3!} + \cdots$$

3. $$\begin{aligned} f'(x) &= \frac{\Im[f(x+ih)]}{h} + \frac{h^2 f'''(x)}{3!} + \frac{h^4 f^{(5)}(x)}{5!} + \cdots \\ &= \frac{\Im[f(x+ih)]}{h} + \mathcal{O}(h^2) \\ &\approx \frac{\Im[f(x+ih)]}{h}. \end{aligned}$$

Figure 1a and 1b

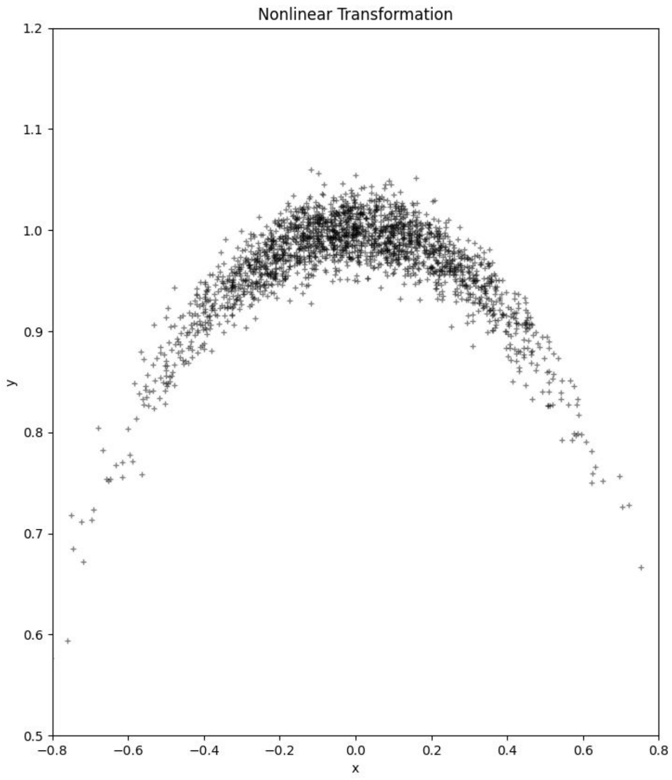
- Target located at (x, y) coordinates (1, 0)
- Converted into polar coordinates using the following:

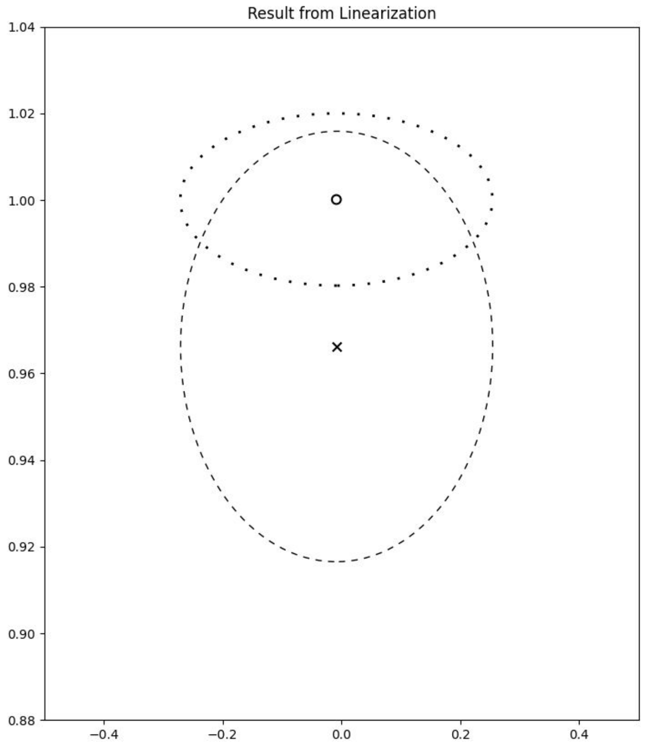
$$x = r\cos\theta$$

$$y = r\sin\theta$$

- Left side (1a) shows plotted data as y versus x
- Right side (1b) shows plots containing the statistical information contained within the data.
- The x symbol indicates the true mean of the data, and the dashed ellipse is the true covariance ellipse.
- The linearized sensor data has a mean indicated by the o symbol, and the dotted ellipse is the corresponding covariance ellipse.

Figure 1a and 1b





Our sinusoidal example

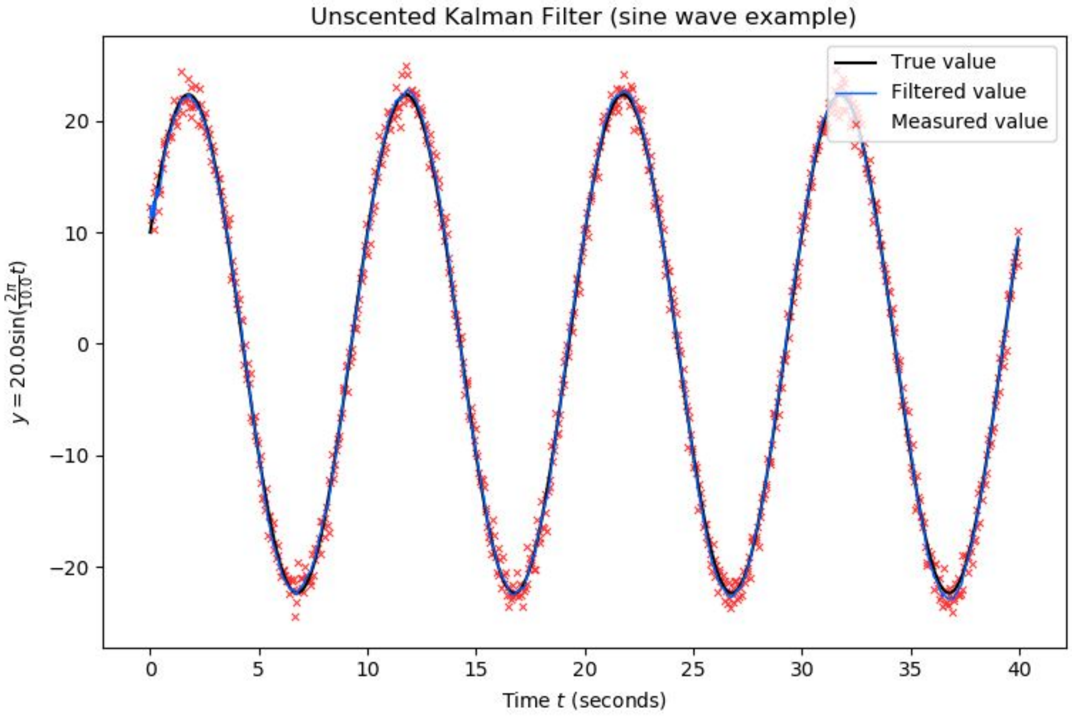
- Equations of motion are as follows:

$$x(t) = \alpha \sin \omega t + \beta \cos \omega t$$

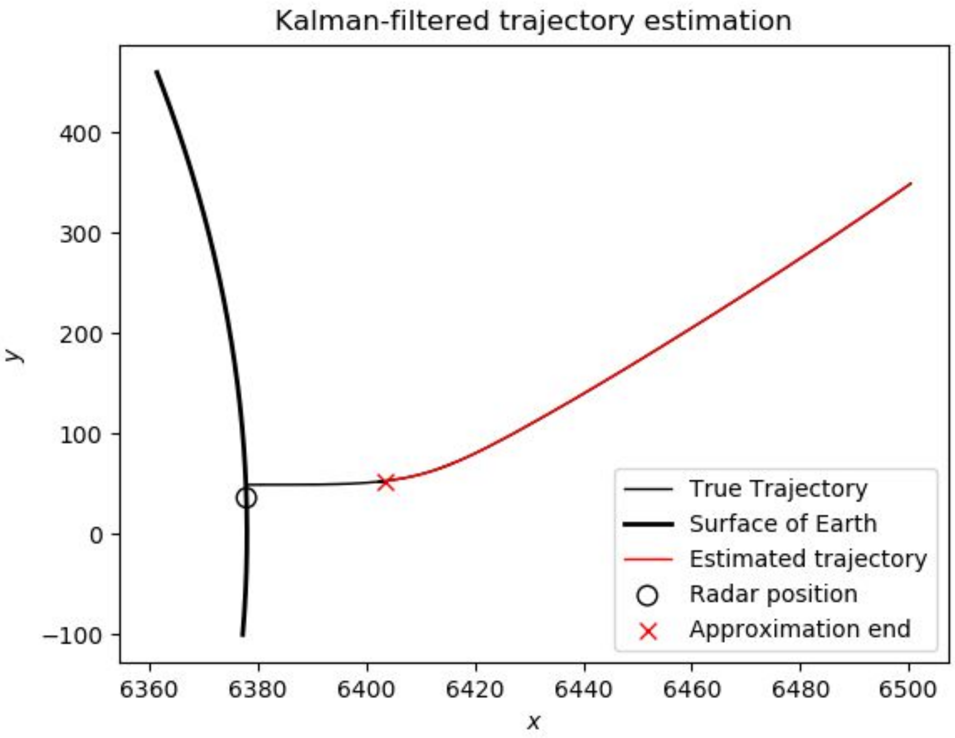
$$\frac{d}{dt}x(t) = \dot{x}(t) = \alpha\omega \cos \omega t - \beta\omega \sin \omega t$$

$$\frac{d^2}{dt^2}x(t) = \ddot{x}(t) = -\alpha\omega^2 \sin \omega t - \beta\omega^2 \cos \omega t$$

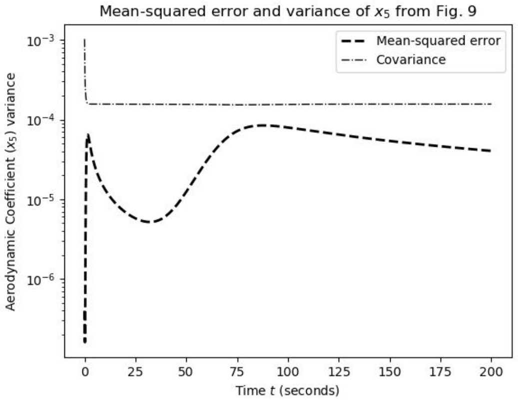
Our sinusoidal example (continued)

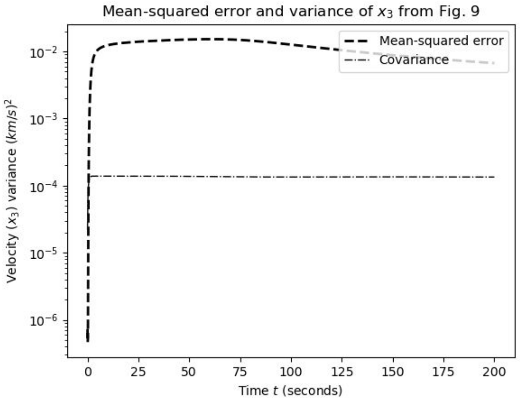


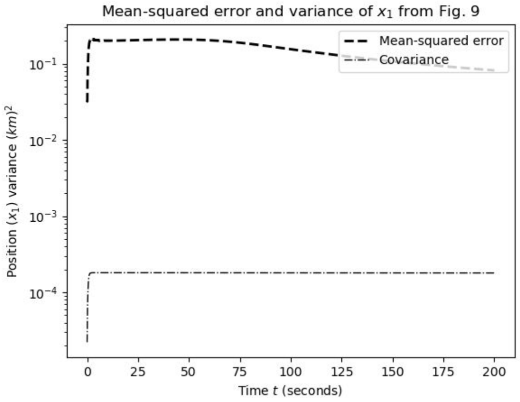
Space trajectory example



Space trajectory example







Future Work

- Separation of the prediction and measurement stages
    - Useful for a sensor with infrequent measurements
    - Also useful for a system with inputs from multiple sensors arriving simultaneously
- Replicating more results of unscented transform from original work
- Extensive direct comparison of the Kalman filters
- Theoretical and practical examples utilizing different Kalman filters

References

- S. J. Julier, and J. K. Uhlmann, ”New extension of the Kalman filter to nonlinear systems,” Proc. SPIE, vol. 3068, pp. 182-193, July 1997, doi: 10.1117/12.280797.
- J. R. R. A. Martins, P. Sturdza, and J. J. Alonso, ”The complexstep derivative approximation,” ACM Trans. Math. Softw., vol. 29, no. 3, pp. 245-262, Sep. 2003, doi: 10.1145/838250.838251.
- G. Welch, and G. Bishop, ”An introduction to the Kalman filter,” Dept. Comp. Sci., Univ. of North Carolina at Chapel Hill, Chapel Hill, NC, USA, Rep. TR 95-041, 2006.